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"Modeling Unawareness in Arbitrary State Spaces"

by

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Modeling Unawareness in Arbitrary State Spaces†

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Abstract

I develop a set-theoretic model of unawareness without making any structural assumptions on the underlying state space. Unawareness is characterized as a measurability constraint that results in players' reasoning about a "coarse" subjective algebra of events. The model is shown to be essentially equivalent to the product model in Li (2007), indicating that such a measurability constraint can be captured by restrictions on the dimensions of the state space without loss of generality. I use a variant of the partition model to examine the case of *partial unawareness*, where the player is aware of a question but unaware of some possible answers to that question, and characterize the player's knowledge hierarchies from his subjective perspective.

Keywords: unawareness, partial unawareness, information, information partition, the state space

JEL Classification: C70, C72, D80, D82, D83

1 Introduction

Alice is planning a trip to Florida. Weather conditions in Florida could be sunny, raining (without a hurricane), or hurricane. However, having spent all her life in Montana, Alice is unaware of the possibility of hurricane: she does not know there could be a hurricane, and she does not know she does not know it. Such unawareness is present in all aspects of our life. For example, in the business world, firms are often unaware of new products or technologies their competitors are developing at the moment; in international relations, nations may be unaware of secret negotiations and pacts between other nations; in science, typically people are unaware of the next theorem to be proved. The importance of developing tractable tools to analyze situations involving unawareness cannot be overestimated.

However, it turns out that the standard state space models are incapable of handling non-trivial unawareness (Dekel, Lipman and Rustichini 1998). Li (**?**) proposes a *product model* that circumvents this problem by modeling information as a pair and exploiting a product structure on the state space. The idea is as follows. Fixing a set of relevant questions and a set of answers to each question, the full state space specifies answers to all of these questions; i.e., it is the Cartesian product of the sets of answers. At any full state, the players receive information as a pair, including awareness information, represented by a set of questions, and factual information, represented by a subset of the full state space just as in the standard models. The players can reason only within their subjective state spaces that specify answers to those questions included in their awareness information, naturally modeled by the Cartesian product of the set of answers to those questions included in the awareness information; and recognize only the factual information concerning answers to questions of which they are aware, conveniently modeled by projections of their factual information on their subjective state spaces.

The product model heavily relies on the product structure on the full state space, especially in the multi-player model. This paper complements the product model by modeling unawareness without making any structural assumptions. The main idea is, fixing an arbitrary full state space S, a relevant question can be represented by a partition over S, where each partition element represents an answer to that question. For example, to model the situation described at the beginning of this article, let the full state space be $S = \{s, r, h\}$, where s is the state where it is sunny in Florida, and similarly for r and h. The question "is there a hurricane?" can be represented by the partition $\{\{s, r\}, \{h\}\}$, where the partition element $\{s, r\}$ corresponds to the answer "there is no hurricane." Using this formulation, one can model a subjective state space by the join of the questions of which the player is aware. For example, Alice's subjective state space, under her unawareness of the hurricane, can be written as $\{\{s\}, \{r, h\}\}$, where each partition element is interpreted as a subjective state. Intuitively, these subjective states consist of the "atoms" in Alice's reasoning. Any event that is true in either r or h but not both is beyond Alice; i.e., her *subjective algebra of events* consists only of events that can be written as unions of her subjective states. The subjective factual information is then modeled by the union of all subjective states intersecting with the factual information the player actually receives in each state. For example, suppose Alice receives the factual information $\{r\}$, i.e., "it rains, it is not sunny and there is no hurricane," but constrained by her unawareness, she perceives it to be only $\{r, h\}$, i.e., "it rains and it is not sunny."

This formulation, dubbed *the partition model*, does not require any structural assumption on the full state space. Unawareness is characterized as a *measurability constraint* in this model: the resulting knowledge hierarchy at each full state is equivalent to the one obtained by removing all knowledge regarding events that are not measurable with respect to the player's subjective state space at that state. I show that in a single-player environment, the partition model is isomorphic to the product model, implying it is without loss of generality to capture such measurability constraints as restrictions on the dimensions of a product state space. The essence of this result extends to the multi-player environment as well.

One particularly interesting case of unawareness is *partial unawareness*, where there is a correlation between the answers to questions of which the player is unaware and the answers to those of which the player is aware. This includes the situation that can be naturally described as being aware of the question but unaware of some of the possible answers. For example, Alice is partially unaware of hurricane, because after all she does ask herself about weather conditions in Florida. This concept is especially relevant in games: in many real-life strategic situations, although players may be aware of their opponents, and hence reason about their strategies, they are often unaware of some actions the opponents can take.

A fundamental asymmetry of partial unawareness is that if one is aware of an event E, then one is necessarily unaware of its negation from a fully aware outside observer's perspective. While Alice is unaware of the event "there is a hurricane" and its negation "there is no hurricane," she *is* aware of an event equivalent to "there is no hurricane," i.e., "it is either sunny or raining." Of course, Alice herself is unaware of this equivalence. In particular, a message such as "it is not sunny" has different implications for players with different awareness.

Thus it is interesting to characterize knowledge *from the player's perspective*. To do this, the key step is to have a subjective state space that reflects the player's view of the world, from the player's own perspective. In the above example, Alice considers only two subjective states: one in which it is sunny, and the other one in which it rains. While the outside observer sees that Alice has confounded the two scenarios "rain" and "hurricane," and hence her subjective state really corresponds to the objective event $\{r, h\}$, from Alice's perspective, it is simply the state "it rains," given her perception that the world consists of only two states, "sunny" and "rainy." Thus, for Alice, her subjective state space is simply $\{s, r\}$, a subset of the full state space. The subjective algebra of events thus can be viewed as a relativization of the objective algebra of events. Using this *interpolated model*, I characterize an interpolated knowledge hierarchy under partial unawareness, which provides a language to talk about erroneous knowledge due to partial unawareness.

This paper belongs to the quickly expanding literature on the epistemic foundation of unawareness (Modica and Rustichini 1994, Modica and Rustichini 1999, Li 2008a, Li 2008b, Heifetz, Meier and Schipper 2006, Heifetz, Meier and Schipper 2007a, Galanis 2006, Board and Chung 2007). In particular, Ely (1998) proposes, in the context of an example, a model where the information structure is represented by a function associating each state with a set of disjoint subsets of the state space, which essentially plays the role of the subjective factual information in the partition model.

The paper is organized as follows. Section 2 presents the partition model. Section 3 shows that the partition model is equivalent to the product model studied in Li (2007) and characterizes unawareness as a measurability constraint, which, without loss of generality, can be captured as a restriction on the dimensions of a product state space. Section 4 discusses partial unawareness and characterizes players' subjective interpolated knowledge hierarchies. Section 5 concludes. Proofs are collected in the Appendix.

2 The Partition Model

2.1 Primitives.

Fix a state space S that is an arbitrary set, with typical elements s, t. An objective event is a subset of S, denoted by upper-case letters such as E, F. *Awareness* is modeled by a partition over S, denoted by lower-case Greek letters such as π, ν. Fixing π, I interpret each partition element in π, which is an objective event in S, as a subjective state in the *subjective state space* of a player with awareness π. Thus, I use awareness level π and the subjective state space π interchangeably. The *full state space* embedding full awareness is identified by the finest partition over the state space and is denoted by S^*: $S^* = \{\{s\} : s \in S\}$. The *trivial state space* embedding complete unawareness is identified by $\{S\}$.

There is a natural partial order on the set of partitions over S: given any partitions π and ν, π is *weakly finer* than ν, denoted by $\pi \trianglerighteq \nu$, if every element in ν can be written as the disjoint union of elements in π. Thus $\trianglerighteq$ corresponds to the notion of "having (weakly) more awareness information than." Let $\triangleright$ denote the asymmetric part of the order.

Definition 1 *Given a state space S, a collection of partitions over S, denoted by $\mathcal{F}$, is a* **frame** *for the full state space S^* if:*

1. $S^* = \bigvee_{\pi \in \mathcal{F}} \pi$;
2. *for all $\nu \in \mathcal{F}$,* $S^* \triangleright \bigvee_{\pi \in \mathcal{F} \setminus \{\nu\}} \pi$;

Each partition in a frame can be thought of as representing a question, and each set in the partition an answer to the question. A frame consists of a *minimal* set of questions one needs to ask in order to differentiate any two states in S. Specifically, condition 1 says, take any two states in S; they must have different answers to at least one question in the frame; condition 2 requires there be no "redundant" question: for each question π, there are at least two states that coincide in their answers to all other questions except π. For example, let $S = \{a, b, c, d\}$, then the following set of partitions consists of a frame for S^*:

$$\{\{\{a, b\}, \{c, d\}\}, \{\{a\}, \{b, c, d\}\}, \{\{d\}, \{a, b, c\}\}\}.$$

I focus attention on the set of subjective state spaces that can be generated by the frame. Let $\bar{\mathcal{F}} = \{\Phi(\mathcal{F}') : \mathcal{F}' \subseteq \mathcal{F}\}$ where Φ is defined by:

$$\Phi(\mathcal{F}') = \begin{cases} \bigvee_{\pi \in \mathcal{F}'} \pi & \text{if } \mathcal{F}' \neq \emptyset, \\ \{S\} & \text{if } \mathcal{F}' = \emptyset. \end{cases} \tag{2.1}$$

Lemma 1 *The function Φ is one-to-one and the inverse function is defined by: for any $\pi \in \bar{\mathcal{F}}$,*

$$\Phi^{-1}(\pi) = \{\nu \in \mathcal{F} : \pi \trianglerighteq \nu\}.$$

Moreover, $\Phi^{-1}(\pi)$ is a frame for π.

The player's information structure is represented by a pair (W, P), where:

- *the awareness function* $W : S \to \bar{\mathcal{F}}$ associates each state with a subjective state space, interpreted as the set of questions of which the player is aware given his information at s;
- *the full possibility correspondence* $P : S \to 2^S \setminus \{\emptyset\}$ associates each state with a nonempty subset of S, interpreted as the factual content of the player's information at s.

Since (W, P) represents the information the player actually receives, it is natural to focus attention on a partitional information structure:

Definition 2 *The pair (W, P) is* **rational** *if*

$$s \in P(s)\ \forall s \in S;\ \textit{and } s \in P(t) \Rightarrow W(t) = W(s),\ P(t) = P(s). \tag{2.2}$$

Let $I = \{1, \cdots, n\}$ denote the set of players, and (W_i, P_i) denote player i's information. The primitive of the partition model is a tuple

$$(S, \mathcal{F}, \mathbf{W}, \mathbf{P})$$

where $\mathbf{W} = (W_1, \cdots, W_n), \mathbf{P} = (P_1, \cdots, P_n)$.

For any $\pi \in \mathcal{F}$, and any $s, t \in S$, let $\pi(s)$ denote the partition element in π that contains s. Thus $W(t)(s)$ denotes the partition element in $W(t)$ containing s. Slightly abusing notation, if $\nu \trianglerighteq \pi$ and $E \in \nu$, then I let $\pi(E)$ denote the element in π that is a weak superset of E, i.e. $E \subseteq \pi(E) \in \pi$.

At s, i's reasoning is contained in his subjective state space $W_i(s)$. Constrained by his awareness, i perceives only *subjective factual information* for each $j \in I$, described by the projection of P_j on $W_i(s)$. For example, at s, i considers the subjective states in $\{W_i(s)(t) : t \in P_i(s)\}$ to be possible. The next condition ensures that every subjective factual information structure is partitional.

Definition 3 *In the partition model, the possibility correspondence P induces a* **nice factual partition** *if P induces an information partition over S, and that for all $\mathcal{F}' \subseteq \mathcal{F}$, and all $s, t \in S$, $\bigcap_{\pi \in \mathcal{F}'} \pi(s) \cap P(t) \neq \emptyset, t' \in P(t) \Rightarrow \bigcap_{\pi \in \mathcal{F}'} \pi(t') \cap P(s) \neq \emptyset$.*

A nice factual partition requires factual information to be decomposable into independent information about answers to each question in the frame. Let $s' \in \bigcap_{\pi \in \mathcal{F}'} \pi(s) \cap P(t)$ and $t' \in \bigcap_{\pi \in \mathcal{F}'} \pi(t) \cap P(s)$. The condition says, if two states (s and s') coincide in their answers to those questions in the set $\mathcal{F}'$, then the player's factual information concerning answers to these questions in these two states must coincide.

Lemma 2 *Fix a full state space and a frame $(S^*, \mathcal{F})$ and let $\pi \in \bar{\mathcal{F}}$. Let P satisfy nice factual partition. Then $s \in \pi(t)$ implies*

$$\{\pi(t') : t' \in P(t)\} = \{\pi(s') : s' \in P(s)\}.$$

I say that a partition model is *nice* if P_i induces a nice factual partition for all $i \in I$. In a nice partition model, at s, *i's subjective possibility correspondence for j*, describing i's perception of j's factual information structure and denoted by $P_j(\cdot|i_s)$ where i_s is shorthand for the pair (i, s), is unambiguously defined as the projection of P_j on $W_i(s)$: for any $E \in W_i(s)$,

$$P_j(E|i_s) = \{W_i(s)(t) : t \in P_j(t'), \text{ where } t' \in E\}. \tag{2.3}$$

Similarly, i reasons about j's awareness within his own awareness. An extra complication arises when there is unawareness of uncertainties in awareness information. At s, i is unaware of the possibility that j could be unaware of the question represented by $\pi \in \mathcal{F}, \pi \trianglerighteq W_i(s)$ if there exist t, u such that $t \in W_i(s)(u)$, $\pi \trianglerighteq W_j(t)$, but $\pi \ntrianglerighteq W_j(u)$. Intuitively, if i is aware of a question and also aware of j, i reasons about j's awareness of this question if and only if i is aware of the possibility that j could be unaware of the question. Thus, it seems natural to assume that whenever i is unaware of j's unawareness of a question of which i himself is aware, i takes it for granted that j is aware of it. This suggests the following definition of *i's subjective awareness function for j*, denoted by $W_j(\cdot|i_s)$: for any $E \in W_i(s)$,

$$W_j(E|i_s) = W_i(s) \wedge \left[\bigvee_{t \in E} W_j(t)\right]. \tag{2.4}$$

Let i's subjective model at s be denoted by $\mathcal{M}(i_s)$, then,

$$\mathcal{M}(i_s) \equiv (W_i(s), \Phi^{-1}(W_i(s)), \mathbf{W}(\cdot|i_s), \mathbf{P}(\cdot|i_s))$$

where $\mathbf{W}(\cdot|i_s) = (W_1(\cdot|i_s), \cdots, W_n(\cdot|i_s))$, $\mathbf{P}(\cdot|i_s) = (P_1(\cdot|i_s), \cdots, P_n(\cdot|i_s))$.

Similarly, one can construct higher-order subjective models that describe i's perception of j's perception of the environment, and so on. The next lemma verifies that in a nice partition model, all first-order subjective models are nice partition models themselves.

Lemma 3 *Let the partition model $(S^*, \mathcal{F}, \mathbf{W}, \mathbf{P})$ be nice. Then for any $i \in I, s \in S$, the subjective model $\mathcal{M}(i_s)$ is nice.*

Below I define the subjective model in general inductively. For notational convenience, I write $W(\delta) \equiv W_i(s)$ where δ denotes (i, s). Let Δ^1 denote the collection of all such pairs, i.e., $\Delta^1 = \{(i_s) : i \in I, s \in S\}$. Let subjective models be defined for all elements in Δ^k. The domain for $k+1$-th order subjective models is:

$$\Delta^{k+1} = \left\{\delta^k + (j_E) : \delta^k \in \Delta^k, j \in I, E \in W(\delta^k)\right\},$$

where "+" denotes concatenation. Fix $\delta \in \Delta^{k+1}, \delta = \delta^k + (j_E)$, I denote the subjective model for δ as

$$\mathcal{M}(\delta) \equiv (W(\delta), \Phi^{-1}(W(\delta)), \mathbf{W}(\cdot|\delta), \mathbf{P}(\cdot|\delta))$$

where $W(\delta) = W_j(E|\delta^k)$ is defined by the induction hypothesis and the rest of the model is defined as follows: for any $j = I, F \in W(\delta)$,

$$W_j(F|\delta) = W(\delta) \wedge \left[\bigvee_{G \subseteq F,\, G \in W(\delta^k)} W_j(G|\delta^k)\right], \tag{2.5}$$

$$P_j(F|\delta) = \left\{W(\delta)(G) : G \in P_j(H|\delta^k) \text{ where } H \in W(\delta^k), H \subseteq F\right\}. \tag{2.6}$$

By Lemma 3, all subjective models are nice. Thus, by Lemma 2, formula (2.6) is well-defined.

2.2 Characterization of knowledge and unawareness.

Let the objective algebra of events in S be denoted by $\mathcal{E} = 2^S$.

Given a subjective state space π, consider those events in $\mathcal{E}$ that are measurable with respect to π, i.e., those events that can be written as a disjoint union of partition elements in π:

$$\mathcal{A}(\pi) = \{E \in \mathcal{E} : E \cap F = \emptyset \text{ or } E \cap F = F \text{ for all } F \in \pi\}.$$

Intuitively, $\mathcal{A}(\pi)$ is the set of events of which a player with awareness π is aware. In this sense, I refer to $\mathcal{A}(W_i(s))$ as i's *subjective algebra* of events at s. Let f_π associate each event with its subjective expression, if possible: for any $E \in \mathcal{E}$,

$$f_\pi(E) = \begin{cases} \{F \in \pi : F \subseteq E\}, \text{ if } E \in \mathcal{A}(\pi), \\ \emptyset, \text{otherwise.} \end{cases} \tag{2.7}$$

Player i is unaware of an event if and only if it is not a subjective event in his subjective state space: for any $E \in \mathcal{E}$,

$$U_i(E) = \{s \in S : E \notin \mathcal{A}(W_i(s))\}. \tag{2.8}$$

Knowledge is defined analogous to the standard model: i knows E if and only if E is true in all *subjective states* i considers possible: for any $E \in \mathcal{E}$,

$$K_i(E) = \left\{s \in S : P_i(W_i(s)(s)|i_s) \subseteq f_{W_i(s)}(E)\right\}. \tag{2.9}$$

Note this definition implicitly requires the player to have a version of E in his subjective state space. In fact, the above definition is equivalent to:

$$K_i(E) = \{s \in S : E \in \mathcal{A}(W_i(s)), P_i(s) \subseteq E\}. \tag{2.10}$$

Higher-order knowledge is obtained by adapting definitions 2.8 and 2.10 to the corresponding subjective models, recursively. More specifically, the knowledge "i^1 knows i^2 knows $\cdots$ i^n knows E" , denoted by $K_{i^1} \cdots K_{i^n}(E)$, is defined as follows:

$$K_{i^1} \cdots K_{i^n}(E) = \left\{s \in S : W_{i^1}(s)(s) \subseteq \tilde{K}^{i^1}_{i^1_s}[\tilde{K}^{i^2} \cdots \tilde{K}^{i^n}]_{i^1_s}(E)\right\}; \tag{2.11}$$

for all $m = 2, 3, \cdots$, and all $\delta \in \Delta \equiv \cup_{k=1}^{\infty} \Delta^k$,

$$[\tilde{K}^{i^1} \cdots \tilde{K}^{i^m}]_\delta(E) = \bigcup \{F \in W(\delta) : W_{i^1}(F|\delta) \subseteq \tilde{K}^{i^1}_{\delta+(i^1_F)}[\tilde{K}^{i^2} \cdots \tilde{K}^{i^m}]_{\delta+(i^1_F)}(E)\} \tag{2.12}$$

where I slightly abuse notation and use $\bigcup \pi$ to denote the union of all the sets in π.[1] Finally, for all $j \in I$, $\delta \in \Delta$,

$$\tilde{K}^j_\delta(E) = \bigcup \left\{F \in W(\delta) : E \in \mathcal{A}(W(\delta)), P_j(F|\delta) \subseteq f_{W(\delta)}(E)\right\}. \tag{2.13}$$

For interpretation, $[\tilde{K}^{i^1} \cdots \tilde{K}^{i^m}]_\delta(E)$ denotes the knowledge "i^1 knows that i^2 knows $\cdots$ knows i^n knows E" in the subjective model $\mathcal{M}(\delta)$.

Similarly, the event "i^1 knows i^2 knows $\cdots$ knows i^n is unaware of E," denoted by $K_{i^1} \cdots U_{i^n}(E)$, can be obtained by replacing all incidents of $\tilde{K}^{i^n}$ in the above definitions by $\tilde{U}^{i^n}$, and adapting 2.8 to the relevant subjective model $\mathcal{M}(\delta)$:

$$\tilde{U}^j_\theta(E) = \bigcup \{F \in W(\delta) : E \notin \mathcal{A}(W_j(F|\delta))\}. \tag{2.14}$$

[1] Here π simply denotes a collection of sets.

3 Equivalence with the Product Model

3.1 The product model.

Here I briefly review the product model studied in Li (2008a). The primitives of the model are a tuple $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ where:

- $\Omega^* = \prod_{q\in Q} D_q \times \{\alpha\}$, where Q is an arbitrary index set of questions and D_q is the set of answers to question q.[2] The symbol α represents "*cogito ergo sum.*"[3] A generic element in Ω^* is denoted by ω^*.
- The awareness function $W_i^* : \Omega^* \to 2^Q$ associates each state with a set of questions of which i is aware.
- The possibility correspondence $P_i^* : \Omega^* \to 2^{\Omega^*} \setminus \{\emptyset\}$ associates each state with the set of states where i receives the same information.

The pair (W^*, P^*) is *rational* if P^* induces an information partition over Ω^*, and for all $\omega_1 \in P^*(\omega^*), W^*(\omega_1^*) = W^*(\omega^*)$. I say P^* satisfies *cylinder factual partition* if there exists a collection of partitions $\{\pi_q\}_{q\in Q}$, π_q is a partition over D_q, such that $P^*(\omega^*) = \prod_{q\in Q} \pi_q(\omega^{*q}) \times \{\alpha\}$ where $\pi_q(\omega^{*q})$ is the partition element in π_q containing ω^{*q}, the q-th coordinate of ω^*.

The subjective models are defined as follows. The subjective state space is naturally defined to be the product of the sets of answers to the questions of which the player is aware. More specifically, for any $Q' \subseteq Q$, the corresponding subjective state space, denoted by $\Omega(Q')$, is simply $\prod_{q\in Q'} D_q \times \{\alpha\}$.

Let $\mathbb{P}^\Omega$ denote the projection operator that yields the projection of an event or a state from a weakly finer space on Ω.[4] I define the subjective models inductively. Let $\Theta^1 = \{((\omega_1, i^1)) : \omega_1 \in \Omega^*, i^1 \in I\}$. The subjective model player i perceives that a full state ω^*, denoted by $\mathcal{M}^p(\theta), \theta = (\omega^*, i)$ is:

$$\mathcal{M}^p(\theta) \equiv (\Omega(\theta), \mathbf{P}(\cdot|\theta), \mathbf{W}(\cdot|\theta)$$

where:[5]

$$\begin{aligned} \Omega(\theta) &= \Omega(W_i(\omega^*)), && (3.1)\\ P_j(\omega|\theta) &= \mathbb{P}^{\Omega(\theta)} P_j^*(\omega_1^*), \text{ where } \mathbb{P}^{\Omega(\theta)}(\omega_1^*) = \omega, \ \forall j, \ \forall \omega \in \Omega(\theta); && (3.2)\\ W_j(\omega|\theta) &= W_i^*(\omega^*) \cap [\bigcup_{\{\omega_1^* :\, \mathbb{P}^{\Omega(\theta)}(\omega_1^*)=\omega\}} W_j^*(\omega_1^*)], \ \forall j, \ \forall \omega \in \Omega(\theta). && (3.3) \end{aligned}$$

[2] Li (2008a) assumes that all questions have binary answers: $D_q = \{1_q, 0_q\}$ for all $q \in Q$. The equivalence of these two formulations is obvious.

[3] This is simply a device to avoid having an empty set as the subjective state space.

[4] I say a subjective state space Ω_1 is weakly finer than Ω_2 if every question specified in Ω_2 is also specified in Ω_1.

[5] The subjective factual information $P_j(\omega|\theta)$ is well-defined under cylinder factual partition.

Suppose the subjective model $\mathcal{M}(\theta^k)$ is defined for all $\theta^k \in \Theta^k$.

$$\Theta^{k+1} = \left\{\theta + (\omega, j) : \theta \in \Theta^k, \omega \in \Omega(\theta^k), j \in I\right\}. \tag{3.4}$$

Fixing $\theta = \theta^k + (\omega, i) \in \Theta^{k+1}$, the relevant subjective model $\mathcal{M}^p(\theta)$ is defined as follows:

$$\begin{aligned}
\Omega(\theta) &= \Omega(W_i(\omega|\theta^k)), & (3.5)\\
P_j(\omega|\theta) &= \mathbb{P}^{\Omega(\theta)} P_j(\omega_1|\theta^k), \text{ where } \mathbb{P}^{\Omega(\theta)}(\omega_1) = \omega, \ \forall j, \ \forall \omega \in \Omega(\theta); & (3.6)\\
W_j(\omega|\theta) &= \Omega(W_i(\omega|\theta^k)) \cap [\bigcup_{\{\omega_1 \in \Omega(\theta^k):\, \mathbb{P}^{\Omega(\theta)}(\omega_1)=\omega\}} W_j(\omega_1|\theta^k)], \ \forall j, \ \forall \omega \in \Omega(\theta). & (3.7)
\end{aligned}$$

Let $\Theta = \bigcup_{k=1}^{\infty} \Theta^k$.

Finally, the set of events in the product model is:

$$\mathcal{E}^p = \bigcup_{Q' \subseteq Q} \left[\{E \subseteq \Omega(Q') : E \neq \emptyset\} \cup \{\emptyset_{Q'}\}\right]. \tag{3.8}$$

where $\emptyset_{Q'}$ is the empty set confined within the space $\Omega(Q')$.[6]

Slightly abusing notation, let q also denote the function mapping each event to the set of questions whose answers are described in the event, i.e., $q(E) \subseteq Q$ is the unique set of questions such that $E \subseteq \Omega(q(E))$.

The first-order knowledge and unawareness operators are defined as follows: for any $E \in \mathcal{E}^p$,

$$\begin{aligned}
U_i^p(E) &= \{\omega^* \in \Omega^* : W_i^*(\omega^*) \not\supseteq q(E)\}; & (3.9)\\
K_i^p(E) &= \left\{\omega^* \in \Omega^* : W_i^*(\omega^*) \supseteq q(E), P_i^*(\omega^*) \subseteq E \times \prod_{q \in Q \setminus q(E)} D_q\right\}. & (3.10)
\end{aligned}$$

As in the partition model, higher-order interactive knowledge is defined through subjective models. Let the objective interactive knowledge "i^1 knows that i^2 knows $\cdots$ knows i^n knows E" be denoted by $[K_{i^1} \cdots K_{i^n}]^p(E)$. For ease of notation, let $\theta = (\omega^*, i^1)$.

$$[K_{i^1} \cdots K_{i^n}]^p(E) = \left\{\omega^* \in \Omega^* : \mathbb{P}^{\Omega(W_{i^1}(\omega^*))}(\omega^*) \in [\tilde{K}^{i^1}]_\theta^p [\tilde{K}^{i^2} \cdots \tilde{K}^{i^n}]_\theta^p(E)\right\}, \tag{3.11}$$

and for all $m = 2, 3, \cdots$, and all $\theta \in \Theta$,

$$[\tilde{K}^{i^1} \cdots \tilde{K}^{i^m}]_\theta^p(E) = \begin{cases} \{\omega \in \Omega(\theta) : \mathbb{P}^{\Omega(\theta^+)}(\omega) \in [\tilde{K}^{i^1}]_{\theta^+}^p [\tilde{K}^{i^2} \cdots \tilde{K}^{i^m}]_{\theta^+}^p(E), \text{ where } \theta^+ = \theta + (\omega, i^1)\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q(E) \subseteq W(\theta), \\ \emptyset_{q(E)}, \text{ otherwise}, \end{cases} \tag{3.12}$$

[6]Here's a more rigorous definition. I introduce a new object $\emptyset_{Q'}$ to be a subset of $\Omega(Q')$, and extend the set operations to this object as follows: for any sets $E, F \in 2^{\Omega(Q')}$, the set inclusion, intersection, union and complement notions are defined in the usual way, except that for disjoint E and F, $E \cap F = \emptyset_{Q'}$ instead of $\emptyset$; for any $E \subseteq \Omega(Q'), E \neq \emptyset$, one has $\emptyset_{Q'} \subseteq E, \emptyset_{Q'} \cup E = E, \ \emptyset_{Q'} \cap E = \emptyset_{Q'}, \ E \setminus \emptyset_{Q'} = E$.

where $W(\theta)$ is shorthand for the set of questions specified in $\Omega(\theta)$, i.e., suppose $\theta = \theta' + (\omega', l)$, then $W(\theta) \equiv W_l(\omega'|\theta')$.

Finally, for all j, for all $\theta \in \Theta$,

$$[\tilde{K}^j]^p_\theta(E) = \begin{cases} \left\{ \omega \in \Omega(\theta) : W_j(\omega|\theta) \supseteq q(E), P_j(\omega|\theta) \subseteq E \times \prod_{q \in W(\theta) \setminus q(E)} D_q \right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } q(E) \subseteq W(\theta), \\ \emptyset_{q(E)}, \text{ otherwise.} \end{cases} \tag{3.13}$$

Similarly, the event "i^1 knows i^2 knows $\cdots$ knows i^n is unaware of E," denoted by $[K_{i^1} \cdots U_{i^n}]^p(E)$, is obtained by replacing $\tilde{K}^{i^n}$ by $\tilde{U}^{i^n}$ in all above definitions and adding:

$$[\tilde{U}^j]^p_\theta(E) = \begin{cases} \{\omega \in \Omega(\theta) : q(E) \nsubseteq W_j(\omega|\theta)\}, & \text{if } q(E) \subseteq W(\theta); \\ \emptyset_{q(E)}, & \text{if } q(E) \nsubseteq W(\theta). \end{cases} \tag{3.14}$$

3.2 The equivalence result.

For any event $E \in \mathcal{E}^p$, let the map $\Psi : \mathcal{E}^p \rightrightarrows \mathcal{E}^p$ yield the set of less detailed descriptions of E, i.e.,

$$\Psi(E) = \left\{ F \in \mathcal{E}^p : F \times \prod_{q \in \mathcal{D}_E \setminus \mathcal{D}_F} D_q = E \right\}.$$

Theorem 4 *The partition model and the product model are equivalent. More specifically,*

1. *Fix a product model $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ where (W_i^*, P_i^*) is rational for all i. Then there exists a partition model $(\Omega^*, \mathcal{F}, \mathbf{W}, \mathbf{P})$ where (W_i, P_i) is rational for all i, and for all $E \in 2^{\Omega^*} \setminus \{\emptyset\}$,*

$$U_i(E) = \bigcap_{F \in \Psi(E)} U_i^p(F), \tag{3.15}$$

$$K_i(E) = \bigcup_{F \in \Psi(E)} K_i^p(F). \tag{3.16}$$

In addition, if P_i^* satisfies cylinder factual partition for all i, then P_i satisfies nice factual partition, and the above equations can be extended to include all higher-order knowledge, i.e., for all n, any $i^1, \cdots, i^n \in I$ and $E \in 2^{\Omega^*} \setminus \{\emptyset\}$,

$$[K_{i^1} \cdots U_{i^n}](E) = \bigcup_{F \in \Psi(E)} [K_{i^1} \cdots U_{i^n}]^p(F), \tag{3.17}$$

and similarly when U_{i^n} in the above formula is replaced by K_{i^n}.

2. *Fix a partition model $(S,\mathcal{F},\mathbf{W},\mathbf{P})$ where (W_i,P_i) is rational for all i. Then there exists a product model $(\Omega^*,\mathbf{W}^*,\mathbf{P}^*)$ and an injection $\Gamma : S \to \Omega^*$ such that for all $i \in I$, (W_i^*,P_i^*) is rational on $\Gamma(S)$; and for any $E \in \mathcal{E}^p$,*

$$U_i^p(E) \cap \Gamma(S) = \Gamma\left(\bigcup_{\omega \in \prod_{q \in q(E)} D_q} U_i(\Gamma^{-1}(\omega \times \prod_{q \in Q \setminus q(E)} D_q) \cap S)\right), \tag{3.18}$$

$$K_i^p(E) \cap \Gamma(S) = \Gamma\left(K_i(\Gamma^{-1}(E \times \prod_{q \in Q \setminus q(E)} D_q) \cap S))\right) \cap \neg U_i^p(E). \tag{3.19}$$

In addition, if P_i satisfies nice factual partition, then P_i^* satisfies cylinder factual partition on $\Gamma(s)$, and the above two equations can be extended to higher-order knowledge.

This theorem says that one can always paraphrase the product model into the partition model, and vice versa. Intuitively, the frame $\mathcal{F}$ in the partition model corresponds to the sets of answers $\{D_q\}_{q\in Q}$ in the product model. In the partition model, one fixes the full state space and represents answers using events, while in the product model, one starts with questions and answers and defines the full state space from them. They are really just two sides of the same coin. Depending on specific applications, either model could be more convenient.

Remark 1. If one restricts attention to own knowledge hierarchies, then the equivalence of the two models is entirely general: the partition model is reduced to the tuple (S,W,P) and the frame is not needed.[7] This is because, in the single-agent case, higher-order reasonings are "local": what the player perceives himself to know or not know in a (subjective) state he excludes is irrelevant for his own knowledge hierarchy. Thus, the assumption that the player perceives the projection of his factual information in all subjective states he considers possible suffices to ensure that the single-agent knowledge hierarchies are well-defined. It is easy to see that this assumption does not depend on any structural assumptions on the state space, and the product structure is without loss of generality.

In the multi-agent environment, players' reasonings about knowledge in a (subjective) state that they themselves exclude are relevant for interactive knowledge hierarchies, which makes it necessary to fully specify the subjective models. The problem arises if i is unaware that j could receive different factual information regarding questions of which i is aware. In the product model, this uncertainty can be resolved by using the natural product order to select the relevant factual information. In an arbitrary state space without any structural assumption, such generality cannot be obtained without enriching the

[7] The restriction on the range of awareness function is not binding: given an arbitrary set of partitions $\mathcal{G}$, one can always find a frame $\mathcal{F}$ such that $\mathcal{G} \subseteq \bar{\mathcal{F}}$.

information structure further. For simplicity, I assume away this complication in this paper by requiring P to satisfy the nice factual partition condition, which restricts the model to situations where there is no unawareness of uncertainties in factual information.

Remark 2. Events are modeled differently in the two models. In the product model, an event is defined by both its factual content *and* its description which reflects the awareness content; while in the partition model, an event is best understood as a factual description in its *coarsest form*, i.e., a minimal collection of factual statements without leaving out any nontrivial facts. Consequently, knowledge and unawareness are interpreted in slightly different ways in the two models: in the product model, "knowing E" means knowing E in the exact form it is described; in the partition model, "knowing E" means knowing the factual content of E in *some* form of description. Similarly, "being unaware of E" means being unaware of E as it is described in the product model, while in the partition model, it means being unaware of *all* possible descriptions of this event. Equations (3.15) - (3.16) describe the connections formally. Apparently, either approach to modeling events can be adopted in either model.

3.3 Common knowledge.

As Li (2008b) shows, introducing unawareness has multiple implications for a set of players to achieve common knowledge of an event. Intuitively, while the possibility of unawareness imposes an additional requirement for a player to know an event and hence makes it harder to achieve common knowledge, being unaware of the opponent's informational uncertainties actually makes it easier to achieve common knowledge. Li (2008b) characterizes common knowledge when where there is no unawareness of informational uncertainties. Using the equivalence result, I give a characterization of common knowledge in the partition model.

Definition 4 *Fix a state space and a frame* $(S, \mathcal{F})$*. I say* W_i *satisfies* **Nice awareness** *if: for any* $\pi \in \mathcal{F}$*,* $t \in \pi(t')$*,* $t \notin \nu(t')$ *for all* $\nu \in \mathcal{F}, \nu \neq \pi$*,* $s \in \pi(s'), t \in W_i(s)(t')$ *implies that* $t \in W_i(s')(t')$*.*

Nice awareness requires that one's awareness of a particular uncertainty depend only on how this uncertainty is resolved. More specifically, this condition says, if the player is unaware of a question π at s, and s' coincides with s in its answer to π, then he must also be unaware of π at s'. In a multi-player environment, this implies that a player is never unaware of the possibility that this opponents could be unaware of an event of which he is aware himself. This condition ensures that all interactive knowledge is true.

Let $CK(E)$ denote the event "E is common knowledge among all players"; that is, all players know E, all players know all players know E, all players know that all players know that all players know E, and so on.

Theorem 5 *In the partition model $(S, \mathcal{F}, \mathbf{W}, \mathbf{P})$, if for all i, (W_i, P_i) is rational and satisfies nice factual partition and nice awareness, then for all $E \in 2^S$,*

$$CK(E) = \left\{ s \in S : E \in \cap_{i=1}^n \bigcap_{t \in \wedge \mathbf{P}(s)} \mathcal{A}(W_i(t)), \wedge \mathbf{P}(s) \subseteq E \right\}, \tag{3.20}$$

where $\wedge \mathbf{P}(s) \equiv [P_1 \wedge \cdots P_n](s)$ denotes the partition element containing s in the finest common coarsening of all information partitions.

3.4 Unawareness as the measurability constraint.

Fix a single-agent partition model $(S, \mathcal{F}, W, P)$ where (W, P) is rational. Let $\hat{K}^n, n = 1, 2, \cdots$ denote the standard knowledge operators associated with the pair (S, P), i.e., for all $E \subseteq S$,

$$\begin{aligned} \hat{K}(E) &= \{s \in S : P(s) \subseteq E\}, \\ \hat{K}^n(E) &= \hat{K}(\hat{K}^{n-1}(E)). \end{aligned}$$

Let K^n denote the n-th order single-agent knowledge operator for the partition model. By the equivalence result and Theorem 1 in Li (2008a), the following holds:

$$K^n(E) = \hat{K}^n(E) \cap \neg U(E). \tag{3.21}$$

Notice that if $W(s) = S^*$ for all $s \in S$, then all subjective models are identical to the full model (S, W, P), and all knowledge operators reduce to the standard ones: $K^n(E) = \hat{K}^n(E)$. Therefore, $\hat{K}$ can be interpreted as the player's "implicit knowledge," knowledge the player *could have entertained were he fully aware at every state.* In this sense, formula (3.21) says unawareness is essentially a "measurability" constraint: at each $s \in S$, only that knowledge in $\hat{K}^n$ concerning events in $\mathcal{A}(W(s))$, events that are *measurable with respect to the player's subjective state space*, becomes explicitly recognized by the player.[8]

This interpretation extends to the multi-player environment.

Proposition 6 *In the partition model $(S, \mathcal{F}, \mathbf{W}, \mathbf{P})$, suppose for all i, (W_i, P_i) is rational and satisfies nice factual partition and nice awareness. Then for all $i, j \in I$ and $E \in \mathcal{E}$,*

$$K_i K_j(E) = \hat{K}_i \hat{K}_j(E) \cap K_i \neg U_j(E) \tag{3.22}$$

Finally, let $\widehat{CK}(E) = \{s \in S : \wedge \mathbf{P}(s) \subseteq E\}$ denote the standard common knowledge operator associated with $(S, \mathbf{P})$, interpreted as "implicit common knowledge." Consider the event:

$$CA(E) = \bigcap_{i=1}^{n} \left[\bigcap_{m=2}^{\infty} \bigcap_{I_i^m \in \mathcal{I}_i^m} KA(E|I_i^m) \bigcap \neg U_i(E) \right].$$

[8] Fagin and Halpern (1988) were the first to study "implicit knowledge" and "explicit knowledge," but they deal with it in an axiomatic model.

It is straightforward to check that this set characterizes the event "everyone is aware of E, and everyone knows everyone is aware of E, and so on" (and hence the notation $CA(E)$). Thus, equation (3.20) is simply

$$CK(E) = \widehat{CK}(E) \cap CA(E).$$

In words, absent an unawareness of informational uncertainties, implicit common knowledge becomes explicit if and only if there is common knowledge that the event is measurable in every player's subjective state space.

4 Partial unawareness and the interpolation model.

A particularly interesting case of unawareness is partial unawareness. Partial unawareness refers to the situation where there is a correlation between the answers to questions of which the player is unaware and the answers to those of which the player is aware, including situations that can be naturally described as being aware of the question but unaware of some of the possible answers. Intuitively, partial unawareness causes the player to fail to recognize the logical connections between events of which one is aware, resulting in erroneous inferences.[9] It is particularly relevant in strategic interactions, where it is natural for a player to be aware of the opponent, but unaware of some actions the opponent can take.

For example, suppose Alice and Bob play a game. Alice can take actions $a1, a2$ or A. Bob is aware that Alice is in the game, but he is aware only of her actions $a1$ or $a2$. Suppose Bob observes whether Alice takes $a2$. One can model this situation as follows. Let the state space $S = \{a1, a2, A\}$, and the frame $\mathcal{F} = \{\{\{a2\}, \{a1, A\}\}, \{\{A\}, \{a1, a2\}\}\}$. In words, the question "what action does Alice take?" is rephrased as two questions, whether Alice takes action $a2$ and whether Alice takes action A, and Bob is unaware of the latter. Thus, Bob's subjective state space can be represented by the partition $\{\{a2\}, \{a1, A\}\}$.

This approach explicitly models how the player confounds unaware states with states of which he is aware. Hence, the knowledge hierarchy in this model reflects the "true" knowledge the player has from the perspective of the modeler, the fully aware outside observer. For example, suppose Bob is fully aware when Alice plays $a2$ and unaware of A otherwise, and consider the following knowledge hierarchy.

$W(a1) = W(A) = \{\{a1, A\}, \{a2\}\}$, $W(a2) = S^*$;
P induces the information partition $\{\{a1, A\}, \{a2\}\}$.

$K(\{a1, A\}) = K(\neg \{a2\}) = \{a1, A\}, K(\{a1\}) = \emptyset, K(\{a1, A, a2\}) = S;$

[9] See Galanis (2006) for a thorough discussion of modeling unawareness of logical inferences in the product model.

$U(\{a1, A, a2\}) = \emptyset, U(\{a1\}) = \{a1, A\}$.

At A, Bob knows $\{a1, A\}$, while he does not know $\{a1\}$; in fact, he is unaware of $\{a1\}$. Here the set $\{a1, A\}$ represents the subjective event $\{\{a1, A\}\} = \neg\{\{a2\}\}$, interpreted as "Alice does not play $a2$," which, from Bob's perspective, is equivalent to "Alice plays $a1$"; while the set $\{a1\}$ represents the event $\{\{a1\}\}$, interpreted as both "Alice plays $a1$" and "Alice plays neither $a2$ nor A." Therefore, one can interpret this knowledge hierarchy as follows: at A, Bob is unaware of the objective content of the event "Alice plays $a1$"; he does have in mind a subjective understanding of this event, in fact, he knows "Alice does not play $a2$," which he subjectively *interpolates* as "Alice plays $a1$."

Similarly, Bob knows $\{a1, A, a2\}$ in both A and $a2$, but interpolates it differently. At A, Bob knows essentially the subjective event $\{\{a1, A\}, \{a2\}\}$, interpreted as "Alice may play $a2$, and she may not," which, from Bob's perspective, is equivalent to "Alice plays $a2$ or $a1$"; while at $a2$, Bob knows $\{\{a1\}, \{A\}, \{a2\}\}$ in the full state space, interpreted as "Alice plays $a1, A$, or $a2$." In a sense, Bob subjectively "knows" different events in the two states, even though from the modeler's perspective, the factual content of Bob's knowledge is really the same: he cannot rule out anything.

It is often more relevant to study the player's knowledge *from the player's own perspective*, taking into account such erroneous inferences resulting from partial unawareness. After all, it is what Bob subjectively "knows" that matters for his decision-making. To capture this, I rephrase the partition model to reflect the player's interpolation of the situation, by replacing each subjective state, which is an objective event, in the partition model with the corresponding "interpolated" state the player has in mind. In the above example, Bob considers only two subjective states: one in which Alice plays $a1$ and another in which Alice plays $a2$. In this sense, Bob interpolates his subjective state space as $\{a1, a2\}$, a subset of the state space. Notice the state $a1$ in state space $\{a1, a2\}$ is a different object than the state $a1$ from the state space $\{a1, a2, A\}$, as they have different negations. Thus, technically, a subjective state could be regarded as a pair, consisting of a state and the universal event the player has in mind constrained by his partial unawareness.

For simplicity, suppose the player is either fully aware or partially aware at every state. Formally, the interpolation model is a tuple (S, W°, P°), where W° associates each state with a subset of the state space S, interpreted as the set of states specifying resolutions of which the player is aware; and P° is the usual possibility correspondence. I define the subjective information as follows: for any $t \in W_i^\circ(s)$,

$$W_j^\circ(t|i_s) = W_i^\circ(s) \cap W_j^\circ(t); \quad (4.1)$$

$$P_j^\circ(t|i_s) = W_i^\circ(s) \cap P_j^\circ(t). \quad (4.2)$$

Let $\Sigma^1 = \{(i, s) : i \in I, s \in S\}$. For notational ease, write $W^\circ(\sigma_1) \equiv W_i^\circ(s), \sigma_1 = (i, s)$. Let subjective models be defined for all elements in Σ^k. The domain for $k+1$-th order subjective models is:

$$\Sigma^{k+1} = \left\{\sigma^k + (j,t) : \sigma^k \in \Sigma^k, j \in I, t \in W(\sigma^k)\right\}.$$

Fix $\sigma \in \Sigma^{k+1}, \sigma = \sigma^k + (i,s)$.The corresponding subjective models are $(W^\circ(\sigma), \mathbf{W}^\circ(\cdot|\sigma), \mathbf{P}^\circ(\cdot|\sigma))$, where $W^\circ(\sigma) = W_i^\circ(s|\sigma^k)$, and for all $j, t \in W^\circ(\sigma)$,

$$\begin{aligned} W_j^\circ(t|\sigma) &= W^\circ(\sigma) \cap W_j^\circ(t|\sigma^k); \\ P_j^\circ(t|\sigma) &= W^\circ(\sigma) \cap P_j^\circ(t|\sigma^k). \end{aligned}$$

It is easy to see the above equations are equivalent to: for any $\sigma = ((i^1, s_1), \cdots, (i^n, s_n))$,

$$\begin{aligned} W_j^\circ(t|\sigma) &= W_j^\circ(t) \cap W_{i^1}^\circ(s_1) \cap \cdots \cap W_{i^n}^\circ(s_n); \\ P_j^\circ(t|\sigma) &= P_j^\circ(t) \cap W_{i^1}^\circ(s_1) \cap \cdots \cap W_{i^n}^\circ(s_n). \end{aligned}$$

Definition 5 *Fix an interpolation model (S, W°, P°). I say (W°, P°) satisfies* **interpolated partial unawareness** *if, for all $s \in S$, $P_i^\circ(s) \cap W_i^\circ(s) \neq \emptyset$.*

Interpolated partial unawareness says the player always has in mind some possible scenario(s). This condition can be regarded as a regularity condition: since the player is aware of the underlying uncertainty, in this case he must consider possible a scenario specifying "none of the above" and be aware of it by definition.

At s, i can reason only about events in his own subjective state space, i.e. $\{E : E \subseteq W_i^\circ(s)\}$, which is a relativization of the objective algebra of events. Fix an event $E \subseteq W_i^\circ(s)$. Notice partial unawareness is necessarily asymmetric: if the player is aware of E under partial unawareness, then he is necessarily unaware of the *objective* negation of E, i.e., $S \setminus E$. On the other hand, the player is certainly aware of a *subjective* negation of E, i.e., $W_i^\circ(s) \setminus E$. One interesting consequence is that this gives rise to a genuine communication failure: intuitively, a message "E is not true“ has different implications for players with different partial awareness.

To capture such subtlety, I separate knowledge of E and knowledge of "not $S \setminus E$." Let $K_i^+(E)$ denote i's "positive knowledge" of E, i.e., "i knows E"; and $K_i^-(E)$ denote i's "negative knowledge" of E, i.e., "i knows 'not E.'" Similarly, let $U_i^+(E)$ represent "i is unaware of E" and $U_i^-(E)$ represent "i is unaware of 'not E.'" In standard models, i has positive knowledge of E if and only if he has negative knowledge of $S \setminus E$, i.e., $K_i^+(E) = K_i^-(S \setminus E)$, so there is no need to have two operators.

Let $\dagger = +, -$. For any $E \in \mathcal{E}$,

$$U_i^\dagger(E) = \{s \in S : E \not\subseteq W_i^\circ(s)\}. \tag{4.3}$$

$$\begin{aligned} K_i^+(E) &= \{s \in S : [P_i^\circ(s) \cap W_i^\circ(s)] \subseteq E \subseteq W_i^\circ(s)\}; &(4.4)\\ K_i^-(E) &= \{s \in S : [P_i^\circ(s) \cap W_i^\circ(s)] \subseteq [W_i^\circ(s) \setminus E], E \subseteq W_i^\circ(s)\}. &(4.5) \end{aligned}$$

To keep in line with standard notation and for simplicity, let $K_iK_j^\dagger(E) = K_i^+K_j^+(E)$, $K_i\neg K_j^\dagger(E) = K_i^-K_j^\dagger(E)$ and so on for all higher-order knowledge. For all $E \in \mathcal{E}$, I define

$$K_iK_j^\dagger(E) \quad = \quad \left\{ s \in S : s \in [\tilde{K}^i]^+_{i_s}[\tilde{K}^j]^\dagger_{i_s}(E) \right\}, \tag{4.6}$$

where $[\tilde{K}^j]^\dagger_{i_s}(E)$ is obtained by applying definitions (4.4) and (4.5) to the subjective interpolation model $(W_i^\circ(s), W_j^\circ(\cdot|i_s), P_j^\circ(\cdot|i_s))$. The interactive knowledge $K_iU_j^\dagger$ and higher-order knowledge are defined analogously.

Consider the following properties on the interpolated knowledge hierarchies.

1. $s \in K_i^+(E) \Leftrightarrow s \in K_i^-(W_i^\circ(s) \setminus E)$: One knows E if and only if one knows its subjective negation is not true;
2. $K_i^-(E) \subseteq S \setminus E$: Negative knowledge of E is always true;
3. if $s \in K_i^+(E)$, then $s \notin E \Leftrightarrow s \notin W_i^\circ(s)$: Suppose one knows E. Then such knowledge is false if and only if one is partially unaware of the current state.

Proposition 7 *In the interpolation model $(S, \mathbf{W}^\circ, \mathbf{P}^\circ)$, suppose (W_i°, P_i°) is rational and satisfies interpolated partial unawareness. Then i's interpolated knowledge hierarchy satisfies the above properties.*

Property 1 makes sure the positive and negative knowledge are defined appropriately. In particular, it says positive knowledge of E is indeed equivalent to negative knowledge of its subjective negation in the subjective, just as in the standard model. Properties 2-3 say the interpolated knowledge hierarchy satisfies a weakening of the truth axiom, which states whenever the player knows E, E must indeed be true.[10] The truth axiom is shown to be equivalent to the requirement that players never exclude the true state (Bacharach 1985), which becomes problematic when the player is partially unaware of the true state, and hence necessarily "excludes" it. Property 2 says the truth of one's negative knowledge is not affected by partial unawareness, while property 3 says false positive knowledge occurs precisely when the player is unaware of the true state, and in which case all his positive knowledge turns out to be false.

The interpolated knowledge hierarchies describe what the players believe they know. This interpretation is best illustrated by revisiting the previous example:

$S = \{a1, a2, A\}$;
$W^\circ(a1) = W^\circ(A) = \{a1, a2\}$, $W^\circ(a2) = S$;
P° induces the information partition $\{\{a1, A\}, \{a2\}\}$.

$K^+(\{a1, A\}) = \emptyset, K^+(\{a1\}) = K^-(\{a2\}) = \{a1, A\}, K^+(\{a1, A, a2\}) = \{a2\};$

[10] The mathematical formula for the truth axiom is $K(E) \subseteq E$.

$$U^+(\{a1, A, a2\}) = \{a1, A\}, U^+(\{a1\}) = \emptyset.$$

In contrast to the partition model, where the modeler observes Bob does not really know the objective event "Alice plays $a1$" because he is unaware of its implication "Alice does not play A," the interpolation model characterizes the knowledge hierarchy from Bob's perspective: at A, Bob knows both "Alice plays $a1$" and "Alice does not play $a2$," while he is unaware of events such as "Alice plays either $a1$ or A" and "Alice plays $a1$, A, or $a2$."[11]

5 Concluding Remarks

Li (2008a, 2008b) and Heifetz et al. (2006) have provided set-theoretic models of unawareness, but both rely on structural assumptions on the state space. In this paper, I construct a model of unawareness in arbitrary state spaces and show it is essentially equivalent to the product model proposed by Li (2008a, 2008b). The equivalence result also sheds light on understanding the effects of unawareness in terms of one's knowledge hierarchy by connecting the characterization of unawareness in the two models. Finally, I explore a special case of unawareness, namely, partial unawareness, where players may make erroneous inferences due to their unawareness, and I show how a variant of the partition model can successfully capture the subtle inference problem in this environment.

6 Appendix

6.1 Proof for Lemma 1.

Proof. To see Φ is one-to-one: take any $\mathcal{F}_1, \mathcal{F}_2 \in 2^{\mathcal{F}}, \mathcal{F}_1 \neq \mathcal{F}_2$. Suppose $\Phi(\mathcal{F}_1) = \Phi(\mathcal{F}_2)$.

Without loss of generality, assume $\mathcal{F}_1 \setminus \mathcal{F}_2 \neq \emptyset$. It follows $[(\mathcal{F} \setminus \mathcal{F}_1) \cup \mathcal{F}_2]$ is a proper subset of $\mathcal{F}$. Now,

$$\begin{aligned} S^* &= \bigvee_{\pi \in \mathcal{F}} \pi \\ &= \left[\bigvee_{\pi \in \mathcal{F}_1} \pi \right] \bigvee_{\pi \in \mathcal{F} \setminus \mathcal{F}_1} \pi \\ &= \left[\bigvee_{\pi \in \mathcal{F}_2} \pi \right] \bigvee_{\pi \in \mathcal{F} \setminus \mathcal{F}_1} \pi \\ &= \bigvee_{\pi \in [(\mathcal{F} \setminus \mathcal{F}_1) \cup \mathcal{F}_2]} \pi. \end{aligned}$$

[11] It is instructive to relate the interpolation model to the product model. Let $\Omega^* = \{0,1\}^3$, and let $\Gamma(a1) = (1,0,0), \Gamma(a2) = (0,1,0)$ and $\Gamma(A) = (0,0,1)$. At A, Bob's subjective factual information is $\{(1,0),(0,0)\}$. But then presumably $(0,0)$ is a contradictory state in the subjective state space $\{0,1\}^2$ and hence is crossed out, leading to the interpolation $\{(1,0)\}$. In some sense, the partial unawareness problem is precisely reflected in the failure of recognizing that $(0,0)$ is in fact a legitimate state.

which contradicts $\mathcal{F}$ being a frame.

To see $g^{-1}(\pi) = \{\nu \in \mathcal{F} : \pi \trianglerighteq \nu\}$: it obviously holds for $\pi = S$; for $\pi \neq \{S\}$, notice the definition of g yields $\mathcal{G} \subseteq \{\nu \in \mathcal{F} : \pi \trianglerighteq \nu\}$, the other direction of set inclusion follows from the observation that if $\pi \trianglerighteq \nu$, then $\pi \vee \nu = \pi$. That $\Phi^{-1}(\pi)$ is a frame for π is obvious. □

6.2 Proof for Lemma 2.

Proof. Let $E \in \{\pi(t') : t' \in P(t)\}$. That is, $E = \pi(t')$ for some $t' \in P(t)$.

Since P induces an information partition, we have $t \in P(t')$.

On the other hand, $t' \in \pi(t) \Rightarrow t \in \pi(t')$, thus, $t \in P(r) \cap \pi(t')$. By nice factual partition, $P(t') \cap \pi(r) \neq \emptyset$. Let $r' \in P(t') \cap \pi(r)$. Then $r' \in P(t')$ means $\pi(r')$ is an element in the set at the right-hand side, but $r' \in \pi(r)$ means $\pi(r') = \pi(r) = E$. This proves $E \in \{\pi(r) : r \in P(t')\}$. The other direction is entirely symmetric, and hence the result. □

6.3 Proof for Lemma 3.

Proof. We need to prove $(W_j(\cdot|i_s), P_j(\cdot|i_s))$ is rational and $P_j(\cdot|i_s)$ satisfies nice factual partition with respect to $(W_i(s), \Phi^{-1}(W_i(s)))$.

1. First I show that for all $E, F \in W_i(s)$, $F \in P_j(E|i_s) \Rightarrow P_j(F|i_s) = P_j(E|i_s)$.

   Pick $t_0 \in E$. By Lemma 2,

   $$P_j(E|i_s) = \{W_i(s)(t) : t \in P_j(t_0)\}.$$

   Thus there exists $t_1 \in P_j(t_0)$ such that $F = W_i(s)(t_1)$. Since (W_j, P_j) is rational, $P_j(t_1) = P_j(t_0)$.

   Since $t_1 \in F$, using Lemma 2 again, we have:

   $$\begin{aligned} P_j(F|i_s) &= \{W_i(s)(t) : t \in P_j(t_1)\} \\ &= \{W_i(s)(t) : t \in P_j(t_0)\} \\ &= P_j(E|i_s) \end{aligned}$$

2. Second I show that $F \in P_j(E|i_s) \Rightarrow W_j(E|i_s) = W_j(F|i_s)$.

   Pick $t_0 \in E$. As argued above, there exists $t_1 \in P_j(t_0)$ such that $F = W_i(s)(t_1)$, or $t_1 \in F$. By rational awareness, $W_j(t_0) = W_j(t_1)$, and hence

   $$\{W_j(t) : t \in E\} \subseteq \{W_j(t) : t \in F\}.$$

Since P_j induces an information partition, $t_0 \in P_j(t_0)$ and hence $E \in P_j(E|i_s)$. By the previous proof, $P_j(E|i_s) = P_j(F|i_s)$ and hence $E \in P_j(F|i_s)$. Applying the arguments in the previous paragraph again, we obtain the other direction of set inclusion, i.e.,

$$\{W_j(t) : t \in E\} \supseteq \{W_j(t) : t \in F\}.$$

Thus we have $\{W_j(t) : t \in E\} = \{W_j(t) : t \in F\}$, and hence,

$$\begin{aligned} W_j(E|i_s) &= W_i(s) \wedge \left[\vee_{t\in E} W_j(t)\right] \\ &= W_i(s) \wedge \left[\vee_{t\in F} W_j(t)\right] \\ &= W_j(F|i_s). \end{aligned}$$

3. Finally, I show $P_j(\cdot|i_s)$ satisfies nice factual partition, that is, for any $\mathcal{F}' \subseteq \Phi^{-1}(W_i(s))$, and any $E, F \in W_i(s)$, if $\left\{G \bigcap_{\pi\in\mathcal{F}'} \pi(E) : G \in W_i(s)\right\} \cap P_j(F|i_s) \neq \emptyset$, then $\left\{G \bigcap_{\pi\in\mathcal{F}'} \pi(F) : G \in W_i(s)\right\} \cap P_j(E|i_s) \neq \emptyset$.

   Let $H \in \left\{G \bigcap_{\pi\in\mathcal{F}'} \pi(E) : G \in W_i(s)\right\} \cap P_j(F|i_s)$. Then we have $H \subseteq \pi(E)$ for all $\pi \in \mathcal{F}'$, and there exist $t, t_0 \in S$ such that $t_0 \in F, t \in P_j(t_0)$ and $H = W_i(s)(t)$. Let $s \in E$. Then $H \subseteq \pi(s)$ for all $\pi \in \mathcal{F}'$. Notice $t \in H \subseteq \pi(s)$, thus $\pi(s) \cap P_j(t_0) \neq \emptyset$ for all $\pi \in \mathcal{F}'$. By nice factual signal condition, we have $\bigcap_{\pi\in \mathcal{F}'} \pi(t_0) \cap P_j(s) \neq \emptyset$. Pick $r \in \bigcap_{\pi\in \mathcal{F}'} \pi(t_0) \cap P_j(s)$. Since $\mathcal{F}' \subseteq \Phi^{-1}(W_i(s))$, it follows $W_i(s)(r) \subseteq \pi(t_0) = \pi(F)$ for all $\pi \in \mathcal{F}'$; $r \in P_j(s), s \in E$ implies $W_i(s)(r) \in P_j(E|i_s)$. It follows that $W_i(s)(r) \in \left\{G \bigcap_{\pi\in\mathcal{F}'} \pi(F) : G \in W_i(s)\right\} \cap P_j(E|i_s)$ and this proves the claim.

$\square$

6.4 Proof for Theorem 4.

6.4.1 Proof for part 1.

1. Construct the partition model.

For any $q \in Q$, let $\pi(q) = \left\{a \times \prod_{q'\in Q, q'\neq q} D_{q'} \times \{\alpha\} : a \in D_q\right\}$ denote the partition over Ω^* generated by the question q. Intuitively, all states in the set $a \times \prod_{q'\in Q, q'\neq q} D_{q'} \times \{\alpha\}$

coincide in their answer to question q, namely a.

Let $\mathcal{F} = \{\pi(q) : q \in Q\}$ denote the collection of partitions generated by all questions. It is straightforward to see $\mathcal{F}$ is a frame for Ω^*.

For any $\omega^* \in \Omega^*$, let

$$W(\omega^*) = \left\{\omega \times \prod_{q\in Q\setminus W^*(\omega^*)} D_q \times \{\alpha\} : \omega \in \prod_{q\in W^*(\omega^*)} D_q\right\}, \tag{6.1}$$

$$P(\omega^*) = P^*(\omega^*). \tag{6.2}$$

2. Structural equivalence of the information structures.

That if (W^*, P^*) is rational, then (W, P) is rational is obvious.

Suppose P^* satisfies cylinder factual partition. Let $\mathcal{F}' \subseteq \mathcal{F}$. Let $s, t \in \Omega^*$ be such that $\bigcap_{\nu\in \mathcal{F}'} \nu(s) \cap P(t) \neq \emptyset$. We need to show $\bigcap_{\nu\in \mathcal{F}'} \nu(t) \cap P(s) \neq \emptyset$.

By construction, there exists a one-to-one map $\Lambda : Q \to \mathcal{F}$. By cylinder factual partition, $P(t) = P^*(t) = \prod_{q\in Q} \pi_q(t^q)$ where π_q is a partition over D_q. Let $a \in \bigcap_{\nu\in \mathcal{F}'} \nu(s) \cap P(t)$. We have:

$$a \in P(t) \Rightarrow a^q \in \pi_q(t^q);$$
$$a \in \bigcap_{\nu\in \mathcal{F}'} \nu(s) \Rightarrow a^q = s^q \text{ for all } q \in \Lambda^{-1}(\mathcal{F}').$$

It follows $t^q \in \pi_q(a^q) = \pi_q(s^q)$ for all $q \in \Lambda^{-1}(\mathcal{F}')$. But then we have:

$$\prod_{q\in \Lambda^{-1}(\mathcal{F}')} \{t^q\} \times \prod_{q\in Q\setminus\Lambda^{-1}(\mathcal{F}')} \pi_q(s^q) \subseteq \prod_{q\in Q} \pi_q(s^q) = P^*(s) = P(s),$$

But since

$$\prod_{q\in \Lambda^{-1}(\mathcal{F}')} \{t^q\} \times \prod_{q\in Q\setminus\Lambda^{-1}(\mathcal{F}')} \pi_q(s^q) \subseteq \bigcap_{\nu\in \mathcal{F}'} \nu(t),$$

we conclude $\bigcap_{\nu\in \mathcal{F}'} \nu(t) \cap P(s) \neq \emptyset$.

3. The equivalence of own knowledge hierarchies.

1. $U(E) = \bigcap_{F\in \Psi(E)} U^p(F)$.

   Let $\omega^* \in U(E)$, and $F \in \Psi(E)$, need to show $\omega^* \in U^p(F)$.

$$\begin{aligned} \omega^* \in U(E) &\Rightarrow E \notin \mathcal{A}(W(\omega^*)), \\ &\Rightarrow \text{ there exists some } G \in W(\omega^*), \omega_1^*, \omega_2^* \in G \text{ such that} \\ &\quad\ \omega_1^* \in E, \omega_2^* \notin E. \end{aligned}$$

But by (6.1), $G = \omega \times \prod_{q\in Q\setminus W^*(\omega^*)} D_q$ for some $\omega \in \prod_{q\in W^*(\omega^*)} D_q$, thus ω_1^*, ω_2^* must differ on the set D_q for some $q \in Q \setminus W^*(\omega^*)$. Now,

$$\begin{aligned} F \in \Psi(E) &\Rightarrow F_{\Omega^*} = F \times \prod_{q\in Q\setminus q(F)} D_q = E, \\ &\Rightarrow j \in q(F), \\ &\Rightarrow q(F) \nsubseteq W^*(\omega^*), \\ &\Rightarrow \omega^* \in U^p(F). \end{aligned}$$

For the other direction, let $\omega^* \in \bigcap_{F\in \Psi(E)} U^p(F)$, need to show $\omega^* \in U(E)$.

Suppose not. Then $E \in \mathcal{A}(W(\omega^*))$, that is, $E = \cup_{k\in K} E_k$ where K is an arbitrary index set and $E_k \in W(\omega^*)$ for all k. By construction, for each k, we can write $E_k = \omega_k \times \prod_{q\in Q\setminus W^*(\omega^*)} D_q$, where $\omega_k \in \prod_{q\in W^*(\omega^*)} D_q$. But then the event $G = \{\omega_k : k \in K\} \in \Psi(E)$ and $q(G) \subseteq W^*(\omega^*)$, which implies $\omega^* \notin U^p(G)$, contradiction.

2. $K(E) = \bigcup_{F\in \Psi(E)} K^p(F)$.

   By construction, for any $E \in \mathcal{A}(W(\omega^*))$, $P^*(\omega^*) \subseteq \prod_{q\in Q\setminus q(E)} D_q$ if and only if $P(\omega^*) \subseteq E$. Now observe $U(E) = \bigcap_{F\in \Psi(E)} U^p(E)$ implies $E \in \mathcal{A}(W(\omega^*)) \Leftrightarrow \omega^* \notin U^p(F)$ for some $F \in \Psi(E)$. The result follows.

4. Equivalence of higher-order interactive knowledge hierarchies.

Fix $\theta = (\omega^*, i)$. Consider the subjective model $\mathcal{M}^p(\theta) = (\Omega(\theta), \mathbf{P}(\cdot|\theta), \mathbf{W}(\cdot|\theta))$ as defined in (3.1) - (3.3). Alternatively, let $\delta = (i_{\omega^*})$ and consider the subjective model $\mathcal{M}(\delta) \equiv (W(\delta), \Phi^{-1}(W(\delta)), \mathbf{W}(\cdot|\delta), \mathbf{P}(\cdot|\delta))$ as defined by formulae (2.3) and (2.4).

Let $\beta : \Omega(\theta) \to 2^{\Omega^*}$ be defined by: $\beta(\omega) = \omega \times \prod_{q\in Q\setminus q(\{\omega\})} D_q \times \{\alpha\}$. The map β is one-to-one and onto on the set $W(\delta)$. It is easy to see that:

1. $q \in W(\theta) \Leftrightarrow \pi(q) \in \Phi^{-1}(W(\delta))$ and hence $q \in W_j(\omega|\theta) \Leftrightarrow \pi(q) \in \Phi^{-1}(W_j(\beta(\omega)|\delta))$ for all j and all $\omega \in \Omega(\theta)$;

2. $\omega' \in P_j(\omega|\theta) \Leftrightarrow \beta(\omega') \in P_j(\beta(\omega)|\delta)$ for all j and all $\omega \in \Omega(\theta)$.

Thus, $\mathcal{M}^p(\theta)$ is isomorphic to $\mathcal{M}(\delta)$. By the previous result, we have:

$$f_{W(\delta)}([\tilde{U}^j]_\delta(E)) = \beta(\bigcap_{F\in \Psi(E)} [\tilde{U}^j]^p_\theta(F)). \tag{6.3}$$

Now,

$$\omega^* \in K_i U_j(E) \Leftrightarrow W_i(\omega^*)(\omega^*) \subseteq [\tilde{K}^i]_\delta[\tilde{U}^j]_\delta(E)$$

if and only if

$$P_i(W_i(\omega^*)(\omega^*)|i_{\omega^*}) \subseteq f_{W(\delta)}([\tilde{U}^j]_\delta(E)).$$

Using equation (6.3), we have:

$$P_i(W_i(\omega^*)(\omega^*)|i_{\omega^*}) \subseteq \beta([\tilde{U}^j]^p_\theta(F))$$

for all $F \in \Psi(E)$. Notice $W_i(\omega^*)(\omega^*) = \beta(\mathbb{P}^{\Omega(\theta)}(\omega^*))$; thus, the above is equivalent to:

$$P_i(\mathbb{P}^{\Omega(\theta)}(\omega^*)|\theta) \subseteq [\tilde{U}^j]^p_\theta(F)$$

which, by definition, is true if and only if $\mathbb{P}^{\Omega(\theta)}(\omega^*) \in [\tilde{K}^i]^p_\theta[\tilde{U}^j]^p(F)$, if and only if $\omega^* \in [K_iU_j]^p(E)$.

The proof for the case of K_iK_j is entirely analogous. The equivalence of higher-order knowledge follows from the recursive structure of the models. □

6.4.2 Proof for part 2.

1. Construct the product state space.

Given $(S^*, \mathcal{F})$, consider the product set:

$$\Omega^* = \{\alpha\} \times \prod_{\pi \in \mathcal{F}} \pi. \tag{6.4}$$

A product state $\omega^* \in \Omega^*$ is a $\#(\mathcal{F}) + 1$-tuple whose coordinates are subsets of S plus α. For example, let $S = \{a, b, c, d\}$ and the frame be given by:

$$\mathcal{F} = \{\{\{a,b\},\{c,d\}\},\{\{a\},\{b,c,d\}\},\{\{d\},\{a,b,c\}\}\},$$

then,

$$\begin{aligned}\Omega^* &= \{\alpha\} \times \{\{a,b\},\{c,d\}\} \times \{\{a\},\{b,c,d\}\} \times \{\{d\},\{a,b,c\}\} \\ &= \{(\alpha,\{a,b\},\{a\},\{d\}),(\alpha,\{a,b\},\{b,c,d\},\{d\}),\cdots\}.\end{aligned}$$

The set $\mathcal{F}$ plays the role of $\{D_q\}_{q\in Q}$ in the product model.

Let the map $\Gamma : S \to \Omega^*$ be defined by

$$\Gamma(s) = \prod_{\pi \in \mathcal{F}} \pi(s).$$

Obviously, Γ is an injection.

2. Construct the information structure.

The idea is, on $\Gamma(S)$, one can simply translate (W, P) into (W^*, P^*) using Γ, i.e., for every $\omega^* \in \Gamma(S)$, define

$$W^*(\omega^*) = \{\alpha\} \cup \Phi^{-1}(W(\Gamma^{-1}(\omega^*))). \tag{6.5}$$

$$P^*(\omega^*) = \left\{\Gamma(t) : t \in P(\Gamma^{-1}(\omega^*))\right\}. \tag{6.6}$$

The complication is that the product structure imposes extra auxiliary states. Thus, we need to make sure that the information structures defined for these states do not interfere with knowledge hierarchies in states in $\Gamma(S)$. This requires some care.

2.1. Suppose (W, P) is rational. Then for $\omega^* \notin \Gamma(S)$, let

$$W^*(\omega^*) = \{\alpha\}; \tag{6.7}$$
$$P^*(\omega^*) = \{\omega^*\}. \tag{6.8}$$

Obviously (W^*, P^*) is rational in the product model.

2.2. Let P satisfy nice factual partition. For any $\pi \in \mathcal{F}$, consider the set:

$$\mathcal{P}(\pi) = \{\{\pi(t) : t \in P(s)\} : s \in S\}.$$

By Lemma 2, $\mathcal{P}(\pi)$ is a partition over π. For every $\omega^* \in \Omega^*$, let $\mathcal{P}(\pi)(\omega^*)$ denote the partition element in $\mathcal{P}(\pi)$ that contains the π-th coordinate of ω^*. Define:

$$P^*(\omega^*) = \{\alpha\} \times \prod_{\pi \in \mathcal{F}} \mathcal{P}(\pi)(\omega^*) \tag{6.9}$$

To check that all subjective models have equivalent information structures: fix $\omega^* \in \Gamma(\omega^*)$. Let $\omega \in \prod_{\pi \in W^*(\omega^*)} \pi$. Let $\Xi(\omega) = [\omega \times \prod_{\pi \in \mathcal{F} \setminus W^*(\omega^*)} \pi] \cap \Gamma(S)$. This is the set of states translated from S into the product space that has the projection ω on the subjective state space $W^*(\omega^*)$. Apparently, either $\Xi(\omega) = \emptyset$ or $\Gamma^{-1}(\Xi(\omega)) \in W(\Gamma^{-1}(\omega^*))$. It is straightforward to verify the following:

$$W^*(\omega|\omega^*) = \begin{cases} \{\alpha\} \cup \Phi^{-1}(W(\Gamma^{-1}(\Xi(\omega))|\Gamma^{-1}(\omega^*))) & \text{if } \Xi(\omega) \neq \emptyset, \\ \{\alpha\} & \text{otherwise;} \end{cases} \tag{6.10}$$

$$P^*(\omega|\omega^*) = \begin{cases} \{\Gamma(G) : G \in P(\Gamma^{-1}(\Xi(\omega))|\Gamma^{-1}(\omega^*))\} & \text{if } \Xi(\omega) \neq \emptyset, \\ \{\omega\} & \text{otherwise;} \end{cases} \tag{6.11}$$

where slightly abusing notation, I let $\Gamma(G)$ denote the product subjective state corresponding to the subjective state G in the partition model, i.e., $\Gamma(G) = \prod_{\pi \in W^*(\omega^*)} \pi(t), t \in G$.

3. Equivalence of knowledge hierarchies.

It suffices to show the equivalence of own knowledge hierarchies. The equivalence of higher-order knowledge follows from the recursive structure of the subjective models and the fact that the subjective models are isomorphic, as shown above.

The set of events in the product model, adapting definition (3.8), is given by:

$$\mathcal{E}^p = \bigcup_{\mathcal{F}' \subseteq \mathcal{F}} \left[\left\{ E \subseteq \{\alpha\} \times \prod_{\pi \in \mathcal{F}'} \pi : E \neq \emptyset \right\} \cup \{\emptyset_{\mathcal{F}'}\} \right], \tag{6.12}$$

where $\emptyset_{\mathcal{F}'}$ is interpreted analogously as $\emptyset_{Q'}$.

Fix $E \in \mathcal{E}^p$ and let $\mathcal{F}_E$ denote the unique subset of $\mathcal{F}$ satisfying $E \subseteq \prod_{\pi \in \mathcal{F}_E} \pi$. Let U^p, K^p be defined as in (3.9) and (3.3) (with obvious adaptation to the new notations, for example, $\mathcal{F}_E$ instead of $q(E)$, π instead of D_q etc.).

1. $U^p(E) \cap \Gamma(S) = \Gamma\left(\bigcup_{\omega \in \prod_{\pi \in \mathcal{F}_E} \pi} U(\Gamma^{-1}(\omega \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi) \cap S) \right)$.

   Let $\omega^* \in U^p(E) \cap \Gamma(S)$. Let $s_0 = \Gamma^{-1}(\omega^*)$. By definition of U^p and W^*, we have $\mathcal{F}_E \nsubseteq \{\alpha\} \cup \Phi^{-1}(W(s_0))$. Thus, there exist $\nu \in \mathcal{F}_E$ and $t, t' \in S$ such that $t \notin \nu(t')$ but $t \in W(s_0)(t')$. Consider the product subjective state $\omega_0 = \prod_{\pi \in \mathcal{F}_E} \pi(t)$. We have:

$$\begin{aligned} \Gamma(t) &\in \omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi, \\ \Gamma(t') &\notin \omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi, \end{aligned}$$

   and hence $\Gamma^{-1}(\omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi) \cap S \notin \mathcal{A}(W(s_0))$, i.e.

$$s_0 \in \bigcup_{\omega \in \prod_{\pi \in \mathcal{F}_E} \pi} U(\Gamma^{-1}(\omega \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi) \cap S),$$

   and hence the "$\subseteq$."

   For the other direction, let $s_0 \in U(\Gamma^{-1}(\omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi) \cap S)$ where $\omega_0 \in \prod_{\pi \in \mathcal{F}_E} \pi$. Then there must exist $t, t' \in S$ such that:

$$\begin{aligned} t &\in W(s_0)(t'), \\ t &\in \Gamma^{-1}(\omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi); \\ t' &\notin \Gamma^{-1}(\omega_0 \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi). \end{aligned}$$

   It follows there must exist $\nu \in \mathcal{F}_E \setminus \Phi^{-1}(W(s_0))$ such that $t \notin \nu(t')$, but this implies $\Gamma(s_0) \in U^p(E)$.

2. $K^p(E) \cap \Gamma(S) = \Gamma\left(K(\Gamma^{-1}(E \times \prod_{\pi \in \mathcal{F} \setminus \mathcal{F}_E} \pi) \cap S) \right) \cap \neg U^p(E)$.

By the definition of K^p and U^p, we have:

$$K^p(E) = \left\{\omega^* \in \Omega^* : P^*(\omega^*) \subseteq E \times \prod_{\pi \in \mathcal{F}\setminus\mathcal{F}_E} \pi\right\} \cap \neg U^p(E).$$

The rest follows from the observation that for all $s \in S$, $P^*(\Gamma(s)) \cap \Gamma(S) = \Gamma(P(s))$.

□

6.5 Proof for Theorem 5.

First, I prove a useful lemma. Let $\triangle$ denote set difference operation.

Lemma 8 *Fix $(S^*, \mathcal{F})$. Let W satisfy nice awareness. Then for any $\pi \in \bar{\mathcal{F}}$, $s \in \pi(s')$ implies*

$$\Phi^{-1}(\pi) \cap [\Phi^{-1}(W(s)) \triangle \Phi^{-1}(W(s'))] = \emptyset.$$

Proof. Suppose not. Let $\nu \in \Phi^{-1}(\pi) \cap [\Phi^{-1}(W(s)) \triangle \Phi^{-1}(W(s'))]$. Without loss of generality, suppose $\nu \notin \Phi^{-1}(W(s'))$.

Since $\nu \in \mathcal{F}$, by Lemma 1, there exist t, t' such that $t \notin \nu(t')$ and $t \in \gamma(t')$ for all $\gamma \in \mathcal{F}, \gamma \neq \nu$. It follows that $t \notin \pi(t'), t \notin W(s)(t')$ but $t \in W(s')(t')$, contradicting nice awareness. □

Proof for the theorem. Define the product set Ω^* as in equation (6.4), and define the factual information P^* on the product set as in formulae (6.9) and (5.8). Define the awareness information W^* on the product set as follows: for any $\omega^* \in \Omega^*$,

$$W^*(\omega^*) = \begin{cases} \{\alpha\} \cup \Phi^{-1}(W(\Gamma^{-1}(\omega_0^*))) & \text{if } \exists\, \omega_0^* \in \Gamma(S) \text{ s.t. } \omega^* \in P^*(\omega_0^*), \\ \{\alpha\} & \text{otherwise.} \end{cases} \tag{6.13}$$

It is easy to see the definition extends (6.5). By construction, (W^*, P^*) is rational. By Lemma 8, (W^*, P^*) is strongly rational in the product model. The result follows from Theorem 9 in Li (**?**) and Theorem 4.

□

6.6 Proof for Proposition 6.

Proof. First I prove "$\subseteq$." By Lemmas 3 and 8, all subjective models have rational information structures. For any $s \in S$, applying the equivalence result and Theorem 1 in Li (**?**) to the subjective model $(W_i(s), \Phi^{-1}(W_i(s)), W_j(\cdot|i_s), P_j(\cdot|i_s))$, we have $K_j(E|i_s) \subseteq \neg U_j(E|i_s)$, and hence $K_iK_j(E) \subseteq K_i\neg U_j(E)$.

Now let $s \in K_iK_j(E)$, we need to show $s \in \hat{K}_i\hat{K}_j(E)$, i.e., $P_i(s) \subseteq \{t \in S : P_j(t) \subseteq E\}$. Let $t \in P_i(s)$. Since $s \in K_iK_j(E)$, we have $W_i(s)(t) \in K_j(E|i_s)$, i.e., $P_j(W_i(s)(t)|i_s) \subseteq$

$f_{W_i(s)}(E)$. By Lemma 2, $P_j(W_i(s)(t)|i_s) = \{W_i(s)(s') : s' \in P_j(t)\}$, which implies $P_j(t) \subseteq E$ and hence $K_i K_j(E) \subseteq \hat{K}_i \hat{K}_j(E)$.

Next, I prove "$\supseteq$." Suppose not. Then there exists $s \in \hat{K}_i \hat{K}_j(E) \cap K_i \neg U_j(E)$ but $s \notin K_i K_j(E)$.

It is immediate that since $s \in K_i \neg U_j(E)$, i is aware of E, i.e., $E \in \mathcal{A}(W_i(s))$. Thus, there must exist $G \in P_i(W_i(s)(s)|i_s)$ such that $G \notin K_j(E|i_s)$. But again since $s \in K_i \neg U_j(E)$, $E \in \mathcal{A}(W_j(G|i_s)$. Thus, we must have $P_j(G|i_s) \nsubseteq f_{W_i(s)}(E)$. Since $G \in P_i(W_i(s)(s)|i_s)$, we have $G = W_i(s)(t_0)$ for some $t_0 \in P_i(s)$. By Lemma 2, $P_j(G|i_s) = \{W_i(s)(t) : t \in P_j(t_0)\}$. But then there must exist $t_1 \in P_j(t_0)$ such that $W_i(s)(t_1) \notin f_{W_i(s)}(E)$, that is, $W_i(s)(t_1) \cap E = \emptyset$. It follows $t_1 \notin E$, and hence $P_j(t_0) \nsubseteq E$, contradicting $s \in \hat{K}_i \hat{K}_j(E)$. $\square$

6.7 Proof for Proposition 7.

Proof. Fix $E \in \mathcal{E}$.

Property 1 easily follows from the definition.

To see $K_i^-(E) \subseteq S \setminus E$: observe $P_i^\circ(s) \cap (S \setminus W_i^\circ(s)) \subseteq S \setminus W_i^\circ(s)$, and hence for any $s \in K_i^-(E)$, we have:

$$\begin{aligned} & [P_i^\circ(s) \cap W_i^\circ(s)] \subseteq [W_i^\circ(s) \setminus E] \\ \Rightarrow\ & [P_i^\circ(s) \cap W_i^\circ(s)] \cup [P_i^\circ(s) \cap (S \setminus W_i^\circ(s))] \subseteq [W_i^\circ(s) \setminus E] \cup [S \setminus W_i^\circ(s)] \\ \Rightarrow\ & P_i^\circ(s) \subseteq S \setminus E. \end{aligned}$$

The conclusion follows from $s \in P_i^\circ(s)$.

To see $s \in K_i^\circ(E) \setminus E \Rightarrow s \notin W_i^\circ(s)$: since $s \in K_i^\circ(E)$, we have $[P_i^\circ(s) \cap W_i^\circ(s)] \subseteq E$, but since $s \notin E$, it follows $s \notin P_i^\circ(s) \cap W_i^\circ(s)$. But then since $s \in P_i^\circ(s)$, we must have $s \notin W^\circ(s)$;

To see $s \in K_i^\circ(E), s \notin W_i^\circ(s) \Rightarrow s \notin E$: observe that $s \in K_i^\circ(E)$ implies $E \subseteq W_i^\circ(s)$, but then since $s \notin W_i^\circ(s)$, it follows $s \notin E$. $\square$

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