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"Multiplicity in General Financial Equilibrium with Portfolio Constraints",
Second Version

by

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Multiplicity in General Financial Equilibrium with Portfolio Constraints

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Abstract

This paper explores the role of portfolio constraints in generating multiplicity of equilibrium. We present a simple financial market economy with two goods and two households, households who face constraints on their ability to take unbounded positions in risky stocks. Absent such constraints, equilibrium allocation is unique and is Pareto efficient. With one portfolio constraint in place, the efficient equilibrium is still possible; however, additional inefficient equilibria in which the constraint is binding may emerge. We show further that with portfolio constraints cum incomplete markets, there may be a continuum of equilibria; adding incomplete markets may lead to real indeterminacy.

JEL Classifications: G12, D52
Keywords: Multiple equilibria, asset pricing, portfolio constraints, indeterminacy, financial equilibrium.

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1. Introduction

Over the past several decades, financial markets have been developing at a mind-boggling pace. Still, no investor can claim to have complete freedom in allocating funds across time and between events. Elaborate risk management practices aimed at limiting the potential negative consequences of investors' information asymmetries, moral hazard problems, or, simply, bad judgement, have been developing hand in hand with financial markets. A complete account of such practices would require a lengthy exposition; but it is fair to say that they may all be summarized as limits on positions in financial assets investors are allowed to take. Thus, abstract constraints on portfolio holdings capture an array of risk-limiting restrictions in reduced form. The question we address in this paper is the general equilibrium consequences of portfolio constraints, and especially their effects on the efficiency of consumption allocation and determinants of asset prices.

This question has been previously raised in the asset pricing literature in Finance, but we approach it from a very different angle. Our main focus is on multiplicity of equilibrium: can the introduction of portfolio constraints increase the number of equilibria in an economy? Can portfolio constraints in tandem with other market frictions expand the set of equilibria even further? We present a number of robust examples of economies in which the answer to both questions is yes.

The issue of the possibility of multiple equilibria has been largely ignored in asset pricing thus far, but we feel that it is very relevant for understanding the variability of asset returns. The presence of multiple equilibria may, for example, shed light on the apparent inability of empirical studies to link many sharp price movements in stock markets to news about economic fundamentals. Indeed, if an economy (with portfolio constraints) admits more than one equilibrium for the same set of fundamentals, a movement in stock prices could be due entirely to the focus of investors' expectations, whereby they coordinate on different equilibria.

Our primary economic setting is a simple two-period, two-good, pure-exchange model with two states of nature and two stocks paying off in units of the goods (a real assets model). Two households differ in their (log-linear) preferences, initial endowments, and investment opportunities, in the sense that one household faces a portfolio constraint on holdings in one of the stocks. Absent this constraint, the model is a familiar workhorse asset pricing model, admitting a unique Pareto efficient equilibrium. With the constraint in place, the efficient equilibrium still obtains; however, additional (inefficient) equilibria in which the constraint is binding may occur. Two features of this result require an elaboration, as it may be unclear how to place it in the context of the *general financial equilibrium* (GFE) literature in Economics.[1] First, the existence of a Pareto efficient equilibrium in which portfolio constraints are fully circumvented even with incomplete asset markets is puzzling. This has to do with the differences between the log-linear, tree-type asset pricing model adopted here and the traditional real assets model of GFE theory. Second, the introduction of portfolio constraints may expand the set of equilibria, even though by itself it does not generate equilibrium indeterminacy (as in the nominal assets model of Balasko, Cass, and Siconolfi (1990)). To our knowledge, this finding is new in the literature. In particular, we demonstrate that when there are (potentially) complete asset markets, there may be a finite number of additional equilibria (in our leading example, always precisely two). In contrast, with (and only with) incomplete asset markets there may be robust real indeterminacy of equilibrium.

To highlight the effects of incomplete markets, we incorporate on additional state into our baseline economy, leaving the number of financial assets unchanged. Absent portfolio constraints, incomplete markets play no significant role in the economy: there is a unique equilibrium corresponding to the original unique Pareto efficient equilibrium (as in Cass and Pavlova (2004)). With portfolio constraints, however, market incompleteness may matter. And, when it does, not only are there multiple equilibria, but there is also a continuum of them, with consumption allocations varying across this continuum.

There are two main strands of literature related to this paper. The first strand is the literature on asset pricing with capital market imperfections in Finance. Detemple and Murthy (1997), Basak and Cuoco (1998), Detemple and Serrat (2003), and Pavlova and Rigobon (2005), among others, all examine the effects of portfolio constraints on equilibrium consumption allocations and asset prices within the basic asset pricing framework featuring no other frictions. However, the issue of non-uniqueness of equilibrium is either not present or not addressed. Within this literature, our analysis is the first to highlight the role of portfolio constraints in generating multiple equilibria. The second strand is the financial equilibrium literature in Economics, and in particular the developments related to real assets models with restricted transactions. Cass, Siconolfi, and Villanacci (2001) demonstrate finite local uniqueness of equilibrium in a real assets model (specifically, a numeraire assets model) with exogenous portfolio constraints while Carosi and Villanacci (2006) extend this

[1] This term refers to the subarea of competitive equilibrium theory which concerns the effects of the interaction of commodity and asset markets on allocation and pricing (of both commodities and assets). When we want to emphasize this aspect of an equilibrium we will refer to as a *financial equilibrium* (FE).

result to encompass endogenous constraints (the distinction depends on whether or not the possible constraints depend on just portfolio strategies or other endogenous variables as well—as in the model we analyze here). However, nothing in their analysis suggests that this result may, in some circumstances, fail when markets are incomplete and, furthermore, that portfolio constraints may expand the number of equilibria. Our baseline framework is similar to theirs, but with one important exception: we consider a Lucas (1978) tree model, in which endowments are specified in terms of shares of stocks, not goods, as in a real assets model. As demonstrated by Cass and Pavlova, who also employ a multiple-good Lucas-tree asset pricing framework, this feature is critical and can cause the two models' implications to be in stark disagreement. Cass and Pavlova establish the existence of a "peculiar financial equilibrium" even with portfolio constraints, which is the analog of the Pareto efficient equilibrium in our model, and highlight the possible role of the goods markets in alleviating portfolio constraints. They also observe that portfolio constraints may give rise to additional equilibria (their Appendix B); however, unlike ours, the example they provide is exceptional – occurring on a measure-zero set of the proposed constraints.

Also related are the papers by Gennotte and Leland (1990), Barlevy and Veronesi (2003), and Hong and Stein (2003) who seek to explain stock market crashes and obtain multiple equilibria. However, these models rely on numerous market imperfections such as asymmetric information and short-sale constraints, as well as distributional assumptions and behavioral biases to generate the desired multiplicity of equilibria. In contrast, our argument relies on a simple portfolio constraint imposed within a canonical Arrow-Debreu framework. In this connection we should also distinguish our results from similar results concerning expansion of the set of equilibria obtained by Cass and Shell (1983) and more recently—in a different context—by DeMarzo, Kaniel, and Kremer (2004). These papers focus on an extreme form of portfolio constraints known as restricted participation, where some agents cannot transact in any financial markets. In our model, all households can trade portfolios of financial instruments, although to an extent limited by portfolio constraints. That is, the fundamental underlying assumptions concerning investors' financial opportunities—as well as many other details of the models—are quite different. Curiously, the analogue of restricted participation in our model does not lead to multiplicity of equilibrium (Cass and Pavlova, Corollary to Proposition 4).

The rest of the paper is organized as follows. Section 2 presents the main ideas of the paper in the simplest possible economic setting, with a specific portfolio constraint. Section 3 explores the effects of incomplete markets. Section 4 discusses the robustness of our main results to alternative economic settings and suggests some extensions. Section 5 provides concluding remarks. The two appendices have the same structure as the body of the paper, but present the complete, formal analysis (in particular, in Appendix A, for our baseline model with a portfolio constraint of general form). We emphasize at the outset that we utilize these appendices—which are, for this reason, self-contained—to present a novel, general, and powerful two step approach for analyzing many second best problems in the guise of our specific analysis of the effects of introducing portfolio constraints on the set of FE.

2. Portfolio Constraints and Multiplicity of Equilibrium: A Leading Example

2.1. Economic Environment

We develop the main ideas of the paper within the most basic pure-exchange economy with two time periods, $t = 0$ and 1. Uncertainty is resolved at time $t = 1$, and is represented by two states of the world, labeled $\omega = u$ ("up"), d ("down"), occurring with probabilities $\pi(\omega)$. It will sometimes be useful to refer to the initial period as state $\omega = 0$. There are two non-storable goods, labeled $g = 1, 2$, with prices $p^g(\omega) > 0$.

Production of each good g is modeled as a Lucas tree, with the exogenously specified stream of output $\delta^g(\omega) > 0$. Financial investment opportunities are given by two risky stocks, with period 0 (ex-dividend) prices $q^g(0)$, which are claims to the outputs of the two trees. Each stock is in constant supply of one unit. The time-1 stock payoff matrix P, representing the returns from the two stocks across the two states, is then given by

$$P = \begin{bmatrix} p^1(u)\delta^1(u) & p^2(u)\delta^2(u) \\ \\ p^1(d)\delta^1(d) & p^2(d)\delta^2(d) \end{bmatrix}. \tag{1}$$

We note that since the spot goods prices p^g are endogenously determined in equilibrium, the invertibility of the payoff matrix P, and hence market completeness, is not immediate. If P is not invertible, the two stocks are perfect substitutes for each other, and hence there are fewer nonredundant investment opportunities than there are states of the world. This technical difficulty is absent in economies with a single good, commonly employed in asset pricing. Under a single-good framework, which lacks spot trade in goods, P can simply be exogenously specified to be invertible by assuming invertibility of $[(\delta^1(\omega),\, \delta^2(\omega)),\, \omega = u, d]$.

The economy is populated by two households, indexed by $h = 1,\, 2$. Each household is endowed with an initial portfolio of the two stocks $s_h(0) = (s_h^1(0),\, s_h^2(0))$, in number of units (so that $s_1(0) + s_2(0) = \mathbf{1}$; we will use $\mathbf{1}$ to denote an appropriate-dimension vector of 1's), and trades in spot markets for goods in periods $t = 0,\, 1$ and the stock market in the initial period $t = 0$. No restrictions are imposed on the households' portfolios (in particular, short sales are permitted), apart from a portfolio constraint on household 2, as specified below. Each household chooses its consumption of the goods, $c_h^g(\omega) > 0$, and terminal portfolio holdings, $s_h(1) = (s_h^1(1),\, s_h^2(1))$, evaluating its actions according to the expectation of a log-linear utility function

$$u_h(c_h^1,\, c_h^2) \equiv a_h \log c_h^1(0) + (1 - a_h) \log c_h^2(0) + \beta \sum_{\omega > 0} \pi(\omega)[a_h \log(c_h^1(\omega)) + (1 - a_h) \log(c_h^2(\omega))],$$

with $a_h \in (0, 1)$, $a_1 > a_2$, and $\beta > 0$. The assumption that $a_1 > a_2$ is not essential for our main results, and is made purely for expositional convenience, as it makes the signs of households' portfolio holdings unambiguous in equilibrium.[2] The assumption that the discount factor β is common across households, on the other hand, is made just for simplicity; again, this restriction is not critical for our analysis. In contrast, the log-linear specification of households' utilities (together with the Lucas-tree specification of households' endowments) is quite important for our main results. An analysis of the robustness of the results to local perturbations of utility functions is left for future research, which is being carried out in a separate project.

The economic environment presented thus far is basically a standard workhorse in asset pricing in Finance. Our only major points of departure come from the introduction of two goods and a portfolio constraint on one of the households. In particular, household 2 faces a portfolio constraint of the form

$$s_2^2(1) \geq f(\cdot),$$

where f depends on various endogenous variables in period 0, as well as various parameters (as, for instance, specified at the beginning of Appendix A). In the body of the paper, when it becomes necessary, we specialize f to represent a lower bound constraint on the fraction of wealth invested in stock 2—the analysis of a constraint of general form is relegated to Appendix A.

[2] We comment on the case where $a_1 < a_2$ at the end of Appendix A. The borderline case where $a_1 = a_2$ is ruled out, because heterogeneity of households' utilities is required for our results.

A Financial Equilibrium (FE) in this economy is defined to be the pair of spot goods-stock prices $(p,\, q)$ and consumption-portfolio choices $(c,\, s)$ such that

(i) each household h maximizes its expected utility over its budget set, taking prices $(p,\, q)$ as given,

$$
\begin{array}{lll}
 & \displaystyle\max_{c_h^1, c_h^2, s_h^1(1), s_h^2(1)} u_h(c_h^1,\, c_h^2) & \text{with multipliers} \\
\text{subj to} & p^1(0)c_h^1(0) + p^2(0)c_h^2(0) + q^1(0)s_h^1(1) + q^2(0)s_h^2(1) & \\
 & \quad = \left(q^1(0) + p^1(0)\delta^1(0)\right) s_h^1(0) + \left(q^2(0) + p^2(0)\delta^2(0)\right) s_h^2(0), & \lambda_h(0) \\
 & p^1(\omega)c_h^1(\omega) + p^2(\omega)c_h^2(\omega) = p^1(\omega)\delta^1(\omega)s_h^1(1) + p^2(\omega)\delta^2(\omega)s_h^2(1), \quad \omega = u,\ d, & \lambda_h(\omega) \\
\text{and} & s_h^2(1) \geq f(\cdot), \quad \text{for } h = 2, & \nu
\end{array}
$$

and (ii) all spot goods and stock markets clear,

$$
\begin{aligned}
c_1^g(\omega) + c_2^g(\omega) &= \delta^g(\omega), \quad \text{and} && (2) \\
s_1^g(1) + s_2^g(1) &= 1, \qquad g = 1,\, 2, \quad \omega = 0,\, u,\, d. && (3)
\end{aligned}
$$

2.2. Preliminaries: Properties of Equilibrium Portfolio Holdings

In this section, we demonstrate that equilibrium portfolio holdings in our economy are exactly the same across all equilibrium allocations that are not Pareto efficient (and can take the same form when they are). We stress that we need not assume any particular type of asset market friction to obtain this result: the only property required of the market friction is that it entails an equilibrium in which the marginal utilities of the households are not colinear.

Consider the period-1 spot budget constraints of one of the households, say, Ms. 1:

$$
p^1(\omega)c_1^1(\omega) + p^2(\omega)c_1^2(\omega) = p^1(\omega)\delta^1(\omega)s_1^1(1) + p^2(\omega)\delta^2(\omega)s_1^2(1). \tag{4}
$$

Substituting the period-1 first-order conditions for the households' problems,

$$
\pi(\omega)\beta a_h / c_h^1(\omega) = \lambda_h(\omega)p^1(\omega) \quad \text{and} \quad \pi(\omega)\beta(1-a_h)/c_h^2(\omega) = \lambda_h(\omega)p^2(\omega), \qquad \omega = u,\ d, \tag{5}
$$

together with the market clearing conditions (2) into this budget constraint and rearranging, we obtain the following simple expression

$$
1 = \left(a_1 + a_2\frac{\lambda_1(\omega)}{\lambda_2(\omega)}\right) s_1^1(1) + \left((1-a_1) + (1-a_2)\frac{\lambda_1(\omega)}{\lambda_2(\omega)}\right) s_1^2(1), \qquad \omega = u,\ d. \tag{6}
$$

At this point it is useful to define the notion of the households' "stochastic weights":

$$\eta_h(\omega) = \begin{cases} \beta/\lambda_h(0), & \omega = 0, \\ 1/\lambda_h(\omega), & \omega = u,\ d. \end{cases}$$

It turns out to be extremely convenient to conduct our structural analysis in terms of these variables rather than, in particular, spot goods prices. For this reason we adopt the normalization $\eta_1(\omega) = 1$, so that we can write simply $\eta_2(\omega) = \eta(\omega)$, for all ω. None of our results depends on this choice, but the ease and elegance of their derivation does (see Appendices A and B). The quantities $\eta(\omega)$ can be interpreted as the weights of Mr. 2's utilities in an auxiliary social planner's problem, a problem defined in a broad sense which makes it convenient for describing equilibrium allocations even if they are not Pareto efficient.[3] Pareto efficiency of the equilibrium allocation requires that $\eta(\omega)$ be a constant weight across all ω. Substituting these weights into equation (6) above, now represented in matrix form, we arrive at the following condition that Ms. 1's equilibrium portfolio holdings $(s_1^1(1),\ s_1^2(1))$ must satisfy:

$$\begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} = \begin{bmatrix} a_1 + a_2\eta(u) & 1 - a_1 + (1 - a_2)\eta(u) \\ \\ a_1 + a_2\eta(d) & 1 - a_1 + (1 - a_2)\eta(d) \end{bmatrix} \begin{pmatrix} s_1^1(1) \\ \\ s_1^2(1) \end{pmatrix}. \tag{7}$$

The 2×2 matrix above is nothing else but the payoff matrix P in equilibrium. The determinant of this matrix P is given by

$$\det(P) = (a_1 - a_2)(\eta(d) - \eta(u)). \tag{8}$$

So, clearly, when the stochastic weights are identical across the two states of the world $\eta(u) = \eta(d)$, the payoff matrix is not invertible. Hence, one of the stocks is redundant, and the portfolio holdings of households in each stock are indeterminate. Note that the condition $\eta(u) = \eta(d)$ implies Pareto efficiency and is equivalent to $\lambda_1(\omega) = \lambda_2(\omega)$, $\omega = u, d$, i.e., identical Lagrange multipliers across

[3] Consider the programming problem of maximizing a fictitious representative agent's utility of the form

$$\sum_h \left[\eta_h(0)[a_h \log c_h^1(0) + (1 - a_h) \log c_h^2(0)] + \beta \sum_{\omega > 0} \pi(\omega)\eta_h(\omega)[a_h \log c_h^1(\omega) + (1 - a_h) \log c_h^2(\omega)] \right]$$

subject to the resource constraints, $c_1^g(\omega) + c_2^g(\omega) = \delta^g(\omega)$, for all g, all ω. This utility assigns weights $\eta_h(\omega)$, $\omega = 0$, u, d, to the households. The weight of a household may take different values across states ω, i.e., formally, it is modeled as a stochastic process. The sharing rules derived from this problem coincide with the consumption allocation arising in competitive equilibrium. As evident from the first-order conditions (5), and with our normalization of weights in mind, the following must be true for the relative weight of Mr. 2 in the representative agent's utility function:

$$\eta(\omega) = \lambda_1(\omega)/\lambda_2(\omega).$$

households or, equivalently, colinear marginal utilities in the two states.[4] Hence, the matrix being invertible, $\eta(u) \neq \eta(d)$, is only possible in an inefficient equilibrium. Finally, assuming that P is invertible in equilibrium, we may explicitly solve for Ms. 1's equilibrium portfolio holdings from (7). For completeness, in the following result we also report the holdings of Mr. 2, which follow from stock market clearing (3).

Result 1. *In any inefficient equilibrium $\eta(u) \neq \eta(d)$, and optimal portfolio holdings of the households are given by*

$$s_1^1(1) = \frac{1-a_2}{a_1-a_2}\,, \qquad s_1^2(1) = -\frac{a_2}{a_1-a_2}\,, \text{ and} \tag{9}$$

$$s_2^1(1) = -\frac{1-a_1}{a_1-a_2}\,, \qquad s_2^2(1) = \frac{a_1}{a_1-a_2}\,. \tag{10}$$

In any efficient equilibrium $\eta(u) = \eta(d)(= k)$, and there is a continuum of optimal portfolio holdings of the form

$$s_1^1(1) = \frac{1-(1-a_1+(1-a_2)k)s_1^2(1)}{a_1+a_2k}\,, \quad s_2^1(1) = \frac{k-(1-a_1+(1-a_2)k)s_2^2(1)}{a_1+a_2k}\,,$$

one of these portfolios is also (9)–(10).

The simplicity of the optimal portfolio holdings that obtain in any inefficient equilibrium is quite striking. They depend on neither the relative weights of the households, nor on the state of the economy. This property is very general and is maintained in a variety of generalized economic settings (see, specifically, Section 3). In particular, the portfolio holdings continue to be of the form (9)–(10) in a multi-period extension of our model.

The derivation above also highlights a curious property of the structure of the system of equilibrium equations, which we report fully in Appendix A: The derivation of the households' portfolio holdings does not employ a number of the equations belonging to this system, such as, for example, the period-0 first-order conditions, the no-arbitrage conditions, or the constraint on portfolio holdings. The methodology for analyzing this economy developed in Appendix A takes full advantage of this separability property, making the local analysis of the equilibrium equations particularly straightforward.

Finally, we remark that Result 1 characterizes equilibrium in which the portfolio constraint is binding (in a nondegenerate way). Of course, not all portfolio constraints would entail the existence

[4] In our model $\eta(u) = \eta(d)$ implies that $\eta(0) = \eta(u) = \eta(d)$, which has to be satisfied for an equilibrium to be Pareto efficient. This can be seen, for example, from equation (A.2) in Appendix A. Therefore, in the sequel, we check equality of the stochastic weights only in the future states $\omega = u,\, d$, and refer to these future states as simply states.

of an inefficient equilibrium. For example, it is clear from this result that any *exogenous* portfolio constraint on the number of units of investment in some stock—e.g., $s_2^2(1) \geq \gamma$, with a constant parameter γ, as in Cass and Pavlova (2004), Appendix B—is going to lead to constrained equilibrium on just a measure-zero set of possible parameters. Hence, we consider below an *endogenous* portfolio constraint, that is, we make the right-hand side of the portfolio constraint, $f(\cdot)$ depend on the endogenous variables of the model.

2.3. Multiplicity of Equilibrium with a Specific Portfolio Constraint

We now focus on a specific example of a portfolio constraint faced by Mr. 2, facilitating fairly simple analysis of the equilibria in the economy. In Appendix A, we present the more comprehensive analysis for an abstract constraint. The analysis there is self-contained: a reader wishing to see a more general analysis than the one presented below may skip directly to that appendix.

We consider a constraint imposed on the fraction of wealth Mr. 2 is permitted to invest in the second stock:

$$q^2(0)s_2^2(1) \geq \gamma W_2(0), \tag{11}$$

where γ is a prespecified constant, and Mr. 2's initial wealth is defined as the value of his portfolio $W_2(0) = (q^1(0) + p^1(0)\delta^1(0))s_2^1(0) + (q^2(0) + p^2(0)\delta^2(0))s_2^2(0)$. Such a minimum investment restriction or concentration constraint is commonplace among institutional investors, pension funds and mutual funds, for which there may be a mandate to maintain a certain proportion of a portfolio in an asset class given the stated investment objective.[5]

Given our financial market friction, at the outset, an inefficient equilibrium is likely. This conjecture is based on the related familiar result that Pareto inefficiency obtains generically in economies with incomplete asset markets and multiple goods in GFE (Duffie and Shafer (1985), Geanakoplos and Polemarchakis (1986), and Geanakoplos, Magill, Quinzii and Dreze (1990)). Moreover, inefficiency of equilibria in security market economies with portfolio constraints and a single good has long been highlighted in the asset pricing literature (e.g., Detemple and Murthy (1997), and Basak and Cuoco (1998)). However, in economies with several goods, many standard asset pricing

[5] For example, many Fidelity funds (www.fidelity.com) explicitly state such a minimum investment restriction in describing their investment strategy. In addition to the obvious select and international funds, many others in different categories and styles incorporate such a restriction. Thus, Blue Chip Growth ($22 billion assets, large growth) invests at least 80% of its assets in "blue chip" companies, Disciplined Equity ($4.2 billion assets, large blend) 80% of its assets in equity securities, Low-Priced Stock ($30 billion assets, small blend) 80% of its assets in securities with price at or below $35 per share.

results involving portfolio constraints may not necessarily go through. This is due to there being additional markets in which investors can trade—spot goods markets. The possibility of trade in these markets has a propensity to alleviate portfolio constraints. Nor do many standard findings of the GFE literature survive in a Lucas-type economy like ours, where endowments are specified in terms of shares of securities. Indeed, in contrast to standard results in GFE, Cass and Pavlova (2004) demonstrate that an efficient equilibrium always exists in a general economic setting with log-linear preferences and a large set of portfolio constraints on the number of shares of a risky stock even with incomplete markets.[6] Result 2 demonstrates that in our setting, we actually get both types of equilibrium: efficient (E), in which the portfolio constraint is completely alleviated, and inefficient (I), in which the portfolio constraint is binding.

Result 2. *For $\gamma \in (\underline{\gamma},\ \overline{\gamma})$, where $0 < \underline{\gamma} < \overline{\gamma}$ are defined in Appendix A, there are multiple equilibria in the economy, falling into two distinct types: type-E (efficient) and type-I (inefficient).*

For $\gamma \notin (\underline{\gamma},\ \overline{\gamma})$, equilibrium is unique and belongs to type-E.[7]

In the absence of portfolio constraints, there is a unique equilibrium allocation in our model. This result is not new, and is to be expected in a model with log-linear preferences. For future reference, we denote quantities corresponding to this equilibrium with an asterisk *, and will sometimes refer to the ensuing allocation as a "good" equilibrium, reflecting the fact that it is Pareto efficient. The purpose of our analysis is to demonstrate that the introduction of portfolio constraints preserves the efficient equilibrium, but also introduces new, "bad" equilibria, which are Pareto inefficient and in which the constraint binds. Figure 1 sketches the positioning of the "bad" equilibria relative to the Pareto frontier in this economy; in anticipation of our results below, the figure depicts not just one, but two "bad" equilibria. Moreover, Result 2 demonstrates that the existence of such "bad" equilibria is a robust phenomenon occurring over a range of model parameter values.[8]

[6] Although the Lucas's tree model employed here and in Cass and Pavlova is a special case of the real assets model in GFE, it is a very special kind. Endowments in the Lucas-tree model are specified in terms of shares of assets, not goods, and hence constitute a measure-zero subset of the commodity endowments examined in the real assets model. Many proofs, which involve perturbation arguments, then simply do not go through, and many implications get reversed.

[7] Of course, as discussed in Section 2.2, portfolio holdings are not unique. But this is an insignificant nonuniqueness stemming simply from the presence of redundant financial assets. More importantly, there exists a set of parameter values in our model for which there exists a continuum of type-E equilibria (see Proposition A.2 in Appendix A). However, this property is non-generic, occurring on a measure zero set of initial stock holdings (and hence commodity endowments). So, in Appendix A we rule these out by assumption (see the Remark following the proof of Proposition A.2.

[8] For expositional reasons, we focus primarily on robustness with respect to one parameter of the model, the lower

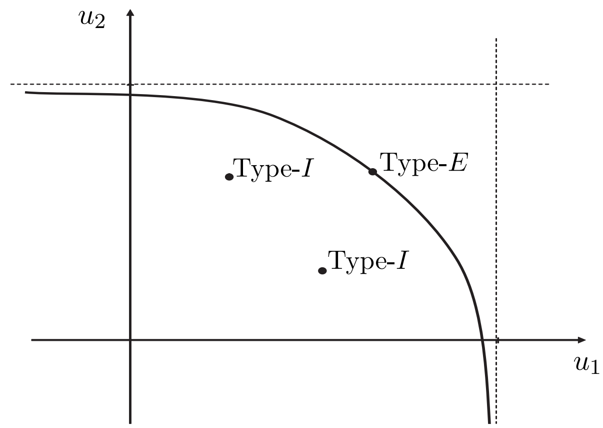

Figure 1: Utility Possibility Set. The thick solid curve depicts the utility possibility frontier.

All equilibria in this economy are rational expectations equilibria, and, provided that the parameter γ falls within a certain range, households' self-fulfilling expectations may give rise to either a "good" or a "bad" equilibrium. This observation could be quite important for understanding why many large movements in financial markets could not be linked to news about economic fundamentals. In our model, a change in fundamentals (e.g., household endowments) does entail a change in equilibrium prices, but so does a shift in the households' expectations, unrelated to fundamentals. Our baseline setting is not sufficiently rich to model a mechanism of the formation of such expectations, but we conjecture that a multi-period model which accommodates some coordination device (e.g., sunspots) can offer such a mechanism, based purely on consistent beliefs.

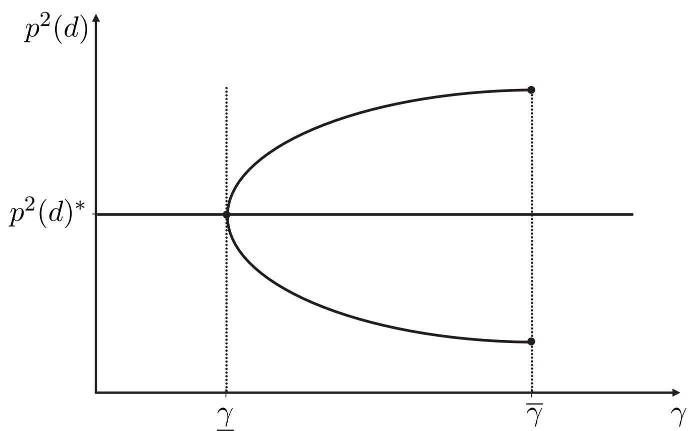

Figure 2: One Projection of the Equilibrium Correspondence. The C-shaped solid curve depicts the set of type-I equilibria. The dashed line is for the type-E equilibria.

bound fraction γ, and highlight just one projection of the equilibrium correspondence. Alternatively, we could have fixed γ and explored the robustness with respect to other parameters, such as endowments or preference weights. Appendix A presents alternative conditions for existence of type-I equilibria, with the main message unaltered: the existence of an inefficient equilibrium is a robust property of our economy.

To understand why there are multiple equilibria in our model, we find it useful to plot the equilibrium correspondence. In Figure 2, we vary the portfolio constraint parameter γ and plot the corresponding prices prevailing in equilibrium. To keep the plot two-dimensional, we report the state-d price of one of the commodities on the vertical axis. A figure with a state-u spot price is analogous. Figure 2 confirms that for any value of the parameter γ, there exists an efficient equilibrium with a corresponding price $p^2(d)^*$. In this equilibrium, the constraint does not bind, and hence the value of γ does not affect any equilibrium prices or quantities. The point corresponding to $\gamma = \underline{\gamma}$ corresponds to the knife-edge case where the equilibrium is still of type-E, the multiplier ν on the constraint is zero, but the constraint holds with equality—the portfolio constraint starts to bind. To the right of this point, the multiplier on the constraint can be distinctly different from zero, and the equilibrium correspondence exhibits a C-shaped set of inefficient equilibria. Moreover, as evident from the figure, corresponding to each $\gamma \in (\underline{\gamma},\, \overline{\gamma})$, there are exactly two distinct equilibria of type-I and one of type-E—three in total.

Result 3 supplements Figure 2 and reports the complete characterization of inefficient equilibria in our economy, which can be easily computed in closed-form.[9]

Result 3. *In the inefficient equilibrium, the consumption allocations, spot good prices, and stock price-dividend ratios are given by*

$$c_1^1(\omega) = \frac{a_1}{a_1 + a_2\eta(\omega)}\delta^1(\omega), \qquad c_2^1(\omega) = \frac{a_2\eta(\omega)}{a_1 + a_2\eta(\omega)}\delta^1(\omega), \tag{12}$$

$$c_1^2(\omega) = \frac{1-a_1}{1-a_1 + a_2\eta(\omega)}\delta^2(\omega), \qquad c_2^2(\omega) = \frac{(1-a_2)\eta(\omega)}{1-a_1 + (1-a_2)\eta(\omega)}\delta^2(\omega), \tag{13}$$

$$p^1(\omega) = \frac{a_1 + a_2\eta(\omega)}{\delta^1(\omega)}, \qquad p^2(\omega) = \frac{1-a_1 + (1-a_2)\eta(\omega)}{\delta^2(\omega)}, \qquad \omega = 0, u, d, \tag{14}$$

$$\frac{q^1(0)}{p^1(0)\delta^1(0)} = \frac{a_1 + a_2 E\left[\eta(\omega)\right]}{a_1 + a_2\eta(0)}, \text{ and } \quad \frac{q^2(0)}{p^2(0)\delta^2(0)} = \frac{1-a_1 + (1-a_2)E\left[\eta(\omega)\right]}{1-a_1 + (1-a_2)\eta(0)}, \tag{15}$$

where

$$\eta(0) = (1+\beta)\left(1 - \frac{(a_1-a_2)\gamma}{a_1(1-a_2)}\right)\eta^* + \beta\frac{1-a_1}{1-a_2}, \tag{16}$$

with $\eta^* = \frac{a_1 s_2^1(0) + (1-a_1)s_2^2(0)}{a_2 s_1^1(0) + (1-a_2)s_1^2(0)}$, *and the period-1 relative weights* $(\eta(u), \eta(d))$ *solve*

$$E\left[\eta(\omega)\right] \equiv \pi(u)\eta(u) + \pi(d)\eta(d) = (1 + 1/\beta)\,\eta^* - \eta(0)/\beta \text{ and} \tag{17}$$

$$E\left[1/\eta(\omega)\right] \equiv \frac{\pi(u)}{\eta(u)} + \frac{\pi(d)}{\eta(d)} = \frac{1}{\eta(0)} + \frac{1+\beta}{\beta}\frac{a_2}{a_1}\left(\frac{\eta^*}{\eta(0)} - 1\right). \tag{18}$$

[9] The analysis of the equilibrium system of equations, which uncovers the fundamental structure of our economy and leads to the expressions in Result 3, is presented in Appendix A. However, one needs to bear in mind that the units in which consumption and prices are expressed in the appendix differ from the original units employed here. The primary reason for the transformation of units adopted in the appendix is that intrinsic uncertainty per se is essentially irrelevant to the fundamental structure of the system of equations defining equilibrium in our model.

The efficient equilibrium is as in the unconstrained benchmark economy with constant relative weights, $\eta(0) = \eta(u) = \eta(d) = \eta^*$, *and*

$$\frac{q^1(0)}{p^1(0)\delta^1(0)} = \frac{q^2(0)}{p^2(0)\delta^2(0)} = 1. \tag{19}$$

Result 3 sheds light on the question of why there are exactly two inefficient equilibria (when such exist). Note that equilibrium consumption allocations and spot goods prices depend on the relative weights of Mr. 2, $\eta(\omega)$, all ω, which in an inefficient equilibrium must vary across states, $\omega = u$, d. Moreover, the system of equations pinning down these two future weights, (17)–(18), is quadratic and, in the region of existence of type-I equilibria, always admits two positive solutions for $(\eta(u), \eta(d))$—hence two inefficient equilibria.

The intuition underlying the multiplicity of equilibrium is the following. Recall that, absent portfolio constraints, equilibrium is unique. Adding a (potentially binding) portfolio constraint on Mr. 2's holdings of stock 2 introduces a wedge in his no-arbitrage condition for this stock. This reflects the fact that, from Mr. 2's perspective, when the portfolio constraint is binding, stock 2 is worth more than simply his (personal) present value of its dividends. But with this wedge in place, i.e., when the multiplier associated with the portfolio constraint is positive, this household's stochastic weights vary across states, implying that the two households' marginal utilities are no longer colinear—hence inefficiency of equilibrium. Together these two cases, depending on whether or not the portfolio constraint is binding, constitute all of the possibilities for solutions to the particular system of equations representing FE in this model (whose entire structure we detail in Appendix A). In this system, the number of equations equals the number of variables, but the equations are nonlinear. So, while it is natural for one to expect that, in general, it will have some solution, it is also natural for one to expect that, in general, this solution may not be unique. Both expectations are rational. A solution which always exists corresponds to the case where the wedge in Mr. 2's no-arbitrage condition for stock 2 is zero—the unique type-E equilibrium coinciding with that in the unconstrained economy. Other solutions correspond to the case where the wedge is positive, and these are type-I equilibria. To solve for such equilibria, we take advantage of Result 1, which characterizes the portfolio holdings in any inefficient equilibrium. Upon substitution of these holdings into the constraint (11), it becomes apparent (with some further analysis) that, associated with each relevant value of the parameter γ, there are exactly two pairs of the weights $(\eta(u), \eta(d))$ which are consistent with inefficiency. We must warn the reader, however, that our finding that there are two equilibria depends critically on the nature of the specific constraint as well as the number of future states (and hence stochastic weights). For example, if we introduce one additional state together with one additional asset (say, a real bond), the number of equilibria may expand beyond three, although remain finite.

Result 3 also reveals that there is variation in the stock prices and price-dividend ratios across type-E and type-I equilibria. In an efficient equilibrium, the price-dividend ratios do not vary with the states of the economy and equal unity. This is a standard result with log-linear utilities. In an inefficient equilibrium, however, they depend on two common factors, $\eta(0)$ and $\eta(\omega)$ which reflect the effects of the portfolio constraint faced by Mr. 2. This is in contrast to analogous results in the continuous-time asset pricing literature investigating the effects of portfolio constraints in the economies populated by investors with log-linear utilities.

3. Incomplete Markets

In our leading example, we assume that markets are potentially complete. It is interesting to investigate whether our central result that portfolio constraints expand the set of equilibria holds under alternative assumptions on the financial markets' structure, in particular, if markets are inherently incomplete. Our goal in this section is to establish that with incomplete markets portfolio constraints may not only lead to multiple inefficient equilibria, but when they do, in fact lead to a continuum of such equilibria.[10]

To demonstrate this, we augment the leading example of Section 2 in one respect: we add a new state of the world, $\omega = m$ ("middle"), so that in period 1, $\omega = u$, m, d, occurring with probabilities $\pi(\omega)$. The rest of the economic environment remains the same, and in particular, there are still two stocks—one fewer than there are states of the world at $t = 1$.

We continue focusing on the specific portfolio constraint introduced in Section 2.3 (equation (11)). In the absence of the constraint, the equilibrium allocation is unaffected by market incompleteness.[11] This means that the stochastic weights are identical across all states of the world, and hence the resulting equilibrium is always of type-E.

We have already established in the leading example, however, that portfolio constraints may give rise to bad, type-I equilibria. As in Section 2.2, we begin the construction of such equilibria

[10]A reader interested in more detailed, rigorous analysis than the one offered in the body of the paper may directly skip to Appendix B, which is self-contained.

[11]This result is not new in the literature (see, for example, Cass and Pavlova (2004)).

here by examining the households' terminal portfolio holdings. The procedure for deriving these portfolios is the same as before, except that now we need to consider three period-1 spot budget constraints for Ms. 1 (corresponding to $\omega = u, m, d$), instead of just the two (corresponding to $\omega = u, d$) in the leading example. Accordingly, the number of stochastic weights extends from 3 to 4—one for period 0 and one for each of 3 possible states ω. The equation determining terminal portfolio holdings, which is an analog of (7), now takes the form

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} a_1 + a_2\eta(u) & 1 - a_1 + (1-a_2)\eta(u) \\ a_1 + a_2\eta(m) & 1 - a_1 + (1-a_2)\eta(m) \\ a_1 + a_2\eta(d) & 1 - a_1 + (1-a_2)\eta(d) \end{bmatrix} \begin{pmatrix} s_1^1(1) \\ \\ s_1^2(1) \end{pmatrix}. \tag{20}$$

It is easy to verify that the households' portfolio holdings reported in Result 1 are the portfolio holdings supporting equilibrium in the economy with incomplete markets. This property is, in fact, quite notable: generally, a non-homogeneous linear system of 3 equations with 2 unknowns does not have a solution. In our case, even when the stochastic weights are different from each other, there is always a unique solution to (20): $s_1(1) = (\frac{1-a_2}{a_1-a_2}, -\frac{a_2}{a_1-a_2})$.[12]

One way to think of the consequences of this observation for the ensuing equilibrium is to notice that the three equilibrium equations in (20) now allow us to determine only 2 variables. This implies that the remaining system of equilibrium equations contains 1 fewer equation than there are unknowns. This observation is the key to the following result.

Result 4. *For $\gamma \in (\underline{\gamma}, \overline{\gamma})$ there is a continuum (i.e., a smooth one-dimensional manifold) of distinct type-I equilibria.*

While the proof of Result 4 is somewhat involved (see Appendix B), the intuition behind it is simple. In the leading example, we have shown that there are two distinct equilibria in which the portfolio constraint binds. This is due to our ability to substitute from the equilibrium equations for all variables except the stochastic weights, and then reduce them to a quadratic system of 2 equations (17)–(18) with 2 unknowns—the stochastic weights in the terminal period. The setting considered in this section preserves this structure, except that we now have 1 "degree of freedom" in that system—1 additional stochastic weight in period 1. In particular, the analogs of equations

[12] The proof of this result parallels that of Proposition A.1 in Appendix A.

(17)–(18) are now

$$E[\eta(\omega)] \equiv \pi(u)\eta(u) + \pi(m)\eta(m) + \pi(d)\eta(d) = (1 + 1/\beta)\,\eta^* - \eta(0)/\beta \text{ and} \tag{21}$$

$$E[1/\eta(\omega)] \equiv \frac{\pi(u)}{\eta(u)} + \frac{\pi(m)}{\eta(m)} + \frac{\pi(d)}{\eta(d)} = \frac{1}{\eta(0)} + \frac{1+\beta}{\beta}\frac{a_2}{a_1}\left(\frac{\eta^*}{\eta(0)} - 1\right), \tag{22}$$

where the quantities η^* and $\eta(0)$ are fully determined by the (exogenous) parameters of the model, as reported in Result 3. Solving these 2 equations with 3 unknowns ($\eta(u)$, $\eta(m)$ and $\eta(d)$), we find a continuum of terminal stochastic weights consistent with equilibrium. This is what is known in the literature as real indeterminacy. Not only do portfolio constraints expand the set of equilibria, but in this case they can also generate a smooth, one-dimensional manifold of equilibria.[13] We illustrate this result in Figure 3, which shows how the utility possibility set changes in the presence of incomplete markets. Note that all characterizations reported in Result 3 continue to hold in any incomplete markets' equilibrium, with ω now taking values in $\{0, u, m, d\}$.

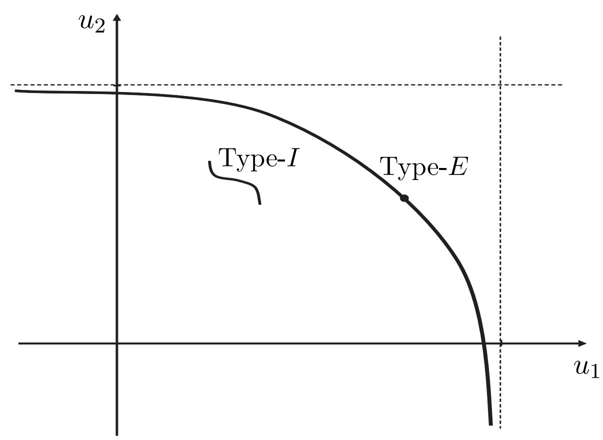

Figure 3: Utility Possibility Set with Incomplete markets. The thick solid curve depicts the utility possibility frontier. The thin curve depicts the set of Type-I equilibria.

Remark. The discussion above readily extends to analyzing an incomplete markets economy with more than 3 states at $t = 1$. If, for example, there are 4 states of the world, the system of equations (20) contains 4 equations with 2 unknowns. Portfolio holdings solving this system in the case when stochastic weights differ from each other are the same as reported in the text. The remaining equilibrium equations can again be reduced to (21)–(22), but with $E[\eta(\omega)]$ and $E[1/\eta(\omega)]$ now being sums of 4 terms. Hence (21)–(22) is now a system of 2 equations with 4

[13]Note that our economy exhibits finite local uniqueness of equilibrium when markets are potentially complete but indeterminacy when markets are inherently incomplete. This result seems to contradict Cass, Siconolfi, and Villanacci (2001), whose analysis shows that generic finite local uniqueness of equilibrium with portfolio constraints obtains independently of the degree of market incompleteness. The difference here is simply that trees give rise to household endowments which are nongeneric.

unknowns. This gives rise to a smooth, two-dimensional manifold of possible inefficient equilibria. In the general case in which the difference between the number of terminal states and the number of stocks equals D, for $\gamma \in (\underline{\gamma}, \overline{\gamma})$ there is a smooth, D-dimensional manifold of type-I equilibria.

4. Extensions

The economic environment in our leading example is, admittedly, very stylized. It is thus of interest to investigate the robustness of our results to richer economic settings. We first consider expanding the number of intrinsic states of the world together with the number of investment opportunities. In particular, in our leading example we introduce an additional state, m, so that now, as in Section 3, $\omega = u$, m, d, and an additional asset, a bond, in zero net supply, paying out one unit of good 1 in each state of the world. Again, we have as many assets as there are states of the world, and so markets are potentially complete. It turns out that equilibria in this economy are very similar to the ones uncovered in Section 2. First, the investors' portfolio holdings are unchanged. The holdings of the two stocks are exactly the same as those presented in Result 1, and no investor holds any shares of the bond. Second, since Result 1 is central to our argument that fleshes out the multiplicity of equilibria in which the portfolio constraint binds, our results on uniqueness of equilibrium readily extend. In particular, in the benchmark economy without portfolio constraints equilibrium allocation is unique and is Pareto efficient, while in the economy with the portfolio constraint there are two types of equilibria: of type-E and of type-I. The exact form of the equilibrium correspondence, however, remains to be completely worked out.

One of the benefits offered by a richer set of investment opportunities is that one can model a larger class of portfolio constraints. Many constraints imposed in practice involve more than one stock. For instance, margin requirements allow investors to use bonds and stocks in lieu of collateral for short positions established in other assets belonging to the portfolio. Borrowing constraints act along similar lines. Additionally, the concentration constraint considered in Section 2 typically involves a number of stocks belonging to the same asset class (e.g., large stocks, small stocks, value stocks). Some of these constraints can be investigated in the three-asset extension of our model, while others may require increasing the number of stocks further. The structure of the problem, however, remains the same. As we demonstrate in Appendix A, the key is to analyze the system of equilibrium equations without the constraint, and then make use of the fact that all these constraints can be written parametrically, for example, as $f(q(0)s_2(1)/W_2(0)) \geq \gamma$, where q is a vector of stock prices, to trace a range of the parameter γ for which multiple equilibria may occur. Another avenue for investigation will be to consider a setting in which portfolio constraints are imposed on more than one household, a more realistic scenario because in practice few investors can claim to be free of portfolio constraints. Even now, little is known about the economic mechanism through which portfolio constraints affect stock returns and their correlations, although there is mounting empirical evidence suggesting that portfolio constraints do matter. Our two step approach is very flexible, and adopting it may prove useful for exploring these questions.

Perhaps most interesting would be to consider a multi-period extension of our leading example. We believe that the bulk of our results extends to this setting, based on partial, preliminary analysis. In a multi-period or an infinite-horizon version of our model, one can meaningfully address questions related to the dynamics of asset returns (e.g., time-varying volatilities and momentum) and their comovement. Another useful feature offered by the multi-period setting is that it allows to model a mechanism through which households form their expectations. Perhaps the easiest way to do so is to introduce extrinsic uncertainty in the model (or a sunspot) that acts as a coordination device in guiding households' (rational) expectations of which equilibrium will occur. However, one needs to first prove the existence of multiple equilibria in a multi-period economy, and this remains an open question.

Finally, one very important potential extension of our analysis here is to investigate the robustness of our results on multiplicity of equilibrium under broader specifications of investors' utilities and endowments. We leave this undertaking for future research.

5. Concluding Remarks

Numerous articles by academics and practitioners in Finance have acknowledged the widespread presence of portfolio constraints in modern financial markets. Yet surprisingly little is known about the possible effects of imposing even one such constraint. In particular, the full extent to which portfolio constraints generate multiplicity or indeterminacy in a general model of financial equilibrium remains largely unexplored. Our paper makes a first step in this direction by constructing a suggestive, transparent leading example in which portfolio constraints may enlarge the set of equilibria, sometimes dramatically. But much more research needs to be done, on investigating this and other potential consequences of restricting the behavior of investors. We believe that such issues provide a fruitful and important research agenda, one which will very likely generate many useful insights into the functioning of modern financial markets.

Appendix A

In this appendix we provide a comprehensive analysis of our leading example – where there are two periods, two states in the second period, two goods (and hence two stocks), and two households – and the second household is constrained in its holding of the second stock. Our aim is to make the appendix self-contained, and hence there is some minor overlap of the analysis here with that presented in the text (in particular, in the derivation of the result labeled Proposition A.1 below), where we want to emphasize and re-emphasize especially important structural features of our model. Before proceeding we need to explain our treatment of the portfolio constraint.

Besides assuming that only Mr. 2 is constrained in his holding of only stock 2, for simplicity we also assume that this constraint takes the form of an endogenous lower bound. Let (the vectors) $p(0)$ and $q(0)$ represent spot goods and stock prices, respectively, in period 0. Also let

$$f : \mathbb{R}^4_{++} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \text{ s.t.}$$

$$((p(0), q(0)), s_2(0), \gamma) \mapsto f(p(0), q(0), s_2(0), \gamma),$$

where $\gamma \in \mathbb{R}$ is a parameter, represent the RHS of Mr. 2's portfolio constraint, so that he is restricted in his transactions in stock 2 by

$$s_2^2(1) \geqq f(\cdot). \qquad \begin{array}{c}\text{with multiplier}\\ \nu\end{array}$$

Such a constraint enters into the system of equations describing equilibrium in just two ways: First, a term ν is added to the second of the two equations which would have represented Mr. 2's lack of arbitrage possibilities were he not constrained; it will be convenient to still refer to these as no-arbitrage conditions. Second, the complementary slackness condition (CSC) associated with the constraint,

$$\min\{s_2^2(1) - f(\cdot), \nu\} = 0,$$

becomes another equation in the system. The critical feature of our approach to uncovering the structure of the solutions to the resulting system of equations is that, except for the fact that we do introduce the multiplier ν at the outset, we ignore the CSC until the very end of the analysis. It is only at that point where we worry about showing how it can be accommodated, essentially by judiciously specifying the dependence of f on γ – the final step in our development, what we will refer to as "tailoring" the constraint.

In principle, the two step approach followed here is not limited to our leading example (in particular, the assumptions that only Mr. 2 faces portfolio constraints, and that only his holding of the second stock is involved), to constraints on lower bounds, or even to analyzing the effects of portfolio constraints in GFE. The analysis of any second best problem which is described by specifying constraints on individual behavior very often involves introducing multipliers into first-order conditions and adding complementary slackness conditions. By specifying these constraints in sufficiently rich parametric form, one can first consider just the effects of the modifications of first-order conditions, second the tailoring of parameters in order to also satisfy complementary slackness conditions. In short, potentially, this two step approach has wide applicability. Of course, in many cases it will be more complicated and difficult to follow than it turns out to be here.

A.I. **The Extended Form Equations (EFE's)**

In order to economize on space and, later on, the number of equations we need to write down, we will switch (as context suggests useful) between notations $(\pi(\omega), \omega = u, d)$ or $(\pi, 1-\pi)$, and $\alpha_h = (\alpha_h^1, \alpha_h^2)$ or $(a_h, 1-a_h)$, $h = 1, 2$. Also, contrary to accepted practice in mathematics, we will find it very useful to treat $\alpha_h = (a_h, 1-a_h)$ and other price-like vectors (i.e., $p(\omega)$ and $q(0)$) as row vectors. All quantity-like vectors, such as $c_h(\omega)$ and $s_h(\omega)$, are column vectors.

The basic system of EFE's describing FE (again, excluding the CSC) consists of the usual first-order conditions (FOC's), no-arbitrage conditions (NAC's), and spot budget constraints (SBC's) for both households, together with the market clearing conditions (MCC's) for goods and stocks. We omit qualifiers (e.g., "$g = 1, 2$" or "$\omega = u, d$"), since they are self-evident.

FOC's $\quad \alpha_h^g / c_h^g(0) = \lambda_h(0) p^g(0)$ and

$$\pi(\omega)\beta\alpha_h^g / c_h^g(\omega) = \lambda_h(\omega) p^g(\omega)$$

NAC's $\quad -\lambda_1(0)q(0) + \sum_{\omega=u,d} \lambda_1(\omega)\left(p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega)\right) = 0$ and

$$-\lambda_2(0)q(0) + \sum_{\omega=u,d} \lambda_2(\omega)\left(p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega)\right) + (0, \nu) = 0$$

SBC's $\quad p(0)c_h(0) = [q(0) + (p^1(0)\delta^1(0), p^2(0)\delta^2(0))]s_h(0) - q(0)s_h(1)$ and

$$p(\omega)c_h(\omega) = (p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega))s_h(1)$$

MCC's $\quad c_1^g(\omega) + c_2^g(\omega) = \delta^g(\omega)$ and

$$s_1^g(1) + s_2^g(1) = 1$$

But for μ and $s_h(1)$ (both unsigned at this point), all variables are strictly positive.

A.II. **The Reduced Form Equations (RFE's)**

The stochastic weights

$$\eta_h(\omega) = \begin{cases} \beta/\lambda_h(0),\ \omega = 0 \\ \\ 1/\lambda_h(\omega),\ \omega = u, d \end{cases}$$

(which we find a more convenient representation than the Lagrange multipliers $\lambda_h(\omega)$) play a central role in our analysis. Since this is a model with real assets (stock payoffs are specified in terms of goods), and there are three spots, there are also three possible price normalizations. We find it extremely useful, at the outset, to choose these as $\eta_1(\omega) = 1$, all ω, so that we can then write $\eta_2(\omega) = \eta(\omega)$, all ω. Given this normalization, we will show that the EFE's can effectively be represented by just 3 RFE's

$$-(1 + 1/\beta)\alpha_1 s_2(0) + [(\eta(0)/\beta) + (\pi\eta(u) + (1-\pi)\eta(d))]\alpha_2 s_1(0) = 0, \tag{A.1}$$

$$\pi(1 - \eta(0)/\eta(u))(a_1 + a_2\eta(u)) + (1-\pi)(1 - \eta(0)/\eta(d))(a_1 + a_2\eta(d)) = 0, \text{ and} \tag{A.2}$$

$$\begin{aligned} &\pi(1 - \eta(0)/\eta(u))[(1-a_1) + (1-a_2)\eta(u)] + \\ &\qquad (1-\pi)(1 - \eta(0)/\eta(d))[(1-a_1) + (1-a_2)]\eta(d) - \mu = 0 \end{aligned} \tag{A.3}$$

in the 4 variables $(\eta, \mu) = ((\eta(\omega), \text{ all } \omega), \mu)$. This is most clearly demonstrated in a series of steps by which we systematically suppress superfluous (from an analytic viewpoint) variables. All this will, of course, depend heavily on the trees and logs structure of the model – though we believe that, for the most part, our central results are robust to local perturbations of utility functions around the specific log-linear functions we employ here. But this conjecture remains to be fully, seriously investigated. The reader who is willing to take on faith the manipulations required in going from the EFE's to the RFE's should skip ahead to the following section, referring back to Proposition A.1 as needed.

Step 1. From inspection of the EFE's it is obvious that the dividends $\delta = ((\delta^g(\omega), g = 1, 2), \text{ all } \omega)$ can be suppressed by a transformation of units of goods to units per dividend (with a corresponding transformation of units of account), so that, say,

$$c_h^g(\omega)/\delta^g(\omega) \mapsto \tilde{c}_h^g(\omega) \text{ and } p^g(\omega)\delta^g(\omega) \mapsto \tilde{p}^g(\omega).$$

Because it is very cumbersome to carry the tildes throughout the analysis in this and the following appendix, hereafter we will rewrite $\tilde{c} = ((\tilde{c}_h^g(\omega), g = 1, 2, \text{ all } \omega),\ h = 1, 2)$ and $\tilde{p} = (\tilde{p}^g(\omega),\ g = 1, 2,$ all $\omega)$ as simply c and p. **The reader should bear in mind that for the purposes of our analysis (in the appendices) we effectively assume that $\delta^g(\omega) = 1$, while for the purposes of our interpretation (in the main text) we assume that, typically, there is intrinsic uncertainty, that is, that $\delta^g(\omega) > 0$ varies with both g and ω.** Thus, for example, $p(u) = p(d)$ here means that, in the original units of account, spot goods prices adjust to completely offset dividend uncertainty.

Step 2. Rewriting the FOC's to highlight the determination of c (bearing in mind that now $\eta_1(\omega) = 1$ and $\eta_2(\omega) = \eta(\omega)$, all ω),

$$c_h^g(\omega) = \begin{cases} (\alpha_h^g/\beta)\eta_h(0)/p^g(0), \ \omega = 0, \\ \\ \pi(\omega)\alpha_h^g\eta_h(\omega)/p^g(\omega), \ \omega = u, d, \end{cases} \tag{A.4}$$

summing over h and using the MCC's for goods, and then solving for p yields

$$p(\omega) = \begin{cases} (\alpha_1/\beta) + (\alpha_2/\beta)\eta(\omega), \ \omega = 0. \\ \\ (\pi(\omega)(\alpha_1 + \alpha_2\eta(\omega)), \ \omega = u, d. \end{cases} \tag{A.5}$$

This entails, first, that (A.5) can be used to solve for p in terms of η, and second, that substituting from (A.5) into (A.4), the resulting equations, for instance,

$$c_1^g(\omega) = \begin{cases} (\alpha_1^g/\beta)/[(\alpha_1^g/\beta) + (\alpha_2^g/\beta)\eta(\omega)], \ \omega = 0, \\ \\ (\pi(\omega)\alpha_1^g)/(\alpha_1^g + \alpha_2^g\eta(\omega)), \ \omega = u, d, \end{cases} \tag{A.6}$$

can be used to solve for c in terms of η, so that both p and c can effectively be ignored (in the subsequent analysis).

Step 3. Using spot-by-spot analogues of Walras' law, Mr. 2's SBC's are redundant, and can just be discarded. And then the MCC's for stocks can be used to solve for $s_2(1)$ terms of $s_1(1)$

$$s_2(1) = \mathbf{1} - s_1(1), \tag{A.7}$$

so that these variables can also effectively be ignored. (We will use $\mathbf{1}$ to denote the appropriate-dimension vector of 1's.)

Step 4. Adding together her SBC's weighted by

$$\lambda_1(\omega) = \begin{cases} \beta/\eta_1(0) = \beta, \ \omega = 0, \\ \\ 1/\eta_1(\omega) = 1, \ \omega = u, d \end{cases}$$

and using her NAC's, after some simplifying and rearranging, Ms. 1's SBC at $\omega = 0$ can be replaced by the overall (Walrasian-like) BC

$$\sum_\omega p(\omega)c_1(\omega) - (p(0) + p(u) + p(d))s_1(0) = 0.$$

But substituting from (A.6) for $p(\omega)c_1(\omega)$, all ω, and from (A.5) for p, then simplifying (in particular, using the fact that $s_1(0) + s_2(0) = \mathbf{1}$ and (A.7) to rewrite $1 - \alpha_1 s_1(1) = \alpha_1 s_2(1)$, and then multiplying the resulting equation by -1) this overall BC becomes the first of the RFE's (A.1).

At this point we're only left to deal with Ms. 1's SBC's at $\omega = u, d$ and the NAC's.

Step 5. Now substituting from (A.6) for just $p(\omega)c_1(\omega)$, $\omega = u, d$, and from (A.5) for just $p(\omega), \omega = u, d$, Ms. 1's SBC's at $\omega = u, d$ become, after some simplifying,

$$(\alpha_1 + \alpha_2\eta(\omega))s_1(1) = 1,\ \omega = u, d. \tag{A.8}$$

These linear equations can be solved for $s_1(1)$ in terms of η, so these variables can also effectively be ignored – that is, ignored until we need them later on (via (A.7)) in order to tailor the portfolio constraint. And their solution(s) have some very special properties which we fully exploit at that point, so may just as well document formally at this. **Note that here and after we will disregard the uninteresting borderline case where $a_1 = a_2$, and concentrate on the case where $a_1 > a_2$.** (We'll comment on the opposite case, where $a_1 < a_2$, at the very end of the appendix.)

Proposition A.1. *If $(\eta(u), \eta(d)) \gg 0$ and $\eta(u) \neq \eta(d)$, then (A.8) has the (same) unique solution*

$$s_1(1) = ((1 - a_2)/(a_1 - a_2), -a_2/(a_1 - a_2)) \tag{A.9}$$

so that, from (A.7),

$$s_2(1) = (-(1 - a_1)/(a_1 - a_2), a_1/(a_1 - a_2)). \tag{A.10}$$

If $\eta(u) = \eta(d) = k > 0$, then (A.8) has the continuum of solutions

$$s_1^1(1) = [1 - ((1 - a_1) + (1 - a_2)k)s_1^2(1)]/[a_1 + a_2k], \tag{A.11}$$

one of which is also (A.9).

Proof of Proposition A.1. Case 1. $(\eta(u), \eta(d)) \gg 0$ and $\eta(u) \neq \eta(d)$.

The equations (A.8) have the form $Ax = \mathbf{1}$, with

$$A = \begin{bmatrix} a_1 + a_2\eta(u) & (1 - a_1) + (1 - a_2)\eta(u) \\ a_1 + a_2\eta(d) & (1 - a_1) + (1 - a_2)\eta(d) \end{bmatrix}.$$

Straightforward calculation shows that

$$\det(A) = (a_1 - a_2)(\eta(d) - \eta(u)) \neq 0,$$

and hence, that

$$\begin{aligned} s_1(1) = A^{-1}\mathbf{1} &= 1/(a_1 - a_2)(\eta(d) - \eta(u)) \times \\ &\quad \begin{bmatrix} (1 - a_1) + (1 - a_2)\eta(d) & -(1 - a_1) - (1 - a_2)\eta(u) \\ -a_1 - a_2\eta(d) & a_1 + a_2\eta(u) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= ((1 - a_2)/(a_1 - a_2), -a_2/(a_1 - a_2)). \end{aligned}$$

Case 2. $\eta(u) = \eta(d) = k > 0$.

(A.11) is obvious (since in this case either of the identical equations in (A.8) yields this continuum of solutions), as is the fact that (A.9) satisfies (A.11) (since the unique solution whenever the equations (A.8) are not colinear must also be a solution when they are).■

Step 6. Ms. 1's NAC's can be used, again in conjunction with (A.5), to solve for $q(0)$ in terms of η,

$$\begin{aligned} q(0) &= p(u) + p(d) \\ &= \pi(\alpha_1 + \alpha_2\eta(u)) + (1-\pi)(\alpha_1 + \alpha_2\eta(d)) \\ &= \alpha_1 + \alpha_2 y. \end{aligned} \tag{A.12}$$

Thus $q(0)$ can be effectively ignored as well. In this connection, we will find it very useful later on to have in hand the auxiliary, expected value variable $y = \pi\eta(u) + (1-\pi)\eta(d)$ introduced in (A.12). Finally, notice that Mr. 2's NAC's give us a second distinct expression for $q(0)$,

$$\begin{aligned} q(0) &= (\lambda_2(u)/\lambda_2(0))p(u) + (\lambda_2(d)/\lambda_2(0))p(d) - (0, \nu/\lambda_2(0)) \\ &= (\eta(0)/\eta(u))\pi(a_1 + a_2\eta(u)) + \\ &\qquad (\eta(0)/\eta(d))(1-\pi)[(1-a_1) + (1-a_2)\eta(d)] - (0, \nu\eta_2(0)/\beta). \end{aligned}$$

Defining $\mu = \nu\eta_2(0)/\beta$, and then subtracting this second expression from the first yields the two remaining RFE's (A.2)-(A.3), and the reduction of the EFE's to the RFE's is complete.

A.III. **Analysis of the RFE's**

Let $\xi = (\eta(0), \eta(u), \mu) \in \Xi = \mathbb{R}^2_{++} \times \mathbb{R}$, $\theta = \eta(d) \in \Theta = \mathbb{R}_{++,}$ and $\Phi : \Xi \times \Theta \to \mathbb{R}^3$, the C^∞ mapping s.t. (just reproducing the LHS's of (A.1)-(A.3))

$$(\xi, \theta) \mapsto \Phi(\xi, \theta) =$$

$$\begin{pmatrix} -(1 + 1/\beta)\alpha_1 s_2(0) + [(\eta(0)/\beta) + (\pi\eta(u) + (1-\pi)\eta(d))]\alpha_2 s_1(0) \\[2ex] \pi(1 - \eta(0)/\eta(u))(a_1 + a_2\eta(u)) + \\ \qquad (1-\pi)(1 - \eta(0)/\eta(d))(a_1 + a_2\eta(d)) \\[2ex] \pi(1 - \eta(0)/\eta(u))[(1-a_1) + (1-a_2)\eta(u)] + \\ \qquad (1-\pi)(1 - \eta(0)/\eta(d))[(1-a_1) + (1-a_2)\eta(d)] - \mu \end{pmatrix}. \tag{A.13}$$

A basic result about this system of equations concerns the existence of solutions when $\mu = 0$, i.e., the existence of equilibria when the portfolio constraint is not binding.

Proposition A.2. *If*

$$\begin{aligned} &\alpha_2 s_1(0) \neq 0 \text{ and} \\ &\eta^* = \alpha_1 s_2(0)/\alpha_2 s_1(0) > 0, \end{aligned} \tag{A.14}$$

then $\Phi(\xi, \theta)_{\mu=0} = 0$ *has the unique positive solution* $\eta(\omega) = \eta^*$, for all ω.

Proof of Proposition A.2. Consider first a candidate solution where $\eta(\omega) = k$, for all ω, with $k \neq 0$ and $\mu = 0$. (A.1) becomes, after simplifying,

$$\alpha_1 s_2(0) - \alpha_2 s_1(0)k = 0, \tag{A.15}$$

while (A.2)-(A.3) are identically satisfied. (A.14) entails that $k = \eta^*$ is the unique solution to (A.15), that is, that $\eta(\omega) = \eta^*$ is the unique positive solution to $\Phi(\xi, \theta)_{\mu=0} = 0$ of this form. What remains to be shown is that no other positive process $\eta(\omega)$, possibly varying across ω, can be a solution to $\Phi(\xi, \theta)_{\mu=0} = 0$. This follows from Proposition 4 in Cass and Pavlova (2004).■

Remark. Proposition A.2 means that, under the hypothesis, there is a unique equilibrium in which the portfolio constraint is not binding. Consider the other possible sign configurations of $\alpha_2 s_1(0)$ and $\alpha_1 s_2(0)$. On the one hand, if $\alpha_2 s_1(0) = \alpha_1 s_2(0) = 0$, then (A.15) has a continuum of positive solutions – and there is a continuum of equilibria in which the portfolio constraint is not binding. On the other hand, if $\alpha_2 s_1(0) \neq 0$ and $\alpha_1 s_2(0)/\alpha_2 s_1(0) \leqq 0$, or $\alpha_2 s_1(0) = 0$ and $\alpha_1 s_2(0) \neq 0$, then (A.15) has no positive solution – and (again in light of Proposition 4 in Cass and Pavlova) there is no equilibrium in which the portfolio constraint is not binding. (These two observations establish that (A.14) is also a necessary condition that $\Phi(\xi, \theta)_{\mu=0} = 0$ has a unique positive solution, i.e., that there is a unique equilibrium in which the portfolio constraint is not binding.) **Because the existence of $\eta^* > 0$ is so central to our analysis of the solutions to (A.1)-(A.3) (to say nothing of interpretive motivation) here and after we assume that (A.14) obtains.**

The ultimate basis for establishing robustness of various properties of the equilibria for our leading example and, to some extent, its extensions, is analysis of the local behavior of the solutions to the equation $\Phi(\xi, \theta) = 0$ around the particular solution η^* (as we will call the point $(\xi, \theta) = (\eta^*, \eta^*, 0, \eta^*)$). Thus, it is a pure bonus that, because of specific structure peculiar to just this minimal model, we are also able to describe their global behavior. This results first, from a rank property of $D\Phi$ (on its whole domain), and second, from explicit calculation of $D_\theta \xi \Big|_{\Phi(\xi,\theta)=0}$ using the Implicit Function Theorem.

Proposition A3. Rank $D_\xi \Phi(\xi, \theta) = 3$.

Proof of Proposition A.3. To begin with, we calculate

$D_\xi \Phi(\xi, \theta) =$

$$
\begin{array}{cccc}
\text{eq.\textbackslash var.} & \eta(0) & \eta(u) & \mu \\
\begin{array}{c}(A.1)\\ \\ (A.2)\\ \\ (A.3)\end{array} &
\multicolumn{3}{c}{\left[\begin{array}{ccc}
\alpha_2 s_1(0)/\beta & \pi\alpha_2 s_1(\beta) & 0 \\
-[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2] & \pi[a_2+a_1\eta(0)/\eta(u)^2] & 0 \\
-[(1-a_1)(\pi/\eta(u)+(1-\pi)/\eta(d))+(1-a_2)] & \begin{array}{c}\pi[(1-a_2)+\\ (1-a_1)\eta(0)/\eta(u)^2]\end{array} & -1
\end{array}\right].}
\end{array}
$$

Then, for $v=(v^1,v^2,v^3)\in\mathbb{R}^3$, and simplifying (using $\alpha_2 s_1(0)\neq 0$ and $\pi>0$), suppose that

$$v^T D_\xi\Phi(\xi,\theta) =$$

$$
\left(\begin{array}{l}
(1/\beta)v^1-[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2]v^2- \\
\qquad\qquad [(1-a_1)(\pi/\eta(u)+(1-\pi)/\eta(d))+(1-a_2)]v^3 \\
v^1+[a_2+a_1\eta(0)/\eta(u)^2]v^2+[(1-a_2)+(1-a_1)\eta(0)/\eta(u)^2]v^3 \\
-v^3
\end{array}\right)=0.
$$

From the third equation, $v^3=0$, which, from the first two equations, implies that

$$
\left[\begin{array}{ll}
(1/\beta) & -[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2] \\
1 & a_2+a_1\eta(0)/\eta(u)^2
\end{array}\right]\left(\begin{array}{c}v^1\\ v^2\end{array}\right)=0,
$$

which in turn (since, in general, $\det\left(\left[\begin{array}{cc}+&-\\+&+\end{array}\right]\right)>0$) implies that $v^1=v^2=0$. Summarizing, $v^T D_\xi\Phi(\xi,\theta)=0$ implies that $v=0$, i.e., that rank $D_\xi\Phi(\xi,\theta)=3$.■

By utilizing Proposition A.3 we obtain a complete characterization of the qualitative properties of the solutions to (A.1)-(A.3).

Proposition A.4 *The derivative of ξ and the auxiliary variable y with respect to θ have the specific form*

$$
D\xi\Big|_{\Phi(\xi,\theta)=0} = -D_\xi\Phi(\xi,\theta)^{-1}\Big|_{\Phi(\xi,\theta)=0} D_\theta\Phi(\xi,\theta)\Big|_{\Phi(\xi,\theta)=0} =
$$

$$
\left(\begin{array}{l}
D_{\eta(d)}\eta(0)\\
D_{\eta(d)}\eta(u)\\
D_{\eta(d)}\mu
\end{array}\right)=\left(\begin{array}{l}
\varphi^1(\eta)(\eta(u)-\eta(d))\\
-\varphi^2(\eta)(1-\pi)\eta(u)/\pi\eta(d)\\
\varphi^3(\eta)(\eta(d)-\eta(u))
\end{array}\right) \textit{ and}
$$

$$D_{\eta(d)}y=\varphi^4(\eta)(\eta(d)-\eta(u)),$$

where $\varphi^i : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$, all i.

Proof of Proposition A.4. In principal this task is routine, in practice a (calculational) nightmare, so we turned it over to Mathematica v. 5.1. A copy of our program is available on request.■

The key implications of this result are that (i) $\eta(u)$ is strictly decreasing, while (ii) $-\eta(0)$, y, and μ are strictly decreasing (resp. strictly increasing) for $\eta(d) \leqq \eta^*$ (resp. $\eta(d) \geqq \eta^*$). We stress, especially, that the last means that

$$\mu \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0 \text{ according as } \eta(d) \left\{ \begin{array}{c} \neq \\ = \end{array} \right\} \eta^*.$$

To complete the picture all that is required is to determine the range of $\eta(d)$, and the limit values for $\eta(0)$, $\eta(u)$, and y at the boundaries of its range. Let $\underline{y} = \eta^* < (1 + 1/\beta))\eta^* = \bar{y}$.

Proposition A.5. *For* every $\eta(d) \in (0, \bar{y}/(1 - \pi))$, *the equation* $\Phi(\xi, \theta) = 0$ *has a solution. At the endpoints of its range,*

$$\lim_{\eta(d)\to 0^+} \eta(0) = \lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} \eta(0) = 0,$$

$$\lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} \eta(u) = 0 < \lim_{\eta(d)\to 0^+} \eta(u) = \bar{y}/\pi, \text{ and}$$

$$\lim_{\eta(d)\to 0^+} y = \lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} y = \bar{y}.$$

Proof of Proposition A.5. We begin by considering the solutions of the system (A.1)-(A.3) as $\eta(d)$ goes to 0.

(a) Since $\eta(u) \geqq \eta^*$ when $\eta(d) \leqq \eta^*$, from (A.2) it follows that $\lim_{\eta(d)\to 0^+} \eta(0) = 0$. Otherwise, (i) if, for some $k > 0$, $\lim_{\eta(d)\to k^+} \eta(0) = 0$, then $\lim_{\eta(d)\to k^+} (RHS(\text{A.2})) > 0$, while (ii) if $\lim_{\eta(d)\to 0^+} \eta(0) = k > 0$, then $\lim_{\eta(d)\to k^+} (RHS(\text{A.2})) = -\infty$, and (A.2) is violated.

(b) Use the definition of y together with (A.14) to simplify (A.1), which becomes

$$y = (1 + 1/\beta)\eta^* - \eta(0)/\beta = \bar{y} + \eta(0)/\beta$$

or, equivalently, for later use,

$$\eta(0) = (1 + \beta)\eta^* - \beta y. \tag{A.16}$$

Then it follows that $\lim_{\eta(d)\to 0^+} y = \lim_{\eta(0)\to 0^+} y = \bar{y}$.

(c) Now use the definition of y to solve for $\eta(u) = y/\pi - [(1-\pi)/\pi]\eta(d)$. Then it follows that $\lim_{\eta(d)\to 0^+} \eta(u) = \bar{y}/\pi$.

Finally, observe that the roles of $\eta(u)$ and $\eta(d)$ are interchangeable, i.e., that either could be chosen as the independent variable. So precisely the same reasoning as the foregoing applied as $\eta(\mu)$ goes to 0 yields the fact that the upper bound of $\eta(d)$ is $(\bar{y}/(1-\pi)$, together with the other two limit values, $\lim_{\eta(d)\to((\bar{y}/(1-\pi))^-} \eta(0) = 0$ and $\lim_{\eta(d)\to((\bar{y}/(1-\pi))^-} y = \bar{y}$.■

The statement (and proof) of these last two propositions make them appear to be much more complicated than they really are, as we try to indicate by means of the diagram in Figure A.1. Note that, since (A.1) entails that $\eta(0)$ is a linear function of y, we needn't plot $\eta(0)$ explicitly.

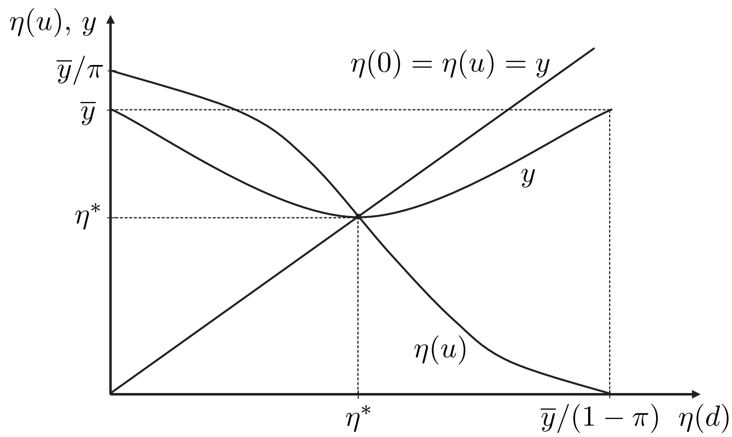

Figure A.1: The Solutions to the Equilibrium Equations.

To end this section we re-emphasize that our leading example is very special in that there is only a single independent variable, and the qualitative behavior of the dependent variables over its whole range is very nice. In short, it is straightforward to go from local to global analysis. Even, say, for the extensions to encompass just three periods, however, without additional specialization, this is no longer the case, and only local analysis is generally possible.

A.IV. **Tailoring the Portfolio Constraint**

Consider the specific portfolio constraint

$$q^2(0)s_2^2(1) \geqq \gamma W_2(0) \text{ with } \gamma > 0, \tag{A.17}$$

where $W_2(0) = (q(0) + p(0))s_2(0)$. When $\mu > 0$, this constraint is binding, and – assuming that $W_2(0) > 0$ – we can use (A.10) in order to solve for γ,

$$\gamma = (a_1/(a_1 - a_2))q^2(0)/W_2(0). \tag{A.18}$$

From (A.12),

$$q^2(0) = (1 - a_1) + (1 - a_2)y,$$

while from (A.12) together with (A.5),

$$W_2(0) = (q(0) + p(0))s_2(0) = [(\alpha_1 + \alpha_2 y) + ((\alpha_1/\beta) + (\alpha_2/\beta)\eta(0))]s_2(0).$$

Substituting for $\eta(0)$ from (A.16) into this last expression and then simplifying yields

$$\begin{aligned} W_2(0) &= [(\alpha_1 + \alpha_2 y) + (\alpha_1/\beta) + (\alpha_2/\beta)((1+\beta)\eta^* - \beta y)]s_2(0) \\ &= (1 + 1/\beta)(\alpha_1 + \alpha_2\eta^*)s_2(0) \\ &= (1 + 1/\beta)\eta^*\alpha_2(s_1(0) + s_2(0)) \\ &= (1 + 1/\beta)\eta^* > 0, \end{aligned}$$

where in deriving the third equality we utilize the definition in (A.14). It then follows that (A.18) is simply a linear mapping

$$\gamma = A + By$$

with

$$A = \frac{a_1(1 - a_1)}{(a_1 - a_2)(1 + 1/\beta)\eta^*} > 0 \text{ and } B = \frac{a_1(1 - a_2)}{(a_1 - a_2)(1 + 1/\beta)\eta^*} > 0.$$

Combining this with what we know from Propositions A.4 and A.5 about the relationship between $\eta(d)$ and the other reduced form variables ($\eta(0)$, $\eta(u)$, μ), and hence y, we can get a very clear picture of the equilibrium correspondence between γ and all the endogenous variables in the model. This is illustrated from one aspect in Figure 2 in the main text, which also illustrates that (A.18) can always be satisfied when $\eta(\omega) = \eta^*$ for all ω and $\mu = 0$ – the second part of Proposition A.1. Thus, for every $\gamma \in (\underline{\gamma}, \bar{\gamma}) = (A + B\underline{y}, A + B\overline{y})$ there are precisely two distinct equilibria in which the portfolio constraint is binding in a nondegenerate way (since the inverse of the mapping such that $\eta(d) \mapsto y$ is one-to-two; the two equations formalizing this property are contained in the statement of Result 3 in the main text). The example is robust (though, obviously, nongeneric) in the parameter γ.

All this can be succinctly summarized (given the portfolio constraint (A.17)).

Proposition A.6. *For every economy $\gamma \in (\underline{\gamma}, \bar{\gamma})$, there are two inefficient equilibria in which the constraint is binding in a nondegenerate way, together with the original efficient equilibrium in which it isn't.*

We note that this particularly simple argument can be extended to cover the case where total wealth, $W_2(0)$, is replaced by portfolio wealth,

$$P_2(0) = q(0)s_2(0)$$

(again, since $q(0)$ – through Ms. 1's NAC's – depends solely on y). The only difference is that now, on the open interval $(\underline{\gamma}, \bar{\gamma})$, γ becomes a strictly increasing, strictly convex (resp. strictly concave) function of y when $\alpha_2 s_1(0) > 0$ (resp. $\alpha_2 s_1(0) < 0$).

Substituting portfolio wealth for wealth in our specific constraint is also the basis for explaining why our leading example requires that stocks pay dividends in period 0 – or equivalently, that households consume in period 0 (as well as period 1). Without this natural specification (a simplification which is often employed for added tractability), the first RFE (A.1) becomes simply

$$-\alpha_2 s_1(0) + (\pi\eta(u) + (1-\pi)\eta(d))\alpha_1 s_2(0) = -\alpha_2 s_1(0) + \alpha_1 s_2(0) y = 0, \tag{A.19}$$

while, as already suggested, wealth necessarily collapses to portfolio wealth

$$W_2(0) = P_2(0). \tag{A.20}$$

(A.19) entails that $y = \eta^*$. Thus, since it is easily verified that here too (A.12) obtains, that is, that $q(0) = \alpha_1 + \alpha_2 y$, (A.20) entails that our specific constraint is nongeneric: (A.18) is only satisfied when

$$\gamma = \gamma^* = (a_1/(a_1 - a_2))((1-a_1)/\eta^* + (1-a_2)) > 0.$$

The fact that $y = \eta^*$ also permits solving the RFE's explicitly, so that the qualitative properties of their solutions can be easily derived by hand. This reveals that, when $\gamma = \gamma^*$, there is a continuum (i.e., a smooth, one-dimensional manifold) of inefficient equilibria. In short, while the analysis of this special case is – given our general approach – definitely simpler, the results of the analysis do not survive accommodating a very basic fact of life, that even the most skilled arbitrageurs on Wall Street eat lunch.

Finally, we should mention that the essence of the two specific constraints considered here can be abstracted and (slightly) generalized in terms of a (local) relationship between $\eta(d)$ and γ, but nothing of much interest or value is learned in the process.

Two Concluding Remarks. 1. It should be absolutely clear that all future spot goods prices and consumption allocations vary across all three equilibria corresponding to $\gamma \in (\underline{\gamma}, \bar{\gamma})$ (since $\eta(\omega)$, $\omega = u, d$, does), though today's, as well as stock prices and portfolio strategies (may) only vary across the type-E and type-I equilibria (since y does).

2. Regarding the maintained assumption that $a_1 > a_2$: For the opposite case, $a_1 < a_2$, the qualitative behavior we have just described is, in most important respects, reversed. For instance, now

$$\mu \left\{ \begin{matrix} < \\ = \end{matrix} \right\} 0 \text{ according as } \eta(d) \left\{ \begin{matrix} \neq \\ = \end{matrix} \right\} \eta^*,$$

the constraint must be an upper rather than lower bound.

Appendix B

In this appendix we explore the role of incomplete markets – together with portfolio constraints – in generating multiplicity of equilibria. In particular, we now extend the leading example by assuming that there is a third state, $\omega = m$, in addition to the original states, $\omega = u$, d. The detailed description of this modified economic setting is presented in Section 3 in the main text. In the presence of such incomplete markets, the effect of introducing the portfolio constraint (A.17) is even more striking: There may be real indeterminacy.[14]

Proposition B. *For every economy $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there is a continuum of distinct type-I equilibria***.**

Remark. It will become apparent that what drives this result is the degree of asset market incompleteness, that there are three states but only two stocks. Our argument easily generalizes to encompass any degree of inherent market incompleteness – the difference between the number of states and the number of stocks – say, $D \geqq 1$, yielding the parallel result that there is a smooth, D-dimensional manifold of type-I equilibria.

Proof of Proposition B. The proof naturally breaks down into several steps.

Step 1. (Characterizing the RFE's and type-E equilibrium). We begin by observing that – aside from the obvious changes in notation necessitated by the proliferation of stochastic weights from 3 to 4 – the analysis which led to the RFE's in Appendix A can be repeated here. In particular, it follows directly from the logic of the proof of Proposition A.1 that Ms. 1's second period SBC's always have the solution $s_1(1)$ such that, for Mr. 2,

$$s_2(1) = \mathbf{1} - s_1(1) = (-(1 - a_1)/(a_1 - a_2), a_1/(a_1 - a_2)),$$

utilized in deriving equation (A.18). Moreover, as before, the logic of the reduction of the EFE's leads to just 3 RFE's,

$$-(1 + 1/\beta)\alpha_1 s_2(0) + ((\eta(0)/\beta) + \sum_{\omega=u,m,d} \pi(\omega)\eta(\omega))\alpha_2 s_1(0) = 0, \tag{B.1}$$

$$\sum_{\omega=u,m,d} \pi(\omega)(1 - \eta(0)/\eta(\omega))(a_1 + a_2\eta(\omega)) = 0, \text{ and} \tag{B.2}$$

[14] We should emphasize at this point that there are now two distinct potential sources of market failure, or market frictions, incomplete markets as well as portfolio constraints. Alone or together (in a specific context) such distinct frictions may have very substantial but quite different effects on market allocations and prices (as is the case here). This point is often obscured by referring to the presence of any financial market imperfection(s) as simply involving "incomplete (financial) markets." A more appropriate generic label would be something like "imperfect (financial) markets," but, regardless of label, in general each specific configuration of market frictions must be considered on its own terms.

$$\sum_{\omega=u,m,d} \pi(\omega)(1-\eta(0)/\eta(\omega))((1-a_1)+(1-a_2)\eta(\omega)) - \mu = 0. \tag{B.3}$$

But now there are 5 rather than 4 basic variables, $(\eta, \mu) = ((\eta(\omega), \text{ all } \omega), \mu)$, the variables which determine all the remaining endogenous prices and quantities (we will again use the fact that both spot goods and stock prices are uniquely determined by the stochastic weights later on). It is also easily verified that the system (B.1)-(B.3) still has the unique positive solution $\eta = \eta^*\mathbf{1}$ for $\mu = 0$.

Step 2. (Picking a "starting point" solution). Pick $\gamma' \in (\underline{\gamma}, \overline{\gamma})$. Then observe that taking $\eta(m) = \eta(d)$ and $\pi = \pi(u)$ (so that $1-\pi = 1-\pi(u) = 1-\pi(m)-\pi(d)$) enables us to reduce the equations and variables of the system (B.1)-(B.3) to those of the leading example already analyzed in Appendix A. Thus, we can pick either of the two type-I equilibria corresponding to γ' in the leading example to define a "starting point" solution, say, (η', μ'), with $\eta(u)' \neq \eta(d)', \eta(m)' = \eta(d)'$, and $\mu' > 0$.

Step 3. (Simplifying the system of equations). Consider again defining the auxiliary variable

$$y = \sum_{\omega=u,m,d} \pi(\omega)\eta(\omega), \tag{B.4}$$

the expected value of Mr. 2's future stochastic weights, and fix both $\eta(0)$ and y at their "starting point" values,

$$\eta(0) = \eta(0)' \text{ and } y = y'. \tag{B.5}$$

Given (B.5), we can ignore both the binding portfolio constraint represented by (A.18) and equation (B.1), since it is easily verified that both only depend on $\eta(0)$ and y (the former through $p(0)$ and $q(0)$, basically in the same way as described in Section A.IV of Appendix A, the latter directly). Finally, notice that since (B.3) in essence defines μ in terms of η, while $\mu' > 0$, we can perturb η around η' and maintain $\mu > 0$.

The upshot is that, for the purposes of local analysis, we are left with just two equations, (B.2) and (B.4), in three variables, $\eta(u), \eta(m)$, and $\eta(d)$. Let this system be represented generally, as before, by

$$\Phi(\xi, \theta) = \begin{pmatrix} (a_1 + a_2 y') - \sum_{\omega=u,m,d} \eta(0)'\pi(\omega)(a_1/\eta(\omega) + a_2) \\ \sum_{\omega=u,m,d} \pi(\omega)\eta(\omega) - y' \end{pmatrix} = 0$$

where $\xi = (\eta(u), \eta(m))$ and $\theta = \eta(d)$, and we have taken into account (B.5).

Step 4. (Establishing existence of a smooth, one-dimensional manifold of distinct type-I equilibria). Differentiating $\Phi(\xi, \theta)$ yields

$$D_\xi \Phi(\xi, \theta) = \begin{bmatrix} (\pi(u)\eta(0)'a_1)/\eta(u)^2 & (\pi(m)\eta(0)'a_1)/\eta(m)^2 \\ \pi(u) & \pi(m) \end{bmatrix}.$$

Since $\eta(u)' \neq \eta(d)' = \eta(m)'$, we have

$$\det(D_\xi \Phi(\xi', \theta')) = (\pi(u)\pi(m)\eta(0)'a_1)(1/\eta(u)'^{\,2} - 1/\eta'(d)'^{\,2}) \neq 0.$$

Hence, it follows directly upon application of the Implicit Function Theorem that, in a neighborhood of $\theta' = \eta(d)'$, these equations define a smooth, one-dimensional manifold such that, for any two points on the manifold, say $(\tilde{\xi}, \tilde{\theta})$, and $(\hat{\xi}, \hat{\theta})$, $\hat{\theta} = \hat{\eta}(d) \neq \tilde{\theta} = \tilde{\eta}(d))$, so that the corresponding consumption allocations differ, and the argument is complete. ■

The result of this construction is illustrated in Figure B.1 (for the case in which (η', μ') lies on the "upper" branch of the correspondence representing type-I equilibria in the leading example). The associated welfare comparisons are illustrated in Figure 3 in the main text. (Establishing these last requires some further detailed analysis of the structure of the solutions to (B.2) and (B.4), analysis which is not reported here because of space limitations.)

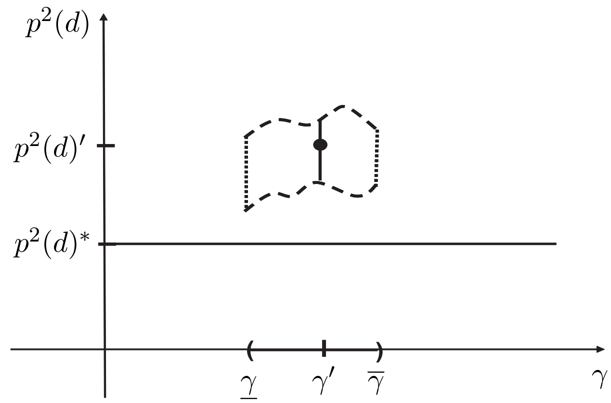

Figure B.1: Real Indeterminacy with Incomplete Markets.

Two Concluding Remarks. 1. This proposition clearly remains valid when our specific portfolio constraint is replaced by its variant, analyzed at the end of Section A.IV of Appendix A, where wealth is replaced by portfolio wealth.

2. It remains to be seen whether this kind of result generalizes to models with more than two goods, or with real assets besides stocks, e.g., real bonds as in Cass and Pavlova (2004), but these are certainly plausible conjectures, since in both cases there are "extra" variables to "play with."

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