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"Multiplicity and Sunspots in General Financial Equilibrium with Portfolio Constrainsts"

by

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Multiplicity and Sunspots in General Financial Equilibrium with Portfolio Constraints

Suleyman Basak, David Cass, Juan Manuel Licari, and Anna Pavlova*

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Abstract

This paper explores the role of portfolio constraints in generating multiplicity of equilibrium. We present a simple asset market economy with two goods and two households, households who face constraints on their ability to take unbounded positions in risky stocks. Absent such constraints, equilibrium allocation is unique and is Pareto efficient. With a single portfolio constraint in place, the efficient equilibrium is still possible; however, additional inefficient equilibria in which the constraint is binding may emerge. We show further that with sunspots, there may be a continuum of equilibria; sunspots may lead to real indeterminacy. Extending our analysis of sunspot phenomena to three periods, we show that our model is also capable of generating moves in stock prices unrelated to so-called fundamentals; such movements are triggered purely by sunspots which only affect investors' (rational) expectations about future market behavior. This provides a simple, coherent explanation for the apparent inability of empirical studies to link many sharp price movements in stock markets to news about economic fundamentals.

JEL Classifications: G12, D52
Keywords: Multiple equilibria, asset pricing, portfolio constraints, sunspots, indeterminacy, general financial equilibrium (GFE).

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1. Introduction

Over the past several decades, financial markets have been developing at a mind-boggling pace. Still, no investor can claim to have complete freedom in allocating funds across time and between events. Elaborate risk management practices aimed at limiting the potential negative consequences of investors' information asymmetries, moral hazard problems, or, simply, bad judgement, have been developing hand in hand with financial markets. A complete account of such practices would require lengthy exposition; but it is fair to say that they may all be summarized as limits on positions in financial assets investors are allowed to take. Thus, abstract constraints on portfolio holdings capture an array of risk-limiting restrictions in reduced form. The question we address in this paper is the general equilibrium consequences of portfolio constraints, and especially their effects on the efficiency of consumption allocation and variability of asset prices.

This question has been previously raised in the asset pricing literature in Finance, but we approach it from a very different angle. Our main focus is on multiplicity of equilibrium: can the introduction of portfolio constraints increase the number of equilibria in an economy? Can portfolio constraints expand the set of equilibria even further, by giving rise to sunspot equilibria—equilibria unrelated to fundamentals and driven purely by investors' expectations? We present a number of robust examples of economies in which the answer to both questions is yes.

The issue of the possibility of multiple and sunspot equilibria has been largely ignored in asset pricing thus far, but we feel that it is very relevant for understanding the variability of asset returns. The presence of sunspot equilibria may, for example, shed light on the apparent inability of empirical studies to link many sharp price movements in stock markets to news about economic fundamentals. Indeed, if an economy (with portfolio constraints) admits more than one equilibrium for the same set of fundamentals, a movement in stock prices could be due entirely to the focus of investors' expectations, whereby they coordinate on different equilibria depending on an extrinsic event.

Our primary economic setting is a simple two-period, two-good, pure-exchange model with two states of nature and two stocks paying off in units of the goods (a real assets model). Two households differ in their (log-linear) preferences, initial endowments, and investment opportunities, in the sense that one household faces a portfolio constraint on holdings in one of the stocks. Absent this constraint, the model is a familiar workhorse asset pricing model, admitting a unique Pareto efficient equilibrium. With the constraint in place, the efficient equilibrium still obtains; however, additional (inefficient) equilibria in which the constraint is binding may occur. Two features of this result require an elaboration, as it may be unclear how to place it in the context of the financial equilibrium literature in Economics. First, the existence of a Pareto efficient equilibrium in which portfolio constraints are fully circumvented even with incomplete asset markets is puzzling. This has to do with the differences between the log-linear, tree-type asset pricing model adopted here and the traditional real assets model of financial equilibrium theory. Second, the introduction of portfolio constraints may expand the set of equilibria, even though by itself it does not generate equilibrium indeterminacy (as in the nominal assets model of Balasko, Cass, and Siconolfi (1990)). To our knowledge, this finding is new in the literature. In particular, we demonstrate that when there are (potentially) complete asset markets, there may be a finite number of additional equilibria (in our leading example, always precisely two). In contrast, with (and only with) incomplete asset markets there may be robust real indeterminacy of equilibrium.

We next examine the role of portfolio constraints in generating sunspot equilibria. Towards this end, we extend our baseline economy to one in which there are incomplete markets by additionally incorporating two extrinsic states, representing uncertainty that does not affect any of the economic fundamentals. Absent portfolio constraints, extrinsic uncertainty plays no significant role in the economy: there is a unique nonsunspot equilibrium corresponding to the original unique Pareto efficient equilibrium. With portfolio constraints, however, sunspots may matter. And, when they do, not only are there multiple sunspot equilibria, but there is also a continuum of them, with consumption allocations varying across this continuum.

To demonstrate the variation in stock prices across (multiple) equilibria, we consider multi-period extensions of our economy. The most instructive, the one we detail here, is a three-period setting in which extrinsic uncertainty gets realized first, households retrade, and then intrinsic uncertainty gets resolved. This extends our baseline economy by adding an intermediate period in which no change in economic fundamentals takes place and no information about future economic fundamentals is revealed—the events observed by households during this period are pure sunspot phenomena. Sunspots revealed in this period are unrelated to fundamentals, but may influence households' expectations. When portfolio constraints bind in the final period, there are, in effect, future multiple equilibria in the economy. ("Future" here means precisely conditional on the choice of portfolios in the initial period.) The revelation of a sunspot acts as an equilibrium selection device: when a "good" sunspot is revealed, the households coordinate on a (conditionally) Pareto efficient equilibrium (which always exists); in contrast, when a "bad" sunspot gets revealed, the households focus on an inefficient equilibrium. We show that stock prices and price-dividend ratios differ across the two types of equilibria. In this extension of our economy, stock price movements need not be associated with news about economic fundamentals: "excess volatility" or "market crashes" may just be (rational) expectations-driven phenomena, with households selecting one equilibrium over another. This interpretation is consistent with the evidence presented by Cutler, Poterba and Summers (1993), who argue that most of the large post-war market moves cannot be explained by releases of economic or other directly relevant information. A prominent example is the famous October 1987 crash, which occurred during a particularly tranquil period and was never linked to any release of economic information. Day-to-day stock market behavior also appears to exhibit similar excess volatility, as argued repeatedly in the literature starting from Shiller (1981), Roll (1984), and French and Roll (1986), who demonstrate that news about fundamentals explain only a small fraction of the variation in asset prices.

There are two main strands of literature related to this paper. The first strand is the literature on asset pricing with capital market imperfections in Finance. Detemple and Murthy (1997), Basak and Cuoco (1998), Detemple and Serrat (2003), and Pavlova and Rigobon (2005), among others, all examine the effects of portfolio constraints on equilibrium consumption allocations and asset prices within the basic asset pricing framework featuring no other frictions. However, the issue of non-uniqueness of equilibrium is either not present or not addressed. Within this literature, our analysis is the first to highlight the role of portfolio constraints in generating multiple equilibria. The second strand is the financial equilibrium literature in Economics, and in particular the developments related to real assets models with restricted participation. Cass, Siconolfi, and Villanacci (2001) demonstrate finite local uniqueness of equilibrium in a real assets model (specifically, a numeraire assets model) with exogenous portfolio constraints while Carosi and Villanacci (2005) extend this result to encompass endogenous constraints (the distinction depends on whether or not the possible constraints depend on just portfolio strategies or other endogenous variables as well—as in the model we analyze here). However, nothing in their analysis suggests that this result fails when markets are incomplete and, furthermore, that portfolio constraints may expand the number of equilibria and give rise to sunspot phenomena. Our baseline framework is similar to theirs, but with one important exception: we consider a Lucas (1978) tree model, in which endowments are specified in terms of shares of stocks, not goods, as in a real assets model. As demonstrated by

Cass and Pavlova (2004), who also employ a multiple-good Lucas-tree asset pricing framework, this feature is critical and can cause the two models' implications to be in stark disagreement. Cass and Pavlova establish the existence of a "peculiar financial equilibrium" even with portfolio constraints, which is the analog of the Pareto efficient equilibrium in our model, and highlight the possible role of the goods markets in alleviating portfolio constraints. They also observe that portfolio constraints may give rise to additional equilibria (their Appendix B); however, unlike ours, the example they provide is exceptional – occurring on a measure-zero set of the proposed constraints. Also related are the works by Gennotte and Leland (1990), Barlevy and Veronesi (2003), and Hong and Stein (2003), who seek to explain stock market crashes and obtain multiple equilibria. However, these models rely on numerous market imperfections such as asymmetric information, short-sale constraints, etc., as well as distributional assumptions and behavioral biases to generate the desired multiplicity of equilibria. In contrast, our argument relies on a simple portfolio constraint imposed within a very standard rational asset-pricing model.

The rest of the paper is organized as follows. Section 2 presents the main ideas of the paper in the simplest possible economic setting, with a specific portfolio constraint. Section 3 demonstrates the role of portfolio constraints in generating sunspot equilibria and presents a multi-period extension of the model. Section 4 discusses the robustness of our main results to alternative economic settings and suggests some extensions. Section 5 provides concluding remarks. The Appendices have the same structure as the body of the paper, but present the complete, formal analysis (in particular, in Appendix A, for our baseline model with a portfolio constraint of general form).

2. Portfolio Constraints and Multiplicity of Equilibrium: A Leading Example

2.1. Economic Environment

We develop the main ideas of the paper within the most basic pure-exchange economy with two time periods, $t = 0$ and 1. Uncertainty is resolved at time $t = 1$, and is represented by two states of the world, labeled $\omega = u$ ("up"), d ("down"), occurring with probabilities $\pi(\omega)$. It will sometimes be useful to refer to the initial period as state $\omega = 0$. There are two non-storable goods, labeled $g = 1$, 2, with prices $p^g(\omega) > 0$.

Production of each good g is modeled as a Lucas tree, with the exogenously specified stream of output $\delta^g(\omega) > 0$. Financial investment opportunities are given by two risky stocks, with period 0 (ex-dividend) prices $q^g(0)$, which are claims to the outputs of the two trees. Each stock is in constant supply of one unit. The time-1 stock payoff matrix P, representing the returns from the two stocks across the two states, is then given by

$$P = \begin{bmatrix} p^1(u)\delta^1(u) & p^2(u)\delta^2(u) \\ \\ p^1(d)\delta^1(d) & p^2(d)\delta^2(d) \end{bmatrix}. \tag{1}$$

We note that since the spot goods prices p^g are endogenously determined in equilibrium, the invertibility of the payoff matrix P, and hence market completeness, is not immediate. If P is not invertible, the two stocks are perfect substitutes for each other, and hence there are fewer nonredundant investment opportunities than there are states of the world. This technical difficulty is absent in economies with a single good, commonly employed in asset pricing. Under a single-good framework, which lacks spot trade in goods, P can simply be exogenously specified to be invertible by assuming invertibility of $[(\delta^1(\omega),\, \delta^2(\omega)),\, \omega = u, d]$.

The economy is populated by two households, indexed by $h = 1,\, 2$. Each household is endowed with an initial portfolio of the two stocks $s_h(0) = (s_h^1(0),\, s_h^2(0))$, in number of units (so that $s_1(0) + s_2(0) = \mathbf{1}$; we will use $\mathbf{1}$ to denote an appropriate-dimension vector of 1's), and trades in spot markets for goods in periods $t = 0,\, 1$ and the stock market in the initial period $t = 0$. No restrictions are imposed on the households' portfolios (in particular, short sales are permitted), apart from a portfolio constraint on household 2, as specified below. Each household chooses its consumption of the goods, $c_h^g(\omega) > 0$, and terminal portfolio holdings, $s_h(1) = (s_h^1(1),\, s_h^2(1))$, evaluating its actions according to the expectation of a log-linear utility function

$$u_h(c_h^1,\, c_h^2) \equiv a_h \log c_h^1(0) + (1 - a_h) \log c_h^2(0) + \beta \sum_{\omega > 0} \pi(\omega)[a_h \log(c_h^1(\omega)) + (1 - a_h) \log(c_h^2(\omega))],$$

with $a_h \in (0, 1)$, $a_1 > a_2$, and $\beta > 0$. The assumption that $a_1 > a_2$ is not essential for our main results, and is made purely for expositional convenience, as it makes the signs of households' portfolio holdings unambiguous in equilibrium.[1] The assumption that the discount factor β is common across households, on the other hand, is made just for simplicity; again, this restriction is not critical for our analysis. In contrast, the log-linear specification of households' utilities

[1] We comment on the case where $a_1 < a_2$ at the end of Appendix A. The borderline case where $a_1 = a_2$ is ruled out, because heterogeneity of households' utilities is required for our results.

(together with the Lucas-tree specification of households' endowments) is quite important for our main results. An analysis of the robustness of the results to local perturbations of utility functions is left for future research, which is being carried out in a separate project.

The economic environment presented thus far is basically a standard workhorse in asset pricing in Finance. Our only major points of departure come from the introduction of two goods and a portfolio constraint on one of the households. In particular, household 2 faces a portfolio constraint of the form

$$s_2^2(1) \geq f(\cdot),$$

where f depends on various endogenous variables in period 0, as well as various parameters (as, for instance, specified in Appendix A). In the body of the paper, when it becomes necessary, we specialize f to represent a lower bound constraint on the fraction of wealth invested in stock 2—the analysis of a constraint of general form is relegated to Appendix A.

A General Financial Equilibrium (GFE) in this economy is defined as a system of spot goods-stock prices $(p,\, q)$ and consumption-portfolio policies $(c,\, s)$ such that

(i) each household h maximizes its expected utility over its budget set, taking prices $(p,\, q)$ as given,

$$\begin{array}{lll}
 & \max\limits_{c_h^1, c_h^2, s_h^1(1), s_h^2(1)} u_h(c_h^1,\, c_h^2) & \text{with multipliers} \\
\text{subj to} & p^1(0)c_h^1(0) + p^2(0)c_h^2(0) + q^1(0)s_h^1(1) + q^2(0)s_h^2(1) & \\
 & \quad = \left(q^1(0) + p^1(0)\delta^1(0)\right) s_h^1(0) + \left(q^2(0) + p^2(0)\delta^2(0)\right) s_h^2(0), & \lambda_h(0) \\
 & p^1(\omega)c_h^1(\omega) + p^2(\omega)c_h^2(\omega) = p^1(\omega)\delta^1(\omega)s_h^1(1) + p^2(\omega)\delta^2(\omega)s_h^2(1), \quad \omega = u,\, d, & \lambda_h(\omega) \\
\text{and} & s_h^2(1) \geq f(\cdot), \quad \text{for } h = 2, & \mu
\end{array}$$

and (ii) all spot goods and stock markets clear,

$$\begin{aligned}
c_1^g(\omega) + c_2^g(\omega) &= \delta^g(\omega), \quad \text{and} &(2)\\
s_1^g(1) + s_2^g(1) &= 1, \qquad g = 1,\, 2, \quad \omega = 0,\, u,\, d. &(3)
\end{aligned}$$

2.2. Preliminaries: Properties of Equilibrium Portfolio Holdings

In this section, we demonstrate that equilibrium portfolio holdings in our economy are exactly the same when equilibrium allocations are not Pareto efficient (and can take the same form when they are). We stress that we need not assume any particular type of asset market friction to obtain this result: the only property required of the market friction is that it entails an equilibrium in which the marginal utilities of the households are not colinear.

Consider the period-1 spot budget constraints of one of the households, say, Ms. 1:

$$p^1(\omega)c_1^1(\omega) + p^2(\omega)c_1^2(\omega) = p^1(\omega)\delta^1(\omega)s_1^1(1) + p^2(\omega)\delta^2(\omega)s_1^2(1). \tag{4}$$

Substituting the period-1 first-order conditions for the households' problems,

$$\pi(\omega)\beta a_h/c_h^1(\omega) = \lambda_h(\omega)p^1(\omega) \quad \text{and} \quad \pi(\omega)\beta(1-a_h)/c_h^2(\omega) = \lambda_h(\omega)p^2(\omega), \qquad \omega = u,\ d, \tag{5}$$

together with the market clearing conditions (2) into this budget constraint and rearranging, we obtain the following simple expression

$$1 = \left(a_1 + a_2\frac{\lambda_1(\omega)}{\lambda_2(\omega)}\right) s_1^1(1) + \left((1-a_1) + (1-a_2)\frac{\lambda_1(\omega)}{\lambda_2(\omega)}\right) s_1^2(1), \qquad \omega = u,\ d. \tag{6}$$

At this point it is useful to define the notion of the households' "stochastic weights":

$$\eta_h(\omega) = \begin{cases} \beta/\lambda_h(0), & \omega = 0, \\ 1/\lambda_h(\omega), & \omega = u,\ d \end{cases}$$

and adopt the normalization $\eta_1(\omega) = 1$, so that we can write $\eta_2(\omega) = \eta(\omega)$, for all ω. The quantities $\eta(\omega)$ can be interpreted as the weights of Mr. 2's utilities in an auxiliary social planner's problem, a problem defined in a broad sense which makes it convenient for describing equilibrium allocations even if they are not Pareto optimal.[2] Pareto optimality of the equilibrium allocation requires that $\eta(\omega)$ be a constant weight across all ω. Substituting these weights into equation (6) above, now represented in matrix form, we arrive at the following condition that Ms. 1's equilibrium portfolio

[2] Consider the programming problem of maximizing a fictitious representative agent's utility of the form

$$\sum_h \left[\eta_h(0)[a_h \log c_h^1(0) + (1-a_h)\log c_h^2(0)] + \beta\sum_{\omega>0}\pi(\omega)\eta_h(\omega)[a_h \log c_h^1(\omega) + (1-a_h)\log c_h^2(\omega)]\right]$$

subject to the resource constraints, $c_1^g(\omega) + c_2^g(\omega) = \delta^g(\omega)$, for all g, all ω. This utility assigns weights $\eta_h(\omega)$, $\omega = 0,\ u,\ d$, to the households. The weight of a household may take different values across states ω, i.e., formally, it is modeled as a stochastic process. The sharing rules derived from this problem coincide with the consumption allocation arising in competitive equilibrium. As evident from the first-order conditions (5), and with our normalization of weights in mind, the following must be true for the relative weight of Mr. 2 in the representative agent's utility function:

$$\eta(\omega) = \lambda_1(\omega)/\lambda_2(\omega).$$

holdings $(s_1^1(1),\, s_1^2(1))$ must satisfy:

$$\begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} = \begin{bmatrix} a_1 + a_2\eta(u) & 1 - a_1 + (1-a_2)\eta(u) \\ \\ a_1 + a_2\eta(d) & 1 - a_1 + (1-a_2)\eta(d) \end{bmatrix} \begin{pmatrix} s_1^1(1) \\ \\ s_1^2(1) \end{pmatrix}. \tag{7}$$

The 2×2 matrix above is nothing else but the payoff matrix P in equilibrium. The determinant of this matrix P is given by

$$\det(P) = (a_1 - a_2)(\eta(d) - \eta(u)). \tag{8}$$

So, clearly, when the stochastic weights are identical across future states $\eta(u) = \eta(d)$, the payoff matrix is not invertible. Hence, one of the stocks is redundant, and the portfolio holdings of households in each stock are indeterminate. Note that the condition $\eta(u) = \eta(d)$ is equivalent to $\lambda_1(\omega) = \lambda_2(\omega)$, $\omega = u, d$, i.e., identical Lagrange multipliers across households or, equivalently, colinear marginal utilities in future states. Hence, the matrix being invertible, $\eta(u) \neq \eta(d)$, is only possible in an inefficient equilibrium. Finally, assuming that P is invertible in equilibrium, we may explicitly solve for Ms. 1's equilibrium portfolio holdings from (7). For completeness, in the following result we also report the holdings of Mr. 2, which follow from stock market clearing (3).

Result 1. *In any inefficient equilibrium $\eta(u) \neq \eta(d)$, and optimal portfolio holdings of the households are given by*

$$s_1^1(1) = \frac{1-a_2}{a_1 - a_2}, \qquad s_1^2(1) = -\frac{a_2}{a_1 - a_2}, \text{ and} \tag{9}$$

$$s_2^1(1) = -\frac{1-a_1}{a_1 - a_2}, \qquad s_2^2(1) = \frac{a_1}{a_1 - a_2}. \tag{10}$$

In any efficient equilibrium $\eta(u) = \eta(d)(= k)$, and there is a continuum of optimal portfolio holdings of the form

$$s_1^1(1) = \frac{1 - (1 - a_1 + (1-a_2)k)s_1^2(1)}{a_1 + a_2 k}, \quad s_2^1(1) = \frac{k - (1 - a_1 + (1-a_2)k)s_2^2(1)}{a_1 + a_2 k},$$

one of these portfolios is also (9)–(10).[3]

The simplicity of the optimal portfolio holdings that obtain in any inefficient equilibrium is quite striking. They depend on neither the relative weights of the households, nor on the state of the economy. This property is very general and is maintained in a variety of generalized economic

[3]In our model $\eta(u) = \eta(d)$ implies that $\eta(0) = \eta(u) = \eta(d)$, which has to be satisfied for an equilibrium to be Pareto efficient. This can be seen, for example, from equation (A.2) in Appendix A.

settings (Sections 3–4). In particular, the portfolio holdings continue to be of the form (9)–(10) in a multi-period extension of our model.

The derivation above also highlights a curious structure of the system of equilibrium equations, which we report fully in Appendix A. Surprisingly, the derivation of the households' portfolio holdings does not employ a number of the equations belonging to this system, such as, for example, the period-0 first-order conditions, the no-arbitrage conditions, or the constraint on portfolio holdings. The methodology for analyzing this economy developed in Appendix A takes full advantage of this separability property, making the local analysis of the equilibrium equations particularly straightforward.

Finally, we remark that Result 1 characterizes equilibrium in which the portfolio constraint is binding (in a nondegenerate way). Of course, not all portfolio constraints would entail the existence of an inefficient equilibrium. For example, it is clear from this result that any *exogenous* portfolio constraint on the number of units of investment in some stock—e.g., $s_2^2(1) \geq \gamma$, with a constant parameter γ, as in Cass and Pavlova (2004), Appendix B—is going to lead to constrained equilibrium on just a measure-zero set of possible parameters. Hence, we consider below an *endogenous* portfolio constraint, that is, we make the right-hand side of the portfolio constraint, $f(\cdot)$ depend on the endogenous variables of the model.

2.3. Multiplicity of Equilibrium with a Specific Portfolio Constraint

We now focus on a specific example of a portfolio constraint faced by Mr. 2, facilitating fairly simple analysis of the equilibria in the economy. In Appendix A, we present the more comprehensive analysis for an abstract constraint. The analysis there is self-contained: a reader wishing to see a more general analysis than the one presented below may skip directly to that appendix.

We consider a constraint imposed on the fraction of wealth Mr. 2 is permitted to invest in the second stock:

$$q^2(0)s_2^2(1) \geq \gamma W_2(0), \tag{11}$$

where γ is a prespecified constant, and Mr. 2's initial wealth is defined as the value of his portfolio $W_2(0) = (q^1(0) + p^1(0)\delta^1(0))s_2^1(0) + (q^2(0) + p^2(0)\delta^2(0))s_2^2(0)$. Such a minimum investment restriction or concentration constraint is commonplace among institutional investors, pension funds and mutual funds, for which there may be a mandate to maintain a certain proportion of a portfolio in an asset class given the stated investment objective.[4]

Given our financial market friction, at the outset, an inefficient equilibrium is to be expected. This conjecture is based on the related familiar result that Pareto inefficiency obtains generically in economies with incomplete asset markets and multiple goods in GFE (Duffie and Shafer (1985), Geanakoplos and Polemarchakis (1986), and Geanakoplos, Magill, Quinzii and Dreze (1990)). Moreover, inefficiency of equilibria in security market economies with portfolio constraints and a single good has long been highlighted in the asset pricing literature (e.g., Detemple and Murthy (1997), and Basak and Cuoco (1998)). However, in economies with several goods, many standard asset pricing results involving portfolio constraints may not necessarily go through. This is due to there being additional markets in which investors can trade—spot goods markets. The possibility of trade in these markets has a propensity to alleviate portfolio constraints. Nor do many standard findings of the GFE literature survive in a Lucas-type economy like ours, where endowments are specified in terms of shares of securities. Indeed, in contrast to standard results in GFE, Cass and Pavlova (2004) demonstrate that an efficient equilibrium always exists in a general economic setting with log-linear preferences and a large set of portfolio constraints on the number of shares of a risky stock even with incomplete markets.[5] Result 2 demonstrates that in our setting, we actually get both types of equilibrium: efficient (E), in which the portfolio constraint is completely alleviated, and inefficient (I), in which the portfolio constraint is binding.

Result 2. *For $\gamma \in (\underline{\gamma},\ \overline{\gamma})$, where $0 < \underline{\gamma} < \overline{\gamma}$ are defined in Appendix A, there are multiple equilibria in the economy, falling into two distinct types: type E (efficient) and type I (inefficient).*

For $\gamma \notin (\underline{\gamma},\ \overline{\gamma})$, equilibrium is generically unique and belongs to type E.[6]

In the absence of portfolio constraints, there is a unique equilibrium allocation in our model.

[4]For example, many Fidelity funds (www.fidelity.com) explicitly state such a minimum investment restriction in describing their investment strategy. In addition to the obvious select and international funds, many others in different categories and styles incorporate such a restriction. For instance, Blue Chip Growth ($22 billion assets, large growth) invests at least 80% of its assets in "blue chip" companies, Disciplined Equity ($4.2 billion assets, large blend) 80% of its assets in equity securities, Low-Priced Stock ($30 billion assets, small blend) 80% of its assets in securities with price at or below $35 per share.

[5]Although the Lucas's tree model employed here and in Cass and Pavlova is a special case of the real assets model in GFE, it is a very special kind. Endowments in the Lucas-tree model are specified in terms of shares of assets, not goods, and hence constitute a measure-zero subset of the commodity endowments examined in the real assets model. Many proofs, which involve perturbation arguments, then simply do not go through, and many implications get reversed.

[6]Of course, as discussed in Section 2.2, portfolio holdings are not unique. But this is an insignificant nonuniqueness stemming simply from the presence of redundant financial assets. More importantly, there exists a set of parameter values in our model for which there exists a continuum of type E equilibria (see Proposition A.2 in Appendix A). However, this property is non-generic, occurring on a measure zero set of initial stock holdings (and hence commodity endowments).

This result is not new, and is to be expected in a model with log-linear preferences. For future reference, we denote quantities corresponding to this equilibrium with an asterisk *, and will sometimes refer to the ensuing allocation as a “good” equilibrium, reflecting the fact that it is Pareto efficient. The purpose of our analysis is to demonstrate that the introduction of portfolio constraints preserves the efficient equilibrium, but also introduces new, “bad” equilibria, which are Pareto inefficient and in which the constraint binds. Figure 1 sketches the positioning of the “bad” equilibria relative to the Pareto frontier in this economy; in anticipation of our results below, the figure depicts not just one, but two “bad” equilibria. Moreover, Result 2 demonstrates that the existence of such “bad” equilibria is a robust phenomenon occurring over a range of model parameter values.[7]

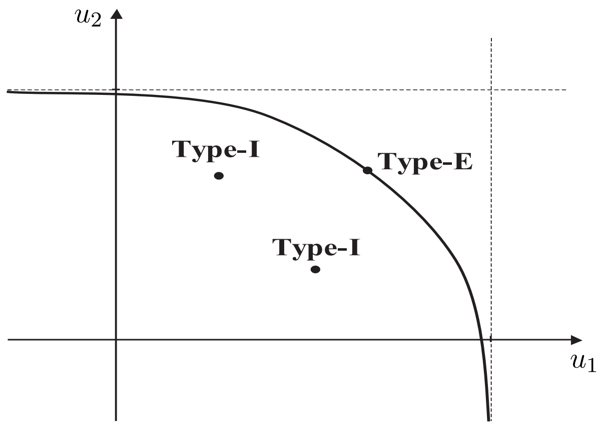

Figure 1: Utility Possibility Set. The solid curve depicts the utility possibility frontier.

All equilibria in this economy are rational expectations equilibria, and, provided that the parameter γ falls within a certain range, households' self-fulfilling expectations may give rise to either a “good” or a “bad” equilibrium. This observation could be quite important for understanding why many large movements in financial markets could not be linked to news about economic fundamentals. In our model, a change in fundamentals (e.g., household endowments) does entail a change in equilibrium prices, but so does a shift in the households' expectations, unrelated to fundamentals. Our baseline setting is not sufficiently rich to model a mechanism of the formation of such expectations, but we revisit this issue later in the paper (Section 3), when we introduce an additional period in which the households observe a sunspot that acts as a coordination device.

[7]For expositional reasons, we focus primarily on robustness with respect to one parameter of the model, the lower bound fraction γ, and highlight just one projection of the equilibrium correspondence. Alternatively, we could have fixed γ and explored the robustness with respect to other parameters, such as endowments or preference weights. Appendix A presents alternative conditions for existence of equilibria of type I, with the main message unaltered: the existence of an inefficient equilibrium is a robust property of our economy.

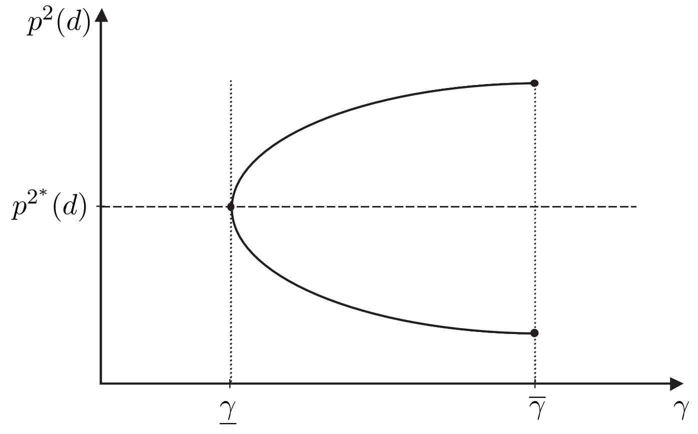

Figure 2: One Projection of the Equilibrium Correspondence. The C-shaped solid curve depicts the set of type-I equilibria. The dashed line is for the type-E equilibria.

To understand why there are multiple equilibria in our model, we find it useful to plot the equilibrium correspondence. In Figure 2, we vary the portfolio constraint parameter γ and plot the corresponding prices prevailing in equilibrium. To keep the plot two-dimensional, we report the state-d price of one of the commodities on the vertical axis. A figure with a state-u spot price is analogous. Figure 2 confirms that for any value of the parameter γ, there exists an efficient equilibrium with a corresponding price $p^{2^*}(d)$. In this equilibrium, the constraint does not bind, and hence the value of γ does not affect any of the equilibrium equations (more precisely, the system of equations reported in Appendix A, Section I). The point corresponding to $\gamma = \underline{\gamma}$ corresponds to the knife-edge case where the equilibrium is still of the efficient type, the multiplier μ on the constraint is zero, but the constraint holds with equality—the portfolio constraint starts to bind. To the right of this point, the multiplier on the constraint can be distinctly different from zero, and the equilibrium correspondence exhibits a C-shaped set of inefficient equilibria. Moreover, as evident from the figure, corresponding to each $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there are exactly two distinct equilibria of type I and one of type E—three in total.

Result 3 supplements Figure 2 and reports the complete characterization of inefficient equilibria in our economy, which can be easily computed in closed-form.[8]

Result 3. *In the inefficient equilibrium, the consumption allocations, spot good prices, and stock*

[8]The analysis of the equilibrium system of equations, which uncovers the fundamental structure of our economy and leads to the expressions in Result 3, is presented in Appendix A. However, one needs to bear in mind that the units in which consumption and prices are expressed in the appendix differ from the original units employed here. The primary reason for the transformation of units adopted in the appendix is that intrinsic uncertainty per se is essentially irrelevant to the fundamental structure of the system of equations defining equilibrium in our model.

price-dividend ratios are given by

$$c_1^1(\omega) = \frac{a_1}{a_1 + a_2\eta(\omega)}\delta^1(\omega), \qquad c_2^1(\omega) = \frac{a_2\eta(\omega)}{a_1 + a_2\eta(\omega)}\delta^1(\omega), \tag{12}$$

$$c_1^2(\omega) = \frac{1 - a_1}{1 - a_1 + a_2\eta(\omega)}\delta^2(\omega), \qquad c_2^2(\omega) = \frac{(1 - a_2)\eta(\omega)}{1 - a_1 + (1 - a_2)\eta(\omega)}\delta^2(\omega), \tag{13}$$

$$p^1(\omega) = \frac{a_1 + a_2\eta(\omega)}{\delta^1(\omega)}, \qquad p^2(\omega) = \frac{1 - a_1 + (1 - a_2)\eta(\omega)}{\delta^2(\omega)}, \qquad \omega = 0, u, d, \tag{14}$$

$$\frac{q^1(0)}{p^1(0)\delta^1(0)} = \frac{a_1 + a_2 E\left[\eta(\omega)\right]}{a_1 + a_2\eta(0)}, \qquad \frac{q^2(0)}{p^2(0)\delta^2(0)} = \frac{1 - a_1 + (1 - a_2)E\left[\eta(\omega)\right]}{1 - a_1 + (1 - a_2)\eta(0)} \tag{15}$$

where

$$\eta(0) = (1 + \beta)\left(1 - \frac{(a_1 - a_2)\gamma}{a_1(1 - a_2)}\right)\eta^* + \beta\frac{1 - a_1}{1 - a_2}, \tag{16}$$

with $\eta^* = \frac{a_1 s_2^1(0) + (1 - a_1)s_2^2(0)}{a_2 s_1^1(0) + (1 - a_2)s_1^2(0)}$*, and the period-1 relative weights* $(\eta(u), \eta(d))$ *solve*

$$E\left[\eta(\omega)\right] \equiv \pi(u)\eta(u) + \pi(d)\eta(d) = \left(1 + 1/\beta\right)\eta^* - \eta(0)/\beta, \tag{17}$$

$$E\left[1/\eta(\omega)\right] \equiv \frac{\pi(u)}{\eta(u)} + \frac{\pi(d)}{\eta(d)} = \frac{1}{\eta(0)} + \frac{1 + \beta}{\beta}\frac{a_2}{a_1}\left(\frac{\eta^*}{\eta(0)} - 1\right). \tag{18}$$

The efficient equilibrium is as in the unconstrained benchmark economy with constant relative weights, $\eta(0) = \eta(u) = \eta(d) = \eta^*$*, and*

$$\frac{q^1(0)}{p^1(0)\delta^1(0)} = \frac{q^2(0)}{p^2(0)\delta^2(0)} = 1. \tag{19}$$

Result 3 sheds light on the question as to why there are exactly two distinct inefficient equilibria (when they exist). Note that the equilibrium consumption allocations and spot good prices depend on the relative weight of Mr. 2, $\eta(\omega)$, which in an inefficient equilibrium varies across states ω. Moreover, the system of equations pinning down this weight, (17)–(18), is quadratic and, in the region of existence of type-I equilibria, always admits two positive solutions for $(\eta(u),\ \eta(d))$—hence the two inefficient equilibria.

Result 3 also reveals that there is variation in the stock prices and price-dividend ratios across type-E and type-I equilibria. In an efficient equilibrium, the price-dividend ratios do not vary with the states of the economy and equal unity. This is a standard result with log-linear utilities. In an inefficient equilibrium, however, they depend on two common factors, $\eta(0)$ and $\eta(\omega)$ which reflect the effects of the portfolio constraint faced by Mr. 2. This is in contrast to analogous results in the continuous-time asset pricing literature investigating the effects of portfolio constraints in the economies populated by investors with log-linear utilities. We further explore the dependence of the price-divided ratios on the fundamental, as well as non-fundamental states of the world in the remainder of the paper, and make a connection to the literature on excess volatility and market crashes.

3. Sunspots and Variability of Stock Prices

In the previous section, we establish that portfolio constraints can give rise to multiple rational expectations equilibria. However, nothing in our analysis suggests a mechanism through which the economy reaches a particular equilibrium. In this section, we focus on the issue of coordination of households' expectations and show how a good or a bad—in welfare terms—equilibrium may occur based purely on consistent beliefs.[9]

3.1. Sunspots and Equilibrium Indeterminacy

Towards this end, we extend the information structure in our leading example to encompass extrinsic uncertainty (or sunspots) along with intrinsic uncertainty. For simplicity, we assume that there are two extrinsic events $\sigma = G$, B ("**g**ood" and "**b**ad"). Hence, there are now four possible states of the world at $t = 1$, which we denote by $\tau = (\sigma, \omega)$, $\sigma = G, B$, $\omega = u, d$. In this context (because we have already assumed expected utility), there are two defining characteristics of the Cass and Shell (1983) concept of sunspots. First, intrinsic uncertainty is independent of extrinsic uncertainty, so that $\pi(\tau) = \pi(\sigma)\pi(\omega)$. Second, stock dividends are independent of extrinsic uncertainty, so that $\delta^g(G,\omega) = \delta^g(B,\omega) = \delta^g(\omega)$, $g = 1,\, 2$, $\omega = u,\, d$. In other words, sunspots affect neither preferences nor payoff streams: nonetheless, they may affect equilibrium.

We continue focusing on the specific portfolio constraint introduced in Section 2.3 (equation (11)). In the absence of the constraint, the equilibrium allocation is immune to sunspots.[10] This means that for all intrinsic states ω, $c_h^g(G,\, \omega) = c_h^g(B,\, \omega)$. The remaining equilibrium variables also exhibit no variation across extrinsic states G and B. Moreover, the resulting equilibrium is always of type E. As we have already established in the leading example with just intrinsic uncertainty, however, portfolio constraints may give rise to bad, type-I equilibria. Here we extend this argument a step further, and show that portfolio constraints can also give rise to equilibria in which sunspots matter (for short, sunspot equilibria).[11]

Similarly to Section 2.2, we begin the construction of such an equilibrium by examining the households' terminal portfolio holdings. The procedure for deriving these portfolios is the same as

[9] A reader interested in more detailed, rigorous analysis than the one offered in the body of the paper may directly skip to Appendices B and C, both of which are self-contained.

[10] This result is not new in the literature (see, for example, Cass and Pavlova (2004)).

[11] This result is also not unexpected: it is an instance of the "Philadelphia Pholk Theorem" (due to Karl Shell, in numerous personal communications).

in Section 2.2, except that now we need to consider four period-1 spot budget constraints for Ms. 1 (corresponding to each τ), instead of just the two (corresponding to each ω) in the leading example. Accordingly, the number of stochastic weights extends from 3 to 5—one for period 0 and one for each of 4 possible τ. The equation determining terminal portfolio holdings, which is an analog of (7), now takes the form

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} a_1 + a_2\eta(G,\, u) & 1 - a_1 + (1-a_2)\eta(G,\, u) \\ a_1 + a_2\eta(G,\, d) & 1 - a_1 + (1-a_2)\eta(G,\, d) \\ a_1 + a_2\eta(B,\, u) & 1 - a_1 + (1-a_2)\eta(B,\, u) \\ a_1 + a_2\eta(B,\, d) & 1 - a_1 + (1-a_2)\eta(B,\, d) \end{bmatrix} \begin{pmatrix} s_1^1(1) \\ \\ s_1^2(1) \end{pmatrix}. \tag{20}$$

It is easy to verify that the households' portfolio holdings reported in Result 1 are the portfolio holdings supporting equilibrium in the sunspot economy. This property is, in fact, quite notable: generally, a non-homogeneous linear system of 4 equations in 2 unknowns does not have a solution. In our case, even when the stochastic weights are different from each other, there is always a unique solution to (20): $s_1(1) = (\frac{1-a_2}{a_1-a_2},\ -\frac{a_2}{a_1-a_2})$.[12]

One way to think of the consequences of this observation for the ensuing equilibrium is to notice that the equilibrium equations in (20) now allow us to determine only 2 variables. This implies that the remaining system of equilibrium equations contains 2 fewer equations than there are unknowns. This observation is the key to the following result.

Result 4. *For $0 < \underline{\gamma} < \bar{\gamma}$, if $\gamma \in (\underline{\gamma}, \bar{\gamma})$, then there is a two-dimensional continuum of distinct sunspot equilibria.*

While the proof of Result 4 is somewhat involved (see Appendix B), the logic behind it is simple. In the leading example, we have shown that there are two distinct equilibria in which the portfolio constraint binds. This is due to our ability to substitute from the equilibrium equations for all variables except the stochastic weights, and then reduce them to a quadratic system of 2 equations (17)–(18) in 2 unknowns—the stochastic weights in the terminal period. The setting considered in this section preserves this structure, except that we now have 2 "degrees of freedom" in that system—2 additional stochastic weights in period 1. This gives rise to an infinite number of terminal stochastic weights consistent with equilibrium. This is what is known in the literature as real indeterminacy. Not only do portfolio constraints expand the set of equilibria, but in this

[12]The proof of this result parallels that of Proposition A.1 in Appendix A.

case they can also generate a two-dimensional continuum of sunspot equilibria. Note that all characterizations reported in Result 3 continue to hold in any sunspot equilibrium, with each intrinsic state ω now replaced by an intrinsic plus extrinsic state $\tau = (\sigma, \omega)$.

Remark 1. The proof that our economy exhibits equilibrium indeterminacy (Appendix B) is carried out for a specific financial market structure entailed by sunspots. However, as becomes clear from our proof, this indeterminacy result is primarily driven by asset market incompleteness, as there are now four states but only two stocks. The system of equations (20) would become a system of 4 equations in 4 unknowns if we introduced 2 additional assets. It would then not be possible to produce 2 additional degrees of freedom—2 unknowns that are not pinned down by the remaining equations of the equilibrium system. We thus may conjecture that our economy exhibits finite local uniqueness of equilibrium for the case of complete markets and indeterminacy when markets are incomplete. This result seems to be in contrast to Cass, Siconolfi, and Villanacci (2001), whose analysis shows that generic finite local uniqueness of equilibrium with portfolio constraints obtains independently of the degree of market incompleteness. The difference here is simply that trees give rise to household endowments which are nongeneric.

3.2. Variability of Stock Prices

Part of our motivation for this paper has been to develop an economic setting in which stock prices and price-dividend ratios can fluctuate without any change in the underlying economic fundamentals. The construction of sunspot equilibria above comes close to achieving this objective in the sense that for the same realization of a fundamental, say $\delta^g(u)$, there could be different equilibrium allocations, $c_h^g(\sigma, u)$, depending on the realization of the sunspot event $\sigma \in \{G, B\}$. However, the stock prices, which are determined in period 0, are the same across all sunspot equilibria. (In period 1, all (ex-dividend) stock prices are zero.) One simple way to create variation in stock prices is to have stocks carry value beyond period 1. In other words, it is sufficient to introduce one additional time period.

We return to our leading example with two intrinsically uncertain terminal states. But additionally, we incorporate an intermediate period in which no intrinsic uncertainty gets resolved, while there is a resolution of extrinsic uncertainty. In particular, in period 1 the households observe an extrinsic event $\sigma \in \{G, B\}$, occurring with probability $\pi(\sigma)$, and in period 2 an intrinsic event $\omega \in \{u, d\}$, occurring with probability $\pi(\omega)$. Again, intrinsic uncertainty is independent of extrinsic uncertainty so that $\pi(\sigma, \omega) = \pi(\sigma)\pi(\omega)$. This timeline is detailed in Figure 3.

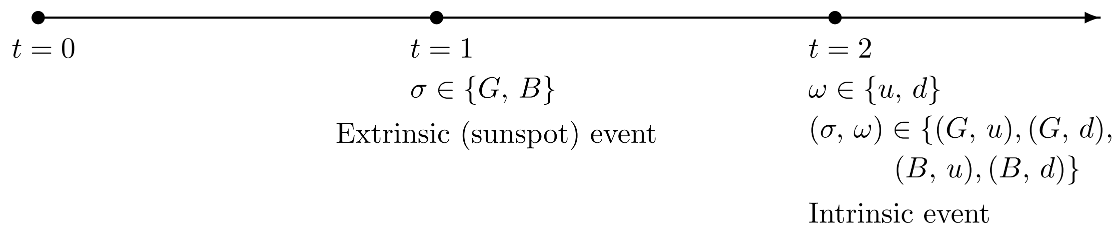

Figure 3: Timeline of the economy. No uncertainty about economic fundamentals is revealed at $t = 1$. Uncertainty about period-2 dividend is revealed at $t = 2$.

The dividends of the stocks in periods 0 and 1 are certain. A sunspot in period 1 does not reveal any information about the (uncertain) dividends in period 2. Households re-trade in period 1, after observing the sunspot state. Note that with the possibility of re-trade in period 1, the economy is back to the setting with (potentially) complete asset markets. The constraint on the portfolio holdings of Mr. 2 is now imposed in periods 0 and 1. That is, his portfolio holdings must satisfy

$$q^2(0)s_2^2(1) \geq \gamma W_2(0) \text{ and } q^2(\sigma)s_2^2(\sigma) \geq \gamma W_2(\sigma),\ \sigma = G, B, \tag{21}$$

where $W_2(0)$ is as defined in Section 2.3 and $W_2(\sigma) = (q^1(\sigma) + p^1(\sigma)\delta^1(\sigma))s_2^1(1) + (q^2(\sigma) + p^2(\sigma)\delta^2(\sigma))s_2^2(1),\ \sigma = G, B$.

The analysis in the leading example readily extends to this case of three time periods. In particular, households' portfolio holdings in any equilibrium in the economy continue to be of the form reported in Result 1. This means that when the portfolio constraint binds, the portfolios of the households are of the "buy-and-hold" type. We believe that this result is fairly general and, in particular, is true in an extension of this economy beyond three periods.

We can thus build on this result, just like we do in the leading example, to show existence of type-I equilibria in which portfolio constraints are binding. For expositional simplicity, we focus on supporting a sunspot equilibrium in which the portfolio constraint binds after the realization of the bad sunspot event and does not bind after the good sunspot. But before we proceed, we need to be clear about the terminology we adopt.

An equilibrium allocation is defined to be *conditionally Pareto efficient* if, conditional on the consumption/portfolio choices made prior to period 1, there in no feasible allocation in periods 1 and 2 that Pareto improves upon it. This weaker variant of Pareto efficiency allows us to capture the notion of an "efficient" equilibrium arising after the economy reaches a particular node in the date-event tree. The following result utilizes this definition and highlights the essence of sunspots-induced variability in consumption allocations and asset prices arising in the presence of the portfolio constraint.

Result 5. *For some* $0 < \underline{\gamma} < \underline{\gamma}^* < \bar{\gamma}^* < \bar{\gamma}$, *if* $\gamma \in (\underline{\gamma}^*, \bar{\gamma}^*)$, *then there is a sunspot equilibrium in which*

(i) consumption allocations, stock prices, and stock price-dividend ratios depend on the realization of a sunspot, that is

$$c_h^g(G) \neq c_h^g(B),\; q^g(G) \neq q^g(B),\; \text{and}\; \frac{q^g(G)}{p^g(G)\delta^g(G)} \neq \frac{q^g(B)}{p^g(B)\delta^g(B)},\quad g = 1, 2,\; h = 1, 2;$$

(ii) equilibrium allocation is conditionally Pareto efficient if $\sigma = G$*; and*

(iii) equilibrium allocation is conditionally inefficient if $\sigma = B$.

As we have established in the leading example, a portfolio constraint may give rise to multiple equilibria: these are of type E, where the constraint does not bind, and of type I, where it does. But which of them emerges as the equilibrium that households coordinate on? Sunspots here offer a natural coordination device. In the equilibrium presented in Result 5, a good, type-E equilibrium is selected after observing the good sunspot, and a bad, type-I equilibrium, after the bad. Of course, type-E is to be understood as being only conditionally Pareto efficient, since the resulting households utilities are in the interior of the (unconditional) utility possibility set (because the constraint is binding in period 0 and in period 1 if the bad sunspot occurs), but are on the conditional utility possibility frontier. On the other hand, in the states occurring after the bad sunspot the portfolio constraint binds and hence they lie in the interior of the conditional utility possibility set. Figure 4 depicts this situation.

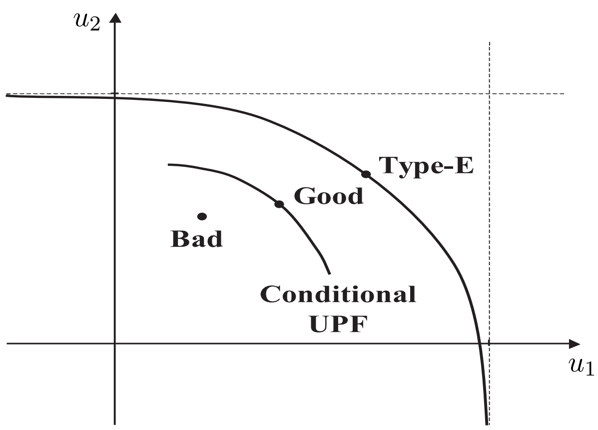

Figure 4: Utility Possibility Set with Sunspots. The solid curves depict the utility possibility frontier and the conditional utility possibility frontier. “Good” (“bad”) corresponds to the equilibrium occurring after a good (bad) sunspot realization.

Note that sunspots do not contain any information about current or future dividends, i.e., they reveal no information about economic fundamentals. However, they matter for equilibrium allocations and asset prices. In particular, stock prices are distinctly different across the sunspot realizations. It is then very easy to compute, within our model, the volatilities (standard deviations) of returns of stocks and price-dividend ratios for the economies with and without the portfolio constraint. The former are necessarily higher than those in the economy without the portfolio constraint. This result provides a simple alternative cause as the explanation for the puzzlingly high variability of stock returns that is typically attributed to irrational behavior of market participants, one which is not recognized in the Finance literature.

Empirical asset pricing literature has long tried to link the variation in prices of financial assets to news about underlying fundamentals (Shiller (1981), Roll (1984), French and Roll (1986), and subsequent research). A strand of this research that tries to identify the determinants of stock market crashes carries particular importance for policymaking. The main finding of this literature is what is known as the "excess volatility" puzzle: news about economic fundamentals explain only a very small fraction of the variation in asset prices. Especially striking is the conclusion reached by Cutler, Poterba and Summers (1993) who argue that most of the biggest post-war market moves were not associated with significant economic news or releases of other relevant information. For example, the famous October 1987 stock market crash was never linked to any significant announcement. Our model sheds light on this phenomenon. As long as market participants face portfolio constraints—and these are obviously widespread in real-life financial markets—there can be multiple equilibria, and in particular, multiple possible stock prices. The mechanism through which the economy reaches a particular financial equilibrium at any point in time may rely on sunspots. Since sunspots do not carry any information about economic fundamentals, the selection of a "bad" over a "good" equilibrium would be unrelated to intrinsic events (which occur only in the terminal period in our model). Result 5 demonstrates that such equilibrium outcomes are robust phenomena in our economic setting, occurring for a large range of parameter values.

4. Extensions

The economic environment in our leading example is, admittedly, very stylized. It is thus of interest to investigate the robustness of our results to richer economic settings. We first consider expanding the number of intrinsic states of the world and the number of investment opportunities.

In particular, in our leading example we introduce an additional state, m, so that now $\omega = u$, m, d, and an additional asset, a bond, in zero net supply, paying out one unit of good 1 in each state of the world. Again, we have as many assets as there are states of the world, and so markets are potentially complete. It turns out that equilibria in this economy are very similar to the ones uncovered in Section 2. First, the investors' portfolio holdings are unchanged. The holdings of the two stocks are exactly the same as those presented in Result 1, and no investor holds any shares of the bond. Second, since Result 1 is central to our argument that fleshes out the multiplicity of equilibria in which the portfolio constraint binds, our results on uniqueness of equilibrium readily extend. In particular, in the benchmark economy without portfolio constraints equilibrium allocation is unique and is Pareto efficient, while in the economy with the portfolio constraint there are two types of equilibria: of type E and of type I. The exact form of the equilibrium correspondence, however, remains to be completely worked out.

One of the benefits offered by a richer set of investment opportunities is that one can model a larger class of portfolio constraints. Many constraints imposed in practice involve more than one stock. For instance, margin requirements or collateral constraints allow investors to use bonds and stocks in lieu of collateral for short positions established in other assets belonging to the portfolio. Borrowing constraints act along similar lines. Additionally, the concentration constraint considered in Section 2 typically involves a number of stocks belonging to the same asset class (e.g., large stocks, small stocks, value stocks). Some of these constraints can be investigated in the three-asset extension of our model, while others may require increasing the number of stocks further. The structure of the problem, however, remains the same. As we demonstrate in Appendix A, the key is to analyze the system of equilibrium equations without the constraint, and then make use of the fact that all these constraints can be written parametrically, for example, as $f(q(0)s_2(1)/W_2(0)) \geq \gamma$, where q is a vector of stock prices, to trace a range of the parameter γ for which multiple equilibria may occur. Another avenue for investigation will be to consider a setting in which portfolio constraints are imposed on more than one household, a more realistic scenario because in practice few investors can claim to be free of portfolio constraints. To this day, little is known about the economic mechanism through which portfolio constraints affect stock returns and their correlations, although there is mounting empirical evidence suggesting that portfolio constraints do matter. Our approach is very flexible, and adopting it may prove useful for exploring these questions.

Furthermore, it would be interesting to consider a multi-period extension of our leading example. We believe that the bulk of our results extends to this setting, as is suggested by the analysis of the three-period extension presented in Section 3.2. In a multi-period or an infinite-horizon version of our model, one can meaningfully address questions related to the dynamics of asset returns (e.g., time-varying volatilities and momentum) and their comovement. However, one needs to first prove the existence of an equilibrium where portfolio constraints bind, and this remains an open question.

Also left for future comprehensive investigation is the role of incomplete asset markets. The real indeterminacy uncovered in Section 3.1, which we attributed to sunspots, is likely to be a general phenomenon occurring in settings with incomplete markets. It is apparent from the analysis in that section that if we add an (intrinsic) state of the world in the leading example without expanding the number of assets, there will be a continuum of equilibria. The robustness of this finding to other settings with incomplete markets, and in particular to models with other real assets (e.g., bonds) and portfolio constraints remains to be verified.

Finally, very important potential extension of our analysis here is to investigate our results on multiplicity of equilibrium under broader specifications of investors' utilities and endowments. In particular, it appears that, for the most part, our central results are robust to local perturbations of utility functions around the specific log-linear functions we employ here. But this conjecture remains to be fully, seriously investigated. We leave it for future research, which is being carried out in a separate project with Yves Balasko.

5. Concluding Remarks

In this paper, we attempt to shed some light on the potential role of portfolio constraints in generating multiplicity of rational expectations equilibrium. Towards this end, we develop a simple asset market economy with two periods, two states, and two goods. Two households with log-linear preferences and real asset endowments make consumption and investment decisions. This economy admits a unique Pareto efficient equilibrium. The introduction of a portfolio constraint on one of the households, however, may give rise to additional inefficient equilibria in which the constraint binds—hence, multiplicity of equilibrium. By introducing extrinsic states into the model, we demonstrate that portfolio constraints may also give rise to sunspot equilibria, and in fact lead to real indeterminacy. The extrinsic uncertainty affects real variables, while absent constraints extrinsic uncertainty plays no significant role in the economy.

Although we make simplifying assumptions on the primitives of the economy, our insights can readily be extended to feature richer asset market structures (to include zero net supply bonds) and richer uncertainty (with more than two intrinsic states), multi-period settings, and additional constraints. Natural generalizations of our analysis would be to increase the number of households or goods in the economy. It would be even more interesting to investigate the robustness of our analysis beyond the trees and logs framework. Our approach here is also well suited for the study of alternative market imperfections.

Appendix A

In this appendix we provide a comprehensive analysis of our leading example – where there are two periods, two states in the second period, two goods (and hence two stocks), and two households – and the second household is constrained in its holding of the second stock. Our aim is to make the appendix self-contained, and hence there is some minor overlap of the analysis here with that presented in the text (in particular, in the derivation of the result labeled Proposition A.1 below), where we want to emphasize and re-emphasize especially important structural features of our model. Before proceeding we need to explain our treatment of the portfolio constraint.

Besides assuming that only Mr. 2 is constrained in his holding of only stock 2, for simplicity we also assume that this constraint takes the form of an endogenous lower bound. Nothing in principle depends on either simplification. Let (the vectors) $p(0)$ and $q(0)$ represent spot goods and stock prices, respectively, in period 0. Also let

$$f : \mathbb{R}^4_{++} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \text{ s.t.}$$

$$((p(0), q(0)), s_2(0), \gamma) \mapsto f(p(0), q(0), s_2(0), \gamma),$$

where $\gamma \in \mathbb{R}$ is a parameter, represent the RHS of Mr. 2's portfolio constraint, so that he is restricted in his transactions in stock 2 by

$$s_2^2(1) \geqq f(\cdot). \qquad \begin{array}{c}\text{with multiplier}\\ \mu\end{array}$$

Such a constraint enters into the system of equations describing GFE in just two ways: First, a term μ is added to the second of the two equations which would have represented Mr. 2's lack of arbitrage possibilities were he not constrained; it will be convenient to still refer to these as no-arbitrage conditions. Second, the complementary slackness condition (CSC) associated with the constraint,

$$\min\{s_2^2(1) - f(\cdot), \mu\} = 0,$$

becomes another equation in the system. The critical feature of our approach to uncovering the structure of the solutions to the resulting system of equations is that, except for the fact that we do introduce the multiplier μ at the outset, we ignore the CSC until the very end of the analysis. It is only at that point where we worry about showing how it can be accommodated, essentially by judiciously specifying the dependence of f on γ – the final step in our development, what we will refer to as "tailoring" the constraint.

In general, nothing in the approach adopted here is necessarily limited to the specific constraint we use for illustrative purposes, or even the leading example analyzed in this appendix and its extensions analyzed in the two following appendices.

A.I. **The Extended Form Equations (EFE's)**

In order to economize on space and, later on, the number of equations we need to write down, we will switch (as context suggests useful) between notations $(\pi(\omega), \omega = u, d)$ or $(\pi, 1 - \pi)$, and $\alpha_h = (\alpha_h^1, \alpha_h^2)$ or $(a_h, 1 - a_h)$, $h = 1, 2$. Also, contrary to accepted practice in mathematics, we will find it very useful to treat $\alpha_h = (a_h, 1 - a_h)$ and other price-like vectors (i.e., $p(\omega)$ and $q(\omega)$) as row vectors. All quantity-like vectors, such as $c_h(\omega)$ and $s_h(\omega)$, are column vectors.

The basic system of EFE's (again, excluding the CSC) consists of the usual first-order conditions (FOC's), no-arbitrage conditions (NAC's), and spot budget constraints (SBC's) for both households, together with the market clearing conditions (MCC's) for goods and stocks. We omit qualifiers (e.g., "$g = 1, 2$" or "$\omega \neq 0$"), since they are self-evident.

FOC's $\quad \alpha_h^g / c_h^g(0) = \lambda_h(0) p^g(0)$ and

$$\pi(\omega)\beta\alpha_h^g / c_h^g(\omega) = \lambda_h(\omega) p^g(\omega)$$

NAC's $\quad -\lambda_1(0)q(0) + \sum_{\omega\neq 0} \lambda_1(\omega)\left(p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega)\right) = 0$ and

$$-\lambda_2(0)q(0) + \sum_{\omega\neq 0} \lambda_2(\omega)\left(p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega)\right) + (0, \mu) = 0$$

SBC's $\quad p(0)c_h(0) = [q(0) + (p^1(0)\delta^1(0), p^2(0)\delta^2(0))]s_h(0) - q(0)s_h(1)$ and

$$p(\omega)c_h(\omega) = (p^1(\omega)\delta^1(\omega), p^2(\omega)\delta^2(\omega))s_h(1)$$

MCC's $\quad c_1^g(\omega) + c_2^g(\omega) = \delta^g(\omega)$ and

$$s_1^g(1) + s_2^g(1) = 1$$

But for μ and $s_h(1)$ (both unsigned at this point), all variables are strictly positive.

A.II. **The Reduced Form Equations (RFE's)**

The stochastic weights

$$\eta_h(\omega) = \begin{cases} \beta/\lambda_h(0),\ \omega = 0 \\ 1/\lambda_h(\omega),\ \omega \neq 0 \end{cases}$$

(which we find a more convenient representation than the Lagrange multipliers $\lambda_h(\omega)$) play a central role in our analysis. Since this is a model with real assets (stock payoffs are specified in terms of goods), and there are three spots, there are also three possible price normalizations. We find it extremely useful, at the outset, to choose these as $\eta_1(\omega) = 1$, all ω, so that we can then write $\eta_2(\omega) = \eta(\omega)$, all ω. Given this normalization, we will show that the EFE's can effectively be represented by just 3 RFE's

$$-(1+1/\beta)\alpha_1 s_2(0) + [(\eta(0)/\beta) + (\pi\eta(u) + (1-\pi)\eta(d))]\alpha_2 s_1(0) = 0, \tag{A.1}$$

$$\pi(1-\eta(0)/\eta(u))(a_1 + a_2\eta(u)) + (1-\pi)(1-\eta(0)/\eta(d))(a_1 + a_2\eta(d)) = 0, \text{ and} \tag{A.2}$$

$$\begin{aligned} \pi(1-\eta(0)/\eta(u))[(1-a_1) + (1-a_2)\eta(u)]+ \\ (1-\pi)(1-\eta(0)/\eta(d))[(1-a_1) + (1-a_2)]\eta(d) - \mu = 0 \end{aligned} \tag{A.3}$$

in the 4 variables $(\eta, \mu) = ((\eta(\omega), \text{ all } \omega), \mu)$. This is most clearly demonstrated in a series of steps by which we systematically suppress superfluous (from an analytic viewpoint) variables. All this will, of course, depend heavily on the trees and logs structure of the model – though we believe that, for the most part, our central results are robust to local perturbations of utility functions around the specific log-linear functions we employ here. But this conjecture remains to be fully, seriously investigated. The reader who is willing to take on faith the manipulations required in going from the EFE's to the RFE's should skip ahead to the following section, referring back to Proposition A.1 as needed.

Step 1. From inspection of the EFE's it is obvious that the dividends $\delta = ((\delta^g(\omega), g = 1, 2), \text{ all } \omega)$ can be suppressed by a transformation of units of goods to units per dividend (with a corresponding transformation of units of account), so that, say,

$$c_h^g(\omega)/\delta^g(\omega) \mapsto \tilde{c}_h^g(\omega) \text{ and } p^g(\omega)\delta^g(\omega) \mapsto \tilde{p}^g(\omega).$$

Because it is very cumbersome to carry the tildes throughout the analysis in the three appendices, hereafter we will rewrite $\tilde{c} = ((\tilde{c}_h^g(\omega), g = 1, 2, \text{ all } \omega), h = 1, 2)$ and $\tilde{p} = (\tilde{p}^g(\omega), g = 1, 2, \text{ all } \omega)$ as simply c and p. **The reader should bear in mind that for the purposes of our analysis (in the appendices) we effectively assume that $\delta^g(\omega) = 1$, while for the purposes of our interpretation (in the text) we assume that, typically, there is intrinsic uncertainty, that is, that $\delta^g(\omega) > 0$ varies with both g and ω.** (When we introduce extrinsic uncertainty or sunspots in the following two appendices this assumption will be modified accordingly.) Thus, for example, $p(u) = p(d)$ here means that, in the original units of account, spot goods prices adjust to completely offset dividend uncertainty.

Step 2. Rewriting the FOC's to highlight the determination of c (bearing in mind that now $\eta_1(\omega) = 1$ and $\eta_2(\omega) = \eta(\omega)$, all ω),

$$c_h^g(\omega) = \begin{cases} (\alpha_h^g/\beta)\eta_h(0)/p^g(0), \ \omega = 0, \\ \\ \pi(\omega)\alpha_h^g\eta_h(\omega)/p^g(\omega), \ \omega \neq 0, \end{cases} \tag{A.4}$$

summing over h and using the MCC's for goods, and then solving for p yields

$$p(\omega) = \begin{cases} (\alpha_1/\beta) + (\alpha_2/\beta)\eta(\omega), \ \omega = 0. \\ \\ (\pi(\omega)(\alpha_1 + \alpha_2\eta(\omega)), \ \omega \neq 0. \end{cases} \tag{A.5}$$

This entails, first, that (A.5) can be used to solve for p in terms of η, and second, that substituting from (A.5) into (A.4), the resulting equations, for instance,

$$c_1^g(\omega) = \begin{cases} (\alpha_1^g/\beta)/[(\alpha_1^g/\beta) + (\alpha_2^g/\beta)\eta(\omega)],\ \omega = 0, \\ \\ (\pi(\omega)\alpha_1^g)/(\alpha_1^g + \alpha_2^g\eta(\omega)),\ \omega \neq 0, \end{cases} \tag{A.6}$$

can be used to solve for c in terms of η, so that both p and c can effectively be ignored (in the following analysis).

Step 3. Using spot-by-spot analogues of Walras' law, Mr. 2's SBC's are redundant, and can just be discarded. And then the MCC's for stocks can be used to solve for $s_2(1)$ terms of $s_1(1)$

$$s_2(1) = \mathbf{1} - s_1(1), \tag{A.7}$$

so that these variables can also effectively be ignored.

Step 4. Adding together her SBC's weighted by

$$\lambda_1(\omega) = \begin{cases} \beta/\eta_1(0) = \beta,\ \omega = 0, \\ \\ 1/\eta_1(\omega) = 1,\ \omega \neq 0 \end{cases}$$

and using her NAC's, after some simplifying and rearranging, Ms. 1's SBC at $\omega = 0$ can be replaced by the overall (Walrasian-like) BC

$$\sum_\omega p(\omega)c_1(\omega) - (p(0) + p(u) + p(d))s_1(0) = 0.$$

But substituting from (A.6) for $p(\omega)c_1(\omega)$, all ω, and from (A.5) for p, then simplifying (in particular, using the fact that $s_1(0) + s_2(0) = \mathbf{1}$ and (A.7) to rewrite $1 - \alpha_1 s_1(1) = \alpha_1 s_2(1)$, and then multiplying the resulting equation by -1) this overall BC becomes the first of the RFE's (A.1).

At this point we're only left to deal with Ms. 1's SBC's at $\omega = u, d$ and the NAC's.

Step 5. Now substituting from (A.6) for just $p(\omega)c_1(\omega)$, $\omega = u, d$, and from (A.5) for just $p(\omega), \omega = u, d$, Ms. 1's SBC's at $\omega = u, d$ become, after some simplifying,

$$(\alpha_1 + \alpha_2\eta(\omega))s_1(1) = 1,\ \ \omega = u, d. \tag{A.8}$$

These linear equations can be solved for $s_1(1)$ in terms of η, so these variables can also effectively be ignored – that is, ignored until we need them later on (via (A.7)) in order to tailor the portfolio constraint. And their solution(s) have some very special properties which we fully exploit at that point, so may just as well document formally at this. **Note that here and after we will disregard the uninteresting borderline case where $a_1 = a_2$, and concentrate on the case**

where $a_1 > a_2$. (We'll comment on the opposite case, where $a_1 < a_2$, at the very end of the appendix.)

Proposition A.1. *If* $(\eta(u), \eta(d)) \gg 0$ *and* $\eta(u) \neq \eta(d)$*, then (A.8) has the (same) unique solution*

$$s_1(1) = ((1 - a_2)/(a_1 - a_2), -a_2/(a_1 - a_2)) \tag{A.9}$$

so that, from (A.7),

$$s_2(1) = (-(1 - a_1)/(a_1 - a_2), a_1/(a_1 - a_2)). \tag{A.10}$$

If $\eta(u) = \eta(d) = k > 0$*, then (A.8) has the continuum of solutions*

$$s_1^1(1) = [1 - ((1 - a_1) + (1 - a_2)k)s_1^2(1)]/[a_1 + a_2 k], \tag{A.11}$$

one of which is also (A.9).

Proof of Proposition A.1. Case 1. $(\eta(u), \eta(d)) \gg 0$ and $\eta(u) \neq \eta(d)$.

The equations (A.8) have the form $Ax = \mathbf{1}$, with

$$A = \begin{bmatrix} a_1 + a_2\eta(u) & (1 - a_1) + (1 - a_2)\eta(u) \\ a_1 + a_2\eta(d) & (1 - a_1) + (1 - a_2)\eta(d) \end{bmatrix}.$$

Straightforward calculation shows that

$$\det(A) = (a_1 - a_2)(\eta(d) - \eta(u)) \neq 0,$$

and hence, that

$$\begin{aligned} s_1(1) = A^{-1}\mathbf{1} &= 1/(a_1 - a_2)(\eta(d) - \eta(u)) \times \\ & \begin{bmatrix} (1 - a_1) + (1 - a_1)\eta(d) & -(1 - a_1) - (1 - a_2)\eta(u) \\ -a_1 - a_2\eta(d) & a_1 + a_2\eta(u) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= ((1 - a_2)/(a_1 - a_2), -a_2/(a_1 - a_2)). \end{aligned}$$

Case 2. $\eta(u) = \eta(d) = k > 0$.

(A.11) is obvious (since in this case either of the identical equations in (A.8) yields this continuum of solutions), as is the fact that (A.9) satisfies (A.11) (since the unique solution whenever the equations (A.8) are not colinear must also be a solution when they are).■

Step 6. Finally, Ms. 1's NAC's can be used, again in conjunction with (A.5), to yield (here we begin to utilize the notation $(\pi, 1 - \pi)$ for probability in earnest)

$$\begin{aligned} q(0) &= \pi(\alpha_1 + \alpha_2\eta(u)) + (1 - \pi)(\alpha_1 + \alpha_2\eta(d)) \\ &= \alpha_1 + \alpha_2 y, \end{aligned} \tag{A.12}$$

where we will find it very useful later on to have in hand the auxiliary (expected value) variable $y = \pi\eta(u) + (1-\pi)\eta(d)$. Thus, (i) (A.12) can be used to solve for $q(0)$ in terms of η, so $q(0)$ can effectively be ignored as well, while (ii) the NAC's themselves can be replaced by their difference (Ms. 1's less Mr. 2's). This yields the two remaining RFE's (A.2)-(A.3), and the reduction of the EFE's to the RFE's is complete.

A.III. **Analysis of the RFE's**

Let $\xi = (\eta(0), \eta(u), \mu) \in \Xi = \mathbb{R}^2_{++} \times \mathbb{R}$, $\theta = \eta(d) \in \Theta = \mathbb{R}_{++}$, and $\Phi : \Xi \times \Theta \to \mathbb{R}^3$, the C^∞ mapping s.t. (just reproducing the LHS's of (A.1)-(A.3))

$$(\xi, \theta) \mapsto \Phi(\xi, \theta) =$$

$$\begin{pmatrix} -(1+1/\beta)\alpha_1 s_2(0) + [(\eta(0)/\beta) + (\pi\eta(u) + (1-\pi)\eta(d))]\alpha_2 s_1(0) \\ \pi(1-\eta(0)/\eta(u))(a_1 + a_2\eta(u)) + \\ \qquad (1-\pi)(1-\eta(0)/\eta(d))(a_1 + a_2\eta(d)) \\ \pi(1-\eta(0)/\eta(u))[(1-a_1) + (1-a_2)\eta(u)] + \\ \qquad (1-\pi)(1-\eta(0)/\eta(d))[(1-a_1) + (1-a_2)\eta(d)] - \mu \end{pmatrix}. \tag{A.13}$$

A basic result about this system of equations concerns the existence of solutions when $\mu = 0$, i.e., the existence of equilibria when the portfolio constraint is not binding.

Proposition A.2. *If*

$$\begin{aligned} &\alpha_2 s_1(0) \neq 0 \text{ and} \\ &\eta^* = \alpha_1 s_2(0)/\alpha_2 s_1(0) > 0, \end{aligned} \tag{A.14}$$

then $\Phi(\xi, \theta)_{\mu=0} = 0$ *has the unique positive solution* $\eta(\omega) = \eta^*$, for all ω.

Proof of Proposition A.2. Consider first a candidate solution where $\eta(\omega) = k$, for all ω, with $k \neq 0$ and $\mu = 0$, (A.1) becomes, after simplifying,

$$\alpha_1 s_2(0) - \alpha_2 s_1(0) k = 0, \tag{A.15}$$

while (A.2)-(A.3) are identically satisfied. (A.14) entails that $k = \eta^*$ is the unique solution to (A.15), that is, that $\eta(\omega) = \eta^*$ is the unique positive solution to $\Phi(\xi, \theta)_{\mu=0} = 0$ of this form. What remains to be shown is that no other positive process $\eta(\omega)$, possibly varying across ω, can be a solution to $\Phi(\xi, \theta)_{\mu=0} = 0$. This follows from Proposition 4 in Cass and Pavlova (2004).■

Remarks. 1. Proposition A.2 means that, under the hypothesis, there is a unique equilibrium in which the portfolio constraint is not binding. Consider the other possible sign configurations of $\alpha_2 s_1(0)$ and $\alpha_1 s_2(0)$. On the one hand, if $\alpha_2 s_1(0) = \alpha_1 s_2(0) = 0$, then (A.15) has a continuum of positive solutions – and there is a continuum of equilibria in which the portfolio constraint is not binding. On the other hand, if $\alpha_2 s_1(0) \neq 0$ and $\alpha_1 s_2(0)/\alpha_2 s_1(0) \leqq 0$, or $\alpha_2 s_1(0) = 0$ and $\alpha_1 s_2(0) \neq 0$, then (A.15) has no positive solution – and (again in light of Proposition 4 in Cass and Pavlova) there is no equilibrium in which the portfolio constraint is not binding. (These two observations establish that (A.14) is also a necessary condition that $\Phi(\xi,\theta)_{\mu=0} = 0$ has a unique positive solution, i.e., that there is a unique equilibrium in which the portfolio constraint is not binding.) **Because the existence of $\eta^* > 0$ is so central to our analysis of the solutions to (A.1)-(A.3) (to say nothing of interpretive motivation) here and after we assume that (A.14) obtains.**

2. (A.1) is simply a very economical form of Ms. 1's overall BC. In fact, this entails that (A.14) is a sufficient condition that her initial wealth

$$W_1(0) = (q(0) + p(0))s_1(0) = (p(0) + p(u) + p(d))s_1(0)$$

(using her NAC to get the second, symmetric representation) be positive. In order that the solutions to (A.1)-(A.3) correspond to equilibria, Mr. 2's initial wealth

$$W_2(0) = (p(0) + p(u) + p(d))s_2(0)$$

must also be positive. It turns out that – as must be the case, obviously – (A.14) is also a sufficient condition for this to be true, as we demonstrate explicitly in the course of tailoring our specific portfolio constraint in section A.4.

The ultimate basis for establishing robustness of various properties of the equilibria for our leading example and, to some extent, its extensions, is analysis of the local behavior of the solutions to the equation $\Phi(\xi,\theta) = 0$ around the particular solution η^* (as we will call the point $(\xi,\theta) = (\eta^*,\eta^*,0,\eta^*)$). Thus, it is a pure bonus that, because of specific structure peculiar to just this minimal model, we are also able to describe their global behavior. This results first, from a rank property of $D\Phi$ (on its whole domain), and second, from explicit calculation of $D_\theta \xi \Big|_{\Phi(\xi,\theta)=0}$ using the Implicit Function Theorem.

Proposition A3. Rank $D_\xi \Phi(\xi,\theta) = 3$.

Proof of Proposition A.3. To begin with, we calculate

$D_\xi \Phi(\xi,\theta) =$

$$\begin{array}{cc}
\text{eq.\textbackslash var.} & \begin{array}{ccc} \eta(0) & \eta(u) & \mu \end{array} \\
\begin{array}{c} (A.1) \\ \\ (A.2) \\ \\ (A.3) \end{array} & \left[\begin{array}{ccc}
\alpha_2 s_1(0)/\beta & \pi\alpha_2 s_1(\beta) & 0 \\
 & & \\
-[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2] & \pi[a_2+a_1\eta(0)/\eta(u)^2] & 0 \\
 & & \\
-[(1-a_1)(\pi/\eta(u)+(1-\pi)/\eta(d))+1-a_2] & \begin{array}{c}\pi[(1-a_2)+ \\ (1-a_1)\eta(0)/\eta(u)^2]\end{array} & -1
\end{array}\right].
\end{array}$$

Then, for $v=(v^1,v^2,v^3)\in\mathbb{R}^3$, and simplifying (using $\alpha_2 s_1(0)\neq 0$ and $\pi>0$), suppose that

$v^T D_\xi \Phi(\xi,\theta) =$

$$\left(\begin{array}{l}
(1/\beta)v^1-[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2]v^2- \\
\qquad\qquad [(1-a_1)(\pi/\eta(u)+(1-\pi)/\eta(d))+(1-a_2)]v^3 \\
 \\
v^1+[a_2+a_1\eta(0)/\eta(u)^2]v^2+[(1-a_2)+(1-a_1)\eta(0)/\eta(u)^2]v^3 \\
-v^3
\end{array}\right)=0.$$

¿From the third equation, $v^3=0$, which, from the first two equations, implies that

$$\left[\begin{array}{ll} (1/\beta) & -[a_1(\pi/\eta(u)+(1-\pi)/\eta(d))+a_2] \\ 1 & a_2+a_1\eta(0)/\eta(u)^2 \end{array}\right]\left(\begin{array}{c} v^1 \\ v^2 \end{array}\right)=0,$$

which in turn (since, in general, $\det\left(\left[\begin{array}{cc} + & - \\ + & + \end{array}\right]\right)>0$) implies that $v^1=v^2=0$. Summarizing, $v^T D_\xi\Phi(\xi,\theta)=0$ implies that $v=0$, i.e., that rank $D_\xi\Phi(\xi,\theta)=3$.■

By utilizing Proposition A.3 we obtain a complete characterization of the qualitative properties of the solutions to (A.1)-(A.3).

Proposition A.4 *The derivative of ξ and the auxiliary variable y with respect to θ have the specific form*

$$D\xi\Big|_{\Phi(\xi,\theta)=0} = -D_\xi\Phi(\xi,\theta)^{-1}\Big|_{\Phi(\xi,\theta)=0} D_\theta\Phi(\xi,\theta)\Big|_{\Phi(\xi,\theta)=0} =$$

$$\left(\begin{array}{l} D_{\eta(d)}\eta(0) \\ D_{\eta(d)}\eta(u) \\ D_{\eta(d)}\mu \end{array}\right)=\left(\begin{array}{l} A(\eta(u)-\eta(d)) \\ -B(1-\pi)\eta(u)/\pi\eta(d) \\ C'(\eta(d)-\eta(u)) \end{array}\right) \textit{ and}$$

$$D_{\eta(d)}y=C''(\eta(d)-\eta(u)),$$

where A, B, C' and C'' are strictly positive (and themselves depend on η).

Proof of Proposition A.4. In principal this task is routine, in practice a (calculational) nightmare, so we turned it over to Mathematica v. 5.1. A copy of our program is available on request.■

The key implications of this result are that (i) $\eta(u)$ is strictly decreasing, while (ii) $-\eta(0)$, y, and μ are strictly decreasing (resp. strictly increasing) for $\eta(d) \leqq \eta^*$ (resp. $\eta(d) \geqq \eta^*$). We stress, especially, that the last means that

$$\mu \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0 \text{ according as } \eta(d) \left\{ \begin{array}{c} \neq \\ = \end{array} \right\} \eta^*.$$

To complete the picture all that is required is to determine the range of $\eta(d)$, and the limit values for $\eta(0)$, $\eta(u)$, and y at the boundaries of its range. Let $\underline{y} = \eta^* < (1 + 1/\beta))\eta^* = \bar{y}$.

Proposition A.5. *For* every $\eta(d) \in (0, \bar{y}/(1-\pi))$, *the equation* $\Phi(\xi, \theta) = 0$ *has a solution. At the endpoints of its range,*

$$\lim_{\eta(d)\to 0^+} \eta(0) = \lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} \eta(0) = 0,$$

$$\lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} \eta(u) = 0 < \lim_{\eta(d)\to 0^+} \eta(u) = \bar{y}/\pi, \text{ and}$$

$$\lim_{\eta(d)\to 0^+} y = \lim_{\eta(d)\to(\bar{y}/(1-\pi))^-} y = \bar{y}.$$

Proof of Proposition A.5. We begin by considering the solutions of the system (A.1)-(A.3) as $\eta(d)$ goes to 0.

(a) Since $\eta(u) \geqq \eta^*$ when $\eta(d) \leqq \eta^*$, from (A.2) it follows that $\lim_{\eta(d)\to 0^+} \eta(0) = 0$. Otherwise, (i) if, for some $k > 0$, $\lim_{\eta(d)\to k^+} \eta(0) = 0$, then $\lim_{\eta(d)\to k^+} (RHS(\text{A.2})) > 0$, while (ii) if $\lim_{\eta(d)\to 0^+} \eta(0) = k > 0$, then $\lim_{\eta(d)\to k^+} (RHS(\text{A.2})) = -\infty$, and (A.2) is violated.

(b) Use the definition of y together with (A.14) to simplify (A.1), which becomes

$$y = (1 + 1/\beta)\eta^* - \eta(0)/\beta = \bar{y} + \eta(0)/\beta$$

or, equivalently, for later use,

$$\eta(0) = (1 + \beta)\eta^* - \beta y. \tag{A.16}$$

Then it follows that $\lim_{\eta(d)\to 0^+} y = \lim_{\eta(0)\to 0^+} y = \bar{y}$.

(c) Now use the definition of y to solve for $\eta(u) = y/\pi - [(1-\pi)/\pi]\eta(d)$. Then it follows that $\lim_{\eta(d)\to 0^+} \eta(u) = \bar{y}/\pi$.

Finally, observe that the roles of $\eta(u)$ and $\eta(d)$ are interchangeable, i.e., that either could be chosen as the independent variable. So precisely the same reasoning as the foregoing applied as $\eta(\mu)$ goes to 0 yields the fact that the upper bound of $\eta(d)$ is $(\bar{y}/(1-\pi)$, together with the other two limit values, $\lim_{\eta(d)\to((\bar{y}/(1-\pi))^-} \eta(0) = 0$ and $\lim_{\eta(d)\to((\bar{y}/(1-\pi))^-} y = \bar{y}$.■

The statement (and proof) of these last two propositions make them appear to be much more complicated than they really are, as we try to indicate by means of the diagram in Figure A.1.

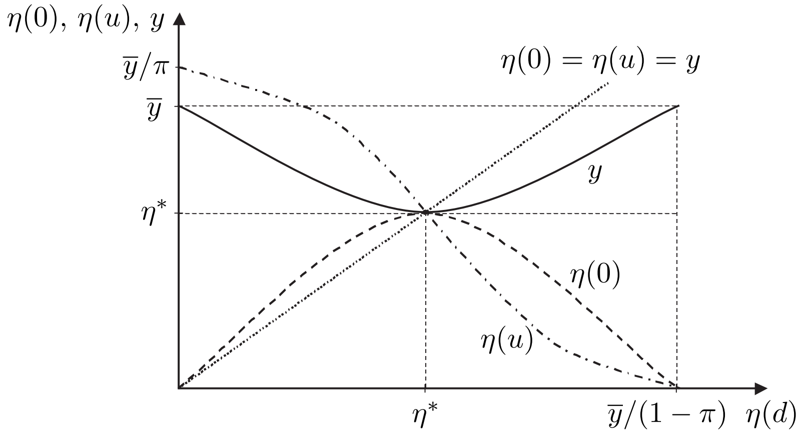

Figure A.1: The Solutions to the Equilibrium Equations.

To end this section we re-emphasize that our leading example is very special in that there is only a single independent variable, and the qualitative behavior of the dependent variables over its whole range is very nice. In short, it is straightforward to go from local to global analysis. Even for the extensions to encompass sunspots and three periods, however, without additional specialization, this is no longer the case, and only local analysis is generally possible.

A.IV. **Tailoring the Portfolio Constraint**

Consider the specific portfolio constraint

$$q^2(0)s_2^2(1) \geqq \gamma W_2(0) \text{ with } \gamma > 0, \tag{A.17}$$

where $W_2(0) = (q(0) + p(0))s_2(0)$. When $\mu > 0$, this constraint is binding, and – assuming that $W_2(0) > 0$ – we can use (A.10) in order to solve for γ,

$$\gamma = (a_1/(a_1 - a_2))q^2(0)/W_2(0). \tag{A.18}$$

From (A.12),

$$q^2(0) = (1 - a_1) + (1 - a_2)y,$$

while from (A.12) and (A.5),

$$W_2(0) = [(\alpha_1 + \alpha_2 y) + ((\alpha_1/\beta) + (\alpha_2/\beta)\eta(0))]s_2(0).$$

Substituting for $\eta(0)$ from (A.16) into this last expression and then simplifying yields

$$\begin{aligned} W_2(0) &= [(\alpha_1 + \alpha_2 y) + (\alpha_1/\beta) + (\alpha_2/\beta)((1 + \beta)\eta^* - \beta y)]s_2(0) \\ &= (1 + 1/\beta)(\alpha_1 + \alpha_2\eta^*)s_2(0) \\ &= (1 + 1/\beta)\eta^*\alpha_2(s_1(0) + s_2(0)) \\ &= (1 + 1/\beta)\eta^* > 0, \end{aligned}$$

where in deriving the third equality we utilize the definition in (A.14). It then follows that (A.18) is simply a linear mapping, $\gamma = \frac{a_1(1-a_1)}{(a_1-a_2)(1+1/\beta)\eta^*} + \frac{a_1(1-a_2)}{(a_1-a_2)(1+1/\beta)\eta^*}\, y$, with a positive intercept and slope. Combining this with what we know from Propositions A.4 and A.5 about the relationship between $\eta(d)$ and the other reduced form variables ($\eta(0)$, $\eta(u)$, μ), and hence y, we can get a very clear picture of the equilibrium correspondence between γ and all the endogenous variables in the model. This is illustrated from one aspect in Figure 2 in the main text, which also illustrates that (A.18) can always be satisfied when $\eta(\omega) = \eta^*$ for all ω and $\mu = 0$ – the second part of Proposition A.1. This result also establishes that, for every

$$\gamma \in (\underline{\gamma}, \bar{\gamma}) = \left(\tfrac{a_1(1-a_1)}{(a_1-a_2)(1+1/\beta)\eta^*} + \tfrac{a_1(1-a_2)}{(a_1-a_2)(1+1/\beta)\eta^*}\, \underline{y}, \tfrac{a_1(1-a_1)}{(a_1-a_2)(1+1/\beta)\eta^*} + \tfrac{a_1(1-a_2)}{(a_1-a_2)(1+1/\beta)\eta^*}\, \bar{y}\right),$$

there are precisely two distinct equilibria in which the portfolio constraint is binding in a nondegenerate way (since the inverse of the mapping such that $\eta(d) \mapsto y$ is two-to-one; the exact form of the expressions detailing this is summarized in Result 3 in the main text). The example is robust (though, obviously, nongeneric) in the parameter γ.

All this can be succinctly summarized, (given the portfolio constraint (A.17)).

Proposition A.6. *For every economy $\gamma \in (\underline{\gamma}, \bar{\gamma})$, there are two inefficient equilibria in which the constraint is binding in a nondegenerate way, together with the original efficient equilibrium in which it isn't.*

We note that this particularly simple argument can be extended to cover the case where total wealth, $W_2(0)$, is replaced by portfolio wealth,

$$P_2(0) = q(0)s_2(0)$$

(again, since $q(0)$ – through Ms. 1's NAC's – depends solely on y). The only difference is that now, on an open interval $(\underline{\gamma}, \bar{\gamma})$, γ becomes a strictly increasing, strictly convex (resp. strictly concave) function of y when $\alpha_2 s_1(0) > 0$ (resp. $\alpha_2 s_1(0) < 0$).

The essence of these two examples can be abstracted and (slightly) generalized in terms of a (local) relationship between $\eta(d)$ and γ, but nothing of much interest or value is learned in the process.

Two Concluding Remarks. 1. It should be absolutely clear that all future spot goods prices and consumption allocations vary across all three equilibria corresponding to $\gamma \in (\underline{\gamma}, \bar{\gamma})$ (since $\eta(\omega)$, $\omega \neq 0$, does), though today's, as well as stock prices and portfolio strategies (may) only vary across the type-E and type-I equilibria (since y does).

2. Regarding the maintained assumption that $a_1 > a_2$: For the opposite case, $a_1 < a_2$, the qualitative behavior we have just described is, in most important respects, reversed. For instance, now

$$\mu \left\{ \begin{array}{c} < \\ = \end{array} \right\} 0 \text{ according as } \eta(d) \left\{ \begin{array}{c} \neq \\ = \end{array} \right\} \eta^*,$$

the constraint must be an upper rather than lower bound.

Appendix B

In this appendix we explore the role of extrinsic uncertainty (or sunspots) in generating multiplicity of equilibrium. In particular, in the leading example we now introduce two extrinsic states, $\sigma = G, B$, in addition to the intrinsic states, $\omega = u, d$. The detailed description of this modified economic setting is presented in Section 3.1. In the presence of extrinsic uncertainty, the effect of introducing the portfolio constraint (A.17) is even more striking.

Proposition B. *For every economy $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there is a continuum (i.e., a smooth, two-dimensional manifold) of distinct sunspot equilibria.*

Remark. It will become apparent that what drives this result is the degree of asset market incompleteness, that there are four states but only two stocks, together with another specific feature of this sunspot economy, that its related certainty economy is nothing more than our leading example.

Proof of Proposition B. The proof naturally breaks down into several steps.

Step 1. (Obtaining the RFE's). We begin by observing that – aside from the obvious changes in notation necessitated by the proliferation of stochastic weights from 3 to 5 – the analysis which led to the RFE's in Appendix A can be repeated here. In particular, it follows directly from Proposition A.1 that Ms. 1's second period BC's always have the solution $s_1(1)$ such that

$$s_2(1) = \mathbf{1} - s_1(1) = (-(1 - a_1)/(a_1 - a_2), a_1/(a_1 - a_2)),$$

utilized in deriving equation (A.18). Moreover, as before, the logic of the reduction of the EFE's leads to just 3 RFE's,

$$-(1+1/\beta)\alpha_1 s_2(0) + ((\eta(0)/\beta) + \sum_{\tau\neq 0}\pi(\tau)\eta(\tau))\alpha_2 s_1(0) = 0, \tag{B.1}$$

$$\sum_{\tau\neq 0}\pi(\tau)(1-\eta(0)/\eta(\tau))(a_1 + a_2\eta(\tau)) = 0, \text{ and} \tag{B.2}$$

$$\sum_{\tau\neq 0}\pi(\tau)(1-\eta(0)/\eta(\tau))((1-a_1) + (1-a_2)\eta(\tau)) - \mu = 0. \tag{B.3}$$

But now there are 6 rather than 4 basic variables, $(\eta, \mu) = ((\eta(\tau), \text{ all } \tau), \mu)$, the variables which determine all the remaining endogenous prices and quantities (we will again use the fact that both spot goods and stock prices are uniquely determined by the stochastic weights later on). It is also easily verified that the system (B.1)-(B.3) still has the unique positive solution $\eta = \eta^*\mathbf{1}$ for $\mu = 0$[13].

Step 2. (Existence of a "starting" nonsunspot equilibrium, SNE). Pick $\gamma' \in (\underline{\gamma}, \overline{\gamma})$. Observe that taking $\eta(G,u) = \eta(B,u) = \eta(u)$, and $\eta(G,d) = \eta(B,d) = \eta(d)$ reduces the equations and the variables to those of the model in Appendix A. Thus, we can pick one (out of the two) inefficient equilibria in that model (with a binding portfolio constraint), say (η', μ'), to define a SNE.

Step 3. (Simplifying the system of equations). Consider again defining the auxiliary expected value

$$y = \sum_{\tau\neq 0}\pi(\tau)\eta(\tau), \tag{B.4}$$

and then fixing this and some variables at their "starting" values,

$$\eta(0) = \eta'(0),\ y = y', \text{ and } s_1(1) = ((1-a_2)/(a_1-a_2), -a_2/(a_1-a_2)). \tag{B.5}$$

Given (B.5), we can ignore both the portfolio constraint (A.17) and equation (B.1), since it is easily verified that both only depend on $\eta(0)$ and y (the former through $p(0)$ and $q(0)$, and hence through $p(\tau)$, all τ, the latter directly). Finally, notice that since (B.3) in essence defines μ, while $\mu' > 0$, we can perturb the SNE locally and maintain $\mu > 0$.

Therefore, we are left with just two equations, (B.2) and (B.4), in the four variables, $\eta(G,u)$, $\eta(B,u)$, $\eta(G,d)$, and $\eta(B,d)$. Let this system be represented generally, as before, by $\Phi(\xi,\theta) = 0$, with $\xi = (\eta(G,u), \eta(G,d))$ and $\theta = (\eta(B,u), \eta(B,d))$.

[13] We should emphasize that all the forgoing remains valid when the additional states σ represent intrinsic rather than extrinsic uncertainty, so that they are not necessarily independent of the original states. But the, more generally, for $\pi(\sigma,\omega) \neq \pi(\sigma,\omega)$, $\sigma = G, B$, $\pi(\omega) = \pi(G,\omega)+ \pi(B,\omega)$, $\omega = u, d$. We also note that (as in Appendix A), the restriction that dividends are invariant across sunspots never enters the picture. All this remains true of in what follows as well—except that independence permits the interpretive distinction between nonsunspot and sunspot equilibria. In short, the logic of the argument here, while focusing on sunspot phenomena for its application, is perfectly general.

Step 4. (Existence of a smooth, two-dimensional manifold of distinct sunspot equilibria). Differentiating $\Phi(\xi,\theta)$ yields

$$D_\xi \Phi(\xi,\theta) = \begin{bmatrix} \pi(G)\pi(u) & \pi(G)\pi(d) \\ \pi(G)\pi(u)\left[a_2 + a_1\eta'(0)/\eta(G,u)^2\right] & \pi(G)\pi(d)\left[a_2 + a_1\eta'(0)/\eta(G,d)^2\right] \end{bmatrix}.$$

Since $\eta'(G,u) \neq \eta'(G,d)$, we have $\det(D_\xi \Phi(\xi',\theta')) = (\pi(G)\pi(u))(\pi(B)\pi(d))((1/\eta'(G,d)^2) - (1/\eta'(G,u)^2)) \neq 0$. Hence, it follows directly upon application of the Implicit Function Theorem that, in a neighborhood of θ', these equations define a smooth, two-dimensional manifold, such that, for any two points on the manifold, say $(\tilde{\xi},\tilde{\theta})$ and $(\hat{\xi},\hat{\theta})$, $\hat{\theta} = (\hat{\eta}(B,u),\hat{\eta}(B,d)) \neq \tilde{\theta} = (\tilde{\eta}(B,u),\tilde{\eta}(B,d))$, so consumption allocations differ. But, in this neighborhood, θ' yields the sole nonsunspot equilibrium, and the argument is complete. ■

Two Concluding Remarks. 1. Obviously, this proposition remains valid with the second portfolio constraint described in the last section of Appendix A.

2. A fortiori, there will be real indeterminacy whenever there is a deficiency of stocks relative to future states – since then there are "extra" variables to "play with". It remains to be seen whether this result extends to models with other real assets, e.g., real bonds as in Cass and Pavlova (2004), but this is certainly a plausible conjecture.

Appendix C

Consider the extension of our leading example to include three periods, while all the other basic assumptions (binomial tree, two commodities, two stocks, and two households who each have log-linear utility functions, with Mr. 2 constrained in his choice of the second stock) are maintained. Extrinsic uncertainty is resolved in period 1, while intrinsic uncertainty is only resolved at the end, in period 2 (this time line is depicted in Figure 2 in the main text).

The purpose of this appendix is to establish that, for this economy, sunspots can have doubly significant effect, that is, that the realization of extrinsic uncertainty can affect both consumption allocation and stock pricing. In particular, we will show that we can support an equilibrium (called the "target" sunspot equilibrium) where

- after observing "good prospects" ($\sigma = G$), consumption allocation is conditionally Pareto efficient ($\eta(G,u) = \eta(G,d) = \eta(G)$), and the portfolio constraint faced by Mr. 2 in his choice of the second stock does't bind (so that the associated Lagrange multiplier, $\mu(G)$, is zero);
- after observing "bad prospects" ($\sigma = B$), consumptionallocation is not even conditionally Pareto efficient ($\eta(B,u) \neq \eta(B,d)$), and his portfolio constraint does bind (so that the associated Lagrange multiplier, $\mu(B)$, is strictly positive); and
- stock prices differ across the two possible events ($q(G) \neq q(B)$).

Thus, most notably – and the reason for this more elaborate example – stock prices may vary even though there are no changes in the fundamentals of the economy, for instance, in prospective stock dividends, but rather simply because households believe that they will.

C.I. **The Intermediate Form Equations (IFE's)**

Retracing steps similar to those followed in Appendix A (see Section A.2), here the system of equations representing GFE can be usefully reformulated as a system of IFE's, a system that still includes variables which will be "solved out" eventually. Again, we omit obvious quantifiers.

$$p(0) - \alpha_1/\beta - (\alpha_2/\beta)\,\eta(0) = 0, \tag{C.1}$$

$$p(\sigma) - \pi(\sigma)\,(\alpha_1 + \alpha_2\eta(\sigma)) = 0, \tag{C.2}$$

$$p\,(\sigma,\omega) - \beta\pi(\sigma)\pi(\omega)(\alpha_1 + \alpha_2\eta\,(\sigma,\omega)) = 0, \tag{C.3}$$

$$-q(0)\beta + \sum\nolimits_{\sigma}(p(\sigma) + q(\sigma)) = 0, \tag{C.4}$$

$$-q(\sigma) + \sum\nolimits_{\omega} p\,(\sigma,\omega) = 0, \tag{C.5}$$

$$\begin{gathered} -\left(q(0)\beta/\eta(0)\right) + \textstyle\sum_\sigma \left[(p(\sigma) + q(\sigma))\,(1/\eta(\sigma))\right] + (0,\mu(0)) + \\ \gamma(\textstyle\sum_\sigma \mu(\sigma)q^1(\sigma), \sum_\sigma \mu(\sigma)) = 0, \end{gathered} \tag{C.6}$$

$$-\left(q(\sigma)/\eta(\sigma)\right) + \sum\nolimits_{\omega}\left(p\,(\sigma,\omega)/\eta\,(\sigma,\omega)\right) + (0,\mu(\sigma)) = 0, \tag{C.7}$$

$$(1/\beta) + (1+\beta) - (p(0) + q(0))\,s_1(0) = 0, \tag{C.8}$$

$$\pi(\sigma)(1+\beta) - (p(\sigma) + q(\sigma))\,s_1(1) = 0, \tag{C.9}$$

$$\beta\pi(\sigma)\pi(\omega) - p\,(\sigma,\omega)\,s_1(\sigma) = 0, \tag{C.10}$$

$$s_1(1) + s_2(1) - \mathbf{1} = 0, \text{ and} \tag{C.11}$$

$$s_1(\sigma) + s_2(\sigma) - \mathbf{1} = 0. \tag{C.12}$$

The variables $\mu = (\mu(0), (\mu(\sigma),\ \text{all}\ \sigma))$ represent the Lagrange multipliers for all the portfolio constraints faced by Mr. 2 in his holdings of the second stock. Note especially that, in this extension of the leading example, there are three portfolio constraints, one for the first period, $q^2(0)s_2^2(1) \geqq \gamma W_2(0)$, and one for each of the two spots in the second period), $q^2(\sigma)s_2^2(\sigma) \geqq \gamma W_2(\sigma), \sigma = u, d$.

We begin the analysis by describing the properties of portfolio strategies. As in the two-period case, budget constraints for spots that come after the first period are linear equations in the portfolios $s_1 = (s_1(1), (s_1(\sigma),\ \sigma = u, d))$. The following lemma describes the choice of $s_1(\sigma)$, $\sigma = u, d$. (Since Proposition C below is the parallel of Proposition B in Appendix B, we will label the preliminary results leading up to this central result simply lemmas.)

Lemma 1. *Consider either sunspot outcome. If* $(\eta(\sigma,u), \eta(\sigma,d)) \gg 0$ *and* $\eta(\sigma,u) \neq \eta(\sigma,d)$, *then (C.10) has the (same) unique solution*

$$s_1(\sigma) = ((1-a_2)/(a_1-a_2), -a_2/(a_1-a_2)) \tag{C.13}$$

so that, from (C.12),

$$s_2(\sigma) = (-(1-a_1)/(a_1-a_2), a_1/(a_1-a_2)). \tag{C.14}$$

On the other hand, if $\eta(\sigma,u) = \eta(\sigma,d) = k > 0$, *then (C.10) has the continuum of solutions*

$$s_1^1(\sigma) = [1 - ((1-a_1) + (1-a_2)k)s_1^2(\sigma)]/[a_1 + a_2 k], \tag{C.15}$$

one of which is also (C.13).

Proof of Lemma 1. Fix σ and use (C.3) to replace $p(\sigma,\omega)$ in (C.10). Then just notice that, after this substitution, equation (C.10) is the same as (A.8), with $\eta(\omega)$ replaced by $\eta(\sigma,\omega)$. ■

No matter whether the the good or the bad sunspot is observed, precisely the same portfolio strategy is optimal (though when $\eta(\sigma,u) = \eta(\sigma,d)$ there are other optimal strategies as well). We show now that exactly the same situation prevails at $t = 0$.

Lemma 2. *Let* $m(\sigma) = \eta(\sigma) + \beta \sum_\omega \pi(\omega)\eta(\sigma,\omega)$. *If* $(\eta(\tau),\ \textit{all}\ \tau > 0)$ *and* $m(G) \neq m(B)$, *then (C.9) has the (same) unique solution*

$$s_1(1) = ((1-a_2)/(a_1-a_2), -a_2/(a_1-a_2)) \tag{C.16}$$

so that, from (C.11),

$$s_2(1) = (-(1-a_1)/(a_1-a_2), a_1/(a_1-a_2)). \tag{C.17}$$

Again, on the other hand, If $m(G) = m(B) = m > 0$, *then (C.9) has the continuum of solutions*

$$s_1^1(1) = \big((1+\beta) - ((1+\beta)(1-a_1) + (1-a_2)m)s_1^2(1)\big)]/((1+\beta)a_1 + a_2 m), \tag{C.18}$$

one of which is also (C.16).

Proof of Lemma 2. Using (C.2), (C.3), and (C.5), equation (C.9) reduces to $As_1(1) = b$, where $b = (1+\beta)\mathbf{1}$, and

$$A = \begin{bmatrix} (1+\beta)a_1 + a_2 m(G) & (1+\beta)(1-a_1) + (1-a_2)m(G) \\ \\ (1+\beta)a_1 + a_2 m(B) & (1+\beta)(1-a_1) + (1-a_2)m(B) \end{bmatrix}.$$

It is then straightforward to show that det $(A) = (1+\beta)(a_1 - a_2)(m(G) - m(B))$. The rest of the proof essentially follows the same path as that for Proposition A.1 in Appendix A. ■

C.II. **The Reduced Form Equations**

In the previous section we only separated out equations (C.9)-(C.12). We now reduce those which remain, (C.1)-(C.8), to a system of 7 equations in the 10 variables, (η, μ)= $((\eta(\tau), \text{all } \tau),$ $(\mu(0), (\mu(\sigma), \text{all } \sigma)))$. For this purpose, merely substitute from (C.1)-(C.5) into (C.6)-(C.8). This then yields, after some simplification and reformulation, the RFE's for this model.

$$(1/\beta) + (1+\beta) - f_1(\eta)s_1^1(0) - f_3(\eta)s_1^2(0) = 0, \tag{C.19}$$

$$\begin{gathered}\textstyle\sum_\sigma([(1+\beta)\alpha_1\pi(\sigma) + \alpha_2 f_2(\eta)][-1/\eta(0) + 1/\eta(\sigma)])+ \\ (0, \mu(0)) + \ \gamma(\textstyle\sum_\sigma \mu(\sigma)q^1(\sigma), \textstyle\sum_\sigma \mu(\sigma)) = 0, \text{ and}\end{gathered} \tag{C.20}$$

$$\beta \sum\nolimits_\omega [\pi(\sigma)\pi(\omega)(\alpha_1 + \alpha_2\eta(\sigma,\omega))] f_4(\eta) + (0, \mu(\sigma)) = 0, \tag{C.21}$$

where,

$$\begin{aligned}
f_1(\eta) &= a_1[(1/\beta) + (1+\beta)] + \\
&\quad\ a_2[(\eta(0)/\beta) + \sum\nolimits_\sigma \pi(\sigma)\eta(\sigma) + \beta \sum\nolimits_\sigma \sum\nolimits_\omega \pi(\sigma)\pi(\omega)\eta(\sigma,\omega)], \\
f_2(\eta) &= \pi(\sigma)\eta(\sigma) + \beta \sum\nolimits_\omega \pi(\sigma)\pi(\omega)\eta(\sigma,\omega), \\
f_3(\eta) &= (1-a_1)[(1/\beta) + (1+\beta)] + (1-a_2)[\eta(0)/\beta + \sum\nolimits_\sigma f_2(\eta)], \text{ and} \\
f_4(\eta) &= [(-1/\eta(\sigma)) + (1/\eta(\sigma,\omega))].
\end{aligned}$$

Our "target" sunspot equilibrium has been described earlier. To establish its existence we start from a solution where

$$\eta(G,u) = \eta(G,d) = \eta(G) \text{ and } \mu(G) = 0, \text{ and} \tag{C.22}$$

$$\eta(B,u) = \eta(B,d) = \eta(B) \text{ and } \mu(B) = 0. \tag{C.23}$$

The first step in our argument will be to show that sunspot equilibria exist under (C.22)-(C.23) (these will be referred to as "starting" sunspot equilibria). We then perturb this solution while maintaining (C.22) but permitting $\eta(B,u)$ and $\eta(B,d)$ to differ from each other, and from $\eta(B)$. After the perturbation, we are then able to find an open set of economies, say $(\underline{\gamma}^*, \overline{\gamma}^*)$, where our "target" sunspot equilibrium exists.

Lemma 3. (Existence of "starting" sunspot equilibria). *Assume that conditions (C.22)-(C.23) hold. For every* $\gamma \in (\underline{\gamma}, \overline{\gamma})$*, there exist sunspot equilibria with* $\mu(0) > 0$, $\mu(G) = 0$, $\mu(B) = 0$,

$\eta(0) < \eta^*$, $\eta(G,u) = \eta(G,d) = \eta(G) \neq \eta^*$, $\eta(B,u) = \eta(B,d) = \eta(B) \neq \eta^*$, and $\eta(G) \neq \eta(B)$. Moreover, $q(G) \neq q(B)$.

Proof of Lemma 3. The specified solution satisfies the NAC's at σ – the equations in (C.21). With (C.22) and (C.23) in place, we are only left with the NAC's at $t = 0$ – the equations in (C.20)– and the overall budget constraint – equation (C.19). This system mimics the RFE's in Appendix A. That is, under (C.22)-(C.23), we are back in the two-period model with the 3 equations (A.1)-(A.3) in the 4 variables $(\eta(0), \eta(G), \eta(B), \mu(0))$. So Proposition A.4 holds and gives us the desired result. ■

With a binding portfolio constraint at $t = 0$, we get $\gamma = \left(q^2(0)s_2^2(1)\right)/W_2(0)$. Using the choice for $s_2(1)$ from equation (C.17), say $\tilde{s}_2 = (-(1-a_1)/(a_1-a_2), a_1/(a_1-a_2))$, we obtain $\gamma = \left(q^2(0)\tilde{s}_2^2\right)/W_2(0)$. Similarly, with a binding contraint at $\sigma = B$, from equation (C.14) we get $\gamma = \left(q^2(B)\tilde{s}_2^2\right)/W_2(B)$. So, substituting for γ we obtain an overall portfolio constraint (OPC) of the form

$$\left(q^2(0)\tilde{s}_2^2\right)/W_2(0) - \left(q^2(B)\tilde{s}_2^2\right)/W_2(B) = 0; \tag{C.24}$$

this must be satisfied if both portfolio constraints are binding (in either a nondegenerate or a degenerate way).

Now let (C.22) hold, but allow for changes in $\eta(B)$, $\eta(B,u)$, and $\eta(B,d)$. This leaves us with 6 equations, (i) the overall budget constraint, (C.19), (ii) the NAC's at $t = 0$, (C.20), (iii) the NAC's at the bad spot $\sigma = B$, (C.21), and (iv) the OPC, (C.24), in the 7 variables, $\eta(0), \eta(G), \eta(B), \eta(B,u), \eta(B,d), \mu(0)$, and $\mu(B)$. As in the two preceding appendices, represent this system of equations generally by $\Phi(\xi,\theta) = 0$. The variables ξ and θ stand for endogenous and exogenous variables, and are given here by $\xi = (\eta(0), \eta(G), \eta(B), \eta(B,u), \mu(0), \mu(B))$ and $\theta = \eta(B,d)$, respectively.

Proposition C. (Existence of "target" sunspot equilibria). *If $(\alpha_1 + \alpha_2\eta^*)\,\tilde{s}_2 \neq (\alpha_1 + \alpha_2\eta^*)\,s_2(0)$, then there is an open set of economies, say, $(\underline{\gamma}^*, \overline{\gamma}^*)$, with $\underline{\gamma} < \underline{\gamma}^* < \overline{\gamma}^* < \overline{\gamma}$, such that, for every $\gamma \in (\underline{\gamma}^*, \overline{\gamma}^*)$, there exist sunspot equilibria with the properties that: (i) $\eta(G,u) = \eta(G,d) = \eta(G)$ and $\mu(G) = 0$, (ii) $\eta(B,u) \neq \eta(B,d)$ and $\mu(B) > 0$, and (iii) $q(G) \neq q(B)$.*

Proof of Proposition C. To reach such a "target" equilibrium from a particular "starting" equilibrium we proceed in four steps[14].

Step 1. (Existence of a "starting" sunspot equilibrium where both constraints bind). From Lemma C.3 we know that for every $\gamma \in (\underline{\gamma}, \overline{\gamma})$, we can find "starting" sunspot equilibria satisfying (C.22)-(C.23). The first step of this proof consists in finding some value $\gamma' \in (\underline{\gamma}, \overline{\gamma})$ such that an

[14]Mathematica v. 5.1 was used for calculations in Steps 2-4. A copy of our program is available on request.

associated "starting" sunspot equilibrium has both portfolio constraints (at $t = 0$ and $\sigma = B$) binding simultaneously (the second in a degenerate fashion since $\mu(B) = 0$).

The analysis carried out in Section A.IV can be applied to tailor the portfolio constraint at $t = 0$, so that $q^2(0)\left(a_1/\left(a_1 - a_2\right)\right) = \gamma W_2(0)$. In other words, define

$$\gamma_0 = q^2(0)\left(a_1/\left(a_1 - a_2\right)\right)/W_2(0) = \frac{\left((1-a_1) + (1-a_2)y(1)\right)\left(a_1/\left(a_1 - a_2\right)\right)}{\left(\alpha_1 + \alpha_2 y(1)\right)s_2(0)}, \tag{C.25}$$

where $y(1) = \pi(G)\eta(G) + \pi(B)\eta(B)$. From Appendix A, we know that (i) $\eta(B)$ bifurcates for $\gamma_0 > \underline{\gamma}$, and that (ii) as $\gamma_0 \to \overline{\gamma}$, one of the branches of $\eta(B)$ goes to zero. Now consider the portfolio constraint at the bad spot, $\sigma = B$. Let

$$\gamma_1 = q^2(B)\left(a_1/\left(a_1 - a_2\right)\right)/W_2(B) = \frac{\left((1-a_1) + (1-a_2)\eta(B)\right)\left(a_1/\left(a_1 - a_2\right)\right)}{\left(\alpha_1 + \alpha_2 \eta(B)\right)\tilde{s}_2}. \tag{C.26}$$

If $\left(\alpha_1 + \alpha_2\eta^*\right)\tilde{s}_2 > \left(\alpha_1 + \alpha_2\eta^*\right)s_2(0)$, then, for $y(1) = \eta(B) = \eta^*$, we have $\gamma_0 > \gamma_1$. As $\eta(B) \to 0$, we get $\gamma_1 \to \infty$ and $\gamma_0 \to \overline{\gamma} < \infty$. Therefore, by the Intermediate Value Theorem, there exists a value $\eta(B)' \in (0, \eta^*)$ (and corresponding $\eta(G)' > \eta^*$) such that $\gamma_0 = \gamma_1 = \gamma'$. A similar argument can be made in the other case, where $\left(\alpha_1 + \alpha_2\eta^*\right)\tilde{s}_2 < \left(\alpha_1 + \alpha_2\eta^*\right)s_2(0)$, with values for $s_2(0)$ close enough to $\tilde{s}_2$. For this case, we would get $\eta(B)' > \eta^*$.

Step 2. (Regularity of "starting" sunspot equilibrium). For any solution to $\Phi(\xi,\theta) = 0$ which satisfies (C.22)-(C.23), $\det\left(D_\xi\Phi(\xi,\theta)\right) = A\alpha_2 s_1(0)$, with $A > 0$. Thus, since $\alpha_2 s_1(0) \neq 0$, $D_\xi\Phi(\xi,\theta)$ has full rank. So any "starting" sunspot equilibrium (in particular, the one found in Step 1) is regular.

Step 3. (Existence of "target" sunspot equilibria). We can use as our "starting" point the sunspot equilibrium found in Step 1, i.e., the values of $\eta(G,u)' = \eta(G,d)' = \eta(G)' \neq \eta(B)' = \eta(B,u)' = \eta(B,d)'$, corresponding to γ'. Note also that $q(G)' \neq q(B))'$. From Step 2, we can apply the Implicit Function Theorem to find a neighborhood of $\eta(B,d)'$, say, $N_{\varepsilon'}\left(\eta(B,d)'\right)$, such that, for $\theta \in N_{\varepsilon'}\left(\eta(B,d)'\right)$, $\xi = F(\theta)$, and F is smooth. Moreover, $D_{\eta(B,d)}\mu(B) = 0$ and $D^2_{\eta(B,d)}\mu(B) > 0$ at $\eta(B,d) = \eta(B,d)'$. Therefore, as we move away from $\eta(B,d)'$, the portfolio constraint at the bad spot still binds but its multiplier becomes strictly positive (the portfolio constraint is no longer degenerate). In addition, for $\eta(B,d) = \eta(B,d)'$, we get $D_{\eta(B,d)}\eta(B,u) < 0$. This means that $\eta(B,u) \neq \eta(B,d)$, for $\eta(B,d)$ different from $\eta(B,d)'$. Thus, for $\eta(B,d) \in \left(\eta(B,d)', \eta(B,d)' + \varepsilon'\right)$, our "target" equilibrium exists.

Step 4. (Robustness of the "target" equilibria). We want to show that the existence of the "target" equilibrium occurs on an open set of parameters, namely $\left(\underline{\gamma}^*, \overline{\gamma}^*\right)$. Notice first that $D_{\eta(B,d)}y(1) = 0$ and $D^2_{\eta(B,d)}y(1) > 0$, for $\eta(B,d) = \eta(B,d)'$. In other words, $y(1)$ attains a (strict) local minimum at $\eta(B,d)'$. Therefore, the interval $\left(\eta(B,d)', \eta(B,d)' + \varepsilon'\right)$ gives us an open neighborhood for $y(1)$, say $(y(1)', y(1)' + \delta')$. Finally, to obtain an open set for γ, just notice that γ is a strictly decreasing smooth funtion of $y(1)$ (applying the argument of Section A.IV to the portfolio constraint at the bad spot). Thus, the image of $(y(1)', y(1)' + \delta')$ is an open interval, say $\left(\underline{\gamma}^*, \overline{\gamma}^*\right)$. ■

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