Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://www.econ.upenn.edu/pier

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"Heterogeneous Firms in a Finite Directed Search Economy"

by

Manolis Galenianos and Philipp Kircher

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Heterogeneous Firms in a Finite Directed Search Economy*

Manolis Galenianos†
Pennsylvania State University

Philipp Kircher‡
University of Pennsylvania

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Abstract

We consider a directed search model for a finite economy with heterogeneous firms in two informational environments. In the first, the productivity of all firms is publicly observed. We prove existence of equilibria in pure posting strategies by firms and show that wage dispersion is driven by fundamentals. That is, more productive firms post higher wages and wage dispersion is absent when firms are homogeneous. When firms have heterogeneous productivities the equilibrium is not constrained efficient. In the second environment the productivity level of each firm is private information. The main results extend to this environment: Equilibria in pure strategies exist; strategies are increasing in productivity; and constrained efficiency does not obtain. When the productivity level of all firms is drawn from the same distribution, symmetric equilibria exist and the ranking of wages equals that of productivity.

1 Introduction

There is a large literature modeling trade that occurs in markets with matching frictions. A strand of this literature which has been very active in recent years is directed search. The main assumption in this class of models is that one side of the market can publicly announce (and commit to) the prices at which it is willing to trade. In the context of the labor market, firms post wages that are observed by all workers before they decide where to apply for a job. Hence, a worker knows the wage that he will receive if he gets the

*We thank Jan Eeckhout, Philip Reny and Shouyong Shi for very helpful comments.
†e-mail: manolis@psu.edu.
‡Corresponding author. e-mail: kircher@econ.upenn.edu.

job that he applies for, but, since firms have a limited number of vacancies, he may get rationed if there are other applicants for the same position. The underlying friction is that workers cannot coordinate their decisions of which job to apply for, which is modeled as a restriction to symmetric application strategies. Hence many workers might apply for the same position even if other vacancies receive no applications at all. Since higher wages attract more applicants on average, the central trade-off for a worker is that he faces a lower probability of getting the job when he applies to a high-wage firm. Firms face the converse trade-off at the wage-posting stage of the game: offering a higher wage increases the probability of hiring, but it also decreases their ex post profit margin.

This idea has been used to analyze a series of applied topics such as investment decisions (Acemoğlu and Shimer, 1999; Shi, 2001), technological change (Shi, 2002), wage dispersion among homogeneous labor (Montgommery, 1991) and labor allocations with two-sided heterogeneity (Shimer, 2005). Two main conclusions arise from this line of research: (1) firms with higher productivity - either due to idiosyncratic ability or due to higher capital investments - post higher wages, and (2) the allocation is constrained efficient.[1] These results are interesting because they provide an interpretation for the well-known fact that observationally identical workers receive higher wages when employed at more profitable (in this context more productive) firms.[2] Furthermore, market frictions mean that dispersion in wages plays an important allocative role as it directs labor towards more productive firms.

These results were obtained for models with a continuum of agents (except for the case of Montgomery (1991) which we discuss below). In this paper, we examine the finite economy version which has received much less attention. The main difference between the continuum and finite versions is that in the latter the action of an individual firm typically affects the payoffs of all other market participants, including other firms. We think that the finite economy is of economic interest because interactions in the labor market at a given point in time, at a given geographical location and for a given profession are likely to only involve a limited number of agents despite the large size of the overall economy. As a result, when these agents act, they may actually take their interdependence into account.

Furthermore, it allows us to assess the extent to which the insights of the continuum economy apply to the finite case. Existing results for finite economies rely on mixed strategies by firms when firms have heterogeneous productivity (Peters 1994, 2000). The randomization of firms may result in outcomes where a low productivity firm posts a higher wage than a firm with high productivity, which runs counter to the characterization results for the

[1]The models of Moen (1997) are Mortensen and Wright (2002) are similar in spirit and results.

[2]Determinants of wage dispersion for homogeneous workers have attracted some interest in the literature. See Mortensen (2003) for a discussion.

continuum case.[3] As a result, we may not be able to interpret wage differences as indicators of productivity differences. Moreover, in such an equilibrium efficiency can be trivially improved by requiring high-productivity firms to post higher wages and hence hire more often. To evaluate whether this is a fundamental problem of finite economies or not, we investigate whether equilibria in pure strategies by the firms exist. Additionally, pure posting strategies seem to be a simpler and more plausible way to recruit. For homogeneous firms, we know from Burdett, Shi and Wright (2001, henceforth BSW) that there exists an equilibrium in pure posting strategies in which all workers offer the same wage.[4] We are interested in whether this implies that wages are determined by fundamentals, or whether there are also other equilibria in which identical firms post different wages.

We analyze a setting with heterogeneous firms and homogeneous workers. Heterogeneous firms allow us to analyze the aspect of productivity on wage announcements. As in Montgomery (1991) and Moen (1997), we reduce the mathematical complexity by assuming homogeneous workers.[5] We consider two informational environments. In the first, the productivity of all firms is publicly observed (in particular, by other firms) as in Peters (1984, 2000) and Montgomery (1991). This may be due to observable investments or due to knowledge obtained during previous coexistence. We establish the existence of an equilibrium in pure strategies for the firms and we characterize it.[6] We prove that higher productivity firms post higher wages and, as a corollary, there is no wage dispersion when there are no productivity differences across firms. Hence, in the case of homogeneous firms, the single wage equilibrium is unique in pure posting strategies and, in the case of heterogeneous firms, the dispersion in wages is due to productivity differentials. We think that our results are interesting because they imply that, even in a finite frictional setting, wage dispersion is driven by fundamentals. This implies that labor is allocated to a larger degree to firms with high productivity. Nevertheless, the market does not achieve constrained efficiency, because market power distorts the difference between wages and thus the application behavior. The exception is the case of homogeneous firms, where dispersion in wages is not present and constrained efficiency is obtained.

In the second environment, the productivity of each firm is a privately observed draw from some known distribution, which is a setting that has not yet been analyzed in the literature. It may arise when a firm faces new competitors or when investments in productivity

[3]Peters (2000) shows that these equilibria approach the continuum outcomes when the economy is replicated and approaches an infinite number of agents. However, in this paper we are interested in outcomes away from that limit.

[4]Existence in pure posting strategies has also been proven in Peters (1994) for homogeneous firms.

[5]Homogeneous labor has been used a lot in the search literature, see Mortensen (2003).

[6]The frictions are introduced by symmetric strategies by workers, which requires mixed strategies on their part. Therefore, our analysis of pure strategies refers only to firms.

are imperfectly observable. In a continuum economy, this environment is identical to the previous one since the (convention about the) law of large numbers implies that there is no uncertainty about the types of firms in the market. However, in a finite model the uncertainty about the realizations of firms' productivity is present. Using recent results on games with incomplete information, we establish the existence of an equilibrium in pure strategies by firms. Furthermore, we show that a firm's wage offer increases in its type. In contrast to the complete information case and to the continuum models, comparisons of wages across firms do not necessarily reveal their ranking in terms of productivity if firms draw their productivity levels from different distributions. Only when the distributions are symmetric can we be certain that a symmetric equilibrium exists, in which case higher wages are indeed posted by higher productivity firms. The inefficiencies observed in the complete information case carry over to this case of incomplete information.

Finite economies entail an effect that we call global market power. It entails that a wage change by a single firm changes the cumulative distribution function of wages in the market. Therefore, firms cannot "mimick" the wage postings of other firms and obtain the same hiring probability, but rather face very different application behavior. It will be instructive to examine this in more detail in the next subsection, and distinguish it from the notion of limited market power that is present even in continuum economies. To do this we will briefly have to review the explicit nature of the frictions that directed search models impose on the workers application behavior. The subsequent sections lay down the formal model and analyze the case of complete and incomplete information.

2 Effects of Global Market Power in Finite Economies

Directed search models are based on the assumption that firms can compete for labor. Nevertheless, there remains a search component since workers do not know how many other workers apply for a job. The presumption is that workers cannot coordinate. This is captured by a restriction to symmetric strategies, in which all workers apply to a given firm with the same probability. This can be viewed as representing the anonymity in the market: players do not need to know their "role" in the game in equilibria in which they all use the same strategy. The plausibility of this assumption is discussed e.g. in Shimer (2005).

If all workers apply to firm j with the same application probability p_j, then a worker who applies there has some probability, say $G(p_j)$, of getting the job. We will lay out the formulas in the next section. In the subgame after observing a wage announcement, worker apply in such a way that they are indifferent between all firms. That is, they apply such that the expected utility $G(p_j)w_j$ is equalized at all firms (except for those firms that might not receive any applications at all). If a firm raises its wage, the probability with which workers apply increases smoothly in the firm's wage. A symmetric equilibrium necessitates mixed application behavior, therefore our statements about pure posting strategies refer to the behavior of the firms. The idea of a subgame in symmetric strategies also translates - with some technical difficulties - to economies with a continuum of agents.[7]

We mentioned in the introduction that *global* market power is a specific feature of finite economies. A single firm that changes its wage affects the cumulative distribution of wages that are obtainable in the market, and thus it changes the expected utility of workers and the expected hiring probability of all other firms. In continuum economies such an effect is not present. Firms still have *limited* market power in the sense that their hiring probability is increasing in their wage announcement. But in a continuum economy, each worker changes his application probability to this firm very little (because otherwise too many workers would apply for the single job), and the workers' expected utilities are not affected. The small changes by many workers sum up to a significant effect for the deviating firm, but since workers do not change their application behavior much, the other firms' profits are also not affected. We will explore the case with and without global market power in Figure 1.

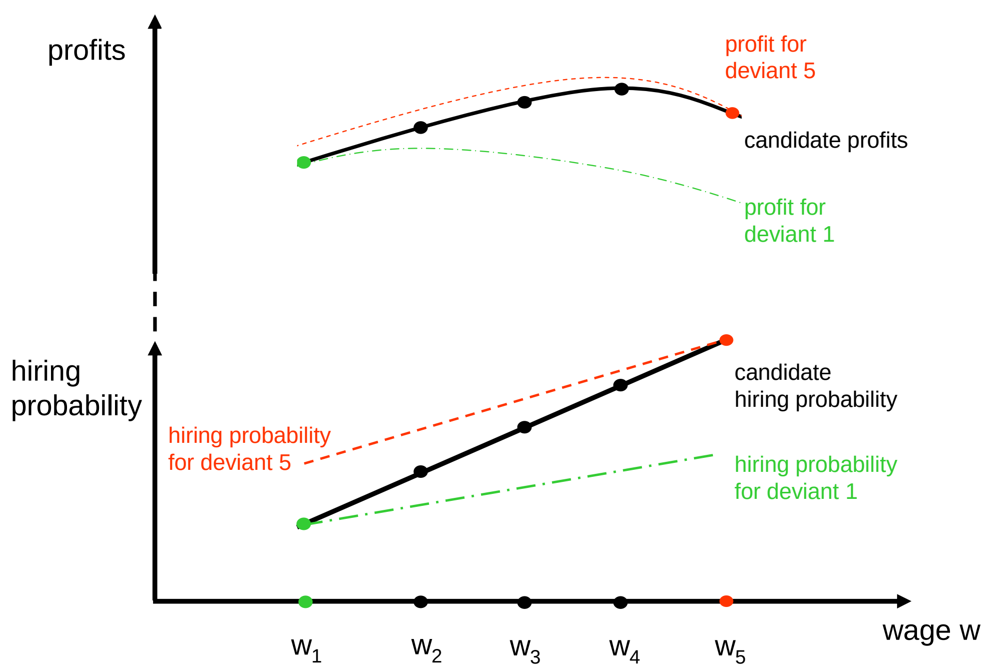

Figure 1: Illustration of global market power.

[7]The technical difficulty arises from the fact that one cannot assign a probability of applying to a firm, since there are too many firms in the market.

The figure shows a candidate equilibrium constellation in which five firms offer five different wages. If workers use symmetric strategies in the subgame, this yields them some expected utility, say $U(\mathbf{w})$, in the subgame. The expected utility depends on the tuple $\mathbf{w} = (w_1, ..., w_5)$ of announced wages. Since workers are indifferent between the firms when they randomize, we have $G(p_j)w_j = U(\mathbf{w})$ at all firms. The application behavior of the workers induces some expected hiring probability for each firm. This is depicted by the dots in the lower part of the picture. Since the expected profit of a firm is

$$Expected\ Profit \ = \ [\ Expected\ Hiring\ Probability\ ]\ [\ Productivity \ - \ Wage\ ] \qquad (1)$$

this induces some expected profit for each firm, as depicted in the upper part of the figure.

If a firm would assume that its own action does not affect the workers utility in the subgame, i.e. firm j believes that $U(\mathbf{w}) = U = constant$ independent of firm j's wage posting w_j, then it would expect to obtain an application behavior of

$$G(p_j)w_j = U = constant, \qquad (2)$$

at each deviation wage w_j. This is the case because workers apply with a probability that makes them just indifferent to the utility they can obtain at other firms. Such a belief induces the solid line that connects the dots. In particular, under this belief, firm 1 thinks that it can obtain the same hiring probability as firm 2 if it deviates and offers the same wage as firm 2. Equation (2) is justified in a continuum economy. Montgomery (1991) assumes such beliefs in a finite economy.[8] Under such beliefs it relatively straightforward to show that identical firms offer the same wage. They all face the same profit function (the higher solid line), and since this function has a unique optimum all firms want to deviate and post this optimal wage. Thus, any equilibrium has to involve all firms posting the same wage. If firms are heterogeneous, each firm will have their own profit function, with the maximum of higher productivity firms being on the right of those firms with lower productivity.

Yet in a finite economy the belief given by (2) is not justified. If firm 1 deviates from the candidate profile and offers the same wage as firm 2, it will get a lower hiring probability than firm 2 obtained in the candidate profile. The reason is simple: Firm 2 faces three high wage competitors and one low wage competitor in the candidate wage announcement. When firm 1 "mimicks" firm 2, after the deviation firm 1 will face four high wage competitors. This means that after the deviation firm 1 has tougher competition than firm 2 had before the deviation. Another way of making the same point is to observe that *after* the deviation

[8] For the case of two workers and two firms Montgomery (1991) also investigates the case of global market power.

firm 2 has fewer lower hiring probability than *before* the deviation, because now firm 1 has become more attractive to workers, i.e. $U(\mathbf{w})$ has increased. After the deviation firm 1 and 2 will both have equal hiring probability, but it will be lower than the hiring probability of firm 2 before the deviation. Therefore, firm 1's hiring probability will always lie below the solid line, as depicted by the lower dotted line. Similarly, firm 5's hiring probability is always above the solid line. Before a deviation the other firms faced a high wage competitor. When firm 5 deviates and offers, say, w_4, there is no other high wage competitor present. This is the same as saying that the workers expected utility $U(\mathbf{w})$ has gone down, and since this indicates that their options have become worse they apply more to wage w_4 (and to all other wages as well). This is represented by the higher of the dotted lines.

The profitability of a deviation now depends on the slope of the hiring probability for the deviant. In the picture, the slope for firm 1 is less than for firm 5. Therefore, it is less profitable to raise the wage for firm 1 than for firm 5. This might lead to profits as depicted by the dotted lines at the top of figure 1. Firm 1's optimal deviation wage is lower than firm 5's optimal deviation wage. The question is whether this can sustain wage dispersion for homogeneous firms. Even if firms are heterogeneous, it is not obvious that firm 1 would want to deviate even if it has a higher productivity than firm 5, since its benefit from deviating is different than the benefit for firm 5. The question arises whether high productivity firms might be locked into lower wage postings. Finally, the global market power effect also complicates the prove of existence substantially, because it is less obvious whether profits are quasi-concave and therefore best responses are convex-valued.

In order to analyze the finite market environment, it will be necessary to analyze the slope of the hiring probability. This requires us to investigate the reaction of workers to a wage change by a firm for any profile of wages by other firms. That is, we need to investigate $\partial p_j/\partial w_j$. One contribution of our work is to provide an explicit solution to these partial derivatives. We expect this to be useful for further work on finite directed search models, because it allows us to proceed analytically. We can sign the second derivative $\partial^2 p_j/\partial w_j^2$ when firm j has positive hiring probability, which allows us to establish quasi-concavity of the profit function and thus existence in pure posting strategies. We can also investigate the first order conditions and show that the difference between the wage postings is always larger than the difference between the best responses for homogeneous firms. It is even smaller when high productivity firms offer the lower wage. Therefore, lower productivity firms cannot offer the higher wage in equilibrium in pure posting strategies.

Figure (1) also illustrates the reason why efficiency fails except for the special case of homogeneous firms. Montgomery (1991) shows that constrained efficiency is obtained when firms hold belief (2), i.e. expect the solid line after a deviation. In a finite economy in which firms realize their global market power, they expect a different response after a deviation, as depicted by the dotted lines. These responses are smaller slope than the solid line, and therefore firms tend to post lower wage since a wage increase yields less of an advantage for their hiring prospects. Wages per se are not important in a transferable utility environment, the question is whether vacancies are filled optimally (given the constraint that workers use symmetric strategies). With homogeneous firms, all firms post the same wage, and are effected by the effects of market power in the same way. That is, they reduce their wages by the same amount compared to the case where they neglect their global market power, and thus the effect does not translate into differences in the workers' application behavior, i.e. workers still apply equally often to all firms and the outcome remains constrained efficient. In the case of heterogeneous firms, firms offer different wages and are effected differently by the effect of global market power. This distorts their wages differently compared to the case where they neglect this effect, workers apply in a different way. We prove for the duopoly case that constrained efficiency is never achieved when firms are heterogeneous.

3 The Model

There is a finite set $M = \{1, ..., m\}$ of firms and a finite set $N = \{1, ..., n\}$ of workers, where $m \geq 2$ and $n \geq 2$. Workers are homogenous. Each firm $j \in M$ has productivity x_j drawn from a distribution with continuous non-zero density on its support $[\underline{x}_j, \bar{x}_j]$. We assume $\underline{x}_j > 0$ for all $j \in M$. Each firm knows its own productivity. We will analyze two informational environments. We will first consider the complete information case where each firm knows the realized productivities of all the other firms. This case has been the focus in the literature, see e.g. Peters (1984, 2000) and BSW. We will also analyze the private information case where firms know their own productivity but only know the distribution over other firms' productivities.

In the first stage of the game, each firm $j \in M$ simultaneously posts a public wage offer $w_j \in [0, x_j]$. In the next stage, workers observe the tuple of announced wages $\mathbf{w} = (w_1, w_2, ..., w_m)$ and decide simultaneously on the firm to which they want to apply.[9] If a firm has multiple applicants, it chooses one of them randomly and employs him at the announced wage. Firm j receives profit $x_j - w_j$ if it hires a worker and zero otherwise. Worker i get's utility w_j if he gets hired by firm j, and zero if he does not get hired. Firms maximize expected profits; workers maximize expected utilities.

[9]The assumption that each applicant auditions for a job only at a single firm is standard in the literature. Multiple applications have recently been analyzed in continuum models by Albrecht, Gautier and Vroman (2006), Galenianos and Kircher (2006) and Kircher (2006). In finite models this leads to severe complications as shown in Albrecht, Gautier, Tan and Vroman (2005).

We will retain the standard assumption of symmetric application behavior in the subgame following the wage postings. This typically requires a mixed strategy on the part of the workers. Let $p_{j,i}(\mathbf{w})$ denote the probability with which worker i applies to firm j after observing $\mathbf{w}$. Symmetric strategies by workers imply that $p_{j,i}(\mathbf{w}) = p_{j,h}(\mathbf{w}) = p_j(\mathbf{w})$ for all $h, i \in N$ and all $j \in M$. When all workers apply to firm j with probability p_j, the probability that the firm has at least one applicant (and is thus able to hire) is given by

$$H(p_j) = 1 - (1 - p_j)^n. \tag{3}$$

With probability $(1 - p_j)^n$ no worker applies, and with the complementary probability at least one worker applies. The probability of an individual worker getting the job at this firm conditional on applying there is

$$G(p_j) = [1 - (1 - p_j)^n]/np_j \tag{4}$$

if $p_j > 0$, and $G(0) = 1$ (see BSW). Since $[1 - (1 - p_j)^n]$ is the probability that firm j hires a worker and there are in expectation np_j workers applying for the job, intuitively an applicant's hiring probability is given by the ration of the two as stated in expression (4).

If workers apply more often to firm j, firm j's hiring probability increases. That is, $h(p_j) = H'(p_j) = n(1 - p_j)^{n-1}$ is strictly positive for $p_j < 1$. If other applicants apply more often to firm j, the probability for an individual worker to obtain a job at firm j decreases conditional on applying, i.e., $g(p_j) = G'(p_j) = -[G(p_j) - (1 - p_j)^{n-1}]/p_j < 0$ for $p_j > 0$ and $g(0) = -(n - 1)/2$. Given a vector of posted wages $\mathbf{w}$ and a symmetric strategy profile by workers $\mathbf{p}(\mathbf{w}) = (p_1(\mathbf{w}), ..., p_m(\mathbf{w}))$, the highest utility that a worker can obtain is

$$U(\mathbf{w}) = \max_{j \in M} G(p_j(\mathbf{w}))w_j. \tag{5}$$

Definition 3.1 (Symmetric Application Response) *A symmetric application response to* $\mathbf{w}$*, i.e. a symmetric equilibrium in the subgame following wage announcements* $\mathbf{w}$*, is a vector* $\mathbf{p}(\mathbf{w})$ *in the* $m - 1$ *dimensional unit simplex such that for all* $j \in M$

$$G(p_j(\mathbf{w}))w_j = U(\mathbf{w}) \text{ if } p_j > 0. \tag{6}$$

In the following we will assume that $\mathbf{p}(\mathbf{w})$ arises from a symmetric application response to the wage offers $\mathbf{w}$. From Proposition 1 in Peters (1984) we know that this response is unique when $w_k > 0$ for some k, and it varies upper hemicontinuously in $\mathbf{w}$. The reason for uniqueness can be seen from the fact that by (6) a fixed U uniquely defines p_j for all j given the announced wage w_j. Increasing U increases p_j for all j and strictly increases p_j for some j, so that there is exactly one U such that $\sum p_j = 1$.

Firms anticipate the response to their wage announcements, and a vector of wage offers $\mathbf{w}$ induces expected profits

$$\pi_j(\mathbf{w}) = H(p_j(\mathbf{w}))(x_j - w_j) \tag{7}$$

for firm j with productivity x_j.[10]

In a directed search environment it is obvious that a firm that offers a wage that is much lower than its competitors will not receive any applications. Firms that never receive any applicants might offer arbitrary wage profiles, which are difficult to handle in the analysis. Therefore, before moving to the main analysis, we will briefly provide a condition under which firms have positive hiring probabilities at least when they offer wages close to their productivity.

Given a wage profile $\mathbf{w}_{-j}$ of other firms, firm j has to offer at least wage $\underline{w}_j$ in order to attract applicants. This minimal wage depends on the utility that workers obtain if they do not apply to firm j. We will define this utility as

$$U_{-j}(\mathbf{w}_{-j}) = U(0, \mathbf{w}_{-j}). \tag{8}$$

If firm j offers a zero wage, workers will not apply to it but rather apply optimally to the other firms, yielding them a utility $U(0, \mathbf{w}_{-j})$ in the subgame after observing the zero wage offer by firm j. If $w_j \leq \underline{w}_j = U_{-j}(\mathbf{w}_{-j})$, firm j will not obtain any applicants, while workers will apply at any wage $w_j > \underline{w}_j$. For later conditions it will be useful to note that $w_j < U_{-j}(\mathbf{w}_{-j})$ is equivalent to $w_j < U(w_j, \mathbf{w}_{-j})$. If w_j is so low that no worker applies, the market utility is not affected by adding this wage to the set of available wages. Also, $w_j = U_{-j}(\mathbf{w}_{-j})$ is equivalent to $w_j = U(w_j, \mathbf{w}_{-j})$. Adding wage w_j to the set of available wages does not alter the utility obtainable for workers, because w_j is only as good as the utility available at other firms if almost surely nobody applies, i.e. $p_j = 0$. If workers do not apply there in the subgame, it does not change their available utility. This immediately implies that $w_j > U_{-j}(\mathbf{w}_{-j})$ is equivalent to $w_j > U(w_j, \mathbf{w}_{-j})$. If w_j is such that workers apply with strictly positive probability, then by (6) the wage is higher than the utility obtainable in the market.

[10]Note that the requirement of a symmetric application response does not define $\mathbf{p}(\mathbf{w})$ uniquely at $\mathbf{w} = \mathbf{0}$. To fix ideas it is convenient to assume $\mathbf{p}(\mathbf{0}) = 1/m$, yet our results hold for any specification of $\mathbf{p}(\mathbf{0})$ in the $m - 1$ dimensional unit simplex.

The following condition ensures that in a symmetric application response workers apply to firm j with strictly positive probability if it offers a wage close to its productivity. It is immediate to see that this condition is non-empty and always holds if the span of the support, that is $\bar{x}_j - \underline{x}_j$, is not too large.

C1 *For all $j \in M$: $\underline{x}_j > U_{-j}(\bar{\mathbf{x}}_{-j})$, where $\bar{\mathbf{x}}_{-j} = (\bar{x}_1, .., \bar{x}_{j-1}, \bar{x}_{j+1}, ..\bar{x}_N)$.*

Imposing this condition will sometimes be convenient as it ensures that all firms in the market have the ability to attract applicants with strictly positive probability, and thus have the ability to make strictly positive profits. Otherwise their choice of wage offer will be arbitrary and it is difficult to talk about wage offers increasing in productivity. Moreover, even if a firm cannot profitably attract applicants is can influence other firms' optimal choices by offering a wage that would attract applicants if another firm would lower its wage. We will explicitly indicate statements where the condition is relevant.

4 The Complete Information Environment

4.1 Heterogeneous Firms

Consider the case where the realization $\mathbf{x} = (x_1, ..., x_m)$ of productivities is known to all firms.[11] We define a directed search equilibrium as a pure strategy Nash Equilibrium in the game among firms with induced payoffs $\pi_j(\mathbf{w})$ given in (7). Letting $\mathcal{W} = \times_{j=1}^{m}[0, x_j]$ denote the firms' strategy space and the vector $(w'_j, \mathbf{w}_{-j}) = (w_1, ..., w_{j-1}, w'_j, w_{j+1}, ..., w_m)$ account for individual deviations, we formally have

Definition 4.1 (Directed Search Equilibrium) *A directed search equilibrium is a vector of wage announcements $\mathbf{w} \in \mathcal{W}$ such that $\pi_j(\mathbf{w}) \geq \pi_j(w'_j, \mathbf{w}_{-j})$ for all $j \in M$ and all $w'_j \in [0, x_j]$.*

Note that our definition involves pure strategies by firms. The existence of such pure strategy equilibria has only been shown for homogeneous productivities (BSW; Peters, 1984). Existence and characterization for the general setup depend on the reaction of the workers'

[11] Following the literature, we take the trading mechanism and its coordination failure as given. For $x_j \neq x_k$ for all $j, k \in M$ the coordination problem could be solved if workers offered contracts, see Coles and Eeckhout (2003). There are various reasons why this might not arise, among them the fact that in an Coles-Eeckhout worker-offer-market firms do not obtain any surplus and would rather enter a firm-offer-market (for a model of competing market sides see Halko, Kultti and Virrankoski (2006)). In general one might think about some firms having equal productivity, so that the coordination problem is not resolved by switching market sides.

application probability p_j to the wage change of firm j. In the next lemma we will analyze this reaction. We will specify this reaction for wage profiles $\mathbf{w} \in \Omega_j = \{\mathbf{w} \in \mathcal{W} | w_j > U(\mathbf{w}), w_k \neq U(\mathbf{w})\ \forall k \in M\}$. Recall from the discussion of equation (8) that $w_k \neq U(\mathbf{w})$ is equivalent to $w_k \neq U_{-k}(\mathbf{w}_{-k})$. The set Ω_j, therefore, excludes those wage profiles with the property that some firm k does not receive applications at wage w_j or higher, but would receive applications if firm j lowered its wage. At the excluded wage profiles first order conditions cannot be applied. For these cases we can define left and right derivatives by taking the appropriate limits, though. We also exclude the uninteresting cases of $w_j < U(\mathbf{w})$ for which firm j does not get any applicants. Let $\mathcal{M}(\mathbf{w}) = \{k \in M | w_k > U(\mathbf{w})\}$ denote the set all of firms that receive applicants, where we drop the argument $\mathbf{w}$ in some equations for brevity. Moreover, let $u_j(\mathbf{w}) = g(p_j(\mathbf{w}))w_j$ denote the marginal benefit of workers from changing the probability of applying to firm j. We will show that the change in application behavior in response to a wage change by a firm depends in a simple way on the probability of obtaining a job at this firm and on a score that reflects the marginal benefits of applying to the various firms:[12]

Lemma 4.1 *For $\mathbf{w} \in \Omega_j$ the workers' reaction to firm j's wage change is given by*

$$\frac{\partial p_j(\mathbf{w})}{\partial w_j} = -\frac{\sum_{s \in \mathcal{M}\setminus\{j\}} \prod_{k \in \mathcal{M}\setminus\{j,s\}} u_k(\mathbf{w})}{\sum_{s \in \mathcal{M}} \prod_{k \in \mathcal{M}\setminus\{s\}} u_k(\mathbf{w})} G(p_j(\mathbf{w})) = -\frac{G(p_j(\mathbf{w}))}{S_j(\mathbf{w})}, \tag{9}$$

where $S_j(\mathbf{w}) = u_j(\mathbf{w}) + [1/\sum_{k \in \mathcal{M}\setminus\{j\}} \frac{1}{u_k(\mathbf{w})}]$.

Proof. Without loss of generality let $\mathcal{M}(\mathbf{w}) = \{1, ..., h\}$ include the first h firms. Since $\mathbf{w} \in \Omega_j$ no firm is on the boundary of getting applicants, i.e. $w_j \neq U(\mathbf{w})$ for all firms. Therefore those firms not in $\mathcal{M}(\mathbf{w})$ have $w_j < U(\mathbf{w})$ and do not get any applicants for wage announcements in the neighborhood of $\mathbf{w}$, and therefore do not enter the analysis. By (6) we have

$$G(p_k(\mathbf{w}))w_k - G(p_h(\mathbf{w}))w_h = 0\ \forall k \in \{1, ..., h-1\} \tag{10}$$

and $\sum_{k \in \mathcal{M}} p_k(\mathbf{w}) - 1 = 0$. Writing the left hand side of these as a system $F(\mathbf{p}; \mathbf{w})$ of h equations with h exogenous parameters $w_1, ..., w_h$ and h endogenous variables $p_1, ..., p_h$ we have an implicit function $F(\mathbf{p}; \mathbf{w}) = \mathbf{0}$. For this system of equations the Jacobian of endogenous

[12] For the relevant matrix algebra see e.g. Korn and Korn (1968).

variables evaluated at $(\mathbf{p}(\mathbf{w}), \mathbf{w})$ is

$$D_{\mathbf{p}}F=\begin{pmatrix} u_1(\mathbf{w}) & 0 & 0 & ... & 0 & 0 & -u_h(\mathbf{w}) \\ 0 & u_2(\mathbf{w}) & 0 & ... & 0 & 0 & -u_h(\mathbf{w}) \\ 0 & 0 & u_3(\mathbf{w}) & ... & 0 & 0 & -u_h(\mathbf{w}) \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & u_{h-2}(\mathbf{w}) & 0 & -u_h(\mathbf{w}) \\ 0 & 0 & 0 & ... & 0 & u_{h-1}(\mathbf{w}) & -u_h(\mathbf{w}) \\ 1 & 1 & 1 & ... & 1 & 1 & 1 \end{pmatrix}.$$

That is, $D_{\mathbf{p}}F$ is the matrix with elements $\alpha_{ss} = u_s(\mathbf{w})$ and $\alpha_{sh} = -u_h(\mathbf{w})$ for $s \in \{1, ..., h-1\}$, $\alpha_{hs} = 1$ for $s \in \{1, ..., h\}$ and $\alpha_{sk} = 0$ otherwise. To calculate the determinant $|D_{\mathbf{p}}F|$ we use Laplace's development to expand the last row and obtain $|D_{\mathbf{p}}F| = \sum_{s=1}^{h} \Lambda_{hs}$, where Λ_{hs} is the cofactor to element α_{hs}. That is, $\Lambda_{hs} = (-1)^{h+s}|A_{hs}|$, where A_{hs} is the matrix resulting from $D_{\mathbf{p}}F$ by elimination of the h'th row and the s'th column. Since A_{hh} is a diagonal matrix we have $|A_{hh}| = \prod_{k\in\mathcal{M}\setminus\{h\}} u_k(\mathbf{w})$. For $s < h$ we expand the $s'th$ row of $|A_{hs}|$ which yields $|A_{hs}| = (-1)^{h-1+s}(-u_h(\mathbf{w}))|B_{hs}|$, where B_{hs} is a $(h-2)^2$-dimensional diagonal matrix with diagonal elements $u_k(w)$ for all $k \in \mathcal{M}\setminus\{s, h\}$. We therefore have $|A_{hs}| = (-1)^{h+s} \prod_{k\in\mathcal{M}\setminus\{s\}} u_k(\mathbf{w})$, which yields $|D_{\mathbf{p}}F| = \sum_{s=1}^{h} \prod_{k\in\mathcal{M}\setminus\{s\}} u_k(\mathbf{w})$. By the definition of $\mathcal{M}$ we have $|D_{\mathbf{p}}F| \neq 0$. By application of the implicit function theorem $\partial p_j(\mathbf{w})/\partial w_j$ exists locally around $\mathbf{w}$, with $D_{\mathbf{w}}\mathbf{p} = -(D_{\mathbf{p}}F)^{-1}D_{\mathbf{w}}F$ defining the matrix of partial derivatives. As an implication of Cramer's Rule $(D_{\mathbf{p}}F)^{-1} = |D_{\mathbf{p}}F|^{-1}C$, where C is the matrix with elements $\gamma_{ls} = \Lambda_{sl}$. The Jacobian with respect to the exogenous variables $D_{\mathbf{w}}F$ evaluated at $(\mathbf{p}(\mathbf{w}), \mathbf{w})$ is simply a diagonal matrix except for the last column, with elements $\beta_{ss} = G(p_s(\mathbf{w}))$ and $\beta_{sh} = -G(p_h(\mathbf{w}))$ for $s \in \{1, .., h-1\}$ and zeros elsewhere. We therefore have $\partial p_j(\mathbf{w})/\partial w_j = -\Lambda_{jj}|D_{\mathbf{p}}F|^{-1}G(p_j(\mathbf{w}))$. This follows immediately for $j \in \{1, .., h-1\}$, and holds for $j = h$ by symmetry which is cumbersome but straightforward to verify analytically. Since the cofactor Λ_{jj} has a similar structure as the determinant $|D_{\mathbf{p}}F|$ only with row and column j missing, we have $\Lambda_{jj} = \sum_{s\in\mathcal{M}\setminus\{j\}} \prod_{k\in\mathcal{M}\setminus\{j,s\}} u_k(\mathbf{w})$, and we obtain the first equality in (9). The second equality follows by simple algebraic manipulations. ■

For the following existence proof we need further properties of the change in the application probability. Recall that we defined that lowest wage at which a firm can attract applicants as $\underline{w}_j = U_{-j}(\mathbf{w}_{-j})$. Since $p_j(w_j, \mathbf{w}_{-j}) = 0$ for $w_j \leq \underline{w}_j$, we can show that $p_j(w_j, \mathbf{w}_{-j})$ is quasi-concave in w_j by showing

Lemma 4.2 *Given $\mathbf{w}_{-j}$ with $w_k > 0$ for some $k \in M\setminus\{j\}$, $p_j(w_j, \mathbf{w}_{-j})$ is strictly concave* *in w_j for $w_j \in [\underline{w}_j, x_j]$.*

Proof. See Appendix. ■

This result will allow us to establish quasi-concavity of the profit function for firms, which will be important to establish existence.[13] To apply fixed point theorems, we have to deal with point of profit discontinuity at $\mathbf{w} = \mathbf{0}$. The discontinuity arises because at $\mathbf{w} = \mathbf{0}$ workers do not get any utility and firms have a probability of hiring less than unity. An individual deviation to any positive wage implies that all workers apply to the deviant for sure, which allows the deviant to hire with certainty and thus yields a jump in profits. To ensure continuity of the payoff function, we will bound the strategy space away from zero to some space $\mathcal{W}_\epsilon = \times_{j=1}^{m}[\epsilon, x_j]$ for $\epsilon > 0$. Lemma 4.3 shows that any equilibrium in the restricted strategy space $\mathcal{W}_\epsilon$ is also an equilibrium in the unrestricted strategy space $\mathcal{W}$, because no firm has an incentive to deviate.

Lemma 4.3 *There exists $\epsilon' > 0$ such that for any $\epsilon \leq \epsilon'$ the following holds: For any wage profile $\mathbf{w} \in \mathcal{W}_\epsilon$ firm j's best response in the unrestricted space $[0, \bar{x}_j]$ includes an element in $[\epsilon, x_j]$.*

Proof. By (6) we can find a number $t > 1$ independent of ϵ such that workers will choose $p_j = 0$ whenever $w_j \leq \epsilon$ and $w_k \geq t\epsilon$ for some $k \in M/\{j\}$. Therefore, whenever some firm chooses a wage higher than $t\epsilon$ firm j's profits are zero on $[0, \epsilon]$, and therefore any wage in $[\epsilon, x_j]$ is at least as good for firm j. On the other hand, assume $\mathbf{w}_{-j} \in [\epsilon, t\epsilon]^{n-1}$. In this case there exists a number z independent of ϵ such that $w_j \geq z\epsilon$ implies $p_k = 0$ for all $k \neq j$ and $p_j = 1$. At $w_j = \epsilon$ it holds that $p_j < 1/m$. Therefore $\pi(\epsilon, \mathbf{w}_{-j}) < \pi(z\epsilon, \mathbf{w}_{-j})$ when ϵ is sufficiently small. Due to strict quasi-concavity of the profit function no choice below ϵ can then yield a higher payoff. ■

The reason why the strategy space can be bounded away from zero – the point of profit discontinuity – is an immediate consequence of the competition in directed search. If some other firm offers a high wage, then a firm that offers a low wage in $[0, \epsilon]$ will not get any applicants because its wage offer is unattractive and so it might as well offer ϵ rather than any lower wage. On the other hand, if all other firms offer low wages, then offering a low wage

[13] The range of low wages that yields zero profits leads to quasi-concavity instead of the strict concavity that obtains at wages at which the firm makes strictly positive profits (given the other firms' wage offers). Since quasi-concavity is not preserved in expectation against other random strategies, we were not able to prove that only pure strategy equilibria exist, i.e. that mixed strategy equilibria would have to be degenerate. Proving a super-modular nature of the interaction by uniformly signing the cross-partials has yet been elusive.

in $[0, \epsilon]$ yields at best an average application probability. Outbidding the other firms with a wage to which all applicants apply for sure is profitable, because it is cheap in absolute terms since the others are offering nearly no utility to the workers and it allows the firm to hire for sure. There are other methods to the discontinuity problem like Reny's (1999) concept of better reply security, which is fulfilled in this environment. The approach taken here will be useful because a similar version applies in the incomplete information environment.

The previous lemmas enable us to show the existence of a directed search equilibrium. Recall that the equilibrium definition entails the focus on pure strategies, which is the main cause of technical difficulty and the main contribution over previous existence results.

Proposition 4.1 *A directed search equilibrium exists.*

Proof. We will restrict the strategy space to $\mathcal{W}_\epsilon$ for ϵ small. We will first establish quasi-concavity of a firm j's profit function given some vector $\mathbf{w}_{-j}$ of other firms wage announcements with $w_k \geq \epsilon$ for all $k \neq j$. For $w_j < \underline{w}_j$ firm j's profit is trivially zero. For $w_j \in [\underline{w}_j, x_j]$ we will establish strict concavity in w_j. We will first show strict concavity locally at all $(w_j, \mathbf{w}_{-j}) \in \Omega_j$. The first derivative of the profit function (7) is given by

$$\frac{\partial \pi_j(\mathbf{w})}{\partial w_j} = -H(p_j(\mathbf{w})) + h(p_j(\mathbf{w}))(x_j - w_j)\frac{\partial p_j(\mathbf{w})}{\partial w_j}. \tag{11}$$

The second derivative is given by

$$\frac{\partial^2 \pi_j}{\partial w_j^2} = -2\ h(p_j)\ \frac{\partial p_j}{\partial w_j} + h'(p_j)\Big(\frac{\partial p_j}{\partial w_j}\Big)^2 (x_j - w_j) + \frac{\partial^2 p_j}{\partial w_j^2}(x_j - w_j), \tag{12}$$

where we suppressed the argument $\mathbf{w}$ for brevity. It is easy to see that $h'(p_i) = -n\ (n-1)\ (1-p_i)^{n-2} < 0$. Since $h(p_i) > 0$, $\partial p_i / \partial w_i > 0$, and $\partial^2 p_i / \partial w_i^2 < 0$ by lemma 4.2, we have strict concavity locally. The set $(w_j, \mathbf{w}_{-j}) \notin \Omega_j$ has only a finite number of elements and we can take left and right derivatives by taking the appropriate limits of (11). As shown in the proof of lemma 4.2 the slope of the workers response $\partial p_j(\mathbf{w})/\partial w_j$ is larger for the left than the right limit, therefore the left derivative of the profit function is larger than the right derivative and strict concavity extends globally.

$\mathcal{W}_\epsilon$ is closed and convex. On $\mathcal{W}_\epsilon$ the vector $\mathbf{p}(\mathbf{w})$ is continuous, therefore the profit functions are continuous, and therefore the best response correspondence of the firms are upper-hemicontinuous by Berge's Theorem. Since profits are quasi-concave the correspondence is convex-valued. This ensures existence by Kakutani's fixed point theorem. ∎

Next, we turn to our main characterization result that higher productivity firms pay higher wages. The result would follow immediately from (11) if the firms would face the same set of competing wages $\mathbf{w}_{-j}$, as then higher productivity firms have a higher first derivative. Since the set of competing wages is in general different for each firm (as each firm does not compete with itself), the proof is more involved. Nevertheless, the insight is similar: A high productivity firm that offers a low wage in a candidate equilibrium has a higher incentive to locally raise its wage compared to a lower productivity firm that considers a deviation around its candidate high wage.

A higher incentive to raise the wage will contradict the case where high productivity firms offer low wages, *if* higher incentives imply profitability of a deviation. This will be the case if optimality is characterized by a first order condition, but might not hold if the profit function is "kinked". Kinks can arise if some firm posts such a wage that it does not attract applicants, but would attract applicants if some other firm would reduce its wage. On the other hand, optimality is characterized by a first order condition if the equilibrium wage profile is in Ω_j for all j. We ensure this by focusing on an environment in which all firms make strictly positive profits. This will be the case when firms' productivities are not too different as ensured by condition (C1).[14] It will also apply if the number of workers is sufficiently large given the number of firms. Alternatively it holds under a standard free entry condition for the firms in M, i.e., if each firm j with productivity x_j first decides whether to actually enter the market at cost K and then the entrants compete as outlined here.

Proposition 4.2 *Under (C1), any directed search equilibrium $\mathbf{w} \in \mathcal{W}$ involves $w_j > w_k$ if $x_j > x_k$ and $w_j = w_k$ if $x_j = x_k$ for $j, k \in M$.*

Proof. Assume $x_1 > x_2$ but $w_1 \leq w_2$. The focus on firms 1 and 2 is without loss of generality. By (C1) all firms make strictly positive profits in equilibrium. By standard arguments there is no equilibrium in which all firms set a wage of zero, as their hiring probability would be less than one and any slight increase in the wage would allow a firm to hire for sure. If some firms set strictly positive wages, then a firm that wants to hire has to offer a strictly positive wage. Since all firms make strictly positive profits, equilibrium wages are characterized by their first order condition which by (11) can be written as

$$-1 + \frac{h(p_l(\mathbf{w}))}{H(p_l(\mathbf{w}))} \frac{\partial p_l(\mathbf{w})}{\partial w_l}(x_l - w_l) = 0 \text{ for } l \in \{1, 2\}. \tag{13}$$

[14] It can be shown that the result holds for those firms making strictly positive profits in any equilibrium $\mathbf{w}$ in which $x_j \neq U(\mathbf{w})$ for all $j \in M$. Only if firms offer a wage equal to their productivity, nevertheless do not get any applicants, but would get applicants with strictly positive probability if other firms lowered their wage, is the first order approach taken here not valid.

The first equality in (9) implies that $\partial p_1(\mathbf{w})/\partial w_1 \geq \partial p_2(\mathbf{w})/\partial w_2$ if and only if $|g(p_2(\mathbf{w}))|w_2 \geq |g(p_1(\mathbf{w}))|w_1$. Since this cannot be assured for arbitrary candidate equilibria $\mathbf{w}$, we will proceed to show $[h(p_j(\mathbf{w}))/H(p_j(\mathbf{w}))][\partial p_j(\mathbf{w})/\partial w_j]$ is weakly higher for firm 1 than for firm 2. Since $x_1 > x_2$ but $w_1 \leq w_2$ optimality for firm 2 implies that firm 1 strictly prefers to raise its wage, providing the desired contradiction.

The denominator of (9) is identical for both firms. Therefore we want to show that

$$\frac{h(p_j(\mathbf{w}))G(p_j(\mathbf{w}))}{H(p_j(\mathbf{w}))} \sum_{s \neq j} \prod_{k \notin \{j,s\}} |g(p_k(\mathbf{w}))|w_k \tag{14}$$

is higher for $j = 1$ than for $j = 2$. Since $p_1(\mathbf{w}) \leq p_2(\mathbf{w})$, we have $h(p_1(\mathbf{w}) \geq h(p_2(\mathbf{w}))$, $H(p_1(\mathbf{w})) \leq H(p_2(\mathbf{w}))$ and $G(p_1(\mathbf{w})) \geq G(p_2(\mathbf{w}))$. The sum in (14) contains the term $\prod_{k \notin \{1,2\}} |g(p_k(\mathbf{w}))|w_k$ that is common to both firm 1 and 2, but is multiplied by a higher factor for firm 1. The other terms of the sum are common except for the fact that $|g(p_1(\mathbf{w}))|w_1$ is important for firm 2 but not for firm 1 and vice versa. Therefore it is sufficient to establish that

$$\frac{h(p_1(\mathbf{w}))G(p_1(\mathbf{w}))}{H(p_1(\mathbf{w}))}|g(p_2(\mathbf{w}))|w_2 \geq \frac{h(p_2(\mathbf{w}))G(p_2(\mathbf{w}))}{H(p_2(\mathbf{w}))}|g(p_1(\mathbf{w}))|w_1. \tag{15}$$

By (6) it holds that $w_2/w_1 = G(p_1(\mathbf{w}))/G(p_2(\mathbf{w}))$. Together with $|g(p_j)| = [G(p_j) - (1 - p_j)^{n-1}]/p_j$ and $G(p_j) = H(p_j)/(np_j)$ inequality (15) reduces to

$$\frac{G(p_2(\mathbf{w})) - (1 - p_2(\mathbf{w}))^{n-1}}{G(p_2(\mathbf{w}))(1 - p_2(\mathbf{w}))^{n-1}} \geq \frac{G(p_1(\mathbf{w})) - (1 - p_1(\mathbf{w}))^{n-1}}{G(p_1(\mathbf{w}))(1 - p_2(\mathbf{w}))^{n-1}}.$$

For this, it will be sufficient to show that $[G(p) - (1 - p)^{n-1}]/[G(p)(1 - p)^{n-1}]$ is strictly increasing in p. The derivative of this expression has the same sign as

$$\frac{n-1}{1-p}G(p) + g(p) - \frac{n-1}{(1-p)^n}g(p)G(p).$$

The last summand is positive, so we only have to establish that $[n - 1]G(p) > -(1 - p)g(p)$. Using $g(p) = -[G(p) - (1 - p)^{n-1}]/p$ this will be the case if $G(p) < (1 - p)^n/[1 - np]$, which is equivalent to $1 - np < (1 - p)^n$. This inequality holds by binomial expansion of $(1 - p)^n$ and establishes the desired contradiction.

For the case $x_1 = x_2$ but $w_1 < w_2$, we have $p_1(\mathbf{w}) < p_2(\mathbf{w})$. Since $h(.)$, $H(.)$ and $G(.)$ are all strictly monotone, a similar argument as above establishes $[h(p_1(\mathbf{w}))/H(p_1(\mathbf{w}))][\partial p_1(\mathbf{w})/\partial w_1] > [h(p_2(\mathbf{w}))/H(p_2(\mathbf{w}))][\partial p_2(\mathbf{w})/\partial w_2]$, which yields the desired contradiction. ■

In section 2 we discussed the difficulty of proving that higher productivity firms post higher wages. Productive firms have an incentive to post high wages because they have a high value from filling their vacancy, yet global market power yields a different application behavior to firms posting different wages. If at low wages it is less desirable to increase the wage, that could outweigh the higher incentives to post higher wages. Proposition 4.2 can be interpreted as the proof that at the first order condition the productivity effect dominates a market power effect. In view of figure 1 it means that the distance between wages w_1 and w_5 is larger than the distance of their best responses. That is, if firm 5 is at its optimum and firm 1 is at least as productive as firm 5, then firm 1 best response is to the left of w_1. Therefore, in equilibrium wage offers have to be indicative of the ranking of underlying productivities by the firms.

4.2 Homogeneous Firms

The last result has immediate consequences for an environment with homogeneous firms: All firms set the same wage in equilibrium. As we discussed in the introduction, the existence of an equilibrium in which all firms post the same wage has been proven in BSW, yet the existence of additional equilibria with wage dispersion had not been considered.[15] One contribution of the present work is the consideration of asymmetric posting strategies, i.e., even for equal productivities firms' equilibrium strategies are allowed to involve different wage offers. Even in this case, despite the finiteness of the market and the presence of search frictions, any directed search equilibrium will only involve identical wage postings. Wage dispersion is driven by fundamentals and is absent when firms are identical.

Corollary 4.1 *If $x_j = x_k$ for all $j, k \in M$, then in the unique directed search equilibrium all firms set the same wage.*

Proof. For equal productivities condition (C1) holds trivially, and therefore proposition 4.2 applies. BSW show that there exists only one directed search equilibrium in which all firms post the same wage. ■

4.3 Lack of Constrained Efficiency

In directed search models the standard notion of efficiency is a notion of constrained efficiency. This notion is based on the idea that the decentralized nature of the application process cannot be overcome, i.e. workers will use symmetric strategies (see e.g. Montgomery (1991), Shimer (2005)). Then the planners problem is to maximize output given the constrained that

[15]For the case of two workers and two firms, BSW consider the set of all equilibria.

workers use a symmetric application behavior. Montgomery (1991) lays out the constrained optimization problem for this setting:

$$\max_{\mathbf{p}\geq 0} \sum_{j=1}^{n} [x_j H(p_j)] \tag{16}$$

such that $\sum_{j=1}^{n} p_j = 1$ and $p_j \geq 0$ for all $j \in M$, where $\mathbf{p} = (p_1, ..., p_n)$ is the vector of application probabilities. This yields the following first order conditions:

$$x_j h(p_j) \leq \lambda, \; = \lambda \text{ if } p_j > 0, \quad \forall j \in M, \tag{17}$$

where λ is the Lagrange multiplier for the problem. This holds only if the marginal output gain from workers applying more is identical for all firms that receive applications with positive probability, i.e. $x_j h(p_j)$ is identical at all these firms. Montomery (1991) shows that this condition holds under belief (2), i.e. when firms neglect their global market power. As we discussed in section 2, the market power in a finite economy yields different wages, and therefore does not necessarily induce constrained efficiency. Montgomery (1991) mentions this possibility without investigating it. While it is obvious that global market power affects wages, it is not clear that it affects the application behavior of workers. It could be possible that the change in wages is such that application behavior is not affected and constrained efficiency might still obtain. We show that this is the case indeed the case for homogeneous firms. For heterogeneous firms this does not happen since firms post different wages and global market power affects them differently. Proving this formally turns out to be difficult due to the implicit nature of the equation describing equilibrium behavior. We prove this point for the case of a duopoly, i.e. $m = 2$, with arbitrary numbers of workers $n \geq 2$, where constrained efficiency is never achieved when firms are heterogeneous.

Proposition 4.3 *If firms are homogeneous, the directed search equilibrium is constrained efficient. If (C1) holds in a duopoly market with heterogeneous firms, i.e.* $M = \{1, 2\}$ *and* $x_1 > x_2$*, then the directed search equilibrium is inefficient.*

Proof. For homogeneous firms this is easy to see. By Corollary 4.1 all firms post identical wages. Then all firms have equal hiring probability, and (17) holds trivially. It is straightforward to verify that this is sufficient for optimality.

Consider now the case of heterogeneous firms, under (C1) the equilibrium is characterized by each firms first order condition, i.e. by (23). In the appendix we show that (23) and optimality condition (17) hold for the two firms only if either $h(p_1)w_1 = h(p_1)w_1$ or $h(p_j)x_j/n =$

$G(p_j)w_j$ for $j \in \{1,2\}$. The first case cannot obtain because in equilibrium expected utilities are equalized, i.e. $G(p_1)w_1 = G(p_2)w_2$, which implies $G(p_1)/h(p_1) = G(p_2)/h(p_2)$, but cannot arise because $G(p)/h(p)$ is a strictly increasing function of p. Since $x_1 > x_2$ by assumption, by Proposition 4.2 we have $w_1 > w_2$ in a directed search equilibrium, and therefore $p_1 > p_2$. This contradicts $G(p_1)/h(p_1) = G(p_2)/h(p_2)$ and condition (17) for optimality cannot be fulfilled. The other condition also leads to a contradiction with the firms optimal behavior, as we show in the appendix. ∎

5 The Incomplete Information Environment

We now consider the environment in which productivities are private information. Each firm j independently draws its productivity from a distribution F_j with non-zero density on $[\underline{x}, \bar{x}]$. Let F denote the joint cumulative distribution function with support $\mathcal{S}$. F is common knowledge. Let $F_{-j} = \prod_{k \neq j} F_k$ denote both the marginal and conditional distribution over $\mathbf{x}_{-j}$ given x_j.

A pure strategy for firm j in this game of incomplete information is an element ϕ_j of the space of functions Φ_j that map each type $[\underline{x}_j, \bar{x}_j]$ into a wage in $\mathcal{W}_j = [0, \bar{x}_j]$. A pure strategy profile is a tuple of functions $\phi = (\phi_1, ..., \phi_m) \in \Phi = \times_{j=1}^{m} \Phi_j$, and correspondingly a pure strategy of the opponents is a tuple $\phi_{-j} = (\phi_1, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m)$. The expected payoffs of firm j are then given by

$$\Pi_j(\phi) = E_{\mathbf{x}} \pi_j(\phi(\mathbf{x})) = \int_{\mathcal{S}} H(p_j(\phi(\mathbf{x})))[x_j - \phi_j(x_j)] dF(\mathbf{x}), \tag{18}$$

where π_j, p_j and H are as defined in the previous section and $E_{\mathbf{x}}$ denotes the expectations operator with regard to $\mathbf{x}$.

Definition 5.1 (DSEII) *A directed search equilibrium with incomplete information (DSEII) is a tuple* $\phi \in \Phi$ *such that* $\Pi_j(\phi) \geq \Pi_j(\phi'_j, \phi_{-j})$ $\forall j \in M, \forall \phi'_j \in \Phi_j$.

Our first result is an analog to lemma 4.3. We want to show that we can restrict the strategy space Φ to $\Phi_\epsilon \subseteq \Phi$ involving only functions that map types into wages weakly above ϵ. We want to show that an equilibrium in the restricted space is also an equilibrium in the larger space.

Lemma 5.1 *There exists* $\epsilon' > 0$ *such that for all* $\epsilon \in [0, \epsilon']$ *the following holds: For any* $\phi \in \Phi_\epsilon$ *every firm* j *has a best response in the unrestricted set* Φ_j *that maps all types into wages in* $[\epsilon, \bar{x}_j]$.

Proof. In lemma 4.3 we showed that a wage $w_j \in [0, \epsilon]$ does not get any applicants when some other firms posts a wage larger than $t\epsilon$, where t is independent of ϵ. If all other firms offer wage in $[\epsilon, t\epsilon]$, the hiring probability for firm j is unity if it posts a wage larger than $z\epsilon$, where z is again independent of ϵ. This is strictly more profitable at any productivity in $[\underline{x}_j, \bar{x}_j]$ than to offer a wage in $[0, \epsilon]$ at which the hiring probability is less than a half – if ϵ is sufficiently small. Therefore $z\epsilon$ weakly payoff dominates any wage in $[0, \epsilon]$ for any wages $\mathbf{w_{-j}} \in [\epsilon, \infty)^{m-1}$ for ϵ small. ■

From now on let ϵ be such that the condition in lemma 4.3 is fulfilled, i.e. $\epsilon \in (0, \epsilon')$ implying wages in $\mathcal{W}_\epsilon = \times_{j=1}^{m}[\epsilon, \bar{x}_j]$. A DSEII entails that for almost all types x_j the announced wage $w_j = \phi_j(x_j)$ maximizes the expected conditional payoff $\Pi_j(w_j, \phi_{-j}|x_j)$, where

$$\Pi_j(w_j, \phi_{-j}|x_j) \;=\; [x_j - w_j] \int H(p_j(w_j, \phi_{-j}(\mathbf{x}_{-j})))dF_{-j}(\mathbf{x}_{-j}). \tag{19}$$

To develop existence arguments along the lines of Athey (2001) we next establish single crossing. Note that under (C1) any type of firm can ensure itself strictly positive profits. That is, given any $\phi \in \Phi$ there exists $\tilde{w}_j \in [0, \underline{x}_j)$ such that any higher wage ensures strictly positive hiring probabilities, i.e. $\tilde{w}_j = \sup\{w \in [0, \underline{x}_j] | E_{\mathbf{x}_{-j}} H(p_j(\phi(x_j, \mathbf{x}_{-j}))) = 0\} < \underline{x}_j$ exists.

Lemma 5.2 *Given $\phi \in \Phi$, the function $\Pi_j(w_j, \phi_{-j}|x_j)$ satisfies the (Milgrom and Shannon, 1994) single crossing property in $(w_j, x_j) \in [0, \bar{x}_j] \times [\underline{x}_j, \bar{x}_j]$. Under (C1) it satisfies strict single crossing in $(w_j, x_j) \in [\tilde{w}_j, \bar{x}_j] \times [\underline{x}_j, \bar{x}_j]$.*

Proof. Consider $w'_j > w_j$ and $x'_j > x_j$. From (19) it trivially follows that $\Pi_j(w'_j, \phi_{-j}|x_j) - \Pi_j(w_j, \phi_{-j}|x_j) \geq (>)0$ implies $\Pi_j(w'_j, \phi_{-j}|x'_j) - \Pi_j(w_j, \phi_{-j}|x'_j) \geq (>)0$. If $w_j \geq \tilde{w}_j$ then $E_{\mathbf{x}_{-j}} H(p_j(w'_j, \phi_{-j}(\mathbf{x}_{-j}))) > 0$. In this case it is easy to see that $\Pi_j(w'_j, \phi_{-j}|x_j) - \Pi_j(w_j, \phi_{-j}|x_j) \geq 0$ implies $\Pi_j(w'_j, \phi_{-j}|x'_j) - \Pi_j(w_j, \phi_{-j}|x'_j) > 0$. ■

This yields immediately an existence result.

Proposition 5.1 *A directed search equilibrium with incomplete information (DSEII) exists in non-decreasing strategies. Under (C1) any DSEII involves strategies that are non-decreasing a.e.*

Proof. For wage profiles in cube $\mathcal{W}_\epsilon$ the profit function $\pi(\mathbf{w})$ is continuous in $\mathbf{w}$. Given the single crossing property established in lemma 5.2, existence in non-decreasing strategies in the restricted strategy space Φ_ϵ follows immediately from Athey (2001) Theorem 1 and

Theorem 2. By lemma 5.1 this equilibrium is also an equilibrium in the larger strategy space Φ.

Any equilibrium strategy ϕ_j maximizes (19) almost everywhere. Under (C1) the range of wages that maximize (19) is a subset of the wages for which we have established strict single crossing in lemma 5.2. Therefore, every selection from the set of maximizing wages (conditional on type) involves a selection that is monotone in the firm's type (see Milgrom and Shannon, 1994). ■

The proposition establishes that in equilibrium every firm increases its wage offer in its type. In general that does not imply that different firms use equilibrium strategies that imply that the firm with the higher realized productivity posts a weakly higher wage. This would arise only if $\phi_j(x) = \phi_k(x)$ for all $x \in suppF_j \cap suppF_k$ and all $j, k \in M$. If $F_j \neq F_k$ it is unlikely to be optimal for each firm to use the same strategy as the other, since the distribution of realized wages that each firm faces will be different.[16] In the case where firms are ex-ante identical, i.e. $F_j = F_k$ for all $j, k \in M$ we can establish the existence of equilibria in symmetric strategies. In this case it is not only true that a firm's wage is increasing in its type, but also that the wage differential between firms yields information about the rank order of their productivities. While it can be verified that Athey's proof readily delivers existence of symmetric equilibria, this is not formally stated and we provide a proof relying on a recent fixed point argument by Reny (2006) which is especially intuitive for Baysian games.

Proposition 5.2 *If firms are symmetric in the sense that $F_j = F_k$ for all $j, k \in M$, there exists a symmetric DSEII in non-decreasing strategies. Under (C1) any symmetric DSEII involves strategies that are non-decreasing a.e..*

Proof. Consider some symmetric strategy in Φ_ϵ played by all firms, with $\hat{\phi}$ denoting the strategy for an individual firm. Due to the single crossing property, the best reply correspondence of firm j to this strategy by the other players has a monotone selection (see Milgrom and Shannon, 1994). Therefore we can restrict the set of best replies to a subset $B(\hat{\phi})$ of the set of increasing functions in Φ_ϵ without loss of optimality. This space of increasing functions in Φ_ϵ is compact under the L_1-norm. Since $\Pi_j(\phi)$ is continuous in $\phi \in \Phi_\epsilon$, by Berge's Theorem the best reply correspondence in increasing functions in Φ_ϵ is non-empty and upper-hemicontinuous.

To use a fixed point argument, we need an additional condition such as convex-valuedness of the best reply correspondence. While this can be established in our context, we will

[16]The intuition is the same as the intuition for different bidding strategies in private value auctions in which buyers have different distributions for the valuations.

rely on the concept of contractible-valuedness proposed in Reny (2006). The best response correspondence $B(\hat{\phi})$ is contractible-valued if there is some $\phi_1 \in B(\hat{\phi})$ and a mapping $h : [0,1] \times B(\hat{\phi}) \to B(\hat{\phi})$ such that $h(0,\phi_0) = \phi_0$ and $h(1,\phi_0) = \phi_1$ for all $\phi_0 \in B(\hat{\phi})$. This simply means that we have to be able to map any best reply into some other best reply in a continuous way without leaving the space of best reply functions. Reny (2006) proposes the function $h(t,\phi_0) = \phi'_t$ for $t \in [0,1]$ with the following property. For $t \in [0,1/2]$ let $\phi'_t(x) = \phi_0(x)$ for $x \leq x_t = \underline{x}_j + (1/2 - t)[\bar{x}_j - \underline{x}_j]$ and $\phi'_t(x) = \max\{\phi_1(x), \phi_0(x)\}$ for $x > x_t$. This function does nothing else then slowly move function ϕ_0 onto ϕ_1 whenever ϕ_1 is higher. It starts with high values of x, and the shift is continuous in the L_1 integral norm. Clearly all ϕ'_t are in $B(\hat{\phi})$ as they are still increasing, and for each type it either assigns $\phi_1(x)$ or $\phi_0(x)$ which are both best replies a.e.. At the end of this part of the process, $\phi'_{1/2}$ is the pointwise maximum of ϕ_1 and ϕ_0. In the second part of the process involving $t \in (1/2, 1]$, this new function $\phi'_{1/2}$ is transformed into ϕ_1 through the specification $\phi'_t(x) = \min\{\phi'_{1/2}(x), \phi_1(x)\}$ for $x \leq x_t = \underline{x}_j + (1-t)[\bar{x}_j - \underline{x}_j]$ and $\phi'_t(x) = \phi'_{1/2}(x)$ for $x > x_t$. Again the transition is continuous and preserves monotonicity and optimality. Clearly, $h(1,\phi_0) = \phi'_1 = \phi_1$, and so $B(\hat{\phi})$ is contraction-valued. Since the set of increasing functions in Φ_ϵ is closed it is also an absolute retract. Then $B(.)$ has a fixed point by Eilenberg and Montgomery's (1946) fixed point theorem. (See Reny (2006) Theorem 5.1.)

By lemma 5.1 the fixed point is a symmetric DSEII in the unrestricted space Φ. The reason why any symmetric DSEII involves non-decreasing strategies under (C1) is the same as in lemma 5.1. ■

Finally, we briefly establish that the inefficiency results from the complete information environment carry over to this environment with incomplete information. Constrained efficiency would be obtained if a social planner that assigns a behavioral strategy to each firm and a symmetric strategy to the workers cannot achieve more matches than the decentralized equilibrium. Consider the case of a duopoly of firms, i.e. $m = 2$ and arbitrary $n \geq 2$. Let F^∞ denote a joint distribution of types for the complete information environment, i.e. $F_j^\infty(x) = 0$ for $x < x_j$ and $F_j^\infty(x) = 1$ otherwise for some productivity x_j of firm j. Assume $x_1 \neq x_2$. Now consider a sequence of joint distributions $\{F^k\}_{k=1}^\infty$ of our incomplete information environment that converges weakly to F^∞. We will show that the equilibrium outcomes of this game of incomplete information are not constrained efficient.[17]

Proposition 5.3 *Assume all F^k have identical support and $C1$ holds. Then, there exists k^* such that for all $k > k^*$ the set of directed search equilibria associated for distribution F^k are*

[17]The proof is a bit more elaborate than the standard purification result by Harsanyi (1973) since we have continuous type and strategy spaces.

not constrained efficient.

Proof. For the limit distribution F^{∞} all directed search equilibria are inefficient. For pure posting strategies we know this from Proposition 4.3. Even if firms would use non-degenerate mixed strategies the equilibrium would be inefficient as (17) would be violated with strictly positive probability. If the set of equilibria of the incomplete information game is upper-hemicontinuous in the type distribution, then the equilibria of the incomplete information game converge to the equilibria of the complete information game, and the inefficiency carries over.

The appropriate notion of upper-hemicontinuity is established in Theorem 2 of Milgrom and Weber (1986). Our environment fulfills their requirement of absolute continuous information because of the independence of the productivity draws. If we restrict the strategy space to Φ_{ϵ}, i.e. allow only wage postings above some $\epsilon > 0$, our environment fulfills also the equicontinuous payoff condition, and we have upper-hemicontinuity under this restriction.

Finally, note that the restriction does not reduce the set of equilibria given condition $C1$. Under $C1$, even the lowest type firm can make strictly positive equilibrium profits. Therefore, every type of firm has to have strictly positive probability of hiring at any wage that it offers. We know from the argument establishing lemma 5.1 that we can find constants t and z such that: If firm 1 offers a wage below ϵ, it has a positive hiring probability only when firm 2 posts a wage below $t\epsilon$. Moreover, if firm 2 posts a wage below $t\epsilon$, firm 1 can be sure to hire a worker by offering a wage of $z\epsilon$. Let $\underline{w}_1$ be the lowest wage firm 1 posts in equilibrium, i.e. $\underline{w}_1 = \inf\{w | \int_{x:\phi_1(x)\geq w} dF_1(x) < 1\}$. Define a similar lowest wage for firm 2. We want to show that each firm offers only wages above ϵ for ϵ small. Assume to the contrary that $\underline{w}_1 < \epsilon$. Offering a wage (close to) $\underline{w}_1$ yields a strictly positive hiring probability for firm 1 only when firm 2 posts a wage below $t\epsilon$. But in this case the hiring probability for firm 1 has to be above the average of $H(1/2)$ (it has to be close to 1 in fact), as otherwise it would be strictly better to offer $z\epsilon$ since the cost of increasing the wage is at most $z\epsilon$ (and thus small compared to the gain in hiring probability). Therefore, firm 2 has to offer wages below $\underline{w}_1$ at least some of the time, i.e. $\underline{w}_2 < \underline{w}_1$. Repeating the analysis from the view of firm 2, it has to hold that $\underline{w}_1 < \underline{w}_2$. This means that firms are only willing to offer very low wages if they outbid their competitor, which cannot be mutually compatible and yields the desired contradiction. Our restriction is therefore not binding. ■

6 Conclusions

In finite markets individual participants have market power. This might be important in a labor market that is segregated by profession, geographical location and time of hiring. We show that equilibria in pure posting strategies exist in a finite directed search economy. This result holds both in the case when firms know each other's productivity levels, and in the case when they only know the distribution from which their competitors draw their productivities. We also prove that, despite the finite frictional nature of our environment, wage differences are driven by productivity differences when firms have complete information and when they have incomplete information drawn from a symmetric distribution. This confirms the results of models where there is no global market power, either because the market is large (Shi (2001), Shimer (2005)) or by assumption (Montgomery (1991)). Furthermore, wage dispersion is absent when firms are identical. The last point expands on the results of Burdett, Shi and Wright (2001) who take global market power into account in a model with homogeneous firms but focus on equilibria where all firms offer the same wage. Our paper clarifies that in their environment there are no other equilibria in pure posting strategies. For homogeneous firms, wages are reduced compared to the case without global market power, yet all firms are equally prone to this reduction and constrained efficient application behavior is obtained. In markets with heterogeneous firms wage dispersion is present and different wages are differently affected by global market power. We prove this for a duopoly case, in which heterogeneity always prevents an efficient application behavior.

7 Appendix

Proof of Lemma 4.2:

The set $\mathcal{A}_j = \{w_j \in (\underline{w}_j, x_j) \mid (w_j, \mathbf{w}_{-j}) \notin \Omega_j\}$ contains only a finite number of elements. By continuity of $p_j(.)$, $u_j(.)$ and $G(.)$ the second equality in (9) readily implies $\lim_{w_j \nearrow \hat{w}_j} \partial p_j(w_j, \mathbf{w}_{-j})/\partial w_j < \lim_{w_j \searrow \hat{w}_j} \partial p_j(w_j, \mathbf{w}_{-j})/\partial w_j$ for all $\hat{w}_j \in \mathcal{A}_j$.

Therefore, it will be sufficient to show strict concavity for all $w_j \in (\underline{w}_j, x_j)\backslash\mathcal{A}_j$. Denoting $S(\mathbf{w}) = u_j(\mathbf{w}) + X_j(\mathbf{w})$ where $X_j(\mathbf{w}) = 1/\sum_{k\in\mathcal{M}\backslash\{j\}} \frac{1}{u_k(\mathbf{w})}$ we obtain the following when differentiating (9)

$$\frac{\partial^2 p_j}{\partial w_j^2} = -\frac{1}{S^2}\left(g(p_j)\frac{\partial p_j}{\partial w_j}[X_j + u_j] - G(p_j)\left[g'(p_j)\frac{\partial p_j}{\partial w_j}w_j + g(p_j) + \frac{\partial X_j}{\partial w_j}\right]\right), \tag{20}$$

where we omitted the argument $\mathbf{w}$ for brevity. We will show that (20) is strictly negative. We will split the round bracket into three parts A, B and C and show that each is non-negative. Moreover, $A = g(p_j)[\partial p_j/\partial w_j]X_j$ is strictly positive because $g(p_j)$ and X_j are strictly negative.

Now consider part B entailing

$$B = g(p_j)\ \frac{\partial p_j}{\partial w_j}\ u_j -\ \ G(p_j)\ \Big[g'(p_j)\ \frac{\partial p_j}{\partial w_j}\ w_j + g(p_j)\Big]. \tag{21}$$

Rearrangements and the use of (9) yields

$$B = G(p_j)w_j[2g(p_j)^2 - g'(p_j)G(p_j)] + X_j\ g(p_j)G(p_j).$$

Since the last summand is positive, B is positive if the square bracket is positive. Let $T(p) = 2g(p)^2 - g'(p)G(p)$ reflect this bracket. Noting that $g'(p) = \frac{1}{np^3}\{2-2(1-p)^n-2np(1-p)^{n-1} - n(n-1)p^2(1-p)^{n-2}\}$, inserting and rearranging yields $T(p) = D(p)/[np^3(1-p)^{n-2}]$, with the numerator $D(p) = -2(1-p) + (n-1)p + 2(1-p)^{n+1} + (n+1)p(1-p)^n$. Since the denominator is positive, we have to show that the numerator is positive. Note that $D(0) = 0$. $D'(p) = n+1-(n+1)(1-p)^n - (n+1)np(1-p)^{n-1}$ and thus $D'(0) = 0$. Then $D''(p) = (n+1)n(n-1)p(1-p)^{n-2} > 0$ proves that $B \geq 0$.

Finally, consider $C = -G(p_j)[\partial X_j/\partial w_j]$. Since

$$\frac{\partial X_j}{\partial w_j} = X_j^2\Big[\sum_{k\in\mathcal{M}\setminus\{j\}} \frac{g'(p_k)}{g(p_k)^2 w_k}\frac{\partial p_k}{\partial w_j}\Big], \tag{22}$$

and clearly $\partial p_k/\partial w_j \leq 0$ we have shown that C is non-negative if $g'(p_k) \geq 0$. Application of L'Hopital's rule yields $g'(0) = (n-1)(n-2)/3 \geq 0$. Since $g''(p) = n(n-1)(n-2)p^2(1-p)^{n-3} \geq 0$ we have $g'(p) \geq 0$ for all $p \geq 0$.

Remaining Proof of Proposition 4.3:

We have to show that for heterogeneous firms the first order condition (23) and the optimality condition (17) imply $h(p_1)w_1 = h(p_2)w_2$. We can use the fact that in a directed search equilibrium $x_1 > x_2$ implies $w_1 > w_2$ and $p_1 > p_2$. The first order condition for both firms implies

$$\frac{h(p_1(\mathbf{w}))}{H(p_1(\mathbf{w}))}\frac{\partial p_1(\mathbf{w})}{\partial w_1}(x_1 - w_1) = \frac{h(p_2(\mathbf{w}))}{H(p_2(\mathbf{w}))}\frac{\partial p_2(\mathbf{w})}{\partial w_2}(x_2 - w_2). \tag{23}$$

By the middle term of (9) the denominators of the partial derivatives cancel, and we obtain

$$h(p_1)\frac{|g(p_2)|w_2}{np_1}(x_1 - w_1) = h(p_2)\frac{|g(p_1)|w_1}{np_2}(x_2 - w_2), \tag{24}$$

where we have suppressed the dependence of p_j on $\mathbf{w}$ and used the relation $G(p_j) = H(p_j)/(np_j)$.

Substituting $|g(p_j)| = [G(p_j) - h(p_j)/n]/p_j$ and rearranging yields

$$h(p_1)(x_1 - w_1)[G(p_2)w_2 - h(p_2)w_2/n] = h(p_2)(x_2 - w_2)[G(p_1)w_1 - h(p_1)w_1/n]. \tag{25}$$

In equilibrium workers apply as to equalize expected utility, i.e. $G(p_1)w_1 = G(p_2)w_2 = C_a$ is constant. Optimality requires $h(p_1)x_1 = h(p_2)x_2 = C_b$ to be constant. We therefore have

$$[-C_a + C_b/n][h(p_1)w_1 - h(p_2)w_2] = 0. \tag{26}$$

This implies that either $C_a = G(p_1)w_1 = G(p_2)w_2 = h(p_1)x_1/n = h(p_2)x_2/n = C_b/n$ or $h(p_1)w_1 = h(p_2)w_2$.

If $h(p_1)w_1 = h(p_2)w_2$, then we can use $G(p_1)w_1 = G(p_2)w_2$ to obtain $G(p_1)/h(p_1) = G(p_2)/h(p_2)$. The function $G(p)/h(p)$ has the derivative

$$\frac{\partial[G(p)/h(p)]}{\partial p} = \frac{(1-p)^{n-1}}{n^2((1-p)^{n-1}p)^2}[-1 + (1-p)^{n-1}p + (1-p)^n + (n-1)(1-p)^{-1}p] \tag{27}$$

The term in square brackets is zero at $p = 0$ and has a derivative $(n-1)[1-(1-p)^n]/(1-p)^2 > 0$, therefore $G(p)/h(p)$ is strictly increasing, and since $p_1 > p_2$ in equilibrium we have $h(p_1)w_1 \neq h(p_2)w_2$.

If $G(p_j)w_j = h(p_j)x_j/n$ for $j \in \{1, 2\}$, we get a contradiction to the first order condition of the firms combined with optimality. Using this relationship to substitute out w_1 from (9) and rearranging yields for firm 1

$$h(p_1)x_1 = \frac{H(p_1)}{\frac{\partial p_1}{\partial w_1}(\frac{h(p_1)}{nG(p_1)} - 1)}. \tag{28}$$

The second term in the denominator can be written as $|g(p_1)|p_1/G(p_1)$. Moreover, $[\partial p_1/\partial w_1] = [|g(p_1)|w_1]/[|g(p_1)|w_1 + |g(p_2)|w_2]$ together with $G(p_1)w_1 = G(p_2)w_2$ yields

$$\frac{\partial p_1}{\partial w_1} = \frac{1}{1 + \frac{|g(p_1)|}{G(p_1} \frac{G(p_2)}{|g(p_2)|}}. \tag{29}$$

Substitution into (31) and rearranging yields

$$h(p_1)x_1 = [\frac{G(p_1)}{|g(p_1)|} + \frac{G(p_2)}{|g(p_2)|}]nG(p_1). \tag{30}$$

where we used the fact that $H(p_1)/[np_1] = G(p_1)$ to obtain the last factor. Similarly, we obtain for firm 2

$$h(p_2)x_2 = [\frac{G(p_1)}{|g(p_1)|} + \frac{G(p_2)}{|g(p_2)|}]nG(p_2). \tag{31}$$

Yet $p_1 > p_2$ implies $G(p_1) < G(p_2)$, and therefore $h(p_1)x_1 < h(p_2)x_2$, which violates the optimality condition (17).

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