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PIER Working Paper 07-027

"Non-parametric counterfactual analysis in dynamic general equilibrium"

by

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http://ssrn.com/abstract=1015525

Non-parametric counterfactual analysis in dynamic general equilibrium*

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September 17, 2007

Abstract

In this paper we examine non-parametric restrictions on counterfactual analysis in a simple dynamic stochastic general equilibrium model. Under the assumption of time-separable expected utility and complete markets all equilibria in this model are stationary, the Arrow-Debreu prices uniquely reveal the probabilities and discount factor and the equilibrium correspondence defined as the map from endowments to stationary (probability-free) state prices, is identical to the equilibrium correspondence in a standard Arrow-Debreu exchange economy with additively separable utility.

We examine observable restriction on this correspondence and give necessary as well as sufficient conditions on profiles of individual endowments that ensure that associated equilibrium prices cannot be arbitrary. While often there are restrictions on possible price changes we also show that in most cases results from a single agent economy do not carry over to a setting with heterogeneous agents.

JEL classification numbers: D50, G10;

Keywords: Dynamic general equilibrium, non-parametric analysis, observable restrictions.

*We thank Don Brown as well as seminar participants at the 2007 SAET conference in Kos/Greece for helpful comments and discussions.

1 Introduction

This paper investigates how equilibrium prices change as profiles of endowments change in a dynamic asset pricing model with heterogeneous agents. In this model all competitive equilibria are stationary if all agents maximize time-separable expected utility and individual endowments follow time-homogeneous Markov chains. In fact, there is a one-to-one relation between the equilibrium correspondence in this model and the equilibrium correspondence of a standard static Arrow-Debreu exchange economy with additively separable utility. We use the non-parametric analysis of Brown and Matzkin (1996) to explore observable restrictions on this equilibrium correspondence. The three main results are as follows. First, we show that equilibrium price changes can be arbitrary if individual endowments change but aggregate endowments are held fixed. Secondly, we show that changes in aggregate endowments always lead to 'predictable' price changes, if in at least one state aggregate endowments weakly decrease while in some other state all individuals' endowments increase. Lastly, we show that restrictions from the single-agent version of the model are neither necessary nor sufficient for restrictions in a heterogeneous agents economy.

Dynamic general equilibrium models play a prominent role in modern macroeconomics, finance and public finance. While stochastic dynamic models with heterogeneous agents have become increasingly important in this literature, there are few general results on counterfactual analysis[1]. In models with complete financial markets, under the assumption that all agents maximize time-separable expected utility, there exists a pricing representative agent: Any given competitive equilibrium price system can be viewed as supporting prices for a single individual who consumes aggregate endowments (see e.g. Constantinides (1982)). However, obviously this fact does not imply that in models with heterogeneous agents, results from counterfactual analysis are similar to the ones in a model with a single agent. In this paper, we investigate if there are any restrictions on global comparative statics in models with several agents and how they compare to the single agent intuition.

For the investigation of counterfactual analysis in a dynamic model with heterogeneous agents, we consider the simplest possible case and ask what happens to equilibrium prices as profiles of individual endowments change. Nachbar (2002) examines this question in an exchange economy with general preferences and gives (very restrictive) conditions under which prices change monotonically with endowments. In most applied work, computational experiments are used to explore the effects of exogenous changes in taxes or transfers on equilibrium allocations and prices, often assuming identical homothetic utility. We regard the consideration of changes in individual endowments as a first step towards understand-

[1] The use of the term "counterfactual analysis" in this paper is inspired by the usage of the term "counterfactual policy analysis" in macroeconomics. We use this term in the sense of a global comparative statics analysis and want to distinguish our analysis from purely local comparative statics or maginal analysis, often used in consumer and producer theory.

ing these more complicated comparative statics exercises. We interpret these changes in endowments as unanticipated policy changes or as a structural break. Either all agents in the economy assign probability zero to this event or the event is completely uninsurable, i.e. there is no asset that pays contingent on the state where the change happens. We largely abstract from the fact that in the presence of long-lived assets, all events are partially insurable and there will be price effects which lead to endogenous changes in the wealth distribution.

The computational results in applied work often seem relatively robust with respect to small changes in preferences. However, from a theoretical standpoint, it might seem that without any assumptions on preferences, almost any counterfactual analysis could be possible. While Brown and Matzkin (1996) successfully challenge the view that without parametric assumptions on preferences 'anything goes' in general equilibrium analysis, there have been few attempts in the literature to characterize the exact form of the observable restrictions. A notable exception is Balasko and Tvede (2005) who, in a standard Arrow Debreu exchange economy, give sufficient conditions on profiles of individual endowments for associated equilibrium prices to be arbitrary. They reach the rather negative conclusion that 'finite collections of data that are not included in any equilibrium manifold make up a set that is certainly not large'. While this seems to suggest that for many global comparative statics exercises general equilibrium imposes no restrictions on observables, the crucial assumption in their analysis is that aggregate endowments remain constant. Therefore, their results are somewhat comparable to Mas-Colell (1977) who shows that without any variations in endowments, the equilibrium set can be arbitrary.

We consider the simplest model of an infinite exchange economy under uncertainty, a version of the Lucas (1978) asset pricing model with heterogeneous investors. All exogenous variables follow a finite Markov chain, all individuals maximize time-separable expected utility with identical beliefs and discounting. When markets are dynamically complete, all asset market equilibria are equivalent to Arrow-Debreu equilibria with stationary consumption allocations (see e.g. Kubler and Schmedders (2003)). We show that the price system reveals uniquely the agents' beliefs and discount factor. If there are S possible shocks, the prices can in fact be decomposed into the $S \times S$ transition probability matrix, a discount factor and S strictly positive probability-free prices that are unique up to a normalization. Under the assumption that beliefs remain constant as endowments change, the first obvious restriction on prices is that after the decomposition only belief-free prices change. The question is then what restrictions exist on the changes of these belief-free prices.

If there is a single agent, it is easy to provide very clear-cut necessary and sufficient conditions on price changes as endowments and therefore the agent's consumptions changes. The purpose of this paper is, first, to explore if it is still possible to make predictions about price changes in a model with heterogeneous agents and, second, to compare these changes with the predictions from the single agent model.

The results depend to some extent on the preference assumptions. The assumption of time separable expected utility leaves open the possibility that felicity functions are shock-dependent. Melino and Yang (2003) and Danthine et al. (2004) emphasize the usefulness of this assumption for explaining standard asset pricing puzzles. The main focus of this paper is on economies where utility functions can be shock dependent. Under this assumption of shock-dependent felicity functions, restrictions on prices exist – even in the single-agent framework – only if in at least one shock aggregate endowments decrease while in some other they increase. With several agents, changes in the income distribution with aggregate endowments held fixed can have arbitrary effects on equilibrium prices. If on the other hand felicity functions are shock-invariant, it is well known that state prices must be negatively co-monotone to aggregate endowments. For completeness, we also discuss this case of shock-independent felicity functions in the context of our heterogeneous agent model and show that while changes in the income distribution can lead to large changes in prices, these cannot be arbitrary and there are restrictions beyond the co-monotonicity condition.

The question of how prices change in response to endowment changes that also vary aggregate endowments is more complicated. On one hand, if there are sufficiently many agents, given any changes of aggregate endowments, one can always construct individual endowments such that prices can be arbitrary, even if utility is shock-independent. On the other hand, we derive a general sufficient condition for restrictions even for the case of shock-dependent utility. If all agents' individual endowments increase in one shock, while aggregate endowments weakly decrease in some other shock, the associated prices cannot be arbitrary. This condition is in line with the intuition that if in some state aggregate endowments decrease while in some other they increase, prices must change in a predictable way. Given this condition, at some prices, there must be agents in the economy whose consumption must change in the same direction as aggregate endowments.

These results raise the question how the restrictions in a model with several agents that arise through changes in aggregate endowments are related to restriction in the single-agent model. We show that there exist changes in endowments for which there exist restrictions on associated prices in a model with several agents even if there are no restrictions in the single-agent model, given the changes in aggregate endowments. Conversely, as already noted above the existence of restrictions in a single-agent economy does not imply restrictions in economies with several agents, if individual endowments can be chosen freely. It turns out that even if individual endowments are collinear and even if all agents have shock-independent utility, the qualitative predictions from the heterogeneous agent model are very different from the ones in the single agent economy.

The remainder of the paper is organized as follows. In Section 2 we introduce the model and show its equivalence to a static Arrow-Debreu model with additively separable utility.

In Section 3 we motivate why we are interested in counterfactual analysis and show how in this model beliefs can be identified separately from prices. Section 4 presents necessary as well as sufficient conditions on changes in endowments for equilibrium price changes not to be arbitrary. In Section 5, we compare the restriction of the model with several agents to restrictions that arise in the single-agent framework. Section 6 concludes.

2 The dynamic exchange economy

We consider a dynamic exchange economy under uncertainty with a single perishable good each period. Time is discrete, $t = 0, 1, \ldots, T \leq \infty$. Uncertainty is driven by exogenous shocks s_t that take values in a finite set set $\mathcal{S} = \{1, 2, \ldots, S\}$ and follow a Markov chain with transition matrix π. Without loss of generality we assume that $s_0 = 1$. A date-event is a finite history of shocks, $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ and the set of all date-events (nodes) of the event tree is denoted by Σ. We write σ for a generic node in the tree and collect all nodes at some time t in $\mathcal{N}_t = \{s^t\}$. We write $\pi(s^t)$ for the period zero probability of node s^t and $\pi(s_{t-1}, s_t)$ for the one-period transition probabilities, that is, the elements of the matrix π.

There are H types of T-period (possibly infinitely) lived agents, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. Individual endowments are a time-invariant function of the exogenous shock alone, $e^h(s^t) = \bar{e}^h(s_t)$ with $\bar{e}^h : \mathcal{S} \to \mathbb{R}_{++}$. Agents have time separable expected utility (EU) preferences represented by the utility function

$$U^h(c) = E_0 \sum_{t=0}^{T} \beta^t u^h(c(s^t), s_t),$$

where $\beta \in (0, 1)$, and for each $s \in \mathcal{S}$, $u^h(\cdot, s) : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, strictly increasing and continuous for $c > 0$. A common assumption in the literature is that utility is shock invariant, i.e. for all $s \in \mathcal{S}$, $u^h(c, s) = u^h(c)$ for some felicity function that is independent of the shock.

2.1 Arrow-Debreu equilibrium

In this paper we abstract from asset markets and simply assume that markets for commodities across all nodes of the event tree are complete (see Kubler and Schmedders (2003) for a formal analysis of the model with asset markets). We can therefore describe the resulting Arrow-Debreu economy simply as a collection of utility functions and individual endowments $(U^h, e^h)^{h \in \mathcal{H}}$. A Walrasian equilibrium (W.E.) for this economy is defined in standard fashion as a collection of state prices $p(\sigma) > 0$ and consumption allocations $c^h(\sigma) \geq 0$ for all nodes $\sigma \in \Sigma$ such that

1. Markets clear, $\sum_{h \in \mathcal{H}} \left(c^h(\sigma) - e^h(\sigma)\right) = 0$ for all $\sigma \in \Sigma$;

2. Agents optimize, $\quad \forall h \in \mathcal{H} \quad (c^h(\sigma))_{\sigma\in\Sigma} \in \arg\max U^h(c) \quad$ s. t.
$$\sum_{\sigma\in\Sigma} p(\sigma)c(\sigma) = \sum_{\sigma\in\Sigma} p(\sigma)e^h(\sigma) < \infty.$$

Walrasian equilibrium exists and both the first and second welfare theorem hold for these economies (Bewley (1972)). The following lemma provides the key to tractable computations in stationary models with many periods. The lemma is well known, see for example the textbook by Duffie (1988) or Kubler and Schmedders (2003). We repeat the proof for completeness.

LEMMA 1 *Given an efficient allocation $(c^h(\sigma))^{h\in\mathcal{H}}_{\sigma\in\Sigma}$, the individual consumptions must be time-invariant functions of the shock alone, i.e. there exist $\bar{c} : \mathcal{S} \to \mathbb{R}^H_+$ such that for all $s^t \in \Sigma$ and all $h \in \mathcal{H}$, $c^h(s^t) = \bar{c}^h(s_t)$.*

Proof: Suppose that there is an equilibrium where for two date-event nodes $\sigma \in \mathcal{N}_t, \sigma' \in \mathcal{N}_{t'}$ with same current shock s, we have $c^{\bar{h}}(\sigma) \neq c^{\bar{h}}(\sigma')$ for some agent $\bar{h} \in \mathcal{H}$. Then we could improve everybody's utility by redistributing consumption at these nodes as follows, let

$$\tilde{c}^h(\sigma) = \frac{\beta^t \pi(\sigma)c^h(\sigma) + \beta^{t'}\pi(\sigma')c^h(\sigma')}{\beta^t\pi(\sigma) + \beta^{t'}\pi(\sigma')}$$

for all $h \in \mathcal{H}$. This convex combination, $\tilde{c}$, is clearly a feasible allocation (since aggregate endowments at σ and σ' are the same) and by strict concavity of $u^{\bar{h}}(\cdot, s)$, agent $\bar{h}$ derives higher utility. Therefore, $c^{\bar{h}}(\sigma) \neq c^{\bar{h}}(\sigma')$ contradicts efficiency. $\Box$

Kubler and Schmedders (2003) show how the model can be reformulated as a model of a "stochastic finance economy" with stock and bond markets (i.e. a Lucas tree economy with heterogeneous agents but dynamically complete markets) and prove that generically in asset payoffs the Arrow-Debreu model and the finance model are equivalent. We focus on the Arrow-Debreu equilibrium in this paper but most of our results extend to the equivalent model. As we will explain below a slight complication arises from the fact that with long-lived assets, changes in endowments also lead to changes in the prices of these assets which affect individuals differently depending on their asset holdings.

The lemma implies that for any Arrow-Debreu equilibrium prices $(p(s^t))$, the expression $p(s^t)/(\beta^t\pi(s^t))$ just depends on the current shock s_t but not on the history. We can therefore define stationary probability-free prices by $\rho(s_t) \equiv \frac{p(s^t)}{\pi(s^t)\beta^t}$ for all $s_t \in \mathcal{S}$.

Based on this definition we can next define the equilibrium correspondence as the map $\omega : \mathbb{R}^{HS}_{++} \to \mathbb{R}^S_{++}$ from HS-dimensional profiles of endowments to S-dimensional equilibrium prices $\rho = (\rho(s))_{s\in\mathcal{S}}$ by

$$\omega(e^h) = \left\{\rho \in \mathbb{R}^S_{++} : (\rho(s_t)\beta^t\pi(s^t))_{s^t\in\Sigma} \text{ are Arrow-Debreu prices } \right\}.$$

2.2 Equivalence to Walrasian model

We can impose the fact that each agent makes stationary consumption choices in any Arrow-Debreu equilibrium directly in the budget constraints. Therefore, we can reformulate an agent's budget constraint $\sum_{\sigma\in\Sigma} p(\sigma)\left(c(\sigma) - e^h(\sigma)\right) = 0$ as follows,

$$\rho(s_0)(c(s_o) - e(s_0)) + \sum_{t=1}^{T} \beta^t \sum_{s^t \in \mathcal{N}_t} \pi(s^t)\rho(s_t)(c(s_t) - e(s_t)) = 0.$$

Note that consumptions and endowments at time t only depend on the current shock s_t and not on any part of the history before t. For $t = 2, \dots, T$ define (recursively) the t-fold product of the Markov transition matrix, $\pi^t = \pi \cdot \pi^{t-1}$. Observe that the probabilities in the first row of π^t yield the distribution for the current state in period t since we assumed w.l.o.g. that the economy starts in state $s_0 = 1$. Next define $\Phi = \sum_{t=0}^{T} \beta^t \pi^t$ and let $\phi = (\phi_1, \dots, \phi_S)$ denote the first row of the matrix Φ. We sometimes write $\phi(\beta, \pi)$ to make explicit the dependence on the discount factor and the transition probabilities.

Each agent's utility maximization problem, being reduced to stationary consumption choices, can now be written as

$$\max \sum_{s=1}^{S} \phi_s u^h(c_s, s) \text{ subject to}$$

$$\sum_{s=1}^{S} \phi_s \rho(s)(c(s) - e(s)) = 0.$$

Observe that an equilibrium is now a vector of prices $\rho \in \mathbb{R}^S_{++}$ and choices $c^h \in \mathbb{R}^S_{+}$ for all $h \in \mathcal{H}$ such that each agent maximizes utility and markets clear. This observation implies that any restriction on the equilibrium correspondence of a Walrasian model with separable utility translates one to one, to restrictions of the intertemporal model (prices are just the Walrasian prices, multiplied by ϕ_s and vice versa). It is useful to introduce the following notation.

$$\phi \circ \rho = (\phi_1\rho_1, \dots, \phi_S\rho_S).$$

3 Non-parametric counterfactual analysis

The purpose of this paper is to examine the possible changes in Walrasian equilibrium prices that may result when agents' individual endowments change. Our objective is to perform this examination in much generality. For this purpose we do not choose particular functional forms for utility functions but instead rely on a non-parametric approach along the lines of the methodology in the seminal paper by Brown and Matzkin (1996). Brown and Matzkin use the so-called Afriat inequalities (Afriat (1967)) to examine whether observations on individual endowments and prices yield restrictions on the Walrasian equilibrium correspondence. In this paper we use the Afriat inequalities to analyze global comparative statics for the dynamic model of the previous section.

Section 3.1 introduces and motivates the non-parametric approach to counterfactual analysis. We maintain the language of the literature started by Brown and Matzkin (1996) even though it may sound unusual at times for a comparative statics analysis. For example, we adopt the term 'observations' to refer to different specifications of endowments together with equilibrium prices. Maintaining the same language is meant to help the reader to relate our approach here to the previous work without having to adjust to a potentially confusing new language. Section 3.2 develops the Afriat inequalities for our model. Lemmas 4 and 5 state simplified necessary and sufficient conditions for the inequalities to hold. Section 3.3 completes the development of our framework with the statement of Lemma 6, a special version of the main result of Brown and Matzkin (1996) for our model. The lemma provides necessary and sufficient conditions for prices to occur in a Walrasian equilibrium for given individual endowments.

3.1 Observable restrictions and counterfactual analysis

We consider N profiles of individual endowments $e^h(i)$ for $h \in \mathcal{H}$ and $i \in \mathcal{E} = \{1, 2, \ldots, N\}$ with $e^h(i) = \left(e_1^h(i), \ldots, e_S^h(i)\right) \in \mathbb{R}_{++}^S$ and say that 'observed' prices $(p_\sigma(i))_{i \in \mathcal{E}, \sigma \in \Sigma}$ are consistent with equilibrium if there are $(\rho(i))_{i \in \mathcal{E}}$, $\beta \in (0,1)$ and a Markov transition matrix π with

$$\rho(s_t) = \frac{p(s^t)}{\pi(s^t)\beta^t} \text{ for all } s_t \in \mathcal{S},$$

and if there is an equilibrium correspondence, ω, such that

$$\rho(i) \in \omega\left((e^h(i))^{h \in \mathcal{H}}\right) \text{ for all } i \in \mathcal{E}.$$

We say that they are consistent with equilibrium with shock-invariant (or shock-independent) utility if there exists an equilibrium correspondence for an economy where all agents maximize shock-invariant utility. Note that since we do not require endowments to be different across observations, we also trivially make statements about the equilibrium set of a given economy.

In the tradition of Brown and Matzkin (1996) we take the Arrow-Debreu equilibrium prices as part of the observations. For our model with a large and possible infinite event tree examining the vector (for finite T) or the sequence (for $T = \infty$) of such prices may at first appear rather daunting. However, the assumption that the observations are generated by an Arrow-Debreu equilibrium of the underlying dynamic exchange economy puts a lot of structure on the equilibrium prices. In particular, we can prove that the Arrow-Debreu prices can be uniquely decomposed into transition probabilities π, the discount factor β and probability free prices ρ.

3.1.1 Beliefs versus prices

The absence of arbitrage and the stationarity of the equilibrium allocations implies a one-to-one relation between Arrow-Debreu prices, $(p(\sigma))_{\sigma\in\Sigma}$, and the prices of 'one-period Arrow securities'. Let $a_{s,s'} = p(s^{t+1})/p(s^t)$ whenever $s_{t+1} = s'$ and $s_t = s$, that is, $a_{s,s'}$ is the price of a one-period security that pays one unit of the consumption good in the next period if state s' occurs and nothing in all other states. Let $A = (a_{s,s'})_{s,s'=1}^S$ be the $S \times S$ matrix of the prices of all such one-period Arrow securities. Theorem 1 states if these prices are equilibrium prices, then the underlying transition probabilities and the discount factor can be recovered uniquely.

THEOREM 1 *For a given matrix of one-period Arrow security prices A, there exists a unique Markov transition matrix π, a unique discount factor $0 < \beta < 1$ and prices $\rho \in \mathbb{R}^S_{++}$, unique up to a normalization, such that for all $s, s' \in \mathcal{S}$,*

$$a_{s,s'} = \beta\pi(s,s')\frac{\rho_{s'}}{\rho_s}.$$

Proof. We prove that for a given positive matrix A the nonlinear system of S^2 equations

$$a_{s,s'}\frac{\rho_s}{\rho_{s'}} = \beta\pi(s,s').$$

in the unknown discount factor β, transition matrix π, and probability-free prices ρ has a unique positive solution. Summing all equations for fixed s and using the property of π that all row elements sum to 1 we obtain

$$\sum_{s'=1}^{S} a_{s,s'}\frac{\rho_s}{\rho_{s'}} = \beta \qquad \text{for all } s \in \mathcal{S}.$$

Defining $\gamma_s = \Pi_{s'\neq s}\rho_{s'}$ and $\gamma = (\gamma_1, \ldots, \gamma_S)^\top$, we obtain the linear system of equations

$$(A - \beta I_{S\times S})\,\gamma = 0,$$

where $I_{S\times S}$ denotes the $S \times S$ identity matrix. Note that this system of linear equations is just the system defining the eigenvalues and eigenvectors of the matrix A! The classical Perron-Frobenius theorem (see e.g. Horn and Johnson (1985)) implies that the positive matrix A has a unique largest real eigenvalue, β^* that is positive and associated with a positive real eigenvector, γ^*. Furthermore all other eigenvalues are associated with eigenvectors that are not non-negative. Therefore, there is only one solution for $\beta > 0$ with associated $\gamma > 0$. Since the elements of the matrix A are generated by our dynamic exchange economy this unique solution must also satisfy $\beta < 1$. Furthermore, note that $\rho_s/\rho_{s'} = \gamma_{s'}/\gamma_s$ and so the prices ρ are also uniquely – up to a normalization – determined which finally leads to a unique transition matrix π. □

The equilibrium conditions of our dynamic exchange economy immediately imply that for a given discount factor β, a given transition matrix π, and probability-free prices ρ there exist uniquely determined Arrow-Debreu equilibrium prices. Theorem 1 establishes the converse of this property. From an observation of Arrow-Debreu equilibrium prices we can recover the transition matrix and agents' discount factor in addition to probability-free state prices. This recoverability of the transition matrix and the discount factor has an important consequence for our analysis in this paper. We do not need to make a case distinction depending on whether (or not) we know the transition probabilities and discount factors. Given the assumption of the observability of Arrow-Debreu prices we can immediately assume that we know β and π.

The result of Theorem 1 resembles the work of Wang (1993), Cuoco and Zapatero (2000) and others on the recoverability problem of preferences and beliefs in a continuous-time infinite-horizon economy with dynamically complete financial markets. The methods in this literature are very different from our application of the Perron-Frobenius theorem here. For example, the analysis of Markovian equilibria in a continuous-time setting requires Cuoco and Zapatero to examine a Riccati differential equation.

3.1.2 Interpretation

Based on Theorem 1 restrictions on global comparative statics can always be viewed in two parts. First, different price systems across multiple observations must reveal the same beliefs and discount factor. Secondly, only the probability-free prices can change across multiple observations. We say that there exist restrictions on prices, if, given N profiles of endowments $(e^h(i))_{i\in\mathcal{E}}^{h\in\mathcal{H}}$, there exist $(\rho(i))_{i\in\mathcal{E}} \gg 0$ that are not consistent with equilibrium. For the investigation of restrictions on the probability-free prices, we consider restrictions on the equilibrium correspondence as defined in Section 2.1. In the following we always assume that prices are strictly positive and that endowments are positive. We often assume that endowments are also strictly positive, but point this out explicitly.

The restrictions on the equilibrium correspondence of the dynamic model must be identical to those of the standard Walrasian model when utility is separable. For general non-separable utility, these restrictions have first been investigated by Brown and Matzkin (1996) who give an example to show that some restrictions exist. Brown and Matzkin interpret their exercise in terms of refutability of the general equilibrium model. They give necessary and sufficient conditions on the equilibrium correspondence and reserve the term 'comparative statics' for necessary conditions only.

In a dynamic general equilibrium model, what is observable is generally only one equilibrium. Testable restrictions on possible equilibrium prices then come from assumptions on preferences (over and above expected time-separable utility, see Kubler (2003)). Here we consider changes in prices as endowments change, i.e. consider equilibria of different economies. As outlined in the introduction, we want to interpret this as unanticipated policy change or a structural break to which all agents in the economy assign probability zero. If at some node s^t, a transfer and lump-sum tax scheme is introduced without agents in the economy anticipating so, equilibrium prices are going to adjust instantly to the new equilibrium corresponding to different endowments.

Alternatively, one can interpret the exercise as a 'structural break'. Either all agents attach zero probability to this structural break or else it is completely uninsurable, i.e. no asset pays contingent on this shock to the economy occurring.

If there are Lucas trees or other multi-period assets in the economy, it is a very strong assumption that the structural break is not insurable, since it amounts to saying that in that state, all trees become worthless, i.e. no longer pay any dividends from there on. The presence of Lucas trees also makes the interpretation of an unanticipated change more complicated. The new endowment-profiles now consist of the actual individual endowments plus the dividends of the Lucas trees an individual held at the time of the change.

3.2 Individual Afriat inequalities

For the characterization of competitive equilibria Brown and Matzkin (1996) use the Afriat inequalities and examine if observed aggregate demand can be expressed as the sum of (unknown) individual choices which satisfy the inequalities and a budget constraint. Brown and Matzkin (1996) give an example showing that these conditions are not vacuous. In a simple model with 2 agents and 2 commodities it is possible to find variations of endowments and prices that are inconsistent with equilibrium. In our framework we need to consider slightly different conditions, since utility is additively separable across states. In particular, this fact implies that the conditions of Brown and Matzkin remain necessary but are no longer sufficient and that there certainly exist restrictions on the the equilibrium correspondence, in the sense that there exit profiles of endowments with associated prices that are not in the equilibrium correspondence.

Afriat (1967) formulates a system of linear inequalities which characterize a finite set of observations of individual choices arising from utility maximization. His techniques can be applied to a wide variety of frameworks. In particular, one can characterize optimal asset demand, savings and demand of goods by Afriat-inequalities (Varian (1983a) and (1983b)).

The basic idea is to assume that the utility function is strictly concave and continuous, to use the Kuhn-Tucker theorem to characterize optimality, and to relate the subgradient of the utility function to prices and to characterize concavity in terms of the subgradients being negatively co-monotone to consumption.

The following lemma states the Afriat inequalities for shock-dependent utility.

LEMMA 2 *Given* $\phi \in \mathbb{R}^S_{++}$ *as well as consumptions and prices* $(c(i), \rho(i))_{i \in \mathcal{E}}$*, the following two* *statements are equivalent.*

1. *There exists a shock-dependent utility function with strictly increasing, strictly concave and continuous* $u(c,s)$, $s \in \mathcal{S}$, *such that*

$$c(i) \in \arg\max_{c \in \mathbb{R}^S_+} \sum_{s=1}^{S} \phi_s u(c_s, s) \text{ s.t. } (\phi \circ \rho(i)) \cdot (c - c(i)) \leq 0$$

2. *Consumptions and prices* $(c(i), \rho(i))_{i \in \mathcal{E}}$ *satisfy the following 'Afriat inequalities'. There exist* $(\lambda(i))_{i \in \mathcal{E}} \gg 0$ *such that for any* $s \in \mathcal{S}$ *and all* i, j

$$(c_s(i) - c_s(j))(\lambda(i)\rho_s(i) - \lambda(j)\rho_s(j)) \leq 0, \tag{1}$$

with strict inequality if $c_s(i) \neq c_s(j)$ *and with* $\lambda(i)\rho_s(i) = \lambda(j)\rho_s(j)$ *if* $c_s(i) = c_s(j) > 0$.

Proof. 1. $\Rightarrow$ 2. The claim follows from convex analysis, see Rockafellar (1970). For all $s \in \mathcal{S}$ the function $u(\cdot, s)$ has a nonempty subdifferential $\partial_{c_s} u(c_s, s)$ with $v_s > 0$ for all subgradients $v_s \in \partial_{c_s} u(c_s, s)$. Optimality of $c(i)$ implies that there exist $v_s(i) \in \partial_{c_s} u(c_s(i), s)$ as well as $\lambda(i) > 0$ such that $\phi_s v_s(i) - \lambda(i)\phi_s\rho_s(i) \leq 0$ and equal to zero if $c_s(i) > 0$. Strict concavity of each $u(\cdot, s)$ and $c_s(i) > c_s(j)$ implies $v_s(i) < v_s(j)$ and thus $\lambda(i)\rho_s(i) < \lambda(j)\rho_s(j)$. Note that $c_s(i) = c_s(j) > 0$ immediately implies $\lambda(i)\rho_s(i) = \lambda(j)\rho_s(j)$. Now the Afriat inequalities follow.

2. $\Rightarrow$ 1. Assume without loss of generality that $c_s(1) \leq c_s(2) \leq \ldots \leq c_s(N)$. Define a positive and strictly decreasing piecewise linear function by setting $u'(c_s(i), s) = \lambda(i)\rho_s(i)$ for $c_s(i) > 0$ and $u'(0, s) = 1 + \max_j \lambda(j)\rho_s(j)$. Moreover for $c > 0$ with $c_s(i) < c < c_s(i+1)$ let $u'(c, s) = u'(c_s(i), s) + \frac{c - c_s(i)}{c_s(i+1) - c_s(i)} (u'(c_s(i+1), s) - u'(c_s(i), s))$ (including the special case $c_s(0) = 0$ if $c_s(1) > 0$). The constructed marginal utility functions are positive and strictly decreasing and thus integrate to strictly increasing and strictly concave utility functions on the respective intervals $[0, k)$ for $k > c_s(N)$. Observe that by construction the necessary and sufficient first-order conditions for the utility maximization problem are satisfied. $\square$

The next lemma states the 'shock-dependent' Afriat inequalities.

Lemma 3 *Given consumptions and prices* $(c(i), \rho(i))_{i \in \mathcal{E}}$ *and a* $\phi \in \mathbb{R}^S_{++}$, *the following two statements are equivalent.*

1. *There exists a shock-invariant utility function with strictly increasing, strictly concave and continuous* $u(\cdot)$, *such that*

$$c(i) \in \arg\max_{c \in \mathbb{R}^S_+} \sum_{s=1}^{S} \phi_s u(c_s) \text{ s.t. } (\phi \circ \rho(i)) \cdot (c - c(i)) \leq 0$$

2. *Consumptions and prices $(c(i), \rho(i))_{i\in\mathcal{E}}$ satisfy the following 'shock-invariant Afriat inequalities'. There exist $(\lambda(i))_{i\in\mathcal{E}} \gg 0$ such that for any two shocks s and s' and any two observations i and j the following conditions hold,*

$$(c_s(i) - c_{s'}(j)) (\lambda(i)\rho_s(i) - \lambda(j)\rho_{s'}(j)) \leq 0, \tag{2}$$

with strict inequality if $c_s(i) \neq c_{s'}(j)$ and with $\lambda(i)\rho_s(i) = \lambda(j)\rho_{s'}(j)$ if $c_s(i) = c_{s'}(j) > 0$.

The proof is identical to the proof of Lemma 1 except that now one has to construct one function which is concave in all $c_s(i)$. The crucial difference between the shock-invariant and the shock-dependent Afriat inequalities lies in the fact that in the former $\lambda(i)\rho_s(i)$ and consumption are negatively co-monotone across observations $i \in \mathcal{E}$ and shocks $s \in \mathcal{S}$, while in the latter this condition must only hold across observations but not shocks.

Unlike in the general case, where versions of the strong axiom of revealed preferences (e.g. GARP) provide an alternative characterization, no equivalent conditions that are quantifier free are known in this case. Varian (1983a) discusses various specifications for which GARP-like restrictions can be derived, but for the case of additively separable utility, he concludes "I have been unable to find a convenient combinatorial condition that is necessary and sufficient for additive separability." While we cannot give conditions that are simultaneously necessary and sufficient we can state simple necessary as well as sufficient conditions for the Afriat inequalities in the presence of additively separable utility functions.

The necessary condition considers the case where in one shock consumption increases while in some other it weakly decreases. In this case, the supporting price of the first shock has do decrease relative to the price of the second shock. The sufficient condition considers the case where in all shocks consumption increases (i.e. there is no shock where it weakly decreases). In this case, there are no restrictions on supporting prices.

Lemma 4 *Necessary and sufficient conditions for consumption and price vectors $(c(i), \rho(i))_{i\in\mathcal{E}}$ to satisfy the Afriat inequalities are as follows.*

(N) If $c_s(i) > c_s(j)$ then the Afriat inequalities imply for all $s' \neq s$,

$$c_{s'}(i) \leq c_{s'}(j) \quad \Longrightarrow \quad \frac{\rho_{s'}(i)/\rho_s(i)}{\rho_{s'}(j)/\rho_s(j)} > 1.$$

(S) If for all i, j, $c(i) \gg c(j)$ (or vice versa), then consumption vectors $(c(i))_{i\in\mathcal{E}} \geq 0$ and any arbitrary price vectors $(\rho(i))_{i\in\mathcal{E}} \gg 0$ satisfy the Afriat inequalities.

Proof. For the proof of the sufficient conditions suppose without loss of generality that $c(1) \ll c(2) \ll \ldots \ll c(N)$. Observe that it is possible to choose $\lambda(1) > \lambda(2) > \ldots > \lambda(N)$ both sufficiently large and different such that $\lambda(i)\rho(i) \gg \lambda(i+1)\rho(i+1)$ for all $i = 1, \ldots, N-1$ and so that conditions (1) of Lemma 1 are satisfied.

Conversely, conditions (1) imply if $c_s(i) > c_s(j)$ for some $s \in \mathcal{S}$ that

$$0 < \lambda(i)\rho_s(i) < \lambda(j)\rho_s(j).$$

Similarly, $c_{s'}(i) \leq c_{s'}(j)$ implies

$$\lambda(i)\rho_{s'}(i) \geq \lambda(j)\rho_{s'}(j) \geq 0.$$

Dividing the second weak inequality by the first strict inequality then yields the inequality of the lemma. $\square$

The next lemma considers the case of shock-invariant utility. The characterization in terms of Afriat inequalities is identical to Varian (1983b). Again the lemma gives necessary and sufficient conditions. The necessary condition follows trivially from the inequalities. The sufficient condition in analogous to the sufficient condition in the shock-dependent case, except that now it is not enough that consumption is strictly ordered but in fact the highest consumption in one observation has to lie below the lowest in the next observation.

Lemma 5 *Necessary and sufficient conditions for consumption and price vectors* $(c(i), \rho(i))_{i\in\mathcal{E}}$ *to satisfy the shock-invariant Afriat inequalities are as follows.*

(N) If consumption vectors $(c(i))_{i\in\mathcal{E}}$ *and prices* $(\rho(i))_{i\in\mathcal{E}} \gg 0$ *satisfy the shock-invariant Afriat inequalities, then for all* i *and each* $s, s' \in \mathcal{S}$ *it holds that*

$$(c_s(i) - c_{s'}(i))(\rho_s(i) - \rho_{s'}(i)) \leq 0,$$

with strict inequality whenever $c_s(i) \neq c_{s'}(i)$ *and with* $\rho_s(i) = \rho_{s'}(i)$ *if* $c_s(i) = c_{s'}(i) > 0$.

(S) Suppose the necessary condition (N) holds. If for all i, j, $\max_s c_s(i) < \min_s c_s(j)$ *(or vice versa) then the shock-invariant Afriat inequalities hold for any arbitrary price vectors.*

Proof. Condition (N) follows simply from the fact that within the same observation i we can divide the shock-invariant Afriat inequalities (2) by $\lambda(i) > 0$.

Under the assumptions of condition (S) we can assume w.l.o.g. that $c(1) \ll c(2) \ll \ldots \ll c(N)$ and then observe that $c_s(j) - c_{s'}(i) < 0$ for any two observations $j < i$ and all $s, s' \in \mathcal{S}$. For any given price vectors $(\rho(i))_{i\in\mathcal{E}}$ we can choose $\lambda(1) > \lambda(2) > \ldots > \lambda(N)$ sufficiently large and different so that in fact $\lambda(i)\min_s \rho_s(i) > \lambda(i+1)\max_s \rho_s(i+1)$ for all $i = 1, \ldots, N-1$. As a consequence it follows that $\lambda(j)\rho_s(j) - \lambda(i)\rho_{s'}(i) > 0$ for any two observations $j < i$ and all $s, s' \in \mathcal{S}$. Equations (2) of Lemma 2 hold for all $i, j \in \mathcal{E}$. $\square$

3.3 Equilibrium

The following lemma specializes Brown and Matzkin's result to the case of additively separable utility.

Lemma 6 (Brown and Matzkin (1996)) *Observations on prices and individual endowments* $(\rho(i), (e^h(i))^{h\in\mathcal{H}})_{i\in\mathcal{E}}$, *are consistent with equilibrium if and only if there exist* $c^h(i) \in \mathbb{R}^S_+$ *for all* $h \in \mathcal{H}$ *and all* $i \in \mathcal{E}$ *such that*

i) *For each* h, $(c^h(i), \rho(i))_{i\in\mathcal{E}}$ *satisfy the Afriat inequalities.*

ii) $(\phi \circ \rho(i)) \cdot (c^h(i) - e^h(i)) = 0$ *for all* $i \in \mathcal{E}$ *and all* $h \in \mathcal{H}$.

iii) $\sum_{h=1}^{H}(c^h(i) - e^h(i)) = 0$ *for all* $i \in \mathcal{E}$.

If for each h, $(c^h(i), \rho(i))_{i\in\mathcal{E}}$ *satisfy the shock-invariant Afriat inequalities the observations are consistent with shock-invariant equilibrium.*

The proof follows directly from Brown and Matkzin (1996): In order for prices to lie on an equilibrium correspondence, there have to exist individual consumptions that are budget feasible and satisfy the Afriat inequalities which characterize choice compatible with utility maximization.

4 Restrictions on counterfactual analysis

As pointed out above, restrictions on counterfactual analysis can be divided into restriction on beliefs and restrictions on probability-free prices ρ. The restrictions on beliefs are clear. Given any two observations on Arrow prices, $A(1), A(2)$, the largest eigenvalue and the eigenvector associated with these prices have to be the same. Furthermore the largest eigenvalue has to be less than 1. Theorem 1 above shows that as far as probabilities and discounting are concerned these are the necessary and sufficient restrictions. The remaining question now concerns what we can say about the associated $\rho(1), \rho(2)$. In the following we refer to restrictions on price changes always as restrictions on ρ assuming that probabilities remain constant.

In this section we derive conditions on profiles of endowments which are necessary in order for price changes not to be arbitrary, as well as conditions that are sufficient for restrictions to exist. It is useful to distinguish two cases. We first assume that individual endowments change but aggregate endowments remain constant, then we move to the case that aggregate as well as individual endowments vary.

4.1 Constant aggregate endowments

Lemma 4 implies that given any fixed aggregate endowments, any finite set of distributions of individual endowments together with any prices are always consistent with equilibrium if utility is shock dependent. This is true because one can take each observation's individual consumption to be collinear to aggregate consumption. Generically, the consumptions will all be different and strictly ordered. More formally, without genericity assumptions, we have the following theorem.

THEOREM 2 *Suppose N observations on individual endowments and prices* $((e^h(i)), \rho(i))_{i\in\mathcal{E}} \geq 0$, $\rho(i) \gg 0$ *for all* $i \in \mathcal{E}$, *satisfy* $\sum_h e^h_s(i) = \sum_h e^h_s(j)$ *for all* $s \in \mathcal{S}$ *and all* $i, j \in \mathcal{E}$. *Then the observations are consistent with equilibrium, independently of* $\phi \gg 0$.

Proof. Given any $\phi \in \mathbb{R}^S_{++}$, for all $h \in \mathcal{H}$ and $i \in \mathcal{E}$ define consumption $c^h(i) \in \mathbb{R}^S_{++}$ by

$$c^h(i) = \frac{(\phi \circ \rho(i)) \cdot e^h(i)}{(\phi \circ \rho(i)) \cdot e(i)} \cdot e(i) + \epsilon^h(i)$$

with perturbations $\epsilon^h(i) \in \mathbb{R}^S$. These perturbations can be chosen arbitrarily small so that $c^h(i) \gg c^h(j)$ (or vice versa) for any $i \neq j$ while also satisfying $(\phi \circ \rho(i)) \cdot \epsilon^h(i) = 0$ and $\sum_{h\in\mathcal{H}} \epsilon^h(i) = 0$. (Note, there are only finitely many observations.) Now Condition (S) of Lemma 4 implies that the Afriat inequalities are satisfied for the constructed consumption vectors $(c^h(i))_{i\in\mathcal{E}}$ and any arbitrary prices $(\rho(i))_{i\in\mathcal{E}} \gg 0$. Moreover, by construction markets clear and the budget constraints are satisfied. Thus, all three conditions of Lemma 6 are satisfied and the theorem follows. $\square$

Balasko and Tvede (2005) derive the same result for an economy with general (not necessarily separable) utility. Given Mas-Colell's (1977) result on the equilibrium set of an exchange economy, it is clear that there cannot be restrictions for sufficiently small variations in endowments, Balasko and Tvede's result extends this intuition to large variations that leave aggregate endowments constant.

While Balasko and Tvede interpret their theorem as a negative result, we do not agree with this interpretation. For example, if one could show that aggregate endowments and prices have to satisfy the weak axiom, one would clearly think that the model produces very clear-cut empirical predictions, yet Theorem 2 would remain valid. But the result shows that under the assumption of shock-dependent utility, a necessary condition for restrictions on counterfactual analysis is that aggregate endowments change. We turn to the case of changes in aggregate endowments after briefly discussing restrictions that arise from assuming shock-invariant utility.

4.1.1 Shock-invariant utility

It is clear that for each observation there are restrictions. In fact, the monotonicity condition

$$e_s(i) \geq e_{s'}(i) \quad \Longrightarrow \quad \rho_s(i) \leq \rho_{s'}(i), \tag{3}$$

with strict inequality whenever $e_s(i) \neq e_{s'}(i)$, is well known, see e.g. Kubler (2003) for a historical overview.

More interestingly, as the following example shows, in general the assumption of shock-invariant utility also implies restrictions on possible comparative statics in the presence of constant aggregate endowments.

EXAMPLE 1 *Suppose there are $S = 3$ states, $H = 2$ agents, and $N = 2$ observations. Individual endowments are identical across both observations and are given by $e_1^1(i) = e_1^2(i) = 1$, $e_2^1(i) = e_2^2(i) = 2$ and $e_3^1(i) = e_3^2(i) = 100$ for $i = 1, 2$. Further suppose $\phi_1 = \phi_2 = \phi_3$. The two price vectors $\rho(1) = (1, 0.9, 0.8)$ and $\rho(2) = (1, 0.5, 0.4)$ cannot be both equilibrium prices although they each satisfy the monotonicity condition (3).*

The individual budget constraints and the market-clearing condition imply $c_2^h(i) < c_3^h(j)$ for $i, j \in \mathcal{E} = \{1, 2\}$ for both agents $h = 1, 2$. Condition (2) in Definition 1 then implies $\lambda^h(2)\rho_2(2) - \lambda^h(1)\rho_3(1) > 0$ for $h = 1, 2$. Prices $\rho_2(2) = 0.5 < 0.8 = \rho_3(1)$ then imply that $\lambda^h(1) < \lambda^h(2)$ for both agents $h = 1, 2$. Again condition (2) of Definition 1 then implies $c_1^h(1) > c_1^h(2)$ for both agents $h = 1, 2$. These last inequalities contradict the market-clearing conditions for $s = 1$.

Example 1 shows that for fixed aggregate endowments the assumption of shock-invariant utility imposes restrictions both on possible equilibrium prices and on global comparative statics. The reason that this result appears to be so different from the statement of Theorem 2 for the shock-dependent case is that here Lemma 4 does not hold; the fact that consumption is strictly ordered does not imply that prices can be arbitrary for shock-independent utility. The question is if Lemma 5 allows us to identify conditions under which price changes can be arbitrary for shock-invariant utility. A simple sufficient condition on individual endowments which ensures that the condition of Lemma 6 holds (at endowments) is given in the following result.

THEOREM 3 *Suppose that for all $h \in \mathcal{H}$ and all $i, j \in \mathcal{E}$,*

$$\min_s e^h(i) > \max_s e^h(j) \text{ or } \min_s e^h(j) > \max_s e^h(i),$$

then there are no restrictions on possible prices beyond Condition (3).

The theorem follows directly from the above discussion together with the fact that agents can always consume their endowments. Note that in the case of no aggregate uncertainty, i.e. $e_s = e_{s'}$, for all s, s', prices must be identical both across states and across observations and there are therefore very strong restrictions.

4.2 Changes in aggregate endowments

Intuitively, even with shock-dependent utility, clear restrictions should arise if aggregate endowments change. If, for example, aggregate endowments become more risky, prices in a representative agent economy become more spread out. One would expect that even with several heterogeneous agents someone has to bear the risk and equilibrium prices must change accordingly. We first show that this intuition depends crucially on what happens to individual endowments. One can always construct changes in individual endowments that destroy this intuition and lead to a situation where price changes can be arbitrary. In the simple benchmark case of constant endowment shares across observations we show that there do in fact exist strong restrictions even for the case of shock-dependent utility. In fact a much weaker assumption suffices to guarantee restrictions. We discuss this issue in detail below.

4.2.1 The role of individual endowments

When individual endowments are allowed to change arbitrarily together with aggregate endowments and if there are sufficiently many different agents in the economy, there are no restrictions on possible price changes. In other words, given changes in aggregate endowments, we can construct individual endowments such that there are no restrictions on associated prices – this is true even when utility is assumed to be shock-invariant. The following theorem formalizes this fact.

THEOREM 4 *Given aggregate endowments $(e(i))_{i\in\mathcal{E}} \gg 0$ and any $\phi \gg 0$, there always exist $H \geq N$ agents with individual endowments $(e^h(i))_{i\in\mathcal{E}}^{h\in\mathcal{H}} \gg 0$ satisfying $\sum_h e^h(i) = e(i)$ such that any arbitrary positive prices $(\rho(i))_{i\in\mathcal{E}} \gg 0$ are consistent with equilibrium. This statement is true even with shock-invariant utility if probability-free prices ρ satisfy the monotonicity condition (3) within each observation.*

Proof. Without loss of generality, we consider $H = N$. For sufficiently small $\epsilon > 0$, let endowments of an individual h in observation i for each shock $s \in \mathcal{S}$ be as follows.

$$e_s^h(i) = \begin{cases} i\epsilon + \delta e_s(i) & \text{if } i \neq h \\ e_s(i) - (H-1)(i\epsilon + \delta e_s(i)) & \text{otherwise.} \end{cases}$$

For sufficiently small $\epsilon > 0$ and $\delta = 0$, individual consumption $c^h(i) = e^h(i)$ is budget feasible, independently of prices. For each agent h the consumption vectors are also strictly ordered and thus satisfy Condition (S) of Lemma 4. Therefore, for each agent the Afriat inequalities are satisfied, too, and all three conditions of Lemma 6 hold.

To prove the statement for shock-independent utility it suffices to show that Condition (S) of Lemma 5 holds. For this purpose we choose a sufficiently small $\delta > 0$ much smaller than $\epsilon > 0$ to ensure that within each observation i individuals' consumptions $c^h(i) = e^h(i)$ are co-monotone to aggregate consumption, that is, $c_s^h(i) < c_{s'}^h(i)$ iff $e_s(i) < e_{s'}(i)$. Since prices $(\rho(i))_{i\in\mathcal{E}}$ satisfy the monotonicity condition (3) the same must be true for all individual consumptions and so Condition (N) of Lemma 5 holds. Also by construction for all

$h \in \mathcal{H}$ and all $i \neq j$ it holds $\max_s c_s^h(i) < \min_s c_s^h(j)$ (or vice versa) and thus Condition (S) of Lemma 5 holds, too. □

Note that the construction hinges critically on $H \geq N$. We do not know if a similar construction works with fewer agents. However, for a fixed number of agents we can always construct examples where price changes cannot be arbitrary.

Example 2 *Consider* $S = 3, H = 2$ *and* $e(1) = (10, 1, 1)$, $e(2) = (1, 10, 1)$, $e(3) = (1, 1, 10)$. *No matter what individual endowments, prices* $\rho(1) = \rho(2) = \rho(3)$ *cannot be rationalized even with shock-dependent utility.*

Note that one of the two agents must consume at least 5 units in the high endowment state in (at least) two observations. W.l.o.g. suppose $c_1^1(1) \geq 5$ and $c_2^1(2) \geq 5$. Thus, $c_2^1(1) < c_2^1(2)$ but

$$\frac{\rho_2(1)/\rho_1(1)}{\rho_2(2)/\rho_1(2)} = 1$$

and so Condition (N) of Lemma 4 is violated. That is, the first agent's consumption imposes restriction on possible price changes.

This discussion points to one possible way how our initial intuition might fail. Changes in aggregate consumption might not be reflected in changes in individual consumption if at the same time incomes change in a way that the changes in aggregate consumption are swamped out and all individuals' consumption are strictly ordered. This observation suggests that one possible way to obtain restrictions would be to restrict the fraction of individual endowments to aggregate endowments to remain constant in each state across all observations, i.e. there are $\kappa_s^h > 0$ such that $e_s^h(i) = \kappa_s^h e_s(i)$ for all observations $i = 1, \ldots, N$. This assumption would guarantee that there are prices for which in fact everybody will have higher consumption in shock s for an observation where aggregate endowments in shock s are higher. Note that this is a much weaker assumption than assuming collinear individual endowments – we return to this assumption in Section 5 below.

4.2.2 Sufficient conditions for restrictions

While constant endowment shares do ensure that there exit restrictions, an assumption which is in the same spirit but much weaker suffices. Price changes are restricted if there are two observations i, j such that in some shock s each individual's endowments increase while in some other shock s' aggregate endowments weakly decrease. Note that the resulting change in aggregate endowments is also a necessary (and sufficient) condition for there to be restrictions in the single agent case.

Theorem 5 *Suppose there are two observations $i = 1, 2$ and shocks $s, s' \in \mathcal{S}$ with $e_s^h(1) > e_s^h(2)$ for all agents $h \in \mathcal{H}$ and $e_{s'}(1) \leq e_{s'}(2)$. Then prices cannot be arbitrary even with shock-dependent utility.*

Proof. Given profiles of endowments, and any $\phi \gg 0$, we show that there exist price vectors $\rho(1), \rho(2) \gg 0$ that are not consistent with equilibrium. We normalize $\rho_{s'}(i) = 1$ for both observations. Because $e_s^h(1) > e_s^h(2)$ we can choose $\epsilon, \delta > 0$ such that $e_s^h(1) - \epsilon > e_s^h(2) + \delta$. Next we can choose prices $\rho_s(2) < \rho_s(1)$ sufficiently large such that for each agent h, the budget constraints

$$(\phi \circ \rho(i)) \cdot c^h(i) \leq (\phi \circ \rho(i)) \cdot e^h(i)$$

yield the following implications: If for an agent h it holds that $c_s^h(1) \leq e_s^h(1) - \epsilon$ then it follows that $c_{s'}^h(1) > e_{s'}(1)$. Similarly, if for an agent h it holds that $c_s^h(2) \geq e_s^h(2) + \delta$ then it follows that $c_{s'}^h(2) < 0$. Market clearing and the non-negativity constraints on consumption thus ensure $c_s^h(1) > c_s^h(2)$ for all agents $h \in \mathcal{H}$. Market clearing also implies for at least one agent h that $c_{s'}^h(1) \leq c_{s'}^h(2)$. But by construction the prices $\rho_s(1) > \rho_s(2)$ satisfy $\frac{\rho_{s'}(1)/\rho_s(1)}{\rho_{s'}(2)/\rho_s(2)} = \frac{1/\rho_s(1)}{1/\rho_s(2)} < 1$ and so Condition (N) of Lemma 4 is not satisfied. The Afriat inequalities cannot hold. □

The theorem seems to contradict Balasko and Tvede's (2005) claim that their result on the absence of restrictions can be extended to an open neighborhood of endowments. While our result does require the strong assumption of separable utility (not made in Balasko and Tvede), the main difference is the order of quantifiers. For any two observations on prices and a give profile of endowments, there might exist a small open neighborhood around that profile of endowments such that prices are consistent with equilibrium. However, for given changes in endowments, no matter how small as long as the satisfy the condition in Theorem 5, there always exist prices that are inconsistent with equilibrium.

Note that for shock-independent utility, these conditions obviously remain sufficient. If price changes cannot be arbitrary with shock-dependent utility then the same must be true for the shock-independent case.

5 Counterfactual analysis and the representative agent

In cases where aggregate endowments change, changes in individual endowments often cannot lead to arbitrary price changes. However, the question is whether there are reasonable assumptions on individual endowments which guarantee that global comparative statics predictions from the representative agent model carry over to a setting with several consumers. In the construction above, we saw that there are some restrictions on prices in the heterogeneous agent case, but it is clear that these restrictions are much milder than the representative agent restrictions. Obviously, one can generally construct cases where the restrictions are the same by making one agent very large and all other agents very small. This observation is the content of the following result.

THEOREM 6 *Given aggregate endowments* $(e(i))_{i\in\mathcal{E}}$*, all different across observations, and prices that are inconsistent with a representative agent there always exist individual endowments* $(e^h(i))^{h\in\mathcal{H}} \gg 0$*,* $\sum_h e^h(i) = e(i)$ *such that prices cannot be equilibrium prices.*

Proof. For $\epsilon > 0$, take $e_s^h(i) = \epsilon$ for all $h = 1, \ldots, H-1$, all i and all s. Since consumer H has to consume on his budget set, eventually, for sufficiently small ϵ, his consumption is arbitrary close to aggregate endowments and hence if prices and choices are inconsistent for aggregate endowments, they must be inconsistent for the representative agent, as well as for agent H whose consumption can be forced to be arbitrarily close to the representative agent's consumption. $\square$

While the proof of the theorem relies on very extreme endowments, this feature is obviously only a sufficient condition. We can easily construct an example where for a large set of individual endowments the counterfactual analysis for a representative agent economy has also to hold in an economy with several agents.

EXAMPLE 3 *Consider* $N = 2$*,* $H = 2$*,* $S = 2$ *and suppose that* $\rho(1) = (1,1)$*,* $e(1) = (2,1)$ *and* $e(2) = (1,2)$*. If* $e_1^1(1) + e_2^1(1) \geq 2$ *and* $e_1^1(2) + e_2^1(2) \geq 2$*, the relative price of shock 2 has decrease from observation 1 to observation 2, just like in the representative agent case. Yet, agent 1 seems far from a 'dominating' agent whose consumption must be close to aggregate endowments.*

5.1 Additional restrictions through heterogeneity?

Although Theorem 6 and Example 3 show that sometimes the restrictions from a single-agent economy carry over to the heterogeneous agent case, we see below that this is an exception. In fact, the above discussion shows that it is easy to construct examples where there are restrictions for a single agent, while there are none for an economy with several agents. Given this, one might conjecture that if aggregate endowments and prices are consistent with a single agent, they must be consistent with an heterogeneous agent economy, no matter what the distribution of income. It turns out that even with shock-dependent utility, this is only true for two observations. In this case, since no restrictions means that endowments are strictly ordered, one can always construct consumptions to be strictly ordered. For more than two observations, there might exist restrictions for the heterogeneous agent economy, although aggregate endowments are strictly ordered and hence there are no restrictions for the representative agent economy.

Suppose $e(i) \gg e(i+1)$ for all $i = 1, \ldots, N-1$. Condition (S) of Lemma 4 implies that price changes can be arbitrary for a representative agent. With heterogeneous agents this is false. The following example proves this fact.

EXAMPLE 4 *Consider $S = 2$ states with $N = 3$ observations with aggregate endowments of $e(1) = (1,1)$, $e(2) = (1.1, 10)$, $e(3) = (1.2, 100)$ and suppose $\phi = (1,1,1)$. For $H = 2$ agents with $e^1(1) = (1-\epsilon, 1-\epsilon)$, $e^1(2) = (1/2, 1)$, and $e^1(3) = (0.9, 50)$ the prices $\rho(1) = \rho(2) = \rho(3) = (1, 1/1000)$ are inconsistent with equilibrium if $\epsilon > 0$ is sufficiently small.*

Condition (N) of Lemma 4 implies that for prices to remain constant across observations consumptions must be strictly ordered. If $c^1(1)$ are $c^1(2)$ ordered then budget feasibility for agent 1 combined with market clearing implies $c^1(1) \gg c^1(2)$. It is impossible that $c^1(3) \gg c^1(1)$, but if $c^1(1) \gg c^1(3)$, by market clearing, it is impossible that $c^2(3) \gg c^2(2)$, in fact, it must be the case that $c_1^2(3) < c_1^2(2)$.

The example shows that agent heterogeneity can in fact impose additional restrictions on global comparative statics, if individual endowments are 'chosen correctly'. We summarize our comparison of representative agent and heterogeneous agents economies in the following theorem.

THEOREM 7 *Restrictions in the representative agent economy are neither necessary nor sufficient for restrictions in the heterogeneous agent economy.*

Proof. Theorem 4 implies that for a representative agent economy with restrictions we can always construct a heterogeneous agents economy without restrictions. Theorem 6 implies that for a representative agent economy with restrictions we can always construct a heterogeneous agents economy having the same restrictions. Example 4 shows a representative agent economy without restrictions and a corresponding heterogeneous agents economy with restrictions. □

5.2 Collinear endowments

It is difficult to find general conditions on individual endowments that imply that representative agent restrictions carry over to the economy with several agents. As an illustration, we consider the simple (but for applications somewhat relevant) case of collinear endowments. Theorem 6 states that there are restrictions for the heterogeneous agents economy whenever there are restrictions for the representative agent economy. Note that the representative agent conditions on prices are sufficient in this case, since each agent can simply consume his endowments. However, the following example shows that while price changes cannot be arbitrary, counter-intuitive price changes that are impossible for the single agent economy are now possible.

Example 5 *Suppose* $H = 2$, $S = 2$, $N = 2$ *with* $e^1(1) = e^2(1) = (1, 2)$, $e^1(2) = e^2(2) = (1, 2.01)$ *and* $\phi = (1, 1, 1)$. *Prices* $\rho(1) = (1, 0.5)$ *and* $\rho(2) = (1, 0.6)$ *are possible in the heterogeneous agent economy, although endowments are identical and these prices are impossible in the representative agent model.*

We show that the three conditions of Lemma 6 are satisfied. Let the two agents' consumption vectors be $c^1(1) = (1.3, 1.4)$, $c^2(1) = (0.7, 2.6)$ and $c^1(2) = (1.31, 2.01 - 0.31/0.6)$, $c^2(2) = (0.69, 2.01+0.31/0.6)$. These consumption vectors satisfy the market clearing conditions and budget equations. Condition (S) of Lemma 4 also holds since $c^1(1) << c^1(2)$ and $c^2(1) >> c^2(2)$. Thus, the Afriat inequalities for shock-dependent utility hold. Actually, even the shock-invariant Afriat inequalities hold.

Now consider the representative agent economy with consumptions and endowments $c(1) = e(1) = (2, 4)$ and $c(2) = e(2) = (2, 4.02)$. Since $c_2(2) > c_2(1)$ and $c_1(2) = c_1(1)$ Condition (N) of Lemma 4 is violated,

$$\frac{\rho_1(2)/\rho_2(2)}{\rho_1(1)/\rho_2(1)} = \frac{1/0.6}{1/0.5} < 1.$$

The Afriat inequalities for shock-dependent utility do not hold. Similarly, equation (2) of Definition 1 is also violated. Since $c_1(1) = c_1(2)$ and $\rho_1(1) = \rho_1(2)$ it must also hold that $\lambda(1) = \lambda(2)$. But now

$$(\rho_2(2) - \rho_2(1))\,(c_2(2) - c_2(1)) = 0.1 \cdot 0.02 > 0$$

and so the shock-invariant Afriat inequalities do not hold.

The reason why the representative agent intuition does not carry over to a model with different tastes is quite simple. If one agent is close to risk neutral, an increase in the price of a shock with large aggregate endowments results in a decrease in consumption in both shocks, the more risk-averse agent's consumption increases in both shocks and resulting aggregate consumption can increase in the shock where the price increased.

6 Conclusion

In this paper we show that under fairly general assumptions price changes that result from changes in individual endowments cannot be arbitrary. For this result, it is crucial that in at least one state each individuals' endowment changes in the same direction. Otherwise it is possible to construct changes in income distributions that 'wash out' changes in aggregate consumption. It is also crucial that aggregate endowments change, without this, there are no restrictions if utility can be shock-dependent.

We also show that while restrictions on price changes are likely to exist, they are rarely the same as those in the representative agent model. But since we show that under very mild assumptions, which are likely to hold in many counterfactual policy analyses considered in practice, there are some restrictions, this opens the way to further research into the exact nature of these restrictions.

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