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"Achievable Outcomes in Smooth Dynamic Contribution Games"

by

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Abstract

This paper studies a class of dynamic voluntary contribution games in a setting with discounting and neoclassical payoffs (differentiable, strictly concave in the public good, and quasilinear in the private good). An *achievable profile* is the limit point of a subgame perfect equilibrium path – the ultimate cumulative contribution vector of the players. A profile is shown to be achievable only if it is in the *undercore* of the underlying coalitional game, i.e., the profile cannot be blocked by a coalition using a component-wise smaller profile. Conversely, if free-riding incentives are strong enough that contributing zero is a dominant strategy in the stage games, then any undercore profile is the limit of achievable profiles as the period length shrinks. Thus, in this case when the period length is very short, (i) the set of achievable contributions does not depend on whether the players can move simultaneously or only in a round-robin fashion; (ii) an efficient profile can be approximately achieved if and only if it is in the core of the underlying coalitional game; and (iii) any achievable profile can be achieved almost instantly.

KEYWORDS: *dynamic games, monotone games, core, public goods, voluntary contribution, gradualism*

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1. Introduction

Define a "dynamic voluntary contribution game" to be a multistage game in which players contribute amounts of a private good to a public project in multiple periods. The contributions are utilized by the project to produce future public benefits. Familiar examples include a fund drive, or a never-ending sequence of fund drives, to finance university buildings, public radio programs, or a presidential campaign. Contributions may take the form of effort or produced inputs, such as the program modules contributed to an open source software project.

Being able to contribute in multiple periods may alleviate the free-rider problem. For example, in the settings considered in Marx and Matthews (2000), equilibria with positive contributions exist if and only if the number of periods in which the players can contribute is sufficiently large. Some of these equilibria achieve efficient outcomes in the limit as the discount factor goes to one. The logic of the result is simple: a player is induced to contribute in early periods because doing so induces others to contribute in future periods. The amount a player contributes in a period must be small so that the others will want to contribute later, which implies that contributions must be made piecemeal over time. The necessity of such "strategic gradualism" has been demonstrated in several related papers, most generally in Compte and Jehiel (2004).

Most of the literature on dynamic contribution focuses on technologies with "threshold provision points," which are aggregate contribution levels at which the produced public good discontinuously increases. (The typical example is a binary project like the building of a bridge.) Because of the discontinuity, once the cumulative contribution is sufficiently close to the final threshold, each player's best reply is to contribute enough to achieve the threshold. Accordingly, backwards induction arguments can be used to characterize equilibria, and equilibrium contributions are raised in only a finite number of periods. This roughly describes much of Admati and Perry (1991), Gale (1995), Marx and Matthews (2000), Compte and Jehiel (2003), Choi, Gale, and Kariv (2006), Yildirim (2006), and Duffy, Ochs, and Vesterlund (2007).

Thresholds are absent, however, in many settings. This is true of a public project with a neoclassical production function, i.e., one that is strictly increasing and concave. With an infinite contributing horizon and no threshold, backwards induction cannot be used. Only a few studies of no-threshold games have appeared. Using a discounting payoff criterion, Marx and Matthews (2000), Lockwood and Thomas (2002), and Pitchford and Snyder (2004) study rather special cases of no-threshold games, showing the existence of equilibria in which contributions are made infinitely often, and which are approximately efficient if the discount factor is close to one. Gale (2001) shows that such games without discounting have fully efficient equilibria.

This paper provides a more complete study of no-threshold, infinite-horizon contribution games with discounting. The goal is to characterize the nature of all equilibrium contribution profiles and payoffs, with an emphasis on distributional as well as efficiency aspects.

Overview of Results

In each period of the games studied in this paper, some players can contribute private good to a public project. Every player is able to move infinitely often, but not necessarily each period. Contributions are irreversible, do not depreciate, and are publicly observed. The project uses the sum of all past contributions to produce a flow of public good. Each player maximizes the discounted sum of her stage-game payoffs. For simplicity and to facilitate comparisons, each player's stage-game payoff function is assumed to be differentiable, quasilinear in the player's private good, and strictly concave in the sum of past contributions currently used to provide public good. The concavity eliminates thresholds.

Attention is restricted to pure strategy subgame perfect equilibria. Any equilibrium generates a nondecreasing path of cumulative contribution profiles. A contribution profile is *achievable* if it is the limit point of an equilibrium path. A profile is efficient if it is Pareto optimal for the stage-game payoff functions. Thus, an outcome path is efficient if and only if it achieves an efficient profile immediately, in the first period possible. We shall see that discounting precludes efficiency. The goal is to characterize the set of achievable profiles and equilibrium payoffs, especially for large discount factors.

The characterizations are in terms of an underlying coalitional game. The notion of "blocking" in this game reflects two features of the dynamic game. First, since every profile on an equilibrium path of the dynamic game must lie below the contribution profile the path achieves, a coalition should only be able to use a smaller profile to block the achieved profile. I thus define a profile to be *underblocked* by a coalition if there exists a smaller profile that each coalition member prefers, and which prescribes zero contributions for the nonmembers. Second, since a player in the dynamic game can raise her contribution any amount whenever she is able to move, blocking is defined using the payoffs the players can obtain by unilaterally raising their contribution from the blocking profile.

The *undercore* is the set of profiles that are not underblocked. Thus, an undercore profile does not require any coalition to contribute a disproportionately large amount. The undercore is typically a strict subset of the individually rational profiles, if there are more than two players. The undercore contains the familiar core, the set of profiles that are not blocked by the usual definition. Indeed, the core is precisely the set of efficient profiles in the undercore.

The first main result is that all achievable profiles are in the undercore. Thus, no coalition can be induced to contribute too much. If the limit of a sequence of achievable profiles is an efficient profile, it must be in the core. Since the undercore (core) is typically a strict subset of the individually rational (and efficient) profiles, this is an "anti-folk-theorem" result.[1]

The second main result is a partial converse of the first: almost any contribution profile in the undercore is achievable if the period length is short enough. Moreover, there is a fixed sequence of profiles converging to the given undercore profile that is an equilibrium path for all small period lengths. This result is obtained under two further assumptions. The first is a weak cyclicity assumption on the move structure satisfied, e.g., by the simultaneous and round-robin structures. The second additional assumption is that the payoffs satisfy the prisoners' dilemma property that in any stage game, not contributing more is a dominant strategy for each player. This is the case in which free riding incentives are strongest.

When both results obtain, the set of achievable contribution profiles converges to the undercore as the period length goes to zero. An efficient profile can be attained in the limit if and only if it is in the core. This is true regardless of whether players can contribute simultaneously each period, or only in a round robin fashion.

The nature of the profile an equilibrium path achieves is, of course, unimportant for payoffs when the convergence is very slow. However, since an achievable profile can be achieved by the same equilibrium path for all small period lengths, the real time required to get close to the achieved profile is negligible when the period length is very small. Thus, in the limit any achievable profile can be achieved in a "twinkling of the eye". Although strategic gradualism is necessary (if the achieved profile is non-autarchic) in the sense that the convergence must be asymptotic, it does not necessarily generate significant inefficiency if the period length is very short. An efficient payoff is the limit of equilibrium payoffs if and only if it is a payoff generated by a profile in the core.

Relationship to the Literature

The dynamic voluntary contribution games of this paper correspond to the "monotone games with positive spillovers" of Gale (2001). The main difference is that a payoff in the latter is not a discounted sum of the stage-game payoffs, but is instead their limit. The stage-game payoff functions in Gale (2001) are more general than those of this paper, being defined on Euclidian spaces of arbitrary dimension and assumed only to be continuous, exhibit positive spillovers,

[1] The folk theorem of Dutta (1995) for stochastic games does not apply here because its "asymptotic state independence" assumptions, (A1) and (A2), are not satisfied.

and satisfy a boundedness property. The main result is that any "strongly minimal positive satiation point" is, in this paper's terminology, achievable. The demonstration in this paper that almost any undercore profile is achievable if the period length is short enough extends Gale's no-discounting result to a class of games with discounting.

As in this paper, Lockwood and Thomas (2002) also consider dynamic contribution games with discounting and no thresholds. They restrict attention to two-player games, with symmetric payoff functions that exhibit the prisoners' dilemma property that each player's payoff decreases in her own contribution. When the payoffs are differentiable, the profile achieved by the most efficient symmetric equilibrium is shown to be inefficient, and to achieve an inefficiently small profile. In this paper the analogous result is obtained for any equilibrium. As the discount factor goes to one, Lockwood and Thomas (2002) show that the most efficient symmetric equilibrium converges to an efficient equilibrium. Pitchford and Snyder (2004) obtain a similar result. These asymptotic results foreshadow the sufficiency result of this paper, that any undercore (and hence core) profile is achievable in the limit as the period length shrinks to zero.

Lockwood and Thomas (2002) also study their model with "linear kinked payoffs" as in Marx and Matthews (2000). The related result they obtain in this case is that the most efficient symmetric equilibrium payoff of the simultaneous move game can be attained also in the alternating move game, in the limit as discounting is taken to zero. This result about the irrelevancy of the move structure foreshadows that of this paper.

Lastly, Bagnoli and Lipman (1989) study the connection between the core of a public goods coalitional game and the equilibrium outcomes of a dynamic game. The public good is discrete, which creates a sequence of thresholds. The main result is the design of a dynamic game, without discounting, that fully implements the core, given an equilibrium refinement criterion combining trembling-hand perfection with successive elimination of dominated strategies. The designed game differs from the usual voluntary contribution game by having a "contribution collector" who proportionately refunds the contribution amounts each period that exceed the largest threshold point that has been reached, and who stops the game and produces the public good once the new contributions in some period are too small to reach the next threshold.

Organization

The games to be studied are described in Section 2, and preliminary results in Section 3. Section 4 is devoted to the coalitional game. Sections 5 and 6 contain the main results, necessary and sufficient conditions for a profile to be achievable. Summarizing theorems are in Section 7, and conclusions in Section 8. Appendices A–E contain proofs.

2. Dynamic Contribution Games

This section contains descriptions of the games to be considered, scenarios from which they arise, and lastly the smoothness assumptions.

Game Description

The set of players is $N = \{1, \ldots, n\}$, with $n \geq 2$. In each stage game, player i will choose a *(cumulative) contribution,* $x_i \in \mathbb{R}_+$. Given a contribution *profile* $x \in \mathbb{R}^n_+$, the *aggregate contribution* is $X = \sum_{i \in N} x_i$. The aggregate of all but player i is $X_{-i} = X - x_i$.

In each period $t = 1, 2, \ldots,$ the players choose a profile $x^t = (x^t_1, \ldots, x^t_n)$. A *path* is a sequence $\vec{x} = \{x^t\}^\infty_{t=0}$ of profiles that starts with $x^0 = (0, \ldots, 0)$. A path generates for player i the payoff

$$U_i(\vec{x}, \delta) := (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(x^t), \tag{1}$$

where $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ is the stage-game payoff function and $\delta \in (0, 1)$ is the discount factor. The stage-game payoffs take the form

$$u_i(x) = v_i(X) - x_i. \tag{2}$$

Each *valuation function* v_i satisfies $v_i(0) = 0$, and other assumptions made below.

The game is monotone in the sense that $x^t \geq x^{t-1}$ for each $t \geq 0$ is required.[2] The *move structure* is a sequence $\vec{N} = \{N_t\}^\infty_{t=1}$ of nonempty subsets of players. Only players in N_t can raise their contributions in period t. Thus, $x^t_i = x^{t-1}_i$ for all $i \notin N_t$ is also required. A *feasible* path is one that satisfies these two requirements.

Each player is able to move infinitely often: $\cup_{\tau \geq t} N_\tau = N$ for all $t \geq 1$. Past contributions are publicly observed. The extensive form game thus defined is denoted $\Gamma(\delta, \vec{N})$.

Contribution Scenarios

The game $\Gamma(\delta, \vec{N})$ arises from at least two simple contribution scenarios.[3] In the first one, the players contribute private good over multiple periods, but consume only at the end of the game. If it ends in period t, the utility of player i is $\hat{u}_i(f(x^t), \omega_i - x^t_i)$, where $f(x^t)$ is the public good produced, ω_i is the player's private good endowment, and $\hat{u}_i$ is the player's utility function for the two goods. The date at which the game ends is a random variable T satisfying

[2] Here, $x \geq x'$ means $x_i \geq x'_i$ for all i; $x > x'$ means $x \neq x'$ and $x \geq x'$; and $x \gg x'$ means $x_i > x'_i$ for all i.

[3] Games similar to $\Gamma(\delta, \vec{N})$ arise also in other, contribution-like settings, such as those in Pitchford and Snyder (2004), Ochs and Park (2004), and Zissimos (2007).

$\Pr(T = t) = \delta^{t-1}(1-\delta)$ for any $t \geq 1$: conditional on the game having not ended before date t, it ends then with probability $1 - \delta$. This random "breakdown" is an external friction that serves both to make the game well-defined and to give teeth to subgame perfection, familiar from the sequential bargaining literature, and used in the related model of Pitchford and Snyder (2004).

In this first scenario, a player's expected utility from a path $\vec{x}$ is as shown in (1), with $u_i(x^t) = \hat{u}_i(f(x^t), \omega_i - x_i^t)$. If the produced public good depends only on the aggregate contribution, and the payoffs take the quasilinear form $\hat{u}_i(f(X), \omega_i - x_i) = \hat{v}_i(f(X)) - x_i + \omega_i$, the stage-game payoffs in (2) are obtained, modulo the constant ω_i.[4]

The second contribution scenario entails ongoing consumption and discounting. Consider an unending sequence of fund drives used to acquire capital (e.g. university buildings) that produces future public goods (education and research). The cost of a contribution is borne when it is made. Participants can contribute any number of times, and are informed of the total amounts contributed to date. If the capital the project uses does not depreciate, $\Gamma(\delta, \vec{N})$ is a model of such a fund drive. To show this, we show how the payoff functions (1) and (2) arise.

Suppose contributions are collected at dates $\tau = \Delta, 2\Delta, 3\Delta$, and so on. At date $t\Delta$, player i makes the incremental contribution $x_i^t - x_i^{t-1} \geq 0$, raising her cumulative contribution to x_i^t. Contributions become the capital of the project, so that the amount available to produce public good in the time interval $[t\Delta, (t+1)\Delta)$ is X^t, the aggregate contribution to date. Player i values this flow of public good at rate $\hat{v}_i(X^t)$, which is normalized so that $\hat{v}_i(0) = 0$. The player's discounted (to date $t\Delta$) utility increment in the period is the discounted sum of this rate of valuation over the period, less the (cost of making the) incremental contribution:

$$\begin{aligned}\hat{u}_i^t &= \int_0^{\Delta} \hat{v}_i(X^t) e^{-r\tau} d\tau - (x_i^t - x_i^{t-1}) \\ &= (1-\delta) r^{-1} \hat{v}_i(X^t) - \left(x_i^t - x_i^{t-1}\right),\end{aligned}$$

where r is her discount rate and $\delta = e^{-r\Delta}$. A path $\vec{x}$ gives her the payoff

$$\sum_{t=1}^{\infty} \delta^{t-1} \hat{u}_i^t = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left[r^{-1} \hat{v}_i(X^t) - x_i^t\right].$$

As desired, this payoff is the same as (1) and (2), letting $v_i(X) = r^{-1}\hat{v}_i(X)$.

In upcoming sections the valuation function v_i is held fixed while taking $\delta \to 1$. If we have this second contribution scenario in mind, it is important to interpret the exercise as taking the period length to zero. It should not be interpreted as taking the interest rate to zero, since

[4] If in $\Gamma(\delta, \vec{N})$ a contribution x_i^t were feasible only if $x_i^t \leq \omega_i$, the results of this paper would hold when each ω_i is large enough that these constraints do not bind in equilibrium.

$v_i(X) = r^{-1}\hat{v}_i(X)$ diverges as $r \to 0$. This reflects the fact that for very small interest rates, a marginal contribution today generates a massively large future benefit in present value terms, since that benefit is received in every future period. In this scenario there is no free rider problem when the discount rate is very small.

Smoothness Assumptions

The valuation functions are assumed to be continuously differentiable, strictly increasing, and strictly concave in X. These are the neoclassical assumptions of perfect divisibility, monotonic convex preferences, and decreasing returns in public good production.

Because v_i is strictly increasing, the following *positive spillover* property holds:

(PS) $u_i(\cdot)$ strictly increases in x_j, for all $i \neq j \in N$.

In order to insure nontriviality, the following assumption is also maintained:

$$\lim_{X\to\infty} \sum_{i\in N} v_i'(X) < 1 < \sum_{i\in N} v_i'(0). \tag{3}$$

The first inequality ensures that $\sum_{i\in N} v_i(X) - X$, the sum of the players' payoffs, has a maximizer. This efficient aggregate contribution is unique, and denoted as Y_N. The second inequality in (3) implies $Y_N > 0$.

The above assumptions are maintained throughout the paper. In contrast, at times it will be useful to assume that $v_i'(0) \leq 1$ for all $i \in N$. This assumption is equivalent, given the strict concavity of each v_i, to the following *prisoners' dilemma* property:

(PD) $u_i(\cdot)$ strictly decreases in x_i, for all $i \in N$.

When (PD) holds, each player's dominant strategy in any stage game, regardless of the history, is to not raise her contribution. Free-riding incentives are the strongest in this case. Much of the related literature, e.g., Lockwood and Thomas (2002) and Pitchford and Snyder (2004), is exclusively concerned with payoffs that satisfy (PD).

3. Equilibrium Paths and Achievable Profiles

We restrict attention to pure strategy subgame perfect equilibria, henceforth referred to simply as "equilibria". Each one gives rise to an *equilibrium path*. The limit of an equilibrium path is an *achievable profile*. This section contains initial observations about these objects. Missing proofs are in Appendix A.

Preliminaries

The *standalone contribution* of player i is the unique maximizer of $v_i(x_i) - x_i$, denoted as Y_i. It is given by $Y_i = 0$ if $v_i'(0) \leq 1$, and otherwise by $v_i'(Y_i) = 1$. Let $\bar{Y} := \max_{i \in N} Y_i$ denote the largest standalone contribution. Note that $\bar{Y} < Y_N$, and $\bar{Y} = 0$ if and only if (PD) holds.

Refer to a profile of the form $(Y_i, 0_{-i})$ as a *solo profile* if $Y_i = \bar{Y}$. Only a player with the largest standalone amount contributes in a solo profile, and she contributes that amount. As is easily shown, the set of equilibria of the one-shot simultaneous contribution game is equal to the convex hull of the set of solo profiles. Since $\bar{Y} < Y_N$, any solo profile is inefficient.

Starting from a profile x, the maximal payoff player i could obtain by raising her contribution, when the others do not raise theirs, is $u_i^*(x) := \max_{x_i' \geq x_i} u_i(x_i', x_{-i})$. Denoting this maximizing contribution as $b_i(x)$, it is given by

$$b_i(x) = x_i + \max(0, Y_i - X). \tag{4}$$

Note that u^* is continuous, and that it satisfies the following positive spillover and weak prisoners' dilemma properties:

(PS*) $u_i^*(\cdot)$ strictly increases in x_j, for all $i \neq j \in N$;

(PD*) $u_i^*(\cdot)$ weakly decreases in x_i, for all $i \in N$.

Any profile x for which $u(x) \geq u^*(0)$ is *individually rational.* It is *strictly* individually rational if $u(x) \gg u^*(0)$.

Any profile x for which $u^*(x) = u(x)$ is a *satiation profile* (Gale, 2001). A satiation profile here is one for which $X \geq \bar{Y}$.

Equilibrium Paths

As usual, a central construct for subgame perfection is the *continuation payoff* that player i receives in period t from a path $\vec{x}$:

$$U_i^t(\vec{x}, \delta) := (1 - \delta) \sum_{s \geq t} \delta^{s-t} u_i(x^s).$$

Note that this is a convex combination of the player's present and future stage-game payoffs.

We now derive two conditions that equilibrium continuation payoffs and paths must satisfy. The conditions are based on the observation that after any history, a player can always choose to never raise her contribution again. Refer to this as her *passive strategy.* Because of (PS), the worst conceivable punishment the other players can impose upon a deviator is to play their passive strategies thereafter.

Suppose player i deviates from an equilibrium path $\vec{x}$ to her passive strategy in period t. If the others maximally punish her by subsequently playing their passive strategies, the deviation yields a degenerate path $\vec{z}$ in which $z^s = (x_i^{t-1}, x_{-i}^t)$ for all periods $s \geq t$. It gives the deviator a continuation payoff of $U_i^t(\vec{z}, \delta) = u_i(x_i^{t-1}, x_{-i}^t)$. Her actual continuation payoff from the deviation cannot be less, and it cannot be more than her equilibrium continuation payoff. The path therefore satisfies the following condition:

$$u_i(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta) \text{ for all } t \geq 1,\ i \in N. \tag{5}$$

The second condition is obtained only for a player $i \in N_t$, i.e., a player who is able to move in period t. If she deviates then from an equilibrium path $\vec{x}$ by raising her contribution to $b_i(x_i^{t-1}, x_{-i}^t)$ and playing her passive strategy thereafter, her continuation payoff will be at least $u_i^*(x_i^{t-1}, x_{-i}^t)$. This implies the following necessary condition:

$$u_i^*(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta) \text{ for all } t \geq 1,\ i \in N_t. \tag{6}$$

Condition (6) is also a sufficient condition for a feasible $\vec{x}$ to be a *Nash* equilibrium path. Define the *passive trigger* strategy profile for $\vec{x}$ as follows: in period t, play x^t if $(x^1, \ldots, x^{t-1})$ was played in the past, but otherwise play the same profile as was played in the previous period. The outcome of this strategy profile is $\vec{x}$, and it is clearly a Nash equilibrium if $\vec{x}$ satisfies (6).

A passive trigger Nash equilibrium need not be subgame perfect, since the passive strategy profile is not an equilibrium of any subgame that starts from a profile with an aggregate less than $\bar{Y}$. In such a subgame, some player's best reply to the passive strategies is to raise the aggregate to her Y_i as soon as possible. However, no subgame of this type exists if (PD) holds, as then $\bar{Y} = 0$, and a passive trigger equilibrium is subgame perfect. Therefore, (6) is sufficient for a feasible path to be an equilibrium path when (PD) holds. Since (5) then implies (6), its necessity allows us to conclude that (5) is both necessary and sufficient in this case. This proves the following lemma.

Lemma 1. *Any equilibrium path satisfies* (5) *and* (6)*. If* (PD) *holds, then any feasible path is an equilibrium path if and only if satisfies* (5).

Remark 1. *If* (PD) *is weakened to the assumption that $Y_i = \bar{Y}$ for all $i \in N$, then* (6) *is still sufficient for a feasible path to be an equilibrium path. The proof of this uses strategies that require a unilateral deviator, whose deviation yields an aggregate $\hat{X} < \bar{Y}$, to alone raise $\hat{X}$ to $\bar{Y}$.*

Unfortunately, if $\bar{Y} > 0$, the assumption that each $Y_i = \bar{Y}$ is not robust to perturbations of the valuation functions. This generalization is, therefore, not pursued here.

The next lemma establishes that every equilibrium path converges, and that its limit is a satiation profile. The convergence is asymptotic if the aggregate is strictly larger than $\bar{Y}$.

Lemma 2. *Every equilibrium path converges to a satiation profile. The convergence does not occur in finite time if the limiting profile satisfies $X > \bar{Y}$.*

A profile x is *achievable* in $\Gamma(\delta, \vec{N})$ if it is the limit of one of its equilibrium paths. The main results of the paper concern the set of achievable profiles.

Achievable Profiles

Solo profiles are often achievable. For example, suppose a player i with $Y_i = \bar{Y}$ is able to move every period. Then the strategy profile in which she chooses $x_i^t = \min(0, Y_i - X^{t-1})$ at every node, and the others play their passive strategies, is an equilibrium. It achieves the solo profile $(Y_i, 0_{-i})$ in the first period.

A routine argument shows that the payoff generated by an achieved profile x is the limit of the continuation payoffs along any path $\vec{x}$ that achieves it: $U^t(\vec{x}, \delta) \to u(x)$ as $t \to \infty$. This convergence may be non-monotonic, since a player's stage-game payoff may decrease when she raises her contribution. However, the next lemma implies that any payoff setback is temporary. The payoff generated by the achieved profile exceeds each stage-game payoff and each continuation payoff.

Lemma 3. *Suppose x is achieved by an equilibrium path $\vec{x}$. Then $u^*(x^s) \leq u(x)$ for each $s \geq 0$, and hence $U^t(\vec{x}, \delta) \leq u(x)$ for each $t \geq 0$.*

Any achievable profile is individually rational, as is verified by setting $t = 0$ in Lemma 3 to obtain $u^*(0) \leq u(x)$. The following lemma establishes a stronger result.

Lemma 4. *Any non-solo achievable profile is strictly individually rational.*

4. The Coalitional Game

This section concerns the underlying coalitional game that will be used to characterize achievable profiles. Missing proofs are in Appendix B.

Blocking, Underblocking, and the Undercore

The coalitional game reflects two features of the dynamic game. First, recall that when a player is able to deviate from an equilibrium path $\vec{x}$, she can insure that her continuation payoff is at least $u_i^*(x_i^{t-1}, x_{-i}^t)$. When considering whether to block a profile by implementing on its own an alternative profile, a coalition should therefore evaluate the alternative using u^*. Hence, letting a *coalition* be a nonempty subset of players, define a profile x to be *blocked* by a coalition S using a profile z if and only if $z_{-S} = 0$ and $u_S^*(z) > u_S(x)$. The set of unblocked profiles is the *core*, denoted as C.

The second relevant feature of the dynamic game is that a player can only deviate from a profile on the equilibrium path, and this path lies below the profile it achieves. We are thus interested in whether an achieved profile can be blocked from a profile that lies below it. Accordingly, define a profile x to be *underblocked* by a coalition S if it blocks x using a profile $z \leq x$. The set of profiles that are not underblocked is the *undercore,* denoted as D.

An underblocked profile is blocked. The core is thus a subset of the undercore: $C \subseteq D$. Since every core profile is Pareto efficient (with respect to u), we see that the undercore contains some efficient profiles. The following lemma shows that the undercore also contains the solo profiles, and that it itself is contained in the set of individually rational satiation profiles.

Lemma 5. (i) *Any solo profile is in the undercore.* (ii) *Any undercore profile is an individually rational satiation profile.*

Familiar definitions are needed to derive the structure of the undercore. For any coalition S,

$$f_S(X) := \sum_{i \in S} v_i(X) - X$$

denotes the sum of the coalition members' payoffs if they contribute X and non-members contribute zero. Our assumptions imply f_S has a unique maximizer, to be denoted as Y_S. (Let $Y_\varnothing := \infty$.) The *value* of S is $V(S) := f_S(Y_S)$. For any profile x, let $X_S := \sum_{i \in S} x_i$.

Proposition 1. *The undercore is the set of satiation profiles satisfying, for all coalitions* S,

$$X < Y_S \text{ or } \sum_{i \in S} v_i(X) - X_S \geq V(S). \tag{7}$$

Equivalently, the undercore is the set of satiation profiles satisfying, for all coalitions S,

$$X_S \leq \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right). \tag{8}$$

If a profile x satisfies the second inequality in (7), it gives the coalition S a total payoff that is not less than the maximal payoff it could obtain on its own. In this case S cannot block x, and hence cannot underblock it either. When x satisfies the first inequality in (7), it requires S to contribute an amount smaller than Y_S. In this case, if S can block x, it can do so only by using a larger contribution, $z_S > x_S$, and so it again cannot underblock x.

From (8) we see that the undercore is the set of satiation profiles satisfying a number of inequalities, each of which bounds a coalition's contribution. This is a "balance" requirement. Like a core profile, an undercore profile must not ask any coalition to contribute more than a certain amount that, in this case, is a nondecreasing function of X.

The inequalities determining the undercore are less restrictive for profiles with smaller aggregates. For example, from (7) we see that if x is a satiation profile satisfying $X < Y_S$ for every non-singleton coalition, then $x \in D$ if and only if it is individually rational. However, if $X = Y_N$, then $x \in D$ if and only if $\sum_{i \in S} v_i(X) - X_S \geq V(S)$ for all coalitions. The set of such undercore profiles is the core, as part (ii) of the following corollary shows. It refers to the coalition that a profile x requires to contribute, $N(x) := \{i \in N : x_i > 0\}$.

Corollary 1. *(i) If $x \in D$, then $X \leq Y_{N(x)}$. (ii) $C = \{x \in D : X = Y_N\}$.*

Example 1. *Let $n = 3$ and each $v_i(X) = 2\sqrt{X}$. The optimal contribution and value of each coalition are then $Y_i = V(\{i\}) = 1$, $Y_{\{i,j\}} = V(\{i, j\}) = 4$, and $Y_N = V(N) = 9$. Satiation profiles are those with $X \geq 1$, and individually rational ones are those with $x_i \leq 2\sqrt{X} - 1$. The undercore is*

$$\begin{aligned} D \;=\; & \left\{x \in \mathbb{R}^3_+ : 1 \leq X \leq 4,\ x_i \leq 2\sqrt{X} - 1\right\} \\ & \cup \left\{x \in \mathbb{R}^3_+ : 4 < X \leq 9,\ x_i \leq 2\sqrt{X} - 1,\ x_i + x_j \leq 4\sqrt{X} - 4\right\}. \end{aligned}$$

Note that D is a strict subset of the set of individually rational satiation profiles. The core is the subset of D for which $X = 9$, and can be written as $C = \left\{x \in \mathbb{R}^3_+ : X = 9,\ 1 \leq x_i \leq 5\right\}$.

Undercore Payoffs

In the next section, only undercore profiles will be shown to be achievable. This is a restrictive result for payoffs only if the set of undercore payoffs, $u(D)$, is not equal to the entire set of individually rational feasible payoffs,

$$R := \left\{\hat{u} \in u(\mathbb{R}^3_+) : \hat{u} \geq u^*(0)\right\}.$$

We have $u(D) \subseteq R$. The following corollary shows that this inclusion is strict so long as a non-singleton, non-grand coalition has a positive value, as is typically true if $n > 2$.

Corollary 2. *If a coalition S exists such that $1 < |S| < n$ and $V(S) > 0$, then $R \backslash u(D)$ contains a nonempty open set of payoffs.*

Denote the set of payoffs that are efficient and individually rational as

$$P := \left\{\hat{u} \in R : \hat{u} \not< u' \text{ for any } u' \in R\right\}.$$

Under the hypothesis of Corollary 2 , a similar (omitted) proof shows that $P \backslash u(C)$ contains a nonempty, relatively open set. Thus, in this case not all payoffs that are efficient and individually rational are core payoffs. This is true for the example above.

Example 1 (con't). *These sets of payoffs in Example 1 are*

$$R = \left\{\hat{u} \in \mathbb{R}^3 : \hat{u}_1 + \hat{u}_2 + \hat{u}_3 \leq 9,\ 1 \leq \hat{u}_i\right\},$$

$$u(D) = \left\{\hat{u} \in R : \hat{u}_1 + \hat{u}_2 + \hat{u}_3 \leq 8\right\} \cup \left\{\hat{u} \in R : \hat{u}_i + \hat{u}_j \geq 4\right\},$$

$$u(C) = \left\{\hat{u} \in \mathbb{R}^3 : \hat{u}_1 + \hat{u}_2 + \hat{u}_3 = 9,\ 1 \leq \hat{u}_i \leq 5\right\}.$$

Note that $u(D)$ is a strict subset of R, and $u(C)$ is a strict subset of

$$\hat{P} = \left\{\hat{u} \in \mathbb{R}^3 : \hat{u}_1 + \hat{u}_2 + \hat{u}_3 = 9,\ 1 \leq \hat{u}_i \leq 6\right\},$$

the subset of P generated by the profiles $x \gg 0$ that are efficient and individually rational.[5]

Lastly, we note for future reference that because the core is equal to the set of efficient subset of the undercore, $u(C) \subseteq P \cap u(D)$. As the reverse is also true,[6] we have

$$u(C) = P \cap u(D). \tag{9}$$

Weak Underblocking

A related notion of underblocking will allow somewhat sharper results. A coalition S will be said to *weakly underblock* a profile x if a profile $z < x$ exists such that $z_{-S} = 0$ and $u^*_S(z) \geq u_S(x)$. This definition differs from that of underblocking in two ways: z cannot equal x, and the coalition members can be indifferent in the sense that $u^*_i(z) = u_i(x)$.

The following lemma records two simple facts. Any profile that is not weakly underblocked is strictly individually rational if it is nonzero, and it is in the undercore if it is a satiation profile.

[5] The larger set P contains payoffs like $u(5,5,0) \approx (1.325, 1.325, 6.325)$, since $x = (5,5,0)$ is efficient and individually rational.

[6] Proof: Let $\hat{u} \in P \cap u(D)$ be the payoff associated with $x \in D$. Then $X \leq Y_N$, by Corollary 1 (i). This and the efficiency of x implies $X = Y_N$. Corollary 1 (ii) now implies $x \in C$, and so $\hat{u} \in u(C)$.

Lemma 6. *Suppose x is not weakly underblocked. Then (i) $u(x) \gg u^*(0)$ if $x > 0$, and (ii) $x \in D$ if x is a satiation profile.*

The following lemma shows that for any $Y \in [\bar{Y}, Y_N]$, a profile exists that is not weakly underblocked and satisfies $X = Y$. Since this profile is a satiation profile, it is in the undercore. It is in the core when $Y = Y_N$, in which case it is the Lindahl contribution profile.

Lemma 7. *For any $Y \in [\bar{Y}, Y_N]$, the profile $x = \big(v_1'(Y_N)Y, \ldots, v_n'(Y_N)Y\big)$ is not weakly underblocked, and hence in D.*

5. Necessary Conditions for Achievability

This section contains two results: any achievable profile is in the undercore, and it is inefficient. Required proofs are in Appendix C.

It is perhaps surprising that an achievable profile must be in the undercore. Indeed, if a coalition S underblocks x using z, and a path $\vec{x}$ converging to x is being played, at some point each coalition member would be better off if they all deviated to z_S. But how can they manage to coordinate thier actions in this way? The answer lies in the dynamics, and the fact that z is below x. In some period τ the path $\vec{x}$ will move into the region above z. Some coalition member i must be pivotal for this movement, so that $x_i^{\tau-1} \leq z_i < x_i^\tau$ and $x_{-i}^\tau > z_{-i}^\tau$ (this requires some proof). Accordingly, under the (counterfactual) supposition that $\vec{x}$ is an equilibrium path, the coalition members $j \neq i$ are induced in equilibrium to raise their contributions to at least z_j, without explicit coordination. From here the argument is straightforward. Properties (PD*) and (PS*) yield $u_i^*(x_i^{\tau-1}, x_{-i}^\tau) > u_i^*(z)$. This and $u_i^*(z) \geq u_i(x)$ show that player i can gain by deviating in period τ. The profile x is therefore not achievable.

The precise statement of this result is the following proposition. The proof of part (i) is a sharpening of the argument just outlined. Part (ii) is the simple statement that any achievable profile is in the undercore, and it follows from part (i). If an achievable profile is solo, it is in the undercore for that reason. If it is not solo, then it is a strictly individually rational satiation profile by Lemmas 2 and 4, and so part (i) and Lemma 6 imply it is in the undercore.

Proposition 2. (i) *Any strictly individually rational achievable profile is not weakly underblocked, and is hence in the undercore.* (ii) *Every achievable profile is in the undercore.*

Remark 2. *The proof of part (i) of Proposition 2 uses only that achievable profiles satisfy (6) (from Lemma 1), that achievable profiles are satiation profiles (from Lemma 2), and that* *undercore profiles are satiation profiles (from Lemma 6). The only payoff assumption the proofs of these parts of these lemmas use is* (PS), *positive spillovers. Part* (i) *of Proposition 2 is thus true for fairly general payoffs: for any* $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ *satisfying* (PS) *and for which* u^* *is well defined, the undercore contains all achievable profiles that are strictly individually rational.*

Proposition 2 leaves open the possibility that an efficient profile can be achieved. This possibility is eliminated by the following proposition.

Proposition 3. *Any achievable* x *is inefficient. In particular, it is either a solo profile, or it is inefficient for the contributing coalition:* $X < Y_{N(x)}$.

A heuristic argument conveys the logic of the proof of Proposition 3. Consider an equilibrium path $\vec{x}$ that achieves x, with $X > \bar{Y}$. To the first order, the date t increase in the equilibrium aggregate, $C^t := X^t - X^{t-1}$, increases the present value of the surplus of the contributing players in periods $s \geq t+1$ by

$$MB := \left(\frac{\delta}{1-\delta}\right)\left(\sum_{i \in N(x)} v_i'(X^{t-1}) - 1\right) C^t.$$

The share of MB received by player i must, since she is willing to increase x_i^{t-1} to x_i^t, exceed her net cost of this increase, $\left(1 - v_i'(X^{t-1})\right)(x_i^t - x_i^{t-1})$. Let MC be the sum over $N(x)$ of these costs. Hence, $MB \geq MC$. Now, MC would be minimal if the entire C^t were contributed by the player with the smallest net unit contribution cost. Hence,

$$MC \geq \min_{i \in N(x)} \left(1 - v_i'(X^{t-1})\right) C^t =: MC^{\min}.$$

We thus have $MB \geq MC^{\min}$. Taking $X^{t-1} \to X$ in this inequality yields

$$\sum_{i \in N(x)} v_i'(X) - 1 \geq \left(\frac{1-\delta}{\delta}\right)\left[1 - \max_{i \in N(x)} v_i'(X)\right].$$

Since $X > \bar{Y}$, the right and hence the left side of this inequality is positive: the marginal social benefit of increasing the aggregate remains positive in the limit. Hence, $X < Y_{N(x)}$.

Remark 3. *Achievable profiles may be efficient if payoffs are not differentiable. Suppose each marginal valuation* v_i' *is positive until it drops to zero at an efficient aggregate amount* X^* *that "completes the project" Then, if* δ *is large, equilibrium paths may exist for which* $X^t \to X^*$. *See Marx and Matthews (2000) and Lockwood and Thomas (2002).*

6. Sufficient Conditions for Achievability

In this section a near converse to Proposition 2 is established: virtually any undercore profile can be achieved if the period length is sufficiently small. Missing proofs are in Appendix D.

The result is obtained under two further assumptions, made to deal with two difficulties caused by discounting. The first one, already discussed, is that a discounting player may want to deviate in a period by contributing too much, e.g., to raise her contribution to Y_i immediately rather than to wait for others to do so in the future. In this case a passive trigger Nash equilibrium may fail to be subgame perfect. This problem is now avoided by assuming (PD). It allows us, by Lemma 1, to focus on equilibrium paths rather than strategies: a feasible path $\vec{x}$ is an equilibrium path if and only if it satisfies (5).

The second difficulty caused by discounting is that a future reward can influence current behavior only if it is not received too far in the future. As this must be true at any date, the interval between the times at which the players can move should not grow too quickly as the game progresses. The following *cyclicity* property ensures this:

(CY) integer $m > 0$ exists such that $i \in N_{(nk+i)m}$ for all $i \in N$ and $k \geq 0$.

Accordingly, player 1 is able to move at date m, player 2 at date $2m$, and so on until the pattern repeats with player 1 able to move at date $(n+1)m$. There are no restrictions on who else can move at dates that are multiples of m, nor on who can move at any other date. Familiar move structures satisfy (CY). With $m = 1$, it is satisfied by both the simultaneous move structure and the round-robin structure defined by $N_t^R := \{t \bmod n + 1\}$.

The following lemma establishes that for any $\vec{N}$ satisfying (CY), any equilibrium path of the round-robin game passes through the same profiles as does an equilibrium path of a game that has the move structure $\vec{N}$ and a certain weakly greater discount factor. This result will allow us to restrict attention to the round-robin structure.

Lemma 8. *Suppose* (PD) *holds,* $\vec{N}$ *satisfies* (CY)*, and* $\vec{x}$ *is feasible for* $\vec{N}^R$*. Then a path* $\vec{z}$ *exists that passes through the same profiles as does* $\vec{x}$*, and has the property that for any* $\delta \in (0, 1)$*, it is an equilibrium path of* $\Gamma(\delta^{1/m}, \vec{N})$ *if* $\vec{x}$ *is an equilibrium path of* $\Gamma(\delta, \vec{N}^R)$*.*

The path $\vec{z}$ in Lemma 8 is obtained by slowing down the the round-robin path $\vec{x}$: player 1 moves in period m instead of period 1, player 2 moves in period $2m$ instead of period 2, and so on. Property (CY) insures that this new path is feasible for $\vec{N}$. Along this new path the future reward a player receives for raising her contribution in the current period is postponed, but it is received for enough periods that raising the discount factor to $\delta^{1/m}$ increases its present value enough to restore incentives.

Given (PD), the necessary conditions obtained in Propositions 2 and 3 reduce to the following: an achievable profile must be inefficient for the contributing coalition, and it must not be weakly underblocked. If $\vec{N}$ satisfies (CY), the following proposition shows that these conditions are sufficient as well as necessary, for large discount factors.

Proposition 4. *Suppose* (PD) *holds and* $\vec{N}$ *satisfies* (CY). *Let* x *be a profile that is not weakly underblocked and satisfies* $X < Y_{N(x)}$. *Then a path* $\vec{x}$ *and a discount factor* $\underline{\delta} < 1$ *exist such that* $\vec{x}$ *is an equilibrium path that achieves* x *for all* $\delta > \underline{\delta}$.

The following is an overview of the proof of Proposition 4. In light of Lemma 8, it only needs to be proved for the round-robin structure.

Given x, the proof begins be finding two profiles, $\bar{x}$ and $\hat{x}$, that satisfy $\bar{x} < \hat{x} < x$ and $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$. These profiles exist because $X < Y_{N(x)}$. The profile $\hat{x}$ is chosen close enough to x that it too is not weakly underblocked. The proof then has three steps.

In Step 1, an infinite round-robin path starting at $\bar{x}$ and converging to x is constructed. This path is a round-robin geometric sequence: each player raises her contribution the same proportional amount towards x when it is her turn to move. The increases are made small enough that $u(x) - u(x^t)$ is bounded above zero. This bound shrinks to zero over time, quickly enough that for all high discount factors, player i's equilibrium continuation payoff is close enough to $u_i(x)$ that she is induced to raise her contribution in the current period. This step uses $X < Y_{N(x)}$ and the concavity of each v_i.

Step 2 uses the fact that $\hat{x}$ is not weakly underblocked. Adapting an argument in Gale (2001), a finite, decreasing sequence from $\bar{x}$ to the origin is constructed, along which the players' payoffs never exceed $u(\hat{x})$. The construction starts with player 1 lowering her contribution from $\bar{x}_1$ as much as possible without allowing her payoff to exceed $u_1(\hat{x})$. This yields the first profile of the sequence. The second is obtained by having player 2 lower her contribution in the same manner. Continuing in round-robin fashion yields a decreasing sequence of profiles that generate payoffs no greater than $u(\hat{x})$. The sequence converges, say to a profile z. The fact that $\hat{x}$ is not weakly underblocked implies $z = 0$: otherwise, $N(z)$ would weakly underblock $\hat{x}$ using z. Since $u(0) \ll u(\hat{x})$ by Lemma 6, the convergence occurs in a finite number of steps: once the sequence is close enough to the origin, a player cannot lower her contribution enough to raise her payoff to $u_i(\hat{x})$.

Step 3 puts together the sequences obtained in Steps 1 and 2 to yield a path $\vec{x}$ that converges to x and is feasible for $\vec{N}^R$. At a date for which $x^t \geq \bar{x}$, the construction of Step 1 insures that the remainder of the path is an equilibrium path of the continuation game if δ is large. At a date for which $x^t < \bar{x}$, $u(x^t)$ is bounded strictly below $u(x)$, and so again their continuation payoff from $\vec{x}$ can be made large enough, by choosing δ large, that the players are induced to play x^{t+1}. The path $\vec{x}$ is thus an equilibrium path if δ is large enough.

7. Synthesis and Discussion

In this section results are put together and implications drawn. Proofs are in Appendix E.

Achievable Profiles

Denote the set of achievable profiles given a move structure $\vec{N}$ as

$$A(\vec{N}) := \left\{ x \in \mathbb{R}^n_+ : x \text{ is achievable in } \Gamma(\delta, \vec{N}) \text{ for some } \delta < 1 \right\}.$$

The results of the previous sections relate the achievable profiles to the set

$$D_0 := \left\{ x \in \mathbb{R}^n_+ : x \text{ is solo} \right\} \cup \left\{ x \in \mathbb{R}^n_+ : \bar{Y} \leq X < Y_{N(x)},\ x \text{ is not weakly underblocked} \right\}.$$

In words, D_0 consists of the solo profiles together with the satiation profiles that are not weakly underblocked, and which have an aggregate that is inefficiently small for the contributing coalition. When payoffs satisfy (PD), D_0 is simply the set of profiles that are not weakly underblocked and satisfy $X < Y_{N(x)}$. Propositions 2 and 3 imply $A(\vec{N}) \subseteq D_0$. Proposition 4 therefore implies $A(\vec{N}) = D_0$ when (PD) and (CY) hold.

The set D_0 is shown in Appendix E to be the same as the undercore up to closure: $c\ell D_0 = D$. This yields the final part of the following summary theorem.

Theorem 1. *Under the maintained assumptions,*

(a) $A(\vec{N}) \subseteq D_0 \subset D$, *and*

(b) $A(\vec{N}) = D_0$ *and* $c\ell A(\vec{N}) = D$ *if* (PD) *and* (CY) *hold.*

Most of Theorem 1 has been discussed already. Part (a) states that any achievable profile is in the undercore. Therefore, any efficient profile that can be approximately achieved is in the core. Part (b) gives a full characterization of the set of achievable profiles when (PD) and (CY) hold, and adds that its closure is then precisely the undercore.

A consequence of (b) is a move structure irrelevancy: when payoffs satisfies (PD), all move structures that satisfy (CY) give rise to the same achievable profiles. In particular, no more profiles are achievable under the simultaneous structure than under the round-robin structure. Note, however, that this is a limiting result obtained as $\delta \to 1$; for a fixed δ the set of achievable profiles generally does depend on the move structure.

The main result not included in Theorem 1 is that any achievable profile x can be achieved, given (PD) and (CY), by the *same* equilibrium path for all large δ (Proposition 4). Thus, if we interpret a decrease in δ as a decrease in the period length Δ, the amount of real time required for the path to enter any fixed neighborhood of x goes to zero as $\Delta \to 0$. Every undercore profile can therefore be approximately achieved instantaneously in the limit. Although strategic gradualism is necessary in so far as profiles with $X > \bar{Y}$ are achieved only asymptotically (Lemma 2), there is no upper bound on the speed of convergence as the period length shrinks.

Equilibrium Payoffs

The set of limits of equilibrium payoffs for a given move structure is

$$W(\vec{N}) := c\ell \left\{ U(\vec{x}, \delta) : 0 < \delta < 1,\ \vec{x} \text{ an equilibrium path of } \Gamma(\delta, \vec{N}) \right\}.$$

Since every payoff in this set is individually rational, the set of efficient payoffs that are limits of equilibrium payoffs is $P \cap W(\vec{N})$. The relationship of this set to $u(C)$, the set of core payoffs, is established in the following theorem.

Theorem 2. *Under the maintained assumptions,*

(i) $P \cap W(\vec{N}) \subseteq u(C)$, *and*

(ii) $P \cap W(\vec{N}) = u(C)$ *if* (PD) *and* (CY) *hold.*

Part (i) of Theorem 2 establishes that any efficient payoff that is approximated by equilibrium payoffs is a core payoff. Its proof is based on the observation that any achievable profile is in the undercore (Proposition 2), and every equilibrium payoff is bounded above by the payoff of the achieved profile (Lemma 3).

Part (ii) establishes the converse for when (PD) and (CY) hold: every core payoff is then the limit of a sequence of equilibrium payoffs. Its proof uses the following stronger result.

Lemma 9. $u(D) \subseteq W(\vec{N})$ *if* (PD) *and* (CY) *hold.*

The proof of Lemma 9 uses the result of Proposition 4 that for almost any $x \in D$ (i.e., any $x \in D_0$), a path $\vec{x}$ exists that converges to it and is an equilibrium path for all large δ. The path spends more and more time near x as the period length shrinks, and so the equilibrium payoff $U(\vec{x}, \delta)$ converges to $u(x)$ as $\delta \to 1$. Therefore, $u(x) \in W(\vec{N})$.

The irrelevancy of the move structure for limits of payoffs is implied by Lemma 9 and, especially, Theorem 2 (ii). Given (PD), the set of limiting equilibrium payoffs generated by any structure satisfying (CY) contains $u(D)$, and the set of efficient limiting equilibrium payoffs is precisely $u(C)$. Neither of these sets depends on the move structure.

8. Conclusion

This paper has characterized the set of achievable contribution profiles and equilibrium payoffs of a certain class of dynamic voluntary contribution games with smooth, discounted payoffs. The first main result is that any achievable profile must be in the undercore of the underlying coalitional game – no coalition can be induced to contribute too much. Unlike the folk theorem for repeated games, this result yields a restriction on the nature of equilibria that may be testable in the field or laboratory. It is also a fairly general result, essentially holding for any payoff function satisfying the positive spillovers property (see Remark 2).

The converse is true in a limiting sense if payoffs satisfy the prisoners' dilemma property (PD), and the move structure satisfies the cyclicity property (CY). Virtually any undercore profile is then achievable if the discount factor is large enough. All core profiles and payoffs are obtained in the limit, but not any other efficient profile or payoff. The limiting set of achievable profiles, and the set of efficient limits of equilibrium payoffs, are the same for all move structures satisfying (CY), including the simultaneous and round-robin ones. Lastly, any achievable profile can be achieved instantly in the limit as the period length shrinks to zero, implying an anti-gradualism result for payoffs.

The payoff assumptions used to obtain the second set of results are strong, even though commonly used. The role of (PD) is particularly interesting. It identifies the economically most problematic case, that in which the incentives to free ride are strongest. One might have thought that this would have caused fewer profiles to be achievable. However, by insuring the existence of continuation equilibria that maximally punish deviators (the passive equilibria), (PD) acutally ensures that the set of achievable profiles is as large as possible. The nature of maximally punishing continuation equilibria in the absence of (PD) is an interesting open question.

The strict concavity of the valuation functions is another assumption it would be interesting to weaken. It is used heavily in the proof that virtually any undercore profile is achievable. This result can fail to hold when threshold provision points exist,[7] and concavity indeed insures that

[7] For example, the binary threshold dynamic contribution game of Compte and Jehiel (2003) has a unique achiev-

thresholds do not exist. However, a plausible conjecture is that the result should remain true under other assumptions that also rule out thresholds, ones for example that allow for initial increasing returns to public good production.

able profile. But it has a continuum of undercore profiles: any individually rational profile that completes the project is in the core and hence undercore.

Appendix A. Proofs Missing from Section 3

Lemma A1. *For any $i \in N$, $t \geq 0$, and equilibrium path $\vec{x}$: an increasing divergent sequence $\{s_k\}_{k=1}^{\infty}$ of dates exists such that $s_1 = t$, $u_i(x^{s_k}) \leq u_i(x^{s_{k+1}})$, and $u_i^*(x^{s_k}) \leq u_i^*(x^{s_{k+1}})$.*

Proof. Set $s_1 = t$. Let $\tau > t$ be the first date larger than t such that $i \in N_\tau$. Then

$$u_i(x^{s_1}) \leq u_i^*(x^{s_1}) \leq u_i^*(x_i^{\tau-1}, x_{-i}^{\tau}) \leq U_i^{\tau}(\vec{x}, \delta),$$

using Lemma 1 to obtain the third inequality. Since $U_i^{\tau}(\vec{x}, \delta)$ is a convex combination of the set $\{u_i(x^s)\}_{s>\tau}$, it is weakly exceeded by at least one of its elements. Hence, $s_2 \geq \tau$ exists such that $u_i(x^{s_1}) \leq u_i(x^{s_2})$ and $u_i^*(x^{s_1}) \leq u_i(x^{s_2})$. The latter implies $u_i^*(x^{s_1}) \leq u_i^*(x^{s_2})$. The desired sequence is obtained by iterating this construction. ■

Proof of Lemma 2. Let $\vec{x}$ be an equilibrium path. Assume it does not converge. Then, as it is monotonic, $X^t \to \infty$. Since $\sum_{i \in N} u_i(x) = \sum_{i \in N} v_i(X) - X$ is strictly concave in X and has a finite maximizer, it diverges to $-\infty$ as $X \to \infty$. Hence, $u_i(x^t) \to -\infty$ for some $i \in N$. This implies $\{u_i(x^t)\}_{t \geq 0}$ does not have a nondecreasing subsequence, contrary to Lemma A1. Hence, $\vec{x}$ converges. Let x be its limit.

To prove x is a satiation profile, fix $i \in N$. At a date t such that $i \in N_t$, Lemma 1 yields

$$u_i^*(x_i^{t-1}, x_{-i}^{t}) \leq (1-\delta) \sum_{s \geq t} \delta^{s-t} u_i(x^s).$$

Take the limit of both sides of this inequality along the unbounded sequence of dates t satisfying $i \in N_t$. Since u^* and u are continuous and $x^t \to x$, its left side converges to $u_i^*(x)$ and its right side to $u(x)$. Hence, $u^*(x) \leq u(x)$, and so x is a satiation profile.

Now suppose $X > \bar{Y}$. Assume $\vec{x}$ converges to x at date $T < \infty : x^{T-1} < x$, and $x^s = x$ for $s \geq T$. Let i be a player for whom $x_i^{T-1} < x_i$ (and so $i \in N_T$). Since $X > Y_i$, $\partial u_i(x)/\partial x_i = v_i'(X) - 1 < 0$. This implies the existence of $z_i \in [x_i^{T-1}, x_i)$ such that $u_i(z_i, x_{-i}) > u_i(x)$. If player i deviates to z_i at date T and plays passively thereafter, her continuation payoff would be at least $u_i(z_i, x_{-i})$. This deviation payoff exceeds her equilibrium continuation payoff $U_i^T(\vec{x}, \delta) = u_i(x)$, contrary to subgame perfection. Thus, $\vec{x}$ converges only asymptotically. ■

Proof of Lemma 3. Let $t \geq 0$ and $i \in N$. Let $\{s_k\}_{k=1}^{\infty}$ be the sequence from Lemma A1. Then $u_i^*(x^t) \leq u_i^*(x^{s_k})$ for all $k \geq 1$. Take $k \to \infty$ to obtain $u_i^*(x^t) \leq u_i^*(x)$. Thus, $u_i^*(x^t) \leq u_i(x)$, since x is a satiation profile by Lemma 2. So we now have $u_i(x^s) \leq u_i^*(x^s) \leq u_i(x)$ for all $s \geq 0$. From this, $U^t(\vec{x}, \delta) \leq u(x)$ for any $t \geq 0$ is immediate. ■

Proof of Lemma 4. Let x be an achievable profile, and suppose $u_i(x) = u_i^*(0)$ for some $i \in N$. We prove the lemma by showing x is solo.

Let $\vec{x}$ be an equilibrium path that achieves x. Let τ be the first date such that $i \in N_\tau$. Then

$$u_i^*(0) \leq u_i^*(0, x_{-i}^\tau) \leq U_i^\tau(\vec{x}, \delta) \leq u_i^*(0),$$

where the first inequality is due to (PS*), the second to Lemma 1, and the third to Lemma 3 and $u_i(x) = u_i^*(0)$. Each displayed inequality is thus an equality. From $u_i^*(0) = u_i^*(0, x_{-i}^\tau)$ we obtain $x_{-i}^\tau = 0$. Recall that $U_i^\tau(\vec{x}, \delta)$ is a convex combination of the terms $u_i(x^s)$, each of which has positive weight and, by Lemma 3, is no more than $u_i(x) = u_i^*(0)$. Hence, $U_i^\tau(\vec{x}, \delta) = u_i^*(0)$ implies $u_i(x^s) = u_i^*(0)$ for each $s \geq \tau$.For $s = \tau$ this is $u_i(x_i^\tau, 0_{-i}) = u_i^*(0)$, which implies $x_i^\tau = Y_i$ since Y_i uniquely maximizes $u_i(\cdot, 0_{-i})$. Thus, $x^\tau = (Y_i, 0_{-i})$.

Assume $x_{-i} > 0$. Then $T > \tau$ exists such that $0_{-i} = x_{-i}^{T-1} < x_{-i}^T$. Thus, (PS) implies $u_i(x_i^{T-1}, x_{-i}^T) > u_i(x^{T-1}) = u_i^*(0)$. But by Lemma 1, since $u_i(x^s) = u_i^*(0)$ for $s \geq T$, we know $u_i(x_i^{T-1}, x_i^T) \leq U_i^T(\vec{x}, \delta) = u_i^*(0)$. This contradiction proves $x_{-i} = 0$. This and $u_i(x) = u_i^*(0)$ imply $x = (Y_i, 0_{-i})$. Since $X \geq \bar{Y}$ by Lemma 2, $Y_i = \bar{Y}$. This proves x is solo. ■

Appendix B. Proofs Missing from Section 4

Recall that a coalition S *underblocks* x using z if $u_S^*(z) > u_S(x)$, $z \leq x$, and $z_{-S} = 0$. The coalition *weakly underblocks* x using z if $u_S^*(z) \geq u_S(x)$, $z < x$, and $z_{-S} = 0$.

Proof of Lemma 5. (i) Let $x = (Y_i, 0_{-i})$ be a solo profile, and let z be any profile satisfying $z \leq x$. Then $z_{-i} = 0_{-i}$ and $u_i^*(z) = u_i(x)$. For $j \neq i$, (PS*) implies $u_j^*(z) \leq u_j^*(x)$. Since x is a satiation profile, $u_j^*(x) = u_j(x)$. Hence, $u^*(z) \leq u(x)$. This proves x is not underblocked.

(ii) Let $x \in D$. Then x is individually rational: $u^*(0) \leq u(x)$ because no singleton coalition $\{i\}$ underblocks x using $z = 0$. And x is a satiation profile: we have $u^*(x) \geq u(x)$ by definition, and so $u^*(x) = u(x)$ because N does not underblock x using $z = x$.

Lemma B1. *Let x be a satiation profile and S a coalition.*

(i) *If S underblocks x, then z exists such that $u_S^*(z) = u_S(z)$, and S underblocks x using z.*

(ii) *If S weakly underblocks x and x is non-solo, then z exists such that $u_S^*(z) = u_S(z)$, and S weakly underblocks x using z.*

Proof. There is nothing to prove if $\bar{Y} = 0$, since then $u^* = u$. So assume $\bar{Y} > 0$. Suppose S underblocks or weakly underblocks x using $\hat{z}$. Let $i \in \arg\max_{j \in S} Y_j$. If $\hat{Z} \geq Y_i$, then $u_S^*(\hat{z}) = u_S(\hat{z})$, and the result holds with $z = \hat{z}$. So assume $\hat{Z} < Y_i$. Define z by $z_{-i} = \hat{z}_{-i}$ and $z_i = b_i(\hat{z})$. Because $\hat{Z} < Y_i$, we see from (4) that $z_i = Y_i - \hat{Z}_{-i} > \hat{z}_i$, and $Z = Y_i \geq Y_j$ for all $j \in S$. Thus, $u_S^*(z) = u_S(z)$. Observe also that $z_{-S} = 0$, since $z_{-i} = \hat{z}_{-i}$ and $\hat{z}_{-S} = 0$.

By (PS*) and $z_i > \hat{z}_i$, we have $u_j^*(z) > u_j^*(\hat{z})$ for all $j \neq i$. Thus, since $u_i^*(z) = u_i^*(\hat{z})$ and $u_S^*(z) = u_S(z)$, two implications hold:

$$u_S^*(\hat{z}) > u_S(x) \implies u_S(z) > u_S(x),$$
$$u_S^*(\hat{z}) \geq u_S(x) \implies u_S(z) \geq u_S(x).$$

The first (second) implication shows that if S underblocks (weakly underblocks) x using $\hat{z}$, then it does so as well with z, so long as $z \leq x$ $(z < x)$, which we now show.

Suppose S underblocks x using $\hat{z}$. Then $\hat{z} \leq x$, and we must prove $z \leq x$. We already have $z_{-i} = \hat{z}_{-i} \leq x_{-i}$. From $v_i(Y_i) - z_i = u_i(z) \geq u_i(x)$ and $X \geq Y_i$, we obtain

$$z_i - x_i \leq v_i(Y_i) - v_i(X) \leq 0.$$

Now suppose x is non-solo and S weakly underblocks x using $\hat{z}$. Then $\hat{z} < x$, and we must prove $z < x$. The previous paragraph still yields $z \leq x$. Assume $z = x$. Then $X = Y_i$. If $j \in S \backslash \{i\}$ exists, then $u_j(z) = u_j^*(z) > u_j^*(\hat{z}) \geq u_j(x)$, contrary to $z = x$. Hence, $S = \{i\}$. From $z = x$, $z_{-i} = 0_{-i}$, and $X = Y_i$, we obtain $x = (Y_i, 0_{-i})$. This implies $Y_i = \bar{Y}$, since X is a satiation profile. Hence, x is solo. This contradiction proves $z < x$. ∎

Lemma B2. *A satiation profile x is underblocked if and only if for some coalition S,*

$$X_S > \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right). \tag{10}$$

Proof. Suppose x is underblocked by a coalition S. Then by Lemma B1, $z < x$ exists such that $z_{-S} = 0$ and $u_S(x) < u_S(z)$. Summing these inequalities over S and using $Z_S = Z$ yields

$$\sum_{i \in S} v_i(X) - X_S < f_S(Z). \tag{11}$$

This and $Z \leq X$ yield $Z < X_S$. As $f_S(Z) \leq V(S)$, (11) also implies $X_S > \sum_{i \in S} v_i(X) - V(S)$, which is half of (10). Now, note that $X_S \leq X$ and (11) imply $f_S(X_S) < f_S(Z)$. Thus, if $X_S \leq Y_S$ were true, the concavity of f_S and $Z < X_S \leq Y_S$ would imply an impossibility, $f_S(X_S) > f_S(Y_S)$. This proves $X_S > Y_S$, the other half of (10).

To prove the converse, suppose (10) holds for coalition S. Then $v_S(X) \geq v_S(X_S) \gg v_S(Y_S)$. Furthermore,

$$\Delta := \frac{V(S) - \left[\sum_{i \in S} v_i(X) - X_S\right]}{|S|} > 0.$$

Define $z \in \mathbb{R}^n$ by $z_{-S} = 0$, and $z_i := x_i - \Delta - v_i(X) + v_i(Y_S)$ for $i \in S$. Then $z_i < x_i$ for all $i \in S$. Summing z_i over S yields $Z = Y_S$. Hence, $\hat{S} := \{i \in S : z_i \geq 0\} \neq \varnothing$. Define $\hat{z} \in \mathbb{R}_+^n$

by $\hat{z}_i := \max(0, z_i)$. Then $\hat{z} \in \mathbb{R}^n_+$, $\hat{z}_{-\hat{S}} = 0$, and $\hat{z} < x$. Because $\hat{Z} \geq Z = Y_S$, and $\hat{z}_i = z_i$ for $i \in \hat{S}$, we have

$$v_i(\hat{Z}) - \hat{z}_i \geq v_i(Y_S) - z_i = v_i(X) - x_i + \Delta > v_i(X) - x_i.$$

for all $i \in \hat{S}$. Hence, $\hat{S}$ can use $\hat{z}$ to underblock x. ■

Proof of Proposition 1. By Lemma 5, D contains only satiation profiles. For a satiation profile x, Lemma B2 implies that $x \in D$ if and only if (10) does not hold for any S, i.e., if and only if x satisfies (8).

It is immediate that (7) implies (8), using $X_S \leq X$. Suppose x satisfies (8). The first case to consider is $Y_S > \sum_{i \in S} v_i(X) - V(S)$. This inequality rearranges to $\sum_{i \in S} v_i(Y_S) > \sum_{i \in S} v_i(X)$. Hence, $Y_S > X$, and x satisfies the first half of (7). The other case is $Y_S \leq \sum_{i \in S} v_i(X) - V(S)$. Then (8) implies $X_S \leq \sum_{i \in S} v_i(X) - V(S)$, and so x satisfies the second half of (7). This proves that (7) and (8) are equivalent. ■

Proof of Corollary 1 (i)**.** If $x = 0$, we $X < Y_{N(x)}$ trivially, since $N(x) = \varnothing$ and $Y_\varnothing := \infty$. So suppose $x \neq 0$, and let $S = N(x)$. Assume $X > Y_S$. Then, since $X = X_S$, from (7) we obtain $f_S(X) \geq V(S) = f_S(Y_S)$. This is impossible, since Y_S is the unique maximizer of f_S. ■

Proof of Corollary 1 (ii)**.** Let $x \in C$. Then (i) implies $X \leq Y_{N(x)} \leq Y_N$. Since x is efficient and $X \leq Y_N$, a standard argument proves $X = Y_N$.

To prove the converse, consider any $x \in D$ with $X = Y_N$. Assume a coalition S blocks x using a profile $\hat{z}$. Then $u^*_S(\hat{z}) > u_S(x)$. Choose $i \in S$ so that $Y_i \geq Y_j$ for all $j \in S$. Let $z_{-i} = \hat{z}_{-i}$ and $z_i = b_i(\hat{z})$. Then $u^*_S(z) \geq u^*_S(\hat{z})$ and $u^*_S(z) = u_S(z)$. Hence, $u_S(z) > u_S(x)$. Summing these inequalities over S and using $Z_S = Z$ and $X = Y_N$ yields $f_S(Z) > \sum_{i \in S} v_i(Y_N) - X_S$. This implies $V(S) > \sum_{i \in S} v_i(Y_N) - X_S$. However, as $x \in D$ and $X = Y_N \geq Y_S$, (7) requires $V(S) \leq \sum_{i \in S} v_i(Y_N) - X_S$. This contradiction proves $x \in C$. ■

Proof of Corollary 2. Because $V(S) > 0$, $Y_S > 0$. This and $|S| > 1$ imply that for any $i \notin S$, substituting i for any $j \in S$ yields a coalition $\hat{S}$ of the same size as S for which $Y_{\hat{S}} > Y_i$. We can thus assume $Y_S > \bar{Y}$.

Let O be the set of profiles $x \in \mathbb{R}^n_{++}$ satisfying

$$Y_S < X < Y_N, \tag{12}$$

$$\sum_{i \in S} v_i(X) - X_S < V(S), \tag{13}$$

$$u^*(0) \ll u(x). \tag{14}$$

The continuity of v implies O is open. By (14), all profiles in O are individually rational. By (12), (13), and Proposition 1, $O \cap D = \varnothing$.

Both O and D (by Corollary 1) are subsets of $\left\{x \in \mathbb{R}^n_+ : X \leq Y_N\right\}$. It is straightforward to show that u is a homeomorphism on this set. Thus, $u(O)$ is an open set and, since $O \cap D = \varnothing$, $u(O) \cap u(D) = \varnothing$. Moreover, (14) implies $u(O) \subset R$. We have thus shown $u(O)$ to be an open subset of $R \backslash u(D)$. It remains to show it is nonempty, i.e., to show $O \neq \varnothing$.

Define a profile x by $x_i = v_i'(Y_S)Y_S$ for $i \in S$, and $x_i = 0$ otherwise. Because $Y_S > 0$, $\sum_{i \in S} v_i'(Y_S) = 1$. Hence, $X = X_S = Y_S$ and $\sum_{i \in S} v_i(X) - X_S = V(S)$. Because $Y_N > 0$ and $S \neq N$, $Y_S < Y_N$. Hence, for all sufficiently small $\varepsilon > 0$, the profile x^ε defined by $x_i^\varepsilon := x_i + \varepsilon$ satisfies (12). Furthermore, since Y_S uniquely maximizes f_S, x^ε also satisfies (13). Now, for any $i \notin S$, $u_i(x) = v_i(Y_S) > v_i(Y_i) - Y_i$, since $Y_S > \bar{Y}$. For $i \in S$, the strict concavity of v_i and $Y_S \neq Y_i$ yield

$$\begin{aligned} u_i(x) &= v_i(Y_S) - v_i'(Y_S)Y_S \\ &> v_i(Y_i) + (Y_S - Y_i)v_i'(Y_S) - v_i'(Y_S)Y_S \\ &= v_i(Y_i) - Y_i. \end{aligned}$$

Thus, x satisfies (14), and so x^ε does too if ε is small. Hence, $\varepsilon > 0$ exists such that $x^\varepsilon \in O$. ■

Proof of Lemma 6. (i) If $u_i(x) \leq u_i^*(0)$, then coalition $\{i\}$ would weakly underblock x using $z = 0 < x$. (ii) If any S were to underblock x using some $z \leq x$, then $u_S^*(z) > u_S(x) = u_S^*(x)$ would imply $z < x$, and so S would weakly underblock x using z.

Proof of Lemma 7. If $Y = \bar{Y} = 0$, then $x = 0$, and hence x is neither weakly underblocked nor, since $\bar{Y} = 0$, underblocked. So we can assume $Y > 0$. Hence, each $x_i = v_i'(Y_N)Y > 0$, and so x is non-solo. Since $\sum_{i \in N} v_i'(Y_N) = 1$, $X = Y$. Thus, $X \geq \bar{Y}$. Now, suppose coalition S weakly underblocks x. Then, since x is a non-solo satiation profile, Lemma B1 (ii) implies the existence of $z < x$ such that $z_{-S} = 0$ and $u_S(x) \leq u_S(z)$. Sum these inequalities to obtain

$$\sum_{i \in S} v_i(X) \leq \sum_{i \in S} v_i(Z) - Z + X_S. \tag{15}$$

However, the strict concavity of each v_i, together with $Z < X \leq Y_N$, implies

$$\begin{aligned} \sum_{i \in S} v_i(X) &> \sum_{i \in S} v_i(Z) + (X - Z) \sum_{i \in S} v_i'(X) \\ &\geq \sum_{i \in S} v_i(Z) + (X - Z) \sum_{i \in S} v_i'(Y_N) \\ &= \sum_{i \in S} v_i(Z) - Z \sum_{i \in S} v_i'(Y_N) + X_S \\ &\geq \sum_{i \in S} v_i(Z) - Z + X_S, \end{aligned}$$

using $X_S = X\sum_{i\in S} v_i'(Y_N)$ to obtain the equality and $\sum_{i\in S} v_i'(Y_N) \le 1$ to obtain the final inequality. This contradiction of (15) proves x is not weakly underblocked. ■

Appendix C. Proofs Missing from Section 5

Proof of Proposition 2. As shown in the text, (i) implies (ii). To prove (i), let x be a strictly individually rational profile achieved by an equilibrium path $\vec{x}$. Assume x is weakly underblocked by some coalition S using a profile z. Then $z < x$, $z_{-S} = 0$, and $u_S^*(z) \ge u_S(x)$.

Suppose S is a singleton, $S = \{i\}$. Then $z = (z_i, 0_{-i})$, and hence $u_i^*(0) \ge u_i^*(z)$ by (PD*). But this implies $u_i^*(0) \ge u_i(x)$, contrary to the strict individual rationality of x. This proves S is not a singleton.

We next prove $z_S \ll x_S$. Assume $z_i = x_i$ for some $i \in S$. This implies $z_{-i} < x_{-i}$. We thus have $u_i^*(z) = u_i^*(x_i, z_{-i}) < u_i^*(x)$, by (PS*). But this implies $u_i^*(z) < u_i(x)$, since x is a satiation profile by Lemma 2. This contradiction proves $z_S \ll x_S$.

Let $\tau := \min\{t \ge 1 : z_S \ll x_S^t\}$. Choose $i \in S$ so that $x_i^{\tau-1} \le z_i < x_i^\tau$. Hence, $i \in N_\tau$. Moreover, since S is not a singleton, $k \in S\backslash\{i\}$ exists such that $z_k < x_k^\tau$. We thus have $z_{-i} < x_{-i}^\tau$, by the definition of τ and the fact that $z_j = 0 \le x_j^\tau$ for $j \notin S$. Now (PD*) and (PS*) together yield $u_i^*(x_i^{\tau-1}, x_{-i}^\tau) > u_i^*(z)$. Hence, $u_i^*(x_i^{\tau-1}, x_{-i}^\tau) > u_i(x)$. Condition (6) thus fails to hold, and so $\vec{x}$ is not an equilibrium path by Lemma 1. This contradiction proves x is not weakly underblocked. This and the fact that x is a satiation profile imply $x \in D$, by Lemma 6. ■

Lemma C1. If x is an achievable profile in $\Gamma(\delta, \vec{N})$, and $X > \max_{i\in N(x)} Y_i$, then

$$\left(\frac{\delta}{1-\delta}\right)\left(\sum_{i\in N(x)} v_i'(X) - 1\right) \ge 1 - \max_{i\in N(x)} v_i'(X). \tag{16}$$

Proof. Suppose x is achieved by an equilibrium path $\vec{x}$. Let $\hat{Y} := \max_{i\in N(x)} Y_i$. Since $X > \hat{Y}$, date $T < \infty$ exists such that $X^{t-1} > \hat{Y}$ for all $t > T$. Fix $i \in N$ and $t > T$. Lemma 1 implies

$$\sum_{s\ge t} \delta^{s-t}\left[v_i(X^s) - v_i(\tilde{X}^t) - (x_i^s - x_i^{t-1})\right] = (1-\delta)^{-1}\left[U_i^t(\vec{x}, \delta) - u_i(x_i^{t-1}, x_{-i}^t)\right] \ge 0, \tag{17}$$

where $\tilde{X}^t := X^t - x_i^t + x_i^{t-1}$. For $s \ge t$, because v_i is concave and $X^{t-1} \le \tilde{X}^t \le X^s$, we have

$$v_i'(X^{t-1})(X^s - \tilde{X}^t) \ge v_i'(\tilde{X}^t)(X^s - \tilde{X}^t) \ge v_i(X^s) - v_i(\tilde{X}^t).$$

Hence, (17) implies

$$\sum_{s\ge t} \delta^{s-t}\left[v_i'(X^{t-1})(X^s - \tilde{X}^t) - (x_i^s - x_i^{t-1})\right] \ge 0.$$

Summing these inequalities over $i \in N(x)$ and replacing $\tilde{X}^t$ by $X^t - x_i^t + x_i^{t-1}$ yields

$$\sum_{s \geq t} \delta^{s-t} \left\{ (X^s - X^t) \sum_{i \in N(x)} v_i'(X^{t-1}) + \sum_{i \in N(x)} \left[v_i'(X^{t-1})(x_i^t - x_i^{t-1}) - (x_i^s - x_i^{t-1}) \right] \right\} \geq 0.$$

Using $\sum_{i \in N(x)} (x_i^s - x_i^t) = X^s - X^t$ and letting $\alpha_t := \sum_{i \in N(x)} v_i'(X^{t-1}) - 1$, this becomes

$$\sum_{s \geq t} \delta^{s-t} \left\{ (X^s - X^t)\alpha_t + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \right\} \geq 0.$$

This rearranges, upon multiplying by $1 - \delta$, to

$$\alpha_t \left[(1 - \delta) \sum_{s \geq t} \delta^{s-t} (X^s - X^t) \right] + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \geq 0.$$

Using the identity $(1 - \delta) \sum_{s \geq t} \delta^{s-t} (X^s - X^t) = \delta \sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s)$, this becomes

$$\delta\alpha_t \left[\sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s) \right] + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \geq 0. \tag{18}$$

Since $X > \hat{Y}$, we have $v_i'(X) < 1$ for all $i \in N(x)$. Choose a number β satisfying

$$\max_{i \in N(x)} v_i'(X) < \beta < 1.$$

Let $T' \geq T$ be a date such that $\beta > v_i'(X^{t-1})$ for any $t > T'$ and $i \in N(x)$. Hence, considering (18) for $t \geq T'$, we can replace each $v_i'(X^{t-1})$ in its last term by β to get

$$\delta\alpha_t \left(\sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s) \right) + (\beta - 1)(X^t - X^{t-1}) \geq 0. \tag{19}$$

Because $X > \hat{Y}$, Lemma 2 implies $\vec{x}$ converges to x asymptotically.[8] Thus, $X^{t-1} < X$. Proposition 2 implies $x \in D$, and so $X \leq Y_{N(x)}$ by Corollary 1 (i). Hence, $X^{t-1} < Y_{N(x)}$. This and the strict concavity of each v_i implies $\alpha_t > 0$. Thus, from (19) we obtain

$$(1 - \delta) \left(\sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s) \right) \geq (X^t - X^{t-1}) \left(\frac{1 - \delta}{\delta} \right) \left(\frac{1 - \beta}{\alpha_t} \right).$$

The left side of this inequality is a convex combination of the terms $X^{s+1} - X^s$, and hence not more than the largest of them. We thus obtain

$$\max_{s \geq t} (X^{s+1} - X^s) \geq (X^t - X^{t-1}) Q_t, \tag{20}$$

where

$$Q_t := \left(\frac{1 - \delta}{\delta} \right) \left(\frac{1 - \beta}{\alpha_t} \right) = \left(\frac{1 - \delta}{\delta} \right) \left[\frac{1 - \beta}{\sum_{i \in N(x)} v_i'(X^{t-1}) - 1} \right].$$

[8] Lemma 2 only states that convergence is asymptotic if $X > \bar{Y}$. But its proof actually shows that convergence is asymptotic under the weaker condition that $X > \max_{i \in N(x)} Y_i$.

Note that Q_t is nondecreasing in t. Hence, if $Q_t \geq 1$, then $Q_s \geq 1$ for all $s \geq t$. But then a recursive application of (20) would yield the contradiction $X^t \to \infty$. Hence, $Q_t < 1$ for all large t. We now have

$$\left(\frac{1-\delta}{\delta}\right)\left[\frac{1-\beta}{\sum_{i \in N(x)} v_i'(X) - 1}\right] = \lim_{t \to \infty} Q_t \leq 1.$$

From this, (16) is obtained by taking $\beta \to \max_{i \in N(x)} v_i'(X)$. ■

Proof of Proposition 3. Let x be achievable. Lemma 2, Proposition 2, and Corollary 1 (i) imply $\bar{Y} \leq X \leq Y_{N(x)}$. If $N(x) = \varnothing$, then $X = 0 = \bar{Y}$, and $x = 0$ is a solo profile. If $N(x) = \{i\}$, then $u_i^*(0) \leq u_i(x) = u_i(x_i, 0_{-i})$ implies $x = (Y_i, 0_{-i})$, and so $Y_i = \bar{Y}$. Thus, in this case x is again a solo profile. Now suppose $|N(x)| > 1$. Then $X > 0$, and so $Y_{N(x)} > 0$. This, since each $v_i' > 0$, implies $Y_{N(x)} > \max_{i \in N(x)} Y_i =: \hat{Y}$. We know $X \geq \bar{Y} \geq \hat{Y}$. If $X = \hat{Y}$, then $X < Y_{N(x)}$. If $X > \hat{Y}$, then Lemma C1 implies (16), and the right side of it is positive. This yields $\sum_{i \in N(x)} v_i'(X) > 1$, and so the concavity of each v_i implies $X < Y_{N(x)}$. ■

Appendix D. Proofs Missing from Section 6

Proof of Lemma 8. Define $\vec{z}$ by letting the players move as in $\vec{x}$, but only at dates that are multiples of m. That is, let $z^t = 0$ for $t = 0, \ldots, m-1$, and for $t \geq m$ let $z^t = x^{nk+i}$, where k and i are the unique integers satisfying $k \geq 0$, $i \in N$, and

$$(nk+i)m \leq t < (nk+i+1)m.$$

In $\vec{z}$ player i moves only at dates $(nk+i)m$, since in $\vec{x}$ she moves only at dates $nk+i$. The path $\vec{z}$ is feasible for $\vec{N}$ by (CY), since $i \in N_{(nk+i)m}$.

Let $\delta \in (0,1)$, and suppose $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$. Let $\hat{\delta} = \delta^{1/m}$. Since (PD) holds, Lemma 1 implies $\vec{z}$ is an equilibrium path of $\Gamma(\hat{\delta}, \vec{N})$ if it and $\hat{\delta}$ satisfy (6). So, letting $t \geq 1$ and $i \in N$, it suffices to show

$$u_i(z_i^{t-1}, z_{-i}^t) \leq (1-\hat{\delta}) \sum_{s \geq t} \hat{\delta}^{s-t} u_i(z^s). \tag{21}$$

(Recall that now $u^* = u$). If $z_i^s = z_i^{t-1}$ for all $s \geq t$, then (PS) implies (21). So suppose a date $\tau \geq t$ exists such that $z_i^{t-1} = z_i^{\tau-1} < z_i^\tau$. This date is a multiple of m, say $\tau = pm$. Furthermore, $z^\tau = x^p$ and $z^{\tau-1} = z^{t-1} = x^{p-1}$. Observe that

$$\begin{aligned} (1-\hat{\delta}) \sum_{s \geq t} \hat{\delta}^{s-t} u_i(z^s) &= (1-\hat{\delta}) \sum_{s=t}^{\tau-1} \hat{\delta}^{s-t} u_i(z_i^{t-1}, z_{-i}^s) + \hat{\delta}^{\tau-t}(1-\hat{\delta}) \sum_{s \geq \tau} \hat{\delta}^{s-\tau} u_i(z^s) \\ &\geq (1-\hat{\delta}^{\tau-t}) u_i(z_i^{t-1}, z_{-i}^t) + \hat{\delta}^{\tau-t}(1-\hat{\delta}) \sum_{s \geq \tau} \hat{\delta}^{s-\tau} u_i(z^s), \end{aligned}$$

since $u_i(z_i^{t-1}, z_{-i}^s) \geq u_i(z_i^{t-1}, z_{-i}^t)$ for each $s = t, \ldots, \tau - 1$. (The overall inequality holds trivially if $\tau = t$.) Hence, (21) holds if

$$(1 - \hat{\delta}) \sum_{s \geq \tau} \hat{\delta}^{s-\tau} u_i(z^s) \geq u_i(z_i^{t-1}, z_{-i}^t), \tag{22}$$

which we now show. The definitions of $\vec{z}$ and $\hat{\delta}$ imply

$$\begin{aligned}
(1 - \hat{\delta}) \sum_{s \geq \tau} \hat{\delta}^{s-\tau} u_i(z^s) &= (1 - \hat{\delta}) \sum_{k=0}^{\infty} \sum_{s=\tau+km}^{\tau+(k+1)m-1} \hat{\delta}^{s-\tau} u_i(z^s) \\
&= (1 - \hat{\delta}) \sum_{k=0}^{\infty} \hat{\delta}^{km} u_i(x^{p+k}) \sum_{s=\tau+km}^{\tau+(k+1)m-1} \hat{\delta}^{s-\tau-km} \\
&= (1 - \hat{\delta}^m) \sum_{k=0}^{\infty} \hat{\delta}^{km} u_i(x^{p+k}) \\
&= (1 - \delta) \sum_{k=0}^{\infty} \delta^k u_i(x^{p+k}).
\end{aligned}$$

Because $\vec{x}$ satisfies (6) at date p, we have

$$\begin{aligned}
(1 - \delta) \sum_{k=0}^{\infty} \delta^k u_i(x^{p+k}) &= (1 - \delta) \sum_{s \geq p} \delta^{s-p} u_i(x^s) \\
&\geq u_i(x_i^{p-1}, x_{-i}^p) \\
&= u_i(z_i^{t-1}, z_{-i}^\tau).
\end{aligned}$$

The two previous displays, with $u_i(z_i^{t-1}, z_{-i}^\tau) \geq u_i(z_i^{t-1}, z_{-i}^t)$, imply (22). ■

Lemma D1. *Given* (PD)*, suppose a profile x is not weakly underblocked. Then, a neighborhood of x exists such that every $\hat{x}$ in it that satisfies $\hat{x} < x$ is also not weakly underblocked.*

Proof. As the lemma is trivially true for $x = 0$, we may suppose $x > 0$. Since x is not weakly underblocked and (PD) implies it is a satiation profile, we have $x \in D$ by Lemma 6 (ii). Hence, $X \leq Y_{N(x)}$ by Corollary 1. Assume the lemma is false. Then an infinite sequence $\{x^k\}$ exists such that $x^k \to x$, $x^k < x$, and each x^k is weakly underblocked, say by a coalition S^k using a profile $z^k < x^k$. By taking a subsequence we may assume $S^k = S$ for all k (as the number of coalitions is finite), and $\{z^k\}$ converges to a profile z (as each z^k is in the compact set $[0, x]^n$). Taking $k \to \infty$ in the inequalities $z^k < x$ and $u_S(z^k) \geq u_S(x^k)$ yields $z \leq x$ and $u_S(z) \geq u_S(x)$. Since $z_{-S}^k = 0$ for all k, $z_{-S} = 0$. Therefore S would weakly underblock x using z if $z < x$. As this is not possible, $z = x$. This implies $N(x) \subseteq S$. Choose k so large that for all $i \in N(x)$, $x_i^k > 0$. Hence, since $N(x) \subseteq S$, we have $X_S^k = X^k$. Because (PD) holds and S uses z^k to weakly underblock x^k, $u_S(z^k) \geq u_S(x^k)$. Summing these inequalities over S yields $f_S(Z^k) \geq f_S(X^k)$. This, the strict concavity of f_S, and

$$X^k < X \leq Y_{N(x)} \leq Y_S,$$

together imply $Z^k \geq X^k$. This contradicts $z^k < x^k$. ■

Proof of Proposition 4. By Lemma 8, it suffices to prove the result for $\vec{N} = \vec{N}^R$. Given (PD), the passive strategy profile is an equilibrium, and so the origin is achievable. So suppose $x > 0$. Define $d \in \mathbb{R}^n_+$ by $d_i := 0$ if $i \notin N(x)$, and

$$d_i := \frac{v_i'(X)}{\sum_{j \in N(x)} v_j'(X)} \text{ for } i \in N(x).$$

Since $X < Y_{N(x)}$ implies $\sum_{j \in N(x)} v_j'(X) > 1$, we have $0 < d_i < v_i'(X)$ for $i \in N(x)$. Choose $\bar{\theta} > 0$ small enough that $\bar{x} := x - \bar{\theta} d \geq 0$. Since x is not weakly underblocked, Lemma D1 implies the existence of $\hat{\theta} \in (0, \bar{\theta})$ such that $\hat{x} := x - \hat{\theta} d$ is not weakly underblocked. Note that $0 \leq \bar{x} < \hat{x} < x$. We also have $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$, since the concavity of each v_i implies that for any $\theta \geq 0$, $\partial u(x - \theta d)/\partial \theta = d - v'(X - \theta) \leq d - v'(X) \ll 0$.

Define $\{x^t\}_{k=0}^{\infty}$ to be a *round-robin sequence* if for each $t > 0$ and $i = t \pmod n$, $x^t_{-i} = x^{t-1}_{-i}$. The rest of the proof consists of three steps.

Step 1. *There exists a nondecreasing round-robin sequence* $\{x^t\}_{t=0}^{\infty}$, *and a discount factor* $\delta' < 1$, *such that* $x^0 = \bar{x}$, $x^t \to x$, *and for all* $t > 0$, $i = t \pmod n$, *and* $\delta \geq \delta'$:

$$u_i(x_i^{t-1}, x_{-i}^t) \leq (1 - \delta) \sum_{s \geq t} \delta^{s-t} u_i(x^s). \tag{23}$$

Proof of Step 1. Since $d_i < v_i'(X)$ for all $i \in N(x)$, and $d_i = 0$ for $i \notin N(x)$, we can find positive numbers a and ε such that

$$\frac{(1+\varepsilon) d_i}{v_i'(X)} < a < 1 \tag{24}$$

for all $i \in N$. Define $\{x^t\}_{t=0}^{\infty}$ by $x^0 := \bar{x}$ and, for $t > 0$,

$$x_i^t := \begin{cases} a x_i^{t-1} + (1-a) x_i & \text{if } i = t \pmod n \\ x_i^{t-1} & \text{otherwise.} \end{cases} \tag{25}$$

This $\{x^t\}_{t=0}^{\infty}$ is a round-robin sequence that starts at $\bar{x}$ and converges to x. Fix $t > 0$, and let $i = t \pmod n$. Let $q \geq 0$ be the integer for which $t = i + qn$. At the end of period $t - 1$, players $j = 1, \ldots, i - 1$ have raised their actions $q + 1$ times, and players $j = i, \ldots, n$ have raised theirs just q times. Hence, since $x - \bar{x} = \bar{\theta} d$,

$$x_j^{t-1} = \begin{cases} x_j - \bar{\theta} a^{q+1} d_j & \text{for } 1 \leq j < i \\ x_j - \bar{\theta} a^q d_j & \text{for } i \leq j \leq n. \end{cases} \tag{26}$$

This implies

$$X^{t-1} = X - \bar{\theta} a^q \left[a \sum_{j=1}^{i-1} d_j + \sum_{j=i}^{n} d_j \right]. \tag{27}$$

Similarly, for any $k \geq 1$,

$$x_j^{t+(k-1)n} = \begin{cases} x_j - \bar{\theta} a^{q+k} d_j & \text{for } 1 \leq j \leq i \\ x_j - \bar{\theta} a^{q+k-1} d_j & \text{for } i < j \leq n, \end{cases} \tag{28}$$

and

$$X^{t+(k-1)n} = X - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right]. \tag{29}$$

Turning to the desired inequality (23), note that it is equivalent to

$$A := \sum_{s \geq t} \delta^{s-t} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right] \geq 0.$$

Observe that $A = \sum_{k=1}^{\infty} \delta^{(k-1)n} A_k$, where

$$A_k := \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right].$$

Each A_k is a sum over n consecutive dates, and player i moves only at the first one, $t+(k-1)n$. Hence, for each of these dates s, $x_i^s = x_i^{t+(k-1)n}$. This implies that

$$\begin{aligned} A_k &= \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^s) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\ &\geq \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\ &= \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right], \end{aligned}$$

where the inequality follows from $X^s \geq X^{t+(k-1)n}$ for $s \geq t+(k-1)n$. Using now the concavity of v_i and $X^{t-1} \leq X^{t+(k-1)n} \leq X$, we obtain

$$A_k \geq \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i'(X) \left(X^{t+(k-1)n} - X^{t-1} \right) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right].$$

This expression can be bounded from below. From (27) and (29) we have

$$\begin{aligned} X^{t+(k-1)n} - X^{t-1} &= \bar{\theta} a^q \left[a \sum_{j=1}^{i-1} d_j + \sum_{j=i}^{n} d_j \right] - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right] \\ &= \bar{\theta} a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + (1-a^k) d_i + (1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right]. \end{aligned}$$

From this, $1-a^k > a(1-a^{k-1})$, and $1-a^{k-1} > a(1-a^{k-1})$, we obtain

$$\begin{aligned}
X^{t+(k-1)n} - X^{t-1} &\geq \bar{\theta}a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + a(1-a^{k-1})d_i + a(1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right] \\
&= \bar{\theta}a^{q+1}(1-a^{k-1}) \sum_{j=1}^{n} d_j \\
&= \bar{\theta}a^{q+1}(1-a^{k-1}).
\end{aligned}$$

From (26) and (28), $x_i^{t+(k-1)n} - x_i^{t-1} = \bar{\theta}a^q \left(1-a^k\right) d_i$. Consequently,

$$A_k \geq \bar{\theta}a^q \left(\frac{1-\delta^n}{1-\delta}\right) \left[v_i'(X)a(1-a^{k-1}) - \left(1-a^k\right) d_i\right].$$

This and (24) imply

$$A_k \geq \bar{\theta}a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \left[\varepsilon - a^{k-1}(1+\varepsilon-a)\right].$$

Therefore,

$$\begin{aligned}
A &\geq \bar{\theta}a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \sum_{k=1}^{\infty} \delta^{(k-1)n} \left[\varepsilon - a^{k-1}(1+\varepsilon-a)\right] \\
&= \bar{\theta}a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \left\{\varepsilon \sum_{k=1}^{\infty} (\delta^n)^{k-1} - (1+\varepsilon-a) \sum_{k=1}^{\infty} (a\delta^n)^{k-1}\right\} \\
&= \left(\frac{\bar{\theta}a^q d_i}{1-\delta}\right) \left\{\varepsilon - \left(\frac{1-\delta^n}{1-a\delta^n}\right)(1+\varepsilon-a)\right\}.
\end{aligned}$$

Thus, $A \geq 0$ for $\delta \geq \delta' := (1+\varepsilon)^{-1/n}$. As δ' does not depend on t, Step 1 is proved. ■

Step 2. *There exists a finite, nonincreasing round-robin sequence $\{x^k\}_{k=0}^K$ such that $x^0 = \bar{x}$, $x^K = 0$, and $u(x^k) \leq u(\hat{x})$ for each $k = 0, \ldots, K$.*

Proof of Step 2. Let $x^0 := \bar{x}$. To define x^1, let $x_{-1}^1 = x_{-1}^0$. Let $x_1^1 = 0$ if $u_1(0, x_{-1}^0) \leq u_1(\hat{x})$. Otherwise, let x_1^1 be the $\tilde{x}_1$ for which $u_1(\tilde{x}_1, x_{-1}^0) = u_1(\hat{x})$; this equation has a unique solution, and it is in the interval $(0, x_1^0)$, since $u_1(\cdot, x_{-1}^0)$ is monotonic and $u_1(x^0) < u_1(\hat{x}) < u_1(0, x_{-1}^0)$. Note that $0 \leq x^1 \leq x^0$, $u_1(x^1) \leq u_1(\hat{x})$, and by (PS), $u_j(x^1) < u_j(\hat{x})$ for $j \neq i$.

Now suppose that for some $k \geq 1$, profiles $x^0, \ldots, x^k$ have been defined, and they satisfy $0 \leq x^k \leq x^{k-1}$ and $u(x^k) \leq u(\hat{x})$. Let $i = k+1 \pmod n$. Define $x_{-i}^{k+1} := x_{-i}^k$. Let $x_i^{k+1} = 0$ if $u_i(0, x_{-1}^k) \leq u_i(\hat{x})$. Otherwise, let x_i^{k+1} be the unique $\tilde{x}_i \in \left(0, x_i^k\right]$ for which $u_i(\tilde{x}_i, x_{-i}^k) = u_i(\hat{x})$. By (PS), we have $u(x^{k+1}) \leq u(\hat{x})$.

This defines a nonincreasing and bounded round-robin sequence $\{x^k\}_{k=0}^{\infty}$. Let z be its limit. We have $z \leq x^k$ for all $k > 0$, and $u(z) \leq u(\hat{x})$.

Assume $z > 0$, so that $N(z)$ is a coalition (nonempty). In addition, assume $u_i(z) < u_i(\hat{x})$ for some $i \in N(z)$. By continuity, $\tilde{x}_i \in (0, z_i)$ exists such that $u_i(\tilde{x}_i, z_{-i}) < u_i(\hat{x})$. Since

$x^k \to z$, there exists k' such that $u_i(\tilde{x}_i, x^k_{-i}) < u_i(\hat{x})$ for all $k > k'$. But then, the construction of the sequence implies that for any $k > k'$ such that $i = k + 1 \pmod n$, $x_i^{k+1} < \tilde{x}_i < z_i$. This contradicts $z_i \leq x_i^{k+1}$. Thus, $u_i(z) = u_i(\hat{x})$ for all $i \in N(z)$. Since $z < \hat{x}$, this shows that $N(z)$ weakly underblocks $\hat{x}$. Since this contradicts the fact that $\hat{x}$ is not weakly underblocked, we conclude that $z = 0$.

We have $u(0) \ll u(\hat{x})$ by Lemma 6 (i), as $u^* = u$ and $\hat{x} > 0$ is not weakly underblocked. Thus, K' exists such that $u_i(0, x^k_{-i}) < u(\hat{x})$ for all $k \geq K'$ and $i \in N$. The construction of the sequence thus implies the existence of $K \leq K' + n$ such that $x^K = 0$. ■

Step 3. *There exists $\underline{\delta} < 1$ and a path $\vec{x}$ converging to x such that $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for $\delta > \underline{\delta}$.*

Proof of Step 3. Reverse the round-robin sequence obtained in Step 2, and add enough copies of 0 to its beginning and $\bar{x}$ to its end to obtain a finite, nondecreasing round-robin path. This yields a path, $\{z^t\}_{t=0}^T$, from $z^0 = 0$ to $z^T = \bar{x}$, that has player 1 moving first and player n moving last ($z^{T-1}_{-n} = \bar{x}_{-n}$). To the end of of this path add the round-robin sequence obtained in Step 1: $z^{T+s} = x^s$ for all integers $s \geq 0$. This yields a path $\vec{z} = \{z^t\}_{t=0}^\infty$ that is feasible for $\vec{N}^R$ and converges to x. To be notationally consistent, relabel the path as $\vec{x} := \vec{z}$.

Let $t \geq 1$ and $i \in N_t^R$, so that $i = t \pmod n$. If $t > T$ and $\delta > \delta'$ Step 1 implies

$$u_i(x_i^{t-1}, x_{-i}^t) \leq U_i(\vec{x}, \delta). \tag{30}$$

If $t \leq T$, then since $x^t_{-i} = x^{t-1}_{-i}$, Step 2 implies

$$u_i(x_i^{t-1}, x_{-i}^t) = u_i(x^{t-1}) \leq u_i(\hat{x}) < u_i(x).$$

Therefore, since $U_i(\vec{x}, \delta) \to u_i(x)$ as $\delta \to 1$, there exists $\delta_t < 1$ such that (30) holds for $\delta > \delta_t$. We conclude that (30) holds for all $t \geq 1$, $i \in N_t^R$, and $\delta > \underline{\delta} := \max(\delta', \delta_1, \ldots, \delta_T)$. Thus, by (PD) and Lemma 1, $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for all $\delta \in (\underline{\delta}, 1)$. ■

Appendix E. Proofs Missing from Section 7

Theorem 1 is immediate from Propositions $2 - 4$ and the following result.

Lemma E1. $c\ell D_0 = D$.

Proof. Since D contains both the solo profiles and the satiation profiles that are not weakly underblocked, $D_0 \subseteq D$. It remains to show that any point in D is a limit point of D_0. So, let

$x^* \in D$. Then $X^* \in [\bar{Y}, Y_N]$. Choose $Y \in (\bar{Y}, Y_N)$ in the following way:

(a) For $X^* = Y_N$, choose $Y < X^*$ so that for all coalitions $S \neq N$, $Y > Y_S$.

(b) For $X^* < Y_N$, choose $Y > X^*$ so that for all coalitions S, if $Y > Y_S$ then $X^* \geq Y_S$.

Define $\hat{x}$ by $\hat{x}_i := v_i'(Y_N)Y$. Then $\hat{x} \in D$, by Lemma 7. Note that $\hat{x} \gg 0$ and $\hat{X} = Y$. Fix $\lambda \in (0, 1)$, and define $x := (1-\lambda)\hat{x} + \lambda x^*$. We shall show x is not weakly underblocked. This will imply, since $x \gg 0$ and $X \in (\bar{Y}, Y_N)$, that $x \in D_0$. This completes the proof, as λ can be arbitrarily close to 1.

So assume x is weakly underblocked, say by a coalition S using a profile z. Then $z < x$, $z_{-S} = 0$, and $u_S^*(z) \geq u_S(x)$. Since x is a non-solo satiation profile, Lemma B1 allows us to assume $u_S^*(z) = u_S(z)$. Hence, $u_S(x) \leq u_S(z)$. Summing these inequalities over S yields

$$\sum_{i \in S} v_i(X) - X_S \leq f_S(Z). \tag{31}$$

Thus, since Y_S maximizes f_S and $f_S(Y_S) = V(S)$,

$$\sum_{i \in S} v_i(X) - X_S \leq V(S). \tag{32}$$

Since $X_S \leq X$, (31) also implies $f_S(X) \leq f_S(Z)$. Hence, since $X > Z$ and f_S is strictly concave,

$$X > Y_S. \tag{33}$$

This implies $S \neq N$, since $X < Y_N$. The remainder of the proof depends on how Y was chosen.

Case (a). In this case $X^* = Y_N > X$, and so (33) yields $X^* > Y_S$. Furthermore, since $S \neq N$, the way $Y = \hat{X}$ was chosen in this case implies $\hat{X} > Y_S$. Hence, because $\hat{x} \in D$ and $x^* \in D$, the first part of Proposition 1 implies

$$\sum_{i \in S} v_i(\hat{X}) - \hat{X}_S \geq V(S) \quad \text{and} \quad \sum_{i \in S} v_i(X^*) - X_S^* \geq V(S). \tag{34}$$

Now, since each v_i is strictly concave, $\lambda \in (0, 1)$, and $\hat{X} \neq X^*$, we have

$$\begin{aligned} \sum_{i \in S} v_i(X) - X_S &= \sum_{i \in S} v_i\left[(1-\lambda)\hat{X} + \lambda X^*\right] - \left[(1-\lambda)\hat{X}_S + \lambda X_S^*\right] \\ &> (1-\lambda)\left[\sum_{i \in S} v_i(\hat{X}) - \hat{X}_S\right] + \lambda\left[\sum_{i \in S} v_i(X^*) - X_S^*\right]. \end{aligned}$$

This and (34) imply $\sum_{i \in S} v_i(X) - X_S > V(S)$, contrary to (32). Hence, x must not be weakly underblocked.

Case (b). In this case $\hat{X} = Y > X^*$, and so $\hat{X} > X$. Now (33) implies $\hat{X} > Y_S$. This and the way $Y = \hat{X}$ was chosen in this case imply $X^* \geq Y_S$. The fact that $\hat{X} > Y_S$ and $\hat{x} \in D$ again imply the first inequality in (34). The second inequality in (34) holds for the same reason if $X^* > Y_S$, and it also holds if $X^* = Y_S$, as then

$$\begin{aligned}\sum_{i \in S} v_i(X^*) - X_S^* &= \sum_{i \in S} v_i(Y_S) - Y_S + X^* - X_S^* \\ &= V(S) + X^* - X_S^* \geq V(S).\end{aligned}$$

So (34) again holds, and the remaining proof is the same as in case (a). ■

Proof of Theorem 2 (i). Let $\hat{u} \in P \cap W(\vec{N})$. Let $\{u^k\}$ be a sequence in $W(\vec{N})$ that converges to $\hat{u}$. Then for each k, $\delta_k < 1$ and an equilibrium path $\vec{x}^k$ of $\Gamma(\delta_k, \vec{N})$ exist such that $u^k = U(\vec{x}^k, \delta_k)$. By Lemma 2, each $\vec{x}^k$ converges. Denote its limit as x^k. Lemma 3 implies $u^k \leq u(x^k)$. Proposition 2 implies $x^k \in D$. Proposition 1 (or Lemma E1) implies D is closed, and Corollary 1 implies it is bounded. So $\{x^k\}$ has a subsequence that converges to some $\hat{x} \in D$. Taking limits on both sides of $u^k \leq u(x^k)$ along the subsequence yields $\hat{u} \leq u(\hat{x})$. This implies $\hat{u} = u(\hat{x})$, since $\hat{u} \in P$. Hence, $\hat{u} \in P \cap D$. From (9) we have $P \cap D = u(C)$. Therefore, $P \cap W(\vec{N}) \subseteq u(C)$. ■

Proof of Theorem 2 (ii). By Lemma 9 (proved below), $u(D) \subseteq W(\vec{N})$. Hence, since $u(C) = P \cap u(D)$, we have $u(C) \subseteq P \cap W(\vec{N})$. Equality follows from part (i). ■

Proof of Lemma 9. Let $x \in D_0$. By Proposition 4, $\underline{\delta} < 1$ and a feasible path $\vec{x}$ exist such that $\vec{x}$ converges to x, and $\vec{x}$ is an equilibrium path for all $\delta > \underline{\delta}$. Hence, $U(\vec{x}, \delta) \in W(\vec{N})$ for all $\delta > \underline{\delta}$. Taking $\delta \to 1$ yields $u(x) \in W(\vec{N})$, since $U(\vec{x}, \delta) \to u(x)$. This proves $u(D_0) \subseteq W(\vec{N})$. Hence, $u(D) \subseteq W(\vec{N})$, since u is continuous, $c\ell D_0 = D$ (Lemma E1), and $W(\vec{N})$ is closed. ■

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