Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://www.econ.upenn.edu/pier

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"Sorting and Decentralized Price Competition"

by

Jan Eeckhout and Philipp Kircher

Sorting and Decentralized Price Competition*

Jan Eeckhout† and Philipp Kircher‡

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Abstract

We investigate under which conditions price competition in a market with matching frictions leads to sorting of buyers and sellers. Positive assortative matching obtains only if there is a high enough degree of complementarity between buyer and seller types. The relevant condition is *root-supermodularity*; i.e., the square root of the match value function is supermodular. It is a necessary and sufficient condition for positive assortative matching under any distribution of buyer and seller types, and does not depend on the details of the underlying matching function that describes the search process. The condition is weaker than log-supermodularity, a condition required for positive assortative matching in markets with random search. This highlights the role competition plays in matching heterogeneous agents. Negative assortative matching obtains whenever the match value function is weakly submodular.

Keywords. Competitive Search Equilibrium. Directed Search. Two-Sided Matching. Decentralized Price Competition. Root-Supermodularity.

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†Department of Economics, University of Pennsylvania, `eeckhout@ssc.upenn.edu`.

‡Department of Economics, University of Pennsylvania, `kircher@econ.upenn.edu`.

1 Introduction

In many economic environments, heterogeneous agents on two sides of the market match to generate gains from trade. In the labor market, firms with diverse technologies hire workers who differ in ability and qualifications; buyers of durables have different preferences over differentiated goods; and in the marriage market men and women differ in appearance and income. Individual and aggregate outcomes depend on who trades with whom. A central question in the literature therefore is to understand the pattern of trade, in particular the conditions under which attractive types on one side of the market trade with attractive types on the other side. In markets with a centralized market clearing institution, the relevant requirement for such positive assortative matching is supermodularity (Becker (1973)). Under supermodularity, or equivalently complementarity, high types have a comparative advantage when matching with other high types.

In many markets, a centralized market clearing institution is absent and price competition is decentralized. Sellers compete in prices, for example by posting wages in the labor market or by posting prices for durables on platforms like Yahoo!, Craigslist or Amazon. Under such decentralized competition – often referred to as Competitive Search or Directed Search – frictions in the trading process prevent perfect market clearing.[1] This arises for example when lack of coordination leads multiple workers to approach the same firm for a single job, which results in structural unemployment. It also arises in goods markets when too many buyers contact a common seller who gets stocked out. This induces a non-trivial probability that agents cannot trade. Lower prices attract more buyers, and thereby affect the probability of trade. For example in the housing market, higher priced houses – ceteris paribus – have a lower sales probability in any given time period. The interaction of the trading probability and price competition is therefore at the heart of our understanding of such environments.

In the presence of such frictions, we investigate under which conditions attractive buyer types match with attractive seller types. To see the role of frictions, consider the complementarity-free case of additive match values. In the frictionless world, any trading pattern is efficient and can be sustained in equilibrium (Becker (1973)). In our setup the frictions lead to an equilibrium allocation that is negatively assorted. The reason is that now there are two distinct mechanisms at work that motivate match formation: One is the intrinsic value of a match, and the other is the likelihood of obtaining one. The latter is the sole driving force of the matching pattern in the absence of complementarities. High valuation buyers are willing to pay a high price in order to minimize the probability of no trade. Some sellers provide "trading security" by attracting few buyers, who then trade with high probability. The low type sellers are those who find it optimal to provide this service for the buyers, as their opportunity cost of not trading is lowest. This results in negative assortative matching: high type buyers match with low type sellers. In the housing market for example, sellers of identical houses but with a low opportunity cost of holding on to it (because they still use their property) charge a high price and wait

[1] See amongst others Peters (1997b) and Moen (1997).

longer to sell, and buyers who have a particular urgency to move will contact exactly these high price sellers because the chance that some other buyer competes for the property is low. Even a moderate degree of supermodularity will not revert the matching pattern into positive assortative matching, as the "trading-security" motive dominates the "match value" motive. Therefore, *sufficient* supermodularity is needed for positive assortative matching to overcome this tendency for negative sorting.

Our main finding is that *root-supermodularity* – i.e. supermodularity of the square root of the match value function – is necessary and sufficient for the dominance of the "match value" motive over the "trading-security" motive. To understand the mechanics behind this requirement, observe that the square-root is a concave transformation. Standard supermodularity implies that the total value of mixed matches (where low types match with high types) is lower than the total value generated by extreme matches (where high types match with high types). In the presence of the "match value" motive only, and no frictions, this leads to positive sorting. In our model with frictions the possibility that trade does not occur is particularly harmful for high types who gain most from trade. Therefore, their concern for "trading security" makes them the first to abandon the positive assortative matching pattern if this increases their matching probability. Positive sorting is therefore only obtained under a stronger condition than in the frictionless environment of Becker (1973). To see this, observe that taking the n-th-root of the match value function concavifies it whenever $n > 1$, thus making it harder for the total value of the extreme matches to dominate that of mixed matches. In our context with decentralized competition, a concavification by taking the square root ($n = 2$) suffices for positive assortative matching, while frictionless trade requires no transformation ($n = 1$).

Compared to random search however, where positive assortative matching requires log-supermodularity ($n \to \infty$) as in Shimer and Smith (2000) and Smith (2006), in our model a lower degree of complementarity is needed. In random search models agents meet each other without any prior information about type or transfer price. In our environment with competition, agents can be selective about price and quality without necessarily having to meet with types that will never be accepted in equilibrium. This selection mitigates some of the frictions and highlights the role of competition in the matching of heterogeneous agents in our model. Our condition for positive assortative matching therefore falls between those for frictionless trade of Becker and random search. Yet, when it comes to negative assortative matching, our results differ substantially. As we argued above, in the absence of any supermodularity the allocation is negatively assorted. In fact, this is the case for any *weakly submodular* match value function. Negative assortative matching obtains only under stronger conditions both in the frictionless case (strict submodularity) and with random search (log-submodularity).

Our requirement of root-supermodularity is necessary and sufficient to obtain Positive Assortative Matching under *any* distribution of types. In particular, it will be binding whenever the ratio of buyers to sellers in some market goes to zero, which is always the case for type distribution where some sellers cannot trade. For some distributions this is not a binding restriction, and in this case there are match value functions that are less than root-supermodular and nonetheless induce positive assortative matching. More interestingly, we show that for some distributions there is negative assortative matching even when the match value function is moderately supermodular. To our knowledge, this is new in the literature.

In Section 5 we analyze three extensions. First, we explore the boundaries of our results when we relax some of the restrictions that we place on the matching function. We focus on the CES matching function which has different properties than the usual micro-founded matching functions and provide a remarkably simple characterization which may prove useful in applications. Second, we show that our condition for assortative matching continues to hold if we extend our model to a dynamic setting, independent of the discount factor.[2] Third, we modify the payoff structure so that *both* sides of the market have preferences over the types they match with, as is the case for example in the marriage market.

The driving force behind our result is the presence of trading frictions. These seem important in the housing or labor market, where waiting times and unemployment duration are a major concern. Nonetheless, it is instructive to link our environment to the large literature on the foundations of competitive equilibria by considering the limit as frictions vanish. In our case the competitive benchmark is Becker (1973), which requires standard supermodularity for positive sorting. We analyze the limit when the discount factor in our dynamic extension approaches unity, and find instead that along the entire sequence root-supermodularity continues to be a necessary and sufficient condition for positive assortative matching. Our condition remains binding locally around those seller types who cannot trade, e.g., due to unequal sizes on each side of the market, yet relaxes to supermodularity elsewhere. It remains binding because a very low probability of trading can offset the increased patience. In the limit, simple supermodularity suffices almost everywhere. Our findings are therefore consistent with the frictionless case of Becker (1973) and contribute to the understanding of the foundations for competitive (matching) markets. In a less orthodox approach to vanishing frictions, we consider a convergent sequence of matching functions in our static economy such that in the limit the short side of the market gets matched with certainty. We confirm the findings that we obtain for the limit of the dynamic economy. To our knowledge, considering vanishing frictions as the limit of a sequence of static matching functions is new in this literature on foundations for competitive equilibrium.

2 Relation to the Literature.

Our model has three key features: two-sided heterogeneity with complementarities, decentralized price competition, and a general specification of the matching frictions.

As mentioned above, our model builds on the frictionless matching model with two-sided heterogeneity due to Becker (1973) where positive assortative matching obtains under strict supermodularity.

[2] A similar observation can be made for random search models where the log-supermodularity conditions are also independent of the discount factor.

Adding random search frictions in this environment with transfers was first done by Shimer and Smith (2000). In random search assortative matching is a set-valued concept, because the random nature of the meetings makes it infeasible to wait for the perfect type. In the absence of complementarities, the opportunity cost of waiting is higher for high types and therefore strong degrees of supermodularity are needed to ensure positive assortative matching. In our model we add frictions to Becker's model but we reserve a special role for prices in the allocation of heterogeneous agents.

The literature on decentralized price competition combines the Walrasian spirit of competition with a notion of frictions that provides a rationale for unemployment in the labor market and waiting times in the product market.[3] Our model falls into the broad class of Walrasian models of contract markets. Each of the contracts, i.e., a price and quality combination, is traded in a separate market. Pricing is competitive in the sense that sellers can affect the amount of trade by changing the offered price. Such Walrasian models of contract markets (Peters (1997b), Mortensen and Wright (2002)) assume that the expectations of traders about out of equilibrium market positions, i.e., in inactive markets, adjust consistent with rational expectations. This assumption basically entails that all traders believe that the number of traders in the inactive markets is large, and that they have common beliefs of what happens in those markets.[4] It is often strengthened by assuming that trading probabilities in all markets, including the inactive ones, lie on the same indifference curve for workers (Moen 1997). We use the broader definition and show as a result that with worker heterogeneity it is the indifference of the worker type who is most eager to trade in each market that is important.

Most existing models of decentralized price competition deal with homogeneous buyers. Exceptions with two-sided heterogeneity include Mortensen and Wright (2002) who analyze a private valuation environment without complementarities under a general specification of the matching frictions. In an economy with free entry of firms and urn-ball matching, Shi (2001) analyzes the efficiency of positive assortative matching of the equilibrium allocation. In his environment firms are ex-ante identical and can choose to enter with different types that have different costs, and he shows that positive assortative matching necessarily requires strong enough complementarities. Shi (2002) and Shimer (2005) consider perfect observability of buyer characteristics and a type-contingent menu of prices and trading priorities.[5] Under urn-ball matching this ex-post selection between different types leads to

[3] Without attempting to be exhaustive, examples include Peters (1991), Acemoglu and Shimer (1999a and b), Shi (2001), Mortensen and Wright (2002), Rocheteau and Wright (2005), Galenianos and Kircher (2006), Kircher (2007), Delacroix and Shi (2006).

[4] Models providing microfoundations often interpret each seller as a market, see e.g. Peters (1997b and 2000) and and Burdett, Shi, and Wright (2001). The strategic foundations for those out-of-equilibrium beliefs are obtained by modeling deviations in the form of a (sub)game with a finite number of homogeneous players. They show that equilibrium allocations in the decentralized Walrasian models of contract markets coincide with the allocations in the limit of finite subgame perfect equilibria when the market size increases. Inactive markets in the limit game can therefore be interpreted as deviations by sellers in large finite games who correctly anticipate the effects of those deviations on buyers' behavior.

[5] Peters (2007) considers observable heterogeneity in the absence of contractibility on the part of the firms.

imperfect sorting in equilibrium.[6]

In our base-line model buyer types are not observed, but nevertheless the price of each seller constitutes a complete contract in the sense that seller valuations are not affected by unobservable buyer types. While our set up is one of price competition in the Walrasian spirit, our model is nonetheless related to the ones of McAfee (1993) and Peters (1997a) who analyze competition in auctions. Without complementarities this seemingly minor difference of using competitive pricing rather than auctions leads to very different equilibrium outcomes. Competitive pricing leads to perfect sorting of buyers and sellers because the prices allow competing sellers to screen different valuation buyers *ex ante.* High valuation buyers are willing to pay more in order to avoid a long queue of potential competing buyers. Thus, buyers reveal their type by "voting with their feet". Auctions lead to random meetings and ex-post screening.[7] The role of auctions under complementarities are yet unexplored.

Finally, we derive our results in the context of a general specification of the matching frictions. While this is common in the random search and labor literature, it is not always used in the context of directed search where the matching function is typically the urn-ball process.[8] We find that our main result does not depend on the specifics of the matching function. This is surprising because the condition for positive assortative matching is completely determined by an expression that depends on the matching function, yet that expression has a common maximum for all functions in the general class we consider. In addition, the general matching function allows us to consider the competitive limit as matching frictions vanish in a static environment.

3 The Model

Players: The economy consists of buyers and sellers. Each seller has one good for sale. Sellers are heterogeneous and indexed by a type $y \in \mathcal{Y}$ that is observable. Let $S(y)$ denote the measure of sellers with types weakly below $y \in \mathcal{Y}$. We assume $\mathcal{Y} = [\underline{y}, \overline{y}] \subset \mathbb{R}_+$, and $S(\overline{y})$ denotes the overall measure of sellers. On the other side of the market there is a unit mass of buyers. Buyers differ in their valuation for the good, which is their private information. Each buyer draws his type x i.i.d. from distribution $B(x)$ on $\mathcal{X} = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$. S and B are C^2 and with strictly positive derivatives s and b, respectively.

Preferences. The value of a good consumed by buyer x and bought from seller y is given by $f(x, y)$,

[6] The imperfect sorting depends on the assumption of the matching technology. It arises under urn-ball matching, but is unlikely to arise for example under bilateral matching. (See also Footnote 7.)

[7] This result of the competing mechanism design literature depends on the exact nature of the matching frictions. Both McAfee (1993) and Peters (1997a) consider (a strategic version of) urn-ball meeting frictions. In Eeckhout and Kircher (2008) we consider a related model with general frictions in which sellers have a choice about the mechanism. In such a setting, price posting constitutes an optimal (equilibrium) sales mechanism under some specifications of the meeting frictions, e.g. when meetings are bilateral.

[8] For discrete time directed search models an exception is the partially directed search model of Menzio (2007). In continuous time competitive search models general matching functions are more common.

where f is a strictly positive function $f : \mathbb{R}^2_+ \to \mathbb{R}_{++}$. Conditional on consuming and paying a price p, the utility of the buyer is $f(x, y) - p$ and that of the seller is p.[9] That is, agents have quasi-linear utilities. We consider indices x and y that are ordered such that they increase the utility of the buyer: $f_x > 0$, $f_y > 0$. We assume that f is twice continuously differentiable in (x, y). The utility of an agent who does not consume is normalized to zero. Clearly, no trade takes place at prices below 0 and above $f(\overline{x}, \overline{y})$, and we define the set of feasible prices as $\mathcal{P} = \mathbb{R}_+$. All agents maximize their expected utility.

Action sets. Sellers decide at which price they want to trade. The sellers trade decisions leads to some joint distribution function $F(p, y)$ over feasible prices and seller types. Distribution F is permissible if its marginal over y coincides with $S(y)/S(\overline{y})$. Buyers decide on the price and the type of seller at which they attempt to trade. This leads to a joint distribution $G(p, x, y)$ over feasible prices and buyer and seller types. Distribution G is permissible if its marginal over x coincides with $B(x)$. Individual rationality is captured by the requirement that the support of G only includes combinations (p, x, y) with $f(x, y) - p \geq 0$.

Matching Technology. Trade is imperfect in the sense that with positive probability an agent does not meet the partner with the characteristics he is looking for and remains unable to trade. We allow for a general formulation of a constant returns to scale matching technology to allocate buyers to sellers. Let number of trades when β buyers and σ sellers interact in a given submarket (p, y) be given by the aggregate matching function $M(\beta, \sigma)$. Because of constant returns, we can write the trading probabilities in terms of the ratio of buyers to sellers at each combination of price and seller characteristic, denoted by $\lambda(p, y)$. This is often referred to as the tightness or queue length in market (p, y). We denote the probability that a seller is matched by $m(\lambda)$. It is equal to $M(\beta, \sigma)/\sigma$, and given constant returns we write $M(\beta, \sigma)/\sigma = M(\beta/\sigma, 1) = m(\lambda)$. The probability that a buyer is matched is $M(\beta, \sigma)/\beta = m(\lambda)/\lambda = q(\lambda)$, with the convention that $q(0) = \lim_{\lambda \searrow 0} q(\lambda)$.

We require the following properties on the entire domain $[0, \infty)$: (bounded functions) $m(\lambda) \leq \min\{\lambda, 1\}$; (strict derivatives) $m'(\lambda) > 0$ and $q'(\lambda) < 0$; (strict curvature) $m''(\lambda) < 0$ and $[1/q(\lambda)]'' > 0$; (bounded derivatives) first and second derivatives of m and q are bounded. These restrictions are motivated by the micro foundations that underlie the matching process, some of which are illustrated in the next paragraph. The requirement of bounded functions ensures that m and q are indeed probabilities with values between zero and one. The requirement of strict derivatives ensures that it is easier for a seller to match when the ratio of buyers to sellers is high. The opposite applies to the buyers. Strict curvature of m ensures that the second order condition of the firm's maximization is satisfied. Strict curvature of $1/q$ ensures that an increase in the sellers' matching probability m decreases the buyers' matching probability more the higher the value of m.[10] In Section 6 we discuss the importance of these assumptions further.

[9] We introduce type-dependent preferences also for the seller in Section 5.

[10] We show this and some other equivalences in Lemma 10 in the Appendix B.

Examples of Matching Technologies. One commonly used specification is the urn-ball matching technology $m_1(\lambda) = 1 - e^{-\lambda}$ (see e.g. Peters (1991), Shi (2001), Shimer (2005)). Variations of this specification arise naturally. If a fraction $1-\alpha$ of all intended trades falls through we obtain $m_2(\lambda) = \alpha(1-e^{-\lambda})$. If a fraction $1-\beta$ of the buyers gets lost on the way we have $m_3(\lambda) = 1-e^{-\beta\lambda}$. Very different matching specifications arise for logarithmic matching where $m_4(\lambda) = 1 - \ln(1 + e^{(1-\lambda)/(1-\delta)})/\ln(1 + e^{1/(1-\delta)})$, with $\delta \in (0,1)$ or under telegraph line matching $m_5(\lambda) = \gamma\lambda/(1-\lambda)$ with $\gamma \in (0,1]$.[11] Examples m_1 to m_4 fulfill all our conditions. For m_5 only a weak version of the curvature of $1/q$ holds, yet most of our analysis still applies as we describe below.

Equilibrium. We take the Walrasian approach to price setting. This approach rests on two assumptions on what agents believe about the tightness in each market. First, traders are supposed to hold well-defined beliefs for every possible market, including those that have no trading in equilibrium, and second, all agents hold the same beliefs about the queue length in each market. The former is equivalent to the complete markets assumption in centralized Walrasian markets.

For the following equilibrium definition it is convenient to define the expected utilities and profits

$$u_x(p,y) = q(\lambda(p,y))(f(x,y)-p) \tag{1}$$

$$\pi_y(p) = m(\lambda(p,y))p. \tag{2}$$

We require optimality on the support of the distributions, where the support of some distribution F is the closure of the set of points with the property that each of their neighborhoods has strictly positive measure under F. We denote the support by a "hat", i.e. $\hat{F}$ in this case. Moreover we require that the queue lengths are consistent.

Definition 1 *An equilibrium is a set of permissible distributions (F, G) such that there exists a queue length function $\lambda : \mathcal{P} \times \mathcal{Y} \to \overline{\mathbb{R}}_+$ that satisfies:*

I. (Profit Maximization) $\forall (p,y) \in \hat{F} : \pi_y(p) \geq \pi_y(p') \quad \forall p' \in \mathcal{P}$.

II. (Utility Maximization) $\forall (p,x,y) \in \hat{G} : u_x(p,y) \geq u_x(p',y') \quad \forall (p',y') \in \mathcal{P} \times \mathcal{Y}$.

III. (Consistency of the Queue Length) For all $y \in \mathcal{Y}$ and all $p \in \mathcal{P}$

$$S(\overline{y}) \int_{[0,p]\times[\underline{y},y]} \lambda(\cdot,\cdot)dF = G(p,\overline{x},y). \tag{3}$$

[11] Matching probability m_1 is based on the idea that buyers choose at random one of the sellers with the appropriate (p,y) combination, in which case $e^{-\lambda}$ reflects the probability that a seller is not visited by any buyer. Matching probability m_5 arises in models of monetary exchange along the lines of Kiyotaki and Wright (1993): agents who want to trade quality y at price p choose an island where they get paired with some randomly chosen other agent (buyer or seller) with probability γ. The probability of a buyer-seller match is $s/(s+b)$ when s is the measure of sellers and b the measure of buyers. In Section 5.1 we show that m_5 is a special case of the CES matching function.

Condition (I) states that the profit at prices at which sellers want to trade is weakly higher than any other profit they could have achieved by setting a different price. Condition (II) states that buyers attempt to trade with type y at price p only if this yields expected utility at least weakly as high as if they had attempted to trade at any other seller-price combination. Condition (III) is a consistency requirement that ensures that the queue length λ indeed reflects the ratio of buyers to sellers. The right hand side specifies the "inflow" of buyers that want to trade up to combination (p, y), while the left hand side specifies the "outflow" of buyers over the sellers that offer the combinations up to that point. This requirement is the analogue of market clearing in competitive markets without frictions.

Note that buyers can always attempt trade at $(0, \overline{y})$ and obtain weakly positive utilities, and sellers obtain weakly positive profits at all non-negative prices. To reflect that some buyers might not be able to trade we have to allow λ in the extended reals and specify $q(\infty) = \lim_{\lambda\to\infty} q(\lambda)$ and $m(\infty) = \lim_{\lambda\to\infty} m(\lambda)$, which are well defined as these functions are monotonic and bounded. To be able to interpret the Consistency Condition (III) with queue lengths in the extended reals, we adopt the convention that for any measurable set A with $\lambda(p, y) = \infty$ for all $(p, y) \in A$ and $\int_A dF(p, y) = 0$, we have $S(\overline{y}) \int_A \lambda(p, y) dF(p, y) = \int_A dG(p, \overline{x}, y)$. In particular, this allows buyers to apply to low prices even if they are not offered by sellers. It leads to a zero probability of trade rather than violating market clearing. We call buyer and seller types *active* if they obtain strictly positive utilities and profits in equilibrium, and denote by x_0 (y_0) the supremum of the inactive buyer (seller) types.[12]

Discussion of the Complete Markets Assumption. As mentioned above, our equilibrium definition entails the complete markets assumption. In our setup, each possible meeting place (p, y) constitutes a separate market and we assume that agents form beliefs $\lambda(p, y)$ about the conditions in all separate markets, including those that are not active in equilibrium. Most work on competitive search markets assumes a particular restriction on the beliefs, the Market Utility (MU) Assumption (see e.g., Montgomery (1991), Moen (1997), Shi (2001), Shimer (2005)). The MU Assumption derives from a sequential notion of market interaction in which sellers post prices and then buyers choose where to trade. For homogeneous buyers Peters (1991, 1997b, 2000) and Burdett, Shi and Wright (2000) have derived those beliefs corresponding to the MU Assumption as approximations obtained in limits of finite equilibria. In the finite economy, deviations and off the equilibrium path beliefs are well-defined and they coincide with that particular complete markets assumption in the limit.

The appropriate extension of the MU assumption to heterogeneous buyers might not be obvious, especially since now some buyers might not be able to trade. We derive an extended MU formulation in the following Lemma 1 from the principle of complete markets. This places our setup within the literature and allows us to the tractability of the MU assumption.[13] Let the Market Utility for buyer

[12]One can think of types x_0 and y_0 as the lowest active types. We define it on the complement of the active types because zero measure of low type sellers could be active without having a trading partner because Consistency (III) only applies for positive measures. When all buyers are active, then $x_0 = \underline{x}$, and if all sellers are active $y_0 = \underline{y}$.

[13]The complete markets assumption can be motivated in many ways, one of which is sellers posting prices.

x, $U(x) = q(\lambda(p,y))(f(x,y) - p)$, be the utility that buyers x get in equilibrium when they use their equilibrium path trading strategy $(p,x,y) \in \hat{G}$. Similarly, $\Pi(y) = m(\lambda(p,y))p$ is the Market Profit for seller y along the equilibrium path $(p,y) \in \hat{F}$. In Lemma 9 in the Appendix we show that $U(x)$ and $\Pi(y)$ are continuous and weakly monotonic on the relevant domain. A buyer is indifferent between trading at any other (p,y) combination if he faced a queue length λ_x such that

$$q(\lambda_x(p,y))[f(x,y) - p] = U(x) \tag{4}$$

whenever $f(x,y) - p \geq U(x)$. For the special case of $f(x,y) - p = U(x) = 0$ define $\lambda_x(p,y) = \infty$, and for $f(x,y) - p < U(x)$ define $\lambda_x(p,y) = 0$. We extend the MU assumption to heterogeneous buyers as follows:[14]

Definition 2 *A queue length function λ fulfills the Market Utility Assumption for given Market Utility $U(\cdot)$ if $\lambda(p,y) = \sup_{x\in\mathcal{X}} \lambda_x(p,y)$.*

The following Lemma shows that any equilibrium with complete markets can be sustained by beliefs that conform to our extended Market Utility Assumption. Since the MU assumption does not change the set of equilibria, we will use it throughout the paper.

Lemma 1 *Any equilibrium (F,G) supported by queue lengths $\lambda(\cdot,\cdot)$ that induce expected utilities $U(\cdot)$ is also an equilibrium under a queue lengths $\tilde{\lambda}(\cdot,\cdot)$ that conform to the MU assumption given $U(\cdot)$.*

Proof. See Appendix A. ■

Assortative Matching. Our main focus is on the sorting of buyers and sellers. We define a matching $\mu : \mathcal{X} \rightrightarrows \mathcal{Y}$ as a correspondence that specifies for each buyer type x the set of all seller types with which this buyer type trades. Some buyer types might not be able to trade, in which case $\mu(x) = \varnothing$. We will refer to the equilibrium matching as μ^*. That is, $\mu^*(x) = \{y \in \mathcal{Y} | (p,x,y) \in \hat{G}$ and $(p,y) \in \hat{F}$ for some price $p \in \mathcal{P}\}$. We are interested in the conditions on $f(x,y)$ that induce sorting, i.e. that generate a matching pattern μ^* that is single-valued and strictly monotonic on $(x_0, \overline{x}]$. In such a case we can treat μ^* as a function on $(x_0, \overline{x}]$. If the function μ^* is strictly increasing on $(x_0, \overline{x}]$, then we say that it exhibits positive assortative matching, if it is strictly decreasing, then we say that it exhibits negative assortative matching.

Degrees of supermodularity. In the analysis we will use the following concepts:

[14] In models with only one homogeneous buyer type x, the Market Utility Assumption states that $\lambda(p,y) = \lambda_x(p,y)$, with the interpretation that an individual seller does not change the buyers' Market Utility $U(x)$ and therefore buyers visit (p,y) until the resulting buyer-seller ratio makes them indifferent to visiting one of the other (p,y) offers in the market. In our environment this notion extends naturally by requiring that the buyer-seller ratio is determined by the buyers that are willing to endure the longest queue: $\lambda(p,y) = \sup_{x\in\mathcal{X}} \lambda_x(p,y)$.

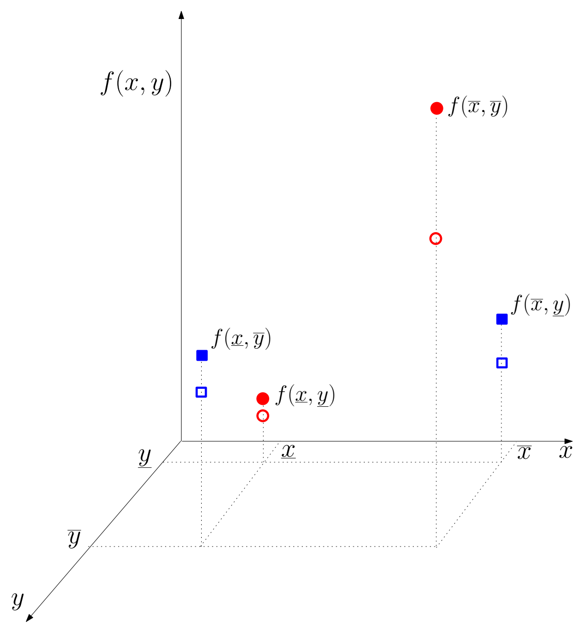

Figure 1: Degrees of supermodularity for two buyer and seller types. Solid circles: match values under positive assortative matching. Solid squares: match values under negative assortative matching. Empty squares and circles: match values after concave transformation.

Definition 3 *The function $f(x, y)$ is:*

1. *supermodular if $f_{xy} \geq 0$ for all (x, y);*
2. *root-supermodular if $\sqrt{f}$ is supermodular;*
3. *n-root-supermodular with coefficient $n \in (1, \infty)$ if $\sqrt[n]{f}$ is supermodular.*
4. *log-supermodular if $\log(f)$ is supermodular;*

n-root-supermodularity requires that the cross-partial derivative of $f^{1/n}$ is non-negative, i.e. that

$$f_{xy}(x, y) \geq \frac{n-1}{n} \frac{f_x(x, y) f_y(x, y)}{f(x, y)}. \tag{5}$$

Supermodularity requires $n = 1$, and log-supermodularity requires $n \to \infty$, and both can therefore be interpreted as the extreme cases of n-root-supermodularity. From inspection of equation (5) it is immediate that root-supermodularity is a weaker requirement than log-supermodularity.

As an illustration of these concepts, consider only the extreme types. Restricted to this set, supermodularity requires $f(\overline{x}, \overline{y}) + f(\underline{x}, \underline{y}) \geq f(\overline{x}, \underline{y}) + f(\underline{x}, \overline{y})$. That is, extreme matches are jointly more valuable than cross-matches. By Jensen's inequality a concave transformation of the match value reduces the extreme values on the left of the inequality more than the intermediate values on the right, which makes it harder to sustain the inequality. This is illustrated in Figure 1, where the solid circles represent the components on the left of the inequality and the solid squares represent the components on the right without the transformation. The empty counterparts represent the values after the transformation. The more concave the transformation, the more difficult it is to fulfill the condition. Since $\log f = g \circ \sqrt{f}$ with $g(z) = 2\log z$, the logarithm is more concave than the square root.

4 The Main Results

The problem that a firm solves is to set prices in order to maximize profits, taking the queue length λ as given. That is, a firm of type y solves

$$\max_{p\in\mathcal{P}} m(\lambda(p,y))p. \tag{6}$$

In equilibrium, buyers obtain some utility $U(.)$, and since the equilibrium can be sustained under the MU assumption, the queue length that the seller faces can be written as

$$\lambda(p,y) = \sup_{x\in\mathcal{X}} \lambda_x(p,y), \tag{7}$$

where λ_x was defined in (4). For given (p,y), since $U(x)$ is continuous it is easy to show that $\lambda_x(p,y)$ is either continuous in x, or there exists $x \in \mathcal{X}$ such that $\lambda_x(p,y) = \infty$. We can therefore write $\lambda(p,y) = \max_{x\in\mathcal{X}} \lambda_x(p,y)$. Now it is straightforward to see that maximizing (6) given (7) yields the same optimal prices and optimal profits as

$$\max_{(p,x)\in\mathcal{P}\times\mathcal{X}} m(\lambda_x(p,y))p, \tag{8}$$

since for given price p the seller would always choose the buyer that gives him the maximal queue length. The problem a firm solves is therefore identical to

$$\begin{aligned} &\max_{(p,x,\lambda)\in\mathcal{P}\times\mathcal{X}\times\overline{\mathbb{R}}_+} && m(\lambda)p \\ &\text{s.t.} && q(\lambda)[f(x,y)-p] = U(x). \end{aligned}$$

After substitution for p, this yields

$$\max_{x\in\mathcal{X},\lambda\geq 0} m(\lambda)f(x,y) - \lambda U(x). \tag{9}$$

Since the buyers' utility $U(\cdot)$ is increasing, it is differentiable almost everywhere. By Consistency (III), almost all of the active sellers trade in equilibrium with buyers in the interior of $[x_0, \overline{x}]$ for which U is differentiable. Also, active sellers make strictly positive profits. For any such seller the necessary first order conditions are

$$m'(\lambda)f(x,y) - U(x) = 0 \tag{10}$$

$$q(\lambda)f_x(x,y) - \lambda U'(x) = 0. \tag{11}$$

Even if $U(.)$ is not differentiable, (10) describes the optimal queue length for seller y conditional on trading with buyer type x. Let $\lambda^*(x,y)$ be defined as the queue length that uniquely solves (10) given $U(x)$, i.e., it is the optimal queue length that a seller of type y would choose conditional on trading with buyer type x. So formally λ^* depends on $U(\cdot)$. In the following we suppress this dependence, and we also suppress the arguments (x,y) for easier readability. Then profits conditional on trading with buyer type x are given by

$$\tilde{\pi}(x,y) = \left[m(\lambda^*) - \lambda^* m'(\lambda^*)\right] f(x,y). \tag{12}$$

We are interested in positive assortative matching, and first we consider the condition to sustain positive assortative matching locally. This will be the case if the marginal increase in profits from trading with a higher buyer type increases in the type of the seller. A necessary condition is that the *induced* profit function $\tilde{\pi}$ is locally supermodular.

Lemma 2 *Consider some $U(\cdot)$ and some optimal choice (x,λ^*) for a seller type y, where (x,λ^*) is characterized by the first order conditions* (10) *and* (11)*. Then $\frac{\partial^2 \tilde{\pi}(x,y)}{\partial x \partial y} > 0$ if and only if*

$$f_{xy}(x,y) > a(\lambda^*) \frac{f_y(x,y) f_x(x,y)}{f(x,y)} \tag{13}$$

where

$$a(\lambda) = \frac{m'(\lambda) q'(\lambda)}{m''(\lambda) q(\lambda)}. \tag{14}$$

Likewise, the statement holds when all strict inequalities are reversed, and when the strict inequalities are replaced by equalities.

Proof. At (x,λ^*) the profit $\tilde{\pi}(x,y)$ coincides with the optimal profit (9). The derivative of profits with respect to y is

$$\frac{\partial \tilde{\pi}(x,y)}{\partial y} = m(\lambda^*) f_y(x,y),$$

where the indirect effect due to a change of the optimal queue length λ^* is zero by the envelope theorem.[15] Taking the derivative with respect to x yields

$$\frac{\partial^2 \tilde{\pi}(x,y)}{\partial y \partial x} = m'(\lambda) \frac{\partial \lambda^*}{\partial x} f_y(x,y) + m(\lambda^*) f_{xy}(x,y), \tag{15}$$

where the optimal queue length is defined by (10). Its derivative is given by implicit differentiation of (10) as

$$\frac{\partial \lambda^*}{\partial x} = \frac{\left[m(\lambda^*) - \lambda^* m'(\lambda^*)\right] f_x(x,y)}{\lambda^* m''(\lambda^*) f(x,y)}.$$

[15] Equivalently, one can differentiate profit equation (12) and substitute the change in λ^* obtained by implicit differentiation of (10).

Substitution into (15) yields a strictly positive cross-partial for the profit $\tilde{\pi}(x, y)$ iff

$$f_{xy}(x, y) > -\frac{m'(\lambda^*)\left[m(\lambda^*) - \lambda^* m'(\lambda^*)\right]}{\lambda^* m''(\lambda^*) m(\lambda^*)} \frac{f_y(x, y) f_x(x, y)}{f(x, y)}. \tag{16}$$

Denote $a(\lambda)$ by the first ratio on the right hand side of (16). Since $q(\lambda) = \frac{m(\lambda)}{\lambda}$, it follows that $q'(\lambda) = \frac{1}{\lambda^2}\left[\lambda m'(\lambda) - m(\lambda)\right]$ and we obtain

$$a(\lambda) = -\frac{m'(\lambda)\left[m(\lambda) - \lambda m'(\lambda)\right]}{\lambda m''(\lambda) m(\lambda)} = \frac{m'(\lambda) q'(\lambda)}{m''(\lambda) q(\lambda)}. \tag{17}$$

which establishes the lemma. ■

Since $a(\lambda) > 0$, the Lemma shows that supermodularity of the induced profit function requires a condition on the match surplus f that is stronger than supermodularity. It turns out that $a(\lambda)$ has a natural interpretation.[16]

Lemma 3 $a(\lambda)$ *is equal to the elasticity of substitution of the aggregate matching function* $M(\beta, \sigma)$.

Proof. The elasticity of substitution of the aggregate matching function $M(\beta, \sigma)$ with β buyers and σ sellers is generally defined as:

$$e = \frac{\frac{d(\beta/\sigma)}{\beta/\sigma}}{\frac{d(M_\sigma/M_\beta)}{M_\sigma/M_\beta}}$$

where $M(\beta, \sigma) = \sigma m(\lambda)$, $M_\beta = m'(\lambda)$, $M_\sigma = m(\lambda) - \lambda m'(\lambda)$, and $\lambda = \beta/\sigma$. Then, suppressing the argument λ, we can write e as:

$$e \quad = \quad \frac{\frac{m - \lambda m'}{m'}}{\lambda \frac{d}{d\lambda} \frac{m - \lambda m'}{m'}} = -\frac{m'(m - \lambda m')}{\lambda m'' m} = a(\lambda)$$

■

The main argument informally. To provide some intuition, we outline a brief graphical argument why supermodularity *per se* is not enough to induce positive assortative matching.

Buyers care about the probability of matching reflected by λ, the price p, and the type of the seller y. Their preferences in equation (1) are over a three-dimensional space (λ, p, y). For clarity of exposition, in Figure 2 we project those preferences into a two-dimensional space (p, λ). The solid concave curves on the left refer to the indifference curves for low buyer type x_1 and high buyer type x_2 when trading with a low seller type y_1. They exhibit single crossing because high valuation buyers value fast trade more, and are therefore willing to pay a higher price to avoid no-trade. The dotted curves on the right represent the same level of utility for the buyers when they match with a higher buyer type $y_2 > y_1$. The thick dotted curves represent the case without complementarities, e.g. when $f(x, y) = x + y$,

[16] We are grateful to John Kennan for pointing this out to us.

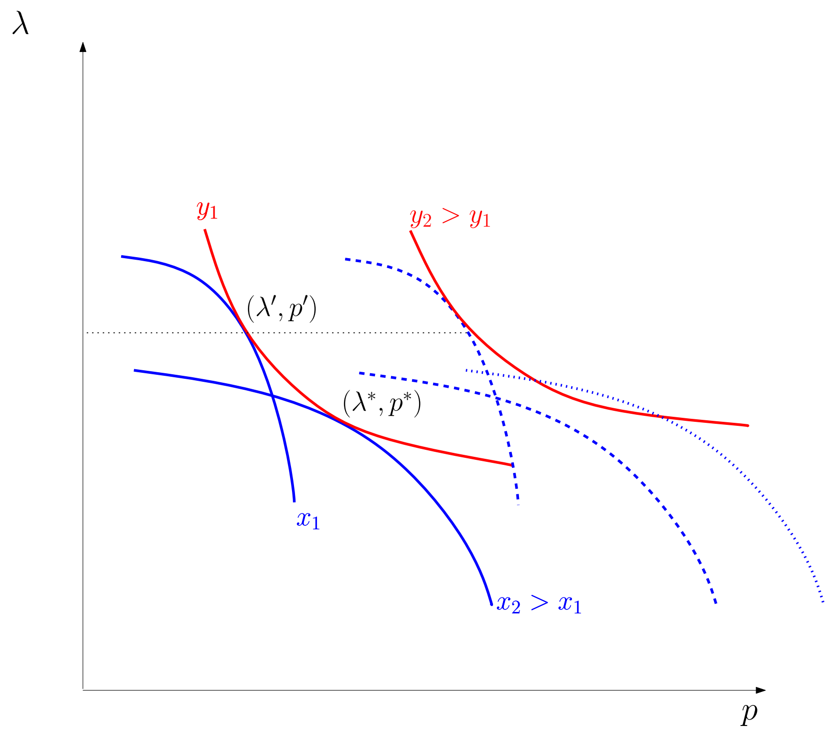

Figure 2: Explanation why sufficient supermodularity is required for PAM. Solid concave curves: Buyers' indifference curves when trading with seller y_1. Thick dotted concave curves: Buyers' indifference when trading with a higher seller type y_2 if the match value has no complementarities. Thin dotted concave curve: High type buyers' indifference curve when trading with high seller type y_2 if the match value has complementarities. Convex curves: Sellers' indifference curves.

while the thin dotted curves represent the case with complementarities. For the latter, the match value increases more for high buyer types when they trade with higher seller types. For a given buyer x, the solid and dotted curves give the same level of utility. This means that along a horizontal line, $f(x, y) - p$ stays constant, which implies a parallel shift of the indifference curve for each x. That shift is proportional to the change in the valuation from an increase in y: $f_y(x, y)$. Moreover, in the absence of complementarities, the change f_y is the same for different x, since $f_{xy} = 0$. As a result, in that case the curves for different buyer types shift by the same amount, as shown by the thick dotted curves.

One can think of an individual seller's problem as taking the utility of the buyers as given and choosing the point that yields the highest iso-profit curve (red convex curves). We have drawn the case where seller type y_1 is indifferent between trading with either buyer. To the right is another iso-profit curve, corresponding to a higher type seller. Since a higher seller type induces higher utility for buyers, he attracts more buyers and therefore obtains higher profits. Because at higher prices and same λ a seller is less willing to reduce the price, higher iso-profit curves are flatter at each λ. The convex curve on the right is therefore not a parallel shift. For the case without complementarities this implies that the highest utility for a type y_2 seller can be achieved when trading with the low seller type x_1. There is negative assortative matching as high types match with low types and vice versa. Observe that in a frictionless environment, the allocation is indeterminate in the absence of complementarities. The market frictions therefore generate a force that leads to negative sorting.

When there is supermodularity in $f(x, y)$, the change f_y is larger for larger x since $f_{xy} > 0$. Therefore, because the shift in the indifference curves is proportional to f_y, buyer type x_2's indifference curve will shift more than that of x_1. In Figure 2, the thin dotted line to the right is x_2's indifference curve in an economy with complementarities. It is moved more than the indifference curve for type x_1, which for simplicity we left unchanged. Only if $f(x, y)$ is sufficiently supermodular will the indifference curve for the high buyer type shift enough to intersect with the iso-profit curve that is illustrated for y_2. Then the high type sellers are better off by trading with the high buyer types, which leads to Positive Assortative Matching. The condition that induces such a large enough shift is root-supermodularity, as we lay out next.

For the interpretation of why the condition for positive assortative matching requires the expression $a(\lambda)$, consider as a starting point equation (12). For a given (x, y) combination in isolation, equation (12) relates the firm's profit to the match value and the change of the aggregate matching function $M(\cdot, \cdot)$ with respect to the number of sellers. If additional sellers incentives to enter this specific (x, y) combination reflect the contribution to match creation. With two-sided heterogeneity and assortative matching, what matters is the relationship between *different* (x, y) combinations. At each market (x, y), there is competition from potential entrants from nearby markets. Because increased trade in a market can be achieved by entry of both sellers and buyers from nearby markets, the elasticity of substitution becomes relevant.

A clear interpretation of our condition for assortative matching in terms of the trading motives mentioned in the introduction can be gained by rewriting the condition (suppressing the arguments) as $f_{xy} - a(\cdot) f_x f_y / f > 0$. The term f_{xy} captures the Beckerian gain in "match value" due to complementarity. It reflects the marginal increase in output from increasing both x and y types. The term $a(\cdot) f_x f_y / f$ captures the opportunity costs induced by not trading, i.e., the "trading security" aspect mentioned in the introduction. It consists of the marginal loss of value from matching a higher type on each side multiplied by the rate of change in the number of matches. Only when the "match value" motive outweighs the costs induced by the "trading security" motive does positive assortative matching arise.

4.1 Positive Assortative Matching under Root-Supermodularity

We now state and prove our main result.

Theorem 1 *There is Positive Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$ if and only if the function $f(x, y)$ is root-supermodular.*

We will prove the theorem by means of two Propositions, one for the sufficient part, one for the necessary part. Each Proposition is preceded by a Lemma. The first Lemma provides the justification for the "root" in the supermodularity condition. With the aid of this Lemma, the first Proposition establishes that f is root-supermodular when we have Positive Assortative Matching for all type distributions. This finding is closely linked to the condition required locally in Lemma 2:

$$f_{xy} > a(\lambda)\frac{f_x f_y}{f}. \tag{18}$$

If this inequality is fulfilled, equilibrium behavior is such that high buyer types gain more by moving towards higher seller types, and therefore positive assortative matching results.

The properties of the function $a(\lambda)$ are key to understanding condition (18). Because distributions B and S can be suitably chosen, this condition must hold for all possible $\lambda \in \overline{\mathbb{R}}_+$. Even at zero, $a(\lambda)$ is well defined, since $q(0) > 0$ and $m''(0) < 0$ and the derivatives are bounded. It turns out that at $\lambda = 0$, the function has a property that holds for any matching function in the class that we consider, namely $a(0) = 1/2$:

Lemma 4 *Under any permissible matching function* $a(0) = 1/2$.

Proof. Constant returns to matching implies $\lambda q(\lambda) = m(\lambda)$, which readily yields after some rearranging that $q'(\lambda) = \frac{m'(\lambda)-q(\lambda)}{\lambda}$ and $q''(\lambda) = \frac{m''(\lambda)-2q'(\lambda)}{\lambda}$. Our class of matching functions also assumes boundedness of first and second derivatives of q. Together this implies $m'(0) = q(0)$ and $m''(0) = 2q'(0)$, since both q' and q'' are divided by λ. Since

$$a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{m''(\lambda)q(\lambda)}$$

it follows that

$$a(0) = \frac{m'(0)\frac{m''(0)}{2}}{m''(0)q(0)} = \frac{1}{2}.$$

■

We are now in a position to prove

Proposition 1 *(Necessary) If there is Positive Assortative Matching of μ^* for all permissible distributions $B(x)$ and $S(y)$, then $f(x,y)$ is root-supermodular.*

Proof. See Appendix A. ■

We proceed to establishing sufficiency of root-supermodularity for Positive Assortative Matching. We know that at a queue length of zero, $a(0) = 1/2$. For root-supermodularity to have enough bite to sustain positive sorting we need $a(\lambda)$ to remain small enough for all queue lengths. That is established in the next Lemma.

Lemma 5 *$a(\lambda) < 1/2$ for all $\lambda \in \mathbb{R}_{++}$ if and only if $q(\lambda)^{-1}$ is strictly convex in λ.*

Proof. Since $\frac{1}{q(\lambda)} = \frac{\lambda}{m(\lambda)}$, we have $(\lambda/m(\lambda))' = \frac{1}{m(\lambda)^2}[m(\lambda) - \lambda m'(\lambda)]$. Therefore [for brevity we suppress the argument of m] :

$$\begin{aligned}(\lambda/m)'' &= \frac{1}{m^4}[m'm^2 - m'm^2 - \lambda m''m^2 - 2mm'(m - \lambda m')] \\ &= \frac{1}{m^3}[-\lambda m''m - 2m'[m - \lambda m']] \\ &= \frac{\lambda^2}{m^3}[-m''q + 2m'q'].\end{aligned}$$

This is strictly positive if and only if $-m''q + 2m'q' > 0$ or $m'q' > \frac{1}{2}m''q$ or $\frac{m'q'}{m''q} < \frac{1}{2}$. The left hand side of the inequality is $a(\lambda)$. ■

Lemma 5 provides a necessary and sufficient condition for $a(\lambda)$ to be strictly less than 1/2. Together with Lemma 2 this establishes that root-supermodularity is sufficient for Positive Assortative Matching. The reason why the proof is more than a corollary has to do with the assumption of differentiability of $U(.)$, and with the fact that we also want to rule out global deviation of the form that the lower half of the seller types trades with the higher buyer types while the upper half of the seller types trades with the lower buyer types. Lemma 2 only considers the local incentives for assortative matching.

Proposition 2 *(Sufficient) If the function $f(x, y)$ is root-supermodular then there is Positive Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

Proof. See Appendix A. ■

We briefly remark here that we either need $a(\lambda) < 1/2$ and root-supermodularity, or $a(\lambda) \leq 1/2$ and a strict form of root-supermodularity in order to guarantee positive assortative matching. Recall that for one of the matching functions we featured as an example, $m_4 = \gamma\lambda/(1-\lambda)$, the term $q(\lambda)^{-1} = \lambda/m(\lambda) = (1 - \lambda)/\gamma$ is only weakly convex and therefore $a(\lambda) = 1/2$ for all λ. Under strict supermodularity Positive Assortative Matching is still guaranteed. Yet if the match value function satisfies *exactly* $f_{xy}(x, y) = \frac{1}{2}\frac{f_x(x,y)f_y(x,y)}{f(x,y)}$ for all (x, y) then any matching pattern can be sustained.

Finally, the next Corollary follows immediately from Proposition 2.

Corollary 1 *If $f(x, y)$ is log-supermodular then the matching pattern exhibits positive assortative matching.*

4.2 Negative Assortative Matching under Weak Submodularity

While "strong enough" supermodularity is needed for positive assortative matching, we now show that negative assortative matching obtains under weak submodularity, i.e. $f_{xy}(x, y) \leq 0$. In the informal graphical argument of Figure 2, we argue that even under a-modularity (e.g. $f(x, y) = x + y$) there is negative assortative matching. While in the frictionless world the equilibrium allocation under a-modularity is indeterminate, the market frictions impose a force towards negative assortative matching. In this sense, there is an asymmetry between positive and negative sorting as there is no need for "strong" submodularity. There is no such asymmetry for example under random search (Shimer and Smith (2000)), as negative assortative matching obtains provided $f(x, y)$ is log-submodular.

We now establish the Theorem with a sufficient condition for Negative Assortative Matching.

Theorem 2 *There is Negative Assortative Matching of μ^* for all permissible type distributions if the function $f(x, y)$ is weakly submodular.*

Proof. By Lemma 2 the condition that implies negative assortative matching locally is $f_{xy}(x, y) < a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}$, where the right-hand side is always positive. Weak submodularity of $f(x, y)$ requires $f_{xy} \leq 0$, implying this is always satisfied. The global argument for negative assortative matching follows along the same lines as in Proposition 2. ■

Weak submodularity is also necessary for Negative Assortative Matching for all type distribution under some matching functions such as m_1 to m_3. These functions have the minimal $a(\cdot)$ at $a(\infty) = 0$, and for buyer types that have a hard time trading the condition becomes binding. Similar to Proposition 1, for these matching functions there always exist distributions such that at a given (x, y) combination, $\lambda^*(x, y)$ goes to infinity. By a similar argument one can show that if $a_{min} \equiv \inf_\lambda a(\lambda) > 0$, then there is Negative Assortative Matching for all type distributions if and only if the match value function $f(x, y)$ is nowhere $\frac{1}{1-a_{min}}$-root-supermodular. Recall that 1-root-supermodularity is standard supermodularity.

Interestingly, it is possible for specific distributions to have negative assortative matching for moderate degrees of supermodularity even if $a_{min} = 0$. To see this, refer back to the initial informal graphical argument of Figure 2. Let there be a very small degree of supermodularity, say $f(x, y) = x+y+\varepsilon xy+1$, with ε small. Then the thick dotted indifference curves representing the case without complementarities will only be slightly to the left of the thin indifference curve representing the case with complementarities. Provided the distributions are such that $a(\lambda)$ is nowhere zero, the iso-profit curve for y_2 will be flatter everywhere and it is more profitable for high seller types to match with low buyer types. Therefore, the equilibrium allocation exhibits negative assortative matching.

Proposition 3 *There exist distributions $B(x), S(y)$ and supermodular functions $f(x, y)$ such that in equilibrium only Negative Assortative Matching of μ^* can obtain.*

Proof. Fix the matching function m. Consider for example a sequence of supermodular functions of the form $f_n(x, y) = x+y+\varepsilon_n xy+1$, with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Consider a sequence of distributions B_n and S_n with support on $[0, \varepsilon_n]$ and a unit measure of sellers for all $n \in \mathbb{N}$. For $n \to \infty$ the maximal equilibrium utility $\max_{x\in[0,\varepsilon_n]} U_n(x)$ in any equilibrium remains bounded below $1-\alpha$ for some $\alpha>0$. To see this, note that the equilibrium utilities of buyers converge, i.e. $\max_{x\in[0,\varepsilon_n]} U_n(x) - \min_{x\in[0,\varepsilon_n]} U_n(x) \to 0$, because the support converges and types can "mimic" each others' strategy. Since the match value f converges to 1, a maximum utility converging to 1 is only possible if all buyer types get matched with probability converging to unity, which is not possible with equal numbers of buyers and sellers. Since $\max_{x\in[0,\varepsilon_n]} U_n(x) \leq 1-\alpha$ for all n large enough, the optimal queue length $\lambda_n^*(x,y)$ remains bounded below some $\overline{\lambda}$. That means that $\min_{(x,y)\in[0,\varepsilon_n]^2} a(\lambda_n^*(x,y))$ remains bounded above some $\overline{a}>0$ in any equilibrium for n sufficiently large. Therefore, $\frac{\partial^2}{\partial x \partial y} f_n(x,y) < a(\lambda_n^*(x,y)) \frac{\partial}{\partial x} f_n(x,y) \frac{\partial}{\partial y} f_n(x,y)/f(x,y)$ holds for all $(x,y)\in[0,\varepsilon_n]^2$ when $\varepsilon_n < \overline{a}/[1+2\varepsilon_n+\varepsilon_n^3]$, which holds for some n large. Therefore, only negative assortative matching can be sustained. ■

For completeness, we state the analogue for Positive Assortative Matching.

Proposition 4 *There exist distributions $B(x), S(y)$ and functions $f(x,y)$ that are not root-supermodular such that there is Positive Assortative Matching of μ^*.*

Proof. See Appendix A. ■

Our results are cast in terms of the monotonicity of the allocation, i.e., assortative matching. The model provides few general predictions in terms of monotonicity of the price schedule. In particular, sellers can be rewarded both through higher prices or better trading probabilities, and if the second effect is large then prices at better sellers can actually fall. Under Negative Assortative Matching (NAM) the trading probabilities for higher types are always higher, which is precisely the reason why high seller types match with low buyer types. In this case prices can both increase or decrease in type, depending on how much better the trading probabilities for high seller types are. Under Positive Assortative Matching the trading probabilities for better seller types might increase or decrease, depending on the relative density in the distribution of seller and buyer types. It increases when the seller types are more dispersed and decreases otherwise, and only in the former case can prices rise.[17]

4.3 Existence

We now establish existence of equilibrium. Existence in our setup is more complicated than in frictionless matching models because we cannot employ the standard measure consistency condition. There can be more agents from one side attempting to trade with the other, because this imbalance is absorbed

[17] Formally, by more dispersed we mean that the support of the distribution of buyer types is very small relative to that of seller types. In this case the derivative $\mu'(x)$ of the assignment is large, and it is easy to see from equation (29) in the next section that in this case the ratio of buyers and sellers is falling in markets where the higher types trade [this is ensured because $-\lambda q(\lambda)$ is bounded by $q(\lambda)$].

through different trading probabilities.[18] The system retains tractability when we impose the sufficient conditions for assortative matching, in which case the following constructive existence proof can be applied.

Proposition 5 *If the function $f(x,y)$ is root-supermodular, there exists an equilibrium for all permissible distributions $B(x), S(y)$.*

Our strategy of proof is the following: 1. we construct an equilibrium that is monotonically increasing (PAM); 2. we show that the solution to the First Order Conditions satisfies a system of two differential equations in λ and μ with the appropriate boundary conditions; 3. we verify the Second Order Condition for seller optimality along the equilibrium allocation μ^* and verify that under root-supermodularity the solution is indeed a local maximum; 4. we consider solutions to the First Order Conditions different from the constructed equilibrium and find that none other exist, thus establishing that the solution is a global maximum.

Proof. Under assortative matching the matching μ is unique and we can treat it as a function. Therefore, the equilibrium is characterized by an assignment function $\mu(x) \in \mathcal{Y} \cup \emptyset$ that specifies the type of seller to which buyer x is matched, where $\mu(x) = \emptyset$ means that the buyer is not matched, as well as a queue length function $\lambda^*(x, \mu(x)) =: \tilde{\lambda}(x)$ along the equilibrium matching. We will construct both in the following. The queue length off the equilibrium matching are then determined according to the Market Utility Assumption. If $y \in \mathcal{Y}$ but $y \notin \mu(\mathcal{X})$ then the seller is not matched to any buyer. We are looking for an assignment function with $\mu'(x) > 0$ for all $x > x_0$, where x_0 is the lowest buyer type that is matched.[19] Similarly, y_0 is the lowest seller type that is matched. A seller of type y maximizes expected profits

$$\max_{x \in \mathcal{X}, \lambda \geq 0} m(\lambda) f(x,y) - \lambda U(x) \tag{19}$$

and the solution for almost all types $y > y_0$ is given by the first order conditions

$$\begin{aligned} m'(\lambda) f(x,y) - U(x) &= 0 & (20) \\ m(\lambda) f_x(x,y) - \lambda U'(x) &= 0. & (21) \end{aligned}$$

[18]Even when we reinterpret our framework as a "many-to-many" matching where in a given market n_b buyers match with n_s sellers to produce $m(\frac{n_b}{n_s})f$ units of output per seller, we cannot apply standard results because the queue length $\lambda = \frac{n_b}{n_s}$ can be any real number and we will have to allow non-finite coalitions which is typically excluded in the literature. Nor can we apply results from the extensive mathematical theory of optimal transportation that followed Kantorovich (1942), because it imposes that the output per seller depends only on the types, but in our case the induced output $m(\frac{n_b}{n_s})f$ depends directly on the assigned measures. Therefore we cannot apply standard linear programming techniques, which is precisely one of the main distinguishing features of our work from standard non-frictional assignment literature, for which it is well-known that the core can be represented as a linear program (see e.g. Shapley and Shubik (1971)).

[19]We defined x_0 as the supremum inactive type, which by Lemma (9) is the infimum of the active types. A similar point applies to y_0 when we apply the MU assumption.

We immediately have the following necessary conditions for an equilibrium

$$
\begin{aligned}
m'(\tilde{\lambda}(x))f(x,\mu(x)) &= U(x) && (22)\\
q(\tilde{\lambda}(x))f_x(x,\mu(x)) &= U'(x). && (23)
\end{aligned}
$$

Moreover, for the boundary types it has to hold by (10) that

$$U(x_0) = m'(\tilde{\lambda}(x_0))f(x_0,\mu(x_0)) \geq 0, \text{ with equality if } x_0 > \underline{x}, \tag{24}$$

and by (12) that

$$\pi(\mu(x_0)) = [m(\tilde{\lambda}(x_0)) - \tilde{\lambda}(x_0)m'(\tilde{\lambda}(x_0))]f(x_0,\mu(x_0)) \geq 0, \text{ with equality if } \mu(x_0) > \underline{y}. \tag{25}$$

Additionally, the queue length function has to be consistent with the assignment function via the relationship

$$\int_{\mu(x)}^{\overline{y}} s(\cdot)dy = \int_{x}^{\overline{x}} \frac{b(\cdot)}{\tilde{\lambda}(\cdot)}dx, \text{ for all } x \geq x_0. \tag{26}$$

Recall that the measure of buyers is normalized to one, whereas the measure of sellers is $S(\overline{y})$. This yields

$$\mu'(x) = \frac{b(x)}{s(\mu(x))\tilde{\lambda}(x)}. \tag{27}$$

An equilibrium is a solution to (22) – (25) and (27). Totally differentiating (22) yields

$$m''(\tilde{\lambda}(x))f(x,\mu(x))\tilde{\lambda}'(x) + m'(\tilde{\lambda}(x))\left[f_x(x,\mu(x)) + f_y(x,\mu(x))\mu'(x)\right] = U'(x).$$

Substituting (23) yields

$$m''(\tilde{\lambda}(x))f(x,\mu(x))\tilde{\lambda}'(x) + m'(\tilde{\lambda}(x))\left[f_x(x,\mu(x)) + f_y(x,\mu(x))\mu'(x)\right] = q(\tilde{\lambda}(x))f_x(x,\mu(x)) \tag{28}$$

which after rearranging and using the fact that $-\lambda q'(\lambda) = q(\lambda) - m'(\lambda)$ gives

$$\tilde{\lambda}'(x) = -\frac{1}{m''(\tilde{\lambda}(x))f(x,\mu(x))}\left[\tilde{\lambda}(x)q'(\tilde{\lambda}(x))f_x(x,\mu(x)) + \mu'(x)m'(\tilde{\lambda}(x))f_y(x,\mu(x))\right]. \tag{29}$$

Together with (27) we have

$$\tilde{\lambda}'(x) = -\frac{1}{\tilde{\lambda}(x)m''(\tilde{\lambda}(x))f(x,\mu(x))}\left[\tilde{\lambda}^2 q'(\tilde{\lambda}(x))f_x(x,\mu(x)) + \frac{b(x)}{s(\mu(x))}m'(\tilde{\lambda}(y))f_y(x,\mu(x))\right]. \tag{30}$$

Equations (27) and (30) together constitute a differential equation system in $\tilde{\lambda}, \mu$. One initial condition is $\mu(\overline{x}) = \overline{y}$. Given a second initial condition $\tilde{\lambda}(\overline{x}) = \overline{\lambda} \in (0,\infty)$ the system is uniquely defined (in the direction of lower x) down to some limit point $x_0(\overline{\lambda})$ where either $x_0(\overline{\lambda}) = \underline{x}$ or $\mu(x_0(\overline{\lambda})) = \underline{y}$ or $\lim_{x\searrow x_0(\overline{\lambda})} \tilde{\lambda}(x) = \tilde{\lambda}(x_0(\overline{\lambda})) = 0$ or $\lim_{x\searrow x_0(\overline{\lambda})} \tilde{\lambda}(x) = \tilde{\lambda}(x_0(\overline{\lambda})) = \infty$, whichever comes first. In the latter cases let $\mu(x_0(\overline{\lambda})) = \lim_{x\searrow x_0(\overline{\lambda})} \mu(x)$. In the Appendix we prove that

Lemma 6 *There exists an initial condition $\overline{\lambda} \in (0,1)$ such that resulting $x_0(\overline{\lambda})$, $\tilde{\lambda}(x_0(\overline{\lambda}))$ and $\mu(x_0(\overline{\lambda}))$ fulfill boundary conditions (24) and (25).*

We now check that no agent has an incentive to deviate under the assignment $\tilde{\lambda}(x)$, $\mu(x)$ that we just constructed. If we can verify that the sellers have no profitable deviation, then their optimization problem (19) implies that buyers do not have a profitable deviation either because the sellers would have exploited it for their benefit. (One can formally show that (30) is exactly the workers envelop condition.) For the sellers we only have to consider types in $[y_0(\overline{\lambda}), \overline{y}]$. If there are seller types below $y_0(\overline{\lambda})$, these types do not have a profitable deviation because by boundary condition (25) $y_0(\overline{\lambda})$ makes zero profits and does not have a profitable deviation despite being a higher type. To check optimality for sellers, we will first check whether the second order condition for firms holds along the equilibrium path. The Hessian of (20) and (21) is:

$$\begin{pmatrix} m''(\lambda)f(x,y) & m'(\lambda)f_x(x,y) - U'(x) \\ m'(\lambda)f_x(x,y) - U'(x) & m(\lambda)f_{xx}(x,y) - \lambda U''(x) \end{pmatrix}.$$

It is negative definite if $m''(\lambda)f(x,y) < 0$, which holds since $m(\lambda)$ is strictly concave, and if

$$m''(\lambda)f(x,y)\left[m(\lambda)f_{xx}(x,y) - \lambda U''(x)\right] - \left[m'(\lambda)f_x(x,y) - U'(x)\right]^2 > 0.$$

Checking this condition locally along the equilibrium path, we can use (23) directly to get

$$m''(\lambda)f(x,y)\left[m(\lambda)f_{xx}(x,y) - \lambda U''(x)\right] - [m'(\lambda) - q(\lambda)]^2 f_x(x,y)^2 > 0. \tag{31}$$

Differentiating (23) we get an expression for $U''(x)$

$$U''(x) = q'(\tilde{\lambda}(x))f_x(x,\mu(x))\tilde{\lambda}'(x) + q(\tilde{\lambda}(x))f_{xx}(x,\mu(x)) + q(\tilde{\lambda}(x))f_{xy}(x,\mu(x))\mu'(x),$$

which inserted in (31) and after rearranging yields along the path

$$-\tilde{\lambda}(x)q'(\tilde{\lambda}(x))f_x(x,\mu(x))\tilde{\lambda}'(x) - m(\tilde{\lambda}(x))f_{xy}(x,\mu(x))\mu'(x) - \frac{[\tilde{\lambda}(x)q'(\tilde{\lambda}(x))f_x(x,\mu(x))]^2}{m''(\tilde{\lambda}(x))f(x,\mu(x))} < 0,$$

where we used $m(\lambda) = \lambda q(\lambda)$, $\lambda q'(\lambda) = m'(\lambda) - q(\lambda)$ and we changed the sign of the inequality after dividing by $m''(\lambda) < 0$. We can use (29) to substitute out $\tilde{\lambda}'(x)$ and get after rearranging

$$\mu'(x)\left[-f_{xy} + \frac{m'(\tilde{\lambda}(x))q'(\tilde{\lambda}(x))}{q(\tilde{\lambda}(x))m''(\tilde{\lambda}(x))}\frac{f_x(x,\mu(x))f_y(x,\mu(x))}{f(x,\mu(x))}\right] < 0.$$

The square bracket is negative if

$$f_{xy}(x,\mu(x)) > a(\tilde{\lambda}(x))\frac{f_x(x,\mu(x))f_y(x,\mu(x))}{f(x,\mu(x))},$$

where $a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{m''(\lambda)q(\lambda)}$, which is fulfilled when f is root-supermodular. We started from the premise that $\mu'(x) > 0$, and as a result, under root-supermodularity and strict convexity of q^{-1} the Hessian is negative definite. The solution to the FOC therefore constitutes a local maximum.

We now establish that the solution is a global maximum. Consider a firm y that is matched to x, i.e. $y = \mu(x)$. Now suppose there is also another partner $\tilde{x}$, different from $\mu^{-1}(y)$, that satisfies the First Order Conditions for optimality. In particular, $\tilde{x}$ solves

$$q(\lambda^*(\tilde{x}, y)) f_x(\tilde{x}, y) - U'(\tilde{x}) = 0, \tag{32}$$

where $\lambda^*(x, y)$ is the optimal queue length, defined implicitly by (11).

Suppose that $\tilde{x} > \mu^{-1}(y)$ which implies $\mu(\tilde{x}) > y$ – the opposite case is analogous. Since $\mu(\tilde{x})$ is matched with $\tilde{x}$ their FOC is satisfied by construction:

$$q(\lambda^*(\tilde{x}, \mu(\tilde{x}))) f_x(\tilde{x}, \mu(\tilde{x})) - U'(\tilde{x}) = 0. \tag{33}$$

We rule out that both (32) and (33) are satisfied simultaneously by showing that the FOC is strictly increasing when y increases for given $\tilde{x}$. Differentiating $q(\lambda^*(\tilde{x}, y)) f_x(\tilde{x}, y) - U'(\tilde{x})$ with respect to y yields

$$q(\lambda^*(\tilde{x}, y)) f_{xy}(\tilde{x}, y)) + q'(\lambda^*(\tilde{x}, y)) f_x(\tilde{x}, y) \frac{\partial \lambda^*(\tilde{x}, y)}{\partial y}. \tag{34}$$

Implicit differentiation of (11) yields

$$\frac{\partial \lambda^*(\tilde{x}, y)}{\partial y} = -\frac{m'(\lambda^*(\tilde{x}, y)) f_y(\tilde{x}, y)}{m''(\lambda^*(\tilde{x}, y)) f(\tilde{x}, y)}.$$

Therefore, (34) is strictly positive if

$$f_{xy}(\tilde{x}, y)) > a(\lambda^*(\tilde{x}, y)) \frac{f_y(\tilde{x}, y) f_x(\tilde{x}, y)}{f(\tilde{x}, y)},$$

which is ensured by root-supermodularity since $a(\lambda) < 1/2$ by Lemma 5. This implies that the solution to the FOC in (22) and (23) is a global maximum. ■

A similar proof can be applied in the case where negative assortative matching is ensured, for example when f is weakly submodular.

5 Extensions and Robustness

In this section we consider some extensions to our framework. We investigate the impact of broadening the class of matching functions, we analyze sorting in a dynamic framework, and we introduce more general payoff structures.

5.1 Broader Classes of Matching Functions

Our setup provides sharp results for a large class of matching functions. Here we briefly consider the effect of relaxing some of the assumptions that we imposed. We do not attempt to relax the assumption of constant returns to scale, though. This assumption, which is pervasive in the search literature, is crucial to all directed and competitive search models that we are aware of because the market size in these models is endogenous as each (p, y) combination is essentially its own market.

We have considered matching functions with $1/q$ strictly convex. In contrast to the other assumptions on strict derivatives, strict curvature and boundedness this assumption is less pervasive in the literature, even though it is common to the directed search literature. If we relax this assumption we lose the sufficiency of root-supermodularity for positive assortative matching. When we replace this assumption by the weaker assumption that the elasticity of the matching function m is strictly decreasing we can still obtain an upper bound on $a(\lambda)$ of unity.

Lemma 7 *If* $\eta_m'(\lambda) < 0$ *then* $a(\lambda) < 1$.

Proof. See Appendix A. ■

Therefore, Corollary 1 still applies as log-supermodularity clearly suffices for Positive Assortative Matching. How far $a(\lambda)$ is bounded below unity depends on the exact nature of the matching function. In general some level of $n-$root-supermodularity with coefficient n between 2 and ∞ will be sufficient to induce Positive Assortative Matching.

Finally, we have assumed that the first and second derivatives of the matching functions are strict at $\lambda = 0$. This condition was important for the necessity of root-supermodularity for positive assortative matching. Without this assumption our approach[20] of assessing the value of $a(\lambda)$ at a queue length of zero would not be valid, and possibly different limit values $a(0)$ could be achieved.

CES. The matching function that is not included in our framework is the Constant Elasticity of Substitution (CES) matching function. The aggregate CES matching function in a discrete time framework for a given number of buyers and sellers β and σ is defined as:

$$M(\beta, \sigma) = (\beta^r + k\sigma^r)^{-1/r}\beta\sigma$$

where $r > 0$ and $k > 1$. The elasticity of substitution is given by $e = \frac{1}{1+r}$. Given constant returns to matching, we can derive the individual matching probability expressed in function of the ratio $\lambda = \beta/\sigma$ from the aggregate matching function and where $m(\lambda) = \sigma M(\beta, \sigma)$. Then:

$$m(\lambda) = (1 + k\lambda^{-r})^{-1/r}.$$

We know from Lemma 3 that the expression for $a(\lambda) = -\frac{m'(m-\lambda m')}{\lambda m'' m}$ is equal to the elasticity of substitution $e = \frac{1}{1+r}$. We can easily verify that this is the case after substituting for m, m' and m''.

The CES matching function does not fulfill our conditions of boundedness and strict derivatives for $\lambda = 0$. When $r < 1$ both m'' and q' converge to -∞ when λ approaches zero. Also, $1/q$ is not convex

[20] In particular, the proof of Lemma 4.

for all r. When $r > 1$ then both m'' and q' attain zero at $\lambda = 0$ and are not strict. Nonetheless, the ratio of the two in $a(\lambda)$ remains bounded. The knife-edge case is when $r = 1$, which corresponds to (a variation of) the telegraph matching function $m_5 = \frac{\lambda}{\lambda+k}$. Only in this case are the derivatives m'' and q' bounded at zero.

To our knowledge, all micro founded matching functions involve bounded and strict derivatives. The notion being that the marginal change in the matching probability is not infinite or zero. Given that the CES matching function satisfies the Inada conditions, there are either infinite or zero returns to matching for the buyers when the buyer to seller ratio is zero, which is typically not derived from micro foundations. Nonetheless, the CES function is a convenient modeling tool, and it turns out that it gives very sharp predictions on the necessary and sufficient conditions for Positive and Negative Assortative Matching. This may prove useful for applications.

Proposition 6 *Let the matching function be CES with elasticity e. Then a necessary and sufficient condition for Positive Assortative Matching is that $f(x, y)$ is n-root-supermodular where $n = \frac{1}{1-e}$.*

Proof. ¿From Lemma 2, along the equilibrium path a necessary and sufficient condition is that $f_{xy} > a(\lambda)\frac{f_x f_y}{f}$. Since from Lemma 3, $a(\lambda)$ is equal to the elasticity of substitution and therefore constant for CES, this condition is equivalent to $f_{xy} > e\frac{f_x f_y}{f}$ and independent of the distribution. By definition, $f(x, y)$ is n-root-supermodular if $f_{xy} > \frac{n-1}{n}\frac{f_x f_y}{f}$, therefore $\frac{n-1}{n} = e$ implies $n = \frac{1}{1-e}$. ∎

It is important to stress that with constant $a(\lambda)$ the result holds independent of which distribution is chosen. The next Proposition immediately follows (the proof is omitted):

Proposition 7 *Let the matching function be CES with elasticity e. Then a necessary and sufficient condition for Negative Assortative Matching is that $f(x, y)$ is nowhere n-root-supermodular where $n = \frac{1}{1-e}$.*

What makes this result for the CES matching function particularly tractable is that the condition holds everywhere independent of the buyer-seller ratio. That implies that there is no gap between the range of match value functions $f(x, y)$ for which PAM is satisfied and the range where NAM is satisfied. No matter what the distribution of types is, if f satisfies the n-root-supermodularity condition, then there is PAM; if it holds nowhere, then there is NAM. These Propositions are thus a counterpart to Theorems 1 and 2 but with a much sharper prediction. Moreover, since Proposition 7 ensures NAM for any given distribution, it also provides a stronger counterpart for our result in Proposition 4 which establishes that there can be NAM even if the match value function exhibits moderate degrees of supermodularity. With CES this is independent of the distribution.

The class of CES matching functions spans the entire range of n-root-supermodularity, from supermodularity to log-supermodularity.

Corollary 2 *Let the matching function be CES with elasticity e. Then a necessary and sufficient condition for PAM is:*

1. *Supermodularity if $e = 0$ (Leontief);*
2. *Root-supermodularity if $e = \frac{1}{2}$ (Telegraph Line Matching);*
3. *Log-supermodularity if $e = 1$ (Cobb-Douglas)*

5.2 Sorting in a Dynamic Framework

The frictions that arise in directed search models are often justified by the cost of delay that arises from miscoordination. Like much of this literature, we have chosen to cast our model in a static form, thereby interpreting the possibility of no trade as a stark form of delay. Since it has not been our intention to derive results on fluctuations or dynamic comparisons, we have thus obtained our results without the additional machinery and notation of a dynamic setting and thereby not unduly distracted attention from the main economic issues.

In this section, we show that all the insights on assortative matching apply also in a dynamic framework with a stationary equilibrium. We first derive the equivalent of our main result on Positive Assortative Matching in a dynamic framework. In Section 6 we use the dynamic framework to discuss the consequences when frictions vanish.

Consider an economy with discrete time and an infinite time horizon. Agents discount the future with factor $\delta \in (0, 1)$. When an agent trades he is removed from the market, and for simplicity we assume that he is replaced by a new agent of identical type.[21] Therefore, distributions $B(x)$ and $S(y)$ are stationary by assumption. When the trading strategies and the queue lengths are stationary, a seller of type y who wants to trade at price p in one period would be willing to do so in all periods. Therefore his expected discounted life-time profit from attempting to trade at p is recursively given by

$$\begin{aligned} \pi_y(p) &= m(\lambda(p,y))p + \delta\left(1 - m(\lambda(p,y))\right)\pi_y(p) \\ \Leftrightarrow \pi_y(p) &= \frac{m(\lambda(p,y))}{1 - \delta\left(1 - m(\lambda(p,y))\right)}p. \end{aligned} \tag{35}$$

[21] When agents who leave the market are replaced by an identical type (often referred to as the "cloning" assumption), the logic of our existence proof extends to the dynamic version with suitably adjusted payoffs. In contrast, when there is an endogenous distribution of types, our constructive proof would have to keep track also of the speed with which the stock of agents of each type adjusts over time. It is well known in models of random search (see for example Shimer and Smith (2000) and Burdett and Coles (1997)) that this substantially complicates proofs of existence. An endogenous pool would not alter the characterization that we provide for positive assortative matching, though, because it is based on incentives of an individual who takes the steady-state conditions (exogenous or endogenous) as given. With an endogenous type distribution we would also require a probability that agents die between two periods to prevent unbounded accumulation of agents who cannot trade and would adjust the discount factor appropriately.

Similarly, the expected discounted life-time utility of a buyer of type x who attempts to trade at combination (p, y) is given by

$$\begin{aligned} u_x(p,y) &= q(\lambda(p,y))(f(x,y)-p)+\delta\,(1-q(\lambda(p,y)))\,u_x(p,y) \\ \Leftrightarrow u_x(p,y) &= \frac{q(\lambda(p,y))}{1-\delta\,(1-q(\lambda(p,y)))}(f(x,y)-p) \end{aligned} \tag{36}$$

With these adjustments of the profit and utility, we can define a stationary equilibrium exactly as in Definition 1.

Again let $U(x)$ be the Market Utility that buyers get in equilibrium, which again has to be increasing and continuous in equilibrium. Similar steps as in the main body reveal that sellers maximize profits as long as they promise the Market Utility to the buyer type they want to trade with. Again let $\tilde{\pi}(x, y)$ be the optimal profit of seller type y who trades with buyer type x, i.e. $\tilde{\pi}(x, y)$ is determined by

$$\max_{\lambda\in\overline{\mathbb{R}}_+} m(\lambda)\left[1-\delta\,(1-m(\lambda))\right]^{-1} p \tag{37}$$

$$\text{s.t.} \quad q(\lambda)\left[1-\delta\,(1-q(\lambda))\right]^{-1}(f(x,y)-p) = U(x), \tag{38}$$

and $\lambda^*(x, y)$ is again the argument that solves this program. In equilibrium the seller chooses the optimal x as well. We are interested in positive assortative matching, for which the crucial insight arises from the analogue to Lemma 2:

Lemma 8 *Consider an equilibrium with Market Utility $U(\cdot)$. For a buyer type y who trades with seller type $x \in (\underline{x}, \overline{x})$ with $U'(x)$ well defined at queue length $\lambda^* \in (0, \infty)$ the equilibrium expected profits satisfy $\frac{\partial^2\tilde{\pi}(x,y)}{\partial x\partial y} > 0$ if and only if*

$$f_{xy}(x,y) \geq A(\lambda^*,\delta)a(\lambda^*)\frac{f_x(x,y)f_y(x,y)}{f(x,y)}$$

where

$$A(\lambda,\delta) = \frac{\left[1-\delta+\delta q(\lambda)+\lambda(1-\lambda)\delta q'(\lambda)\right]\left[1-\delta\right]}{\left[1-\delta+m(\lambda)\delta\right]\left[1-\delta+\delta q(\lambda)\right]}$$

and as before

$$a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{m''(\lambda)q(\lambda)}.$$

Proof. See Appendix A. ■

Almost all active sellers make strictly positive profits, and $U(\cdot)$ increasing implies that it is differentiable almost everywhere. Therefore, the condition holds for almost all active sellers. For our purposes the following results are important and can easily be verified.[22]

[22] To see part 1, observe that with $q' \leq 0$ and $m \geq 0$, each of the factors in the numerator is smaller than the one of the factors in the denominator: $1-\delta+\delta q(\lambda)+\lambda(1-\lambda)\delta q'(\lambda) \leq 1-\delta+\delta q(\lambda)$ and $1-\delta \leq 1-\delta+m(\lambda)\delta$.

Corollary 3 *It holds that*

1. $A(\lambda, \delta) \in [0, 1]$ *for all* $(\lambda, \delta) \in \overline{\mathbb{R}}_+ \times [0, 1)$,

2. $\lim_{\lambda \to 0} A(\lambda, \delta) = 1$ *for all* $\delta \in [0, 1)$,

3. $\lim_{\delta \to 1} A(\lambda, \delta) = 0$ *for all* $\lambda > 0$.

As a result, the product $A(0, \delta)a(0) = 1/2$. Therefore, root-supermodularity is still the necessary condition for Positive Assortative Matching because it is again necessary when λ is close to zero. Moreover, $A(\lambda, \delta) \leq 1$ guarantees that root-supermodularity is also sufficient for Positive Assortative Matching. We discuss the case of vanishing frictions $\delta \to 1$ in more detail in the discussion section.

5.3 More General Payoff Structures

In the basic model we considered a setup in which buyers have private information about their type and the benefits of the match accrues only to them. We think that this setup is particularly relevant in goods market contexts, and our analysis shows that in equilibrium sellers are able to differentiate between buyers who sort according to price. It is straightforward to augment our basic setup by allowing sellers' payoffs to depend on their own type. That is, assume the same setup as in Section 3 but a seller of type y obtains payoff $f^s(y) + p$ that depends on the price and on his type.[23] Therefore, his expected payoff is $\pi_y(p) = m(\lambda(p, y)) [f^s(y) + p]$. Similar to the derivation that led to (9) we now arrive at an optimization problem for the seller of type y

$$\max_{x \in \mathcal{X}, \lambda \geq 0} m(\lambda) [f(x, y) + f^s(y)] - \lambda U(x). \tag{39}$$

Treating $\hat{f}(x, y) = f(x, y) + f^s(y)$ as the match value, our analysis carries over exactly as above.

Our results carry over even to a setup in which the buyer's type influences the sellers payoff, but only if we give up the assumption that buyers' types are unobservable. If one interprets the buyer's type as his productivity in a match with a seller, then in order to avoid "lemon" problems the price of the seller has to condition on the buyer's type. For such an extension, we undertake the following changes to our setup: The payoff for a seller type y who matches with a buyer type x at price p is $f^s(x, y) + p$, and the sellers strategy F is a distribution function over $\mathcal{P} \times \mathcal{X} \times \mathcal{Y}$. Also, the queue length depends on the price and the seller type, but also on the buyer type, i.e., we consider $\lambda(p, x, y)$.[24] Expected

[23] Hoppe, Moldovanu and Sela (2008) consider unobservable seller types in a very different model. They analyze signalling by sellers.

[24] The important feature is that the market conditions have to be contingent on all payoff-relevant information for each side of the market. If sellers' payoffs are not directly influenced by the buyer's type, then λ does not have to directly depend on the buyer's type, and therefore the buyer's type can be private information as in our baseline specification. A similar idea has recently been proposed by Mailath, Postlewaite and Samuelson (2006) for matching markets without frictions.

payoffs are then

$$
\begin{aligned}
u_x(p,y) &= q(\lambda(p,x,y))(f(x,y)-p), &(40)\\
\pi_y(p,x) &= m(\lambda(p,x,y))(f^s(x,y)-p), &(41)
\end{aligned}
$$

and we arrive naturally at the following extended equilibrium definition

Definition 4 *An equilibrium is a set of permissible distributions (F,G) such that there exists a queue length function $\lambda : \mathcal{P} \times \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}_+$ such that:*

1. *(Profit Maximization) $\forall (p,x,y) \in \hat{F} : \pi_y(p,x) \geq \pi_y(p',x') \quad \forall (p',x') \in \mathcal{P} \times \mathcal{X}$.*
2. *(Utility Maximization) $\forall (p,x,y) \in \hat{G} : u_x(p,y) \geq u_x(p',y') \quad \forall (p',y') \in \mathcal{P} \times \mathcal{Y}$.*
3. *(Consistency of the Queue Length) For all $p \in \mathcal{P}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$*

$$
S(\overline{y}) \int_{[0,p]\times[\underline{y},y]\times[\underline{x},x]} \lambda(\cdot,\cdot,\cdot)dF = G(p,x,y). \tag{42}
$$

For a given Market Utility $U(x)$ the MU Assumption now requires that the queue length is given by the indifference $q(\lambda(p,x,y))[f(x,y)-p] = U(x)$ if $f(x,y)-p \geq U(x)$. Again this condition does not have bite when $f(x,y)-p = U(x) = 0$, for which we define $\lambda(p,x,y) = \infty$. If $f(x,y)-p < U(x)$ then buyers would not like to trade, which is captured by $\lambda(p,x,y) = 0$. Under the market utility assumption, similar steps to those in (6) to (9) lead to a maximization problem for the seller of type y:

$$
\max_{x\in\mathcal{X},\lambda\geq 0} m(\lambda)\left[f(x,y)+f^s(x,y)\right] - \lambda U(x). \tag{43}
$$

Because of this structure, all our subsequent derivations carry over using the joint match value $\hat{f}(x,y) = f(x,y) + f^s(x,y)$.

6 Discussion

The main interest of this paper has been to provide a characterization of the patterns of sorting in a world with non-neglegible frictions. Nonetheless, a common theme in the literature is to investigate whether those economies converge to the Walrasian outcome as frictions vanish. In the context of the dynamic version of our model, we can investigate whether the limit with vanishing discounting frictions converges to the competitive allocation. This is the approach taken in the large literature on sequential bargaining and competition (see amongst many others Rubinstein and Wolinsky (1985), Gale (1986) and recently Lauermann (2007)). Alternatively, even in the static economy we can consider matching functions that embody relatively little frictions, and consider sequences of those functions that converge to the frictionless case as in Becker (1973). Here, we will consider each case in turn, starting with the limit of the dynamic economy.

For our purposes, the question is whether in the competitive limit we can reconcile the fact that root-supermodularity is the requirement for Positive Assortative Matching in the model with frictions, with the fact that in the frictionless model (Becker 1973) that requirement is mere supermodularity.

The limit of the dynamic economy with vanishing frictions. In the dynamic version of our model, the requirement for Positive Assortative Matching is

$$f_{xy}f \geq A(\lambda,\delta)a(\lambda)f_xf_y,$$

where the strength of the condition depends on the value of $A(\lambda,\delta)a(\lambda)$. From Corollary 3, for given $\lambda > 0$ we have, when the discount factor converges to unity, that:

$$\lim_{\delta\to 1} A(\lambda,\delta)a(\lambda) = 0.$$

Therefore, whenever all sellers can trade with non-negligible probability, simple supermodularity is sufficient to obtain Positive Assortative Matching with very patient players. In those dynamic markets with supermodular match value functions, the standard condition by Becker applies for the allocation of types.

Nevertheless, root-supermodularity remains a stringent requirement to guarantee Positive Assortative Matching for all active agents even when players are long-lived and become infinitely patient. It remains binding in economies where some buyers cannot trade due to uneven market sides. In particular, for any δ we have shown in Corollary 3 that

$$\lim_{\lambda\to 0} A(\lambda,\delta)a(\lambda) = \frac{1}{2}.$$

Therefore if some sellers remain without trading opportunities, root-supermodularity is necessary to achieve Positive Assortative Matching of those seller types with small trading probabilities. The intuition why root-supermodularity remains important is immediate: If some sellers cannot trade, then even with very high discount factors those sellers who are close to not being able to trade face increasingly long waiting times that outweigh even high levels of patience.

The limit of the static economy with vanishing frictions. Our condition for positive assortative matching depends on the properties of the general matching function m because it defines $a(\lambda)$. A frictionless model à la Becker can be represented by the frictionless matching function $m_B(\lambda) = \min\{\lambda, 1\}$ indicating that when sellers are on the long side of the market ($\lambda < 1$), a match for the sellers realizes with probability λ while buyers trade with probability 1. When sellers are on the short side ($\lambda > 1$), a match for the sellers realizes with probability 1. This frictionless matching probability is weakly increasing and weakly convex.

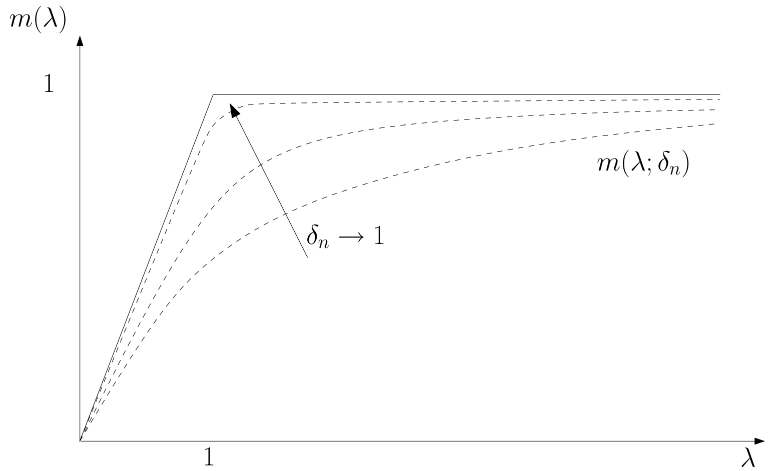

Figure 3: Vanishing Frictions for the Static Matching Function

In our setup we require smoothness of the matching function. In particular we require m to be strictly increasing and strictly convex. We can nonetheless consider the limit of a sequence of matching functions that converges to m_B as illustrated in Figure 3. For example, in the case of the logarithmic matching function $m_4(\lambda) = 1 - \ln(1 + e^{(1-\lambda)/(1-\delta)})/\ln(1 + e^{1/(1-\delta)})$ with $\delta \in (0,1)$ it can readily be shown that $\lim_{\delta\to 1} m_4(\lambda) = \min\{\lambda, 1\}$. Our condition for positive assortative matching

$$f_{xy}f \geq a(\lambda,\delta) f_x f_y$$

entails a factor $a(\lambda,\delta)$ that depends on δ since the matching function depends on δ. The strength of the condition depends on the value of $a(\lambda,\delta)$: if it is zero, then PAM obtains under supermodularity, if it is 1/2 then PAM obtains under root-supermodularity. Some algebra establishes that

$$a(\lambda,\delta) = \left(1 + e^{\frac{1-\lambda}{1-\delta}}\right)\frac{1-\delta}{\lambda} - e^{\frac{1-\lambda}{1-\delta}}\left(\ln(1 + e^{\frac{1}{1-\delta}}) - \ln(1 + e^{\frac{1-\lambda}{1-\delta}})\right)^{-1}.$$

Since m_4 fulfills our assumptions for any $\delta \in (0,1)$, we have

$$\lim_{\lambda\to 0} a(\lambda,\delta) = 1/2 \quad \text{for all } \delta \in (0,1),$$

and therefore root-supermodularity remains a necessary and sufficient condition for Positive Assortative Matching for any sequence of δ's such that $m_4 \to m_B$ pointwise.

This shows the robustness of our root-supermodularity condition. The condition arises from considering $\lambda \to 0$. For given frictions – i.e., a given matching function – we need strong supermodularity conditions for all queue length because $a(\lambda) > 0$ for all $\lambda \in (0,\infty)$. With vanishing frictions the strength of the root-supermodularity condition comes from the region where $\lambda \approx 0$ only. That is, it arises only when some sellers are not able to trade even when the frictions disappear, for example due to uneven market sides. If this is not the case, i.e., if all sellers can trade with probability bounded away from zero along a sequence of δ's such that $m_4 \to m_B$, then the standard supermodularity condition of Becker's frictionless analysis emerges. In particular, some tedious application of De l'Hôpital's rule reveals

$$\lim_{\delta \to 1} a(\lambda, \delta) = 0 \quad \text{for all } \lambda > 0.$$

Therefore, when the trading probability of all sellers remains bounded away from zero simple (strict) supermodularity induces Positive Assortative Matching for δ sufficiently large.

7 Appendix

7.1 Appendix A: Proofs

Lemma 9 *In equilibrium $U(x)$ is continuous, weakly increasing, and strictly increasing on $(x_0, \overline{x}]$. $\Pi(y)$ equals zero a.e. on $[\underline{y}, y_0)$, and is strictly increasing and continuous on $(y_0, \overline{y}]$.*

Proof. Equilibrium condition (II) can be rewritten as maximizing (1) over (p, y) for given parameter x and we can apply the envelope theorem. This implies continuity since $f(x, y)$ is continuous. $U(x)$ is increasing since type $x' > x$ can trade at exactly the same (p, y) as type x, but $f(x, y)$ is strictly increasing. This relationship is strict when buyers make strictly positive profits.

For $\Pi(y)$ the argument is more complicated as seller type y faces different queue length $\lambda(p, y)$ than a seller y' that offers the same price. To prove the claim, assume that strictly increasing profits are violated, i.e. there exist (p, y) and (p', y') in $\hat{F}$ such that $y > y' > y_0$ but $m(\lambda(p, y))p \leq m(\lambda(p', y'))p'$. $y > y_0$ implies $0 < m(\lambda(p, y))p$ which implies that queue length and price are strictly positive. By Profit Maximization $m(\lambda(p', y))p' \leq m(\lambda(p, y))p$, and so $m(\lambda(p', y))p' \leq m(\lambda(p', y'))p'$, which in turn implies $\lambda(y, p') \leq \lambda(y', p')$. But this violates Utility Maximization (II) for the buyer type x' who trades with (p', y'), because x' could trade with a better seller type at a weakly lower queue at equal price. [Such a buyer type exists: Since (p', y') is in the support of F, any neighborhood of (p', y') has positive measure under F. Since the firms make strictly positive profits in a neighborhood of y' (by $y' > y_0$) their queue lengths are strictly positive, so by Consistency (III) there has to exist $x' \in \mathcal{X}$ such that $(p', x', y') \in \hat{G}$.] A discontinuity in profits at type y compared to nearby types y' can be ruled out by a similar argument: $\lambda(y, p')$ must be discontinuously lower than $\lambda(y', p')$ for nearby types, which makes it strictly preferable for buyers to trade with type y. ■

Proof of Lemma 1:

Before proving the result, we briefly lay out the intuition. For buyers almost any equilibrium combination $(p, x, y) \in \hat{G}$ has a queue that delivers the Market Utility [where exceptions might be due to inactive types], and therefore the queue lengths λ and $\tilde{\lambda}$ coincide at almost all (p, y) in the support of their trading strategy. By the definition of Market Utility no buyer type can obtain a utility larger than $U(.)$ in the market, and therefore Utility Maximization (II) holds. Since active sellers trade with buyers, they have the same support and again λ and $\tilde{\lambda}$ coincide at almost all (p, y) combinations in $\hat{F}$. Since the queue lengths coincide a.e. on the support, we have Consistency (III). Moreover, the MU assumption leads to the lowest queue length that can support an equilibrium because any lower queue would lead a buyer to deviate: $\tilde{\lambda}(p, y) \leq \lambda(p, y)$ holds a.e. for all (p, y). Therefore sellers have less incentives under $\tilde{\lambda}$ to deviate from the equilibrium support than under λ and Profit Maximization (III) holds. The presence of inactive types implies the qualification of "almost everywhere", and the main difficult of the proof is to show that the result applies everywhere.

Proof. Consider a trading constellation (F, G) that satisfies equilibrium conditions (I) - (III) under queue length function λ and that yields equilibrium profits and utilities $\Pi(\cdot)$ and $U(\cdot)$. We will show that (F, G) also satisfies equilibrium conditions (I) - (III) under queue length function $\tilde{\lambda}$ conforming to the MU Assumption. The proof is structured in a sequence of steps. Let $\mathcal{G}(p, y) = G(p, \overline{x}, y)$ be the marginal distribution over (p, y).

Step 1. a) We show: If $U(x) = 0$ for some $x \in \mathcal{X}$, then $\Pi(y) > 0$ $\forall y \in \mathcal{Y}$.
Proof: Assume $U(x) = 0$ for some $x \in \mathcal{X}$. Since f is strictly positive for any $y \in \mathcal{Y}$, there exists strictly positive $p \in \mathcal{P}$ such that $f(x, y) - p > 0$. Then $U(x) = 0$ only if $\lambda(p, y) = \infty$. But this implies $\Pi(y) > 0$.

b) We show: If $\Pi(y) = 0$ for some $y \in \mathcal{Y}$, then $U(x) > 0$ $\forall x \in \mathcal{X}$.
Proof: Assume $\Pi(y) = 0$ for some $y \in \mathcal{Y}$. By contradiction, assume there exists $x \in \mathcal{X}$ such that $U(x) = 0$. But since $f(x, y) - p' > 0$ for some $p' > 0$, we have to have $\lambda(p', y) = \infty$ and therefore $\Pi(y) > 0$.

Step 2. a) Note that $\Pi(y) > 0$ $\forall y \in \mathcal{Y}$ implies that $\hat{F} \subseteq \hat{\mathcal{G}}$ by Consistency (III) of the queue length and the fact that all $(p, y) \in \hat{F}$ have strictly positive queue length by $\Pi(y) > 0$. Similarly, if $U(x) > 0$ for all $x \in \mathcal{X}$ then $\hat{\mathcal{G}} \subseteq \hat{F}$.

b) If $\Pi(y) = 0$ for some $y \in \mathcal{Y}$ such a relationship still holds for firms with $y > y_0$. Let $F_{>y_0}$ be the distribution conditional on values being in $\{(p, y) \in \mathcal{P} \times \mathcal{Y} | y > y_0\}$. It holds that $\hat{F}_{>y_0} \subseteq \hat{\mathcal{G}}$ again by Consistency (III). Similarly, let $\mathcal{G}_{>x_0}$ be the conditional distribution of G, conditional on $x > x_0$. Then $\hat{\mathcal{G}}_{>x_0} \subseteq \hat{F}$.

Step 3. We show: $\lambda(p, y) \geq \tilde{\lambda}(p, y)$ $\forall (p, y)$; except possibly when $U(x_0) = 0$ and (p, y) is such that $f(x_0, y) - p = 0$. [The latter possibility is a special case because any queue length at (p, y) yields type x_0 zero profits, and the queue length at (p, y) is not tied down by possible deviations of lower buyer types as lower types find the price unattractive at any queue length, nor it is necessarily tied down by deviations by higher buyer types.]
Proof: Consider some $(p, y) \in \mathcal{P} \times \mathcal{Y}$.

Case 3.1: There exists $x \in \mathcal{X}$ with $f(x, y) - p \geq U(x)$ and $U(x) > 0$. Then $\lambda(p, y) \geq \tilde{\lambda}(p, y)$ as otherwise $q(\lambda(p, y))[f(x, y) - p] > q(\tilde{\lambda}(p, y))[f(x, y) - p] = U(x)$, which means that Buyer Optimality (II) is violated as the buyers equilibrium utility is lower than what he could have gotten by trading at (p, y).

Case 3.2: There exists $x \in \mathcal{X}$ with $f(x, y) - p > 0$ and $U(x) = 0$. For $U(x) = 0$ Buyer Optimality (II) requires $\lambda(p, y) = \infty$. Moreover $\lambda_x(p, y) = \infty$ and $\tilde{\lambda}(p, y) \geq \lambda_x(p, y)$ by definition.

Case 3.3: There exists $x \in \mathcal{X} \backslash \{x_0\}$ with $f(x, y) - p = 0$ and $U(x) = 0$. By continuity of $U(x)$ and definition of x_0 we have $x < x_0$. But by $f_x > 0$ there exists $x' \in (x, x_0)$ such that $f(x', y) - p > 0$ and $U(x') = 0$, which is Case 3.2.

Case 3.4: For all $x \in X$ it holds that $f(x_0, y) - p < U(x)$. Then $\lambda_x(p, y) = 0$ for all $x \in \mathcal{X}$ and thus $\tilde{\lambda}(p, y) = 0$, which is trivially weakly smaller than $\lambda(p, y)$. This exhausts all cases.

Step 4. We show: If $(p, x, y) \in \hat{G}$, then $\lambda_x(p, y) = \tilde{\lambda}(p, y) = \lambda(p, y)$.
Proof: Consider some $(p, x, y) \in \hat{G}$.

Case 4.1: $U(x) = 0$ and $f(x, y) - p > 0$. By Case 3.2 it holds that $\lambda(p, y) = \infty$. Moreover, by definition $\lambda_x(p, y) = \infty$, and $\tilde{\lambda}(p, y) \geq \lambda_x(p, y)$ then yields $\tilde{\lambda}(p, y) = \infty$.

Case 4.2: $U(x) = 0$ and $f(x, y) - p = 0$ and $x < x_0$. By Case 3.3 this reduces to Case 4.1.

Case 4.3: $U(x) = 0$ and $f(x, y) - p = 0$ and $x = x_0$. We show that $\lambda(p, y) = \infty$. By Step 2b) $\hat{\mathcal{G}}_{>x_0} \subseteq \hat{F}$ and therefore (by the closure property of the support and the fact that each type has zero measure) $\hat{\mathcal{G}}_{\geq x_0} \subseteq \hat{F}$. Therefore, there exists $(p', y') \in \hat{F}$ such that $(p', x_0, y') \in \hat{G}$. By Step 1a) we have $\Pi(y') > 0$, and by definition $q(\lambda(p', y'))p' = \Pi(y')$. We know that at all prices p'' strictly below p' it holds that $\lambda(p'', y') = \infty$ for buyer type x_0 to make zero utility. Therefore, $\lambda(p', y') = \infty$, as otherwise $q(\lambda(p', y'))p' < q(\lambda(p'', y'))p'' = q(\infty)p''$ for some $p'' < p'$, violating Profit Maximization (I).

Case 4.4: $U(x) > 0$. Since $U(x) = q(\lambda(p, y))[f(x, y) - p]$ this uniquely identifies $\lambda(p, y)$ as a function of $U(x)$, and the same relationship determines $\lambda_x(p, y)$, yielding $\lambda(p, y) = \lambda_x(p, y)$. By definition, $\tilde{\lambda}(p, y) \geq \lambda_x(p, y)$. Moreover, by Step 3 $\lambda(p, y) \geq \tilde{\lambda}(p, y)$ $\forall (p, y)$; except possibly when $U(x_0) = 0$ and (p, y) such that $f(x_0, y) - p = 0$.

If that exception is indeed present we know that $\lambda_{x_0}(p, y) = \tilde{\lambda}(p, y) = \infty$, and by a logic similar to Case 4.1 we can show that $\lambda(p, y) = \infty$, in which case the proof is complete. To see the latter note that by Step 1a) the seller type y makes strictly positive profits and by Step 2a) $(p, y) \in \hat{F}$. For $p' < p$ we have $\lambda(p', y) = \infty$ for buyer type x_0 to obtain zero utility, and therefore again $\lambda(p, y)$ finite will violate Profit Maximization (I) of seller type y as he will have a discrete jump in trading probability by trading at $p - \varepsilon$ rather than at p if $\lambda(p, y)$ finite.

Step 5: Checking Utility Maximization (II) under $\tilde{\lambda}$.
By Step 4 for any $(p, x, y) \in \hat{G}$ the queue length $\tilde{\lambda}(p, y)$ and $\lambda(p, y)$ coincide, and therefore the utility on the equilibrium path is the same. There is no profitable deviation for buyers because by construction of $\tilde{\lambda}(p, y)$ no buyer can obtain a higher expected utility than $U(x)$ under $\tilde{\lambda}(p, y)$.

Step 6: Checking Profit Maximization (I) under $\tilde{\lambda}$.

Case 6.1: $\Pi(y) > 0$ $\forall y \in \mathcal{Y}$. By Step 2a) $\hat{F} \subseteq \hat{\mathcal{G}}$, and therefore by Step 4 we have $\tilde{\lambda}(p, y) = \lambda(p, y)$ for all $(p, y) \in \hat{F}$. Therefore profits at these combinations are identical under either queue length. By Step 3 at any other (p', y) we have $\lambda(p', y) \geq \tilde{\lambda}(p', y)$ and therefore these "deviation" prices are even less attractive under $\tilde{\lambda}$.

The last argument does not apply to (p', y) if $f(x_0, y) - p' = 0$ and $U(x_0) = 0$. In this situation $\tilde{\lambda}(p', y) = \infty$ and a deviation for the seller would be optimal if $q(\infty)p' > \Pi(y)$. Yet for any $p'' < p$ it holds that $\tilde{\lambda}(p'', y) = \infty$ [since $f(x_0, y) - p'' > 0$ and $U(x_0) = 0$]. If indeed $q(\infty)p' > \Pi(y)$, then

$q(\infty)p'' > \Pi(y)$ for p'' close enough to p'. Yet we have shown in the previous paragraph that p'' cannot be a profitable deviation.

Case 6.2: $\Pi(\tilde{y}) = 0$ for some $\tilde{y} \in \mathcal{Y}$. Consider Profit Maximization (I) for some $y \in \mathcal{Y}$.

Sub-Case 6.2i) $y > y_0$. By Step 2b) $y \in \hat{F}_{>y_0} \subseteq \hat{\mathcal{G}}$, and the argument follows Case 4.1.

Sub-Case 6.2ii) $y < y_0$; or $y = y_0$ and $\Pi(y_0) = 0$. There exists $y' \in (y, y_0]$ such that $\Pi(y') = 0$ under λ. Therefore $\lambda(p, y') = 0$ $\forall p > 0$. By the combination of Step 1b) and Step 3 we also have $\tilde{\lambda}(p, y') = 0$ $\forall p > 0$. Since $\tilde{\lambda}(p, y')$ is increasing in y, we have $\tilde{\lambda}(p, y) = 0$ $\forall p > 0$. Therefore, type y makes zero profit at any combination (p, y) and therefore Profit Maximization (I) is fulfilled.

Sub-Case 6.2iii) $y = y_0 > \underline{y}$ and $\Pi(y_0) > 0$. Because of the compactness of $\mathcal{P} \times \mathcal{Y}$ there exists a sequence $(p_n, y_n) \in \hat{F}_{\geq y_0}$ such that $y_n > y_0$ and $\lim_{n\to\infty} y_n = y_0$ and such that $\lim_{n\to\infty} p_n = p'$, where $\hat{F}_{\geq y_0}$ is the support of the conditional distribution of F conditional on combinations (p, y) with $y \geq y_0$. By Step 2b) $\hat{F}_{>y_0} \subseteq \hat{\mathcal{G}}$, and by the closedness of the support also $(p', y_0) \in \hat{F}_{\geq y_0} \subseteq \hat{\mathcal{G}}$. But then $\lambda(p', y_0) \in \hat{F}$, and therefore $\Pi(y_0) > 0$ implies $\lambda(p', y_0) > 0$. Yet for any $\varepsilon > 0$ there exist $y' \in (y' - \varepsilon, y')$ such that $\lambda(p', y') = 0$, and therefore the buyers trading $(p', y_0) \subseteq \hat{\mathcal{G}}$ would make higher utility trading at (p', y'). This violates Utility Maximization (II), and therefore this sub-case cannot arise when λ supports G in equilibrium.

Sub-Case 6.2iv) $y = y_0 = \underline{y}$ but $\Pi(y_0) > 0$. This coincides with Case 4.1.

Step 7: Checking Consistency (III) under $\tilde{\lambda}$.
By Step 4 the queue lengths on $\hat{\mathcal{G}}$ are identical, and by Step 2 they are identical on $\hat{F}_{>y_0}$. On $\hat{F}_{<y_0}$ the queue lengths are almost everywhere equal to zero under both λ and $\tilde{\lambda}$. Since queue lengths coincide almost everywhere on the support F and $\mathcal{G}$, Consistency (III) is fulfilled. ■

Proof of Proposition 1:

Proof. Consider some match value function f that is not root-supermodular at some $(\hat{x}, \hat{y})$. We will show that there are distributions for which positive assortative matching cannot be an equilibrium outcome. By the smoothness properties of f there exists $\varepsilon > 0$ such that f is not root-supermodular anywhere on $Z_\varepsilon = [\hat{x}-\varepsilon, \hat{x}+\varepsilon] \times [\hat{y}-\varepsilon, \hat{y}+\varepsilon]$. In fact, ε can be chosen such that $f_{xy}(x, y) - \alpha \frac{f_x(x,y) f_y(x,y)}{f(x,y)} < 0$ for all $(x, y) \in Z_\varepsilon$ for some $\alpha < 1/2$. By the continuity of $a(\lambda)$ we can find $\hat{\lambda}$ such that $a(\lambda) > \alpha$ for all $\lambda \in [0, \hat{\lambda}]$. The proof is complete once we show that there are type distributions for which a positive measure of seller types in $[\hat{x} - \varepsilon, \hat{x} + \varepsilon]$ trades with buyer types in $[\hat{y} - \varepsilon, \hat{y} + \varepsilon]$ at queues below $\hat{\lambda}$, because for some of these types their choices have to be characterized by their first order conditions and the condition for Positive Assortative Matching in Lemma 2 cannot be satisfied.

Consider a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, $0 < \varepsilon_n < \varepsilon$, that monotonically converges to zero. Let B_n and S_n be associated sequences of distributions of buyer and seller types. Let B_n be uniform with support on $[\underline{x}_n, \bar{x}_n] = [\hat{x} - \varepsilon_n, \hat{x} + \varepsilon_n]$ and unit mass: $B_n(\hat{x} + \varepsilon_n) = 1$. Let S_n be uniform with support on $[\underline{y}_n, \bar{y}_n] = [\hat{y} - \varepsilon_n, \hat{y} + \varepsilon_n]$ with mass $S_n(\hat{y} + \varepsilon_n) = \frac{1}{\varepsilon_n}$ (observe that the mass is increasing in the sequence). By construction the buyer-seller pairs that match are within Z_ε for any n. We want to show that for some n a strictly positive measure of sellers trade at a queue length in $(0, \hat{\lambda})$. The remainder of the proof has two parts. First, we show that the measure of sellers that trade at queues above $\hat{\lambda}$ goes to zero. Second, we show that the measure of sellers that trade at a queue above zero remains bounded away from zero. This completes the proof.

We show the first part by contradiction: Assume there exists a subsequence such that the measure of sellers that trade at queue length above $\hat{\lambda}$ is larger than some $\xi > 0$. Then, by Consistency (III), there is a non-vanishing measure of buyers that trades with these sellers. These buyers make at most a utility $q(\hat{\lambda})f(\bar{x}_n, \bar{y}_n)$, which converges to $q(\hat{\lambda})f(\hat{x}, \hat{y})$. But then *all* sellers are able to ensure themselves profits bounded away from zero: They can offer a price $p_n > 0$ and queue length $\frac{\hat{\lambda}}{2}$ that provides at least this utility to buyers, for example by choosing p_n such that even for the worst types $q(\frac{\hat{\lambda}}{2})\,[f(\underline{x}_n, \bar{x}_n) - p_n] = q(\hat{\lambda})f(\hat{x}, \hat{y})$. Such a strictly positive p_n exists for n large since the support converges. Therefore, all sellers could ensure themselves profits that are bounded from zero despite their increasing mass by the MU assumption. But this requires a queue for all sellers that is bounded away from zero, which is not possible because there are few buyers relative to sellers and Consistency (III) would be violated.

We show the second part also by contradiction: Assume that the measure of sellers who trade at a strictly positive queue goes to zero. As the measure of active sellers becomes much smaller than unity, the trading probability for some buyer types must go to zero, which implies that their Market Utility goes down to zero. [Mathematically, Consistency (III) can only be satisfied when $\lambda(p, y)$ goes to infinity for some (p, y) on the support of $G(., \bar{x}, .)$.] But then any seller can profitably attract these buyers by the MU assumption. Therefore, profits (and thus queue lengths) of all sellers remain bounded away from zero, yielding a contradiction to the assumption. ■

Proof of Proposition 2:

Proof. Strict convexity of $1/q$ is equivalent to $a(\lambda) < 1/2$ when $\lambda > 0$ from Lemma 5. Fix an equilibrium with associated buyer utility $U(.)$ and seller profit $\Pi(.)$. We focus on active buyers as only these are relevant for assortative matching. Almost all of these trade at an interior queue length in $[0, \infty]$, and therefore almost all of their trading partners are active sellers. [Seller types below y_0 have zero queue length a.e. as shown in the proof of the previous proposition.] We can therefore focus on active seller types.

Given $U(\cdot)$ sellers solve maximization problem (9). For an active seller type y let M_y be the set of buyer types who maximize his profits (when combined with the optimal queue length $\lambda^*(x, y)$). Since the sellers' problem is continuous, $\max M_y$ and $\min M_y$ exist.

We consider the global incentives for assortative matching by showing that $y' > y$ implies $\min M_{y'} \geq \max M_y$. This then implies $\min M_{y'} > \max M_y$ – i.e. strictly positive assortative matching. Otherwise all types (y', y) have $\max M_y$ as their only maximizer, which implies zero profits for all these seller types because by Consistency (III) their queue length is zero and violates that these sellers are active.

To see the result, let $\hat{x} = \max M_y$. To avoid discussions of differentiability of the buyers' utility, we will endow buyer types in $[\underline{x}, \hat{x})$ with a hypothetical utility $\tilde{U}(x)$ such that $\pi(y) = \max_{(x,\lambda)} m(\lambda)f(x,y) - \lambda\tilde{U}(x)$. [Note that $\tilde{U}(x)$ might be negative for some types. Let $\check{x}$ be the lowest type with $\tilde{U}(x) \geq 0$.] That means that under $\tilde{U}(x)$ seller type y is indifferent between all x weakly below $\hat{x}$. Moreover, since by construction $U(x) \geq \tilde{U}(x)$ all x in $[\check{x}, \hat{x}]$ are more attractive under utility $\tilde{U}(x)$ than under $U(x)$ for *any* seller type. Note that $\tilde{U}(x)$ is differentiable by construction on $[\check{x}, \hat{x}]$. Now we can apply Lemma 2 given $\tilde{U}(x)$, establishing that under root-supermodularity higher type strictly prefer $\hat{x}$ over any type in $[\check{x}, \hat{x})$. ■

Proof of Proposition 4:

Proof. Fix the matching function m. Consider for example a sequence of functions f_n of the form $f_n(x,y) = (x+y+1)^2 - \varepsilon_n$, with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. As in the proof of Proposition 3, consider the sequence of distributions B_n and S_n with support on $[0, \varepsilon_n]$ and a unit measure of sellers for all $n \in \mathbb{N}$. This function is weakly root-supermodular in the limit but not for any finite n. By the same logic as in Proposition 3 the buyers' utilities are bounded below $1 - \alpha$ for some $\alpha > 0$. This implies again that the optimal queue length $\lambda_n^*(x,y)$ remains bounded below some $\overline{\lambda}$. It also implies that profits are bounded below by some $\beta > 0$ as an immediate consequence of the sellers' maximization problem (9). Then then queue length λ_n^* are bounded away below by some $\underline{\lambda} > 0$ for all (x,y). Since q^{-1} is strictly convex, $a(\lambda) < 1/2$ for $\lambda > 0$ and by continuity $\overline{a} = \max_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} a(\lambda) < 1/2$. Therefore, $\frac{\partial^2}{\partial x \partial y} f_n(x,y) < a(\lambda_n^*(x,y)) \frac{\partial}{\partial x} f_n(x,y) \frac{\partial}{\partial y} f_n(x,y) / f(x,y)$ holds when $\frac{\partial^2}{\partial x \partial y} f_n(x,y) < \overline{a} \frac{\partial}{\partial x} f_n(x,y) \frac{\partial}{\partial y} f_n(x,y) / f(x,y)$ for all $(x,y) \in [0, \varepsilon_n]^2$, which holds for some n large. Therefore, only positive assortative matching can be sustained. ■

Proof of Lemma 6:

Proof. Equations (27) and (30) together constitute a differential equation system $z' = F(z,x)$ where $z' = (\mu', \tilde{\lambda}')$, $z = (\mu, \tilde{\lambda})$ and F is defined by the right hand sides of equations (27) and (30). Let F^1 be the right hand side of (27) and F^2 the right hand side of (30). Boundary conditions are $\mu(\overline{x}) = \overline{y}$ and $\tilde{\lambda}(\overline{x}) = \overline{\lambda} \in (0, \infty)$. Since F is C^1 on $\mathcal{Y} \times (0,\infty) \times \mathcal{X}$ it is locally Lipschitz around the initial condition, and a unique solution $z(x)$ exists in some interval around the initial condition by Picard's existence theorem. Let $z(x)$ in fact be the unique maximal solution, defined on the maximal set $(x_0\left(\overline{\lambda}\right), \overline{x}]$. The lower bound $x_0\left(\overline{\lambda}\right)$ arises when one of the variables leaves the bound of its domain or diverges. It is given by the first occurrence of either one of the following conditions (see De la Fuente (2000), Theorem 6.12):

(i) $x_0\left(\overline{\lambda}\right) = \underline{x}$, or

(ii) $\mu(x_0\left(\overline{\lambda}\right)) = \underline{y}$, or

(iii) $\tilde{\lambda}(x_0\left(\overline{\lambda}\right)) := \lim_{x \searrow x_0(\overline{\lambda})} \tilde{\lambda}(x) = 0$, or

(iv) $\tilde{\lambda}(x_0(\overline{\lambda})) := \lim_{x \searrow x_0(\overline{\lambda})} \tilde{\lambda}(x) = \infty$,

where more than one condition can apply at the same time. Note that $x_0(\overline{\lambda})$ and $z(x)$ are functions of $\overline{\lambda}$, i.e., $x_0(\overline{\lambda})$ and $z(x,\overline{\lambda})$. Since F is continuous and locally Lipschitz on $\mu \in \mathcal{Y}$, $\tilde{\lambda} \in (0,\infty)$ and $x \in \mathcal{X}$, $z(x,\overline{\lambda})$ is continuous in $\overline{\lambda}$ on $(x_0(\overline{\lambda}), \overline{x}]$, and therefore $x_0(\overline{\lambda})$ is continuous in $\overline{\lambda}$ (De la Fuente (2000), Theorem 6.20). Let $R(\overline{\lambda}) \subseteq \{(i),(ii),(iii),(iv)\}$ be the subset of the conditions (i), (ii), (iii) and (iv) that hold at $x_0(\overline{\lambda})$. Continuity of $z(x,\overline{\lambda})$ on $(x_0(\overline{\lambda}), \overline{x}]$ implies upper-hemicontinuity of $R(\overline{\lambda})$.

$R(\overline{\lambda})$ is non-empty because the differential equation terminates latest because of (i). A necessary condition for an equilibrium is that the differential equation either pairs all buyers and sellers, i.e. $R(\overline{\lambda}) \in \{\{(i),(ii)\},\{(i),(ii),(iii)\},\{(i),(ii),(iv)\}\}$, or the sellers get rationed and the boundary seller makes zero profits, i.e. $R(\overline{\lambda}) = \{(i),(iii)\}$, or the buyers get rationed and the boundary buyer makes zero utility, i.e. $R(\overline{\lambda}) = \{(ii),(iv)\}$.[25] The remaining part of the proof in the main body presents the existence proof under the condition that one of the joint boundary conditions can be satisfied. We therefore have to show that for some $\overline{\lambda}$ we have

$$R(\overline{\lambda}) \in \{\{(i),(ii)\},\{(i),(iii)\},\{(ii),(iv)\},\{(i),(ii),(iii)\},\{(i),(ii),(iv)\}\}.$$

Showing this takes several steps.

Step 1: Observe that there exists $\underline{r}$ such that for all $\tilde{\lambda} < \underline{r}$, $\mu \in \mathcal{Y}$ and $x \in \mathcal{X}$ we have $F^2(\tilde{\lambda},\mu,x) > 0$, i.e. we have $\tilde{\lambda}'$ strictly positive. Similarly, observe that there exists $\overline{r}$ such that for all $\lambda > \overline{r}$, $\mu \in \mathcal{Y}$ and $x \in \mathcal{X}$ we have $F^2(\tilde{\lambda},\mu,x) < 0$. The first observation follows immediately since $m'(0) > 0$, $q'(0) < 0$ and $m''(0) < 0$ and $f_x, f_y, f, b(.)$ and $s(.)$ are bounded away from zero. The second observation follows from the fact that the elasticities of the matching function add to unity, i.e. $\lambda m'(\lambda)/m(\lambda) - \lambda q'(\lambda)/q(\lambda) = 1$, which implies $\lambda m'(\lambda) < m(\lambda)$. The assumption that $1/q$ is convex ensures that the elasticity $\lambda m'(\lambda)/m(\lambda)$ is decreasing, and therefore $\lambda m'(\lambda) < m(\lambda)$ holds strictly in the limit when λ grows large. Since $m(\lambda) = \lambda m'(\lambda) - \lambda^2 q'(\lambda)$, we have $-\tilde{\lambda}^2 q'(\tilde{\lambda})$ bounded away from zero. Since $m'(\lambda)$ converges to zero for λ large as $m(\lambda)$ is bounded, this ensures the second observation.

Step 2: For $\overline{\lambda}$ sufficiently small we have $R(\overline{\lambda}) \subseteq \{(ii),(iii)\}$. This arises because at $\bar{x}$ the term $\tilde{\lambda}'$ is positive and therefore the queue length declines even more at lower x, and for low market tightness μ' is large and therefore seller types are matched very quickly. Therefore, clearly neither (i) nor (iv) can be the limit to which the differential equation can be extended.

Similarly, for sufficiently high $\overline{\lambda}$ we have $R(\overline{\lambda}) \subseteq \{(i),(iv)\}$. This arises because at $\bar{x}$ the term $\tilde{\lambda}'$ is negative and therefore the queue length rises even higher at lower x, and for high market tightness μ' is small and therefore seller types are matched very slowly. Therefore, clearly neither (ii) nor (iii) can be the limit to which the differential equation can be extended. Neither $\{(ii),(iii)\}$ nor $\{(i),(iv)\}$ can support an equilibrium, though.

[25] Both boundary types making zero profits cannot be an equilibrium, as shown in Step 1 of the proof of Lemma 1.

Step 3: Consider the lowest initial condition for which $\{(i),(iv)\}$ hold, i.e. $\overline{\lambda}_{\text{inf}} = \inf\{\overline{\lambda}|(i) \in R(\overline{\lambda})$ or $(iv) \in R(\overline{\lambda})\}$. By the definition of the infimum there exists a sequence $\{\overline{\lambda}_n\}_{n=1}^{\infty}$ with limit $\overline{\lambda}_{\text{inf}}$ such that $(i) \in R(\overline{\lambda}_n)$ or $(iv) \in R(\overline{\lambda}_n)$. By the upper-hemicontinuity of $R(\overline{\lambda}_n)$, $(i) \in R(\overline{\lambda}_{\text{inf}})$ or $(iv) \in R(\overline{\lambda}_{\text{inf}})$. Similarly, there exists a sequence $\{\overline{\lambda}_n\}_{n=1}^{\infty}$ with limit $\overline{\lambda}_{\text{inf}}$ such that $(ii) \in R(\overline{\lambda}_n)$ or $(iii) \in R(\overline{\lambda}_n)$. By the upper-hemicontinuity of $R(\overline{\lambda}_n)$, $(ii) \in R(\overline{\lambda}_{\text{inf}})$ or $(iii) \in R(\overline{\lambda}_{\text{inf}})$. Since by Step 2 (iii) and (iv) cannot hold simultaneously, we have

$$R(\overline{\lambda}_{\text{inf}}) \in \{\{(i),(ii)\},\{(i),(iii)\},\{(ii),(iv)\}\{(i),(ii),(iii)\},\{(i),(ii),(iv)\}\}.$$

This completes the proof. ■

Proof of Lemma 7

Proof. Since $\eta_m(\lambda) = \lambda m'(\lambda)/m(\lambda)$ we have

$$\begin{aligned}
\eta_m'(\lambda) < 0 \quad &\Leftrightarrow \quad m(\lambda)^{-2}\left[m(\lambda)'m(\lambda) + \lambda m''(\lambda)m(\lambda) - \lambda m'(\lambda)^2\right] < 0 \\
&\Leftrightarrow \quad m'(\lambda)\lambda^{-2}\left[m(\lambda) - \lambda m'(\lambda)\right] + m''(\lambda)q(\lambda) < 0 \\
&\Leftrightarrow \quad m'(\lambda)q'(\lambda) + m''(\lambda)q(\lambda) < 0 \\
&\Leftrightarrow \quad a(\lambda) < 1.
\end{aligned}$$

■

Proof of Lemma 8

Proof. In an equilibrium of the dynamic version sellers maximize the following program similar to (37) and (38):

$$\max_{x\in\mathcal{X},\lambda\in\overline{\mathbb{R}}_+} \; m(\lambda)\left[1-\delta\left(1-m(\lambda)\right)\right]^{-1}p \tag{44}$$

$$\text{s.t.} \quad q(\lambda)\left[1-\delta\left(1-q(\lambda)\right)\right]^{-1}\left(f(x,y)-p\right) = U(x). \tag{45}$$

The necessary first order conditions are

$$\begin{aligned}
\frac{m'(\lambda)f(x,y) - \left[1-\delta+\delta q(\lambda)\right]U(x) - \lambda\delta q'(\lambda)U(x)}{1-\delta+m(\lambda)\delta} - \frac{m(\lambda)f(x,y) - \lambda\left[1-\delta+\delta q(\lambda)\right]U(x)}{\left[1-\delta+m(\lambda)\delta\right]^2}\delta m'(\lambda) &= 0. \\
\frac{m(\lambda)f_x(x,y) - \lambda\left[1-\delta+\delta q(\lambda)\right]U'(x)}{1-\delta+m(\lambda)\delta} &= 0.
\end{aligned}$$

These conditions can be simplified substantially. After some algebra we get:

$$m'(\lambda)f(x,y) - [1-\delta+\delta q(\lambda) + \lambda(1-\lambda)\delta q'(\lambda)]U(x) = 0. \tag{46}$$

$$q(\lambda)f_x(x,y) - \left[1-\delta+\delta q(\lambda)\right]U'(x) = 0. \tag{47}$$

Conditional on matching with buyer type x, we can express the optimal profit $\tilde{\pi}(x,y)$ of seller y by solving the constraint (45) for p and substituting into (44) as

$$\tilde{\pi}(x,y) = \max_{\lambda\in\overline{\mathbb{R}}_+} \frac{m(\lambda)f(x,y) - \lambda\left[1-\delta+\delta q(\lambda)\right]U(x)}{1-\delta+m(\lambda)\delta}.$$

The optimal queue length λ for this restricted problem is still determined by optimality condition (46), and we denote the implicit solution to (46) by $\lambda^*(x,y)$. We want to see when $\frac{\partial^2 \tilde{\pi}}{\partial x \partial y} > 0$. The first derivative with respect to y is

$$\frac{\partial \tilde{\pi}}{\partial y} = \frac{m(\lambda^*) f_y(x,y)}{1-\delta+m(\lambda^*)\delta} \tag{48}$$

where by the envelope theorem the effect through $\frac{\partial \lambda^*}{\partial y}$ is zero.[26] Next, the derivative with respect to x is

$$\begin{aligned} \frac{\partial^2 \tilde{\pi}}{\partial x \partial y} &= \frac{m(\lambda^*) f_{xy}(x,y)}{1-\delta+m(\lambda^*)\delta} + \left[\frac{m'(\lambda^*) f_y(x,y)\left[1-\delta+m(\lambda^*)\delta\right] - \delta m(\lambda^*) m'(\lambda^*) f_y}{\left[1-\delta+m(\lambda^*)\delta\right]^2}\right] \frac{\partial \lambda^*}{\partial x} \geq 0 \\ \Leftrightarrow & \quad \frac{1-\delta+m(\lambda)\delta}{1-\delta} m(\lambda) f_{xy}(x,y) + m'(\lambda) f_y(x,y) \frac{\partial \lambda}{\partial x} \geq 0. \end{aligned}$$

To determine $\frac{\partial \lambda^*}{\partial x}$, implicitly differentiate (46) to obtain

$$\begin{aligned} \frac{\partial \lambda^*}{\partial x} &= -\frac{m'(\lambda^*) f_x(x,y) - [1-\delta+\delta q(\lambda^*) + \lambda^*(1-\lambda^*)\delta q'(\lambda^*)] U'(x)}{m''(\lambda^*) f(x,y) - \delta\left[q'(\lambda^*) + (1-2\lambda^*) q' + \lambda^*(1-\lambda^*) q''(\lambda^*)\right] U(x)} \\ &= -\frac{m'(\lambda^*) f_x(x,y) - [1-\delta+\delta q(\lambda^*) + \lambda^*(1-\lambda^*)\delta q'(\lambda^*)] U'(x)}{m''(\lambda^*) f(x,y) - \delta(1-\lambda^*) m''(\lambda^*) U(x)} \end{aligned}$$

after simplifying and using the fact that $m'' = \lambda q'' + 2q'$.[27] Using (47) to substitute for $U'(x)$ and (46) to substitute for $U(x)$ and we get

$$\frac{\partial \lambda^*}{\partial x} = -\frac{\left[m'(\lambda^*) - q(\lambda^*) \frac{[1-\delta+\delta q(\lambda^*) + \lambda^*(1-\lambda^*)\delta q'(\lambda^*)]}{[1-\delta+\delta q(\lambda)]}\right] f_x(x,y)}{\left[\frac{1-\delta+\delta \lambda^* q(\lambda^*)}{1-\delta+\delta q(\lambda^*) + \lambda^*(1-\lambda^*)\delta q'(\lambda^*)}\right] m''(\lambda^*) f(x,y)}.$$

Therefore $\frac{\partial^2 \tilde{\pi}}{\partial x \partial y} \geq 0$ if

$$f_{xy}(x,y) \geq A(\lambda) a(\lambda) \frac{f_x(x,y) f_y(x,y)}{f(x,y)}$$

where

$$A(\lambda) = \frac{\left[1-\delta+\delta q(\lambda) + \lambda(1-\lambda)\delta q'(\lambda)\right]\left[1-\delta\right]}{\left[1-\delta+m(\lambda)\delta\right]\left[1-\delta+\delta q(\lambda)\right]}, \quad a(\lambda) = \frac{m'(\lambda) q'(\lambda)}{q(\lambda) m''(\lambda)}.$$

■

[26] The complete first order condition has the following additional term on the right hand side of (48):

$$\left[\frac{m'(\lambda^*) f(x,y) - \left[1-\delta+\delta q(\lambda^*)\right] U(x) - \lambda^* \delta q'(\lambda^*) U(x)}{1-\delta+m(\lambda^*)\delta} - \frac{m(\lambda^*) f(x,y) - \lambda\left[1-\delta+\delta q(\lambda^*)\right] U(x)}{\left[1-\delta+m(\lambda^*)\delta\right]^2} \delta m'(\lambda^*)\right] \frac{\partial \lambda^*}{\partial y}$$

where the square bracket is zero due to the first of the two first order condition.

[27] We will use the following relationships: $m = q/\lambda$, $q' = \frac{1}{\lambda}[m' - q]$ and $q'' = \frac{1}{\lambda}[m'' - 2q']$.

7.2 Appendix B: Additional Results

Lemma 10 *Let η_g denote the elasticity of function g with respect to its argument. q^{-1} strictly convex is equivalent to*

1. *The elasticity of the derivative of the matching function being larger for buyers: $\eta_{q'} > \eta_{m'}, \forall \lambda \in \overline{\mathbb{R}}_+$.*

2. *$x(M)$ is strictly concave; where $x(M)$ is the buyers' matching probability if sellers match with probability M. That is, for given M there is λ_M such that $m(\lambda_M) = M$, and x is defined as $x(M) = q(\lambda_M)$.*

Proof. For statement 1, observe that $m'(\lambda)q'(\lambda)m''(\lambda)^{-1}q(\lambda)^{-1} < 2^{-1}$ is equivalent to $q''(\lambda)q'(\lambda)^{-1} > m''(\lambda)m'(\lambda)^{-1}$. To see this, observe the following, where we make use of $\lambda m(\lambda) = q(\lambda)$ and the relationship of the derivatives resulting from this

$$
\begin{aligned}
& - \quad 2q'(\lambda) < -m''(\lambda)q(\lambda)m'(\lambda)^{-1} \\
& \Leftrightarrow \quad m''(\lambda) - 2q'(\lambda) < m''(\lambda) - m''(\lambda)q(\lambda)m'(\lambda)^{-1} \\
& \Leftrightarrow \quad \lambda^{-1}\left[m''(\lambda) - 2q'(\lambda)\right] < m''(\lambda)\left[m'(\lambda) - q(\lambda)\right]\lambda^{-1}m'(\lambda)^{-1} \\
& \Leftrightarrow \quad q''(\lambda) < m''(\lambda)q'(\lambda)m'(\lambda)^{-1} \qquad (49) \\
& \Leftrightarrow \quad q''(\lambda)q'(\lambda)^{-1} > m''(\lambda)m'(\lambda)^{-1}.
\end{aligned}
$$

For statement 2, since $x(m(\lambda)) = q(\lambda)$ we have $q'(\lambda) = x'(m)m'(\lambda)$ and $q''(\lambda) = x'(m)m''(\lambda) + x''(m)m'(\lambda)$. Therefore $x''(m) < 0$ is equivalent to

$$
\begin{aligned}
& x''(m)m'(\lambda) < 0 \\
\Leftrightarrow \quad & q''(\lambda) - x'(m)m''(\lambda) < 0 \\
\Leftrightarrow \quad & q''(\lambda) - q'(\lambda)m'(\lambda)^{-1}m''(\lambda) < 0
\end{aligned}
$$

which coincides with (49). ■

Relation between log-supermodularity of f and the conditions in Shimer and Smith (2000)

Shimer and Smith (2000) consider a match value function that ensures Positive Assortative Matching with the following properties: weak supermodularity of f, weak log-supermodularity of f_x and f_y, and weak log-supermodularity of f_{xy}.[28] They also assume that $f \geq 0$ and $f_y(0, y) \leq 0 \leq f_y(1, y)$ for all y. We will show that this implies weak log-supermodularity of f under the additional restriction that higher types are better, i.e. $f_x(x, y) \geq 0$ (and by symmetry $f_y(x, y) \geq 0$).

[28] They consider a symmetric function f such that $f\,(x, y) = f(y, x)$.

Log-supermodularity arises when $\log f$ is supermodular, or equivalently if for all (x, y) the following holds (where we suppress the argument):

$$f_{xy} f - f_x f_y \geq 0. \tag{50}$$

This condition holds whenever $f_x = 0$ because of supermodularity ($f_{xy} > 0$) and $f > 0$.

Now we establish that condition (50) holds for any $f_x > 0$. First, we observe that the condition holds at $(0, 0)$ under the assumptions above. Then we show that at any (x, y) at which (50) holds, the left hand side of (50) increases in x.[29] This then establishes log-supermodularity at all (x, y). The left hand side increases in x if

$$f_{x^2y} f + f_{xy} f_x - f_{x^2} f_y - f_x f_{xy} \geq 0. \tag{51}$$

Now log-supermodularity of f_x is assumed, which implies

$$f_{x^2y} f_x - f_{x^2} f_{xy} \geq 0. \tag{52}$$

Whenever $f_x > 0$, equation (52) implies that (51) holds if

$$\frac{f_{x^2} f_{xy}}{f_x} f + f_{xy} f_x - f_{x^2} f_y - f_x f_{xy} \geq 0. \tag{53}$$

Since (50) holds, (53) holds if

$$\frac{f_{x^2} f_x f_y}{f_x} + f_{xy} f_x - f_{x^2} f_y - f_x f_{xy} \geq 0,$$

which holds trivially.

[29]The argument applies symmetrically for increases in y, which establishes the result.

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