



Penn Institute for Economic Research  
Department of Economics  
University of Pennsylvania  
3718 Locust Walk  
Philadelphia, PA 19104-6297  
[pier@econ.upenn.edu](mailto:pier@econ.upenn.edu)  
<http://www.econ.upenn.edu/pier>

## ***PIER Working Paper 05-010***

“Generic Uniqueness of the Solutions to a Continuous Linear Programming Problem”

by

Nicola Persico

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# Generic Uniqueness of the Solutions to a Continuous Linear Programming Problem

Nicola Persico\*

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## Abstract

Let  $f : R \rightarrow R$  be strictly increasing. We are interested in the set of probability distributions  $\mu$  on the interval  $[0, S]$  that solve the linear programming problem  $\max_{\mu} \int_0^S f(p) d\mu(p)$  subject to  $\int_0^S g(p) d\mu(p) \leq C$ . We provide a sufficient condition on the pair  $(f, g)$  for the solution to the linear programming problem to be unique and show that this sufficient condition is satisfied “generically.”

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## 1 The problem

Let  $[0, S]$  be an interval of the real line. Let the function  $f : [0, S] \rightarrow [0, 1]$  be continuous and strictly increasing. Let the function  $g : [0, S] \rightarrow \mathbf{R}$  be continuous. Let  $\mathcal{M}$  denote the set of probability distributions on  $[0, S]$ . We wish to characterize the solutions to the following linear problem:

$$\begin{aligned} \max_{\mu \in \mathcal{M}} \int_0^S f(p) d\mu(p) & \quad (\text{P}) \\ \text{s.t. } \int_0^S g(p) d\mu(p) & \leq C. \end{aligned}$$

To guarantee existence of a solution, in what follows we assume that  $\min_{p \in [0, S]} g(p) \leq C$ . We will prove the following result showing that, “generically,” the solution to problem P is unique.

**Proposition 1** *The solution to problem P is unique and puts positive probability on at most two points whenever  $g \circ f^{-1}$  belongs to a set which is a dense  $G - \delta$  set in the space of all continuous functions equipped with the supnorm.*

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\*Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia PA 19104; email: persico@ssc.upenn.edu. Web: <http://www.ssc.upenn.edu/~persico>.

## 2 Characterization of the solutions to problem P

Let  $\mathcal{T}$  denote the set of continuous functions  $x(t) : [0, S] \rightarrow \mathbf{R}$  with the property that at least one of the hyperplanes that supports  $x(t)$  from below is tangent to  $x(t)$  at more than two points. In what follows, the superscript  $C$  indicates set complements.

**Lemma 2** *If  $g \circ f^{-1}$  belongs to  $\mathcal{T}^C$ , any solution to problem P places all the probability on one or two points in  $[0, S]$ .*

**Proof:** If  $\mu^*$  is a solution to problem P, there exists a number  $\lambda \geq 0$  such that  $\mu^*$  maximizes the Lagrangean

$$\mathcal{L}(\mu, \lambda) = \int_0^S [f(p) - \lambda g(p)] d\mu(p) + \lambda C.$$

The case  $\lambda = 0$  is easily disposed of, since in that case the (unique) solution requires all the mass to be placed on  $S$ . Let us turn, then, to the case  $\lambda > 0$ . If  $\mu^*$  is to maximize the Lagrangean, the set of  $p$ 's on which  $\mu^*$  may place positive probability is given by

$$\mathcal{A} = \arg \max_p [f(p) - \lambda g(p)].$$

By definition of  $\mathcal{A}$ , there is a number  $M$  such that

$$\begin{aligned} f(p) - \lambda g(p) &= M & \text{for } p \in \mathcal{A} \\ f(p) - \lambda g(p) &< M & \text{for } p \notin \mathcal{A} \end{aligned}$$

If  $\mu^*$  puts positive mass on more than two points, then the cardinality of  $\mathcal{A}$  would have to exceed 2. Consider the transformation  $\varphi(p) = f^{-1}(p/S)$ . The function  $\varphi$  is a one-to-one mapping of  $[0, S]$  onto itself. We can therefore write

$$\begin{aligned} f(\varphi(p)) - \lambda g(\varphi(p)) &= M & \text{for } \varphi(p) \in \mathcal{A} \\ f(\varphi(p)) - \lambda g(\varphi(p)) &< M & \text{for } \varphi(p) \notin \mathcal{A}, \end{aligned}$$

or, with the obvious meaning of symbols,

$$\begin{aligned} \frac{p}{S} - \lambda g(\varphi(p)) &= M & \text{for } p \in \varphi^{-1}(\mathcal{A}) \\ \frac{p}{S} - \lambda g(\varphi(p)) &< M & \text{for } p \notin \varphi^{-1}(\mathcal{A}). \end{aligned}$$

Note that the set  $\varphi^{-1}(\mathcal{A})$  has the same cardinality of  $\mathcal{A}$ . Thus, if  $\mathcal{A}$  has cardinality greater than 2, it means that the two numbers  $\lambda$  and  $M$  are such that the straight line identified by  $\frac{1}{\lambda} \left( \frac{p}{S} - M \right)$  is tangent to the function  $g(f^{-1}(p/S))$  at more than two points and never exceeds it. This means that there is a tangent hyperplanes to the set

$$Y = \{(p, y) : y \leq g(f^{-1}(p/S))\}$$

which makes contact with the set  $Y$  at more than two points.  $\diamond$

**Lemma 3** *If  $g \circ f^{-1}$  belongs to  $\mathcal{T}^C$ , the solution to problem  $P$  is unique.*

**Proof:** If the constraint is not binding then clearly  $\mu^*$  is unique. If the constraint is binding, Lemma 2 guarantees that there are two points, call them  $p_L$  and  $p_H$  (here we allow for the possibility that  $p_L = p_H$ ), such that

$$\mu^*(p_L) + \mu^*(p_H) = 1, \quad (1)$$

where the constraint in problem (P) reads

$$g(p_L) \cdot \mu^*(p_L) + g(p_H) \cdot \mu^*(p_H) = C. \quad (2)$$

The system of equations given by (1) and (2) identifies a unique pair  $\mu^*(p_L), \mu^*(p_H)$  unless the system has less than full rank, i.e., unless  $g(p_L) = g(p_H)$ . But in that case, since  $f$  is strictly increasing, the (unique) solution requires all the mass to be placed on  $\max\{p_L, p_H\}$ .  $\diamond$

### 3 Characterization of the set $\mathcal{T}$

We now show that the set  $\mathcal{T}$  is “small” in this topological sense: that its complement  $\mathcal{T}^C$  is a dense  $G - \delta$  set, i.e., it is dense and the countable intersection of sets that are open, in the space of continuous functions equipped with the supnorm.

To this end, let  $C(x(\cdot))$  denote the convex hull of  $x(\cdot)$ , i.e., the function that is given by the intersection of all the epigraphs of the affine linear functions that lie below the function  $x(t)$  and are tangent to it. Consider all open intervals  $\iota$  of the real line such that for all  $t \in \iota$ , the convex hull  $Cx(t)$  coincides with a linear affine function, and such that this property is not enjoyed by any open interval containing  $\iota$ . Among those, select all those intervals  $\iota'$  such that  $x(t) - Cx(t) = 0$  for some  $t \in \iota'$ . If there are no such  $\iota'$ , then  $x \notin \mathcal{T}$ . If there are such intervals, let the extremes of the largest among these be denoted by  $a_x$  and  $b_x$  (if there are several largest intervals, pick one at random.) Clearly,  $x(t)$  and  $C(x(t))$  agree at  $t = a_x, b_x$ . Furthermore, denote

$$m_x = \arg \min_{t \in (a_x, b_x)} \left| t - \frac{a_x + b_x}{2} \right|$$

s.t.  $x(t) - C(x(t)) = 0$ .

Clearly, if  $x \in \mathcal{T}$  we have  $a_x < m_x < b_x$ . The point  $m_x$  represents the point of tangency of the supporting hyperplane that is closest to the middle of the interval  $(a_x, b_x)$ . Denote  $I_x = [a_x, b_x]$  For  $x \in \mathcal{T}$ ,  $I_x$  represents the largest interval at which any tangent hyperplane to  $x$  is tangent at more than two points.

Denote

$$\mathcal{T}_{n,m} = \left\{ x \in \mathcal{T} \text{ such that } (b_x - a_x) \geq \frac{1}{n} \text{ and } \min \{m_x - a_x, b_x - m_x\} \geq \frac{1}{m} \right\}.$$

Clearly,

$$\mathcal{T} = \bigcup_{\substack{n \geq 1 \\ m \geq n}} \mathcal{T}_{n,m},$$

and so to obtain the required characterization for  $\mathcal{T}^C$  we need to show that for each  $n, m$ , the set  $(\mathcal{T}_{n,m})^C$  is open and that the set  $\mathcal{T}^C$  is dense in the set of all continuous functions endowed with the supnorm. We start with showing dense.

**Lemma 4** *The set  $\mathcal{T}^C$  is dense in the set of continuous functions endowed with the supnorm.*

**Proof:** We show that in the supnorm neighborhood of every continuous function there is a function  $x' \in \mathcal{T}^C$ . Fix, then, an arbitrary continuous function  $x$ . If  $x \in \mathcal{T}^C$  there is nothing to prove, so we can focus on the case  $x \in \mathcal{T}$ . Our task is to show that there is a function  $\tilde{x}$  close to  $x$  with the property that no supporting hyperplane to  $\tilde{x}$  has more than two contact points with  $\tilde{x}$ . Let  $\mathcal{H}$  denote the set of hyperplanes that have more than two contact points with  $x$ . Elements of  $\mathcal{H}$  are identified by their slope  $h$ . For every hyperplane  $h \in \mathcal{H}$ , take the sup and the inf of all its contact points  $t$  and call those  $a_h$  and  $b_h$ . Consider now a continuous function  $d_g(t)$  which is equal to 0 for every  $t$  unless  $t \in (a_h, b_h)$  for some  $h \in \mathcal{H}$ , in which case  $d_g(t)$  assumes values strictly between zero and 1. Let  $\tilde{x}_\varepsilon(t) \equiv [1 + \varepsilon \cdot d_g(t)] \cdot x(t)$ . For any  $\varepsilon > 0$ , the function  $\tilde{x}_\varepsilon(t)$  has exactly the same set of supporting hyperplanes as  $x(t)$ . Moreover, by construction no hyperplane is tangent to  $\tilde{x}_\varepsilon(t)$  at more than two points. Since the function  $\tilde{x}_\varepsilon(t)$  can be made arbitrarily close to  $x(t)$  in the supnorm by choosing  $\varepsilon$  to be small, the set  $\mathcal{T}^C$  is shown to be dense, as desired.  $\diamond$

**Lemma 5** *For each  $n \geq 1, m \geq n$ , the set  $(\mathcal{T}_{n,m})^C$  is open in the set of continuous functions endowed with the supnorm.*

**Proof:** The result will follow if we show that  $\mathcal{T}_{n,m}$  is closed. To this end, let's fix an arbitrary sequence  $\{x_i\}$  that converges to some  $x$  in the supnorm. For each  $x_i$ , consider the set  $D_i =$

$\{a_{x_i}, m_{x_i}, b_{x_i}\}$ . Since the interval  $[0, S]$  is compact in the Euclidean distance, the space of subsets of  $[0, S]$  endowed with the Hausdorff distance is compact (see Theorem 3.71 in Aliprantis and Border 1999). The sequence of sets  $\{D_i\}$  therefore contains a convergent subsequence  $\{D_j\}$ . Consequently, the limits (in the euclidean distance) of  $a_{x_j}, m_{x_j}$ , and  $b_{x_j}$  for  $j \rightarrow \infty$  exist. Denote these limit points by  $a, m$ , and  $b$  respectively. It is clear that  $(b - a) \geq \frac{1}{n}$  and  $\min\{m - a, b - m\} \geq \frac{1}{m}$ .

Since the mapping  $C(\cdot)$  is continuous in the supnorm, we have

$$\lim_{j \rightarrow \infty} C(x_j(a)) = C(x(a)) \quad \text{and} \quad \lim_{j \rightarrow \infty} C(x_j(b)) = C(x(b))$$

For each  $j$ , the function  $C(x_j(t))$  is linear affine for  $t \in [a_{x_j}, b_{x_j}]$ . Thus, for  $t \in [a, b]$ , the function  $C(x(t))$  is the limit in the supnorm of linear affine functions, hence a linear affine function itself. To finish the proof, we use the continuity of the mapping  $C(\cdot)$  to write

$$x(m) = \lim_{j \rightarrow \infty} x_j(m_{x_j}) = \lim_{j \rightarrow \infty} C(x_j(m_{x_j})) = C(x(m)).$$

We have shown that  $C(x(t))$  is linear affine for  $t \in [a, b]$ , and that  $x(m) = C(x(m))$ . This shows that  $x \in \mathcal{T}_{n,m}$ , and thus that  $\mathcal{T}_{n,m}$  is closed. ◇

## 4 Conclusion

We have shown that  $\mathcal{T}^C = \bigcap_{\substack{n \geq 1 \\ m \geq n}} (\mathcal{T}_{n,m})^C$  is dense, and is the countable intersection of sets that are open, in the space of continuous functions endowed with the supnorm. This fact, in conjunction with Lemma 3, proves Proposition 1.

## References

- [1] Aliprantis, Charalambos D., and Border, Kim C. *Infinite Dimensional Analysis*. 2nd ed. Springer Verlag, Heidelberg, 1999.