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# "Utility-Based Utility"
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# Utility-Based Utility*

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**Abstract**

A major virtue of von Neumann-Morgenstern utilities, for example, in the theory of general financial equilibrium (GFE), is that they ensure intertemporal consistency: consumption-portfolio plans (for the future) are in fact executed (in the future) – assuming that there is perfect foresight about relevant endogenous variables. This note proposes an alternative to expected utility, one which also delivers consistency between plan and execution – and more. In particular, it turns out that one special case is in fact simply discounted (subjective) expected utility. Moreover, this alternative formulation affords an extremely natural setting for introducing extrinsic uncertainty. The key idea behind my approach is to divorce the concept of filtration (of the state space) from any considerations involving probability (on the state space), and then concentrate attention on nested utilities of consumption looking forward from any date-event: utility today depends only on consumption today and prospective utility of consumption tomorrow, utility tomorrow depends only on consumption tomorrow and prospective utility of consumption the day after tomorrow, and so on.

JEL classification: D61, D81, D91

Key words: Utility theory, Expected utility, Intertemporal consistency, Extrinsic uncertainty, Cass-Shell Immunity Theorem

*Interaction with the very able TA's helping me with (carrying?) the first year equilibrium theory course at Penn during the fall of 2007 – Matt Hoelle and Soojin Kim – spurred me into pursuing this research. They are not responsible for the trail I followed, however. After having searched my memory for personal antecedents, I realized that the main offshoot cultivated here – a more pleasing (to me) development of the basic concept of extrinsic uncertainty – has been germinating for a long time, most likely being a cutting taken from conversations I had with Yves Balasko in the past, and then later with Herakles Polemarchakis, concerning Yves's clever generaliztion, reported in [2].

## I. Introduction

In the presence of uncertainty, a necessary condition for realized consumption to accord with planned consumption is *intertemporal consistency* of utility, which has two distinct aspects: First, utility at any date-event must only depend on consumption at that date-event and prospective utilities at successor date-events. Second, utility in the initial period must be strictly increasing in prospective utility at every subsequent date-event. It is well-understood that discounted expected utility (EU) is sufficient for intertemporal consistency. It seems to me that it is also widely believed that something like the converse must be true. Such a belief is false. The purpose of this note is to provide an alternative, general formulation of utility, which I have labeled utility-based utility (UBU). Besides embodying intertemporal consistency, this alternative is founded on a simple observation: the concept of a state space is distinct from any notion of probability.

In order to build on this base, I focus first on the leading case, where there are only two periods, today and tomorrow, with uncertainty about what environment will prevail during the second. Then, after defining UBU in terms of consumption today and prospective utilities tomorrow, I describe primitive assumptions under which it displays standard regularity, monotonicity, and convexity properties required, for example, to prove existence of a GFE when there are (complete or incomplete) markets for nominal assets. I also relate this formulation to the more familiar discounted EU hypothesis, which turns out to be the special case where utility today depends linearly on prospective utilities tomorrow. Besides entailing intertemporal consistency (which follows directly from its definition) UBU provides an especially congenial setting for specifying the concept of extrinsic uncertainty (which Cass and Shell originally specified in terms of EU; pp. 196-198 in [3]). My specification here involves two specializations of the UBU hypothesis: invariance of the utility indices for consumption at future date-events and symmetry between them. In this context, the usefulness of introducing such a symmetry property was first recognized, and then exploited by Balasko, and I have adapted his Axiom 2 (p. 205 in [2]) for my purposes here. Finally, after describing the extension of UBU from 2 to $2 < T+1 < \infty$ periods, I briefly discuss the relationship of my approach to the seminal analyses of Arrow [1] and Debreu [4], contributions which ushered the Wald-Savage viewpoint about uncertainty into economics.

Searching the literature (after this note was almost completed, as is my wont) I confirmed a vague memory that the closest previous work is a nice note by Johnsen and Donaldson [5]. They concentrate on the leading case, permit path dependence (which I don't, since otherwise optimization smacks of choosing which habits to form), and assume strict monotonicity. Their main analysis concludes with the analogue of my postulated representation (1), which they show is necessary as well as sufficient for intertemporal consistency. So, basically, my analysis starts where theirs leaves off.

**II. Leading Case**

Let $s \in \mathcal{S} = \{1, 2, \cdots, S\}$ with $S < \infty$ denote the possible *states* of the world tomorrow (and, for convenience, $s = 0$ denote today – so that $\{0\} \cup \mathcal{S}$ are all the possible *spots* at which economic activity might take place), $c \in \mathcal{C} = \{1, 2, \cdots, C\}$ with $C < \infty$ the distinct commodities (say, in terms of their physical characteristics), and $x = (x(0), (x(s), s \in \mathcal{S}))$ a consumption vector. A representative household is described by his consumption set $X \subset \mathbb{R}_+^{C(S+1)}$, utility function $u : X \to \mathbb{R}$, and endowment $e = (e(0), (e(s), s \in \mathcal{S})) \in X$ (this last will not be used until the following section). My basic assumption is that $u$ takes the general form

$$u(x) = v^0(x(0), (v^s(x(s)), s \in \mathcal{S})), \tag{1}$$

where, for $V^s \subset \mathbb{R}, s \in \mathcal{S}, v^0 : \mathbb{R}_+^C \times_{s\in\mathcal{S}} V^s \to \mathbb{R}$ is the household's utility as perceived from spot 0, and, for $s \in \mathcal{S}, v^s : \mathbb{R}_+^C \to V^s$ is his utility as perceived tomorrow from spot $s > 0$ – after today has become history.

It is easily seen that if, for all $s, v^s$ is continuous and increasing, then so is $u$. Assume, in addition, that $v^0$ is strictly increasing in $v^s, s \in \mathcal{S}$. Then it is obvious that (1) entails intertemporal consistency: an optimal plan in period 0 must evolve into an optimal choice in state $s \in \mathcal{S}$. So only convexity properties of $u$ require explicit justification. I show that if, for all $s, v^s$ is (strictly) concave, then $u$ is (strictly) concave: for $x'', x' \in X, x'' \neq x'$, and $0 < \theta < 1$,

$$\begin{aligned} u((1-\theta)x'' + \theta x') &= v^0((1-\theta)x''(0) + \theta x'(0), (v^s((1-\theta)x''(s) + \theta x'(s)), s \in \mathcal{S})) \\ &\geqq v^0((1-\theta)x''(0) + \theta x'(0), ((1-\theta)v^s(x''(s)) + \theta v^s(x'(s)), s \in \mathcal{S})) \\ &\geqq (1-\theta)v^0(x''(0), (v^s(x''(s)), s \in \mathcal{S})) + \theta v^0(x'(0), (v^s(x'(s)), s \in \mathcal{S})) \\ &= (1-\theta)u(x'') + \theta u(x'), \end{aligned}$$

with at least one of the inequalities strict when all the component mappings are strictly concave.

But what about the EU hypothesis (and thus the various axiom systems used to justify it)? **In blunt terms, this conventional formulation emerges as simply one of many possible specializations – after it is been properly understood what future prospects must actually represent.** To see this clearly, assume that $v^0$ is linear in $v^s, s \in \mathcal{S}$. Then, without log,

$$\begin{aligned} u(x) &= v^0(x(0)) + \textstyle\sum_{s\in\mathcal{S}} a^s v^s(x(s)) \\ &= v^0(x(0)) + (1+\delta)^{-1} \textstyle\sum_{s\in\mathcal{S}} \pi^s v^s(x(s)) \end{aligned} \tag{2}$$

where $\delta > -1$ is a discount rate, and $\pi^s > 0, s \in \mathcal{S}$, with $\sum_{s\in\mathcal{S}} \pi^s = 1$ are, generally, subjective probabilities. In other words, discounted (subjective) EU (hereafter, again simply EU) is just a special linear version of UBU. And, aside from ease of analysis (or maybe the intellectual laziness which comes from familiarity), is there a single, substantive and convincing reason why the household's discount rate (or for that matter, when viewed from today, its marginal rates of substitution between utilities in different states) should be constant? Not one that I can easily imagine. So, it seems to me that EU leaves much to be desired.

**III. Extrinsic Uncertainty**

Let $h \in \mathcal{H} = \{1, 2, \cdots, H\}$ with $H < \infty$ denote the households populating a Walrasian economy. These are described by $X_h, u_h$ satisfying (1), and $e_h, h \in \mathcal{H}$. Extrinsic uncertainty, as Karl Shell and I have described it in general terms originally, is uncertainty which does not affect the fundamentals of an economy. In this setting (with pure distribution), the fundamentals are the housholds' certainty utilities and their endowments, and extrinsic uncertainty is defined by two properties, for $h \in \mathcal{H}$,

**Invariance**, of endowments,

$$e_h(s) = \bar{e}_h, s \in \mathcal{S}\,, \tag{3}$$

and of future utility,

$$v_h^s = v_h, s \in \mathcal{S}\,, \tag{4}$$

together with

**Symmetry** of present utility $v_h^0$ in terms of invariant future utility $v_h$, that is, for every permutation of $\mathcal{S}, \sigma : \mathcal{S} \to \mathcal{S}$,

$$v_h^0(x_h(0), (v_h(x_h(\sigma(s)), s \in \mathcal{S})) = v_h^0(x_h(0), (v_h(x_h(s)), s \in \mathcal{S})). \tag{5}$$

It is clear what (3) means: extrinsic uncertainty has no affect whatsoever on the households' endowments. Less obvious is that (4)-(5) mean, in effect, that $v_h$ is basically just certainty utility in the second period. This follows from the observation that, given invariance of future utility, symmetry reduces to the property that, if $x_h(s) = \bar{x}_h(1), s \in \mathcal{S}$, then the labeling of states is immaterial. So I can write $v_h^0(x_h(0), (v_h(x_h(s)), s \in \mathcal{S}))$ as simply $v_h^0(x_h(0), v_h(\bar{x}_h(1)))$.

Given the additional structure (3)-(5), it can be shown (the same result follows from Balasko's reformulation) that the Cass-Shell Immunity Theorem remains valid. This argument seems well worth presenting explicitly, since the theorem provides a useful benchmark (as well as substantive validation for my specific definition of extrinsic uncertainty). In order to avoid the uninteresting cases which may arise when there are flats, assume that, for $h \in \mathcal{H}, v_h$ is strictly concave.

**Immunity to Extrinsic Uncertainty.** Under the same assumptions (implicit as well as explicit) required for the FBWT, every Walrasian or general equilibrium (GE) allocation is state-invariant (or as Karl and I described it, in more catchy terms, "sunspots don't matter").

**Proof.** Suppose that $(x_h^*, h \in \mathcal{H})$ is a GE allocation s.t., for some $h^*$ and $s'', s' \in \mathcal{S}$, $x_{h^*}^*(s'') \neq x_{h^*}^*(s')$. I will show that the average allocation

$$\bar{x}_h = (x_h^*(0), (\bar{x}_h(1), s \in \mathcal{S}) \text{ with } \bar{x}_h(1) = (1/\mathcal{S}) \sum_{s \in \mathcal{S}} x_h^*(s), h \in \mathcal{H},$$

is (i) a feasible allocation, and (ii) Pareto dominates the supposed GE allocation in which future consumption varies for some household. This contradicts the FBWT.
(feasibility) Summing $\bar{x}_h(1)$ over $h$, interchanging the order of summation, and then using spot market clearing for $s \in \mathcal{S}$ and invariance of endowments (3) yields materials balance in each state

$$\begin{aligned} \textstyle\sum_{h\in\mathcal{H}} \bar{x}_h(1) &= (1/S) \textstyle\sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} x_h^*(s) \\ &= (1/S) \textstyle\sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} e_h(s) \\ &= (1/S) \textstyle\sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} \bar{e}_h = \sum_{h\in\mathcal{H}} \bar{e}_h. \end{aligned}$$

So, since spot market clearing also yields materials balance today, $\bar{x} = (\bar{x}_h, h \in \mathcal{H})$ is a feasible allocation.
(Pareto dominance) Using invariance of future utility (4), and then symmetry (5) for the particular, say, *circular permutations* $\sigma(s'), s' \in \mathcal{S}$, s.t.

$$s \mapsto \sigma(s, s') = \begin{cases} s' + (s-1), & s' + (s-1) \leqq S, s \in \mathcal{S}, \\ s' + (s-1) - S, & s' + (s-1) > S \end{cases}$$

yields, for $h \in \mathcal{H}$,

$$\begin{aligned} u_h(x_h^*) &= \textstyle\sum_{s'\in\mathcal{S}} (1/S) u_h(x_h^*) \\ &= \textstyle\sum_{s'\in\mathcal{S}} (1/S) v_h^0(x_h^*(0), (v_h(x_h^*(\sigma(s,s'))), s \in \mathcal{S})) \\ &\leqq v_h^0(x_h^*(0), (\textstyle\sum_{s'\in\mathcal{S}} (1/S) v_h(x_h^*(\sigma(s,s'))), s \in \mathcal{S})) \\ &\left\{ \begin{array}{c} < \\ \leqq \end{array} \right\} v_h^0(x_h^*(0), (v_h(\bar{x}_h(1)), s \in \mathcal{S}) = u_h(\bar{x}_h) \text{ according as } h \left\{ \begin{array}{c} = \\ \neq \end{array} \right\} h^*, \end{aligned}$$

and hence $u_h(\bar{x}_h) \geqq u_h(x_h^*), h \in \mathcal{H}$, with strict inequality for $h = h^*$, and the argument is complete.■

**Remarks.** 1. A fortiori, the proof remains valid under the weaker assumptions that only aggregate resources $r = \sum_{h\in\mathcal{H}} e_h$ rather than individual endowments $(e_h, h \in \mathcal{H})$

are invariant, and that only the circular perturbations $((\sigma(s,s'), s \in \mathcal{S}), s' \in \mathcal{S}\,)$ have no effect on overall utility.
2. With EU, symmetry means equiprobability in (2) – $\pi^s = 1/S, s \in \mathcal{S}$ – the only case in which the original Cass-Shell definition of extrinsic uncertainty coincides with that which accords with UBU. In fact, for me it is obvious now that UBU is better suited to specifying that preferences are unaffected by extrinsic utility, precisely because this formulation avoids a host of awkward questions concerning probabilities – in particular, the question of why they should be identical across households.

## IV. Many Periods

Let $\mathcal{S}_t, 0 \leqq t \leqq T$ be a filtration of $\mathcal{S}$ over periods $t = 0, 1, \cdots, T$ with $T < \infty$, that is, a finite sequence of partitions of $\mathcal{S}$ s.t., for $0 < t \leqq T, \mathcal{S}_t$ is a finer partition of $\mathcal{S}_{t-1}$, and $\mathcal{S}_0 = \{\mathcal{S}\}$ and $\mathcal{S}_T = \mathcal{S}$. A typical element of $\mathcal{S}_t$ is $s_t$. Also, for $0 < t \leqq T$, let $\mathcal{S}_t(s_{t-1})$ be the successors of $s_{t-1}$, that is, the partition of $s_{t-1}$ in $\mathcal{S}_t$. The generalization of (1) for this extension is straightforward (as is the verification that it is continuous, increasing, and [strictly] concave provided that all the component mappings, $v^{s_t}, s_t \in \cup_{t=0}^{t=T} \mathcal{S}_t$, are),

$$\begin{aligned} u(x) \;\; = \;\; & v^{s_0}(x(s_0), (v^{s_1}(x(s_1), (v^{s_2}(x(s_2)), (v^{s_3}(x(s_3), \cdots, (v^{s_T}(x(s_T)), s_T \in \mathcal{S}_T(s_{T-1})), \\ & s_{T-1} \in \mathcal{S}_{T-1}(s_{T-2})), s_{T-2} \in \mathcal{S}_{T-2}(s_{T-3})), \cdots, s_1 \in \mathcal{S}_1(s_0)). \end{aligned}$$

As before, intertemporal consistency follows – after assuming that, in addition, $v^{s_t}$ is strictly increasing in $v^{s_{t+1}}$– using a backward induction argument most familiar from the "hot potato" problem in monetary theory (or perhaps, for the more recently educated, from the definition of subgame perfection in game theory).

Regarding the finite horizon: It is an open question whether there is a relatively straightforward way of extending this general case to an infinite horizon, but this seems unlikely. However, it is clear that some special cases can be. In particular, this is true for EU. Thus, for anyone who believes that postulating infinite-lived households leads to constructing useful models for interpreting real world phenomena, this is a very welcome parameterization. But I don't. Rather, I find it much more interesting (as well as very gratifying) that the rationale underlying UBU also provides a natural way for evaluating a vaguely uncertain future beyond the terminus, namely, inclusion of an estimate of the utility which will be derived from terminal stocks: even for $T$ (in conventional units of time) relatively small, my formulation admits consistent treatment of both direct and indirect utility.

## V. Historical Note

What I aim to do here is elaborate how my formulation of utility is related to the original Arrow-Debreu formulation of the state-of-the-world approach to modeling uncertainty in economics. I take some liberty in interpreting Arrow's analysis according to the later development of GFE based on it. Moreover my criticism of Debreu requires recognition of the importance of intertemporal consistency, whose need only really became quite apparent later on. In other words, my critique relies heavily on perfect hindsight. So I must emphasize that it is designed only to illuminate (certainly not to denigrate) the crucial contributions of both to the modern development of equilibrium theory.

Arrow's ingenious paper presents his fundamental Equivalency Theorem (AET). Again for the leading case, consider two market structures: The first postulates spot markets for commodities at every spot $s \geqq 0$, together with a market for nominal assets (i.e., assets whose payoffs are specified in units of account) at spot 0 (Arrow). In contrast, the second postulates a single overall market for contingent commodities at spot 0 (Debreu). Let $p = (p(s), s \geqq 0) \in \mathbb{R}_+^{C(S+1)} \backslash \{0\}$ represent spot prices, $\lambda = (\lambda(s), s \in \mathcal{S}) \in \mathbb{R}_{++}^S$ state prices (i.e., the values of wealth in the future relative to wealth today) and $p' = (p'(s), s \geqq 0) \in \mathbb{R}_+^{C(S+1)} \backslash \{0\}$ contingent commodity prices. Then AET states that if there is a complete asset market, and equilibrium prices with the second market structure are related to those with the first by the formula

$$p'(s) = \begin{cases} p(s), & s = 0 \\ \lambda(s)p(s), & s \in \mathcal{S}, \end{cases} \tag{6}$$

then the set of allocations corresponding to GE is identical to that corresponding to GFE. The essential requirement is the presence of a complete asset market, where there are $S$ independent assets (in terms of their payoffs), and therefore, given state prices, unique asset prices (determined by no-arbitrage considerations). The proof of the theorem consists in showing that, focusing on just consumption, the relationship (6) implies that the budget sets for the two market structures are the same. This means that – except for a weak spot-by-spot monotonicity assumption for some household (in order to justify no-arbitrage) – AET does not depend in any way on the households' utility functions: the theorem is consistent with UBU. Since I've shown that EU is merely a special case, it is therefore not required per se for the theorem's validity.

This last claim seems contradicted by Arrow's concern with concavity of the certainty utility function (and hence quasi-concavity of the EU function; p. 95 in [1]). There is no conflict. Arrow mixes his equivalency result into a proof of the SBWT when there are spot markets for commodities and assets, a proof in which there is need for convexity. And since his argument also relies on intertemporal consistency, he used the only construct then available to guarantee this. Note that he and I agree on the need for the component mappings defining overall utility to be concave, though he shows that this property is necessary as well as sufficient (for quasi-concavity of EU), while I don't. It is an open question whether, in some sense, concavity (together with monotonicity) is necessary for quasi-concavity of UBU, though this is a plausible conjecture.

How does all this reflect on Debreu's careful exposition of the notion of a filtration of the state space – in order to justify his claim that uncertainty represented by date-events is just another commodity characteristic? Well, while the concept of contingent commodities available at future date-events is itself extremely useful, the additional concept of a single overall market for contingent commodities is just a useful fiction; it only makes sense in light of AET, which in turn only makes sense when utility functions are intertemporally consistent. This belies Debreu's confident assertion that his approach is compatible with utility functions of the same generality as those in any model of GE (p. 98 in [4]). While the assertion is (in Debreu's own words) "formally" correct, it is misleading. As I've claimed throughout, much more is required, in particular, the intertemporal consistency provided by UBU, which is indeed (again in Debreu's own words) "free from any probability concept" – a property which I too strongly commend.

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