Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://www.econ.upenn.edu/pier

PIER Working Paper 06-017

"A Three-Factor Yield Curve Model:
Non-Affine Structure, Systematic Risk Sources, and
Generalized Duration"

by

Francis X. Diebold, Lei Ji, and Canlin Li

http://ssrn.com/abstract=910218

A Three-Factor Yield Curve Model: Non-Affine Structure, Systematic Risk Sources, and Generalized Duration*

Francis X. Diebold
University of Pennsylvania and NBER

Lei Ji
University of Pennsylvania

Canlin Li
University of California-Riverside

March 9, 2004

Abstract

We assess and apply the term-structure model introduced by Nelson and Siegel (1987) and re-interpreted by Diebold and Li (2003) as a modern three-factor model of level, slope and curvature. First, we ask whether the model is a member of the affine class, and we find that it is not. Hence the poor forecasting performance recently documented for affine term structure models in no way implies that our model will forecast poorly, which is consistent with Diebold and Li's (2003) finding that it indeed forecasts quite well. Next, having clarified the relationship between our three-factor model and the affine class, we proceed to assess its adequacy directly, by testing whether its level, slope and curvature factors do indeed capture systematic risk. We find that they do, and that they are therefore priced. Finally, confident in the ability of our three-factor model to capture the pricing relations present in the data, we proceed to explore its efficacy in bond portfolio risk management. Traditional Macaulay duration is appropriate only in a one-factor (level) context; hence we move to a three-factor generalized duration, and we show the superior performance of hedges constructed using it.

Contact author:
Professor Canlin Li
Graduate School of Management
University of California
Riverside, CA 92521
canlin.li@ucr.edu
telephone (909) 787 2325
fax (909) 787 3970

* *Acknowledgments:* This paper is dedicated to the memory of our teacher and colleague, Albert Ando. We thank Lawrence R. Klein for helpful editorial comments, but we alone are responsible for any remaining errors. We are grateful to the Guggenheim Foundation and the National Science Foundation for research support.

1 Introduction

We assess and apply the term-structure model introduced by Nelson and Siegel (1987) and re-interpreted by Diebold and Li (2003) as a modern three-factor model of level, slope and curvature. Our assessment and application has three components. First, we ask whether the model is a member of the recently-popularized affine class, and we find that it is not. Hence the poor forecasting performance recently documented for affine term structure models (e.g., Duffee, 2002) in no way implies that our model will forecast poorly, which is consistent with Diebold and Li's (2003) finding that it indeed forecasts quite well.

Second, having clarified the relationship between our three-factor model and the affine class, we proceed to assess its adequacy directly, by asking whether its level, slope and curvature factors capture systematic risk. We find that they do, and that they are therefore priced. In particular, we show that the cross section of bond returns is well-explained by the sensitivities (loadings) of the various bonds to the level, slope and curvature state variables (factors).

Finally, confident in the ability of our three-factor model to capture the pricing relations present in the data, we proceed to explore its use for bond portfolio risk management. Traditional Macaulay duration is appropriate only in a one-factor (level) context; hence we move to a three-factor generalized duration vector suggested by our model. By matching all components of generalized duration, we hedge against level risk, slope risk, and curvature risk. Traditional Macaulay duration hedging, which hedges only level risk, emerges as a special and highly-restrictive case. We compare the hedging performance of our generalized duration to that of several existing competitors, and we find that it compares favorably.

We proceed as follows. In section 2 we review affine term structure models and study their relationship to our three-factor model. In section 3 we test whether our level, slope and curvature factors represent priced systematic risks, and we ask whether they successfully explain the cross section of bond returns. In section 4 we extend Macaulay duration to a generalized vector duration and study its properties and immunization performance. We conclude in section 5.

2 Is the Three-Factor Model Affine?

Affine term structure models have recently gained great popularity among theorists (e.g., Dai and Singleton, 2000), due to their charming simplicity. Ironically, however, Duffee (2002) shows that the restrictions associated with affine structure produce very poor forecasting performance. A puzzle arises: the Nelson-Siegel (1987) and Diebold-Li (2003) three-factor term structure model appears affine, yet it forecasts well. Here, we resolve the puzzle.

2.1 Background

A little background on Nelson-Siegel (1987) is required to understand what follows. Nelson and Siegel proposed the parsimonious yield curve model,

$$y_t(\tau) = b_{1t} + b_{2t}\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - b_{3t}e^{-\lambda_t \tau}. \tag{1}$$

where $y_t(\tau)$ denotes the continuously-compounded zero-coupon nominal yield at maturity τ, and b_{1t}, b_{2t}, b_{3t}, and λ_t are (time-varying) parameters. The Nelson-Siegel model can generate a variety of yield curve shapes including upward sloping, downward sloping, humped, and inversely humped, but it can not generate yield curves with two or more local minima/maxima that are sometimes (though rarely) observed in the data.

Diebold and Li (2003) reformulated the original Nelson-Siegel expression as

$$y_t(\tau) = f_{1t} + f_{2t}\frac{1 - e^{-\lambda \tau}}{\lambda \tau} + f_{3t}\left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau}\right). \tag{2}$$

The advantage of the Diebold-Li representation is that we can easily give economic interpretations to the parameters f_{1t}, f_{2t}, and f_{3t}. In particular, we can interpret them as a level factor, a slope factor, and a curvature factor, respectively. To see this, note that the loading on f_{1t} is 1, a constant that doesn't depend on the maturity. Thus f_{1t} affects yields at different maturities equally and hence can be regarded as a level factor. The loading associated with f_{2t} is $(1 - e^{-\lambda\tau})/\lambda\tau$, which starts at 1 but decays monotonically to 0. Thus f_{2t} affects primarily short-term yields and hence changes the slope of the yield curve. Finally, factor f_{3t} has loading $(1 - e^{-\lambda\tau})/\lambda\tau - e^{-\lambda\tau}$, which starts at 0, increases, and then decays. Thus f_{3t} has largest impact on medium-term yields and hence moves the curvature of the yield curve. In short, based on (2), we can express the yield curve at any point of time as a linear combination of the level, slope and curvature factors, the dynamics of which drive the dynamics of the entire yield curve.

In the original Nelson-Siegel formulation, the parameter λ_t may change with time. But as argued in Diebold and Li (2003), we can treat it as fixed with little degradation of fit (they fix λ at 0.0609 with maturities measured in months). This treatment greatly simplifies the estimation procedure, and even more importantly, it sharpens economic intuition, because λ_t has no obvious economic interpretation. After fixing λ_t, it is trivial to estimate f_{1t}, f_{2t}, and f_{3t} from equation (2) via ordinary least squares (OLS) regressions.

In the empirical work that follows we use the Center for Research in Security Prices (CRSP) monthly treasury file to extract the three factors. For liquidity and data quality considerations, we keep only bills with maturity longer than 1 month and notes/bonds with maturity longer than 1 year (see also Bliss, 1997). In the first step, as in Fama and Bliss (1987), we use a bootstrap method to infer zero bond yields from available bill, note, and bond prices. In the second step, we treat the factor loadings in the above equation as regressors and we fix $\lambda = 0.0609$ to calculate the regressor values for each zero bond. In the third step, we run a cross-sectional regression of the zero yields on the calculated regressor values. The regression coefficients $\widehat{f_{1t}}$, $\widehat{f_{2t}}$, and $\widehat{f_{3t}}$ are the estimated factor values. We do this in each month to get the time series of three factors, which we display in Figure 1 from 1972 to 2001. In Figure 2, we show a few selected term structure scenarios. Both the bootstrapped zero yields and the three-factor fitted yield curves are included. From the graph, it's clear that the fitted curves can reproduce raw zero yields very well, at both the short and long ends of the curve[1].

2.2 Analysis

Even a casual look at equation (2) reveals that the yield is affine in the three factors. This raises a natural question: is our model related to the affine term structure models popular in the literature? In particular, can we show that the Nelson-Siegel model is an affine term structure model (ATSM), in the sense of Duffie and Kan (1996), Dai and Singleton (2000), and Piazessi (2002)? As in Dai and Singleton (2000), an N-factor general ATSM has the following elements:

The state variable $Y(t)$, an $N \times 1$ vector, follows the affine diffusion,

$$dY(t) = \widetilde{K}(\widetilde{\theta} - Y(t))dt + \Sigma\sqrt{S(t)}d\widetilde{W}(t), \tag{3}$$

where $\widetilde{W}(t)$ is an N-dimensional independent standard Brownian motion, $\widetilde{K}$ and Σ are $N \times N$ matrices, $\widetilde{\theta}$ is an $N \times 1$ vector, and $S(t)$ is a diagonal matrix with the ith diagonal element

$$[S(t)]_{ii} = \alpha_i + \beta_i' Y(t).$$

The time t price of a zero-coupon bond with maturity τ is

$$P(t,\tau) = \exp(A(\tau) - B(\tau)'Y(t)), \tag{4}$$

[1] Note that Fama and Bliss consider only yields on bonds with maturities up to five years. As a result, much information contained in the long end of the yield curve is lost. Similarly, Diebold and Li (2003) consider only yields with maturities up to ten years. Because we want to distill as much information as possible from the entire yield curve, in this paper we use all non-callable government bonds in the bootstrapping exercise.

where $A(\tau)$ and $B(\tau)$ are solutions to the ordinary differential equations (ODEs),

$$\frac{dA(\tau)}{d\tau} = -\widetilde{\theta}' \widetilde{K}' B(\tau) + \frac{1}{2} \sum_{i=1}^{N} \left(\Sigma' B(\tau)\right)_i^2 \alpha_i - \delta_0 \tag{5}$$

$$\frac{dB(\tau)}{d\tau} = -\widetilde{K}' B(\tau) - \frac{1}{2} \sum_{i=1}^{N} \left(\Sigma' B(\tau)\right)_i^2 \beta_i + \delta_y, \tag{6}$$

with initial conditions

$$A(0) = 0; \quad B(0) = 0. \tag{7}$$

From (4), it is straightforward to express bond yields as

$$y_t(\tau) = -\frac{1}{\tau} \ln(P(t,\tau)) = -\frac{A(\tau)}{\tau} + \frac{B(\tau)' Y(t)}{\tau}. \tag{8}$$

Dai and Singleton (2000) give a canonical representation for the ATSM, in which each ATSM with N factors can be uniquely classified as $A_m(N)$, where m is the number of state variables entering the variance of the diffusion term $S(t)$. The most general (maximal) form of the canonical representation is

$$K = \begin{pmatrix} K_{m\times m} & 0_{m\times(N-m)} \\ K_{(N-m)\times m} & K_{(N-m)\times(N-m)} \end{pmatrix} \tag{9}$$

$$\theta = \begin{pmatrix} \theta_{m\times 1} \\ 0_{(N-m)\times 1} \end{pmatrix}$$

$$\Sigma = I$$

$$\beta = \begin{pmatrix} I_{m\times m} & \beta_{m\times(N-m)} \\ 0_{(N-m)\times m} & 0_{(N-m)\times(N-m)} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 0_{m\times 1} \\ 1_{(N-m)\times 1} \end{pmatrix}. \tag{10}$$

Several important points arise. First, Σ is equal to the identity matrix in the canonical representation, which significantly simplifies the differential equations (5) and (6). Second, note that $\widetilde{K}$ and $\widetilde{\theta}$ in (5) and (6) are parameters under the equivalent martingale measure, while the parameters in the canonical representation are all under the physical measure. Therefore, we have to convert $\widetilde{K}$ and $\widetilde{\theta}$ to K and θ using the transformations

$$\widetilde{K}\widetilde{\theta} = K\theta - \begin{pmatrix} \lambda_1\alpha_1 \\ \vdots \\ \lambda_N\alpha_N \end{pmatrix}$$

$$\widetilde{K} = K + \begin{pmatrix} \lambda_1\beta_1' \\ \vdots \\ \lambda_N\beta_N' \end{pmatrix}, \tag{11}$$

where λ_i is the market price of risk for factor i. Finally, the canonical representation puts several restrictions on the parameters of the model in order to guarantee admissibility[2] and stationarity. As we will see, these restrictions have significant implications for our purposes. In Table 1, we list all parameter restrictions required by ATSM for various $N = 3$ subclasses.

To check whether our three-factor model is nested within ATSM, we employ the following approach. First, without loss of generality we identify (f_{1t}, f_{2t}, f_{3t}) in equation (2) as $Y(t) = (Y_{1t}, Y_{2t}, Y_{3t})$ in equation (8), such that the initial conditions are satisfied, and then we derive the implied functional forms for $A(\tau)$ and $B(\tau)$ from (2). Next, we insert these implied expressions into the differential equations (5) and (6), from which we can derive equalities/inequalities required by these two equations. Finally we compare parameter

[2] Admissibility means that the specification of the model parameters is such that each element of $S(t)$ in equation (3) is strictly positive over the range of Y.

restrictions thus derived with those given in Table 1 to see whether the two sets of restrictions are consistent. If our restrictions are not consistent with those imposed by ATSM, then our model is not nested within ATSM. We perform this verification procedure for each three-factor ATSM sub-class: $A_0(3)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$.

Before exploring each sub-class, note that from (2) we have

$$y_t(\tau) = f_{1t} + f_{2t}\frac{1-e^{-\lambda\tau}}{\lambda\tau} + f_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

and from (8), we have

$$y_t(\tau) = -\frac{A(\tau)}{\tau} + \frac{B_1(\tau)}{\tau}Y_1(t) + \frac{B_2(\tau)}{\tau}Y_2(t) + \frac{B_3(\tau)}{\tau}Y_3(t).$$

Now we peoceed to identify factor f_{it} with factor $Y_i(t)$;hence the $A(\tau)$ and $B(\tau)$ implied by (2) are

$$\begin{aligned} A(\tau) &\equiv 0 \\ B_1(\tau) &= \tau \\ B_2(\tau) &= \frac{1-e^{-\lambda\tau}}{\lambda\tau} \\ B_3(\tau) &= \frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}. \end{aligned} \tag{12}$$

Note that $A(\tau)$ and $B(\tau)$ thus defined satisfy the initial condition (7). At this stage, it seems that our three-factor model is just a special case of the three-factor ATSM. Inserting the implied "solutions" (12) into the ODEs (5) and (6), and using the specification (9) and the transformation (11), we obtain

$$0 = -(\widetilde{K}\widetilde{\theta})'B(\tau) + \frac{1}{2}\sum_{i=1}^{3} B_i(\tau)^2\alpha_i - \delta_0 \tag{13}$$

$$\begin{pmatrix} 1 \\ e^{-\lambda\tau} \\ \lambda\tau e^{-\lambda\tau} \end{pmatrix} = -\widetilde{K}' B(\tau) - \frac{1}{2}\sum_{i=1}^{3} B_i(\tau)^2 \beta_i + \delta_y. \tag{14}$$

We then use the four equations (13) and (14) implied by our 3-factor model to derive parameter restrictions. We include detailed derivations for the four sub-classes $A_0(3)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$ in the appendix. Our results show that there exists no three-factor ATSM which can generate our three-factor model of the term structure. This reconciles (1) the poor forecasting performance recently documented for affine term structure models, and (2) the good forecasting performance recently documented for the Nelson-Siegel model.

3 Assessing the Three-Factor Model

Diebold and Li (2003) show that the three-factor model (2) performs well in both in-sample fitting and out-of-sample forecasting. In this section, with an eye toward eventual use of the three-factor model for bond portfolio risk management, we study its cross-sectional performance. In particular, we ask whether the three factors represent systematic risk sources, which are priced in the market. In other words, we ask whether the cross section of bond returns is explained by the loadings of the various bonds on the three factors, in a fashion that parallels Ross's (1976) well-known Arbitrage Pricing Theory (APT) model, which has been used in equity contexts for many years. Interestingly, the APT has been applied to fixed income assets by only a few authors, notably Elton, Gruber, and Blake (1995).

3.1 Factor Extraction

Traditional empirical tests of the APT typically rely on statistical methods, such as principal components and factor analysis[3], to get the factors. For example, Lehmann and Modest (1988) and Connor and Korajczyk (1988) apply these methods to stocks, while Litterman and Scheinkman (1991) and Knez, Litterman, and Scheinkman (1994) apply them to bonds. In particular, Litterman and Scheinkman (1991) and Knez, Litterman, and Scheinkman (1994) use principal components and factor analysis to extract factors in bond returns. They conclude that a large portion (up to 98%) of bond return variation can be explained by the first three principal components or factors. But neither paper considers bond pricing, which is the focus of this section. In addition, Knez, Litterman, and Scheinkman (1994) consider only the very short end of yield curve (money market assets). In contrast, we consider the entire yield curve.

We will compare our three factors with factors extracted from principal components analysis. We use the CRSP monthly treasury file from Dec. 1971 to Dec 2001. We use the bootstrapping method outlined in section 2 to infer zero-coupon yields in each time period. At each t, we consider a set of fixed maturities (measured in months): 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 168, 180, 192, 216, 240, 264, 288, 312, 336, 360. These cover the range of bond maturities, and they also reflect the different trading volumes at different maturities. In particular, at the short end where bonds concentrate, we include more fixed maturities, and at the long end where there are fewer bonds, we include fewer fixed maturities. In addition, we cap these fixed maturities with the joint longest maturity within the time range of interest. For example, from Dec. 1971 to Jan. 1985 with a total of 158 months, we calculate the longest available bond maturities in the market month by month to get $(l_{t=1}, l_{t=2}, ..., l_{t=158})$. The minimum of these longest maturities forms the common longest maturity, and we keep only those fixed maturities smaller than this value in this time range. We obtain the zero yield at each fixed maturity by reading off or linearly interpolating from

[3] See also Chen, Roll, and Ross (1986). Elton, Gruber, and Blake (1995), and Ang and Piazzesi (2003).

the bootstrapped zero yield curve. Finally, we calculate the principal components from the time series of fixed-maturity yields.

In Figure 3 we plot the first three principal components from Dec. 1971 to Dec. 2001. Comparing it with Figure 1, we see that in general the principal components behave in a fashion similar to our three factors. Because Litterman and Scheinkman (1991) have shown that the three principal components account for more than 90% of bond return variation, we wish to know whether our three factors span a space similar to that of the three principal components. For this purpose, we regress the three principal components $\widehat{pc_{1t}}$, $\widehat{pc_{2t}}$, and $\widehat{pc_{3t}}$ on $\widehat{f_{1t}}$, $\widehat{f_{2t}}$, and $\widehat{f_{3t}}$,

$$\widehat{pc_{it}} = \alpha_i + \beta_{i1}\widehat{f_{1t}} + \beta_{i2}\widehat{f_{2t}} + \beta_{i3}\widehat{f_{3t}}, \quad i = 1, 2, 3. \tag{15}$$

We run this regression in selected sample periods and examine the regression R^2.

We consider four sample periods: Dec. 1971 to Dec. 2001, Dec. 1971 to Jan. 1985, Feb. 1985 to Dec. 2001, and Oct. 1979 to Oct. 1982, which corresponds to the time when the Fed targeted non-borrowed reserves rather than interest rates.

In Table 2 we report the regression results. The regression R^2 for the first principal component is very close to one (larger than 0.999) in all cases. The R^2 for the second principal component is also close to one (larger than 0.98) in all cases. The R^2 for the third principal component is high during the overall period (equal to 0.8098), especially during the post-1985 period (equal to 0.9249), but it decreases as we move to the pre-1985 period (between 0.65 and 0.8). The overall result from the regression exercise is that the three factors do span approximately the same space as the first three principal components, in a variety of time ranges. This is so even during unusual periods such as Oct. 1979 to Oct. 1982. As a result, we expect the three factors to account for a high fraction of bond return variation.

3.2 An APT Test

Assumptions and procedures for APT tests are well-documented in the literature. We will follow the usual steps, as for example in Campbell, Lo, and MacKinlay (1997). In particular, we assume the return generating process,

$$R_t = a + BF_t + \varepsilon_t; \quad E(\varepsilon|F) = 0; \quad E(\varepsilon\varepsilon'|F) = \Sigma, \tag{16}$$

where R_t is an N-by-1 vector of returns on the N assets and F_t is a K-by-1 vector of K factors. We also assume that factors account for the common variation in returns. Under a no-arbitrage condition, the above structure implies that

$$\mu = \ell\lambda_0 + B\lambda_K, \tag{17}$$

where μ is an N-by-1 vector of expected returns, ℓ is a vector of 1's, B is an N-by-K factor loading matrix, and λ_K is a K-by-1 vector of factor risk premia. This is the factor pricing model we want to estimate and test using maximum likelihood (ML). For that purpose, we assume returns are dynamically *i.i.d.* and cross-sectionally jointly normal. We use a likelihood ratio test statistic, as defined in Campbell, Lo, and MacKinlay(1997),

$$J = -(T - N/2 - K - 1)\left(\log|\widehat{\Sigma}| - \log|\widehat{\Sigma}^*|\right) \sim \chi^2(r), \tag{18}$$

where $\widehat{\Sigma}$ refers to the ML estimator of the residual covariance without constraints, and $\widehat{\Sigma}^*$ refers to the ML estimator of the residual covariance with constraints imposed by the pricing model. Finally, r is the number of restrictions under the null hypothesis.

In our case, the factor pricing model (17) produces the constrained model

$$R_t = \ell\lambda_0 + B(\lambda_K - E(F_{Kt})) + \varepsilon_t, \tag{19}$$

where the constraint is $a = \ell\lambda_0 + B(\lambda_K - E(F_{Kt}))$. This constraint can be tested using the J statistic in (18), which in the current context is distributed as $\chi^2(N-K-1)$ under the null that the constraint (and hence the APT) holds.

Note that there is a subtle difference between the three factors in (2) and the return-generating factors in the APT test used here. Recall that the definition of holding period return as of time $t+1$ for purchasing a bond with maturity of τ months at time t is

$$r_{t+1}(\tau) = \tau y_t(\tau) - (\tau-1)y_{t+1}(\tau-1).$$

Combining this definition with (2), it is straightforward to show that

$$\begin{aligned} r_{t+1}(\tau) &= -[(\tau-1)f_{1,t+1} - \tau f_{1,t}] - \left[\frac{1-e^{-\lambda(\tau-1)}}{\lambda} f_{2,t+1} - \frac{1-e^{-\lambda\tau}}{\lambda} f_{2,t}\right] \\ &\quad - \left\{\left[\frac{1-e^{-\lambda(\tau-1)}}{\lambda} - (\tau-1)e^{-\lambda(\tau-1)}\right] f_{3,t+1} - \left[\frac{1-e^{-\lambda\tau}}{\lambda} - \tau e^{-\lambda\tau}\right] f_{3,t}\right\}. \end{aligned}$$

Because the length of time passed by is small (one month) compared with maturities of most of the assets, we can ignore the difference between τ and $\tau - 1$. As a result, the return-generating factors can be regarded as $-(f_{1,t+1} - f_{1,t})$, $-(f_{2,t+1} - f_{2,t})$, and $-(f_{3,t+1} - f_{3,t})$. In other words, the return generating factors are the first differences of the original Nelson-Siegel yield factors as calculated in section 2.

For the test assets, we use the Fama Maturity Portfolio Returns file from CRSP. This file consists of two sets of portfolios: the first set uses six-month maturity intervals to construct bond portfolios and has twelve portfolios, while the second set uses 12-month maturity intervals and has seven portfolios. We use both sets of portfolios in the subsequent testing. We also consider different sample periods as a robustness check. In line with the Fama Maturity Portfolio file, the longest sample is Jan. 1975 to Dec. 2001.

We report the test results in Table 3, Panel A (for 12 test assets) and Panel B (for 7 test assets). In the table, J_3 denotes the statistic defined in (18), and p_3 is the associated p-value.

For most of the sample periods considered, p_3 is large enough so that we fail to reject APT at the 5% significance level. However, in certain sample periods, we get small p_3 values. For example, when we use the entire sample period, the p_3 value becomes small. Structural shifts may be responsible for this result. As argued by Diebold and Li (2003), 1985 is a potential break point, after which the term structure is more forecastable. Another possible break point is 1990. Note that when we use twelve test assets, the p_3 value during Jan. 1985 to Dec. 1989 is 0.243, but it drops sharply to 0.098 during Jan. 1985 to Dec. 1990. The same pattern emerges when we use seven test assets: the p_3 value drops from 0.281 (from Jan. 1985 to Dec. 1989) to 0.118 (from Jan. 1985 to Dec. 1990). This pattern suggests that 1990 is possibly a structural break. Because the overall sample includes at least two possible breaks, it is no surprise that the p_3 value is small, as indicated in the table. However, in the sample periods that exclude the three possible break dates, our three factors price the test assets well.

Finally, to check whether our model prices bonds well even in unusual market conditions, we consider the Fed's monetary experiment of the early 1980's, the stock market crash of 1987, and the Asian crisis of 1997-1998. To have a sufficient number of data points for a formal test, we a window of several years around each of the three stress dates: Oct. 1979 to Oct. 1982 for the Fed's monetary experiment, Jan. 1985 to Dec. 1989 for the 1987 stock market crash, and Jan. 1996 to Dec. 2001 for the Asian crisis. It is interesting to note that the three-factor pricing model holds in each of these three sub-periods: p_3 is larger than 0.2 in all cases, whether we use twelve test assets or seven test assets. This suggests the validity of the three-factor model even in extraordinary market conditions.

We also report the J statistics and associated p-values when we use only the first two factors (J_2 and p_2) or only the first factor (J_1 and p_1). Surprisingly, most of the time we fail to reject the APT when we use only two factors or one factor, perhaps due to low power of the tests. But to evaluate the relative performance of the 3-factor, 2-factor, and 1-factor models, we need to consider them together rather than in isolation. Because the models are nested, we use a likelihood ratio test to compare a K-factor model and an L-factor model ($K > L$),

$$m(\log|\widehat{\Sigma_L}| - \log|\widehat{\Sigma}_K|) \sim \chi^2(q), \tag{20}$$

where $\widehat{\Sigma_K}$ and $\widehat{\Sigma_L}$ are the ML estimators of the residual covariance for the K-factor model and the L-factor model, respectively, q is the number of additional restrictions imposed on the K-factor model by the L-factor model, and $m = T - K - \frac{1}{2}(N - (K - L) + 1)$ is the small-sample correction proposed by Jobson and Korkie (1982) and Anderson (1958).

We calculate test statistics for twelve test assets and seven test assets, in different subsamples. All of the the associated p-values are zero, which provides strong evidence against the null that the two models perform equally well. In other words, three factors are always better than two factors or one factor in terms of pricing the test assets. This holds true for all the market conditions and sub-periods we have studied[4].

In summary, in this section we have formally tested and verified the pricing implications of our three-factor model. The results support the hypothesis that the three factors represent systematic risks priced by the market. The cross section of bond returns does depend on the bonds' sensitivities to the three factors. In addition, the three factors represent different sources of risks that can not be captured by only one or two factors, as revealed by the outcome of the formal hypothesis testing.

4 A Generalized Duration Measure and its Performance in Bond Portfolio Risk Management

An implication of our three-factor yield curve model is the natural movement that it suggests away from traditional Macaulay duration and toward "vector" duration measures. In this section we derive the generalized duration measure and evaluate its hedging performance.

[4]Because the p-values are virtually zero for all subperiods we considered, we do not include them in the table.

4.1 Alternative Duration Measures

The concept of duration has a long history in the fixed income literature. Since Macaulay introduced this concept in 1938, it has been used mostly as a measure of bonds' sensitivities to interest rate risk, and it has been widely used in bond portfolio management. But it was subsequently shown that the Macaulay duration measure works well only when the yield curve undergoes parallel shifts and when the shifts are small. It breaks down for more complicated yield curve movements. To overcome this problem, efforts have been made to extend the duration concept in different directions.

Early work by Cooper (1977), Bierwag (1977), Bierwag and Kaufman (1978), and Khang (1979) extends the duration measure by presuming various specific characteristic movements of the term structure. For a brief summary, see Gultekin and Rogalski (1984). Despite the improvements, the extended duration measures remain one-factor in nature, and as Gultekin and Rogalski (1984) show, they are very similar. Hence we lump them together and consider only the Macaulay duration. In another direction, Cox, Ingersoll, and Ross (1985) develop a general equilibrium model of the term structure, widely known as the CIR model. One by-product of the model is the stochastic duration measure proposed in Cox, Ingersoll, and Ross (1979), who argue that the new duration is superior to Macaulay duration and that the latter overstates the basis risk of bonds.

Other authors are not satisfied with either of these two approaches. They argue that we need to consider not only movements in the level of the yield curve, but also movements in other aspects of the curve. That is, we need to use a vector duration whose components capture different characteristics of movements in the curve. Based on this idea, Chambers, Carleton, and McEnally (1981) and Garbade (1996) have proposed the use of so-called duration vectors. Because the elements of their duration vectors derive from polynomial expansions, we call them polynomial-based duration vectors. However, polynomial-based duration vectors are unappealing at long maturities, because polynomials diverge at long maturities, which degrades performance.

Motivated by the above considerations, in the next sub-section we take a different approach to bond portfolio risk management. Our three-factor term structure model immediately suggests generalized duration components corresponding to the level, slope, and curvature risk factors. In keeping with the exponential form of the Nelson-Siegel model, and to distinguish it from the earlier-discussed polynomial model, we call our generalized duration the exponential-based duration vector, to which we now proceed.

4.2 Generalized Duration

In general we can define a bond duration measure as follows. Let the cash flows from the bond be $C_1, C_2, ..., C_I$, and let the associated times to maturity be τ_1,τ_2,...,τ_I. Assume also that the zero yield curve is linear in some arbitrary factors f_1, f_2, and f_3,

$$y_t(\tau) = B_1(\tau)f_{1t} + B_2(\tau)f_{2t} + B_3(\tau)f_{3t} \tag{21}$$

$$dy_t(\tau) = B_1(\tau)df_{1t} + B_2(\tau)df_{2t} + B_3(\tau)df_{3t}. \tag{22}$$

Then, assuming continuous compounding, the price of the bond can be expressed as

$$P = \sum_{i=1}^{I} C_i e^{-\tau_i y_t(\tau_i)}.$$

Note that we need to use the corresponding zero yield $y_t(\tau_i)$ to discount cash flow C_i. Then, for an arbitrary change of the yield curve, the price change is

$$dP = \sum_{i=1}^{I} \left[\partial P / \partial y_t(\tau_i)\right] dy_t(\tau_i) = \sum_{i=1}^{I} \left[C_i e^{-\tau_i y_t(\tau_i)}(-\tau_i)\right] dy_t(\tau_i),$$

where we have treated $y_t(\tau_i)$ as independent variables. Therefore

$$\begin{aligned} -\frac{dP}{P} &= \sum_{i=1}^{I} \left[\frac{1}{P} C_i e^{-\tau_i y_t(\tau_i)} \tau_i \right] dy_t(\tau_i) \\ &= \sum_{i=1}^{I} \left[\frac{1}{P} C_i e^{-\tau_i y_t(\tau_i)} \tau_i \right] \sum_{j=1}^{3} B_j(\tau_i) df_{jt}, \end{aligned}$$

where we have used (21) in the second equality. Now, rearranging terms, we can express the percentage change in bond price as a function of changes in the factors

$$\begin{aligned} -\frac{dP}{P} &= \sum_{j=1}^{3} \left\{ \sum_{i=1}^{I} \left[\frac{1}{P} C_i e^{-\tau_i y(\tau_i)} \tau_i \right] B_j(\tau_i) \right\} df_{jt} \\ &= \sum_{j=1}^{3} \left\{ \sum_{i=1}^{I} w_i \tau_i B_j(\tau_i) \right\} df_{jt}, \end{aligned} \tag{23}$$

where w_i is the weight associated with C_i.

In (23), we have decomposed the change of bond price into changes in risk factors. Hence we can define the duration component associated with each risk factor as

$$D_j = \sum_{i=1}^{I} w_i \tau_i B_j(\tau_i); \quad j = 1, 2, 3.$$

In particular, the vector duration based on our three-factor model of yield curve for any coupon bond is

$$(D_1, D_2, D_3) = \left(\sum_{i=1}^{I} w_i \tau_i, \sum_{i=1}^{I} w_i \frac{1 - e^{-\lambda \tau_i}}{\lambda}, \sum_{i=1}^{I} w_i \left(\frac{1 - e^{-\lambda \tau_i}}{\lambda} - \tau_i e^{-\lambda \tau_i} \right) \right). \tag{24}$$

Although this duration vector has been proposed independently in Willner (1996), our derivation and discussion are more rigorous, and our subsequent hedging performance analysis will also be more realistic, using actual bond returns. Note that the first element of the duration vector is exactly the traditional Macaulay duration, and that it is straightforward to verify the following additional properties of this vector duration measure:

- D_1, D_2, and D_3 move in the same direction, because they are all increasing in τ
- D_1, D_2, and D_3 decrease with coupon rate
- D_1, D_2, and D_3 decrease with yield to maturity
- Portfolio Property: D_1, D_2, and D_3 for a bond portfolio is equal to the average of individual bonds' D_1, D_2, and D_3, where the weight assigned to each bond is equal to the portion of the bond value in the whole portfolio.

As an illustration of how the duration measure changes with coupon rate and time to maturity, in Figure 4 we plot the three components of the generalized duration measure as functions of these characteristics (we fix the yield at 5%). The three components increase with time to maturity and decrease with coupon rate, consistent with our previous conclusion. As time to maturity increases, D_2 and D_3 first increase sharply, but they quickly flatten out around 10 years. By contrast, D_1 continues to increase as time to maturity increases. Note that although D_2 and D_3 look similar in the plots, they are quite different for maturities up to 3 years.

In Figure 5, we plot the duration components as functions of coupon rates and yield to maturity (we fix the time to maturity at 10 years). D_1, D_2, and D_3 all exhibit similar behavior, decreasing in both coupon rates and yields.

4.3 Comparing Alternative Duration Measures

As discussed above, the performance of our duration measure relative to Macaulay duration, stochastic duration, and polynomial-based duration will determine its usefulness as a practical risk management tool.

First, Macaulay duration is simply the first element of our exponential-based duration vector.

Second, the stochastic duration introduced by Cox, Ingersoll, and Ross (1979) is

$$D_s = G^{-1}\left(\frac{\sum_{i=1}^{I} C_i P_i G(\tau_i)}{\sum_{i=1}^{I} C_i P_i}\right) \tag{25}$$

$$G^{-1}(x) = \frac{2}{\gamma}\coth^{-1}\left(\frac{2}{\gamma x} + \frac{\pi - \beta}{\gamma}\right), \tag{26}$$

where P_i is the price of a zero bond maturing at τ_i and π is the liquidity premium parameter[5]. Using the annualized estimated values for the parameters in the above two equations from Cox, Ingersoll, and Ross (1979) ($\pi = 0$, $\mu = 5.623\%$, $\beta = 0.692$, and $\sigma^2 = 0.00608$), we can calculate the stochastic duration for different bond characteristics. In Figure 6 we plot the stochastic duration as a function of bond coupon and time to maturity. As shown in the graph, the behavior of D_s is dramatically different for low-coupon bonds and high-coupon bonds. Cox, Ingersoll, and Ross (1979) did not find this strange behavior because they considered only three high coupon rates: 4%, 6%, and 8%, all of which fall into the relatively flat region of the graph. The unusual behavior of stochastic duration indicates that it will be of limited use as a measure of bond risk, so we we will not consider it further.

Finally, the polynomial-based vector duration is

$$(D'_1, D'_2, D'_3) = \left(\sum_{i=1}^{I} w_i\tau_i, \left(\sum_{i=1}^{I} w_i\tau_i^2\right)^{1/2}, \left(\sum_{i=1}^{I} w_i\tau_i^3\right)^{1/3}\right). \tag{27}$$

Note that the first element of D' is the same as the Macaulay duration. As a visual illustration, in Figure 7 we plot D'_2 as a function of coupon rate and time to maturity. The plot of D'_3 is very similar to that of D'_2, so we do not show it separately. As we can see from Figure 7, D'_2 increases with coupon rates and time to maturity in a fashion similar to Figure 4(a). The plots of D'_2 and D'_3 as functions of coupon rates and yields are very similar

[5] $\coth(x)$ is the hyperbolic cotangent of x. It's defined as: $\coth(x) = (e^x + e^{-x}) / (e^x - e^{-x})$.

to Figure 5(a), and thus we omit them to save space.

4.4 Risk Management Based on Alternative Duration Measures

Having introduced the definitions and basic properties of Macaulay duration, polynomial-based vector duration, and exponential-based duration, we apply these measures as immunization tools in a practical bond portfolio management context. Immunization is the strategy used to protect the value of the bond portfolio against interest rate changes by controlling some characteristics of the bond portfolio. In our case, the characteristics for each duration measure are Macaulay duration D_m, polynomial-based duration $D' = (D'_1, D'_2, D'_3)$, and exponential-based duration $D = (D_1, D_2, D_3)$.

Our empirical immunization design is similar to that in Elton, Gruber, and Nabar (1988). First we need to specify some bond as the target asset, whose payoffs (returns) we wish to match (immunize). For each duration measure, we will construct a bond portfolio such that its duration measure matches that of the target asset. To the extent the duration measure is a good measure of interest rate risks, the hedging portfolio should have realized returns approximately equal to those of the target asset. The difference between realized target returns and realized hedging-portfolio returns (i.e., the hedging errors) will be used to compare the different duration measures. For a chosen target asset, we can construct virtually an infinite number of portfolios whose duration matches the target. To narrow our choice, we use the arguments of Ingersoll (1983) to impose the following minimization objective

$$\min \sum_{n=1}^{N} w_n^2, \tag{28}$$

subject to two constraints:

$$(1). \qquad \sum_{n=1}^{N} w_n = 1 \tag{29}$$

$$(2). \quad \sum_{n=1}^{N} w_n D_n = D_{target} \quad or \quad \sum_{n=1}^{N} w_n D_n^k = D_{target}^k, \tag{30}$$

where w_n is the portfolio weight of bond n, D_n is the duration measure of bond n, and D_{target} is the duration of the target bond. Note that the second constraint applies to Macaulay duration and exponential-based vector duration, whereas the third constraint applies to the polynomial-based duration, because in the latter case the portfolio property applies only to the moment of the bond, not to the polynomial-based duration per se. Minimizing the sum of squared weights guarantees diversification of the portfolio. In addition, short sales are allowed in our design.

We use the CRSP monthly treasury data from Dec. 1971 to Dec. 2001. Because the three duration measures reduce to the same measure for zero-coupon bonds, we exclude T-Bills from the immunization universe. As before, for liquidity considerations, we keep only notes and bonds with maturity longer than one year. But there are still too many bonds concentrated at the short end. To avoid having too many assets in our portfolio (in any case, we have at most three duration components to match), we follow Chambers, Carleton, and McEnally (1988) and consider only government notes maturing on February 15, May 15, August 15, and November 15.

To use the true market price data, our target asset is chosen from existing bonds rather than an artificially constructed bond as in Willner (1996). We set the target bond's maturity at 5 years. At each point of time, it is virtually impossible to find a bond whose time to maturity is exactly 5 years, so as a compromise we choose as our target asset the bond with maturity closest to 5 years. We then calculate the duration for the target asset, and we construct the immunization portfolio from bonds excluding the target bond. In addition, in portfolio construction, we use only those bonds with return data throughout the holding period. We solve for the portfolio weights using (28), (29) and (30).

In Table 4 we list the number of assets in the hedging portfolio in different sample periods and for different holding horizons. In the table, N_{mean} denotes the average number of assets in the hedging portfolios during that time range. The maximum number of assets goes from 24 to 31, while the minimum number of assets goes from 9 to 22, which suggests that the number of asets in our hedging portfolio is reasonable. We list in Table 5 the difference between the target asset's actual maturity and our ideal target bond maturity, i.e. 5 years. The average difference D_{mean} varies from 0.03 year to 0.09 year, a small difference relative to 5 years. The maximum difference $D_{\max}$ is 0.37 year, still a reasonably small value.

Once the portfolio is formed, we hold it and the target asset for certain time, which we set to 1 month, 3 months, 6 months, and 12 months in the current context. We compare the realized target returns and portfolio returns and record the hedging errors, and we repeat month-by-month to get the time series of hedging errors.

In Table 6 we list the hedging performance for the three duration measures. In the table, M refers to hedging using Macaulay duration, E refers to the hedging using exponential-based duration, and P refers to hedging using polynomial-based duration. As in Chambers, Carleton, and McEnally (1988), we are mainly interested in the mean absolute hedging error (MAE) and the standard deviation of the hedging error (Std). The better the hedge, the smaller these two quantities will be. We also report the mean hedging error. A positive mean suggests that hedging portfolio is earning a higher average return than the target asset and is thus a desirable property.

Panel A of Table 6 reveals that, for a 1-month holding period, the exponential-based duration consistently outperforms Macaulay duration in terms of generating smaller average absolute hedging error and smaller standard deviation of hedging errors. This is true even during the pre-1985 period and during the structure-break period of early eighties. In addition, the average error from exponential-based hedging is consistently larger than that from the Macaulay hedging. This fact is promising because it implies the exponential-based hedging can generate higher mean returns while maintaining smaller variation. The comparison between exponential-based and polynomial-based hedging is less clear-cut. In most cases they have very similar performance. One exception is the monetary experiment period in the early eighties (10/1979-10/1982). One possible reason is that during that period the yields were very volatile and noisy as the market were trying to learn the Federal Reserve's new policy.

Panel B of Table 6 extends the holding period from one month to one quarter. The superior performance of exponential-based hedging over Macaulay hedging is again obvious: we have smaller average absolute error and smaller standard deviation while always keeping a higher average mean error. Also, this superior performance is more marked than when the holding period is 1 month. Compared with polynomial-based hedging, exponential-based hedging still has similar performance. But in the unusual market conditions of the early eighties, exponential hedging again proves to be superior than polynomial hedging: the average absolute errors are 0.004 and 0.0044, respectively, and the standard deviations are 0.0049 and 0.0054.

Panels C and D of Table 6 consider even longer holding periods of 6 and 12 months. The comparative performance of exponential-based hedging vs. Macaulay hedging remains unchanged: the former is consistently superior to the latter in all samples considered. As for exponential-based vs. polynomial-based hedging, the results are mixed. On the one hand, exponential hedging seems to have higher average absolute errors in most cases, but on the other hand, its hedging errors always have smaller standard deviation. Combining these two features, it is hard to say which hedging strategy is better. Depending on the specific needs of the investor, some might prefer exponential-based hedging and others might prefer polynomial-based hedging. But as before, during the early eighties, exponential-based hedging proves best: it has the smallest average absolute error and the smallest standard deviation[6].

Overall, Table 6 suggests that hedging based on our vector duration outperforms hedging based on Macaulay duration in almost all samples and all holding periods. It performs

[6]Our hedging exercise considers only a single 5 year bond, which has simple cash flows, as the target. For target bonds with more complicated cash flows, say a portfolio of 1 year, 5 year and 10 year bonds, or for times of high yield level or high volatility, the exponential-based duration might provide a larger hedging improvement over the other duration measures.

similarly to hedging based on polynomial vector duration in normal times but outperforms during unusual market situations such as the monetary regime of the early eighties. All told, our vector duration measure seems to be an appealing risk management instrument for bond portfolio managers.

5 Conclusion

We have assessed and applied the term-structure model introduced by Nelson and Siegel (1987) and re-interpreted by Diebold and Li (2003) as a modern three-factor model of level, slope and curvature. First, we asked whether the model is a member of the affine class, and we found that it is not. This reconciles (1) the poor forecasting performance recently documented for affine term structure models, and (2) the good forecasting performance recently documented for the Nelson-Siegel model. Next, having clarified the relationship between our three-factor model and the affine class, we proceeded to assess its adequacy directly, by testing whether its level, slope and curvature factors do indeed capture systematic risk. We found that they do, and that they are therefore priced. Finally, confident in the ability of our three-factor model to capture the pricing relations present in the data, we proceeded to explore its efficacy in bond portfolio risk management, and we found superior performance relative to hedges constructed using traditional measures such as Macaulay duration, which account only for level shifts.

Appendix

Here we prove that our three factor model does not belong to ATSM. We will check one-by-one all four sub-classes of three-factor ATSM: $A_0(3)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$.

- $A_0(3)$

In this case, the fourth equation from (13) and (14) reduces to

$$
\begin{aligned}
\eta\tau e^{-\eta\tau} &= -\frac{1}{\eta}k_{33}(1-e^{-\eta\tau}) + k_{33}\tau e^{-\eta\tau} + \delta_3 \Longrightarrow \\
0 &= -\frac{1}{\eta}k_{33} + \delta_3 + \frac{1}{\eta}k_{33}e^{-\eta\tau} + \tau e^{-\eta\tau}(k_{33}-\eta) \Longrightarrow \\
k_{33} &= \eta\delta_3; k_{33} = 0; k_{33} = \eta.
\end{aligned}
$$

The restrictions on k_{33} are contradictory. Hence class $A_0(3)$ of ATSM is inconsistent with our three factor model.

- $A_1(3)$

In this case, the first equation from (13) and (14) reduces to

$$
\begin{aligned}
0 = {} & -k_{11}\theta_1\tau - (k_{21}\theta_1 - \lambda_2 + k_{31}\theta_1 - \lambda_3)\frac{1}{\eta} + \frac{1}{\eta^2} - \delta_0 \\
& + \left(k_{21}\theta_1 - \lambda_2 + k_{31}\theta_1 - \lambda_3 - \frac{2}{\eta}\right)\frac{1}{\eta}e^{-\eta\tau} \\
& + \left(k_{31}\theta_1 - \lambda_3 - \frac{1}{\eta}\right)\tau e^{-\eta\tau} + \frac{1}{\eta^2}e^{-2\eta\tau} + \frac{1}{2}\tau^2 e^{-2\eta\tau} + \frac{1}{\eta}\tau e^{-2\eta\tau}.
\end{aligned}
$$

This equation requires that $k_{11}\theta_1 = 0$, which contradicts the admissibility condition for $A_1(3)$ in Table 1. Therefore, the $A_1(3)$ sub-class of ATSM is inconsistent with our three factor model.

- $A_2(3)$

In this case, the fourth equation in (13) and (14) reduces to

$$
\begin{aligned}
\eta\tau e^{-\eta\tau} &= -\frac{1}{\eta}k_{33}(1-e^{-\eta\tau}) + k_{33}\tau e^{-\eta\tau} + \delta_3 \Longrightarrow \\
0 &= -\frac{1}{\eta}k_{33} + \delta_3 + \frac{1}{\eta}k_{33}e^{-\eta\tau} + \tau e^{-\eta\tau}(k_{33}-\eta) \Longrightarrow \\
k_{33} &= \eta\delta_3; k_{33} = 0; k_{33} = \eta.
\end{aligned}
$$

Similar to the $A_0(3)$ case, the restrictions on $A_2(3)$ violates the stationarity condition in Table 1. Therefore sub-class $A_2(3)$ of ATSM is inconsistent with our three factor model.

- $A_3(3)$

In this case, the first equation in (13) and (14) reduces to

$$\begin{aligned} 0 \quad = \quad & (k_{11}\theta_1 + k_{12}\theta_2 + k_{13}\theta_3)\tau + \frac{1}{\eta}(k_{21}\theta_1 + k_{22}\theta_2 + k_{23}\theta_3 + k_{31}\theta_1 + k_{32}\theta_2 + k_{33}\theta_3) \\ & -\frac{1}{\eta}(k_{21}\theta_1 + k_{22}\theta_2 + k_{23}\theta_3 + k_{31}\theta_1 + k_{32}\theta_2 + k_{33}\theta_3)e^{-\eta\tau} \\ & -\frac{1}{\eta}(k_{31}\theta_1 + k_{32}\theta_2 + k_{33}\theta_3)\tau e^{-\eta\tau}. \end{aligned}$$

This equation alone implies $k_{11}\theta_1 + k_{12}\theta_2 + k_{13}\theta_3 = 0$, which contradicts the admissibility of ATSM in Table 1. Therefore sub-class $A_3(3)$ of ATSM is inconsistent with our three factor model.

References

Anderson, T.W., 1958, *An Introduction to Multivariate Statistical Analysis*, John Wiley and Sons, Inc.

Ang, A., and M. Piazzesi, 2003, "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50, 4, 745-787.

Bierwag, G.O., 1977, "Immunization, Duration and the Term Structure of Interest Rates," *Journal of Financial and Quantitative Analysis*, 12, 725-742.

Bierwag, G.O., and G. Kaufman, 1978, "Bond Portfolio Strategy Simulations: A Critique," *Journal of Financial and Quantitative Analysis*, 13, 519-526.

Bliss, R., 1997, "Testing Term Structure Estimation Methods," *Advances in Futures and Options Research*, 9, 197-231.

Bliss, R., and D. Smith, 1997, "The Stability of Interest Rate Processes," working paper, Federal Reserve Bank of Atlanta.

Campbell, J., A. Lo, and C. MacKinlay, 1997, *The Econometrics of Financial Markets*, Princeton University Press.

Chambers, D., W. Carleton, and R. McEnally, 1988, "Immunizing Default-Free Bond Portfolios with a Duration Vector," *Journal of Financial and Quantitative Analysis*, 23, 89-104.

Chen, N., R. Roll, and S. Ross, 1986, "Economic Forces and the Stock Market," *Journal of Business*, 59, 383-404.

Connor, G., and R. Korajczyk, 1988, "Risk and Return in an Equilibrium APT: Application of a New Test Methodology," *Journal of Financial Economics*, 21, 255-289.

Cooper, I., 1977, "Asset Values, Interest Rate Changes, and Duration," *Journal of Financial and Quantitative Analysis*, 12, 701-724.

Cox, J., J. Ingersoll, and S. Ross, 1979,"Duration and the Measurement of Basis Risk," *Journal of Business*, 52, 51-61.

Cox, J., J. Ingersoll, and S. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-408.

Dai, Q., and K. Singleton, 2000, "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, 55, 1943-1978.

Diebold, F.X., and C. Li, 2003, "Forecasting the Term Structure of Government Bond Yields," working paper, University of Pennsylvania.

Duffee, G., 2002, "Term Premia and Interest Rate Forecasts in Affine Models," *Journal of Finance*, 57, 405-443.

Duffie, D., and R. Kan, 1996, "A Yield-Factor Model of Interest Rates," *Mathematical Finance*, 6, 379-406.

Elton, E., M. Gruber, and P. Nabar, 1988, "Bond Returns, Immunization and the Return Generating Process," *Studies in Banking and Finance*, 5, 125-154.

Elton, E., M. Gruber, and C. Blake, 1995, "Fundamental Economic Variables, Expected Returns, and Bond Fund Performance," *Journal of Finance*, 50, 1229-1256.

Fama, E., and R. Bliss, 1987, "The Information in Long-Maturity Forward Rates," *American Economic Review*, 77, 680-692.

Garbade, K., 1996, *Fixed Income Analytics*, MIT Press.

Gultekin, B., and R. Rogalski, 1984, "Alternative Duration Specifications and the Measurement of Basis Risk: Empirical Tests," *Journal of Business*, 57, 241-264.

Ingersoll, J., 1983, "Is Duration Feasible? Evidence from the CRSP Data," in G.O. Bierwag, G. Kaufman, A. Toves, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization*, JAI Press.

Jobson, J.D., and B. Korkie, 1982, "Potential Performance and Tests of Portfolio Efficiency," *Journal of Financial Economics*, 10, 433-466.

Khang, C., 1979, "Bond Immunization when Short-Term Rates Fluctuate more than Long-Term Rates," *Journal of Financial and Quantitative Analysis*, 13, 1085-1090.

Knez, P., R. Litterman, and J. Scheinkman, 1994, "Explorations into Factors Explaining Money Market Returns," *Journal of Finance*, 49, 1861-1882.

Lehmann, B., and D. Modest, 1988, "The Empirical Foundations of the Arbitrage Pricing Theory," *Journal of Financial Economics*, 21, 213-254.

Litterman, R., and J. Scheinkman, 1991, "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, June, 54-61.

Nelson, C.R., and A.F. Siegel, "Parsimonious Modeling of Yield Curve," *Journal of Business*, 60, 473-489.

Piazzesi, M., 2002, "Affine Term Structure Models," working paper, UCLA Anderson School of Management.

Ross, S., 1976, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341-360.

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177-188.

Willner, R., 1996, "A New Tool For Portfolio Managers: Level, Slope, and Curvature Durations," *Journal of Fixed Income*, June, 48-59.

Table 1: Parameter Restrictions for the Three-Factor ATSM

We list parameter restrictions imposed on canonical (maximal) three-factor ATSM models: $A_0(3)$, $A_1(3)$, $A_2(3)$, and $A_3(3)$. "eigen" refers to eigenvalues of the corresponding matrix.

Model	Existence/Admissibility	Stationarity
$A_0(3)$	$\delta_1 \geq 0$ $\delta_2 \geq 0$ $\delta_3 \geq 0$	$k_{11} \geq 0$ $k_{22} \geq 0$ $k_{33} \geq 0$
$A_1(3)$	$\delta_2 \geq 0; \delta_3 \geq 0$ $k_{11}\theta_1 > 0; \theta_1 \geq 0$ $\beta_{12} \geq 0; \beta_{13} \geq 0$	$k_{11} \geq 0$ $eigen \begin{pmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{pmatrix} > 0$
$A_2(3)$	$\delta_3 \geq 0$ $k_{11}\theta_1 + k_{12}\theta_2 > 0$ $k_{21}\theta_1 + k_{22}\theta_2 > 0$ $k_{21} \leq 0; k_{12} \leq 0$ $\theta_1 \geq 0; \theta_2 \geq 0$	$k_{33} \geq 0$ $eigen \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} > 0$
$A_3(3)$	$k_{11}\theta_1 + k_{12}\theta_2 + k_{13}\theta_3 > 0$ $k_{21}\theta_1 + k_{22}\theta_2 + k_{23}\theta_3 > 0$ $k_{31}\theta_1 + k_{32}\theta_2 + k_{33}\theta_3 > 0$ $k_{21} \leq 0; k_{12} \leq 0$ $k_{23} \leq 0; k_{32} \leq 0$ $k_{13} \leq 0; k_{31} \leq 0$	$eigen \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} > 0$

Table 2: Regressions of Three Principal Components on the Three Factors

For each sample period, we report the coefficients from regressions of each principal component pc_i on a constant and three Nelson-Siegel factors (f_1, f_2, and f_3), with corresponding t statistics in parenthesis. The last column contains the regression R^2.

	Constant	f_1	f_2	f_3	R^2
	Sample period: 12/1971 - 12/2001				
pc_1	-0.077	1.008	0.489	0.222	0.9998
	(-1090.4)	(1257.4)	(560.8)	(267.9)	
pc_2	-0.143	1.052	-2.917	0.218	0.9885
	(-94.6)	(61.4)	(-156.9)	(12.3)	
pc_3	-0.105	1.451	1.022	-3.531	0.8098
	(-13.4)	(16.3)	(10.6)	(-38.4)	
	Sample period: 02/1985 - 12/2001				
pc_1	-0.068	0.992	0.345	0.183	0.9997
	(-631.8)	(643.7)	(223.7)	(158.3)	
pc_2	-0.199	0.826	-5.148	-0.959	0.9997
	(-85.9)	(24.9)	(-155.2)	(-38.7)	
pc_3	-0.045	0.768	0.942	-0.904	0.9249
	(-25.9)	(30.5)	(37.4)	(-48.0)	
	Sample period: 12/1971 - 01/1985				
pc_1	-0.092	1.009	0.489	0.222	0.9998
	(-711.7)	(764.9)	(299.7)	(120.5)	
pc_2	-0.133	0.989	-3.602	0.217	0.9910
	(-54.1)	(39.6)	(-116.5)	(6.2)	
pc_3	-0.026	1.407	-2.321	-11.405	0.6464
	(-0.544)	(2.9)	(-3.8)	(-16.6)	
	Sample period: 10/1979 - 10/1982				
pc_1	-0.128	1.000	0.495	0.221	0.9995
	(-201.6)	(193.8)	(158.9)	(75.7)	
pc_2	-0.128	1.035	-1.111	0.260	0.9946
	(-27.9)	(27.9)	(-49.8)	(12.4)	
pc_3	-0.176	1.582	-0.794	-1.537	0.7946
	(-5.6)	(6.2)	(-5.2)	(-10.7)	

Table 3: APT Test of the Three Factor Model

We report APT-J test statistics using different test assets and for different sample periods. The J_1, J_2, J_3 columns contain the J-statistics as defined in (18) when we use the first factor, the first two factors, and all three factors, respectively. The p_1, p_2, p_3 columns contain the corresponding p-values. Panel A uses Fama Maturity Portfolios with a 6-month maturity interval as the test assets, while Panel B uses Fama Maturity Portfolios with a 12-month maturity interval as the test assets.

Sample Period	J_3	p_3	J_2	p_2	J_1	p_1
Panel A: 6-month maturity interval test assets						
01/1975 - 12/2001	28.331	0.001	29.871	0.001	32.590	0.000
01/1975 - 12/1984	10.824	0.212	11.508	0.243	12.079	0.279
01/1985 - 12/2001	15.879	0.044	22.331	0.008	66.442	0.000
01/1985 - 12/1989	10.322	0.243	14.315	0.112	26.088	0.004
01/1985 - 12/1990	13.405	0.098	18.215	0.033	33.068	0.000
01/1990 - 12/2001	10.064	0.261	13.650	0.135	54.143	0.000
10/1979 - 10/1982	6.050	0.642	6.304	0.709	6.670	0.756
01/1996 - 12/2001	10.093	0.259	12.741	0.175	27.304	0.002
Panel B: 12-month maturity interval test assets						
01/1972 - 12/2001	7.137	0.068	7.230	0.124	10.638	0.059
01/1972 - 12/1984	2.259	0.521	2.298	0.681	3.167	0.674
01/1985 - 12/2001	5.012	0.171	6.391	0.171	27.164	0.000
01/1985 - 12/1989	3.829	0.281	3.919	0.417	5.255	0.386
01/1985 - 12/1990	5.865	0.118	6.001	0.199	8.929	0.112
01/1990 - 12/2001	2.230	0.526	3.263	0.514	24.532	0.000
10/1979 - 10/1982	0.751	0.861	1.134	0.888	1.856	0.869
01/1996 - 12/2001	0.750	0.861	1.907	0.753	9.907	0.078

Table 4: Number of Assets in the Hedging Portfolios

We report the average ($Nmean$), maximum ($Nmax$), and minimum ($Nmin$) number of assets in hedging (immunization) portfolios for different holding periods and during different sample periods.

Time	$Nmean$	$Nmax$	$Nmin$
1-month holding period			
12/1971 - 12/1984	20	29	12
01/1985 - 12/2001	28	32	22
12/1971 - 12/2001	25	32	12
3-month holding period			
12/1971 - 12/1984	20	28	12
01/1985 - 12/2001	28	31	21
12/1971 - 12/2001	24	31	12
6-month holding period			
12/1971 - 12/1984	18	26	11
01/1985 - 12/2001	27	30	21
12/1971 - 12/2001	23	30	11
12-month holding period			
12/1971 - 12/1984	16	24	9
01/1985 - 12/2001	25	28	20
12/1971 - 12/2001	21	28	9

Table 5: Divergence Between the Target Asset's Maturity and the Five-Year Target Maturity

We present the average (*Dmean*), the maximum (*Dmax*), and the minimum (*Dmin*) difference, measured in years, between the empirically chosen target asset's maturity and our theoretical target bond maturity of five years.

Time	*Dmean*	*Dmax*	*Dmin*
12/1971 - 12/1984	0.09	0.37	0.03
01/1985 - 12/2001	0.03	0.21	0
12/1971 - 12/2001	0.06	0.37	0

Table 6: Hedging Performance Comparison

We present the hedging performance for the three hedging strategies based on the Macaulay duration (denoted M), the exponential-based duration vector (denoted E), and the polynomial-based duration vector (denoted P). We measure hedging performance by the mean absolute hedging error (MAE), the mean (Mean), and the standard deviation (Std) of the hedging errors.

Time	MAE (%)			Std (%)			Mean (%)		
	M	E	P	M	E	P	M	E	P
Panel A: 1-month holding period									
12/1971-12/2001	0.17	0.15	0.15	0.25	0.23	0.23	0.02	0.03	0.02
01/1985-12/2001	0.13	0.11	0.11	0.17	0.13	0.13	0.05	0.07	0.06
12/1971-12/1984	0.23	0.20	0.21	0.33	0.30	0.31	-0.03	-0.03	-0.03
07/1993-12/2001	0.12	0.10	0.10	0.15	0.12	0.12	0.04	0.06	0.05
01/1985-06/1993	0.14	0.13	0.12	0.19	0.15	0.14	0.05	0.08	0.08
12/1971-09/1979	0.17	0.15	0.14	0.23	0.20	0.20	-0.02	-0.02	-0.03
10/1979-10/1982	0.42	0.39	0.41	0.55	0.52	0.54	-0.05	-0.04	-0.02
Panel B: 3-month holding period									
12/1971-12/2001	0.23	0.21	0.21	0.32	0.27	0.28	0.03	0.08	0.06
01/1985-12/2001	0.20	0.18	0.18	0.25	0.20	0.20	0.09	0.06	0.14
12/1971-12/1984	0.28	0.24	0.25	0.38	0.32	0.33	-0.03	-0.02	-0.04
07/1993-12/2001	0.17	0.15	0.14	0.20	0.15	0.15	0.09	0.12	0.11
01/1985-06/1993	0.22	0.22	0.21	0.28	0.23	0.23	0.08	0.09	0.09
12/1971-09/1979	0.22	0.19	0.19	0.29	0.25	0.24	-0.01	-0.01	-0.04
10/1979-10/1982	0.47	0.40	0.44	0.58	0.49	0.54	-0.11	-0.07	-0.06
Panel C: 6-month holding period									
12/1971-12/2001	0.29	0.27	0.26	0.37	0.33	0.33	0.07	0.15	0.12
01/1985-12/2001	0.26	0.26	0.25	0.31	0.26	0.27	0.14	0.16	0.22
12/1971-12/1984	0.31	0.27	0.27	0.42	0.35	0.35	-0.01	0.01	-0.02
07/1993-12/2001	0.22	0.20	0.19	0.25	0.18	0.19	0.13	0.19	0.15
01/1985-06/1993	0.31	0.34	0.32	0.37	0.30	0.32	0.15	0.32	0.30
12/1971-09/1979	0.23	0.20	.20	0.31	0.27	0.27	-0.01	-0.00	-0.03
10/1979-10/1982	0.53	0.41	0.41	0.66	0.50	0.51	0.02	0.01	0.01
Panel D: 12-month holding period									
12/1971-12/2001	0.34	0.32	0.29	0.44	0.37	0.38	0.12	0.14	0.16
01/1985-12/2001	0.32	0.30	0.28	0.36	0.29	0.30	0.18	0.32	0.25
12/1971-12/1984	0.37	0.30	0.31	0.51	0.42	0.45	0.06	0.04	0.04
07/1993-12/2001	0.29	0.28	0.21	0.27	0.22	0.22	0.21	0.27	0.18
01/1985-06/1993	0.35	0.40	0.35	0.43	0.33	0.35	0.15	0.30	0.31
12/1971-09/1979	0.25	0.21	0.22	0.32	0.25	0.28	0.06	0.04	0.05
10/1979-10/1982	0.60	0.49	0.51	0.71	0.61	0.66	0.31	0.28	0.30

Figure 1: Time Series Plot of the Three Factors

We present the time series of the level, slope, and curvature factors. We extract these factors from zero-coupon bond yields using OLS regressions of (2) with λ fixed at 0.0609.

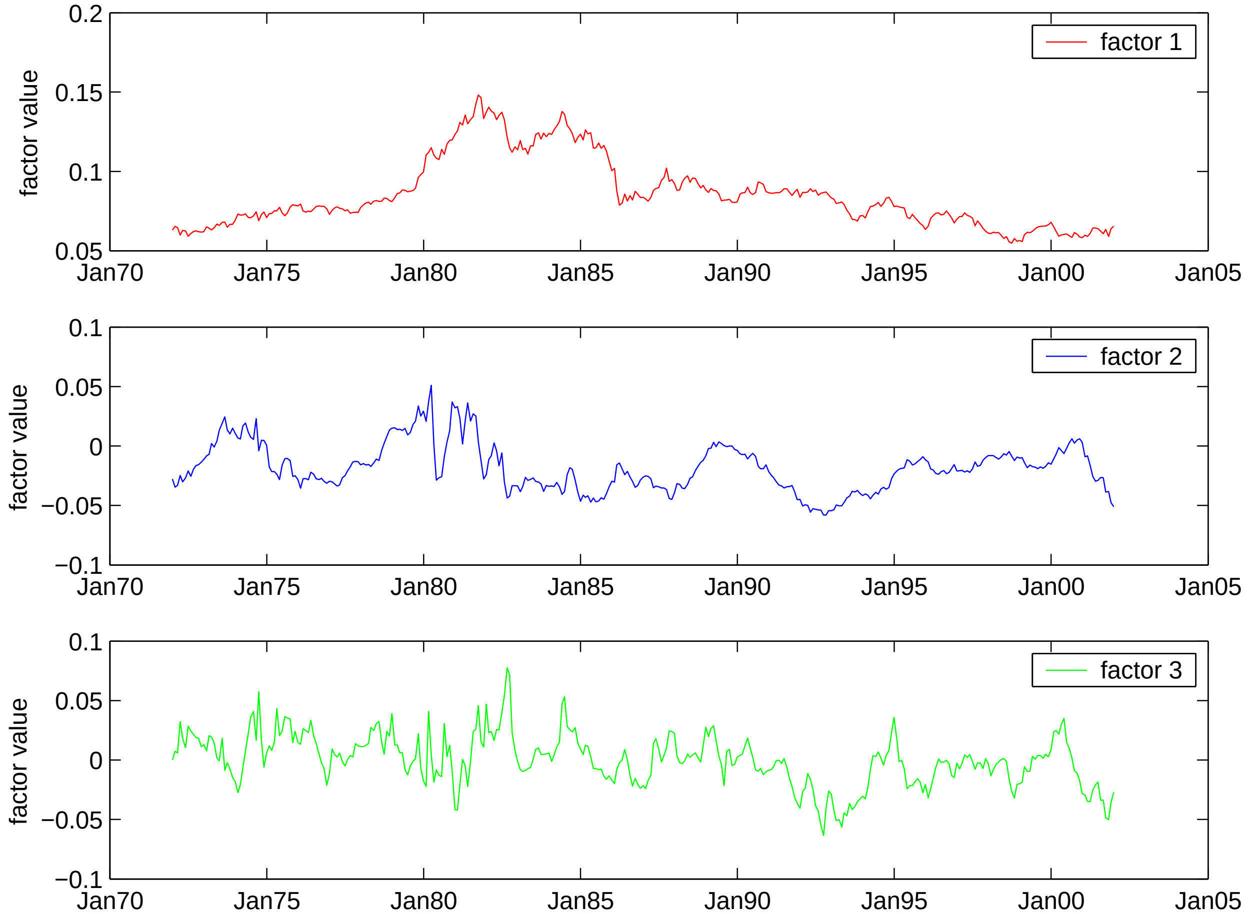


Figure 2: Selected Model-Based Yield Curves

We plot bootstrapped raw zero yields and model-implied zero yields on the following days: 02/28/1997, 04/30/1974, 09/30/1981, and 12/29/2000. We select these days to represent different yield curve shapes: increasing, decreasing, humped, and inverted humped.

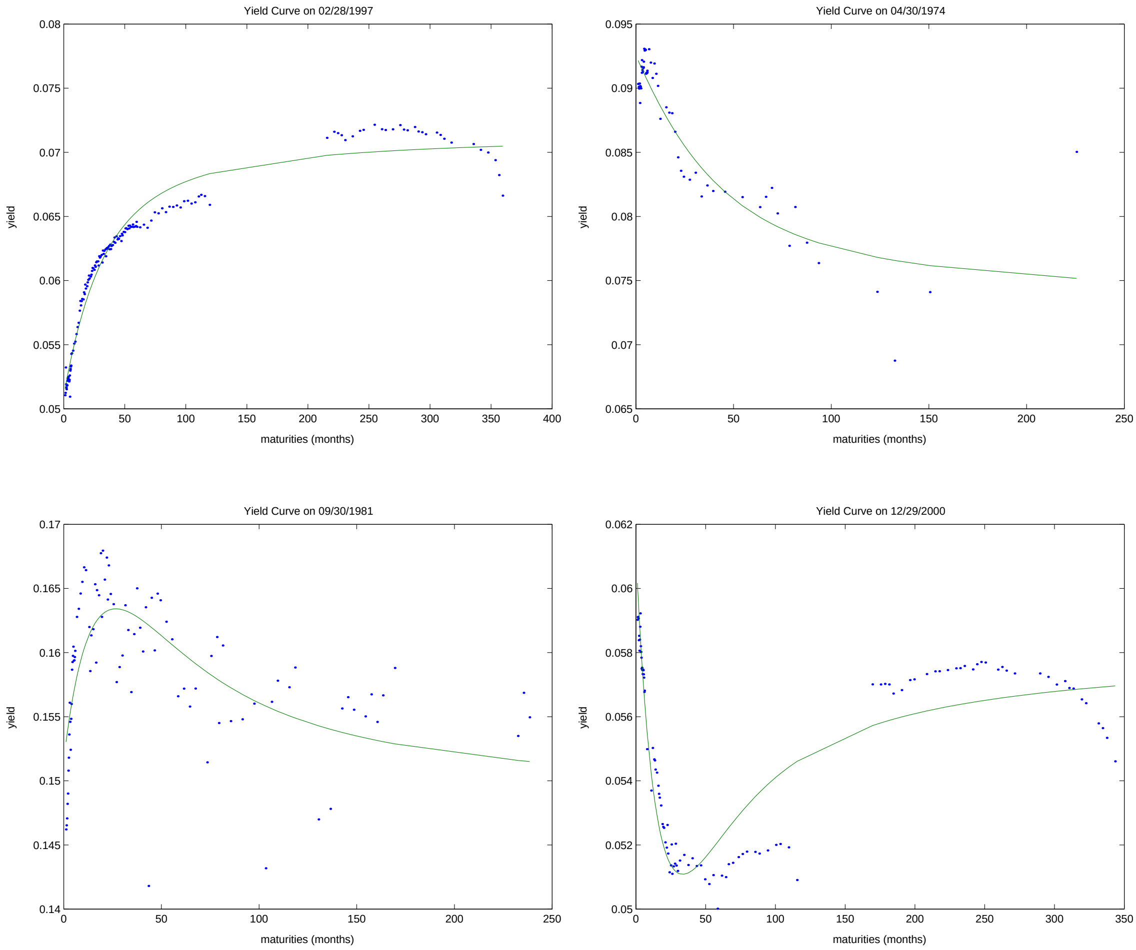


Figure 3: Time Series Plot of First Three Principal Components

We present the time series of the first three principal components extracted from fixed-maturity zero-coupon bond yields inferred from all bond data. We choose these fixed maturities so that yield data exist throughout Dec 1971 to Dec 2001.

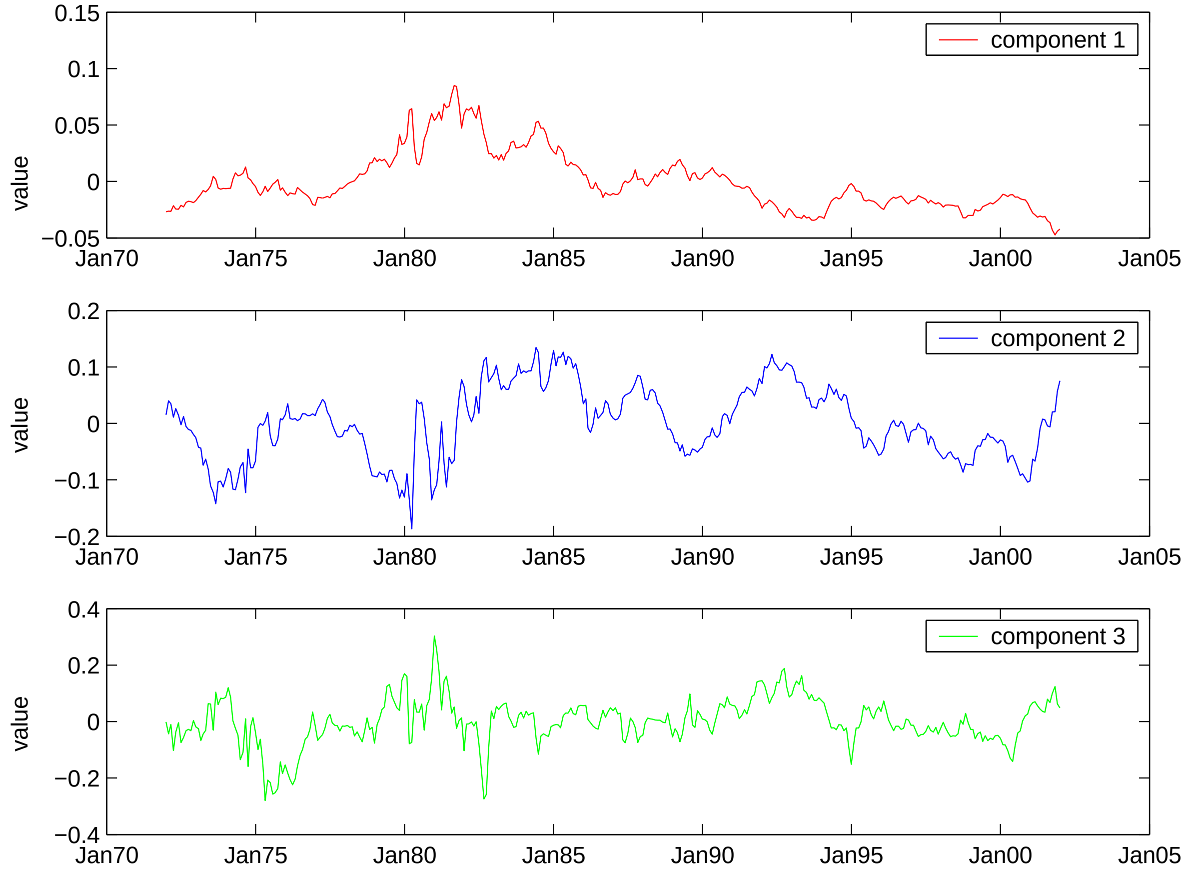
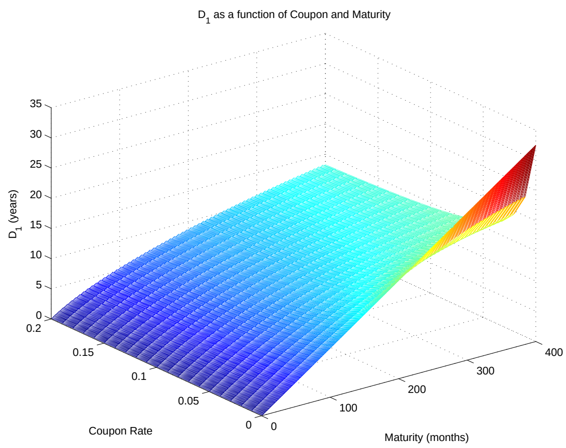
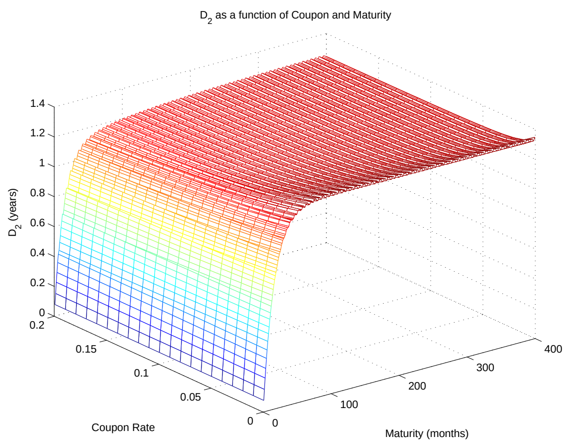


Figure 4: Exponential-Based Duration Vector

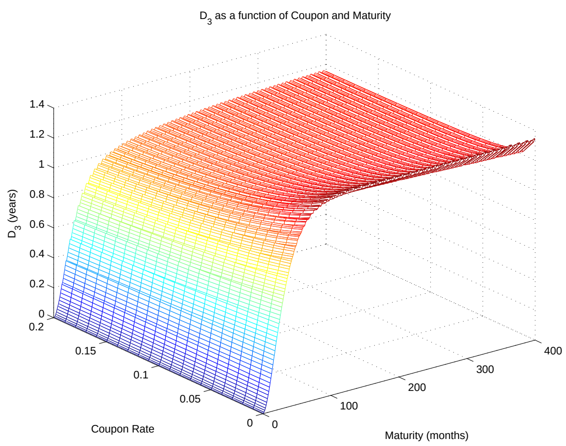
We plot the three elements of the exponential-based duration vector D, defined in equation (24), as a function of coupon rates and time to maturity. The first element is the Macaulay duration. We fix the bond yield at 5%.



(a)



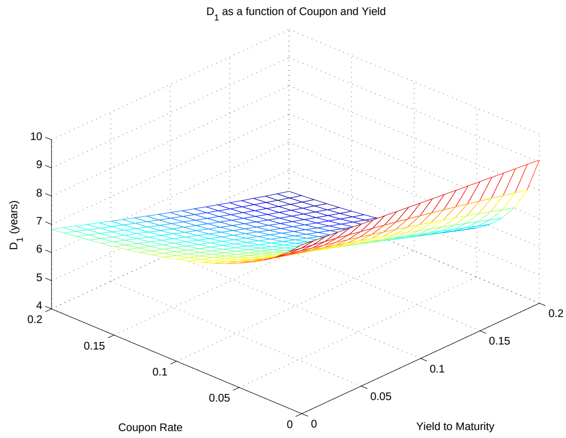
(b)



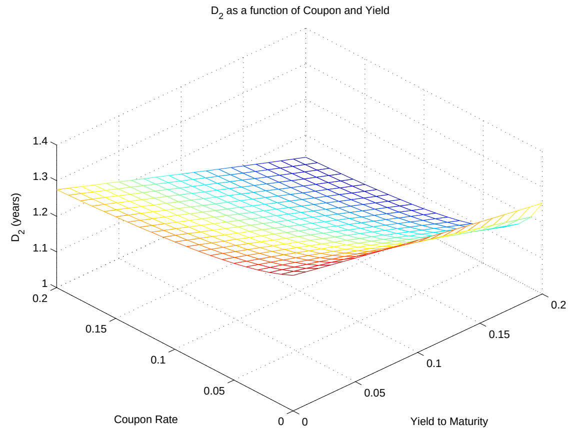
(c)

Figure 5: Exponential-Based Duration Vector

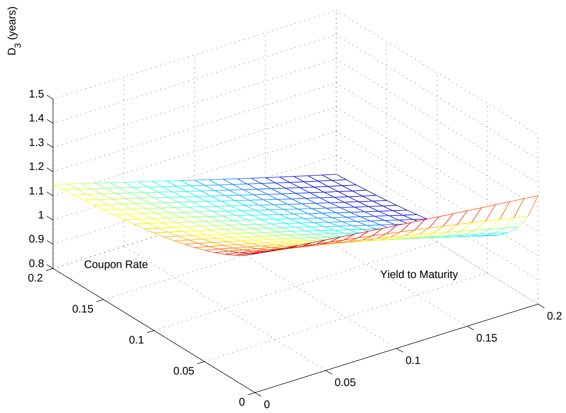
We plot the three elements of the exponential-based duration vector D, defined in equation (24), as a function of coupon rates and yield to maturity. The first element is the Macaulay duration. We fix the time to maturity at 10 years.



(a)



(b)



(c)

Figure 6: Stochastic Duration

We plot the stochastic duration D_s, defined in equations (25) and (26), as a function of coupon rates and time to maturity. We fix the bond yield at 5% and use the following parameter values: $\pi = 0$, $\mu = 5.623\%$, $\beta = 0.692$, and $\sigma^2 = 0.00608$.

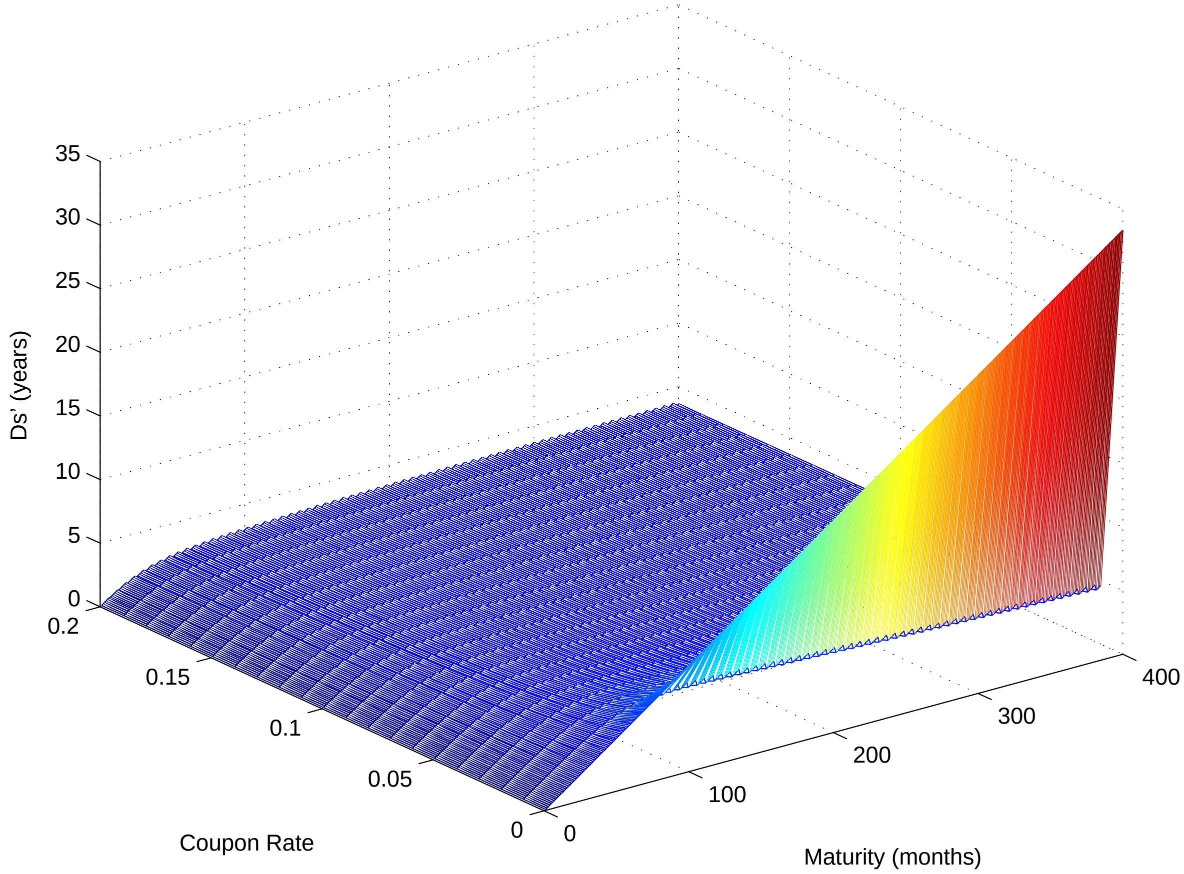


Figure 7: Second Element of the Polynomial-Based Duration Vector

We plot the second element of polynomial-based duration vector D_2', defined in equation (27), as a function of coupon rates and time to maturity. We fix the bond yield at 5%.