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"Interdependent Durations"

by

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Interdependent Durations *

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Abstract

This paper studies the identification of a simultaneous equation model where the variable of interest is a duration measure. It proposes a game theoretic model in which durations are determined by strategic agents. In the absence of strategic motives, the model delivers a version of the generalized accelerated failure time model. In its most general form, the system resembles a classical simultaneous equation model in which endogenous variables interact with observable and unobservable exogenous components to characterize a certain economic environment. In this paper, the endogenous variables are the individually chosen equilibrium durations. Even though a unique solution to the game is not always attainable in this context, the structural elements of the economic system are shown to be semiparametrically point identified. We also present a brief discussion of estimation ideas and a set of simulation studies on the model.

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1 Introduction

This paper investigates the identification of a simultaneous equation model where the variables of interest are a duration measure. We present a game theoretic setting in which spells are determined by multiple optimizing agents in a strategic way. As a special case, our proposed structure delivers the familiar proportional hazard model. In a more general setting nonetheless, the system resembles a classical simultaneous equation model in which endogenous variables interact with observable and unobservable exogenous components to characterize a certain economic environment. In our case, the endogenous variables are the individually chosen equilibrium durations. In this context, a unique solution to the game is not always attainable. In spite of that, the structural elements of the economic system are shown to be semiparametrically point identified. We also present a set of estimation ideas and apply them to a specific application. The results presented here have connections to the literatures on simultaneous equations and statistical duration models as well as to the recent research on incomplete econometric models that result from structural (game theoretic) economic models. It also adds to the research on time-varying explanatory variables in duration models. In that literature the time varying explanatory variable is considered to be exogenous. One can think of the contribution of this paper as providing one framework that allows it to be endogenous.

One frequently observes situations in which two or more durations interact with each other. Park and Smith (2006), for instance, cite circumstances in which late rushes in market entry occur as some pioneer firm creates a market for a new service or good. In our model, the decision by the pioneer is understood as having an impact in the attractiveness of the market as seen by other potential entrants. In another related example, Fudenberg and Tirole (1985) examine technology adoption by a certain set of players. The adoption time by one agent affects the preferred timing chosen by the other player in possibly many ways. Under certain circumstances, a "diffusion" equilibrium arises, in which players adopt the new technology sequentially. For other parametric configurations, concomitant adoption occurs and there are many equilibrium times at which this occurs. Our model allows for similar results where sequential timing choices arise under some realizations of our game and concurrent spells occur as multiple equilibria for other realizations.

Many other illustrations involve some manifestation of peer effects. In de Paula (2006), for example, soldiers in the Union Army during the American Civil War tended to desert in groups. Mass desertion could be thought as lowering the costs of desertion, direct and indirect, as well as reducing the military efficiency of a company. Another example involves the decision by adolescents to first consume alcohol, drugs or cigarettes or to drop out of high school. In this case, the timing chosen by one individual would have, for potentially many reasons, an effect on the decisions of others in a given reference group.

As pointed out before, these illustrations typically display concurrent timing decisions with positive probability. From a statistical viewpoint and considering only two individuals, one might specify a reduced form model for the conditional distributions as

$$\mathbb{P}(T_i \leq t | T_j = t_j) = \begin{cases} F_i(t)(1 - \pi_i(t_j)) & \text{if } t < t_j \\ F_i(t)(1 - \pi_i(t_j)) + \pi_i(t_j) & \text{otherwise.} \end{cases}$$

where $i \neq j$, $F_i(\cdot)$ is a continuous cdf and $\pi_i(\cdot)$ is between 0 and 1. In other words, conditional on T_j, T_i has a continuous distribution, except that there is a point mass at T_j. In biology, one can motivate such a distribution by a model in which tree types of events occur. The first two "fatal events" lead to terminations of the spells for individuals 1 and 2, respectively, and the third will lead both spells to terminate. These "shock" models, for which an early reference is Marshall and OlkinMarshall and Olkin (1967), have been used in industrial reliability and biomedical statistical applications (see for example Klein, Keiding, and Kamby (1989)). In these models the relationship between the durations is driven by the unobservables, but no direct relationship exists between them. This brings them closer to a "seemingly unrelated regressions" framework. In economics, it is interesting to consider models in which durations depend on each other in a structural way, allowing for an interpretation of estimated parameters closer to economic theory. This is the aim of our paper and, as such, the difference between Marshall and Olkin's model and ours is similar to the difference between seemingly unrelated regressions and structural simultaneous equations models.

To achieve this we set up a very simple game theoretic model with complete information where players make decisions about the timing at which to switch from one state to another. Our analysis bears some resemblance to previous studies in the empirical games literature, such as Bresnahan and Reiss (1991) and, more recently, Tamer (2003). Bresnahan and Reiss (1991), building on pioneering work such as Heckman (1978), analyze a simultaneous game with a discrete number of possible actions for each agent. A major pitfall in such circumstances is that "when a game has multiple equilibria, there is no longer a unique relation between players' observed strategies and those predicted by the theory." Given large enough supports for the unobservable components in the economic model, this situation is pervasive for the class of games they analyze. Tamer (2003) characterizes this particular issue as an "incompleteness" in the model and shows that this nuisance does not necessarily preclude point identification of the deep parameters in the model under some conditions. Our model also possesses multiple equilibria and, like Tamer, we also obtain point identification of the main structural features. This is possible because certain realizations of the stochastic game we analyze deliver unique equilibrium outcomes with sequential timing choices while multiplicity occurs if and only if spells are concomitant. We are then able to obtain point identification using arguments not unfamiliar to the identification of duration models (see for example Elbers and Ridder (1982)) on the events for which one attains uniqueness of the equilibrium solution.

Since the econometrician observes outcomes for two agents, our model is a multiple duration model. If multiple durations for a given individual were recorded, such as unemployment spells for workers or time intervals between transactions for assets, panel duration observations would provide leverage both in terms of identification and subsequent estimation (see Honoré (1993), Horowitz and Lee (2004) and Lee (2003)). Whereas there subsequent spells are observed for a given individual, here parallel individual spells[1] are recorded for a given game, and some elements in our analysis can be made game-specific (such as the function $Z(\cdot)$ to be defined later) mirroring the appearance of individual specific effects in the panel duration literature.

Our setting is a continuous-time one. This corresponds to the traditional approach in econometric duration studies and statistical survival analysis. Many game theoretic models of timing are also set in continuous time. The framework can be understood as the limit of a discrete time game. As the frequency of interactions increases, the setting converges to our continuous time framework, which can in turn be seen as an approximation to the discrete time model. The exercise is thus in line with the early theoretical analysis by Simon and Stinchcombe (1989), Bergin and MacLeod (1993) and others and with most of the econometric analysis of duration models (e.g. Elbers and Ridder (1982), Heckman and Singer (1984), Honoré (1990), Hahn (1994), Ridder and Woutersen (2003), Abbring and van den Berg (2003)). See also Van den Berg (2001).

The remainder of the paper proceeds as follows. In the next section we present the economic model. Section 3 presents a few simulation exercises to illustrate the consequences of ignoring the endogeneity problem introduced by the interaction or mistaken choices for the equilibrium selection mechanism. The fourth section inves-

[1] SeeHougaard (2000) and Frederiksen, Honoré, and Hu (2007).

tigates the identification of the many structural components in the model. Section 5 briefly discusses estimation strategies and the subsequent section deals with the case of discrete (grouped) observations. We conclude in the last section.

2 The Economic Model

The economic model consists of a system of two individuals who interact in continuous time. Information is complete for the individuals. Each individual i chooses how long to take part in a certain activity by selecting a termination time $T_i, i = 1, 2$. Agents start at an activity that provides an utility flow given by the positive random variable $K_i \in \mathbb{R}_+$. At any point in time, an individual can choose to switch to an alternative activity which provides him or her a flow utility $U(t, \mathbf{x}_i)$ where the vector $\mathbf{x}_i$ denotes a set of covariates. This utility flow is incremented by a factor e^{δ} when the other agent switches to the alternative activity. We assume that $\delta \geq 0$. Since only the difference in utilities will ultimately matter for the decision, the utility flow in the initial activity is normalized to be a random variable independent of $\mathbf{x}_i$.

In order to facilitate the link of our study to the analysis of duration models it will be convenient to adopt a multiplicative specification for $U(t, \mathbf{x}_i)$ as $Z(t)\varphi(\mathbf{x}_i)$ where $Z : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing, absolutely continuous function such that $Z(0) = 0$. Assuming an exponential discount rate ρ, individual i's utility for taking part in the initial activity until time t_i given the other agent's timing choice T_j is:

$$\int_0^{t_i} K_i e^{-\rho s} ds + \int_{t_i}^{\infty} Z(s).\varphi(\mathbf{x}_i) e^{\mathbb{I}_{(s \geq T_j)}\delta}.e^{-\rho s} ds$$

The first order condition for maximizing this with respect to t_i is:

$$K_i.e^{-\rho t_i} - Z(t_i).\varphi(\mathbf{x}_i).e^{\mathbb{I}_{(t_i \geq T_j)}.\delta}.e^{-\rho t_i}$$

This may not be equal to zero for any t_i since it is discontinuous at $t_i = T_j$. Given the opponent's strategy, the optimal behavior of an agent in this game consists of monitoring the (un-discounted) marginal utility $K_i - Z(t).\varphi(\mathbf{x}_i).e^{\mathbb{I}_{(t \geq T_j)}.\delta}$ at each moment of time t. As long as this quantity is positive the individual participates at the initial activity, and he or she switches as soon as the marginal utility becomes less than or equal to zero.

Formally, the appropriate concept for optimality is that of mutual best responses. We start by considering the best response function of individual i given that individual j has chosen T_j:

$$\begin{aligned} b_1(T_2) &= \inf\{t_1 : K_1 - Z(t_1).\varphi(\mathbf{x}_1).e^{\mathbb{I}(t_1 \geq T_2)\cdot\delta} < 0\} \\ b_2(T_1) &= \inf\{t_2 : K_2 - Z(t_2).\varphi(\mathbf{x}_2).e^{\mathbb{I}(t_2 \geq T_1)\cdot\delta} < 0\} \end{aligned}$$

A Nash Equilibrium for this game is given by a fixed point to the profile of best response functions: $b(T_1, T_2) = (b_1(T_2), b_2(T_2))$. Existence of an equilibrium can be established using the usual Debreu-Glicksberg-Fan results for games with continuous action spaces.[2]

In the absence of external influence ($\delta = 0$), the individual switches at $T_i = Z^{-1}(K_i/\varphi(\mathbf{x}_i))$ or

$$\ln Z(T_i) = -\ln \varphi(\mathbf{x}_i) + \underbrace{\epsilon_i}_{\equiv \ln k_i}$$

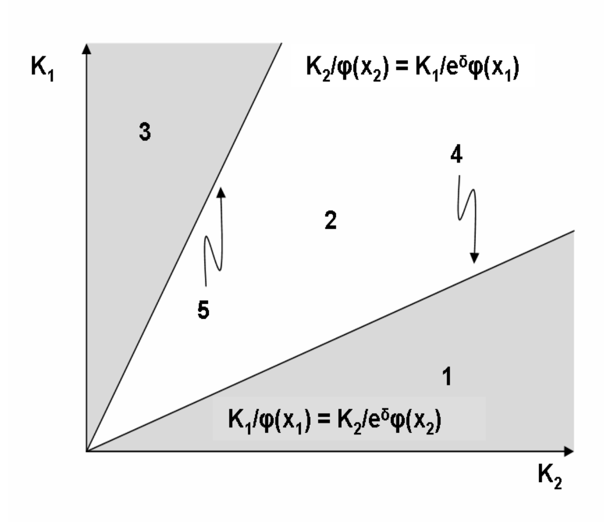
which is a semi-parametric Generalized Accelerated Failure Time (GAFT) model like the ones discussed in RidderRidder (1990). For example, if $Z(t) = \lambda s^{\alpha_i}$, $\varphi(\mathbf{x}_i) = \exp(\mathbf{x}_i'\beta)$ and $K_i \sim \exp(1)$, the cumulative distribution function is given by

$$\begin{aligned} F_{T_i}(t) &= \mathbb{P}[(K_i e^{-\mathbf{x}_i'\beta}/\lambda)^{1/\alpha_i} \leq t] \\ &= \mathbb{P}(K_i \geq t^{\alpha_i}\lambda e^{-\mathbf{x}_i'\beta}) \\ &= 1 - \exp(-t^{\alpha_i}\lambda \exp(\mathbf{x}_i'\beta)) \end{aligned}$$

and the model corresponds to a proportional hazard duration model with a Weibull baseline hazard.

For the remaining of this section, we characterize the equilibrium behavior in the game. Depending on the realization of K_1 and K_2, the model may be consistent with multiple equilibria. Indeed, we can identify five distinct regions on the $K_1 \times K_2$ space:

[2] Formally these results would require that the strategy space be compact. It can be seen though that the agents in our game will switch states in finite time regardless of the action taken by the other agent. So, for a given realization of the game ($K_i,\ i = 1, 2$), we can always bound the action space.



The regions are characterized and obtained as follows.

Region 1: $T_1 < T_2$ *and the equilibrium is unique.* This region is such that $K_1/\varphi(\mathbf{x}_1) < K_2 e^{-\delta}/\varphi(\mathbf{x}_2)$ and hence $Z^{-1}(K_1/\varphi(\mathbf{x}_1)) < Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$. Here, for any t less than $Z^{-1}(K_1/\varphi(\mathbf{x}_1))$, $K_2 - Z(t)\varphi(\mathbf{x}_2)e^{\delta}$ is greater than zero and agent 2 has no incentive to switch even if agent 1 has already switched. Also both $K_1 - Z(t)\varphi(\mathbf{x}_1)$ and $K_1 - Z(t)\varphi(\mathbf{x}_1)e^{\delta}$ are greater than zero, and agent 1 would therefore not switch regardless of whether the other agent has switched or not. Once $t > Z^{-1}(K_1/\varphi(\mathbf{x}_1))$, then $K_1 - Z(t)\varphi(\mathbf{x}_1)$ is strictly less than 0 and agent one will prefer to have switched earlier no matter what action second agent might take. It is therefore optimal for agent 1 to switch at $T_1 = Z^{-1}(K_1/\varphi(\mathbf{x}_1))$. This in turn induces agent 2 to switch at $T_2 = Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2)) > T_1$.

Region 2: $T_1 = T_2$ *and there are multiple equilibria.* This region is given by $Z^{-1}(K_1/\varphi(\mathbf{x}_1)) > Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$ and $Z^{-1}(K_2/\varphi(\mathbf{x}_2)) > Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1))$.

To see that individuals will stop simultaneously and there are many equilibria, let

$$\underline{T} = \max\left(Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1)), Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))\right)$$

and

$$\overline{T} = \min\left(Z^{-1}(K_1/\varphi(\mathbf{x}_1)), Z^{-1}(K_2/\varphi(\mathbf{x}_2))\right)$$

Because $Z^{-1}(K_1/\varphi(\mathbf{x}_1)) > Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$ and $Z^{-1}(K_2/\varphi(\mathbf{x}_2)) > Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1))$, we have that $\underline{T} \leq \overline{T}$. We now consider three cases depending on t's location relative to $\underline{T}$ and $\overline{T}$.

For $t < \underline{T}$, let j be the agent such that $\underline{T} = Z^{-1}(K_j e^{-\delta}/\varphi(\mathbf{x}_j))$. Since $t < \underline{T} = Z^{-1}(K_j e^{-\delta}/\varphi(\mathbf{x}_j))$, $K_j - Z(t)\varphi(\mathbf{x}_j) > 0$ and he would not be willing to switch regardless of the action of the other agent, whom we denote by i. Also $K_i - Z(t)\varphi(\mathbf{x}_i) > 0$ and this individual will not switch either given that individual j does not switch. Hence no agent switches in this region.

For $\underline{T} \leq t \leq \overline{T}$, $Z(t)\varphi(\mathbf{x}_i)e^{\delta} - K_i \geq 0$ and $Z(t)\varphi(\mathbf{x}_i) - K_i \leq 0$ for each agent. At each point in time in the interval, an agent can therefore do no better than the alternative activity if the other agent has already switched. Hence, any profile such that $\underline{T} \leq T_1 = T_2 \leq \overline{T}$ will be an equilibrium.

Finally, for $\overline{T} < t$, $Z(t)\varphi(\mathbf{x}_i) - K_i > 0$ for both individuals and each has an incentive to decrease his or her switching time towards $\overline{T}$ regardless of what the other agent does.

Hence, simultaneous switching at any t in the interval $[\underline{T}, \overline{T}]$ is an equilibrium.

Region 3: $T_2 < T_1$ *and the equilibrium is unique.* This region is such that $Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1)) > Z^{-1}(K_2/\varphi(\mathbf{x}_2))$. The reasoning is similar to that of Region 1.

Region 4: $T_1 = T_2$ *and the equilibrium is unique.* This region is given by $Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1)) = Z^{-1}(K_2/\varphi(\mathbf{x}_2))$. Here $Z^{-1}(K_1/\varphi(\mathbf{x}_1)) \geq Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1)) = Z^{-1}(K_2/\varphi(\mathbf{x}_2)) \geq Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$. For $t < Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$ no one would be willing to switch regardless of the opponent's action. For $Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2)) \leq t \leq Z^{-1}(K_2/\varphi(\mathbf{x}_2))$, agent 2 would like to switch if agent 1 did. If $t < Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1))$, agent 1 does not want to switch even if agent 2 does. When $t = Z^{-1}(K_2/\varphi(\mathbf{x}_2)) = Z^{-1}(K_1 e^{-\delta}/\varphi(\mathbf{x}_1))$, agent 2 switches and agent 1 follows. For $t > Z^{-1}(K_2/\varphi(\mathbf{x}_2))$ agent 2 would have already switched.

Region 5: $T_1 = T_2$ *and the equilibrium is unique* This region is given by $Z^{-1}(K_1/\varphi(\mathbf{x}_1)) = Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$. Here the reasoning is analogous to that of Region 4.

Regions 1,3,4 and 5 all result in an unique equilibrium. In Region 2, a simultaneous switch at any t in $[\underline{T}, \overline{T}]$ would be an equilibrium. This interval will be degenerate if δ is equal to zero.

We end this section with a brief discussion on the multiple equilibria encountered in Region 2. In our approach, we are agnostic as to which of these equilibria is selected. Some of the solutions in that region may be singled out by different selection criteria nevertheless. The Nash solution concept we use is equivalent to that of an open-loop equilibrium (as discussed for example in Fudenberg and Tirole (1991), Section 4.7): one in which individuals condition their strategies on calendar time only and hence commit to this plan of action at the beginning of the game. If individuals can react to events as time unfolds, a closed-loop solution concept which here would be equivalent to subgame perfection would single out the earliest of the Nash equilibria, in which individuals switch at $\underline{T}$. Intuitively, an optimal strategy in Region 2 contingent on the game history would prescribe switching simultaneously at any time between $\underline{T}$ and $\overline{T}$. Faced with an opponent carrying such (closed-loop) strategy, an individual might as well switch as soon as possible to maximize his or her own utility flow. This outcome also corresponds to the Pareto-dominant equilibrium. Under this information structure, the equilibria displayed in our analysis would still be Nash, but not necessarily subgame-perfect. In selecting one of the multiple equilibria that may arise, the early equilibrium is nevertheless a compelling equilibrium and we give it special consideration in the simulation exercises performed later in the paper.

Other selection mechanisms may nonetheless point to later equilibria among the many Nash solutions available. Since the switching decision is irreversible, risk dominance-type considerations could for example lead to a later switching time.[3] For this reason, we remain agnostic as to which Nash equilibrium is selected.

3 The Effect of Misspecifications

In this section we examine the effect of misspecifications in the economic model or equilibrium selection process on the estimation of the parameters of interest through a few simulation exercises. In the following experiments, we assume that time is observed at a high frequency so no interval censoring occurs.

3.1 Ignoring Endogeneity

This subsection investigates the consequences of treating an opponent's decision as exogenous in a parametric version of our model. The first data generating process is

[3] This is illustrated by the following quote:

> It is usually the essence of mob formation that the potential members have to know not only where and when to meet but just when to act so that they act in concert. (. . .) In this case the mob's problem is to act in unison without overt leadership, to find some common signal that makes everyone confident that, if he acts on it, he'll not be acting alone. Schelling (1960))

defined by

$$\begin{aligned} Z(t) &= t^{\alpha} \\ \varphi(\mathbf{x}_i) &= \exp(\beta_0 + \beta_1 \mathbf{x}_i) \\ (\alpha, \beta_0, \beta_1, \delta) &= (1.0, -3.0, 0.3, 0.3) \end{aligned}$$

and

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)$$

When the model gives rise to multiple equilibria (and hence simultaneous exit) a specific duration is drawn from a uniform distribution over the possible duration times.[4] Tables 1, 2, and 3 present the results based on 1000 replications of datasets of size 1000. Table 1 is based on a correctly specified likelihood that groups all ties occurring in realizations of Region 2 in the previous discussion of the model. Table 2 presents results from maximum likelihood estimation for agent 1 taking agent 2's action as exogenous.

TABLE 1: Incorporating Endogeneity

	True Value	Bias	RMSE	Median Bias	Median Abs.Err.
α	1.000	0.001	0.019	0.000	0.013
β_0	−3.000	0.000	0.067	−0.001	0.045
β_1	0.300	0.000	0.018	0.000	0.012
δ	0.300	−0.001	0.023	−0.001	0.016

TABLE 2: Weibull. Dependent variable T_1

	True Value	Bias	RMSE	Median Bias	Median Abs.Err.
α	1.000	−0.079	0.084	−0.080	0.080
β_0	−3.000	0.076	0.116	0.078	0.087
β_1	0.300	−0.005	0.027	−0.005	0.019
δ	0.300	0.523	0.530	0.524	0.524

[4] We experimented with different selection rules and these made no appreciable difference to the results we present here.

As expected, the maximum-likelihood estimator that incorporates endogeneity performs well, whereas the Weibull estimator that assumes that the other agent's action is exogenous performs poorly. Specifically, the effect of the opponent's decision is grossly over-estimated. Treating the other agent's action as exogenous also bias estimates toward negative duration dependence. Both of these are expected. In the first case, δ is biased because the estimation does not take into account the multiplier effect caused by the feedback between T_1 and T_2. The assumption of exogeneity also leads to a downward bias on duration dependence as duration lengths reinforce themselves: a shock leading to a longer duration by one agent will tend to lengthen the opponent's duration and hence further reduce the hazard for the original agent.

The results in Tables 1 and 2 assume symmetry between the two agents in the model. The next design changes this by changing the joint distribution of $(\mathbf{x}_1, \mathbf{x}_2)$ to

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \sim N\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)$$

This makes the first agent likely to move first. When multiple equilibria were possible, an equilibrium was selected as in the previous exercise. The overestimation bias on δ is of a similar magnitude as before. The effect on the estimation of α is different for each individual given the asymmetry in the distribution of the $\mathbf{x}$s.

TABLE 3: Incorporating Endogeneity

	True Value	Bias	RMSE	median bias	median abs.err.
α	1.000	0.000	0.019	0.000	0.012
β_0	-3.000	0.000	0.067	0.000	0.045
β_1	0.300	0.000	0.017	0.000	0.011
δ	0.300	0.000	0.024	0.000	0.017

TABLE 4: Weibull. Dependent variable T_1

	True Value	Bias	RMSE	median bias	median abs.err.
α	1.000	-0.065	0.071	-0.066	0.066
β_0	-3.000	0.049	0.107	0.052	0.075
β_1	0.300	-0.002	0.026	-0.002	0.018
δ	0.300	0.523	0.530	0.524	0.524

TABLE 5: Weibull. Dependent variable T_2

	True Value	Bias	RMSE	median bias	median abs.err.
α	1.000	-0.095	0.099	-0.095	0.095
β_0	-3.000	0.083	0.121	0.083	0.087
β_1	0.300	-0.007	0.027	-0.008	0.018
δ	0.300	0.530	0.537	0.531	0.531

3.2 Equilibrium Selection

In this subsection we examine the effect of equilibrium selection assumptions in the estimation of a parametric version of the model. The data generating process for all the results below are based on:

$$\begin{aligned} Z(t) &= t^{\alpha} \\ \varphi(\mathbf{x}_i) &= \exp(\beta_0 + \beta_1 \mathbf{x}_{1i} + \beta_2 \mathbf{x}_2) \\ (\alpha, \beta_0, \beta_1, \beta_2, \delta) &= (1.35, -4.00, 1.00, 0.50, 1.00) \end{aligned}$$

where $\mathbf{x}_{i1},\ i = 1, 2$ represents an individual specific covariate and $\mathbf{x}_2$, a common covariate. These three variables are independent standard normal random variables. A total of 1000 replications with sample sizes of 2000 observations (games) were generated.

Tables 6 through 10 differ in the way equilibrium is selected when there are multiple equilibria. Aside from the column indicating the value of each of the parameters, each of the tables presents Median Bias and Median Absolute Error for three alternative estimators: the maximum likelihood estimator that pools equilibria without selecting the equilibrium; a maximum likelihood estimator that assumes the earliest equilibrium ($\underline{T}$) is played when there are multiple equilibria; and a maximum likelihood estimator that takes the latest equilibrium ($\overline{T}$) as the selected equilibrium in case there are many equilibria.

In Table 6, the latest equilibrium ($\overline{T}$) is selected. As expected, the estimator corresponding to the results in the last two columns performs the best, as it assumes the correct selection rule generating the data. Pooling equilibria in the estimation seems to do an appreciably better job than the estimator that incorrectly assumes the equilibrium selection criterion as the earliest possible equilibrium: although the estimates for β_1 and δ present similar median bias and absolute error, the other parameters appear to present much less bias in the estimator that pools the equilibria.

The estimator for the constant term β_0 seems to be particularly biased down when $\underline{T}$ is assumed to be selected. This makes sense: by assuming an earlier selection scheme the constant is below the true parameter, lowering the hazard and thus increasing the durations to match the data.

TABLE 6: $\overline{T}$ Selected

		Pools Ties		*Assumes $\underline{T}$*		*Assumes $\overline{T}$*	
	True Value	Median Bias	Median Absolute	Median Bias	Median Absolute	Median Bias	Median Absolute
α	1.350	0.018	0.053	-0.025	0.046	0.011	0.041
Constant	-4.000	-0.036	0.160	-0.168	0.189	-0.028	0.129
δ	1.000	-0.003	0.060	-0.001	0.059	0.001	0.054
β_1	1.000	0.014	0.059	-0.015	0.052	0.005	0.046
β_2	0.500	0.006	0.043	-0.033	0.043	0.006	0.038

Table 7 displays a design where the earliest equilibrium ($\underline{T}$) is picked. Here the middle estimator, which correctly assumes the selection scheme generating the data, is as expected the best of the three. The improvement of the pooling estimator over the one that wrongfully assumes the selection mechanism seems even more compelling than in the previous case. The effect of mistaken equilibrium selection on the constant term is again fairly large: in order to accommodate an equilibrium selection rule that chooses later equilibria than the ones actually played is to increase the hazard so that durations are lowered and the estimation matches the data.

TABLE 7: $\underline{T}$ Selected

		Pools Ties		*Assumes $\underline{T}$*		*Assumes $\overline{T}$*	
	True Value	Median Bias	Median Absolute	Median Bias	Median Absolute	Median Bias	Median Absolute
α	1.350	0.007	0.049	0.008	0.040	-0.014	0.042
Constant	-4.000	-0.017	0.158	-0.012	0.125	0.321	0.321
δ	1.000	0.005	0.062	0.005	0.062	-0.137	0.137
β_1	1.000	0.006	0.058	0.007	0.046	-0.013	0.046
β_2	0.500	0.003	0.042	0.002	0.038	0.006	0.039

In Table 8, an equilibrium is randomly selected according to a uniform distribution on $[\underline{T}, \overline{T}]$ as was the case in the previous subsection. The performance of the pooling estimator is noticeably better in comparison to the two other estimators except for the estimation on α, the Weibull parameter.

TABLE 8: $U[\underline{T}, \overline{T}]$ **Selected**

		Pools Ties		*Assumes* $\underline{T}$		*Assumes* $\overline{T}$	
	True Value	Median Bias	Median Absolute	Median Bias	Median Absolute	Median Bias	Median Absolute
α	1.350	0.010	0.048	-0.001	0.041	0.006	0.040
Constant	-4.000	-0.025	0.152	-0.125	0.154	0.116	0.150
δ	1.000	0.005	0.062	0.008	0.060	-0.065	0.071
β_1	1.000	0.011	0.060	0.003	0.046	0.007	0.045
β_2	0.500	-0.002	0.044	-0.020	0.041	0.002	0.038

Table 9 shows the case in which the earliest equilibrium is selected when the common variable $\mathbf{x}_2$ is greater than zero whereas the latest equilibrium is picked when $\mathbf{x}_2$ is less then zero — this amplifies the effect of this variable on the hazard beyond the impact already present in the multiplicative $\varphi(\cdot)$ term. In this case, the pooling estimator fares better across all the parameters.

TABLE 9: $\underline{T}\mathbf{1}(\mathbf{x}_2 > 0) + \overline{T}\mathbf{1}(\mathbf{x}_2 \leq 0)$ **Selected**

		Pools Ties		*Assumes* $\underline{T}$		*Assumes* $\overline{T}$	
	True Value	Median Bias	Median Absolute	Median Bias	Median Absolute	Median Bias	Median Absolute
α	1.350	0.009	0.051	-0.015	0.043	-0.007	0.042
Constant	-4.000	-0.032	0.154	-0.095	0.146	0.161	0.177
δ	1.000	0.002	0.057	0.005	0.058	-0.069	0.075
β_1	1.000	0.008	0.059	0.085	0.086	0.065	0.070
β_2	0.500	0.007	0.042	-0.016	0.040	0.006	0.037

Finally, Table 10 displays results for a selection mechanism that picks $\underline{T}$ when this quantity is greater than 10 and selects $\overline{T}$ when $\underline{T}$ is less than 10. Again the pooling estimator seems to be the superior one when comparing median bias and median absolute error for the parameters of interest.

TABLE 10: $\underline{T}\mathbf{1}(\underline{T} > 10) + \overline{T}\mathbf{1}(\underline{T} \leq 0)$ Selected

		Pools Ties		*Assumes $\underline{T}$*		*Assumes $\overline{T}$*	
	True Value	Median Bias	Median Absolute	Median Bias	Median Absolute	Median Bias	Median Absolute
α	1.350	0.014	0.048	0.057	0.059	0.051	0.056
Constant	-4.000	-0.030	0.143	-0.253	0.254	0.020	0.129
δ	1.000	0.009	0.067	-0.006	0.061	-0.091	0.095
β_1	1.000	0.012	0.061	-0.039	0.056	-0.024	0.048
β_2	0.500	0.001	0.042	-0.023	0.041	0.002	0.038

Since either ignoring the strategic interaction in the model by assuming exogeneity or misspecifying the equilibrium selection mechanism may lead to erroneous inference, our next section studies the identifiability of the structural components of this model without assuming a particular equilibrium selection procedure.

4 Identification

The previous section illustrates how misspecifications disregarding the strategic nature of decisions or imposing an erroneous selection rule for the solution may generate misguided inferences. In this section we ask what aspects of the model can be identified by the data once one recognizes the endogeneity of choices and abstains from an equilibrium selection rule. The proof strategy is similar to that in for example Elbers and Ridder (1982) and Heckman and Honoré (1989).

The subsequent analysis relies on the following assumptions:

Assumption 1 *K_i is independent across i and identically distributed according to $G(\cdot)$, where $G(\cdot)$ is a continuous distribution with full support on $\mathbb{R}_+$. Furthermore, the probability density function $g(\cdot) = G'(\cdot)$ is bounded away from zero and infinity in a neighborhood of zero.*

Assumption 2 *The function $Z(\cdot)$ is differentiable with positive derivative.*

Assumption 3 *At least one component of $\mathbf{x}_i$, $\mathbf{x}_{ik}$, is such that supp($\mathbf{x}_{ik}$) contains an open subset of $\mathbb{R}$.*

Assumption 4 *The range of $\varphi(\cdot)$ is $\mathbb{R}_+$ and it is continuously differentiable with nonzero derivative.*

The following results establish that these assumptions are sufficient (though not necessary in many cases) for the identification of the different components in the model. We begin by analyzing $\varphi(\cdot)$.

Theorem 1 (Identification of $\varphi(\cdot)$) *Under Assumptions 1 and 2, the function $\varphi(\cdot)$ is identified up to scale if* $\mathrm{supp}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{supp}(\mathbf{x}_1) \times \mathrm{supp}(\mathbf{x}_2)$.

Proof. Consider the absolutely continuous component of the conditional distribution of (T_1, T_2), the switching times for the agents, given the covariates $\mathbf{x}_1, \mathbf{x}_2$. Using the fact that $T_1 = Z^{-1}(K_1/\varphi(\mathbf{x}_1))$ and $T_2 = Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))$ when $T_1 < T_2$ and the Jacobian method we can obtain that the probability density function for this component on the set $\{(t_1, t_2) \in \mathbb{R}^2_+ : t_1 < t_2\}$ is given by:

$$
\begin{aligned}
f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(t_1, t_2|\mathbf{x}_1, \mathbf{x}_2) = \lambda(t_1)\varphi(\mathbf{x}_1)g(Z(t_1)\varphi(\mathbf{x}_1)) \\
\times\lambda(t_2)\varphi(\mathbf{x}_2)e^{\delta}g(Z(t_2)\varphi(\mathbf{x}_2)e^{\delta})
\end{aligned}
$$

where

$$
Z(t) = \int_0^t \lambda(s)ds, i = 1, 2.
$$

Given two sets of covariates $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{x}'_1, \mathbf{x}_2)$ we obtain that

$$
\begin{aligned}
\lim_{\substack{(t_1,t_2)\to(0,0) \\ t_1<t_2}} \frac{f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(t_1, t_2|\mathbf{x}'_1, \mathbf{x}_2)}{f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(t_1, t_2|\mathbf{x}_1, \mathbf{x}_2)} &= \lim_{\substack{(t_1,t_2)\to(0,0) \\ t_1<t_2}} \frac{\varphi(\mathbf{x}'_1)g(Z(t_1)\varphi(\mathbf{x}_1))}{\varphi(\mathbf{x}_1)g(Z(t_1)\varphi(\mathbf{x}'_1))} \\
&= \frac{\varphi(\mathbf{x}'_1)}{\varphi(\mathbf{x}_1)}
\end{aligned}
$$

where the last equality uses the fact that $\lim_{t\to 0} Z(t) = 0$. So, $\varphi(\cdot)$ is identified up to a scale transformation. ■

The assumption that K_i is independent across i is stronger than necessary for the identification of $\varphi(\cdot)$ and can be relaxed. This is also the case with the condition that $\mathrm{supp}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{supp}(\mathbf{x}_1) \times \mathrm{supp}(\mathbf{x}_2)$. In order to identify $\varphi(\mathbf{x}_1)/\varphi(\mathbf{x}'_1)$ all we need is to be able to find $\mathbf{x}_2$ such that $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{x}'_1, \mathbf{x}_2)$ are in the support. The proof strategy also allows $\varphi(\cdot)$ to depend on i. Finally, the identification of $\varphi(\cdot)$ would still hold even if the players shared the same covariates $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$ as long as $\varphi(\cdot)$ is the same for both.

Theorem 2 (Identification of $Z(\cdot)$) *The function $Z(\cdot)$ is identified up to scale under Assumptions 1-4.*

Proof. On the set $\{s < t_2\}$, consider the function

$$\begin{aligned} h(s, t_2, \mathbf{x}_1, \mathbf{x}_2) &= \int_0^s f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(t_1, t_2|\mathbf{x}_1, \mathbf{x}_2)dt_1 \\ &= G(Z(s)\varphi(\mathbf{x}_1)) \times \underbrace{\lambda(t_2)\varphi(\mathbf{x}_2)e^{\delta}g(Z(t_2)\varphi(\mathbf{x}_2)e^{\delta})}_{\equiv c(t_2,\mathbf{x}_2)} \end{aligned}$$

Then notice that

$$\frac{d\ln h/ds}{\partial \ln h/\partial \mathbf{x}_{1k}} = \frac{\lambda(s)\varphi(\mathbf{x}_1)}{Z(s)\partial_k\varphi(\mathbf{x}_1)}$$

and we have that

$$CZ(s)^{\varphi(\mathbf{x}_1)/\partial_k\varphi(\mathbf{x}_1)}$$

is identified where C is a constant. Given the identification of $\varphi(\cdot)$ up to scale, one obtains that $Z(\cdot)$ is also identified up to scale (the constant C). ■

The mechanics of the proof suggests that we can also allow $Z(\cdot)$ to depend on i as is the case with $\varphi(\cdot)$, but both the characterization of the equilibrium in section 2 and the identification argument for the δs below assume $Z(\cdot)$ to be the same for both individuals. Also in contrast to $\varphi(\cdot)$, we make use of the independence assumption on K_i to show that $Z(\cdot)$ is identified. As in the previous result, the identification would still hold were the covariates for the two agents identical for a given draw of the game ($\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$). We finalize by establishing the identification of δ.

Theorem 3 (Identification of δ) *δ is identified under Assumptions 1-4 .*

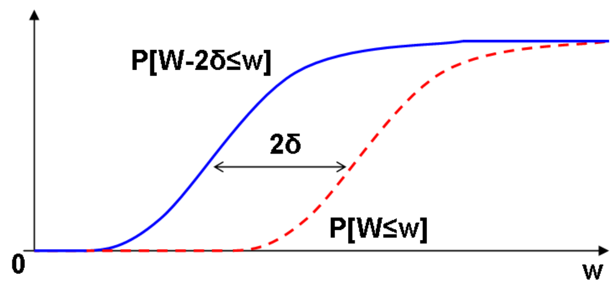
Proof. Consider the probability

$$\mathbb{P}(T_1 < T_2|\mathbf{x}) = \mathbb{P}(\ln K_1 - \ln K_2 + \delta < \ln \varphi(\mathbf{x}_1)/\varphi(\mathbf{x}_2)).$$

Since $\varphi(\cdot)$ is identified up to scale, as one varies $\mathbf{x}_1$ and $\mathbf{x}_2$, the probability above traces the cumulative distribution function for the random variable $W = \ln K_1 - \ln K_2 + \delta$. Likewise, the probability

$$\mathbb{P}(T_1 > T_2|\mathbf{x}) = \mathbb{P}(\ln K_1 - \ln K_2 - \delta > \ln \varphi(\mathbf{x}_1)/\varphi(\mathbf{x}_2))$$

traces the survivor function (and consequently the cumulative distribution function) for the random variable $\ln K_1 - \ln K_2 - \delta = W - 2\delta$. Since this is basically the random variable W displaced by 2δ, this difference is identified as the (horizontal) distance between the two cumulative distribution functions which are identified from the data (the events $T_1 > T_2$ and $T_1 < T_2$ conditioned on $\mathbf{x}$). The figure below illustrates this idea:



From this argument, the parameter δ is identified. ■

Again, the assumption of independence is unnecessary for the identification of δ. Assumption 1 is invoked to guarantee the identification of $\varphi(\cdot)$. If this function is identified we can dispense with this assumption.

In the remainder of this section, we discuss results for some variations on the model depicted on Section 2.

Under certain circumstances, such as in interactions between husband and wife, the players in the games sampled may be easily labelled, say $i = 1, 2$. In this case, one may consider different δs for different players: $\delta_i,\ i = 1, 2$. The previous result would render identification for $\delta_1 + \delta_2$. The following establishes the identification of $\delta_1 - \delta_2$ and hence of $\delta_i,\ i = 1, 2$.

Theorem 4 (Identification of $\delta_i,\ i = 1, 2$) *$\delta_i,\ i = 1, 2$ are identified under Assumptions 1-4.*

Proof. The sum $\delta_1 + \delta_2$ is identified according to the arguments in the previous theorem. Define

$$\begin{aligned}
c_1 &\equiv \frac{\lim_{\substack{s\to 0\\ k>1}} f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(s, ks|\mathbf{x}_1, \mathbf{x}_2)}{\lim_{\substack{s\to 0\\ k>1}} f_{T_1,T_2|\mathbf{x}_1,\mathbf{x}_2}(ks, s|\mathbf{x}_1, \mathbf{x}_2)} \\
&= \frac{\lim_{\substack{s\to 0\\ k>1}} \lambda(s)\lambda(ks)\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)e^{\delta_2}}{\lim_{\substack{s\to 0\\ k>1}} \lambda(ks)\lambda(s)\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)e^{\delta_1}} \\
&= \frac{\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)e^{\delta_2}}{\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)e^{\delta_1}} \times \lim_{\substack{s\to 0\\ k>1}} \frac{\lambda(s)\lambda(ks)}{\lambda(ks)\lambda(s)} \\
&= e^{\delta_2 - \delta_1}
\end{aligned}$$

which identifies $\delta_2 - \delta_1$. This and the previous result identify $\delta_i,\ i = 1, 2$. ■

As remarked, independence between $K_i,\ i = 1, 2$ is not a necessary condition for the identification of many aspects of the model. This allows for some dependence in the latent utility flow obtained in the initial activity. Another source of correlation though may be represented by the arrival of a common shock that drives both individuals to the outside activity concurrently. Even under such extreme circumstances, some aspects of the structure remain identified.

A natural way to introduce this non-strategic shock in the model would follow the one delineated by Cox and Oakes (1984) and assume that the common shock which drives both spells to termination concomitantly happens at a random time $V \geq 0$ characterized by a probability density function given by $h(\cdot)$. Individuals switch for two possible reasons: either they deem the decision to be optimal as in the original model; or they are driven out of the initial activity by the common shock. If both individuals are still in the initial activity when the shock arrives, they both switch simultaneously. If one of them switches before the shock arrives, the second one is driven out of the initial activity earlier than he or she would have voluntarily chosen.[5] In keeping with the notation used so far, let T_i be the switching time chosen by individual i and $\tilde{T}_i = \min\{T_i, V\}$, the switching time observed by the econometrician. The following result holds:

Theorem 5 (Identification of $\varphi(\cdot)$ with Common Shocks) *Suppose Assumptions 1 and 2 hold and* $\mathrm{supp}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{supp}(\mathbf{x}_1) \times \mathrm{supp}(\mathbf{x}_2)$*. Furthermore assume that the common shock V, is independent of* $\mathbf{x}_i, K_i,\ i = 1, 2$*. Then the function $\varphi(\cdot)$ is identified up to scale.*

Proof. The proof is similar to that of Theorem 1. Consider the absolutely continuous component of the conditional distribution of $(\tilde{T}_1, \tilde{T}_2)$, the observed switching times for the individuals, given the covariates $\mathbf{x}_1, \mathbf{x}_2$. Like in the proof for Theorem 1 and using the definition of $\tilde{T}_i = \min\{T_i, V\}$, we can obtain that the probability density function for this pair on the set $\{(\tilde{t}_1, \tilde{t}_2) \in \mathbb{R}^2_+ : \tilde{t}_1 < \tilde{t}_2\}$ is given by:

$$
\begin{aligned}
f_{\tilde{T}_1,\tilde{T}_2|\mathbf{x}_1,\mathbf{x}_2}(\tilde{t}_1, \tilde{t}_2|\mathbf{x}_1, \mathbf{x}_2) = {} & \lambda(\tilde{t}_1)\varphi(\mathbf{x}_1)g(Z(\tilde{t}_1)\varphi(\mathbf{x}_1))\lambda(\tilde{t}_2)\varphi(\mathbf{x}_2)e^{\delta}g(Z(\tilde{t}_2)\varphi(\mathbf{x}_2)e^{\delta}) \times \\
& \times\mathbb{P}(V > \tilde{t}_2) + \lambda(\tilde{t}_1)\varphi(\mathbf{x}_1)g(Z(\tilde{t}_1)\varphi(\mathbf{x}_1))h(\tilde{t}_2)\mathbb{P}(T_2 > \tilde{t}_2)
\end{aligned}
$$

where

$$Z(t) = \int_0^t \lambda(s)ds, i = 1, 2.$$

[5]The optimal switching times derived in Section 2 would still hold. Should the realizations of V happen after that chosen time, the individual would have no incentives to wait. If v arrives earlier than the optimal time, there would be no incentive to anticipate the switch nor would there be anything to be done about it after the shock.

Given two sets of covariates $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{x}'_1, \mathbf{x}_2)$ we can again obtain that

$$\lim_{\substack{(\tilde{t}_1,\tilde{t}_2)\to(0,0)\\ \tilde{t}_1<\tilde{t}_2}} \frac{f_{\tilde{T}_1,\tilde{T}_2|\mathbf{x}_1,\mathbf{x}_2}(\tilde{t}_1,\tilde{t}_2|\mathbf{x}'_1,\mathbf{x}_2)}{f_{\tilde{T}_1,\tilde{T}_2|\mathbf{x}_1,\mathbf{x}_2}(\tilde{t}_1,\tilde{t}_2|\mathbf{x}_1,\mathbf{x}_2)} = \frac{\varphi(\mathbf{x}'_1)}{\varphi(\mathbf{x}_1)}$$

using the assumption that $\lim_{t\to 0} Z(t) = 0$. So, $\varphi(\cdot)$ is identified up to a scale transformation. ■

The assumption that K_i is independent across i is again stronger than necessary, as is the case with the condition that $\mathrm{supp}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{supp}(\mathbf{x}_1) \times \mathrm{supp}(\mathbf{x}_2)$. The proof strategy also allows $\varphi(\cdot)$ to depend on i.

5 Estimation Strategies

Consider first the case where $G(\cdot)$ is known. In the absence of interaction effects (δ) and when $G(\cdot)$ is a unit exponential, this would correspond to a classical proportional hazard model. The following characterization can then be obtained for the event $\{T_1 < T_2\}$:

$$\begin{aligned}\mathbb{P}(T_1 < T_2|\mathbf{x}_1,\mathbf{x}_2) &= \mathbb{P}(K_1\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1) < K_2|\mathbf{x}_1,\mathbf{x}_2)\\ &= \int_0^{+\infty}[1 - G(k\varphi(\mathbf{x}_2)e^{\delta_2}/\varphi(\mathbf{x}_1))]dG(k)\end{aligned}$$

and a similar characterization would hold for $\{T_2 < T_1\}$. Assume that $\varphi(\cdot)$ and $Z(\cdot)$ are modelled up to the (finite-dimensional) parameters β and λ, respectively ($\varphi(\cdot) \equiv \varphi(\cdot;\beta)$ and $Z(\cdot) \equiv Z(\cdot;\lambda)$). Given data on the realization of the game analyzed in this paper and the previous results, we then obtain the likelihood function for this problem as

$$
\begin{aligned}
\mathcal{L}(\beta,\lambda,\delta) \equiv\;& \Pi_{t_1<t_2}\Big\{\partial_t Z(t_1;\lambda)\varphi(\mathbf{x}_1;\beta)\partial_t Z(t_2;\lambda)\varphi(\mathbf{x}_2;\beta)e^{\delta}\times \\
& \times g(Z(t_1;\lambda)\varphi(\mathbf{x}_1;\beta))g(Z(t_2;\lambda)\varphi(\mathbf{x}_2;\beta)e^{\delta})\Big\}\times \\
& \times\Pi_{t_1>t_2}\Big\{\partial_t Z(t_1;\lambda)\varphi(\mathbf{x}_1;\beta)\partial_t Z(t_2;\lambda)\varphi(\mathbf{x}_2;\beta)e^{\delta}\times \\
& g(Z(t_1;\lambda)\varphi(\mathbf{x}_1;\beta)e^{\delta})\times g(Z(t_2;\lambda)\varphi(\mathbf{x}_2;\beta))\Big\}\times \\
& \times\Pi_{t_1=t_2}\Big\{1-\textstyle\int_0^{+\infty}[1-G(k\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1))]dG(k) \\
& -\textstyle\int_0^{+\infty}[1-G(k\varphi(\mathbf{x}_1)e^{\delta}/\varphi(\mathbf{x}_2))]dG(k)\Big\}
\end{aligned}
$$

where $\Pi_{t_1<t_2}, \Pi_{t_1>t_2}$ and $\Pi_{t_1=t_2}$ denote the product over the observations for which $t_1 < t_2$, $t_1 > t_2$ and $t_1 = t_2$. We use the fact that, for sequential switching ($t_1 < t_2$ or $t_1 > t_2$), it is possible to date the termination time, but not for the event in which termination times coincide. Under standard assumptions, this likelihood function provides us with an estimator for the parameters of interest in this model. We conjecture that a sieves approach or the ideas contained in Ai (1997), for instance, may be adapted to obtain a more general estimation procedure.

The characterization above can also be used to obtain an estimator for $\varphi(\cdot;\beta)$ and δ without the assumption that $Z(\cdot)$ is the same across games as long as it is the same for players within the same game. Assume initially that $G(\cdot)$ is the cdf for a unit exponential distribution: $G(t) = (1-e^{-t})\mathbb{I}_{t\geq 0}$. In particular, we can focus on the event $\{T_1 < T_2\}$ (or $\{T_1 > T_2\}$) and use the probability

$$
\begin{aligned}
\mathbb{P}(T_1 < T_2|\mathbf{x}_1,\mathbf{x}_2) &= \mathbb{P}(Z^{-1}(K_1/\varphi(\mathbf{x}_1)) < Z^{-1}(K_2 e^{-\delta}/\varphi(\mathbf{x}_2))|\mathbf{x}_1,\mathbf{x}_2) \\
&= \mathbb{P}(K_1\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1) < K_2|\mathbf{x}_1,\mathbf{x}_2) \\
&= \int_0^{\infty} e^{-k_1}\int_{k_1\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1)}^{\infty} e^{-k_2}dk_2 dk_1 \\
&= \int_0^{\infty} e^{-k_1-k_1\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1)}dk_1 \\
&= \frac{1}{1+\varphi(\mathbf{x}_2)e^{\delta}/\varphi(\mathbf{x}_1)} \\
&= \frac{1}{1+e^{\delta+\log\varphi(\mathbf{x}_2)-\log\varphi(\mathbf{x}_1)}}
\end{aligned}
$$

Taking $\varphi(\mathbf{x};\beta) = \exp(\beta'\mathbf{x})$, for example, the above becomes

$$\frac{1}{1+e^{\delta+\beta'(\mathbf{x}_2-\mathbf{x}_1)}} = \frac{e^{\beta'(\mathbf{x}_1-\mathbf{x}_2)-\delta}}{1+e^{\beta'(\mathbf{x}_1-\mathbf{x}_2)-\delta}}.$$

An analogous expression can be obtained for $\{T_2 < T_1\}$. Define then the variable Y by

$$\begin{aligned} Y = 1 &\quad \text{if} \quad T_1 < T_2 \\ Y = 2 &\quad \text{if} \quad T_1 = T_2 \\ Y = 3 &\quad \text{if} \quad T_1 > T_2 \end{aligned}$$

It can be seen that

$$\begin{aligned} \mathbb{P}(Y \leq 1|\mathbf{x}_1,\mathbf{x}_2) &= \Lambda((\mathbf{x}_1-\mathbf{x}_2)\beta-\delta) \\ \mathbb{P}(Y \leq 2|\mathbf{x}_1,\mathbf{x}_2) &= \Lambda((\mathbf{x}_1-\mathbf{x}_2)\beta+\delta) \end{aligned}$$

where $\Lambda(\cdot)$ is the cdf for the logistic distribution. This corresponds to an ordered logit on Y with explanatory variables $\mathbf{x}_1 - \mathbf{x}_2$ and cutoff points at $-\delta$ and δ. If we take $G(\cdot)$ to be the cdf for a log-normal distribution, an ordered probit is obtained.

When $G(\cdot)$ is unknown this becomes

$$\begin{aligned} \mathbb{P}(Y \leq 1|\mathbf{x}_1,\mathbf{x}_2) &= H((\mathbf{x}_1-\mathbf{x}_2)\beta-\delta) \\ \mathbb{P}(Y \leq 2|\mathbf{x}_1,\mathbf{x}_2) &= H((\mathbf{x}_1-\mathbf{x}_2)\beta+\delta) \end{aligned}$$

where $H(w) = \mathbb{P}(\ln K_1 - \ln K_2 \leq w)$. Various authors have proposed alternative estimation procedures for the estimation of this semiparametric ordered choice model (for instance, Chen and Khan (2003), Coppejans (2007), Klein and Sherman (2002), Lee (1992), Lewbel (2003) and Honoré and de Paula (2007)).

Finally we note that, if $G(\cdot)$, and hence $H(\cdot)$, is known, δ is identified even if $\mathbf{x}_1 = \mathbf{x}_2$ since

$$\delta = -H^{-1}(\mathbb{P}(T_1 < T_2|\mathbf{x})).$$

6 Conclusion

In this article we have provided a novel motivation for simultaneous duration models that relies on strategic interactions between agents. The paper thus relates to previous literature on empirical games. We presented an analysis of the possible Nash equilibria in the game and noticed that it displays multiple equilibria, but in a way that still permits point identification of structural objects.

The maintained assumption in the paper is that agents can exactly control their duration. Heckman and Borjas (1980), Honoré (1993) and Frijters (2002) consider statistical models in which the hazard for one duration depends on the outcome of a previous duration and Rosholm and Svarer (2001) consider a model in which the hazard for one duration depends on the simultaneous hazard for a different duration. It would be interesting to investigate whether a strategic economic model in which agents can control their hazard subject to costs, will generate incomplete econometric models and what the effect of this would be on the identifiability of the key parameters of the model.

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