Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://www.econ.upenn.edu/pier

PIER Working Paper 08-023

"Interactive Knowledge with Unawareness"

by

Jing Li

http://ssrn.com/abstract=1157319

Interactive Knowledge with Unawareness†

Jing Li

Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104
E-mail: jing.li@econ.upenn.edu

June 2008

†I am indebted to Bart Lipman and Larry Samuelson for invaluable guidance and discussions on this project. I am grateful for Pierpaolo Battigalli, Eddie Dekel, Malcolm Forster, George Mailath and Bill Sandholm and seminar participants at BU, CMU, Georgia State University, Northwestern, Penn, Penn State, Queen's, Toronto, UCLA, UIUC and UW-Madison for suggestions and comments. Financial support from the Richard E. Stockwell Fellowship is gratefully acknowledged.

Abstract

This paper extends Li (2008b) to the multi-agent environment, where players reason about each other's awareness as well as knowledge, subject to their own awareness constraints. I characterize the interactive knowledge hierarchies under unawareness, which significantly differ from those in the standard information partition model by allowing for false interactive knowledge. Aumann's classic characterization of common knowledge does not immediately apply in this environment, even if there is "common awareness" of the event involved. An alternative characterization of common knowledge is provided.

Keywords: unawareness, state space models, interactive knowledge, common knowledge
JEL Classification: C70, C72, D80, D82, D83

1 Introduction

A person is unaware of an event E if he does not know E, and he does not know that he does not know E, and so on. Li (2008b) proposes a product model that generalizes Aumann's information partition model to accommodate non-trivial unawareness. This paper extends Li (2008b) to the multi-agent environment, where players reason about each other's awareness as well as knowledge, subject to their own awareness constraints. In particular, player i could be unaware that player j might be unaware of an event E, of which i is aware himself. It seems plausible to assume, in this case, i would take it for granted that j reasons about E. I characterize interactive knowledge hierarchies where a player has false interactive knowledge if and only if the player is unaware of others' unawareness.[1]

The concept of common knowledge has been playing an important role in game-theoretic analysis. An event E is common knowledge if everybody knows E, and everybody knows everybody knows E, and so on. To analyze any strategic situation where unawareness is involved, it is critical to understand how the presence of unawareness affects common knowledge among players. Intuitively, introducing the possibility of being unaware of an event obviously makes it harder for players to arrive at common knowledge. On the other hand, the presence of unawareness may also reduce higher-order uncertainties among players. In general, Aumann's classic characterization of common knowledge in the standard information partition model plus an additional clause that captures "'common knowledge' of awareness of E" are sufficient to deliver common knowledge but not necessary. In particular, at the presence of unawareness, Aumann's classic formula is not necessary for common knowledge: it is possible for players to have common knowledge of an event under unawareness, even if they would only have mutual knowledge had all been fully aware. However, once one rules out unawareness of uncertainties in information structures, the additional "common knowledge of awareness" clause does characterize all implications of unawareness when it comes to common knowledge.

2 The Primitives

I consider a set of questions Q^* that summarizes all relevant uncertainties in the environment. Without loss of generality, for each question $q \in Q^*$, let there be two possible answers, denoted by 1_q and 0_q. The *full state space* describing all possible scenarios is denoted by Q^* and is defined by:

$$Q^* = \Pi_{q \in Q^*} \{1_q, 0_q\} \times \{\Delta\}$$

where Δ stands for "*cogito ergo sum*." Let $N = \{1, \cdots, n\}$ be the set of players. Player i's information structure is represented by a pair (W_i^*, P_i^*), where W_i^* maps each full state

[1] Li (2008a) models unawareness in both the single-agent environment and the multi-agent environment without imposing the product structure.

to a subset of Q^*, and P_i^* maps each full state to a subset of Q^*. The interpretation is, at ω^*, player i is aware of questions in the set $W_i^*(\omega^*)$ and $P_i^*(\omega^*)$ contains all full states that are indistinguishable from ω^* for i. Thus W_i^* represents i's awareness information structure and is called the full awareness function, and P_i^* represents i's factual information structure is called the full possibility correspondence.[2] Let $\mathbf{W}^* = (W_1^*, \cdots, W_n^*)$ and $\mathbf{P}^* = (P_1^*, \cdots, P_n^*)$ denote the vector of awareness functions and full possibility correspondences.

Analogous to the single-agent case, at each full state, an agent $i \in N$ has a *multi-agent subjective model* that describes his perception of the world, including everyone's information structure. Comparing to the single-agent case, there are two complications when multiple agents are involved. First, now subjective knowledge at counterfactual states, i.e. (subjective) states i excludes, matters for higher-order interactive knowledge, because in i's subjective model, another player, say, j, may find them possible. This requires i's subjective possibility correspondence to be defined on the entire subjective space, not just subjective states i considers possible as in the single-agent case.[3] The second complication arises from interactive reasoning about each other's awareness: to allow for i's reasoning about j's awareness, i's subjective model has to be extended to include i's perception of j's awareness information at every subjective state as well as j's factual information. In other words, i's subjective model needs to be a product model itself, equipped with not only a subjective possibility correspondence but also a *subjective awareness function.*

3 Deriving Subjective Models

3.1 Subjective possibility correspondence.

Fix $\omega^* \in \Omega^*, i, j \in N$. Let i's subjective state space at ω^* be denoted by $\Omega(i_{\omega^*})$. The symbol i_{ω^*} is to be understood as the pair (ω^*, i). Let $P_j(\cdot|i_{\omega^*})$ denote j's possibility correspondence in i's subjective model at ω^*. That is, at ω^*, i perceives j's possibility set at $\omega \in \Omega(i_{\omega^*})$ to be $P_j(\omega|i_{\omega^*})$. It seems the natural definition for $P_j(\cdot|i_{\omega^*})$ would be the projection of j's factual information specified in the full model on i's subjective state space. However, this leaves an indeterminacy for counterfactual states. More specifically, the indeterminacy occurs when there are $\omega_1^*, \omega_2^* \in \Omega^*$ such that, (1), $\mathbb{P}^{\Omega(i_{\omega^*})}(P_i^*(\omega_1^*)) \neq \mathbb{P}^{\Omega(i_{\omega^*})}(P_i^*(\omega_2^*))$, i.e., in ω_1^* and ω_2^*, j receives different factual information regarding questions of which i is aware at ω^*; and (2) they are both projected to $\omega \in \Omega(i_{\omega^*}) \setminus \mathbb{P}^{\Omega(i_{\omega^*})}(P_j^*(\omega^*))$, i.e., i excludes ω at ω^*, while is unaware of the distinction between ω_1^*

[2] See Li (2008b) for more discussions.

[3] For instance, in Example 1, at $(1_r, 0_p, \Delta)$, Charlie excludes the subjective state $(0_r, \Delta)$. His (subjective) factual information at $(0_r, \Delta)$ is irrelevant to his own knowledge hierarchy at $(1_r, 0_p, \Delta)$. However, suppose there is another player Dorothy, who cannot tell whether it rains; then Charlie's subjective knowledge and hence his subjective factual information at $(0_r, \Delta)$ matters for his interactive knowledge hierarchy at $(1_r, 0_p, \Delta)$, such as his knowledge about Dorothy's knowledge about his own knowledge.

and ω_2^*. In other words, the indeterminacy occurs when i is unaware of uncertainties in j's factual information at counterfactual states.

I resolve this indeterminacy by using the product structure to select, for each ω^* and each i, a particular factual information set. Formally, for each pair (ω^*, i), I define a "label" which consists of the answers to questions of which i is unaware at ω^*:

$$u(i_{\omega^*}) = \mathbb{P}^{\Omega(Q^* \setminus W_i^*(\omega^*))}(\omega^*).$$

Slightly abusing notation, I let $\omega \times u(i_{\omega^*})$ denote the full state whose projection on $\Omega(i_{\omega^*})$ is ω, and whose answers to questions of which i is unaware coincide with ω^*. That is, $\omega \times u(i_{\omega^*})$ is the full state ω_1^* such that $\mathbb{P}^{\Omega(i_{\omega^*})}(\omega_1^*) = \omega$ and $\mathbb{P}^{\Omega(Q^* \setminus W_i^*(\omega^*))}(\omega_1^*) = u(i_{\omega^*})$. I define:

$$P_j(\omega|i_{\omega^*}) = \begin{cases} \mathbb{P}^{\Omega(i_{\omega^*})} P_j^*(\omega^*) & \text{for } \omega \in \mathbb{P}^{\Omega(i_{\omega^*})} P_j^*(\omega^*), \\ \mathbb{P}^{\Omega(i_{\omega^*})} P_j^*(\omega \times u(i_{\omega^*})) & \text{otherwise.} \end{cases} \tag{3.1}$$

This definition says that at ω^*, the subjective factual information structure is the projection of the factual information sets at those full states selected using the "label" $u(i_{\omega^*})$.[4,5]

Definition (3.1) does not impose real restrictions on the model, as one can always introduce "auxiliary" questions to manipulate the order of full states. Figure 1 illustrates this idea. Boxes represent factual information, and the intersecting ovals represent the awareness information. Consider the full model depicted in the unshaded area on the left. The agent's subjective models are depicted in the shaded area. In particular, at $(1_a, 0_b, \Delta)$, definition (3.1) yields a non-partitional factual information structure in the agent's subjective model. Alternatively, one may wish to model the situation in which the agent has the information partition $\{\{(1_a, \Delta)\}, \{0_a, \Delta\}\}$ in the subjective model at $(1_a, 0_b, \Delta)$. This can be achieved by adding an auxiliary question c to reorder the full states, so the "label" picks up the desired factual information sets, as shown in the graph on the right.

3.1.1 Subjective awareness function.

Let $W_j(\cdot|i_{\omega^*})$ denote j's awareness function in i's subjective model at ω^*. That is, at ω^*, i perceives j's awareness information at $\omega \in \Omega(i_{\omega^*})$ to be $W_j(\omega|i_{\omega^*})$. Of course, i can

[4] The label yields "consistent" selection of factual information sets. Suppose $W_i^*(\omega^*) \supseteq W_k^*(\omega^*)$, then the following is true:

$$\{\omega \times u(i_{\omega^*}) : \omega \in \Omega(i_{\omega^*})\} \supseteq \{\omega \times u(k_{\omega^*}) : \omega \in \Omega(k_{\omega^*})\}.$$

It follows that, take any subjective state $\omega_k \in \Omega(k_{\omega^*}) \setminus \mathbb{P}^{\Omega(k_{\omega^*})}(P_j^*(\omega^*))$, i.e. ω_k is excluded by j's factual information at ω^*, then there must exist a $\omega_i \in \Omega(i_{\omega^*}) \setminus \mathbb{P}^{\Omega(i_{\omega^*})}(P_j^*(\omega^*))$ such that $\omega_k \times u(k_{\omega^*}) = \omega_i \times u(i_{\omega^*})$.

[5] The use of the label makes crucial use of the product structure. Thus, although the product structure in the single-agent model is without loss of generality, it is not in the general case of the multi-agent model.

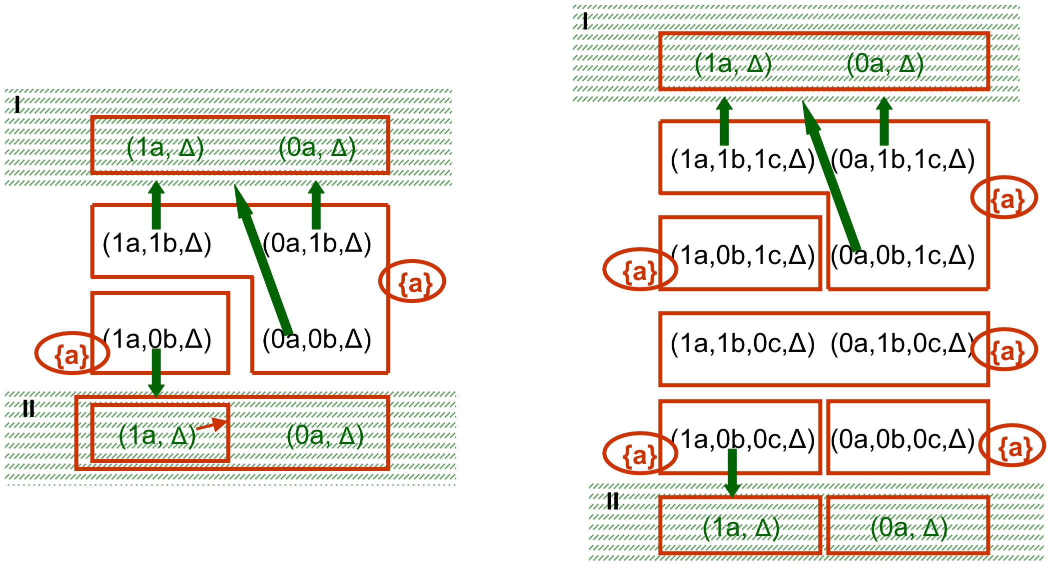

Figure 1: Subjective possibility correspondence.

only reason about j's awareness within i's own awareness. Thus, for any $\omega \in \Omega(i_{\omega^*})$, a natural definition for $W_j(\omega|i_{\omega^*})$ is to take the intersection of $W_i^*(\omega^*)$ and $W_j^*(\omega_1^*|i_{\omega^*})$ for some $\omega_1^* \in \Omega^*$ that is projected to ω. But as before, there seems to be an indeterminacy if there are two full states ω_1^*, ω_2^*, both projected to $\omega \in \Omega(i_{\omega^*})$, such that

$$W_i^*(\omega^*) \cap [W_j^*(\omega_1^*) \bigtriangleup W_j^*(\omega_2^*)] \neq \emptyset,$$

where $\bigtriangleup$ denotes the symmetric difference of two sets. Intuitively, in this case, for any $q \in W_i^*(\omega^*) \cap [W_j^*(\omega_1^*) \bigtriangleup W_j^*(\omega_2^*)]$, i is unaware that j could be unaware of question q, of which i is aware himself. In other words, i is unaware of uncertainties in j's awareness information structure. However, unlike the case of unawareness of uncertainties in the factual information structure, it seems here the *only* plausible scenario is that i should take it for granted that j reasons about q. After all, i considers q a relevant question and j a relevant player, if he is unaware that j could be unaware of q, then how could he not reason about j's reasoning about q? This prompts the following definition:

$$W_j(\omega|i_{\omega^*}) = W_i^*(\omega^*) \cap \Big[\bigcup_{\{\omega_1^*:\ \mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*)=\omega\}} W_j^*(\omega_1^*)\Big]. \tag{3.2}$$

Let $\mathbf{W}(\cdot|i_{\omega^*}) = (W_1(\cdot|i_{\omega^*}), \cdots, W_n(\cdot|i_{\omega^*}))$ and $\mathbf{P}(\cdot|i_{\omega^*}) = (P_1(\cdot|i_{\omega^*}), \cdots, P_n(\cdot|i_{\omega^*}))$. For ease of notation, let $\theta^1 = (\omega^*, i)$, and write $\Omega(\theta^1) \equiv \Omega(i_{\omega^*})$. Then i's subjective model at ω^* is:

$$(\Omega(\theta^1), \mathbf{W}(\cdot|\theta^1), \mathbf{P}(\cdot|\theta)).$$

Let it be denoted by $\mathcal{M}(\theta^1)$. Intuitively, this is the "full" model from i's own perspective. The second-order subjective models that describes i's perception of every player's perception of the world can be constructed by applying (3.1) and (3.2) to $\mathcal{M}(\theta^1)$ for every subjective state $\omega \in \Omega(\theta^1)$ and every player $j \in N$, and so on for higher-order subjective models.

Formally, first let $\Theta^1 = \{((\omega_1, i^1)) : \omega_1 \in \Omega^*, i^1 \in N\}$. Suppose the subjective model $\mathcal{M}(\theta^k)$ is defined for all $\theta^k \in \Theta^k, k = 1, \cdots, n-1$. Fixing $\theta^k \in \Theta^k$, let $\theta^l, l \leq k$ denote the tuple that consists of the first l elements of θ^k. That is, suppose $\theta^k = ((\omega_1, i^1), \cdots, (\omega_k, i^k))$, then $\theta^l = ((\omega_1, i^1), \cdots, (\omega_l, i^l)), l \leq k$. Of course, $\theta^l \in \Theta^l$. Let $+$ denote concatenation. I define the n-th order subjective models inductively as follows. First, the set of relevant n-th order reasoning sequences is:

$$\Theta^n = \left\{\theta + (\omega, j) : \theta \in \Theta^{n-1}, \omega \in \Omega(\theta), j \in N\right\}. \tag{3.3}$$

For ease of notation, I write $P(\theta^k) \equiv P_{i^k}(\omega_k|\theta^{k-1})$ and $W(\theta^k) \equiv W_{i^k}(\omega_k|\theta^{k-1}), k = 1, \cdots, n$. Then for every $\theta^n \in \Theta^n$, the relevant higher-order subjective state space is:

$$\Omega(\theta^n) = \prod_{q \in W(\theta^n)} \{1_q, 0_q\} \times \{\Delta\}. \tag{3.4}$$

Now for all $\omega \in \Omega(\theta^n)$ and any $j \in N$,

$$W_j(\omega|\theta^n) = W(\theta^n) \cap [\bigcup_{\left\{\omega' \in \Omega(\theta^{n-1}):\, \mathbb{P}^{\Omega(\theta^n)}(\omega') = \omega\right\}} W_j(\omega'|\theta^{n-1})], \tag{3.5}$$

$$P_j(\omega|\theta^n) = \begin{cases} \mathbb{P}^{\Omega(\theta^n)} P_j(\theta^n) & \text{for } \omega \in \mathbb{P}^{\Omega(\theta^n)} P_j(\theta^n), \\ \mathbb{P}^{\Omega(\theta^n)} P_j(\omega \times u(\theta^n)|\theta^{n-1}) & \text{otherwise,} \end{cases} \tag{3.6}$$

where $u(\theta^n) = \mathbb{P}^{\Omega(W(\theta^{n-1}) \setminus W(\theta^n))}(\omega_n)$. Notice $\omega \times u(\theta^n)$ is the subjective state in $\Omega(\theta^{n-1})$ where the uncertainties of which i^n is aware are resolved as in ω and the uncertainties of which i^n is unaware are resolved according to the "label" $u(\theta^n)$.

The tuple $\mathcal{M}(\theta^n) = (\Omega(\theta^n), \mathbf{W}(\cdot|\theta^n), \mathbf{P}(\cdot|\theta^n))$ describes i^1's perception at ω_1 of i^2's perception at ω_2 of $\cdots$ of i^n's subjective model at ω_n. Since players can only reason within their own awareness, their subjective models of different orders are "nested": for any n and any $\theta^n \in \Theta^n$, $W(\theta^k) \subseteq W(\theta^{k-1})$ for all $k = 1, \cdots, n$.

The tuple $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ along with definitions (3.4)-(3.6) describes the full multi-agent product model. A full state in Ω^* completely resolves all uncertainties in the environment, including what each player is aware of, what they believe others are aware of, and so on. Thus a full state generalizes the concept of a state in standard state-space models.

3.1.2 Richness and product factual partition conditions.

I am most interested in situations where all subjective models have rational information structures. However, that information structures are rational in the full model does not guarantee all information structures are rational in subjective models. Hence, some additional conditions are needed.

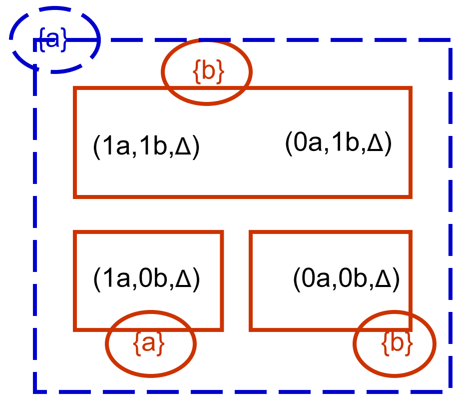

Figure 2: A full state space that is not rich.

In Figure 2, the solid lines represent Alice's information structure and the dash lines represent Bob's information structure. Both information structures are rational. At $(1a, 1b, \Delta)$, Alice is unaware of question a, and unaware that she is unaware of a, while at $(1a, 0b, \Delta)$, she is not only aware of a, but also aware that she would be unaware of a had $0a$ been true.[6] Moreover, Alice's factual information regarding a also differs depending on the answer to question b. Thus, question b plays multiple roles: it resolves uncertainties in Alice's awareness of a, Alice's awareness of possible unawareness of a, and Alice's factual information about a. As a consequence, Bob cannot be unaware of Alice's possible unawareness of a when $1a$ is true without also perceiving Alice to be aware that she would be unaware of a had $0a$ been true, and Bob must perceive Alice's subjective factual information set to be $\{(1a, \Delta), (0a, \Delta)\}$ at states $(1a, 1b, \Delta)$ and $(0a, 1b, \Delta)$. But then it follows Alice's information structure is irrational in Bob's subjective model.

The problem is, the full state space in Figure 2 simply does not have enough "dimensions" to accommodate higher-order interactive reasonings to their full extent. Thus I require different orders of unawareness to be coded using different questions in situations like this.

Definition 1 *The product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *is* **rich** *if, for all* $i \in N$ *and* $\Omega \in \mathcal{S}$, $\mathbb{P}^{\Omega}(\omega_1^*) = \mathbb{P}^{\Omega}(\omega_3^*), \omega_2^* \in P_i^*(\omega_1^*), q \in q(\Omega) \cap [W_i^*(\omega_3^*) \setminus W_i^*(\omega_1^*)]$ *imply there exists some* ω_4^* *such that* $\mathbb{P}^{\Omega}(\omega_2^*) = \mathbb{P}^{\Omega}(\omega_4^*)$ *and* $q \in W_i^*(\omega_4^*)$.

In the context of the above example, the richness condition requires the full state space to contain states in which $1a$ is true and Alice is aware of a, but unaware that she would be unaware of a had $0a$ been true. Let $\Theta = \cup_{n=1}^{\infty} \Theta^n$. The following result shows that the richness condition ensures all subjective models have "partitional" awareness information when the full information structures are rational:

[6] At $(1a, 0b, \Delta)$, Alice's subjective model has two subjective states, $(1a, \Delta)$, in which she is aware of a, and $(0a, \Delta)$, in which she is unaware of a.

Lemma 1 *Let the product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be rich and rational. Then for any* $\theta \in \Theta$ *and any* $j \in N$, $\omega' \in P_j(\omega|\theta)$ *implies* $W_j(\omega'|\theta) = W_j(\omega|\theta)$.

It follows that all players know their own subjective models, and all players know all players know their own subjective models, and so on. In this sense, there is "common knowledge" of *subjective models* among all players who are aware of them, which generalizes the case in the standard model that information structures are "common knowledge" among all players.[7]

Corollary 2 *Suppose the product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *is rich and rational. Then for any* $\theta = ((\omega_1, i^1), \cdots, (\omega_n, i^n)) \in \Theta$, *the tuple* $\theta' = ((\omega_1, i^1), \cdots, (\omega_n, i^n), (\omega, i^n))$ *where* $\omega \in P_{i^n}(\mathbb{P}^{\Omega(\theta)}(\omega_n)|\theta)$ *is an element in* Θ, *and the two subjective models are identical:*

$$\mathcal{M}(\theta') = \mathcal{M}(\theta).$$

Next, I consider a condition that ensures factual information in *all* possible subjective models is partitional.

Definition 2 *The possibility correspondence* $P^* : \Omega^* \to 2^{\Omega^*} \setminus \{\emptyset\}$ *satisfies* **product factual partition** *if it induces an information partition over* Ω^*, *and that for all* $\omega^* \in \Omega^*$, $P^*(\omega^*)$ *can be written as a product set. That is, for every* ω^*,

$$P^*(\omega^*) = \left\{\omega_0^* \in \Omega^* : \mathbb{P}^{\{1_q, 0_q\}}(\omega_0^*) \in \pi_{\omega^*}^q(\mathbb{P}^{\{1_q, 0_q\}}(\omega^*)),\ \forall q \in Q^*\right\},$$

where $\pi_{\omega^*}^q$ *is a partition over the set* $\{1_q, 0_q\}$ *and* $\pi_{\omega^*}^q(\omega)$ *is the partition element containing* ω, *for* $\omega = 1_q, 0_q$.

This condition says all factual information consists of independent pieces of information regarding the answers to each question. It rules out information sets such as $\{(1a, 1b, \Delta), (0a, 1b, \Delta), (0a, 0b, \Delta)\}$ in Figure 1, and hence ensures that for *every* subjective state space, the "label" selects factual information sets that are projected to a partition.

Lemma 3 *Fix a product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$. *If* P_j^* *satisfies product factual partition, then for all* $\theta \in \Theta$, $P_j(\cdot|\theta)$ *induces an information partition over* $\Omega(\theta)$.

4 Interactive knowledge hierarchy.

Analogous to the single-agent case, the objective interactive knowledge hierarchy is derived by tracking the "true" subjective states in the subjective interactive knowledge

[7]See, for example, Aumann (1987) and Brandenburger and Dekel (1993).

hierarchy in the corresponding subjective models. The complication is, since the multi-agent subjective models are product models themselves, knowledge in them may need to be further derived from yet higher-order subjective models.

For example, the objective event "i knows j knows E," denoted by $K_iK_j(E)$, is obtained by tracking the corresponding subjective version of this knowledge in i's subjective model. In particular, the event "j knows E" should also be the subjective version in i's subjective model, and is obtained by tracking j's subjective knowledge "I know E" in the second-order subjective model describing i's perception of j's perception. Let $\tilde{K}^j_{i_{\omega^*}}(E)$ denote the event "j knows E" in i's subjective model.

$$\tilde{K}^j_{i_{\omega^*}}(E) = \left\{\omega \in \Omega_i(\omega^*) : P_j(\omega|i_{\omega^*}) \subseteq E_{\Omega_i(\omega^*)},\ W_j(\omega|i_{\omega^*}) \supseteq q(E)\right\}$$

if $q(E) \subseteq W_i^*(\omega^*)$ and $\tilde{K}^j_{i_{\omega^*}}(E) = \emptyset_E$ otherwise.

The objective interactive knowledge, analogous to the objective knowledge, is defined by:

$$K_iK_j(E) = \left\{\omega^* \in \Omega^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega^*) \in \tilde{K}^i_{i_{\omega^*}}\tilde{K}^j_{i_{\omega^*}}(E)\right\}.$$

Similarly, the objective event "i knows j is unaware of E," denoted by $K_iU_j(E)$, is defined by:

$$K_iU_j(E) = \left\{\omega^* \in \Omega^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega^*) \in \tilde{K}^i_{\omega^*}\tilde{U}^j_{i_{\omega^*}}(E)\right\},$$

where $\tilde{U}^j_{i_{\omega^*}}(E)$ is the event "j is unaware of E" in i's subjective model at ω^*, and is defined by:

$$\tilde{U}^j_{i_{\omega^*}}(E) = \{\omega \in \Omega_i(\omega^*) : q(E) \not\subseteq W_j(\omega|i_{\omega^*})\}$$

if $q(E) \subseteq W_i^*(\omega^*)$ is set to be $\emptyset_E$ otherwise.

Extending the above to the general case, the objective interactive knowledge "i^1 knows that i^2 knows $\cdots$ knows i^n knows E," denoted by $K_{i^1}\cdots K_{i^n}(E)$, is obtained by recursively computing the relevant knowledge in the corresponding subjective models.

$$K_{i^1}\cdots K_{i^n}(E) = \left\{\omega^* \in \Omega^* : \mathbb{P}^{\Omega_{i^1}(\omega^*)}(\omega^*) \in \tilde{K}^{i^1}_{i^1_{\omega^*}}[\tilde{K}^{i^2}\cdots\tilde{K}^{i^n}]_{i^1_{\omega^*}}(E)\right\}, \tag{4.1}$$

and for all $m = 2, 3, \cdots$, and all $\theta \in \Theta$,

$$[\tilde{K}^{i^1}\cdots\tilde{K}^{i^m}]_\theta(E) = \begin{cases} \{\omega \in \Omega(\theta) : \mathbb{P}^{\Omega_{i^1}(\omega|\theta)}(\omega) \in \\ \qquad \tilde{K}^{i^1}_{\theta+(\omega,i^1)}[\tilde{K}^{i^2}\cdots\tilde{K}^{i^m}]_{\theta+(\omega,i^1)}(E)\}, & \text{if } q(E) \subseteq W(\theta), \\ \emptyset_E, & \text{if } q(E) \not\subseteq W(\theta), \end{cases} \tag{4.2}$$

and finally, for all $j \in N$, $\theta \in \Theta$,

$$\tilde{K}^j_\theta(E) = \begin{cases} \left\{\omega \in \Omega(\theta) : P_j(\omega|\theta) \subseteq E_{\Omega(\theta)}, W_j(\omega|\theta) \supseteq q(E)\right\}, & \text{if } q(E) \subseteq W(\theta), \\ \emptyset_E, & \text{if } q(E) \not\subseteq W(\theta). \end{cases} \tag{4.3}$$

Here, $[\tilde{K}^{i^1}\cdots\tilde{K}^{i^m}]_\theta(E)$ denotes the knowledge "i^1 knows that i^2 knows $\cdots$ knows i^n knows E" in the subjective model $\mathcal{M}(\theta)$. It is worth pointing out that the difference between subjective knowledge and objective knowledge is subtle in the multi-agent model. For example, the knowledge $[\tilde{K}^{i^2}\cdots\tilde{K}^{i^n}]_{i^1_{\omega^*}}(E)$ is subjective with respect to the full model, in the sense that it is the event "i^2 knows $\cdots$ knows i^n knows E" from i's perspective at ω^*. On the other hand, it is objective *within i's own subjective model at* ω^*, as it takes into account all players' unawareness at each (subjective) state in $\Omega(i_{\omega^*})$, given i's own awareness at ω^*. The subjective perspective of this knowledge is reflected in the second line of (4.2), while the objective perspective is reflected in the the first line of (4.2), which is simply formula (4.1) adapted to the subjective model $\mathcal{M}(\theta)$.

Similarly, the event "i^1 knows i^2 knows $\cdots$ knows i^n is unaware of E," denoted by $K_{i^1}\cdots U_{i^n}(E)$, can be obtained by replacing all incidents of $\tilde{K}^{i^n}$ in the above definitions by $\tilde{U}^{i^n}$, and using the single-agent characterization result proved in Li (2008b) adapted to the subjective model $\mathcal{M}(\theta)$:

$$\tilde{U}^j_\theta(E) = \begin{cases} \{\omega \in \Omega(\theta) : q(E) \subseteq W(\theta), q(E) \nsubseteq W_j(\omega|\theta)\}, & \text{if } q(E) \subseteq W(\theta); \\ \emptyset_E, & \text{if } q(E) \nsubseteq W(\theta). \end{cases} \tag{4.4}$$

The following theorem characterizes the properties of interactive knowledge hierarchies in the product model when all subjective models have rational information structures.

Theorem 4 *Let the product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be rich and rational, and* P_i^* *satisfy product factual partition for all* i*. Then for any* $E \in \mathcal{E}^p$*,*

IK1 $i^k = i^{k-1}$ *for some* $1 < k \leq n \Rightarrow K_{i^1}\cdots K_{i^n}(E) = K_{i^1}\cdots K_{i^{k-1}}K_{i^{k+1}}K_{i^n}(E)$*;*

IK2 $K_iU_j(E) = \bigcap_{n=1}^\infty K_i(\neg K)^n_j(E)$*;*

IK3 $K_iK_j(E) \subseteq K_j(E) \cup [U_j(E) \cap K_i\neg U_j(E)]$.

$IK1$ says every agent knows his own knowledge, and everybody knows that everybody knows his own knowledge, and so on. $IK2$ says i knows j is unaware of E if and only if i knows the knowledge of E is lacking from j's knowledge hierarchy at all levels.

$IK3$ says two things: first, interactive knowledge could be "false," in the sense that it could be the case that i "knows" j knows E while j actually does not know E; second, the only situation in which this could happen is one in which j is in fact unaware of E, while i is unaware that j is unaware of E. For example, suppose Holmes is unaware that Watson is unaware of the possibility of no intruder, then, knowing there was no intruder, Holmes "knows" Watson knows there was no intruder, too, due to the public nature of their factual information.[8] It is also possible for i to be aware that j may be unaware of E while being unaware that k could be unaware of E.

[8]The presence of false interactive knowledge is not inconsistent with the axiom of knowledge, which still holds in every subjective model. Whenever i knows j knows E, j indeed knows E *in i's subjective model.* The false knowledge is a consequence of an incorrect model, which arises whenever there is unawareness of unawareness.

5 Common knowledge.

An event E is common knowledge if everybody knows it, everybody knows everybody knows it, and so on. Formally, fix $i \in N$, let $\mathcal{I}_i = \cup_{m=2}^{\infty} \{(i, i^2, \cdots, i^m) : i^2, \cdots, i^m \in N\}$ and let a typical element in this set be denoted by I_i. For notational ease, take $I = (i^1, \cdots, i^m)$, I write $K_I(E) \equiv K_{i^1} \cdots K_{i^m}(E)$. Then the event "$E$ is common knowledge among all players in N" is simply:

$$CK(E) \equiv \bigcap_{i=1}^{n} \bigcap_{I_i \in \mathcal{I}_i} K_{I_i}(E). \tag{5.1}$$

Common knowledge plays a critical role in economic analysis, especially in game-theoretic models. For example, it is the key assumption in various "no-trade" type results.[9] The formal analysis of common knowledge in the economics literature greatly benefits from Aumann's classic characterization of this concept (Aumann 1976): In the standard model $(\Omega^*, \mathbf{P}^*)$, for any $E \subseteq \Omega^*$,

$$CK(E) = \left\{\omega^* \in \Omega^* : \ \wedge_{j=1}^{n} P_j^*(\omega^*) \subseteq E\right\}, \tag{5.2}$$

where $\wedge_{j=1}^{n} P_j^*$ denotes the meet of the information partition generated by the agents' possibility correspondences, and $\wedge_{j=1}^{n} P_j^*(\omega^*)$ is the partition element containing ω^* in the meet.

My goal is to characterize common knowledge in the presence of interactive unawareness. Formula (5.2), with E_{Ω^*} replacing E, is obviously too weak. A natural candidate in the environment with unawareness seems to be the following formula containing a "common awareness of E" clause in addition to (5.2):

$$\underline{CK}(E) = \left\{\omega^* \in \Omega^* : \ q(E) \subseteq \cap_{j=1}^{n} \bigcap_{\omega_1^* \in \wedge_{j=1}^{n} P_j^*(\omega^*)} W_j^*(\omega_1^*), \ \wedge_{j=1}^{n} P_j^*(\omega^*) \subseteq E_{\Omega^*}\right\} \tag{5.3}$$

Indeed, the following theorem says $\underline{CK}(E)$ implies E is common knowledge.

Theorem 5 *In the product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$, *if* P_i^* *satisfies product factual partition for all* $i \in N$, *then for all* $E \in \mathcal{E}^p$,

$$\underline{CK}(E) \subseteq CK(E).$$

However, (5.3) turns out to be hardly necessary. Consider the following two examples. First, recall the hearing problem example. Suppose there is a second player Dorothy, who is unaware that Charlie has a hearing problem, and her full information

[9] See Geanakoplos (1992) for more discussion on the role of common knowledge in the economics literature.

partition is $\{\{(1_r, 1_p, \Delta), (1_r, 0_p, \Delta)\}, \{(0_r, 1_p, \Delta), (0_r, 0_p, \Delta)\}\}$. It is easy to check that at $(1_r, 0_p, \Delta)$, both Dorothy and Charlie know it rains and know that both know it rains, and so on. In this case, common knowledge is achieved without satisfying the right-hand side of even (5.2):[10] At $(1_r, 0_p, \Delta)$, Dorothy's full factual information does not exclude $(1_r, 1_p, \Delta)$, where Charlie does not know it rains.

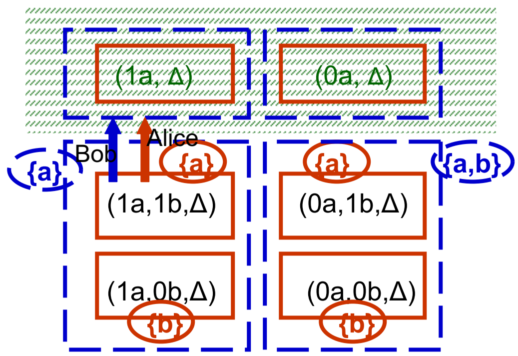

Figure 3: Common knowledge.

For the second example, consider the product model illustrated in the unshaded area of Figure 3. Alice's information structure is depicted in solid lines, while Bob's information structure is depicted in dash lines. At $(1_a, 1_b, \Delta)$, both Bob and Alice are only aware of question a, their first-order subjective models coincide and it is illustrated in the shaded area.[11] It is easy to check that the event $\{1_a\}$ is common knowledge at $(1_a, 1_b, \Delta)$. However, $(1_a, 1_b, \Delta) \notin \underline{CK}(\{1_a\})$, as the "common knowledge of awareness" clause in (5.3) is violated: at $(1_a, 1_b, \Delta)$, Bob's factual information does not exclude $(1_a, 0_b, \Delta)$, in which Alice is unaware of question a.

In both cases, unawareness makes it *easier* for players to arrive at common knowledge by reducing higher-order uncertainties in information structures. Had Dorothy been fully aware at $(1_r, 0_p, \Delta)$, her uncertainty about Bob's factual information would have prevented her from knowing Bob knows it rains and hence "it rains" cannot be common knowledge. Similarly, had Bob been fully aware at $(1_a, 1_b, \Delta)$, his uncertainty about Alice's awareness information would have prevented him from knowing Alice knows the answer to question a is "yes" and hence $\{1_a\}$ cannot be common knowledge.[12] These observations yield a tight upper bound for common knowledge, that is, mutual knowledge of E among players:

$$CK(E) \subseteq \{\omega^* : P_i^*(\omega^*) \subseteq E_{\Omega^*}, W_i^*(\omega^*) \supseteq q(E)\ \forall i\}.$$

[10] In other words, in an environment with unawareness, an event can be common knowledge without containing a self-evident event for all players.

[11] The subjective awareness information in all subjective states is $\{a\}$, which I omit in the picture for simplicity.

[12] Consider the full state $(0_a, 1_b, \Delta)$ and the event $\{0_a\}$ in Figure 3.

In light of the above, to obtain a characterization of common knowledge at the presence of unawareness, I focus attention on cases where there is no unawareness of uncertainties in information structures.

Definition 3 *The full possibility correspondence P^* satisfies* **cylinder factual partition** *if it induces an information partition over Ω^*, and that for all $\omega^* \in \Omega^*$, $P^*(\omega^*)$ is a cylinder event. That is, for every $\omega^* \in \Omega^*$,*

$$P^*(\omega^*) = \left\{\omega_0^* \in \Omega^* : \mathbb{P}^{\{1_q,0_q\}}(\omega_0^*) \in \pi^q(\mathbb{P}^{\{1_q,0_q\}}(\omega^*)),\ \forall q \in Q^*\right\},$$

where π^q is a partition over $\{1_q, 0_q\}$.

Cylinder factual partition strengthens product factual partition by requiring the decomposition of factual information regarding the answers to each question to be independent of the full states. Under this condition, in all full states projected to the same subjective state, the factual information sets have the same projection in that subjective state space. Thus, this condition rules out unawareness of uncertainties in the factual information structure.[13]

Definition 4 *The awareness function W^* satisfies* **nice awareness** *if, fixing $\omega_1^*, \omega_2^* \in \Omega^*$, suppose $q \in Q^*$ satisfies $\mathbb{P}^{\{1_q,0_q\}}(\omega_1^*) = \mathbb{P}^{\{1_q,0_q\}}(\omega_2^*)$, then*

$$q \notin [W^*(\omega_1^*) \triangle W^*(\omega_2^*)].$$

Nice awareness says, if two full states coincide in their answers to a question q, then the agent is either aware of q in both states or unaware of q in both states. This condition rules out unawareness of uncertainties in the awareness information structure: in all full states projected to the same subjective state, the agent's awareness information never differs on the set of questions specified in that subjective state space. Thus, i is never unaware that j could be unaware of a question of which i is aware himself. It is also worth noting that the richness condition is vacuous when the nice awareness condition is satisfied.

Definition 5 *The pair $(\mathbf{W}^*, \mathbf{P}^*)$ is* **strongly rational** *if (W_i^*, P_i^*) is rational and satisfies cylinder factual partition and nice awareness conditions for all $i \in N$.*

The next theorem confirms that indeed, as long as there is no unawareness of uncertainties in information structures, formula (5.3) characterizes common knowledge.

Theorem 6 *Let $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ be strongly rational. Then for all $E \in \mathcal{E}^p$,*

$$\underline{CK}(E) = CK(E).$$

[13] Under this condition, the "label" in the definition of the subjective possibility correspondence is vacuous. Thus, to model situations where all players are aware of all uncertainties in the factual information structure, the product structure is again without loss of generality. Li (2008a) has more details.

Remark. Consider the full state $(1_a, 0_b, \Delta)$ in Figure 3. At this state, Bob's subjective model is identical to the one illustrated in the shaded area, in which $\{1_a\}$ is common knowledge at the subjective state $(1_a, \Delta)$ by Aumann's characterization. Thus, *from Bob's perspective*, $\{1_a\}$ is common knowledge between himself and Alice. Intuitively, at $(1_a, 0_b, \Delta)$, Bob is unaware that Alice is unaware of question a, and hence has "false" knowledge of Alice's knowledge as discussed in property $IK3$, leading to the "false" common knowledge from his perspective. This is a novel feature in the environment with interactive unawareness.

Consider definition (5.2). Notice that $\omega^* \in \bigcap_{I_i \in \mathcal{I}_i} K_{I_i}(E)$ if and only if

$$\mathbb{P}^{\Omega_i(\omega^*)}(\omega^*) \in \bigcap_{j=1}^{n} \bigcap_{I_j \in \mathcal{I}_j} [\tilde{K}^{I^j}]_{i_{\omega^*}}(E).$$

But the latter says precisely E is common knowledge in i's subjective knowledge hierarchy at ω^*. In this sense, the event $\bigcap_{I_i \in \mathcal{I}_i} K_{I_i}(E)$ can be interpreted as i's *subjective common knowledge* of E. Let it be denoted by $CK_i(E)$. Then definition (5.1) can be rewritten as:

$$CK(E) = \cap_{i=1}^{n} CK_i(E).$$

That is, an event E is common knowledge if and only if it is every player's subjective common knowledge. In the standard information partition model, this formula is vacuous by the truth axiom: an event is common knowledge if and only if any player knows it is common knowledge. The fact that with interactive unawareness, there could be a discrepancy between common knowledge and subjective common knowledge seems to raise intriguing issues, as ultimately it is subjective common knowledge that matters in individual decision-making.

6 Appendix.

6.1 Proof of Lemma 1.

Proof. Fix $i, j \in N$ and $\omega^* \in \Omega^*$. It suffices to show the claim is true for $\theta = (\omega^*, i)$, and that the subjective model $\mathcal{M}(\theta)$ is rich.

First, fix $\omega \in \Omega_i(\omega^*)$, and let $\omega' \in P_j(\omega | i_{\omega^*})$. Need to show $W_j(\omega' | i_{\omega^*}) = W_j(\omega | i_{\omega^*})$. By (3.1), there exist two full states ω_1^*, ω_2^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega$, $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*) = \omega'$, and $\omega_2^* \in P_j^*(\omega_1^*)$.

Suppose $q \in W_j(\omega | i_{\omega^*})$. Recall that $W_j(\omega | i_{\omega^*}) = W_i^*(\omega^*) \cap [\bigcup_{\{\omega_0^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega_0^*) = \omega\}} W_j^*(\omega_0^*)]$.

There are two cases to consider.

1. $q \in W_i^*(\omega^*) \cap W_j^*(\omega_1^*)$. Then since (W_j^*, P_j^*) is rational, $W_j^*(\omega_1^*) = W_j^*(\omega_2^*)$ and hence $q \in W_i^*(\omega^*) \cap W_j^*(\omega_2^*)$. It follows that $q \in W_j(\omega' | i_{\omega^*})$;

2. $q \in W_i^*(\omega^*) \cap [W_j^*(\omega_3^*) \setminus W_j^*(\omega_1^*)]$, where ω_3^* is projected to ω on $\Omega_i(\omega^*)$, i.e. $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_3^*) = \omega$. But then by richness, there exists ω_4^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_4^*) = \mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*) = \omega'$ and $q \in W_j^*(\omega_4^*)$. Again it follows that $q \in W_j(\omega'|i_{\omega^*})$.

This proves $W_j(\omega|i_{\omega^*}) \subseteq W_j(\omega'|i_{\omega^*})$. The other direction is completely symmetric.

Next, I show the subjective model $\mathcal{M}(\theta)$ is rich. Fix $\Omega \in \mathcal{S}$ that is weakly coarser than $\Omega_i(\omega^*)$. Let $\omega_k, k = 1, 2, 3$ be such that $\mathbb{P}^\Omega(\omega_1) = \mathbb{P}^\Omega(\omega_3), \omega_2 \in P_j(\omega_1|i_{\omega^*})$. Suppose $q \in q(\Omega) \cap [W_j(\omega_3|i_{\omega^*}) \setminus W_j(\omega_1|i_{\omega^*})]$. Need to show there exists some $\omega_4 \in \Omega_i(\omega^*)$ such that $\mathbb{P}^\Omega(\omega_2) = \mathbb{P}^\Omega(\omega_4)$ and $q \in W_j(\omega_4|i_{\omega^*})$.

Since $\omega_2 \in P_j(\omega_1|i_{\omega^*})$, there exist ω_2^*, ω_1^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*) = \omega_2, \mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega_1$, and $\omega_2^* \in P_j^*(\omega_1^*)$. Since $q \in W_j(\omega_3|i_{\omega^*})$, there must exist ω_3^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_3^*) = \omega_3$ and $q \in W_j^*(\omega_3^*)$. On the other hand, $q \notin q(\Omega) \cap W_j(\omega_1|i_{\omega^*}), q(\Omega) \subseteq W_i^*(\omega^*)$ imply $q \notin W_j(\omega_1|i_{\omega^*})$, which in turn implies that for all ω_0^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_0^*) = \omega_1$, we must have $q \notin W_j^*(\omega^*)$. It follows that, $q \notin W_j^*(\omega_1^*)$. Thus $q \in q(\Omega) \cap [W_j^*(\omega_3^*) \setminus W_j^*(\omega_1^*)]$ while $\omega_2^* \in P_j^*(\omega_1^*)$. By richness, there exists $\omega_4^* \in \Omega^*$ such that $\mathbb{P}^\Omega(\omega_4^*) = \mathbb{P}^\Omega(\omega_2)$, and $q \in W_j^*(\omega_4^*)$. Let $\omega_4 = \mathbb{P}^{\Omega_i(\omega^*)}(\omega_4^*)$. Then we have $q \in W_j(\omega_4|i_{\omega^*})$, as desired. □

6.2 Proof of Corollary 2.

Proof. Let $\theta = ((\omega_1, i^1), \cdots, (\omega_n, i^n)) \in \Theta$, and $\theta' = ((\omega_1, i^1), \cdots, (\omega_n, i^n), (\omega, i^n))$ where $\omega \in P_{i^n}(\mathbb{P}^{\Omega(\theta)}(\omega_n)|\theta)$. Need to show $\Omega(\theta) = \Omega(\theta')$, and for all $j \in N, \omega' \in \Omega(\theta)$, $(W_j(\omega'|\theta), P_j(\omega'|\theta)) = (W_j(\omega'|\theta'), P_j(\omega'|\theta'))$.

To see $\Omega(\theta) = \Omega(\theta')$: since $\omega \in P_{i^n}(\mathbb{P}^{\Omega(\theta)}(\omega_n)|\theta)$, by Lemma 1, $W_{i^n}(\omega|\theta) = W_{i^n}(\mathbb{P}^{\Omega(\theta)}(\omega_n)|\theta)$. Let θ^{n-1} denote the tuple consisting of the first $n-1$ elements in θ, that is, $\theta^{n-1} = ((\omega_1, i^1), \cdots, (\omega_{n-1}, i^{n-1}))$, we have:

$$\begin{aligned}
W(\theta') &= W_{i^n}(\omega|\theta) \\
&= W_{i^n}(\mathbb{P}^{\Omega(\theta)}(\omega_n)|\theta) \\
&= W_{i^n}(\omega_n|\theta^{n-1}) \cap [\bigcup_{\{\omega' \in \Omega(\theta^{n-1}): \mathbb{P}^{\Omega(\theta)}(\omega') = \mathbb{P}^{\Omega(\theta)}(\omega_n)\}} W_{i^n}(\omega'|\theta^{n-1})] \\
&= W_{i^n}(\omega_n|\theta^{n-1}) \\
&= W(\theta).
\end{aligned}$$

Therefore, $\Omega(\theta) = \prod_{q \in W(\theta)} \{1_q, 0_q\} \times \{\Delta\} = \prod_{q \in W(\theta')} \{1_q, 0_q\} \times \{\Delta\} = \Omega(\theta')$.

Now fixing $j \in N$ and $\omega' \in \Omega(\theta')$, we have:

$$\begin{aligned}
W_j(\omega'|\theta') &= W(\theta') \cap [\bigcup_{\omega'' \in \Omega(\theta): \mathbb{P}^{\Omega(\theta')}(\omega'') = \omega'} W_j(\omega''|\theta)] \\
&= W(\theta) \cap W_j(\omega'|\theta) \\
&= W_j(\omega'|\theta);
\end{aligned}$$

Next, for the subjective possibility correspondence:

$$P_j(\omega'|\theta') = \begin{cases} \mathbb{P}^{\Omega(\theta')} P_j(\omega|\theta) & \text{for } \omega' \in \mathbb{P}^{\Omega(\theta')} P_j(\omega|\theta), \\ \mathbb{P}^{\Omega(\theta')} P_j(\omega' \times u(\theta')|\theta) & \text{otherwise.} \end{cases}$$

But note $P_j(\cdot|\theta) \subseteq \Omega(\theta) = \Omega(\theta')$, and $W(\theta) = W(\theta')$ implies $\omega' \times u(\theta') = \mathbb{P}^{\Omega(W(\theta)\setminus W(\theta'))}(\omega) = \omega'$, therefore we have:

$$P_j(\omega'|\theta') = \begin{cases} P_j(\omega|\theta) & \text{for } \omega' \in P_j(\omega|\theta), \\ P_j(\omega'|\theta) & \text{otherwise.} \end{cases}$$

But this simply says $P_j(\omega'|\theta') = P_j(\omega'|\theta)$. □

6.3 Proof of Lemma 7.

Lemma 7 *Fix a product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$. *If* P_j^* *satisfies product factual partition, then for all* $\theta \in \Theta$, $P_j(\cdot|\theta)$ *induces an information partition over* $\Omega(\theta)$.

Proof. Let $\theta = (\omega^*, i)$, it suffices to show $P_j(\cdot|\theta)$ satisfies product factual partition. For any $\omega \in \Omega_i(\omega^*)$,

$$P_j(\omega|i_{\omega^*}) = \begin{cases} \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) & \text{for } \omega \in \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*), \\ \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega \times u_i(\omega^*)) & \text{otherwise.} \end{cases}$$

Now suppose $\omega \in \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*)$. Since P_j^* is an information partition, $\omega^* \in P_j^*(\omega^*)$; since P_j^* is a product set, $\omega \times u_i(\omega^*) \in P_j^*(\omega^*)$. Therefore $P_j^*(\omega \times u_i(\omega^*)) = P_j^*(\omega^*)$ since P_j^* is a partition. But this means for all $\omega \in \Omega_i(\omega^*)$, we have

$$\begin{aligned} P_j(\omega|i_{\omega^*}) &= \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega \times u_i(\omega^*)) \\ &= \mathbb{P}^{\Omega_i(\omega^*)} \left\{ \omega_0^* \in \Omega^* : \mathbb{P}^{\{1_q, 0_q\}}(\omega_0^*) \in (\pi_j)_{\omega^*}^q (\mathbb{P}^{\{1_q, 0_q\}}(\omega \times u_i(\omega^*))) \right\} \\ &= \left\{ \omega_0 \in \Omega_i(\omega^*) : \mathbb{P}^{\{1_q, 0_q\}}(\omega_0) \in (\pi_j)_{\omega^*}^q (\mathbb{P}^{\{1_q, 0_q\}}(\omega_0)), \forall q \in W_i^*(\omega^*) \right\}, \end{aligned}$$

which proves the claim. □

6.4 Proof of Theorem 4.

Proof of IK1. Fix $k, n \in \mathbb{N}$ such that $1 < k \leq n$, and $i^1, \cdots, i^n \in N^n$ such that $i^k = i^{k-1}$. Need to show $K_{i^1} \cdots K_{i^n}(E) = K_{i^1} \cdots K_{i^{k-1}} K_{i^{k+1}} K_{i^n}(E)$. For simplicity, let $i^k = i^{k-1} = i$. By the definition of interactive knowledge (equations (4.1) - (4.3)), need to show for any $\theta = ((\omega_1, i^1), (\omega_2, i^2), \cdots, (\omega_{k-2}, i^{k-2})) \in \Theta^{k-2}$, the following holds:

$$[\tilde{K}^i \tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E) = [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E). \tag{6.1}$$

Let $\omega \in [\tilde{K}^i \tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E)$. By definition (4.2), we have

$$\mathbb{P}^{\Omega_i(\omega|\theta)}(\omega) \in \tilde{K}^i_{\theta+(\omega,i)}[\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta+(\omega,i)}(E).$$

Let $\theta' = \theta + (\omega, i)$, and let $s(\theta') \equiv \mathbb{P}^{\Omega(\theta')}(\omega)$, by definition (4.3), we have:

$$P_i(s(\theta')|\theta') \subseteq [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E).$$

Since $P_i(\cdot|\theta')$ induces an information partition over $\Omega(\theta')$ (Lemma 7), $s(\theta') \in P_i(s(\theta')|\theta')$, and hence:

$$\begin{aligned} s(\theta') &\in [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E); \\ &= \left\{\omega' \in \Omega(\theta') : \mathbb{P}^{\Omega_i(\omega'|\theta')}(\omega') \in \tilde{K}^i_{\theta'+(\omega',i)}[\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'+(\omega',i)}(E)\right\}. \end{aligned}$$

It follows

$$\mathbb{P}^{\Omega_i(s(\theta')|\theta')}(s(\theta')) \in \tilde{K}^i_{\theta'+(s(\theta'),i)}[\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'+(s(\theta'),i)}(E). \tag{6.2}$$

Using Corollary 2, we have $\mathcal{M}(\theta') = \mathcal{M}(\theta' + (s(\theta'), i))$, and hence equation (6.2) is simply:

$$s(\theta') \in \tilde{K}^i_{\theta'}[\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E),$$

which implies $\omega \in [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E)$.

For the other direction, let $\omega \in [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E)$.
To show $\omega \in [\tilde{K}^i \tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E)$, we need to show

$$\mathbb{P}^{\Omega_i(\omega|\theta)}(\omega) \in \tilde{K}^i_{\theta+(\omega,i)}[\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_{\theta+(\omega,i)}(E),$$

which is equivalent to the following (definition (4.3)): let $\theta' = \theta + (\omega, i)$, and $s(\theta') = \mathbb{P}^{\Omega(\theta'}(\omega)$,

$$P_i(s(\theta')|\theta') \subseteq [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E), \tag{6.3}$$

$$W_i(s(\theta')|\theta') \supseteq q(E). \tag{6.4}$$

Now since $\omega \in [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^n]_\theta(E)$, by definition, we have

$$s(\theta') \in \tilde{K}^i_{\theta'}[\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E),$$

and hence:

$$P_i(s(\theta')|\theta') \subseteq [\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E), \tag{6.5}$$

$$W_i(s(\theta')|\theta') \supseteq q(E). \tag{6.6}$$

Thus we only need to prove equation (6.3). Let $\omega' \in P_i(s(\theta')|\theta')$. The goal is to show $\omega' \in [\tilde{K}^i \tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E)$, which amounts to proving:

$$P_i(s(\theta'')|\theta'') \subseteq [\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta''}(E), \tag{6.7}$$
$$W_i(s(\theta'')|\theta'') \supseteq q(E), \tag{6.8}$$

where $\theta'' = \theta' + (\omega', i)$ and $s(\theta'') = \mathbb{P}^{\Omega(\theta'')}(\omega')$.

By Corollary 2, $\mathcal{M}(\theta'') = \mathcal{M}(\theta')$. Thus $\Omega(\theta'') = \Omega(\theta')$, which means $s(\theta'') = \omega'$, and the above two equations reduce to:

$$P_i(\omega'|\theta') \subseteq [\tilde{K}^{i^{k+1}} \cdots \tilde{K}^{i^n}]_{\theta'}(E), \tag{6.9}$$
$$W_i(\omega'|\theta') \supseteq q(E), \tag{6.10}$$

By Lemma 1, $W_i(\omega'|\theta') = W_i(s(\theta')|\theta')$; and by Lemma 7, $P_i(\omega'|\theta') = P_i(s(\theta')|\theta')$. Thus equations (6.9) - (6.10) follow from equations (6.5) - (6.6). □

Proof of IK2. Fix $\omega^* \in \Omega^*$. Notice $\tilde{U}^j_{i_{\omega^*}}$ and $[\tilde{K}^j]^n_{i_{\omega^*}}$ are simply the single-agent knowledge operator defined in (**??**) and the objective knowledge operator defined in (**??**) adapted to the subjective model $\mathcal{M}(i_{\omega^*})$. By Lemmas 1 and 7, $(W_j(\cdot|i_{\omega^*}), P_j(\cdot|i_{\omega^*}))$ is rational, and hence Lemma **??** applies to $\mathcal{M}(i_{\omega^*})$. It follows $\tilde{U}^j_{i_{\omega^*}}(E) = [(\neg\tilde{K})^j]^n_{i_{\omega^*}}(E)$ for all n, and hence the result. □

Proof of IK3. Let $\omega^* \in K_iK_j(E)$. Then $q(E) \subseteq W^*_i(\omega^*)$ and $\mathbb{P}^{\Omega_i(\omega^*)}(\omega^*) \equiv s_i(\omega^*) \in \tilde{K}^i_{i_{\omega^*}}\tilde{K}^j_{i_{\omega^*}}(E)$.

It is easy to see that $\tilde{K}^j_{i_{\omega^*}}(E) \subseteq \neg\tilde{U}^j_{i_{\omega^*}}(E)$, thus $s_i(\omega^*) \in \tilde{K}^i_{i_{\omega^*}}\tilde{U}^j_{i_{\omega^*}}(E)$, which implies $\omega^* \in K_iU_j(E)$.

On the other hand, $s_i(\omega^*) \in \tilde{K}^i_{i_{\omega^*}}\tilde{K}^j_{i_{\omega^*}}(E)$ implies $s_i(\omega^*) \in \tilde{K}^j_{i_{\omega^*}}(E)$, which is equivalent to the following:

$$P_j(s_i(\omega^*)|i_{\omega^*}) \subseteq E_{\Omega_i(\omega^*)}, \tag{6.11}$$
$$W_j(s_i(\omega^*)|i_{\omega^*}) \supseteq q(E). \tag{6.12}$$

By (3.1), $P_j(s_i(\omega^*)|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)}P^*_j(\omega^*)$, and hence (6.11) implies $P^*_j(\omega^*) \subseteq E_{\Omega^*}$. Then if $q(E) \subseteq W^*_j(\omega^*)$ then $\omega^* \in K_j(E)$; and if $q(E) \nsubseteq W^*_j(\omega^*)$ then $\omega^* \in U_j(E)$. □

6.5 Proof of Theorem 5.

For any $\theta \in \Theta$, and any $\omega \in \Omega(\theta)$, let $R(\omega|\theta) = \wedge^n_{j=1}P_j(\omega|\theta)$ denote the set of reachable states from ω.

Lemma 8 *Fix a product model $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ where P^*_i satisfies product factual partition for all $i \in N$. Then for any $E \in \mathcal{E}^p$, $\omega^* \in \underline{CK}(E)$ implies that for any n and any*

$\theta^n = ((\omega_1, i^1), \cdots, (\omega_n, i^n)) \in \Theta^n$ such that $\omega_1 = \omega^*$, and for all $m = 2, \cdots, n$, $\omega_m \in P_{i^{m-1}}(\mathbb{P}^{\Omega(\theta^{m-1})}(\omega_{m-1})|\theta^{m-1})$, where θ^m denotes the string consisting of first m elements in θ^n for all $m \leq n$, the following holds:

$$R(\mathbb{P}^{\Omega(\theta^n)}(\omega_n)|\theta^n) \subseteq E_{\Omega(\theta)}, \tag{6.13}$$

$$\bigcap_{j \in N} \bigcap_{\omega \in R(\mathbb{P}^{\Omega(\theta^n)}(\omega_n)|\theta)} W_j(\omega|\theta^n) \supseteq q(E). \tag{6.14}$$

Proof. It suffices to prove the case for $\theta = (\omega^*, i)$ for any $i \in N$, and the rest follows from an induction argument on the length of θ. For notational ease, let $s_i(\omega^*) \equiv \mathbb{P}^{\Omega_i(\omega^*)}(\omega^*)$.

Since $\omega^* \in \underline{CK}(E)$, $W_i^*(\omega^*) \supseteq q(E)$ and hence $E_{\Omega_i(\omega^*)}$ is well-defined.

For all j, $P_j(s_i(\omega^*)|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*)$, thus,

$$\begin{aligned} R(s_i(\omega^*)|i_{\omega^*}) &= \wedge_{j=1}^n \mathbb{P}^{\Omega_i(\omega^*)} P_j(s_i(\omega^*)|i_{\omega^*}) \\ &= \wedge_{j=1}^n \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) \\ &\subseteq \mathbb{P}^{\Omega_i(\omega^*)} \wedge_{j=1}^n P_j^*(\omega^*) \\ &\subseteq E_{\Omega_i(\omega^*)}. \end{aligned}$$

Let $\omega \in R(s_i(\omega^*)|i_\omega^*)$. By definition, there exists $\omega_1^* \in \wedge_{j=1}^n P_j^*(\omega^*)$ such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega$. By hypothesis, $q(E) \subseteq W_j^*(\omega_1^*)$ for all j. Therefore,

$$\begin{aligned} W_j(\omega|i_{\omega^*}) &= W_i^*(\omega^*) \cap [\bigcup_{\{\omega_2^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*) = \omega\}} W_j^*(\omega_2^*)] \\ &\supseteq W_i^*(\omega^*) \cap W_j^*(\omega_1^*) \\ &\supseteq q(E). \end{aligned}$$

□

Proof of Theorem 5. Let $\omega^* \in \underline{CK}(E)$, and fix a set of players $i^1, \cdots, i^n \in N$. It suffices to show:

$$\omega^* \in K_{i^1} K_{i^2} \cdots K_{i^n}(E). \tag{6.15}$$

Now fix a reasoning string $\theta^{n-1} = ((\omega_1, i^1), \cdots, (\omega_{n-1}, i^{n-1}))$ such that $\omega_1 = \omega^*$, and for all $m = 2, 3, \cdots, n-1$, $\omega_m \in P_{i^{m-1}}(\mathbb{P}^{\Omega(\theta^{m-1})}(\omega_{m-1})|\theta^{m-1})$, where θ^m denotes the string consisting of first m elements in θ^n for all $m \leq n-1$.

Let $\bar{\omega} = \mathbb{P}^{\Omega(\theta^{n-1})}(\omega_{n-1})$ and $\omega \in P_{i^{n-1}}(\bar{\omega}|\theta^{n-1}) \subseteq R(\bar{\omega}|\theta^{n-1})$. By Lemma 8,

$$\begin{aligned} R(\bar{\omega}|\theta^{n-1}) &\subseteq E_{\Omega(\theta^{n-1})}. \\ W_{i^n}(\omega|\theta^{n-1}) &\supseteq q(E). \end{aligned}$$

Since $P_{i^n}(\omega|\theta) \subseteq R(\bar{\omega}|\theta^{n-1})$, it follows that $\omega \in \tilde{K}_{\theta^{n-1}}^{i^n}(E)$.

Now this is true for all $\omega \in P_{i^{n-1}}(\bar{\omega}|\theta^{n-1})$. Thus we have:

$$P_{i^{n-1}}(\bar{\omega}|\theta^{n-1})) \subseteq \tilde{K}^{i^n}_{\theta^{n-1}}(E),$$

thus, $\bar{\omega} \in \tilde{K}^{i^{n-1}}_{\theta^{n-1}} \tilde{K}^{i^n}_{\theta^{n-1}}(E)$, which in turn yields:

$$\omega_{n-1} \in [\tilde{K}^{i^{n-1}}_{\theta^{n-2}} \tilde{K}^{i^n}_{\theta^{n-2}}]_{\theta^{n-2}}(E).$$

(Note that since subjective awareness information is nested with respect to the order of reasoning, $W_{i^n}(\omega|\theta^{n-1}) \supseteq q(E)$ ensures all the awareness clauses are satisfied in the definition of higher-order subjective knowledge.)

Repeat the above for all $\theta = \theta^{n-2} + (\omega, i^{n-1})$, where $\omega \in P_{i^{n-2}}(\mathbb{P}^{\Omega(\theta^{n-2})}(\omega_{n-2})|\theta^{n-2})$. Then we have:

$$P_{i^{n-2}}(\mathbb{P}^{\Omega(\theta^{n-2})}(\omega_{n-2})|\theta^{n-2}) \subseteq [\tilde{K}^{i^{n-1}} \tilde{K}^{i^n}]_{\theta^{n-2}}(E),$$

which yields

$$\mathbb{P}^{\Omega(\theta^{n-2})}(\omega_{n-2}) \in \tilde{K}^{i^{n-2}}_{\theta^{n-2}} [\tilde{K}^{i^{n-1}} \tilde{K}^{i^n}]_{\theta^{n-2}}(E),$$

and hence

$$\omega_{n-2} \in [\tilde{K}^{i^{n-2}} \tilde{K}^{i^{n-1}} \tilde{K}^{i^n}]_{\theta^{n-3}}(E).$$

Similarly, applying the above reasoning for every $\theta^m, m \leq n-1$ yields the theorem.
□

6.6 Proof of Theorem 6.

Proof. Only need to prove necessity, i.e. $CK(E) \subseteq \underline{CK}(E)$. I show both clauses in the definition of $\underline{CK}(E)$ are necessary.

Let $\omega_1^* \notin \underline{CK}(E)$.
Case 1: Suppose there exists $\bar{\omega}^* \in \wedge_{j=1}^n P_j^*(\omega_1^*)$ such that $\bar{\omega}^* \notin E_{\Omega^*}$. Suppose $\bar{\omega}^*$ is reachable from ω_1^* as follows:

$$\begin{array}{rcl} \bar{\omega}^* & \in & P^*_{i^n}(\omega^*_n), \\ \omega^*_n & \in & P^*_{i^{n-1}}(\omega^*_{n-1}), \\ & \cdots & \\ \omega^*_2 & \in & P^*_{i^1}(\omega^*_1). \end{array}$$

Let $\theta^1 = (\omega_1^*, i^1)$, and for $m = 2, \cdots, n$, let $\theta^m = \theta^{m-1} + (\mathbb{P}^{\Omega(\theta^{m-1})}(\omega^*_m), i^m)$. If $W(\theta^m) \not\supseteq q(E)$ for some $m < n$, then $[\tilde{K}^{i^{m+1}} \cdots \tilde{K}^{i^n}]_{\theta^m}(E) = \emptyset_E$ and hence $\omega^* \notin K_i \cdots K_{i^n}(E)$. So suppose $W(\theta^n) \supseteq q(E)$.

Extending the proof of Lemma 7, we see that for any $\omega \in \Omega_i(\omega^*)$,

$$P_j(\omega|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega_1^*),$$

where ω_1^* is such that $\mathbb{P}^{\Omega(\omega^*)}(\omega_1^*) = \omega$. In particular, this implies

$$\wedge_{j=1}^{n} \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) = \mathbb{P}^{\Omega_i(\omega^*)} \wedge_{j=1}^{n} P_j^*(\omega^*).$$

It follows that $\mathbb{P}^{\Omega(\theta^n)}(\bar{\omega}^*) \notin E_{\Omega(\theta^n)}$, which implies $P_{i^n}^*(\mathbb{P}^{\Omega(\theta^n)}(\omega_n^*)|\theta^n) \nsubseteq E_{\Omega(\theta^n)}$, and hence $\mathbb{P}^{\Omega(\theta^{n-1})}(\omega_n^*) \notin \tilde{K}_{\theta^{n-1}}^{i^n}(E)$. Applying this argument recursively for all $\theta^m, m < n$ yields $\omega_1^* \notin K_{i^1} \cdots K_{i^n}(E)$.

Case 2: suppose there exists $\bar{\omega}^* \in \wedge_{j=1}^{n} P_j^*(\omega^*)$ such that $q(E) \nsubseteq W_j^*(\bar{\omega}^*)$ for some $j \in N$.

Suppose $\bar{\omega}^* \neq \omega_1^*$. (Otherwise $\omega_1^* \notin K_j(E)$ and the theorem is proved.) Let $\bar{\omega}^*$ be reachable from ω_1^* through $i^k, \omega_k^*, k = 1, \cdots, n$ as in case 1. Define $\theta^m, m = 1, 2, \cdots, n$ as above.

By nice awareness, for any $\Omega \in \mathcal{S}$ and $\omega \in \Omega$, we have

$$q(\Omega) \cap [W_j^*(\omega_1^*) \triangle W_j^*(\omega_2^*)] = \emptyset, \forall j \in N, \ \forall \omega_1^*, \omega_2^* \text{ such that } \mathbb{P}^{\Omega}(\omega_1^*) = \mathbb{P}^{\Omega}(\omega_1^*).$$

Thus we have: for all $m = 1, 2, \cdots, n-1$,

$$\begin{aligned} W(\theta^{m+1}) &= W_{i^{m+1}}(\mathbb{P}^{\Omega(\theta^m)}(\omega_{m+1}^*)|\theta^m) \\ &= W(\theta^m) \cap [\bigcap_{\left\{\omega \in \Omega^{\theta^{m-1}} : \mathbb{P}^{\Omega(\theta^m)}(\omega) = \mathbb{P}^{\Omega(\theta^m)}(\omega_{m+1}^*)\right\}} W_{i^{m+1}}^*(\omega)], \end{aligned}$$

where $\left\{\omega \in \Omega^{\theta^{m-1}} : \mathbb{P}^{\Omega(\theta^m)}(\omega) = \mathbb{P}^{\Omega(\theta^m)}(\omega_{m+1}^*)\right\}$ is the set of states in $\Omega(\theta^{m-1})$ that has the same projection as Ω_{m+1}^* in $\Omega(\theta^m)$. It follows $W(\theta^{m+1}) = W(\theta^m) \cap W_{i^{m+1}}(\omega_m^*)$. Now $W(\theta^1) = W_{i^1}(\omega_1^*)$, thus we have:

$$W(\theta^n) = \cap_{k=1}^{n} W_{i^k}^*(\omega_k^*).$$

But then:

$$\begin{aligned} & W_j(\mathbb{P}^{\Omega(\theta^n)}(\bar{\omega}^*)|\theta^n) \nsupseteq q(E) \\ \Rightarrow \ & \mathbb{P}^{\Omega(\theta^n)}(\bar{\omega}^*) \notin \tilde{K}_{\theta^n}^{j}(E) \\ \Rightarrow \ & \mathbb{P}^{\Omega(\theta^n)}(\omega_n^*) \notin \tilde{K}_{\theta^n}^{i^n} \tilde{K}_{\theta^n}^{j}(E) \\ \Rightarrow \ & \mathbb{P}^{\Omega(\theta^{n-1})}(\omega_n^*) \notin [\tilde{K}^{i^n} \tilde{K}^{j}]_{\theta^{n-1}}(E), \end{aligned}$$

and so on for all orders of subjective knowledge. It follows that $\omega_1^* \notin K_{i^1} \cdots K_{i^n} K_j(E)$. $\square$

References

Aumann, Robert J., "Agreeing to Disagree," *Annals of Statistics*, 1976, *76* (4), 1236–1239.

——, "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica*, 1987, *55* (1), 1–18.

Brandenburger, Adam and Eddie Dekel, "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 1993, *59*, 189–198.

Geanakoplos, John, "Common Knowledge," *Journal of Economic Perspectives*, 1992, *6* (4), 53–82.

Li, Jing, "Modeling Unawareness in Arbitrary State Spaces," 2008a. PIER working paper 08-021.

——, "Information Structures with Unawareness," 2008b. PIER working paper 08-024.