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“A Note on Unawareness and Zero Probability”

by

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# A Note on Unawareness and Zero Probability

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## Abstract

I study how choice behavior given unawareness of an event differs from choice behavior given subjective belief of zero probability on that event. Depending on different types of unawareness the decision-maker suffers, behavior under unawareness is either incomparable with that under zero probability (in the case of pure unawareness), or drastically different (in the case of partial unawareness). The key differences are (1) partial unawareness permits dynamically inconsistent choice, while zero probability beliefs do not; and (2) there are unforeseen options in an unawareness environment that are necessarily modeled as dominated options in zero probability models.

*Keywords:* unawareness, zero probability, dynamic consistency, unforeseen contingency, unforeseen options

*JEL Classification:* C70, C72, D80, D82, D83

# 1 Introduction

It is a well-recognized fact that people may be unaware of some relevant uncertainties when making decisions. For example, most insurance companies in the 1970s were unaware of the harmful effects lead-based paint had on the human body, which subsequently resulted in millions of dollars in compensation. The war against terrorism has been an extremely difficult endeavor precisely because we are unaware of many possible strategies the terrorists could employ. Thus, understanding decision-making under unawareness is of great interest.

There are two interesting notions of unawareness. The first is unawareness of a specific event, in the sense of not knowing it, and not knowing not knowing it, and so on. The second is (un)awareness of the general issue of unawareness itself, i.e., whether one is aware that there may exist *some* event of which one is unaware. In principle, these are separate issues. For example, while the insurance companies in the 1970s were unaware of the harmful effects of lead-based paint, they could have been aware that they were unaware of *something*. On the other hand, modeling the latter necessarily requires embedding the former in the model. In this note, I focus attention on the case where one is not only unaware of some specific events, but also essentially unaware of such specific unawareness.

A frequently raised question is whether such unawareness is observationally equivalent to having full awareness with zero probability beliefs. The question arises from the observation that while one cannot take into account what happens in events of which one is unaware, neither does one care about what happens in events to which one assigns zero probability, as long as one is *rational*.<sup>1</sup> Indeed, this is the approach used in many papers to model agents with unawareness, for example, Modica, Rustichini and Tallon (?), and Ozbay (?), among others. On the other hand, there is also a clear conceptual distinction between unawareness and zero probability that prompts many economists to explore models of unawareness explicitly. Nonetheless, all existing models of unawareness deliver results (in terms of behavior and outcomes) that can be obtained in models with zero-probability beliefs. Thus, the goal of this note is two-fold: I explore whether zero-probability belief systems are reasonable approximations of beliefs under unawareness; and what characterizes behavior under unawareness.

Epistemically, unawareness clearly differs from having zero probability beliefs (?). One is unaware of an event if one doesn't know it, and doesn't know he doesn't know it, and so on, while assigning zero probability to an event *requires* being aware of it. In a decision context, this distinction first translates to the availability of corresponding bets. While one cannot bet on an event of which one is unaware, one can certainly bet on an event to which one assigns zero probability. However, this observation has no behavioral content: a rational decision-maker (DM) never bets on an event to which he assigns zero probability anyway.

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<sup>1</sup>In this context, rationality refers to expected utility maximization.

The second observation is that unawareness has a natural symmetry property that cannot be satisfied in a probability system: one is unaware of an event if and only if one is unaware of the negation of it, but one cannot assign zero probability to both an event and its negation. It follows that while the DM can bet neither on nor against an event of which he is unaware, he would *always* want to bet against an event to which he assigns zero probability. However, this observation, too, has minimal behavioral content: testing it requires asking the DM to rank bets involving the event in question, which necessarily makes the DM aware of it. Moreover, in many situations, the DM could bet on an event that is equivalent to an event of which he is unaware. For example, imagine there are red, black and white balls in an urn, but Alex is unaware of the red balls. Although Alex cannot bet on either “the ball is red” or “the ball is not red,” he could, and presumably would, bet on the event “the ball is either black or white,” just as he would if he assigned zero probability to having red balls in the urn.<sup>2</sup> Therefore, I turn to examine the DM’s choice behavior with respect to those bets of which he is aware.

An immediate problem is that, if the DM is unaware of any relevant uncertainty, then he is also necessarily unaware of all feasible bets. A bet in its regular usage specifies an unambiguous outcome for each deterministic scenario, and in that sense, it is objectively well-specified. Call it an *objective bet*. In the previous example, an objective bet specifies what Alex receives for each color of the ball drawn from the urn. If Alex is unaware of the red balls, then he must be unaware of all such objective bets. However, arguably Alex can conceive bets on events such as “black ball” or “white ball.” I call such “bets” perceived by the DM under unawareness the *subjective bets*.

But a subjective bet is in fact not well-defined from a fully aware outside observer’s perspective: there is at least one scenario the DM has in mind when he has unawareness that is not deterministic. Thus, the DM’s ranking of subjective bets reflects both his likelihood assessments of those events of which he is aware and his *perception* of the outcome of the bet in each scenario he has in mind. Thus the effects of unawareness are reflected in how the subjective bets are connected with the objective ones. Consider the following two-period decision process. In the first period, the DM ranks all subjective bets; in the second period, the DM is informed of all relevant uncertainties and then ranks objective bets. Intuitively, each subjective bet the DM considers in the first period corresponds to *some* objective bet he becomes aware of in the second period, except that under his unawareness in the first period, the DM is unable to express those subjective bets precisely. Thus, one would expect the DM to rank those bets in the same way in both periods.

There are two plausible interpretations of subjective bets. For example, suppose there are two relevant uncertainties, whether it rains and whether there is an earthquake, but Bob is unaware of the possibility of an earthquake. Consider the subjective

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<sup>2</sup>Of course, in the former case, Alex is unaware that his bet is equivalent to one wagering on “the ball is not red,” while in the latter case he knows it.

bet that says Bob receives \$10 if it rains and pays \$10 if it does not rain. What does Bob have in mind when he contemplates such a bet? One possibility is that he assumes he is to receive \$10 *whenever* it rains and pay \$10 *whenever* it does not rain. In this case, the DM simply neglects the details of which he is unaware in evaluating subjective bets. I refer to this case as “pure unawareness.” It is not hard to see that the only implication of pure unawareness is the DM’s inability to bet on or form beliefs about events of which he is unaware. Notice that since one cannot assign zero probability to both an event and its negation, pure unawareness of an event is incomparable with assigning zero probability to it.

A second possibility is that Bob implicitly assumes there is no earthquake in the first period. Thus, each subjective bet corresponds to some objective bet that coincides with the subjective bet in states where there is no earthquake and yields a lottery consisting of only the prizes \$10 or -\$10 otherwise.<sup>3</sup> I refer to this case as “partial unawareness.” Unlike pure unawareness, in this case, the DM not only neglects some relevant uncertainties in the environment, but also implicitly assigns probability one to some particular resolution of those uncertainties. Moreover, the DM is unaware of his making such implicit assumptions. It is this case that begs a comparison with zero probability.

Notice that under partial unawareness, there is a continuum of objective bets extending the same subjective bet. However, it seems there is no general rule that selects a particular objective bet for all partial unawareness. How the DM’s partial unawareness affects his likelihood assessments about events of which he is aware depends on the specific nature of his unawareness. This can be seen clearly in the urn example, where there is a natural correlation between realizations of different uncertainties, and hence the DM’s unawareness is necessarily partial unawareness.<sup>4</sup> Suppose Alex is unaware of the red balls because he suffers from color blindness that makes him unable to distinguish red and black. Then upon being told about the red balls, Alex would realize the event “the ball is black” that he perceives in the first period actually confounds two objective events: “the ball is black” and “the ball is red.” Consequently, in comparison to his valuation of the objective bets in the second period, his valuation of the subjective bets in the first period must have a systematic bias towards more weights on the consequences of the state where the ball is black. More specifically, suppose Alex is indifferent between not betting and taking the bet “getting \$10 if the ball is black, paying \$10 if the ball is white” in the first period; then he must be indifferent between not betting and taking the bet “getting \$10 if the ball is black *or red*, paying \$10 if the ball is white” in the second period. Alternatively, suppose Alex’s color blindness makes him unable to distinguish between red and white instead. Then it seems that in the second period Alex would be indifferent between not betting and

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<sup>3</sup>An implicit assumption here is that there are no unforeseen consequences. I discuss this assumption later.

<sup>4</sup>Since a ball can only be either black, white, or red, if Alex is unaware of the red balls, he must implicitly assume no ball is red.

taking the bet “getting \$10 if the ball is black, paying \$10 if the ball is white *or red*” instead.

On the other hand, partial unawareness does impose some restrictions on the DM’s behavior. Arguably Alex would always prefer to receive the better prize in states of which he was unaware before in the second period, regardless of the nature of his unawareness. Thus, in general, it seems plausible for the following axiom to be satisfied in the case of partial unawareness: the DM’s valuation of a subjective bet under partial unawareness should fall between his valuation of a best-scenario extension of the subjective bet and a worst-scenario extension of it upon updating his awareness.<sup>5</sup> For example, the axiom says, regardless of what causes Alex’s unawareness of the red balls, his valuation of the subjective bet “receives \$10 if the ball is black and pays \$10 if the ball is white” under partial unawareness falls between his valuation of the objective bets “receives \$10 if the ball is black or red and pays \$10 if the ball is white” and “receives \$10 if the ball is black and pays \$10 if the ball is white or red” under full awareness.

This axiom leads to the result that the DM’s subjective beliefs regarding events of which he is aware under partial unawareness are bounded below by those under full awareness. More specifically, let  $\mu_1$  and  $\mu_2$  denote the DM’s beliefs under partial unawareness and under full awareness, respectively. Then for any event  $E$  that is measurable with respect to both  $\mu_1$  and  $\mu_2$ , one has  $\mu_1 \geq \mu_2$ .

Such beliefs encompass assigning zero probabilities to the corresponding events under full awareness. Notice that as long as there is full awareness, new information must be interpreted as information about which state obtains. I refer to such information as *factual information*. Then one would expect such information to only affect behavior regarding bets that differ in states for which the new information and the old information have different implications. In other words, the DM’s preferences should satisfy an “event consistency” axiom: as long as two bets coincide in states for which he has different information, the DM’s ranking of these bets should be the same despite the different information. Myerson (?) shows that this is the key axiom that characterizes the *conditional probability system (CPS)*, which dictates that for all  $E$  that receives positive probabilities given different information, the relative likelihood remains the same.

In particular, notice that behavior under such beliefs is *dynamically consistent*: conditional on the same event, preferences given different information coincide. In contrast, partial unawareness permits dynamic inconsistency: under different awareness

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<sup>5</sup>It is worth noting that the dichotomy of pure unawareness and partial unawareness is only meaningful in the context of decision-making. Epistemically, partial unawareness is not fundamentally different from pure unawareness. However, given correlations between realizations of uncertainties, unawareness of an uncertainty could entail the additional complication of unawareness of logical deductions. For example, not only is Alex unaware of both “the ball is red” and “the ball is not red,” but he is also unaware that “the ball is black or white” (of which he is aware) is equivalent to “the ball is not red.” See Galanis (?) for a thorough discussion of unawareness of logical deductions.

levels, the DM may bet differently *conditional on an event of which he is always aware*. Intuitively, expanding the probability space imposes little restrictions on the DM's relative likelihood assessments on the common set of measurable events because of the possible correlation between the unforeseen scenarios and the foreseen scenarios. As a result of becoming aware of the unforeseen scenarios, the DM also becomes aware of these "hidden" correlations, which may cause him to revise his relative likelihood assessments regarding events of which he was aware before, and hence reverse his preferences.

It is worth emphasizing that such dynamic inconsistency due to different awareness is very different from that in the standard non-expected utilities models (?). First, intuitively a DM who anticipates that he will be dynamically inconsistent tomorrow under an expanded awareness would prefer to have tomorrow's preferences instead of today's if possible, while in the standard models, dynamic inconsistency is always a problem the DM wants to avoid. In other words, dynamic inconsistency due to different awareness generates a preference for flexibility instead of a preference for commitment in the standard case. Second, under partial unawareness, even though a sophisticated DM may anticipate that he will be dynamically consistent tomorrow *in an abstract sense*, he can never anticipate the specific dynamic inconsistency he will be subject to and hence cannot fully integrate it into today's decision.

Finally, while unawareness of an event is not behaviorally equivalent to assigning zero probability to it, it is always possible to *model* behavior under partial unawareness by *constructing a model involving updating on zero probability*. More specifically, one can take the state space to be the disjoint union of  $S_1$  and  $S_2$ , let the DM's preferences conditional on the information  $S_1$  and  $S_2$  (appropriately extended to the expanded domain) match the two period preferences in the unawareness case respectively, and the preferences conditional on the entire state space reflect a subjective belief of zero probability on  $S_2$ . Under this construction, the DM is fully aware of all states, including both the subjective states and the objective states, but first assigns probability one to  $S_1$  in the first period, and then upon receiving the information that true state lies in the prior zero probability event  $S_2$ , the DM updates to assign probability one to  $S_2$ . The caveat of this approach is that beliefs are disjoint by construction and hence are entirely unrestrictive.

There have few attempts to explore beliefs under unawareness from a decision-theoretic perspective. Ahn and Ergin (?) explore a model in which the DM's subjective beliefs for different sub-algebras of events are solicited, and the connection between these beliefs is interpreted as reflecting the effects of different framings, possibly embedding different awareness. In the context of the above example, Ahn and Ergin compare Bob's subjective beliefs for "rain" and "no rain" with his beliefs for "rain, earthquake," "rain, no earthquake" and "no rain," and interpret the discrepancy, if any, between the probability for "rain" in the first belief system and the sum of the probabilities for "rain, earthquake" and "rain, no earthquake" in the second belief system as evidence that Bob was unaware of earthquake when he ranked only bets measurable with respect



to only the events “rain” and “no rain.” In contrast, in this paper, I compare Bob’s beliefs for “rain” when he is unaware of the earthquake and when he is aware of the earthquake.

The note is organized as follows. Section ?? investigates how unawareness affects the DM’s beliefs regarding events of which he is aware. I discuss both the benchmark case of pure unawareness and the main interest, the case of partial unawareness. Section 3 contrasts the case of unawareness with zero probability models. Section 4 discusses caveats of the decision-theoretic approach and potential interesting issues in this environment. I conclude in Section 5.

## 2 Beliefs under Unawareness

Let  $Z$  denote an arbitrary set of prizes and  $\Delta(Z)$  the set of simple lotteries over  $Z$ .<sup>6</sup> Let  $S_i$  denote the state space for period  $i, i = 1, 2$ . Let  $S_2$  be finite. Let  $\succsim^i$  denote the DM’s preference orderings over the period  $i$  choice set  $C_i = (\Delta(Z))^{S_i}$ . Let  $\succ^i$  and  $\sim^i$  denote the asymmetric and symmetric parts of  $\succsim^i$ , respectively. Let  $l$  denote a generic element in  $\Delta(Z)$ . Slightly abusing notation, I also use  $l$  to denote the constant act that yields  $l$  in every state. As usual, the convex combination of acts is defined by taking the convex combination state-wise: for all  $\alpha \in [0, 1]$  and any  $f, g \in C_i, [\alpha f + (1 - \alpha)g](s) = \alpha f(s) + (1 - \alpha)g(s)$  for all  $s \in S_i$ .

Fixing  $f, g \in C_i$  and  $E \subseteq S_i$ , I say the DM *prefers  $f$  to  $g$  conditional on  $E$* , denoted by  $f \succsim_E^i g$ , if  $f' \succsim^i g'$  where  $f'(s) = g'(s)$  for all  $s \notin E$  and  $f'(s) = f(s)$  and  $g'(s) = g(s)$  for all  $s \in E$ . An event  $E \subseteq S_i$  is *Savage-null* under  $\succsim^i$  if  $f \sim_E^i g$  for all  $f, g \in C_i$ . A state  $s$  is said to be *non-null* if  $\{s\}$  is not Savage-null.

First I postulate  $\succsim^i$  satisfies the standard Anscomb-Aumann axioms.

AA.1 (weak order):  $\succsim^i$  is transitive and complete;

AA.2 (continuity): for all  $g \in C_i$ , the sets  $\{f \in C_i : g \succsim^i f\}$  and  $\{f \in C_i : f \succsim^i g\}$  are closed;

AA.3 (independence): for all  $f, g, h \in C_i, f \succ^i g, \alpha \in (0, 1)$  implies  $\alpha f + (1 - \alpha)h \succ^i \alpha g + (1 - \alpha)h$ ;

AA.4 (non-triviality): there exist  $f, g \in C_i$  such that  $f \succ^i g$ ;

AA.5 (state-independence): for all non-null  $s, t \in S_i, l_1 \succsim_{\{s\}}^i l_2$  if and only if  $l_1 \succsim_{\{t\}}^i l_2$  for all constant acts  $l_1, l_2 \in \Delta(Z)$ .

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<sup>6</sup>A simple lottery is a lottery that has finite support. For concreteness, think of the set  $Z$  as the real line, representing the monetary payoffs.

**Proposition 1 (Anscombe and Aumann (1963)):** *The axioms AA.1-5 are necessary and sufficient for  $\succsim^i, i = 1, 2$  to have the following representation: for all  $f \in C_i$ ,*

$$V_i(f) = \sum_{s \in S_i} \mu_i(s) u_i(f(s)), \quad (2.1)$$

where  $u_i : \Delta(Z) \rightarrow \mathbb{R}$  is linear in probabilities and unique up to affine transformation, and  $\mu_i : 2^{S_i} \rightarrow [0, 1]$  is the unique subjective probability on  $S_i$ .

I refer to  $\mu_2$  as the DM's *latent beliefs* under full awareness and use it as the reference point in discussing the DM's subjective beliefs under unawareness,  $\mu_1$ .

Next I discuss axioms that connect the DM's ranking of subjective bets and his ranking of objective bets. The first axiom postulates there are no unforeseen consequences.

U.1 (foreseen consequences): for all  $l_1, l_2 \in \Delta(Z), l_1 \succsim^1 l_2 \iff l_1 \succsim^2 l_2$ .

This axiom says that the induced ranking over all monetary payoffs are independent of the DM's awareness level. Combined with the fact that the set of prizes  $Z$  is fixed across periods, this axiom amounts to isolating the effects of the DM's unawareness of uncertain events on the DM's beliefs. One way to think of the model is that it presents a thought experiment in which beliefs under different awareness levels are solicited using monetary bets. In this context the foreseen consequence axiom seems natural.

Adding axiom U.1 to AA.1-5 results in setting  $u_1 = u_2 = u$  in the above representations, which makes it possible to have a direct comparison of beliefs under different awareness levels.<sup>7</sup>

The key axiom concerns how the DM's subjective bets relate to the objective ones, for which the most important piece is how the DM's subjective state space relates to the full state space. I discuss two plausible cases of unawareness.

## 2.1 Pure unawareness.

Given a subjective bet  $f$ , if the DM interprets  $f$  as the bet that yields  $f(s)$  whenever the situation described in the subjective state  $s \in S_1$  is true without making any other assumptions, explicitly or implicitly, then I say the DM has *pure unawareness*.

To model pure unawareness, I let  $S_2 = S_1 \times U$ , where  $U$  is a non-singleton set. The interpretation is  $U$  contains all possible resolutions of those uncertainties of which the DM is unaware. For example, one can represent Bob's subjective state space by  $S_1 = \{r, nr\}$ , where  $r, nr$  represent "rain" and "no rain," respectively. Bob

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<sup>7</sup>Technically, the DM's Bernoulli utility for lotteries and his beliefs over the state space are jointly identified from his preferences over acts. Fixing the utility numbers are necessary to pin down the DM's beliefs.

is unaware of the uncertainty of whether there is earthquake, for which there are two possible resolutions and can be denoted by  $e, ne$ . The full state space is represented by  $S_2 = \{r, nr\} \times e, ne$ . Intuitively, the singleton subjective event  $\{r\}$  is a coarser description of the event  $\{r\} \times \{e, ne\}$  in the full state space.

Each subjective bet is identified by one and only one objective bet. Let  $G : C_1 \rightarrow C_2$  be defined as follows: for any  $f \in C_1$ ,  $G(f)((s, u)) \equiv f(s)$  for all  $s \in S_1$  and  $u \in U$ . Thus, pure awareness amounts to the following axiom:

U.2 (pure unawareness): for any  $f, g \in C_1$ ,  $f \succsim^1 g \Leftrightarrow G(f) \succsim^2 G(g)$ .

**Proposition 2** *Axioms AA.1-5 and U.1-2 are necessary and sufficient for  $\succsim^i, i = 1, 2$ , to be represented as in (??), and in addition, (1)  $u_1 = u_2$ ; (2) for all  $E \subseteq S_1$ ,  $\mu_1(E) = \mu_2(E \times U)$ .*

The proof is straightforward and hence omitted.

Proposition ?? says pure unawareness is equivalent to a measurability constraint: the DM's preferences over the subjective bets are identified by a subset of his preferences over the objective bets. The DM's subjective beliefs under pure unawareness are the restriction of his latent beliefs on events of which he is aware. In terms of the DM's choice behavior, the only implication of pure unawareness is the incompleteness of his choice set. Pure unawareness is simply not comparable to assigning zero probability.

## 2.2 Partial unawareness.

Alternatively, the DM's unawareness may be associated with some implicit assumptions about the underlying state of the world, and hence the DM perceives the lottery  $f(s)$  to realize in some particular situation where not only the description of  $s$  is true but also the implicit assumptions are true. In the earthquake example, Bob may implicitly assume there is no earthquake. In the urn example, Alex is necessarily assuming the ball is not red. In a sense, each subjective state the DM has in mind corresponds to a particular full state, and hence the DM's subjective state space can be viewed as a subset of the full state space, i.e., the set of states where his implicit assumptions are true. I refer to this case as partial unawareness.<sup>8</sup>

Let  $S_2 = S_1 \cup U_p$ , where  $U_p \neq \emptyset$ . Upon revelation of the full state space in the second period, the DM becomes aware of his own implicit assumption and reassesses every event with respect to the expanded universal event. Fix a subjective bet  $f \in C_1$ . From an outside observer's perspective,  $f$  leaves consequences in states in  $U_p$  unspecified, while from the unaware DM's perspective, all scenarios have been

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<sup>8</sup>The DM is actually unaware of his implicit assumptions, so strictly speaking, the DM's subjective state space should be disjoint with the full state space, just as in the pure unawareness case. A subjective state, technically, is a pair  $(s, S_1)$ , consisting of the state and the "subjective state space" that represents the implicit assumption under partial unawareness. See Li (?) for details.

considered. Since in this model, the DM evaluates monetary bets on the likelihood of uncertain events, even though there is unawareness of the uncertainties, there is no unawareness of the possible monetary payoffs. The DM *knows* that by choosing  $f$  he will receive a lottery in the set  $\{f(s) : s \in S_1\}$ . Intuitively, the DM implicitly “confounds” scenarios described in  $U_p$  when reasoning about his subjective states in  $S_1$ . Thus, I say an objective bet  $g \in C_2$  is an *extension* of  $f$ , or  $g$  *extends*  $f$ , if  $g(s) = f(s)$  for all  $s \in S_1$  and  $g(s) \in \{f(s) : s \in S_1\}$  for all  $s \in U_p$ . Absent additional assumptions about the nature of the DM’s partial unawareness, that the DM perceives each subjective bet to be *some* objective extension of it seems to be the only thing that one can say.

Let  $l_1[E]l_2[S_i \setminus E]$  denote the (subjective or objective) bet that yields the lottery  $l_1$  on the event  $E$  and  $l_2$  on its complement  $S_i \setminus E$ . Given any  $f \in C_i$ , let  $\bar{f}$  denote its certainty equivalent under  $\succsim^i$  (which exists by the Anscomb-Aumann axioms), i.e.,  $\bar{f} \in \Delta(Z)$  is the constant act such that  $f \sim^i \bar{f}$ .

U.3 (partial unawareness): for all  $E \subseteq S_1$ ,  $l_1 \succ^1 l_2$ ,

$$l_1[E \cup U_p]l_2[S_1 \setminus E] \succ^2 \overline{l_1[E]l_2[S_1 \setminus E]} \succ^2 l_1[E]l_2[S_2 \setminus E]. \quad (\dagger)$$

This axiom essentially says that the DM’s valuation of a subjective bet falls between its best-scenario extension and worst-scenario extension. To see this, notice  $\overline{l_1[E]l_2[S_1 \setminus E]}$  is the certainty equivalent of the subjective bet  $l_1[E]l_2[S_1 \setminus E]$  and serves as a “price tag” for the latter that can be taken to the second period for comparison. The objective bet  $l_1[E \cup U_p]l_2[S_1 \setminus E]$  is the best-scenario extension of  $l_1[E]l_2[S_1 \setminus E]$ , assigning the better lottery  $l_1$  to the DM in all states in  $U_p$ , while  $l_1[E]l_2[S_2 \setminus E]$  is the worst-scenario extension, assigning the worse lottery  $l_2$  to the DM in all states in  $U_p$ .

**Proposition 3** *Axioms AA.1-5, U.1 and U.3 are necessary and sufficient for  $\succsim^i, i = 1, 2$ , to be represented as in (??), and in addition, (1)  $u_1 = u_2$ ; (2) for all  $E \subseteq S_1$ ,  $\mu_1(E) \geq \mu_2(E)$ . Moreover,  $\mu_1 = \mu_2$  if and only if  $U_p$  is Savage-null under  $\succsim^2$ .*

**Proof:** Only need to prove sufficiency for 2. Let  $V_i(f) = \sum_{s \in S_i} \mu_i(s)u(f(s))$  represent  $\succsim^i$ .

Fix  $s \in S_1$  and let  $l_1 \succ^1 l_2$ . By U.3, we have:

$$\mu_1(s)u(l_1) + (1 - \mu_1(s))u(l_2) \geq \mu_2(s)u(l_1) + (1 - \mu_2(s))u(l_2).$$

The expression is monotonic in the probabilities, and hence we have  $\mu_1(s) \geq \mu_2(s)$ . This is true for every  $s \in S_1$ .

Suppose  $U_p$  is Savage-null under  $\succsim^2$ . Then for all  $E \subseteq S_1$ ,  $l_1[E \cup U_p]l_2[S_1 \setminus E] \sim^2 l_1[E]l_2[S_2 \setminus E]$ . By axiom U.3, we have  $\overline{l_1[E]l_2[S_1 \setminus E]} \sim^2 l_1[E]l_2[S_2 \setminus E]$ , which then indicates  $\mu_1(E)u(l_1) + (1 - \mu_1(E))u(l_2) = \mu_2(E)u(l_1) + (1 - \mu_2(E))u(l_2)$ . It follows  $\mu_1(E) = \mu_2(E)$ .

For the converse, observe that  $\mu_2(S_1) = \mu_1(S_1) = 1$  implies  $S \setminus S_1$  is Savage-null under  $\succsim^2$ .  $\square$

Thus, absent additional assumptions regarding the nature of unawareness, beliefs under partial unawareness are rather unrestrictive. For example, suppose Bob's latent beliefs are such that each color has probability  $\frac{1}{3}$ ; then, being partially unaware of red balls, Bob's subjective beliefs can be anything that assigns a probability between  $\frac{1}{3}$  and  $\frac{2}{3}$  to the ball being black and to the ball being white, respectively.

On the flip side, beliefs under unawareness have little implication for how the DM updates his beliefs upon expanding his awareness level, too. The DM's beliefs under partial unawareness only put upper bounds on his beliefs under an expanded awareness. In particular, the relative likelihood of events is completely unrestrictive. Among other things, this means the DM will in general be *dynamically inconsistent*, in the sense that upon expanding awareness but conditional on the set of foreseen scenarios, the preferences could reverse. More specifically, as long as there exist two non-null states  $s, t \in S_1$  such that  $\frac{\mu_2(s)}{\mu_1(s)} \neq \frac{\mu_2(t)}{\mu_1(t)}$ , then  $\succsim_{S_1}^2 \neq \succsim^1$ , that is, there exist  $f, g \in C_2$ ,  $f(s) = g(s)$  for all  $s \in U_p$ ,  $g|_{S_1} \succsim^1 f|_{S_1}$ , but  $f \succ^2 g$ .

### 3 Partial Unawareness and Zero Probability Beliefs

Partial unawareness begs the comparison with zero probability belief under full awareness; i.e., the DM is always aware of  $S_2$  but simply assigns zero probability to  $U_p$  in the first period. Then the key question is how one interprets the informational content of the signal  $S_2$ . I consider three cases.

First, consider the standard interpretation that  $S_2$  is the signal that all states in  $S_2$  are possible. Given the DM's full awareness of  $S_2$ , such information is trivial. To recast the previous model in this story, one can view the first-period preferences  $\succsim^1$  over  $C_1$  as resulting from omitting the null-event  $U_p$ , while in the second period, the DM is asked to explicitly rank all available bets in  $S_2$ . Obviously, partial unawareness encompasses this case of naive probability zero.

One can argue that the above is not the relevant comparison, because in the case of unawareness,  $S_2$  is an informative signal, while in the above story,  $S_2$  is not. Thus, a more desirable comparison is to reinterpret  $S_2$  as new information that somehow induces the DM to update his beliefs and assign a positive probability to  $U_p$ . Note that, in this case,  $S_2$  is not information in the standard usage of this term: it *expands* the set of non-null states instead of narrowing it. However, no matter what interpretation one gives to the signal  $S_2$ , it differs from  $S_1$  only in the event  $U_p$ . Thus, for any two bets  $f, g$  that coincide on  $U_p$ , the new information has no implications, and hence the DM should rank them in the same way.

Z.1 (event consistency): let  $f, g \in C_2$  be such that  $f(s) = g(s)$  for all  $s \in U_p$ , then  $f|_{S_1} \succsim^1 g|_{S_1} \Rightarrow f \succ^2 g$ .

**Proposition 4 (Myerson (?)):** *Axioms AA.1-5 and Z.1 are necessary and sufficient for  $\succ^i, i = 1, 2$ , to be represented as in (??), and in addition,  $u_1 = u_2; \mu_1(s) = \mu_2(s|S_1)$ .*

That is, the first-period beliefs are conditional probabilities. Note Z.1 implies U.3, so again partial unawareness encompasses this case. In terms of behavior, Z.1 characterizes dynamic consistency: conditional on events in  $S_1$ , the DM's preferences remain the same regardless of his information.

How to understand this? Intuitively, there is a fundamental conceptual difference between updating on new *factual* information, i.e., information about which events obtain, and updating on new *awareness* information, i.e., information on which events should be included in the probability space. Given state-independent preferences for lotteries, choices are naturally additively separable in events under different factual information, due to the fact that states are mutually exclusive. However, when there is partial unawareness, the problem is precisely the DM's inability to separate states and hence events. New awareness information is not about revealing relevant facts, but rather, it is about revealing the hidden correlation between those states of which the DM is aware. Consequently, while dynamic consistency characterizes rational choice behavior under different factual information, dynamic inconsistency tends to arise in environments involving partial unawareness.

Notice that the dynamic inconsistency due to partial unawareness is very different from the one in standard non-expected utility models. In standard models, dynamic inconsistency tends to be an undesirable consequence of lack of self-control that the DM would like to avoid, while in the case of partial unawareness, if anything, the DM would like to make room for such dynamic inconsistency to occur because these are the choices made with better information. Using the language of the multi-selves story, in the first case, today's DM prefers to make decisions for his future selves, while in the second case, today's DM prefers to be his future self and would like to leave room for his future self to override his own decision today. The first case leads to a preference for a commitment device, while the second case leads to a preference for flexibility. Second, in an unawareness environment, a sophisticated DM may anticipate himself to be dynamically inconsistent, but he will not be able to predict specifically how his preferences will change with expanded awareness. Awareness information, by definition, is beyond the DM. Thus, there is necessarily uncertainty as to how his preferences will change tomorrow, and the DM is necessarily "fuzzy" about such uncertainty.<sup>9</sup> In contrast, in standard models, the DM perceives all possible scenarios and anticipates precisely how his preferences change in each scenario.

Finally, I note that while unawareness of an event differs from assigning zero probability to an event per se, one can always write down *some* model of zero probability

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<sup>9</sup>In a seminal paper, Kreps (?) studies a DM who exhibits preference for flexibility, anticipating possible changes of preferences, and later reinterprets the work as a model of unforeseen contingencies (?).

that *matches* the behavior under unawareness.<sup>10</sup> The idea is to construct a state space encompassing both  $S_1$  and  $S_2$  and reinterpreting the DM's preferences in the two periods as conditioning on disjoint events. More specifically, given  $\succsim^i$  over  $C_i, i = 1, 2$ , I can define another model à la Myerson (?) as follows. Let  $S$  be the disjoint union of  $S_1$  and  $S_2$ , and  $C = (\Delta(Z))^S$  be the choice set for both periods. For all  $f, g \in C$ , define  $f \succsim_S g$  if and only if  $f|_{S_1} \succsim^1 g|_{S_1}$ , and  $f \succsim_{S_2} g$  if and only if  $f|_{S_2} \succsim^2 g|_{S_2}$ . Then one can view the two-period belief system as a *conditional probability system* (CPS)  $\mu : 2^S \times \{S, S_2\} \rightarrow [0, 1]$ , such that  $\mu(S_2|S) = 0$ .<sup>11</sup> Intuitively, this model describes the following: On the expanded state space  $S$ , the DM has a prior belief of zero probability on  $S_2$ . In the second period, the DM receives the zero-probability signal  $S_2$  and updates his beliefs to assign probability 1 to  $S_2$  instead.

The caveat of this approach is that, given disjoint beliefs in the two periods, behavior can be made to match anything one wishes. Therefore, to give this model any content, one needs to impose additional structures that capture the essence of unawareness, for example, the counterparts of U.2-3. But doing this necessarily requires one to examine a model of unawareness, which defeats the very purpose of writing a zero probability model for unawareness.

## 4 Discussion

In this section, I discuss the limitations of this model, in particular those imposed by the standard decision-theoretic framework itself, as well as some conceptual issues of unawareness.

### 4.1 Unawareness and (bounded) rationality.

In this paper, unawareness is treated as an informational constraint. There is a sense in saying that the DM in this model is perfectly rational with respect to his own awareness: He forms probabilistic beliefs about relevant uncertain events and maximizes subjective expected utilities under such beliefs. He is also Bayesian within the boundary of his awareness. In particular, unlike the standard model, dynamic inconsistency under partial unawareness reflects the DM's rationality rather than bounded rationality: Whenever there is correlation between resolutions of aware uncertainties and unaware uncertainties, states in the subjective state space are either not mutually exclusive or not exhaustive, which the DM, if rational, should recognize upon expanding his awareness.

<sup>10</sup>Note this construction surpasses both pure unawareness and partial unawareness.

<sup>11</sup>Fix a finite state space  $\Omega$ . A CPS is any function  $\mu : 2^\Omega \times 2^\Omega \setminus \{\emptyset\} \rightarrow [0, 1]$  such that, for all  $X, Y, Z \subseteq \Omega$ ,  $Z \neq \emptyset$ ,  $\mu(\cdot|Z)$  is a probability measure over  $\Omega$  such that (1)  $\mu(\Omega|Z) = \mu(Z|Z) = 1$ ; (2)  $X \subseteq Y \subseteq Z$ ,  $Y \neq \emptyset \Rightarrow \mu(X|Z) = \mu(X|Y)\mu(Y|Z)$ .

## 4.2 Unforeseen consequences.

As pointed out earlier, this model is best viewed as a thought experiment designed to solicit beliefs under different awareness levels. A key device is the use of monetary bets, which removes unforeseen consequences from consideration. Notice this assumption does impose extra burden on another axiom, namely, the state-independence axiom, by postulating that the DM's preferences for money do not depend on the states, even the unforeseen ones.

It is worth mentioning that this model does not provide a choice theory foundation for decision-making under unawareness. The main problem is that the choice objects in real life are usually *options*, not acts. The mapping between options and acts is typically taken as given in standard models. For example, in a typical model, the DM chooses between a set of risky projects, and the outcomes of these projects in each (subjective) state are known. This is a reasonable simplification when full awareness is assumed, but it becomes problematic when unawareness is involved. Intuitively, unforeseen contingencies give rise to unforeseen consequences. For example, suppose Bob's decision problem is to choose a vacation plan in California. Being unaware of the possibility of an earthquake, presumably he would also be unaware of the influence of an earthquake (or the lack of) on his vacation plans. Upon expanding his awareness, Bob also updates his perception of the travel plans in terms of the corresponding acts in his choice set. In particular, this implies in general that the act representing an option when the DM has more awareness is not an extension of the act representing the same option when the DM has less awareness. In other words, real life choices tend to confound both unawareness of contingencies and unawareness of consequences, which makes it difficult to disentangle beliefs from preferences for final outcomes.

## 4.3 Unforeseen options.

While the model focuses on the implications of unawareness with respect to the set of subjective bets, or foreseen options, the most salient feature in the unawareness environment is actually an incomplete choice set, i.e., the unavailability of bets on events of which the DM is unaware.<sup>12</sup> For a simple example, suppose Charlie is contemplating the choice problem of whether to go for a walk in the park. If he is unaware of the possibility of rain, then the option "take a walk in the park with an umbrella," which could well be the option that he prefers the most had he been aware of rain, will simply not occur to him. The presence of such unforeseen options gives rise to a novel issue that is entirely different from those in the standard model; that is, *ex ante*, it is not possible to specify a contingent plan that yields the best option in each scenario. In contrast, in any standard model with full awareness, the choice set is complete, and hence the

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<sup>12</sup>One immediate implication is that, even if the DM is irrational, whatever it means, he will never bet on an event of which he is unaware, but an irrational DM could certainly bet on an event to which he assigns zero probability.



best option is always conceivable *ex ante*. In this sense there are true dynamics in an unawareness environment: optimal decisions can only be made *after* the arrival of new information. However, since the choice set is the primitives of any standard decision theory model, modeling a DM who contemplates the incompleteness of his choice set may require a drastic departure from the standard approach.

## 5 Conclusion

While unawareness is a realistic cognitive restriction that seems to have broad implications in many important economic situations, it is commonly regarded to be behaviorally equivalent to having zero probability beliefs in economic models. In this paper, I carefully compare, in an axiomatic decision theory framework, the case of unawareness and models of zero probability beliefs. I show that, on the one hand, unawareness does have implications for one's beliefs with respect to events of which one is aware; on the other hand, such restrictions are rather weak, yielding beliefs that have much less structure than those involving assigning zero probabilities to corresponding events. The axiomatic framework also enables me to organize novel issues brought up by unawareness, as well as point out which implicit assumptions of the standard decision theory models will have to be relaxed in order to address them. Dekel, Lipman and Rustichini (?) point out the need of “(finding) an ‘Ellsbergian paradox’ that indicates behaviors that are due to unforeseen contingencies and that conflict with standard subjective (non)expected utility models, thus helping focus the search for alternatives.” While this note does not provide such a “Ellsbergian paradox,” it points out structural assumptions in the standard models that preclude producing such a paradox, and possible directions where one may find one.

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