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PIER Working Paper 05-006

"Identification and Estimation in Highway Procurement Auctions Under Unobserved Auction Heterogeneity"

by

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http://ssrn.com/abstract=651901

Identification and Estimation in Highway Procurement Auctions Under Unobserved Auction Heterogeneity

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May 24, 2004

Abstract

This paper proposes a semi-parametric method to uncover the distribution of bidders' private information in the market for highway procurement when unobserved auction heterogeneity is present. I derive sufficient conditions under which the model is identified and show that the estimation procedure produces uniformly consistent estimators of the distributions in question.

The estimation procedure is applied to data from Michigan highway procurement auctions. I estimate that 75% of the variation in bidders' costs may be attributed to the factors known to all bidders and only 25% may be generated by private information. My results suggest that failing to account for unobserved auction heterogeneity may lead to overestimating uncertainty that bidders face when submitting their bids. As a result both inefficiency of the auction mechanism and mark-ups over the bidders' costs may be overestimated.

Keywords: First-Price Auctions, unobserved auction heterogeneity, highway procurement

JEL Classification: L0, L1, L2, L8, M3

[*]This paper consists of various chapters of my dissertation.

[†]I would like to thank my advisors Martin Pesendorfer, Steve Berry and Hanming Fang for their help and encouragement. I also thank Don Andrews, Han Hong, Philip Hail and Katsumi Shimotsu for helpful conversations. I am grateful to the Michigan Department of Transportation for providing data for my research and to Mr. Wayne Roe in particular for answering my questions and helping me to understand the industry better.

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1 Introduction

In auction markets, the performance of a particular mechanism crucially depends on the degree of uncertainty faced by market participants. In private values environments, a growing literature started by Laffont, Ossard, Vuong (1995) and Guerre, Perrigne, Vuong (2000) uses the equilibrium relationship between bids and bidders' costs to uncover the distribution of private information. After controlling for observed auction characteristics, the estimation procedures proposed in the literature typically assume that remaining variation in bids is generated by variation in private information. Existing procedures do not allow for the possibility that bidders take into account some characteristics of the auction that the researcher cannot observe. Therefore, they can significantly overestimate the magnitude of private information if unobserved auction heterogeneity is present.

This paper constructs a model that incorporates both private information and unobserved auction heterogeneity. It develops a semi-parametric estimation method to recover distributions of private information and unobserved auction heterogeneity from submitted bids. It also establishes sufficient conditions under which these distributions are identified and shows uniform consistency of the estimators. The estimation method is applied to data from Michigan highway procurement auctions to quantify the importance of private information in this market and to study the biases that result from ignoring unobserved auction heterogeneity.

I assume that project cost for any particular bidder equals the product of a common and an individual component. The common component consists of cost attributes known to all bidders, some of which may not be observed by the researcher. The individual component consists of additional cost attributes privately observed by each bidder. This costs structure implies that the distribution of costs may vary across projects even after all project characteristics known to the researcher are held constant. In addition, I allow bidders to be asymmetric, so that the distribution of individual cost component may vary with observable characteristics of the bidder.

I exploit dependence between bids submitted in the same auction to recover the distributions of the common and individual components of bids. In particular, I show that the distributions of components are identified from the joint distribution of two arbitrary bids submitted in the same auction when the individual cost components are independently distributed across bidders and are independent from the common component. Further, the distributions of individual bid components are used to uncover the distributions of individual cost components. This new identification result provides insight into sources of identification for more general models with unobserved auction heterogeneity. The estimation procedure proposed in the paper follows the steps of the identification argument.

I conduct a Monte Carlo study to analyze small sample behavior as well as sensitivity of the estimation procedure to the assumptions of the model. Simulation analysis shows that this procedure behaves well in samples of moderate size. It also correctly recovers individual components and a degenerate common component when applied to data generated by the model with independent private values and no unobserved auction heterogeneity.

I also study the consequences of misspecifying the model and failing to account for unobserved auction heterogeneity. I find that, when applied to data with unobserved auction heterogeneity, the estimation procedures based on the assumptions of independent private values or affiliated private values tend to recover bid functions that are much flatter than the true bid function and to predict mark-ups that are significantly higher than the ones implied by the true distribution. They also predict a higher chance of inefficient outcomes, i.e., when projects are not assigned to the lowest cost bidder. Also, the recovered distributions of costs have higher variances than the true distributions.

The proposed method is implemented using data for highway maintenance projects auctioned by the Michigan Department of Transportation between February 1997 and December 2003. This set of highway procurement projects has features consistent with unobserved auction heterogeneity. For example, the bidders participating in such auctions have access to a detailed description of the project and travel to the project site. Therefore they likely to have an advantage over the researcher in recognizing the differences in the distributions of costs across projects. On the other hand, these projects are precisely specified and quite simple, so that this market is well described by the assumption of private values.

Descriptive analysis of the data indicates that unobserved auction heterogeneity may be present. In particular, fixed and random effects regressions show that a large component of bids' variation could be attributed to the so-called "between variation" or variation across auctions.

I use the estimation procedure developed in this paper to estimate the bidding strategies and the distributions of individual and common cost components. Results indicate that 85% of the variation in costs is explained by the variation in the common component. The estimated bid function implies an average mark-up over the bidders' costs of around 7%. In contrast, the model with affiliated private values predicts average mark-ups of 11%, whereas the model with independent private values predicts 15%. The difference amounts to \$33,000 in the case of affiliated private values and \$61,000 for the model with independent private values. I estimate that there is a 28% chance of an inefficient outcome, which is lower than that obtained under alternative procedures. I also estimate expected distribution of total costs. The variance of the cost distribution estimated under the assumption of unobserved auction heterogeneity is about 25% lower that the variance of the cost distribution estimated under the assumption of affiliated private values and 35% lower than the variance of the cost distribution estimated under the assumption of independent private values. Finally, I perform several robustness checks for the assumptions of the model.

The paper proceeds as follows. Section 2 describes the model. Section 3 discusses identification, testable implications and some extensions of the model. Section 4 details the estimation procedure and summarizes results of the simulation study. Section 5 presents results of estimation and section 6 concludes.

1.1 Related Literature

This paper adds to the literature on estimation of auction models that aims to uncover distribution of bidders' private information from the bids submitted in the auction. In particular, Donald and Paarsch (1993, 1996) and Laffont, Ossard and Vuong (1995) develop parametric methods to recover the distribution of cost from the observed distribution of bids. Elyakime, Laffont, Loisel and Vuong (1994, 1997) propose a nonparametric method to estimate distribution of cost. Guerre, Perrigne, and Vuong (2000) study identification of the First-Price auction model with symmetric bidders. They establish that the distribution of bidders' valuations can be identified from bid data if and only if the empirical inverse bid function is increasing. They propose a uniformly consistent estimation procedure. Li, Perrigne, and Vuong (2000, 2002) extend the result to the affiliated private values and the conditionally independent private values models. Campo, Perrigne, and Vuong (2001) prove identification and develop a uniformly consistent estimation procedure for first-price auctions with asymmetric bidders and affiliated private values. These papers rely on the assumption of no unobserved auction heterogeneity, i.e., they explicitly use a one-to-one mapping between distribution of bidders' costs and distribution of observed bids that arises in such environments.

The few papers that indirectly address the issue of unobserved auction heterogeneity include Campo, Perrigne, and Vuong (2001), Bajari and Ye (2003) and Hong and Shum (2002). The first two papers rely on the assumption that the number of bidders can serve as a sufficient statistic for the unobserved auction heterogeneity. Hong and Shum (2002) account for unobserved auction heterogeneity by modelling the median of the bid distribution as a normal random variable with a mean that depends on the number of bidders. In this paper, I allow for unobserved auction heterogeneity to vary even within the subset of auctions with the same number of bidders. I estimate non-trivial auction specific component after controlling for the number of bidders, which implies that these papers may underestimate common information available to all bidders. In contrast to this literature, I also study identification of the model and implications of failing to account for unobserved auction heterogeneity.

To the best of my knowledge, only a couple of papers directly address the issue of unobserved auction heterogeneity. Athey and Haile (2001) study unobserved auction heterogeneity in the context of second-price and English auctions. Chakraborty and Deltas (1998) assume that the distribution of bidders' valuations belongs to a two-parameter distribution family. They use this assumption to derive small sample estimates for the corresponding parameters of the auction-specific valuation distributions. The estimates are later regressed on the observable auction characteristics to determine the percent of values variation that could be attributed to unobserved auction heterogeneity. The methodology is applied to data for packages of real estate loans. They find a significant auction-specific component in their data. In this sense, my results are consistent with their findings.

Highway procurement auctions have already been studied in the literature. Porter and Zona (1993) find evidence of collusion in Long Island highway procurement auctions. Bajari and Ye (2003) reject the hypothesis of collusive behavior in the procurement auctions conducted in Minnesota, North Dakota, and South Dakota. Jofre-Bonet and Pesendorfer (2003) find evidence of capacity constraints in California highway procurement auctions. Hong and Shum (2002) find some evidence of common values in the bidders' costs in the case of New Jersey highway construction auctions.

2 The Model

This section describes the first-price auction model under unobserved auction heterogeneity and summarizes properties of the equilibrium bidding strategies.

The seller offers a single project for sale to m bidders. Bidder $i's$ cost is equal to the product of two components: one is common and known to all bidders; the other is individual and private information of the firm i. Both the common and the individual cost components are random variables, and they are denoted by the capital letters Y and X respectively. The small letters y and x denote realizations of the common component and the vector of individual components. The two random variables (Y, X) are distributed on $[\underline{y}, \overline{y}] \times [\underline{x}, \overline{x}]^m$, $\underline{y} > 0$, $\underline{x} > 0$, according to the probability distribution function H,

$$\Pr(Y \leq y_0, X \leq x_0) = H(y_0, x_0).$$

Asymmetries between bidders: I assume that there are two types of bidders: m_1 bidders are of type 1, and m_2 bidders, $m_2 = (m - m_1)$, are of type 2. Thus, the vector of independent cost components is given by $X = (X_{11}, .., X_{1m_1}, X_{2(m_1+1)}, .., X_{2m})$. The model and all the results can easily be extended to the case of m types. I focus on the case of two types for the sake of expositional clarity. Types are defined from the observable characteristics of bidders.

Assumptions $(D_1) - (D_4)$ are maintained throughout the paper.

(D_1) Y and X_j's are mutually independent and distributed according to

$$H(y_0, x_{10}, .., x_{m0}) = H_Y(y_0) \prod_{j=1}^{j=m_1} H_{X_1}(x_{j0}) \prod_{j=m_1+1}^{j=m} H_{X_2}(x_{j0}),$$

where H_Y, H_{X_1}, and H_{X_2} are marginal distribution functions of Y, X_{1j}, and X_{2j} respectively.The supports of H_Y and H_{X_k} are given by $S(H_Y) = [\underline{y}, \overline{y}]$, $\underline{y} > 0$, $\underline{y} \leq \overline{y}$; $S(H_k) = [\underline{x}, \overline{x}]$, $\underline{x} > 0$, $\underline{x} \leq \overline{x}$, for $k \in \{1, 2\}$.

(D_2) The probability density functions of the individual cost components, h_{X_1} and h_{X_2}, are continuously differentiable and bounded away from zero on every closed subset of $(\underline{x}, \overline{x})$.

(D_3) $EX_{1j} = 1$ for all $j = 1, ..., m_1$.

(D_4) (a) The number of bidders is common knowledge;

(b) There is no binding reservation price.

Assumption (D_2) ensures the existence of equilibrium. The identification result relies on assumptions (D_1) and (D_3). In particular, assumption (D_3) is used to fix the scale of the distribution of the individual cost component for a bidder of type 1. (D_4) summarizes miscellaneous assumptions about the auction environment.

The auction environment can be described as a collection of auction games indexed by the different values of the common component. An auction game corresponding to the common component equal to y, $y \in [\underline{y}, \overline{y}]$, is analyzed below.

The cost realization of bidder i is equal to $x_i * y$, where x_i is the realization of the individual cost component. The information set of bidder i is given by $P_{yi} = \{x_i | x_i \in [\underline{x}, \overline{x}]\}$. A bidding strategy of bidder i is a real-valued function defined on $[\underline{x}, \overline{x}]$

$$\beta_{yi}: \; [\underline{x}, \overline{x}] \rightarrow [0, \infty].$$

I use a small Greek letter β with subscript yi to denote the strategy of bidder i as a function of the individual cost components and a small Roman letter b to denote the value of this function at a particular realization x_i.

Expected profit. The profit realization of the bidder i, $\pi_{yi}(b_i, b_{-i}, x_i)$, equals $(b_i - x_i * y)$ if bidder i wins the project and zero if he loses. The symbol b_i denotes the bid submitted by bidder i, and the symbol b_{-i} denotes the vector of bids submitted by bidders other than i. At the time of bidding, bidder i knows y and x_i but not b_{-i}. The bidder who submits the lowest bid wins the project. The interium expected profit of bidder i is given by

$$E[\pi_{yi}|X = x_i, Y = y] = (b_i - x_i * y) * \Pr(b_i \leq b_j, \forall j \neq i|Y = y).$$

A Bayesian Nash Equilibrium is then characterized by a vector of functions $\beta_y = \{\beta_{y1}, ..., \beta_{ym}\}$ such that $b_{yi} = \beta_{yi}(x_i)$ maximizes $E[\pi_i|X = x_i, Y = y]$, when $b_j = \beta_{yj}(x_j)$, $j \neq i$, $j = 1, .., m$; for every $i = 1, .., m$ and for every realization of X_i.

LeBrun (1999) and others establish that, under assumptions $(D_1) - (D_2)$, a vector of equilibrium bidding strategies $\beta_y = \{\beta_{y1}, ..., \beta_{ym}\}$ exists. The strategies are strictly monotone and differentiable. Maskin and Riley (2000) show that under these assumptions there is a unique vector of equilibrium strategies, $\beta_y = \{\beta_{y1}, ..., \beta_{ym}\}$, which satisfy the following boundary condition: for all i $\beta_{yi}(\overline{x}) = \overline{x}$, and there exists $d_{yi} \in [\underline{x}, \overline{x}]$ such that $\beta_{yi}(\underline{x}) = d_{yi}$.

These results accordingly establish equilibrium existence and uniqueness in the game where the common cost component equals y.

Next, I characterize a simple property of the equilibrium bidding strategies.

Proposition 1

If $(\alpha_1(.), ..., \alpha_m(.))$ *is a vector of equilibrium bidding strategies in the game with* $y = 1$, *then the vector of equilibrium bidding strategies in the game with* y, $y \in [\underline{y}, \overline{y}]$, *is given by* $\beta_y = \{\beta_{y1}, ..., \beta_{ym}\}$, *such that* $\beta_{yi}(x_i) = y * \alpha_i(x_i)$, $i = 1, ..., m$.

The proposition shows that the bid function is multiplicatively separable into a common and an individual bid component, where the individual bid component is given by $\alpha_i(.)$. The proof of this proposition is based on the comparison of the two sets of first-order conditions and follows immediately from the assumption that costs are multiplicatively separable and that the common component is known to all bidders.

Next, I characterize the necessary first-order conditions for the set of equilibrium strategies when $y = 1$. Note that $\alpha_i(.)$ denotes a strategy of bidder i as a function of the individual cost component and a_i the value of this function for a particular realization of X_i. The equilibrium inverse bid function of the individual bid component for a type k bidder is denoted by ϕ_k. Since the function $\alpha_k(.)$ is strictly monotone and differentiable, the function $\phi_k(.)$ is well-defined and differentiable.

The probability of winning in this game can be expressed as

$$\Pr(a_j \geq a_i, \forall j \neq i) = [(1 - H_{X_{k(i)}}(\phi_{k(i)}(a_i)))]^{(m_{k(i)}-1)}[(1 - H_{X_{-k(i)}}(\phi_{-k(i)}(a_i)))]^{m_{-k(i)}},$$

where $k(i)$ denotes bidder $i's$ type and "$-k(i)$" denotes the complementary type.

The necessary first-order conditions are, then, given by

$$\frac{1}{a - \phi_{k(i)}(a)} = (m_{k(i)} - 1)\frac{h_{X_{k(i)}}(\phi_{k(i)}(a))\phi'_{k(i)}(a)}{1 - H_{X_{k(i)}}(\phi_{k(i)}(a))} + m_{-k(i)}\frac{h_{X_{-k(i)}}(\phi_{-k(i)}(a))\phi'_{-k(i)}(a)}{1 - H_{X_{-k(i)}}(\phi_{-k(i)}(a))}, \quad (1)$$

where $\phi'_k(.)$ denotes the derivative of $\phi_k(.)$.

Equation (1) characterizes the equilibrium inverse individual bid function when $y = 1$. It describes a trade-off the bidder faces when choosing a bid: an increase in the markup over the cost may lead to a higher ex-post profit if bidder i wins, but it reduces the probability of winning. The bid a is chosen in such a way that the marginal effects of an infinitesimal change in a bid on the winner's profit and the probability of winning sum to zero.

The next section uses properties of the equilibrium bidding functions to show how the primitives of the first-price auction model can be recovered from the submitted bids in the presence of unobserved auction heterogeneity.

3 Identification and Testable Implications

The first part of this section formulates an identification problem and provides conditions under which a first-price auction model with unobserved auction heterogeneity is identified. The second part describes restrictions this model imposes on the data. The third part discusses possible extensions.

3.1 Identification

I assume that the econometrician has access to bid data, based on n independent draws from the joint distribution of (Y, X). The observable data are in the form $\{b_{ij}\}$, where i denotes the identity of the bidder, $i = 1, .., m$; and j denotes project, $j = 1, ..., n$. If data represent equilibrium outcomes of the model with unobserved auction heterogeneity, then

$$b_{ij} = \beta_{y_j k(i)}(x_{ij}) \tag{2}$$

(i.e., b_{ij} is a value of bidder i's equilibrium bidding strategy corresponding to y_j evaluated at the point x_{ij}).

As was shown in the previous section, b_{ij} depends on the realizations of the common and individual components as well as on the joint distribution of the individual cost components. This section examines under what conditions on available data there exists a unique triple $\{\{x_{ij}\}, \{y_j\}, H_X\}$ that satisfies 2, i.e., under what conditions the model from a previous section is identified.

Guerre, Perrigne, and Voung (2000) obtain an identification result by transforming the first-order conditions for optimal bids to express a bidder's cost as an explicit function of the submitted bid, the bid probability density function, and the bid distribution function. Under unobserved auction heterogeneity, the necessary first-order condition yields an expression for $x_{ij} \cdot y_j$ as a function of b_{ij} and the conditional bid probability density function and the conditional bid distribution function conditional on $Y = y_j$. The econometrician does not observe the realization of Y and, consequently, does not know the conditional distribution of bids for $Y = y_j$. Hence, it is not possible to establish identification based on the above first-order conditions.

The idea of my approach is to focus on the joint distributions of bids submitted in the same auction instead of the marginal bid distributions in order to identify the model with unobserved auction heterogeneity.

I use B_i to denote the random variable that describes the bid of bidder i with distribution function $G_{B_{k(i)}}(.)$ and the associated probability density function $g_{B_{k(i)}}(.)$; b_{ij} denotes the realization of this variable in the auction j. The econometrician observes the joint distribution function of $(B_{i_1}, .., B_{i_l})$ for all subsets $(i_1, ..., i_l)$ of $(1, ..., m)$.

Proposition 1 establishes that

$$b_{ij} = y_j * a_{ij},$$

where a_{ij} is a hypothetical bid that would have been submitted by bidder i if y were equal to one. I use A_i to denote the random variable with realizations equal to a_{ij}. The associated distribution function is denoted by $G_{A_{k(i)}}(.)$ with the probability density function $g_{A_{k(i)}}(.)$. Notice that the econometrician does not observe y_j and neither therefore a_{ij}. The distribution of A_i is latent.

My identification result is established in two steps. First, it is shown that the probability density functions of Y, A_i's can be uniquely determined from the joint distribution of two bids that share the same cost component. Second, monotonicity of the inverse bid function is used to establish identification of the probability density functions H_{X_1} and H_{X_2} from the distributions of the individual bid components, G_{A_1} and G_{A_2}.

The following theorem is the main result of this section. It formulates sufficient identification conditions for the model with unobserved heterogeneity.

Theorem 1

If conditions $(D_1)-(D_4)$ are satisfied, then probability density functions $h_Y(.)$, $h_{X_1}(.)$ and $h_{X_2}(.)$ are identified from the joint distribution of (B_{i_1}, B_{i_2}), where (i_1, i_2) is any pair such that $i_1 \in \{1, .., m_1\}$; $i_2 \in \{m_1 + 1, .., m\}$.

Theorem 1 states that the distribution functions of cost components $H_{X_k}(.)$ and $H_Y(.)$ are identified. The proof of this theorem consists of two steps and is given in Part A of the Appendix. In the first step, a statistical result by Kotlarski[1] (1966) is applied to the log-transformed random variables B_{i_1} and B_{i_2} given by

$$\begin{aligned}\log(B_{i_1}) &= \log(Y) + \log(A_{i_1}),\\ \log(B_{i_2}) &= \log(Y) + \log(A_{i_2}).\end{aligned}$$

Kotlarski's result is based on the fact that the characteristic function of the sum of two independent random variables is equal to the product of characteristic functions of these variables. This property allows us to find the characteristic functions of $\log(Y)$, $\log(A_{i_1})$, and $\log(A_{i_2})$ from the joint characteristic function of $(\log(B_{i_1}), \log(B_{i_2}))$. It leads to the following three equations:

$$\begin{aligned}\Phi_{\log(Y)}(t) &= \log(\int_0^t \frac{\Psi_1(0, u_2)}{\Psi(0, u_2)} du_2), \qquad (3)\\ \Phi_{\log(A_1)}(t) &= \frac{\Psi(t, 0)}{\Phi_{\log Y}(t)},\\ \Phi_{\log(A_2)}(t) &= \frac{\Psi(0, t)}{\Phi_{\log Y}(t)},\end{aligned}$$

[1] See Rao (1992).

where $\Psi(.,.)$ and $\Psi_1(.,.)$ denote the joint characteristic function of $(\log(B_{i_1}), \log(B_{i_2}))$ and the partial derivative of this characteristic function with respect to the first component respectively. Since there is a one-to-one correspondence between the set of characteristic functions and the set of probability density functions, the probability density functions of Y, A_{i_1}, A_{i_2} can be uniquely deduced from the characteristic functions of $\log(Y)$, $\log(A_{i_1})$, and $\log(A_{i_2})$ since $\log(.)$ is a strictly increasing function; $\alpha_k(.)$, $k = 1, 2$, are increasing functions of x; $[\underline{x}, \overline{x}] \subset (0, \infty)$, $[\underline{y}, \overline{y}] \subset (0, \infty)$. Notice that the marginal distribution of a single bid per auction may not allow us to identify the distribution functions of Y, A_{i_1}, A_{i_2} because there is no unique decomposition of the sum (or product) into its components. The second step in the proof establishes that the distribution of the individual cost component is identified with (possibly) asymmetric bidders and independent private values. It is similar to the argument given in Laffont and Vuong (1996).

A related question concerns identification of specific realizations x_{ij} and y_j corresponding to a particular bid b_{ij}. In this case, the answer is negative: x_{ij} and y_j cannot be separately identified. The reason is that for every value of y from the support of the distribution $H_Y(.)$, we can find values $\{x_{ij}\}$, $i = 1, .., m$, such that a vector $(x_{1j}, .., x_{mj}, y)$, together with the distribution functions $H_{X_k}(.)$, $k = 1, 2$, rationalizes the vector of bids $\{b_{ij}\}$, $i = 1, .., m$. More details are provided in Part A of the Appendix after the proof of Theorem 1.

Theorem 1 establishes that identification of the model with unobserved auction heterogeneity crucially relies on the assumption of independence of individual components across bidders and from the common cost component. Next, we show how validity of these assumptions can be evaluated within a framework of the model with unobserved auction heterogeneity.

3.2 Testable Implications

Notice that instead of $\log(B_{i_1})$ and $\log(B_{i_2})$, Kotlarski's result can be applied to the variables $\log(\frac{B_{i_1}}{B_{i_3}})$ and $\log(\frac{B_{i_2}}{B_{i_3}})$, since $\log(\frac{B_{i_1}}{B_{i_3}}) = \log(A_{i_1}) - \log(A_{i_3})$ and $\log(\frac{B_{i_2}}{B_{i_3}}) = \log(A_{i_2}) - \log(A_{i_3})$. Here $\log(A_{i_3})$ plays the role of a common component whereas $\log(A_{i_1})$ and $\log(A_{i_2})$ remain individual components. If the individual cost components X_{i_1}, X_{i_2} and X_{i_3} are independently distributed, then so are $\log(A_{i_1})$, $\log(A_{i_2})$, and $\log(A_{i_3})$. The characteristic functions of these variables can be computed using the joint characteristic function of $(\log(\frac{B_{i_1}}{B_{i_3}}), \log(\frac{B_{i_2}}{B_{i_3}}))$, which I denote by $\Theta(.,.)$, according to a formula similar to equation (3).[2] Specifically,

[2] The symbol $\Theta_1(.,.)$ denotes the partial derivative of $\Theta(.,.)$ with respect to the first argument.

$$\Lambda_{\log(A_{i_3})}(t) = \exp(\int_0^t \frac{\Theta_1(0,u_2)}{\Theta(0,u_2)} du_2), \quad (4)$$

$$\Lambda_{\log(A_{i_1})}(-t) = \frac{\Theta(t,0)}{\Lambda_{\log(A_{i_3})}(t)}.$$

Two observations can be made at this point. First, if B_{i1} and B_{i3} are submitted by bidders of the same type and the assumption about independence of individual components holds, then $\Lambda_{\log(A_{i_3})}(t)$ and $\Lambda_{\log(A_{i_1})}(t)$ should be equal. Second, I have relied only on the functional form and the independence of the individual cost components assumptions to obtain $\Lambda_{\log(A_{i_k})}(.)$. The assumption of independence of Y and X then implies that $\Lambda_{\log(A_{i_3})}(.)$ and $\Lambda_{\log(A_{i_1})}(.)$ have to coincide with the functions given by (3). These observations are summarized by conditions (W_1) and (W_2).

(W_1) For any triple (i_1, i_2, i_3) such that $\{i_1 = 1, .., m_1 \textit{ and } i_3 = 1, .., m\}$, or $\{ i_1 = m_1 + 1, .., m \textit{ and } i_3 = m_1 + 1, .., m\}$,

$$\Lambda_{\log(A_{i_1})}(t) = \Lambda_{\log(A_{i_3})}(t)$$

for every $t \in [-\infty, \infty]$.

(W_2) For any triple (i_1, i_2, i_3),

$$\Phi_{\log(A_{i_3})}(t) = \Lambda_{\log(A_{i_3})}(t),$$
$$\Phi_{\log(A_{i_1})}(t) = \Lambda_{\log(A_{i_1})}(t)$$

for every $t \in [-\infty, \infty]$. Here $\Phi_{\log(A_{i_3})}(t)$ and $\Phi_{\log(A_{i_1})}(t)$ denote the characteristic functions of the *log* of the individual bid components defined earlier in the identification section.

Independence of individual cost components further implies condition (W_3).

(W_3) For any quadruple $(i_1, i_2, i_3, i_4) \subset \{1, ..., m\}$, $\frac{B_{i_1}}{B_{i_2}}$ and $\frac{B_{i_3}}{B_{i_4}}$ are independently distributed.

Proposition 2 describes implications of the independence assumptions.

Proposition 2

Let bidder i's cost for the project j be given by $c_{ij} = x_{ij} * y_j$.

(1) If the individual cost components are independent, then (W_1) *has to be satisfied.*

(2)If the individual cost components are independent, then (W_3) *has to be satisfied.*

(3) Further, if Y *is independent of* X, *then* W_2 *holds.*

The proof of proposition 2 is given in Part A of the Appendix.

3.3 Rationalization

The identification section derives conditions under which primitives of a model with unobserved auction heterogeneity can be uniquely recovered from data generated within the framework of this model. Next, I address an issue of model rationalization, i.e., I identify properties of data that allow us to conclude that a particular data set could have been generated by the model with unobserved auction heterogeneity. Then we discuss if and how the model with unobserved heterogeneity can be distinguished from other models consistent with private values environment.

Conditions below describe a set of joint restrictions imposed on data by all the assumptions of the model with unobserved auction heterogeneity.

$(W_{\mathbf{4}})$ For every pair (i_l, i_p), $i_l = 1, ..., m_1$; $i_p = m_1+1, .., m$, the functions $\Phi_{\log(Y)}(.)$, $\Phi_{\log(A_{i_l})}(.)$, $\Phi_{\log(A_{i_p})}(.)$ given by (2) represent characteristic functions of real-valued variables.

(W_5) The characteristic functions $\Phi_{\log(Y)}(.)$, $\Phi_{\log(A_{i_l})}(.)$ and $\Phi_{\log(A_{i_p})}(.)$ do not depend on the pair of (i_l, i_p), $i_l = 1, ..., m_1$; $i_p = m_1 + 1, .., m$, which was used to derive them.

(W_6) The inverse bid functions

$$\phi_k(a) = a - \frac{(1 - G_{A_k}(a))(1 - G_{A_{-k}}(a))}{(m_k - 1)g_{A_k}(a)(1 - G_{A_{-k}}(a)) + m_{-k}g_{A_{-k}}(a)(1 - G_{A_k}(a))}, \ k = 1, 2,$$

are strictly increasing in a.

Proposition 3

If available data satisfy conditions $(W_4) - (W_6)$, *then there exists a model with unobserved heterogeneity that could have generated the data.*

The first condition guarantees that two independent random variables Y and A_i exist with the property $B_i = Y * A_i$. The third condition ensures that $A_i^{\prime}s$ are consistent with the equilibrium behavior under the independent private values assumption. The first and second assumptions guarantee that bidders within each of the types are identical.

The model with independent private values is nested in the model with independent private values. Condition (W_7) provides the basis for Proposition 4, which shows how the null of independent private values can be tested against the alternative of independent private values.

(W_7) The distribution of $\log(Y)$ is degenerate.

Proposition 4

If conditions $(W_{\mathbf{4}})$ *and* (W_7) *are satisfied then data are generated by the model with independent private values.*

While I do not have a formal proof, my conjecture is that the model with unobserved auction heterogeneity and affiliated individual components cannot be identified from bid data. Li, Perrigne and Vuong (2002) outline conditions for the rationalization of the model with affiliated private values. If conditions of proposition 3 are satisfied simultaneously with conditions in Li, Perrigne and Vuong (2002), then the three models - the model with unobserved auction heterogeneity and independent individual components, the model with unobserved auction heterogeneity and affiliated individual components, and the model with affiliated private values - are observationally equivalent.

3.4 Extensions

The model with unobserved auction heterogeneity assumes that bidder $i^{\prime}s$ cost of completing the project equals the product of the common and the individual cost components. This functional form emerges when the cost distribution for a particular project is scaled by a project-specific common variable, in which case mean and variance vary with the common component in a coordinated way.

A more general model may allow for the common component to have distinct effects on the mean and variance of the cost distribution function. Such a model can be constructed using a two-dimensional project heterogeneity. Bidder $i's$ cost of the project is, then, equal to

$$c_{ij} = y_{1j} + y_{2j} * x_{ij},$$

where (y_{1j}, y_{2j}) is a realization of a two-dimensional cost component that is common knowledge among all bidders; x_{ij} is a realization of an individual component, which is private information of firm i. This specification has the following interpretation: the average cost of the project j equals y_{1j}, and the individual cost deviations have auction-specific scale. It can be shown that the described model is identified under conditions similar to those in Theorem 1. The exact conditions and the proof are given in Part A of the Appendix.

4 Estimation

This section describes the estimation method, derives properties of the estimators, and discusses practical issues related to the estimation procedure.

4.1 Estimation Method

The econometrician has data for n auctions. For each auction j, $(m_j, \{b_{ij}\}_{i=1}^{i=m_j}, z_j)$ are observed, where m_j is the number of bidders in the auction j, with m_{j1} bidders of type 1 and m_{j2} bidders of type 2; $\{b_{ij}\}_{i=1}^{i=m_j}$ is a vector of bids submitted in the auction j; and z_j is a vector of auction characteristics. The estimation procedure is described for the case of discrete covariates. It can be extended to the case of continuous z_j.

The estimates are obtained conditional on the number of bidders, $m_j = m_0$, $m_{1j} = m_{01}$, and $z_j = z_0$. Let n_0 denote the number of auctions that satisfy these restrictions. The estimation procedure closely follows the identification argument described in the proof of Theorem 1. It consists of two steps. First, the joint characteristic function of two log transformed bids is estimated for every (t_1, t_2) as a sample average of the $\exp(it_1 \cdot B_{lj} + it_2 \cdot B_{pj}))$, where the average is taken across auctions with $m_j = m_0$, $m_{1j} = m_{01}$ and $z_j = z_0$. Then the joint characteristic function is used to compute the characteristic functions of the logs of the common and the individual bid components according to the formulas given by (3). The inversion formula is used to recover the probability density functions for the logs of the common and the individual bid components from the characteristic functions. Finally, the probability density functions of logs are used to recover the probability density functions of the common and the individual bid components.

In the second step, the probability density functions of the individual bid components are used to obtain estimates of the probability density function of the individual cost component. For that, a sample of pseudo-bids is drawn form the probability density function of the relevant individual bid component. This sample is then used to obtain the sample of pseudo-costs with the help of the corresponding inverse bid function. Finally, the sample of pseudo-costs is used to non-parametrically estimate the probability density function of the individual cost component.

To estimate the probability density function of the total cost of the bidder i at a point c, I compute an integral of the function $h_{X_i}(\frac{c}{y}) * h_Y(y)$ with respect to y over the interval $[\underline{y}, \overline{y}]$. To evaluate this integral, I perform Monte Carlo integration with respect to $h_Y(.)$.[3] The value of an average inverse bid function at a point b is estimated as the mean of the value of the individual bid function at a point $\frac{b}{y}$ multiplied by y with respect to the distribution of y. Again, Monte Carlo integration methods are used to compute the mean.

The details of the estimation procedure are outlined in the Appendix.

4.2 Properties of the Estimator

This subsection shows that the estimation procedure yields uniformly consistent estimators of the relevant distributions. I use the result from Li and Vuong (1998) to establish uniform consistency of the first stage estimators. Their argument applies if probability distributions of bid components satisfy following restrictions on the tail behavior of characteristic functions.

(D_5) The characteristic functions $\phi_{LY}(.)$ and $\phi_{LA_k}(.)$ are ordinary smooth[4] with $\varkappa > 1$.

This property holds, for example, when cumulative probability functions of cost components admit up to R, $R > 1$, continuous derivatives on the support interior such that M of them, $1 \leq M \leq R$, can be continuously extended to the real line. The uniform consistency of the first stage estimators is used to establish uniform consistency of the estimator of individual cost component distribution.

Proposition 5 summarizes properties of the estimator.

[3] Judd (2000) provides a detailed explanation of a Monte-Carlo integration method.

[4] Following Fan (1991),

Definition 1 The distribution of random variable Z is *ordinary-smooth of order* $\varkappa$ if its characteristic function $\phi_z(t)$ satisfies

$$d_0|t|^{-\varkappa} \leq |\phi_z(t)| \leq d_1|t|^{-\varkappa}$$

as t$\rightarrow \infty$ for some positive constants d_0,d_1,$\varkappa$.

Proposition 5

If conditions (D_1)*-*(D_5) *are satisfied, then* $\widehat{h}_Y(.)$ *and* $\widehat{h}_{X_k}(.)$ *are uniformly consistent estimators of* $h_Y(.)$ *and* $h_{X_k}(.)$, $k = 1, 2$, *respectively.*

The proof of Proposition 5 is presented in Part A of the Appendix.

4.3 Practical Issues

Several important comments must be made about the first step. First, to reduce the error in the characteristic function estimation, I scale bids to fit into the interval $[0, 2\pi]$. Second, as noted by Diggle and Hall (1993) and Li, Perrigne and Vuong (2000), the estimators for $\widehat{h}_{LY}(.)$ and $\widehat{h}_{LA}(.)$, which are obtained by truncated inverse Fourier transformation, may have fluctuating tails.[5] This feature can be alleviated by adding a damping factor to the integrals in $\widehat{h}_{LY}(.)$ and $\widehat{g}_{LA}(.)$. Following Diggle and Hall (1993) and Li, Perrigne and Vuong (2000), I introduce a damping factor defined as

$$d_T(t) = \left\{ \begin{array}{c} 1 - \frac{|t|}{T},\ if\ |t| \leq T \\ 0,\ otherwise \end{array} \right\}.$$

Thus, the estimators are generalized to

$$\begin{aligned} \widehat{g}_{LA}(a) &= \frac{1}{2\pi} \int_{-T}^{T} d_T(t) \exp(-ita) \widehat{\Phi}_{LA}(t) dt, \\ \widehat{h}_{LY}(y) &= \frac{1}{2\pi} \int_{-T}^{T} d_T(t) \exp(-ity) \widehat{\Phi}_{LY}(t) dt. \end{aligned}$$

Third, the smoothing parameter T should be chosen to diverge slowly as $n \to \infty$, so as to ensure uniform consistency of the estimators. However, the actual choice of T in finite samples has not yet been addressed in the literature. I choose T through a data-driven criterion. In particular, I use the bid data to obtain estimates of the means and variances for distributions[6] of LY and LA, $\widehat{\mu}_{LY}$, $\widehat{\mu}_{LA} = 0$, $\widehat{\sigma}_{LY}$, $\widehat{\sigma}_{LA}$. These estimates are then used to choose a value of T. Specifically, I try different values of T and obtain estimates of $h_{LY}(.)$ and $h_{LA}(.)$. From each estimated density I compute the means and variances $\widetilde{\mu}_{LY}$, $\widetilde{\mu}_{LA}$, $\widetilde{\sigma}_{LY}$,

[5] Li,Perrigne and Vuong (2000) encountered this problem as well and dealt with it in a similar way.

[6] The estimates for the first two moments of the distributions of LY, LA_1 and LA_2 can be obtained as follows: $\widehat{\mu}_{LY} = \frac{\sum \log(b_i)}{n*m}$, $\widehat{\mu}_{LA} = 0$, $\widehat{\sigma}_{LA} = \frac{\sum(\log(b_{i_1}) - \log(b_{i_2}))^2}{2*n*m}$, $\widehat{\sigma}_{LY} = \frac{\sum(\log(b_i))^2}{2*n*m} - (\widehat{\mu}_{LY})^2 - \widehat{\sigma}_{LA}$.

$\widetilde{\sigma}_{LA}$, respectively. This gives goodness-of-fit criterion $|\widehat{\mu}_{LY} - \widetilde{\mu}_{LY}| + |\widehat{\sigma}_{LY} - \widetilde{\sigma}_{LY}|$ for LY, and similarly for LA. The value of T that I choose minimizes the sum of these errors in percentage of $\widehat{\sigma}_{LY}$ and $\widehat{\sigma}_{LA}$. In the estimation, the optimal T equals 50.

The second step in the estimation involves taking random draws from the estimated density. I use a rejection method.[7] In this method random pairs (z_j, a_j) are drawn from the uniform distribution on $[0, r] \times [\underline{a}, \overline{a}]$, where r is the maximum value that $\widehat{h}_A(.)$ attains on the support of the distribution of A. Then, a_j is added to the sample of pseudo-bids if $z_j \leq \widehat{h}_A(a_j)$. The resulting sample of pseudo-bids is distributed according to $\widehat{h}_A(.)$.

The second step of the estimation involves non-parametric estimation of the density and distribution function. In the density estimation a tri-weight kernel is used, because it satisfies conditions of compact support and continuous differentiability on the support, including the boundaries.[8] The tri-weight kernel is defined as

$$K(u) = \frac{35}{32}(1-u^2)^3 1(|u| \leq 1).$$

I follow Guerre, Perrigne, and Vuong(2000) in my choice of bandwidth, $\delta_g = d_g(L)^{-\frac{1}{6}}$, where d_g is computed according to a "rule of thumb." Specifically, I use $d_g = 2.978 \times 1.06\widehat{\sigma}_a$, where $\widehat{\sigma}_a$ is the standard deviation of the logarithm of $(1 + bids)$, and 2.978 follows from the use of tri-weight kernel.[9]

Confidence intervals for the estimates are obtained through a bootstrap procedure.

4.4 Monte Carlo Study

This section describes results of the Monte Carlo study. It consists of two parts. The first part studies small sample properties of the estimation procedure that accounts for the presence of unobserved auction heterogeneity. The second part investigates the direction and magnitude of the bias that arises when procedures that ignore unobserved auction heterogeneity are applied to data that possess this feature. The results of the first part are summarized in tables 1*b* and 2*b*. In particular, table 1*b* describes the data generating process for each of the experiments, whereas table 2*b* summarizes estimation results. For each data generating process I report the chosen variances of the cost components, the variance of the total costs, the correlation between common and individual components, the correlation of individual components across bidders,

[7] The rejection method was proposed by Newmann (1951). We need to know the support of the distribution in question to apply this method. A procedure for the supports estimation is described in the Part A of the Appendix.

[8] Conditions given in Li, and Vuong (1998) ensure uniform consistency of the second-stage estimator.

[9] See Hardle, 1991.

the correlation of total costs across bidders and the expected mark-up over the bidders' costs implied by the equilibrium bid function. The estimation results include the variances of the estimated distributions of components, the variance of the estimated distribution of total costs as well as the estimated expected mark-up over the bidders' cost. Table 3*b* summarizes results of the second part. The study is performed for the case of symmetric bidders, i.e. all bidders are assumed to draw their costs from the same distribution.

The simulated data sets are generated as follows. The cost of bidder i is assumed to equal the sum of the common and the individual cost components, $c_i = y + x_i$. The individual cost component is distributed according to the uniform distribution with mean fixed at zero. The common component is chosen to be distributed according to the power distribution with the exponent equal to three and mean fixed at 10.5. To study the effect of an increase in the correlation of bids on the performance of estimation procedures, I use power distributions with different variances. The distributions are chosen so that analytical expression for bidding strategies can be derived in each case.[10] To create a typical data set describing n procurement auctions with k bidders, $k * n$ independent draws from the uniform distribution are combined with n draws from the corresponding power distribution ($\alpha = 3$), such that

$$\{c_{ij}, c_{ij} = y_j + x_{ij}, i = 1, ., k;\ j = 1, .., n\}$$

is a matrix of simulated costs. The matrix of associated bids is calculated according to the equilibrium bid function. The values of n and k are set to equal 400 and 2, respectively. I replicate each experiment 500 times and illustrate the resulting distributions of the estimators.

Independent Costs. First, I apply a procedure that accounts for unobserved auction heterogeneity to the bid data based on the independent costs draws, i.e., data with a degenerate common component. This experiment corresponds to case 1 in the tables 1*b* and 2*b*. Figures 1*b* and 2*b* present the results of the estimation. As figure 2*b* shows, the estimated density of the common component is close to degenerate. It is not exactly degenerate due to the estimation error. The estimates indicate that the error is always confined to the interval of an order of 0.2. Figure 1*b* shows that at the same time, the density of the individual component is estimated quite precisely. In particular, the true density function lies within a 95% pointwise confidence interval of the corresponding estimator. It is also true for the bid function, expected bid function and expected density.

Non-trivial Unobserved Auction Heterogeneity Component. Figures 3*b* and 4*b* demonstrate the estimation results when the procedure is applied to the data with non-trivial unob-

[10] I considered combinations of several distributions to analyze the behavior of the estimation procedure, and the results were very similar to those presented here.

served auction heterogeneity component. This experiment corresponds to case 2 in the tables $1b$ and $2b$. In particular, I consider two sub-cases: in the first sub-case the common component is chosen to be relatively small (case $2a$) and in the second sub-case it is relatively large (case $2b$). The results of estimation indicate that estimation procedure performs equally well in both cases. In general, the estimation results are somewhat less precise as compared to the case 1. The experimentation with different distributions shows that the estimator is the least precise around the mode of the distribution. In the case of the uniform distribution, it is the least precise around the ends of the support; for the power distribution, the precision is worst at the right end of the support. However, the estimates for both the density of the common component and the density of the individual components are still reasonably close to the true densities. In particular, the true densities lie within a 95% pointwise confidence interval of the estimators. The same is true for the estimators of the individual bid function, expected bid function, and the estimated density of total costs.

Affiliated Private Values. Next, I evaluate the performance of the estimation procedure when data are generated by the affiliated private values model. This experiment corresponds to case 3 in the tables $1b$ and $2b$. As before, the cost realizations are equal to the sum of draws from two distributions: X and Y, where X has a uniform distribution and Y is distributed according to power distribution. However, in this case I assume that bidder i observes only her cost realization, $c_i = x_i + y$, but not x_i or y. I derive an analytical expression for the bidding function according to the characterization result stated in Milgrom and Weber (1982) and use it to compute bid values corresponding to the sample of cost draws. I consider two sub-cases: in the first sub-case the Y component is chosen to be relatively small (case $3a$) and in the second sub-case it is relatively large (case $3b$). The results of estimation presented in table $2b$ show that the recovered expected cost density is tighter than the true cost distribution. The recovered expected bid function is steeper than the true bid function. On average, estimated expected mark-up is smaller than the true one.

Correlated Individual Components. Another experiment analyzes the performance of the estimation procedure when individual cost components are correlated (case 4). The underlying cost realizations are generated as a sum of draws from three distributions: X_0, X_1, and Y. In particular, $c^{1j} = x^{1j} + y^j$ and $c^{2j} = x^{2j} + y^j$, where $x^{1j} = x_0^{1j} + x_1^j$ and $x^{2j} = x_0^{2j} + x_1^j$, are realizations of individual components, and y^j is a realization of the common cost component. The distributions of X_0 and X_1 are uniform and the distribution of the common cost component Y is power with exponent three. The bidding function is derived taking into account the correlation between individual components, and bid realizations are computed by evaluating the bid function at cost realizations. As before, I consider two sub-cases: in the first sub-case the X_1 component is chosen to be relatively small and thus affiliation is rather weak (case

$4a$), and in the second sub-case it is relatively large (case $4b$). In the case 4, the estimation procedure tends to overestimate the variance of the common component and underestimate the variance of the individual component. This effect decreases as the variance of X_1 decreases and practically vanishes when the correlation is about 5% of the variance of X. In fact, it seems that a small amount of correlation between individual cost components improves the performance of the procedure. The estimated individual bid function tends to underestimate the true mark-ups but the estimated expected bid function produces almost unbiased estimates of the true expected bid function. The last effect arises because variance of Y is overestimated. Because of this effect the estimated distribution of total costs is also an almost unbiased estimator of the true distribution of expected total costs. the bias increases as the strength of affiliation increases.

Correlated Common and Individual Components. Finally, I investigate the performance of the procedure when individual components are correlated with the common component but not with each other (case 5). The underlying cost realizations are generated as a sum of draws from distributions of three random variables: X_0, X_1, and Y_0, namely, $c^{1j} = x^{1j} + y^j$ and $c^{2j} = x^{2j} + y^j$, where $x^{1j} = x_0^{1j} + x_1^{1j}$ and $x^{2j} = x_0^{2j} + x_1^{2j}$ are individual components, and $y^j = x_1^{1j} + x_1^{2j} + y_0^j$ is a common component. The distributions are chosen as follows: the distributions of X_0 and X_1 are uniform and the distribution of Y_0 is a power distribution with exponent three. The bidding function is derived for the bidders' beliefs corresponding to the distribution of $X_0 + X_1$. As before, I consider two sub-cases: in the first sub-case the X_1 component is chosen to be relatively small (case $5a$) and in the second sub-case it is relatively large (case $5b$). The results of estimation presented in the table 2b show that the recovered distribution of the common component tends to have larger variance than the true distribution of Y, and the recovered distribution of X tends to have smaller variance than the true distribution. Changes in variances are consistent with the variance of the joint part of X and Y being added to the variance of Y and subtracted from the variance of X. Therefore, it confirms our intuition that the estimation procedure decomposes the sum of random variables into the orthogonal components. This effect decreases as the correlation between X and Y decreases and practically vanishes when the correlation is about 5% of the variance of X. In fact, it seems that a small amount of correlation between X and Y improves the performance of the procedure. Similar to the case of correlated individual components, the estimated individual bid function tends to underestimate the true mark-ups, but the estimated expected bid function and the estimated distribution of total costs produce almost unbiased estimates of the true expected bid function and the true distribution of total costs.

The second part of the study aims to evaluate the magnitude and direction of the bias that arises when estimation procedures that ignore the presence of unobserved auction heterogeneity are applied to the data generated by the model with unobserved auction heterogeneity. In particular, I obtain estimates under the assumption of independent private values and affiliated private values respectively. In this analysis I allow the ratio between the variance of the individual cost component and the variance of the common cost component to vary. Estimation results are summarized in the table 3*b*. I use average mark-ups over the bidders' costs to compare the bidding function's estimators across different estimation procedures. Estimation results indicate that bid functions estimated under the assumption of independent private values or affiliated private values tend to underestimate bidders' costs and therefore overestimate the mark-ups over the bidders' costs. The bias increases as the ratio between variances of the common and individual components increases, i.e., as the correlation between costs increases. In particular, this effect becomes significant when the correlation coefficient is around or exceeds 0.3. The bid functions estimated by the IPV and APV procedures are flatter than the true inverse bid function for most of the support. In particular, the IPV estimator gets increasingly flatter at the lower end of the support. Intuitively, the presence of the known common component leads to a bid distribution with very thin tails. Under the assumption of independent private values, such a distribution reflects extreme bid shading at the lower end of the support, or an extremely flat bid function.

Table 3*b* also presents estimates of the mean and variance of the cost distribution across estimation procedures and for different values of the variance of the common component. The results suggest that the cost distributions estimated under both the independent and affiliated private values assumptions tend to have lower means and higher variances compared to the true cost distribution.

To summarize, results of the simulation study show that the procedure performs reasonably well when applied to data generated by the model with unobserved auction heterogeneity and also when a small amount of correlation between individual components and/or a correlation between the individual component and common cost component is present. They also show that if an estimation procedure that relies on the assumption of no unobserved auction heterogeneity is applied to data with a significant amount of unobserved auction heterogeneity, it can lead to substantially biased estimates of bidding function, mark-ups and the probability density function of bidders' costs.

So far, I have described identification conditions, proposed an estimation procedure, and discussed small sample properties of the estimator using a Monte-Carlo study.

5 Michigan Highway Procurement Auctions

This section describes characteristics of the Michigan highway procurement auctions. Section 5.1 and 5.2 present the data and report descriptive evidence of auction-specific variation in the bids distribution. Section 5.3 describes estimation results for the model with unobserved heterogeneity and compares them to the estimates obtained under the assumption of independent and affiliated private values. The estimates suggest that unobserved auction heterogeneity may account for a large part of bid variation. If unobserved auction heterogeneity is present, estimators obtained under alternative assumptions may substantially exaggerate bidders' mark-ups and misrepresent the shape of the cost distribution.

5.1 Market Description

The Michigan Department of Transportation (DoT) is responsible for construction and maintenance of most roads within Michigan. The Department of Transportation identifies work that has to be done and allocates it to companies in the form of projects through a first-price sealed bid auction. The project usually involves a small number of tasks, such as resurfacing, or replacing the base or filling in cracks.

Letting process. The Department of Transportation advertises projects 4 to 10 weeks prior to the letting date. Advertisement usually consists of a short description of the project, including the location, completion time and a short list of the tasks involved. Companies interested in the project can obtain a detailed description from the DoT.

Estimated cost. The DoT constructs a cost estimate for every project. This estimate is based on the engineer's assessment of the work required to perform each task and prices derived from the winning bids for similar projects let in the past. The costs are then adjusted through a price deflator.

Federal law requires that the winning bid should be lower than 110% of the engineer's estimate. If a state decides to accept a bid that is higher than this threshold, it has to justify this action in writing. In this case the engineer's estimate has to be revised and verified for any possible mistake. In my data set, I observe a number of bids higher than 110% of the engineer's estimate. On multiple occasions, the winning bid is higher than this threshold. These facts suggest that bidders consider the probability of an event when this restriction comes into effect to be rather small. The assumption of no reserve price is justified in this environment.

Number of bidders. It is unclear if the auction participants have a good idea about the number of their competitors. The existing literature on highway procurement auctions tends to argue that this is a small market where participants are well informed about each other and can accurately predict the identities of auction participants.[11] I follow this tradition and assume that the number of actual bidders is known to auction participants.

5.2 Descriptive Statistics

I use data for highway procurement auctions held by the Michigan Department of Transportation (DoT) between February 1997 and December 2003. In particular, I focus on highway maintenance projects with bituminous resurfacing as the main task. The data set consists of a total of 3,947 projects. My information includes the letting date, the completion time, the location, the tasks involved, the identity of all the bidders, their bids, an engineer's estimate, and a list of planholders.

My choice of the projects' type is motivated by two objectives. First, I want to ensure that auction environment is characterized by private rather than common values. Second, I am looking for an environment that is likely to have unobserved auction heterogeneity. Highway maintenance projects are usually precisely specified and relatively simple. It is likely that bidders can predict their own costs for the project quite well. The existing uncertainty is, therefore, associated with variation in costs across firms, which is consistent with the private values environment. This variation is generated by differences in opportunity costs and input prices faced by different firms. Further, although highway maintenance projects are rather simple, their costs can be substantially affected by local conditions such as elevation and curvature of the road; traffic intensity; age and quality of the existing surface. Information about these features may not be available to the researcher. On the other hand, firms' representatives usually travel to the project site and therefore are likely to collect this information and incorporate it in their bids. Hence, I expect to find unobserved auction heterogeneity.

The paving companies participating in the maintenance auctions mostly differ by their size (employment, number of locations), specialization (single vs. multiple tasks), and frequency of participation in the DoT market. Each of these three features may potentially imply cost differences. The size maybe important if economies of scale are present. For example, larger companies are likely to own their equipment instead of renting it, which maybe cost reducing. Specialization may be important because projects in my data set usually involve some auxiliary work, such as marking or landscaping. A firm that specializes in paving may have to subcontract these tasks, whereas a firm with multiple specializations may be able to perform all the tasks internally. This may result in cost differences. Frequency of participation reflects experience with DoT projects. Construction Business Handbook notes that "...in most government contracts, a body of standard specifications have developed over the years. ...

[11] See, for example, Bajari and Ye (2003).

A bidder is required to learn a whole new and separate body of specifications..." Therefore companies that infrequently participate in the DoT market may have higher costs due to the limited experience with DoT projects. On the other hand, infrequency of participation maybe a manifestation of some inherent cost disadvantages or dynamic strategies of the company. Since size, specialization and frequency are observable to all market participants it is important to allow for the possibility that market participants have different beliefs about the distribution of costs for groups of companies that differ according to these features. Therefore, I allow for asymmetries between bidders. In particular, I distinguish between two types of bidders: regular (large) bidders and fringe bidders. The set of regular bidders is defined to include large companies that frequently participate in auctions held by the DoT and have multiple specializations. In particular, it includes firms that submit at least 12 bids per year, are within the top 10% of firms according to the employment size, and specialize in three or more types of tasks. The set of fringe bidders includes all other companies. It may be worth noting that the set of regular bidders is completely defined by the first two conditions. All large companies submitted at least 12 bids per year and specialized in multiple tasks. However, some of the companies that submitted many bids are quite small as well as may specialize in only one task. Companies that specialize in many tasks tend to be large.

In my data, the number of bidders per project varies between 1 and 11. More than 85% of projects attracted between 2 to 6 bidders with the mean number of bidders equalling 3.4 and standard deviation of 1.3. About 75% of the projects have an engineer's estimate between $100,000 and $1,000,000; 5% are below $100,000 and 20% are above $1,000,000.

Table 1c provides summary statistics of several important variables by the number of bidders. It shows that the mean of the engineer's estimate does not change significantly across groups of projects that attracted different number of bidders. The tabulation of the winning bid indicates that the difference between the engineer's estimate and the winning bid is positive and increases with the number of bidders, which implies that the engineer's estimate may not be a good indicator of the costs of the project. An important statistic of the data is "money left on the table" as represented by the difference between the lowest and second-to-lowest bid normalized by the engineer's estimate. This variable is usually taken to indicate the extent of uncertainty present in the market. "Money left on the table" is, on average, equal to 7% of the engineer's estimate and decreases with the number of bidders. The magnitude of the "money left on the table" variable is similar to the findings of other studies.[12] It indicates that cost uncertainty may be substantial. Table 1c also shows that the number of regular bidders is usually between 1 and 3 and increases only slightly with the total number of bidders.

Next, I explore if there is a scope for unobserved auction heterogeneity in my data.

[12] See, for example, Jofre-Bonnet and Pessendorfer (2003).

Table 2c presents results of the regression analysis of bid levels. It includes results of simple OLS regression, OLS regression with auction dummies and random effects regression. Each regression controls in a flexible way for firm and auction features observable to a researcher. In particular, I estimate the relationship between bid levels and the engineer's estimate, time to complete the project (duration), the smallest distance between firm's locations and the project site (Distance), the firm's backlog[13] at the time of bidding, number of potential bidders and indicator whether firm is a regular bidder. I use logarithms of bids, the engineer's estimate, and the backlog variable. These variables are measured in hundreds of thousands of dollars. Distance is measured in tens of miles. Regression analysis is performed conditional on the main task of the project and number of bidders. presented results correspond to the bituminous resurfacing projects with four bidders. OLS regression indicates that the engineer's estimate, distance to the project site, and backlog variable have a positive and statistically significant impact on the submitted bid. The effect of distance to the project on the bid level is quite small. It is about \$300 for every 10 extra miles. The time of completion does not produce a significant impact on the bid level. On the other hand, fringe bidders bid, on average, \$20,000 more than regular bidders. As expected the number of potential bidders has negative effect on the bid level. The second-order and interaction terms are not statistically significant. This analysis produces R^2 equal to 0.81 which indicates that variables included in the regression describe factors affecting bid levels quite well. However, when auction dummies are added to the set of explanatory variables R^2 increases to 0.89 which indicates that substantial amount of inter-auction variation cannot be explained by the variables available to the researcher. This result suggests that unobserved auction heterogeneity maybe present. The results of the random effects regression are very similar to the first two sets of results. They show that about 48% of the residual variation in the log's of bids may be across auctions. Thus, the regression analysis provides strong evidence for the importance of unobserved auction heterogeneity in Michigan highway procurement auctions.

5.3 Estimation Results

Estimation results presented below correspond to the set of projects with an engineer's estimate between \$500,000 and \$900,000 and time to completion between 6 and 9 months that attracted two regular and two fringe bidders. This set consists of 487 projects. The results for different values of engineer's estimate, duration, and the number of bidders are qualitatively similar.

In the estimation, the mean of the high type is normalized to be equal to one. Figures

[13] Backlog of the firm i is computed as a sum of the projects won by firm i that have not reached the completion deadline weighted by the ratio of time remaining before the project deadline to the total time allocated for the project.

1c and 2c present estimated distributions of the common and individual cost components. The recovered distribution of the common component has a mean equal to \$654,000 and a standard deviation of \$31,700. The recovered distributions of individual components for high and low types are very similar. The individual cost component of fringe type has a slightly higher mean and variance than the individual cost component of the regular type. The mean of the fringe type distribution is 1.2 . Standard deviations of the regular and fringe type distributions are 0.22 and 0.23, respectively.

Variance decomposition. Recall that bidder $i's$ cost for project j is given by $c_{ij} = y_j * x_{ij}$. Taylor approximation applied to C(.,.) as a function of X and Y allows us to approximate the variance of C in the following way:

$$Var(c) = (EY)^2 Var(X) + (EX)^2 Var(Y).$$

If $(EY)^2 Var(X)$ and $(EX)^2 Var(Y)$ are taken to represent parts of the cost variation generated by the variation in the individual and common cost components respectively, then it can be calculated that the individual cost component accounts for almost 15% of variation in the cost.[14]

Mark-ups over the bidders' costs. The estimated inverse bid functions are used to compute mark-ups over the bidders costs. The normalized mark-up, $\frac{b-c}{c} = \frac{a-x}{x}$, $x = \phi(a)$, ranges from 0.1% to 25% and, on average, is equal to 7.5% for the regular bidder. Mark-ups for the fringe type bidders range between 0.1% to 18% and, on average, are equal 6%. Mark-ups for the winning bid are, on average, equal to 16% and 14% respectively.

Inefficient outcomes. When bidders are asymmetric, it is possible that the project is not awarded to the lowest cost bidder, i.e., the auction outcome is not efficient. To compute the probability of such event for the selected set of projects, I use the estimated distributions of cost components to create a pseudo-sample of cost vectors, where for each vector two draws are taken from the cost distribution of regular type and two draws from the cost distribution of the fringe type. Then, for each cost draw, the bid value is calculated on the basis of the

[14] Note that this decomposition does not depend on our choice of mean normalization. Suppose that X_0 and Y_0 are true random variables representing the individual and common cost components, respectively. Due to normalization we are working with $X = \frac{1}{k} X_0$ and $Y = kY_0$, for some $k > 0$. Then

$$(EY)^2 Var(X) = k^2 (EY_0)^2 \frac{1}{k^2} Var(X_0),$$

i.e.

$$(EY)^2 Var(X) = (EY_0)^2 Var(X_0).$$

Similarly,

$$(EX)^2 Var(Y) = (EX_0)^2 Var(Y_0).$$

estimated bidding function. Finally, the fraction of the auctions in which the lowest bids do not correspond to the lowest costs is computed. This exercise is repeated 100 times. I find that the estimated probability of inefficient outcome is, on average, equal to 28% with 95% quantile range given by [26.6, 30.3].

Comparison to alternative auction models. Figure 4c compares the expected bid function estimated under the assumption of unobserved auction heterogeneity to the bid function recovered under the affiliated private values (APV) and independent private values (IPV) assumptions, respectively.[15] Both the IPV and APV procedures estimate total costs that are substantially lower than the expected costs estimated under the unobserved auction heterogeneity assumptions both for low and high type. In particular, the model with unobserved auction heterogeneity implies an average mark-up over the bidders' costs of around 7%. In contrast, the model with affiliated private values predicts average mark-ups of 11%, whereas the model with independent private values predicts 15%. The difference amounts to $33,000 in the case of affiliated private values and $61,000 for the model with independent private values. In both cases, confidence intervals for the IPV and APV estimates intersect the confidence interval constructed under the null of unobserved heterogeneity only for a very small part near the upper end of the support. Finally, the models with private and affiliated values predict a higher probability of inefficient outcome: 34% and 38%, respectively. These results suggest that the APV and IPV models may lead to significant overestimation of mark-ups and thus erroneous policy conclusions if the data are generated by the model with unobserved auction heterogeneity.

Figure 4c compares the expected density function of the cost distribution estimated under the assumption of unobserved auction heterogeneity to the cost density functions recovered under APV and IPV assumptions.[16] The IPV and APV densities are much flatter relative to the density function estimated under the assumption of unobserved auction heterogeneity. In both cases, confidence intervals for the IPV and APV estimates intersect the confidence interval constructed under the null of unobserved auction heterogeneity only for a very small part near the upper end of the support. The variance of the cost distribution estimated under the assumption of unobserved auction heterogeneity is about 25% lower that the variance of the cost distribution estimated under the assumption of affiliated private values and 35% lower than the variance of the cost distribution estimated under the assumption of independent private values. Therefore, if data are generated by the model with unobserved auction heterogeneity, then the models that fail to account for unobserved auction heterogeneity tend

[15]To compute the value of the expected inverse bid function at a point b, I first derived total costs for every value of the common component that could have resulted in a bid b and then computed an expectation of total costs with respect to the distribution of the common component.

[16]The total cost density function is computed as a convolution of the density functions of the common and the individual cost component.

to overestimate the uncertainty present in the market.

Measuring fit. To measure the fit of the model, I generate a sample of pseudo-bids on the basis of the model with unobserved auction heterogeneity and draws from the estimated distributions of the cost components. I then construct an estimate of the pseudo-bid densities for two types of bidders and compare them to the densities estimated from the real bids. Results for regular type bidders' are presented in Figure 5c. It shows that for both types, densities of pseudo-bids are very similar to the estimated bid densities. The largest discrepancy occurs near the mode of distribution, which is a typical feature of this estimation procedure as we have observed in the Monte Carlo section.

Evaluating assumptions of the model. The identification and estimation of the model with unobserved auction heterogeneity relies on the assumption about functional relation between total costs and cost components and the assumption that individual cost components are independent from each other and from the common cost component. Proposition 2 from the identification section allows us to evaluate validity of these assumptions in the data. It exploits the observation that the same methodology which we used to recover the distributions of individual bid and common cost components from the joint distribution of $(log(B_1), log(B_2))$ can be used to recover the distributions of the individual bid components of B_1, B_2, B_3 from the joint distribution of $(log(B_1) - log(B_3))$, $(log(B_2) - log(B_3))$. This suggests two checks for the internal consistency of the model. Let us suppose that B_1 and B_3 represent bids submitted by bidders of the regular type, whereas B_2 represent bids submitted by bidders of the fringe type. Then, if functional relation assumption and independence of individual components hold in the data, the distributions of individual bid components A_1 and A_3 recovered from the joint distribution of $(log(B_1) - log(B_3))$, $(log(B_2) - log(B_3))$ should look very similar. On the other hand if functional form assumption is very imprecise then the dependence in $(log(B_1) - log(B_3))$ and $(log(B_2) - log(B_3))$ is generated not only by $log(A_3)$ but also by some random variable that captures difference between the true and assumed functional form. In this case, we are likely to recover different distributions for the A_1 and A_3. Also, if individual cost components, and thus individual bid components, are not independent, then the estimate for the distribution of the A_1 is likely to underestimate the variance of the individual bid component of B_1, whereas an estimate for the distribution of A_3 is likely to overestimate the variance of individual bid component of B_3. This provides the first check on the model. The second check arises from the fact that we do not use an assumption about independence of the individual cost components from the common cost component when recovering distributions of A_1, A_2 and A_3 from the joint distribution of $(log(B_1) - log(B_3))$, $(log(B_2) - log(B_3))$. Therefore, if this assumption holds in the data then the distributions of A_1 and A_2 recovered from joint distribution of $(log(B_1) - log(B_3))$, $(log(B_2) - log(B_3))$ should be very similar to the distributions of A_1

and A_2 recovered from joint distribution of $(log(B_1), log(B_2))$. On the other hand if this assumption is violated then the estimated distributions of A_1 and A_2 produced by the second procedure are likely to have smaller variances than the the estimated distributions of A_1 and A_2 produced by the first procedure.

Table 3c presents estimation results for the two methodologies when B_1 and B_3 represent bids submitted by the regular type. In particular, it shows variances of the estimated distributions. The formal test that would allow us to establish if two sets of estimates are statistically different has not yet been developed in the literature. However, the estimates for the distribution of individual bid component look reasonably similar for the random variables A_1 and A_3 as well as across methodologies. This gives us confidence that the assumptions of the model hold at least approximately.

Robustness check. I perform several robustness checks to verify if my estimates are sensitive to some of the assumptions about auction environment.

The model of bidding behavior that I take to the data assumes that firms' bidding decisions are independent across auctions. This assumption maybe violated if bidders' decisions are affected by dynamic considerations. In particular, when company is capacity constrained it has to take into account the effect of winning project today on its ability to explore profitable opportunities tomorrow. If dynamic links between auctions are substantial in magnitude our estimates of the characteristic function of joint distribution of two bids submitted in the same auction may be biased which in turn would lead to biased estimates for the distributions of cost components. To evaluate the effect of dynamic links on the performance of the estimation procedure I re-estimate the model for the subset of projects, such that all regular firms bidding for the projects in this subset have their backlog variable between 30% and 75% of the maximum of backlog variable for this firm observed in our data. Even though this exercise substantially reduces the number of available projects and therefore leads to less precise estimates, the results of the estimation are quantitatively similar. In particular, findings about variance decomposition and biases from misspecification hold.

Presence of unobserved auction heterogeneity manifests itself through the correlation between bids submitted in the same auction. It is possible, however, that the correlation between bids is generated through some other mechanism. For example, it may arise if the auction environment has common values features. It may also arise if participating companies are systematically engaged in collusive behavior. I deal with the first issue by restricting my attention to maintenance projects that are unlikely to have any project-related uncertainty that could lead to a common values effect.

It is much harder to reject a possibility of collusion, since all the tests proposed in the literature depend on the particular collusion scheme employed. I use the test proposed by Porter in Zona (1993), which is based on the assumption that if there is a collusion scheme, then only the winning bid corresponds to a real cost realization and all other bids are "phony,", i.e., unsubstantiated by any cost realization. I use a procedure described in Athey and Haile (2001) to recover distribution of high and low type bids from the distribution of the winning bid. I then compare these distributions to the ones estimated from the losing bids. Distributions estimated through these two procedures appear to be similar, which gives us confidence that the data do not reflect the outcome of collusive behavior.

6 Conclusion

This paper proposes a semi-parametric procedure to recover the distribution of bidders' private information in the market for highway procurement when unobserved auction heterogeneity is present. I derive sufficient conditions under which the model is identified and show that the estimation procedure produces uniformly consistent estimators of the distributions in question. Using data for highway maintenance projects collected by the Michigan Department of Transportation, I estimate that conditional on the number of bidders, the type of the project as defined by the main task, and the size and duration bracket, about 75% of the project cost is a common knowledge among bidders. Therefore, private information accounts for only 25% of the project costs. Results of the estimation further reveal that the estimation procedures that ignore unobserved auction heterogeneity tend to estimate lower costs and higher mark-ups. They also tend to overestimate the variance of the distribution of bidders' costs.

My estimation procedure relies on the assumptions of the independence of individual cost components across bidders and from the common cost component. These assumptions seem consistent with the data. The downside of the first assumption is that the procedure cannot recover the distribution of private information if individual components are affiliated. Therefore, I cannot distinguish between the model with unobserved auction heterogeneity and the model with affiliated private values - the two models that generate correlated bids. However, these two models represent two extremes in the environment with private values. The model with unobserved auction heterogeneity attributes all correlation to the common knowledge of the bidders about the project costs, whereas the model with affiliated values interprets the correlation in bids as the correlation of bidders' costs that occurs due to factors unknown to the bidders. Therefore, the model with unobserved auction heterogeneity gives us the minimal and the model with affiliated values gives us the maximum amount of private information, which is consistent with the data.

The analysis is performed conditional on bidders' decisions to participate in the auction and on the number of bidders. The next step in the analysis of unobserved auction heterogeneity is to explicitly account for the participation decision of bidders. I address this issue in a different paper.

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A Proofs of Theoretical Results

Proof of Proposition 1:

LeBrun (1999) and Maskin and Riley (2000) establish that $(\alpha_{1y}(.), \alpha_{2y}(.))$ constitutes a unique vector of equilibrium strategies conditional on $Y = y$ if and only if functions $\alpha_{ky}(.)$ satisfy a system of differential equations for every a

$$\frac{1}{a - y * x} = \frac{\frac{(m_k-1)}{y} h_{X_k}(\frac{\alpha_{ky}^{-1}(a)}{y})}{(1 - H_{X_k}(\frac{\alpha_{ky}^{-1}(a)}{y}))\alpha'_{ky}(\alpha_{ky}^{-1}(a))} + \frac{\frac{m_{-k}}{y} h_{X_k}(\frac{\alpha_{ky}^{-1}(a)}{y})}{(1 - H_{X_k}(\frac{\alpha_{ky}^{-1}(a)}{y}))\alpha'_{ky}(\alpha_{ky}^{-1}(a))} \tag{5}$$

with boundary conditions given by

(A_1) $\alpha_{ky}(y * \overline{x}) = \overline{x}$ and (2) there exists $d_{ky} \in [y * \underline{x}, y * \overline{x}]$ such that $\alpha_{ky}(y * x) = d_{ky}$.

The vector of equilibrium strategies conditional on $y = 1$ satisfies the system of differential equations

$$\frac{1}{a - x} = \frac{(m_k - 1)h_{X_k}(\alpha_{k1}^{-1}(a))}{(1 - H_{X_k}(\alpha_{k1}^{-1}(a)))\alpha'_{k1}(\alpha_{k1}^{-1}(a))} + \frac{m_{-k}h_{X_k}(\alpha_{k1}^{-1}(a))}{(1 - H_{X_k}(\alpha_{k1}^{-1}(a)))\alpha'_{k1}(\alpha_{k1}^{-1}(a))}. \tag{6}$$

Let us consider a set of functions $(\gamma_{1y}, \gamma_{2y})$, $\gamma_{ky} : [y * \underline{x}, y * \overline{x}] \to (0, \infty)$ such that

$$\begin{aligned} \gamma_{ky}(z) &= y * \alpha_k(\frac{z}{y}), \\ \gamma_{ky}(y * \overline{x}) &= y * \overline{x}, \\ \gamma_{ky}(y * \underline{x}) &= y * d_{y1}. \end{aligned}$$

Then

$$\alpha_k^{-1}(\frac{a}{y}) = \frac{\gamma_{ky}^{-1}(a)}{y}$$

and

$$(\alpha_k^{-1}(z))' = (\frac{\gamma_{ky}^{-1}(y * z)}{y})' = (\gamma_{ky}^{-1}(y * z))'.$$

If we substitute functions γ_{1y}, γ_{2y} in the system of equations (5), then (5) can be transformed to

$$\frac{1}{\frac{a}{y} - x} = \frac{(m_k - 1)h_{X_k}(\alpha_{k1}^{-1}(\frac{a}{y}))}{(1 - H_{X_k}(\alpha_{k1}^{-1}(\frac{a}{y})))\alpha'_{k1}(\alpha_{k1}^{-1}(\frac{a}{y}))} + \frac{m_{-k}h_{X_k}(\alpha_{k1}^{-1}(\frac{a}{y}))}{(1 - H_{X_k}(\alpha_{k1}^{-1}(\frac{a}{y})))\alpha'_{k1}(\alpha_{k1}^{-1}(\frac{a}{y}))}. \tag{7}$$

Replacing $\frac{a}{y}$ with z in the system of equations (7), we are back to the system of equations (7), which we know is satisfied by $\alpha_{k1}(z)$, which implies that γ_{1y}, γ_{2y} satisfy the system of equations (5). They also satisfy corresponding boundary conditions by definition if d_{ky} is set equal to $y * d_{y1}$. Since the solution to the system (5) that satisfies boundary conditions (A_1) is unique and constitutes the set of equilibrium functions, γ_{1y}, γ_{2y} coincide with α_{1y}, α_{2y}. Thus

$$\alpha_{ky}(z) = y * \alpha_{k1}(\frac{z}{y}).$$

When $Y = y$, $z = y * x$, where $x \in [\underline{x}, \overline{x}]$, then

$$\alpha_{ky}(z) = y * \alpha_{k1}(x).$$

Proof of Theorem 1:

(a) I start by establishing a statistical result that I use to prove Theorem 1. Namely,

Lemma 1

Let X be a random variable with the probability density function f(.) and support $[\underline{x},\overline{x}]$, *then the characteristic function of variable X is non-vanishing, i.e. it does not turn into zero on any non-empty interval of the real line.*

Proof

The idea of a proof is to consider the extension of the characteristic function $\varphi_X(t) = \int\limits_{\underline{x}}^{\overline{x}} e^{itx} f(x)dx$ to the complex domain. In particular, I consider function $\widetilde{\varphi}_X(.)$ defined as $\widetilde{\varphi}_X(z) = \int\limits_{\underline{x}}^{\overline{x}} e^{izx} f(x)dx$ at an arbitrary complex point z. It is straightforward to show that $\widetilde{\varphi}_X(.)$ is an entire function, i.e. it is infinitely complex differentiable at every finite point of the complex plane. Therefore, it can only be equal to zero in a countable number of points. Thus the number of points where $\varphi_X(t)$ is equal to zero cannot be more than countable which means that $\varphi_X(t)$ is non-vanishing.

Finally, $\widetilde{\varphi}_X(.)$ is an entire function because

$$\widetilde{\varphi}_X^{(k)}(z) = \int\limits_{\underline{x}}^{\overline{x}} (ix)^k e^{izx} f(x)dx.$$

Notice that for every k $\widetilde{\varphi}_X^{(k)}(z)$ is well defined due to the boundedness of the X' support. That concludes the proof of Lemma 1.

(b) Random variables Y, A_i, $\log(Y)$ and $\log(A_i)$ have bounded supports and therefore have non-vanishing characteristic functions.

(c) As has been established in Proposition 1, $B_{kij} = y_j * a_{kij}$, where a_{kij} is an individual bid component. Two bids per auction produce two relationships $B_1 = Y * A_1$ and $B_2 = Y * A_2$. Since Y and A_k's take only positive values, these relationships can be rewritten as

$$\begin{aligned} \log(B_1) &= \log(Y) + \log(A_1), \\ \log(B_2) &= \log(Y) + \log(A_2). \end{aligned}$$

As we noted before characteristic functions of $\log(Y)$ and $\log(A_i)$ are non-vanishing. Therefore theorem by Kotlarski (1966)[17] applies directly to this environment and ensures that distributions of $\log(Y)$, $\log(A_1)$, and $\log(A_2)$ are identified up to a constant. To fix the constant we assume that $E(\log(A_1)) = 0$. Then, since the distribution functions of $\log(Y)$, $\log(A_1)$, and $\log(A_2)$ are uniquely identified, and, since $\log(.)$ is a strictly monotone function, then the distribution functions of Y, A_1, and A_2 are uniquely identified as well.

Since the individual bid components represent bids that would have been submitted in the auction game without unobserved heterogeneity and with asymmetric bidders, then the identification of the distribution of the individual cost component from the distribution of the individual bid component follows according to the results established by Laffont and Vuong (1996).

Remark

The realizations of the common component and individual cost component corresponding to bid b_{ij} are not uniquely identified. In particular, let us denote by $b_j = \{b_{ij}\}$ the vector of bids submitted in the auction j and by $x_j = \{x_{ij}\}$ a vector of individual cost component draws in the auction j. We will show now that for a generic b_j, (i.e., $b_j = \{b_{ij}$, $\underline{y} * \alpha_i(\underline{x}) < b_{ij} < \overline{y} * \alpha_i(\overline{x})\}$) there exist multiple pairs (y_j, x_{ij}) such that $b_{ij} = \beta_{y_j}(x_{ij})$.

Consider $\{[\underline{y_i^0}, \overline{y_i^0}],\ \underline{y_i^0} = \max(\frac{b_{ij}}{\alpha_i(\overline{x})}, \underline{y}),\ \overline{y_i^0} = \min(\frac{b_{ij}}{\alpha_i(\underline{x})}, \overline{y})\}$.

If data are generated by the model with unobserved heterogeneity, then $b_{ij} = y_j * \alpha_{y_j}(x_{ij})$ and

$$\max(\frac{y_j * \alpha_{y_j}(x_{ij})}{\alpha_i(\overline{x})\}}, \underline{y}) \leq y_j \leq \min(\frac{y_j * \alpha_{y_j}(x_{ij})}{\alpha_i(\underline{x})}, \overline{y}),$$

with the equality on either side occurring only if $x_{ij} = \underline{x}$ or $x_{ij} = \overline{x}$. Since the event where $x_{ij} = \underline{x}$ or $x_{ij} = \overline{x}$ for some i has a probability of zero, generally

$$\max(\frac{y_j * \alpha_{y_j}(x_{ij})}{\alpha_i(\overline{x})\}}, \underline{y}) < y_j < \min(\frac{y_j * \alpha_{y_j}(x_{ij})}{\alpha_i(\underline{x})}, \overline{y}).$$

[17] See Rao (1992).

Thus,

$$\underline{y^0} = \max(y_i^0) < y_j < \min(y_i^0) = \overline{y^0}.$$

For any $y \in [\underline{y^0}, \overline{y^0}]$, let us define $a_{iy} = \frac{b_{ij}}{y}$, $i = 1, .., m$. Notice that $\alpha_i(x) \leq x_{iy} \leq \alpha_i(x)$ by construction. This means that the inverse bid function from (D_6) could be used to find x_{iy} such that $a_{iy} = \alpha_i(x_{iy})$. Thus, I have shown that there are multiple pairs $(y, \{x_{iy}\})$ that rationalize b_j.

Proof of proposition 2

(1) The proof follows from the property of independent variables: if the random variables Z_1 and Z_2 are independent, then so are $f(Z_1)$ and $f(Z_2)$, for any function $f(.)$.

(2) If X_i's are independent, then so are $\log(X_i)$. The structure of the bidder's cost, $c_i = y * x_i$, implies that $\log(\frac{B_{1i_1}}{B_{1i_1}}) = \log(A_{1i_1}) - \log(A_{2i_2})$ and $\log(\frac{B_{1i_1}}{B_{2i_2}}) = \log(A_{1i_3}) - \log(A_{2i_2})$. Then by Kotlarski (1966) theorem the characteristic function of $\log(A_3)$ is given by

$$f_3(t) = \int_0^t \frac{\Theta_1(0, u_2)}{\Theta(0, u_2)} du_2.$$

and characteristic function of $\log(A_3)$ by

$$f_1(-t) = \frac{\Theta(t, 0)}{\Lambda_{\log(A_{i_3})}(t)}.$$

If bidders 1 and 3 are of the same type then characteristic functions of $\log(A_1)$ and $\log(A_3)$ should be the same, i.e.

$$f_1(t) = f_3(t).$$

(3) If Y and X_i's are independent, the cost structure is given by $c_{ij} = y_j * x_{ij}$, then Kotlarski (1966) theorem applied to $(\log(B_{i_1j}), \log(B_{i_2j}))$ implies that the characteristic function of $\log(A_{1i_1j})$ are given by the function $\Phi_{\log(A_1)}(t)$. Kotlarski (1966) theorem applied to $\log(\frac{B_{1i_1j}}{B_{1i_1j}})$ and $\log(\frac{B_{1i_1j}}{B_{2i_2j}})$ implies that the characteristic function of $\log(A_{1i_1j})$ is given by $f_1(t)$. Thus, the following equality has to hold

$$\Phi_{\log(A_1)}(t) = f_1(t).$$

Proof of proposition 3:

According to (W_2), $\Phi_{\log(Y)}(.)$, $\Phi_{\log(A_{1i_l})}(.)$ and $\Phi_{\log(A_{2i_p})}(.)$ are the same for all pairs (i_l, i_p) such that $i_p = 1, .., m_1$ and $i_l = m_1 + 1, .., m_2$. This implies that i indices can be dropped, so that we can focus on just three functions: $\Phi_{\log(Y)}(.)$, $\Phi_{\log(A_1)}(.)$, and $\Phi_{\log(A_2)}(.)$.

If (W_1) is satisfied, then there exist independent random variables Y, A_1, and A_2, such that the characteristic functions of $\log(Y)$, $\log(A_1)$, and $\log(A_2)$ are given by $\Phi_{\log(Y)}(.)$, $\Phi_{\log(A_1)}(.)$, $\Phi_{\log(A_2)}(.)$, respectively. Kotlarski (1966) shows that

$$\Phi_{\log(Y)}(t_1 + t_2)\Phi_{\log(A_1)}(t_1)\Phi_{\log(A_2)}(t_2) = \Psi(t_1, t_2).$$

This equality implies that $(\log(Y)+\log(A_1), \log(Y)+\log(A_2))$ are distributed the same as $(\log(B_1), \log(B_2))$.

Let us consider $X_k = \phi_k(A_k)$. Then Y, X_1, and X_2 define the model with unobserved heterogeneity that rationalizes the data.

Two-dimensional model with unobserved heterogeneity

The cost of bidder i is equal to $c_i = y_1 + y_2 * x_i$, where $Y = (Y_1, Y_2)$ is a random vector representing the common cost component and $X = (X_{11}, .., X_{1m_1}, X_{2,m_1+1}, .., X_{2m})$. Random variables (X, Y) are distributed on $[\underline{y_1}, \overline{y_1}] \times [\underline{y_2}, \overline{y_2}] \times [\underline{x}, \overline{x}]^m$ according to the probability distribution function $H(., .., .)$ with the associated probability density function $h(., .., .)$.

Assumptions $(F_1) - (F_6)$ are analogous to $(D_1) - (D_6)$ of the one-dimensional case.

(F_1) The components of Y and X are independent:

$$H(y_{10}, y_{20}, x_{10}, .., x_{m0}) = H_{Y_1}(y_{10})H_{Y_2}(y_{20}) \prod_{j=1}^{j=m_1} H_{X_1}(x_{j0}) \prod_{j=m_1+1}^{j=m} H_{X_2}(x_{j0}),$$

where H_{Y_1}, H_{Y_2}, H_{X_1} and H_{X_2} are marginal distribution functions of Y_1, Y_2, X_{1j}, and X_{2j}, respectively.

(F_2) The probability density functions of the individual cost component, h_{X_1} and h_{X_2}, are continuously differentiable and bounded away from zero on $[\underline{x}, \overline{x}]$.

(F_3) $EX_{1j} = 1$ for all $j = 1, ..., m_1$.

(F_4) (a) The number of bidders is common knowledge;

(b) There is no binding reservation price;

The condition (F_2) ensures the existence and uniqueness of the equilibrium in the auction game corresponding to the realization (y_1, y_2) of (Y_1, Y_2). The conditions (F_1),(F_3), and (F_4) provide a basis for the identification of the probability density functions of Y_1,Y_2, A_1, A_2, X_1 and X_2.

Theorem 1a

If conditions (F_1)$-$(F_4) are satisfied, then the probability density functions h_{Y_1},h_{Y_2},h_{X_1}, h_{X_2} are uniquely identified from the joint distribution of four arbitrary bids $(B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4})$.

Sketch of the proof

(a) Applying Kotlarski's argument to the log-transformed random variables $(B_{i_1} - B_{i_2})$ and $(B_{i_3} - B_{i_4})$ allows identification of the probability density function h_{Y_2}.

(b) The joint characteristic function of $((B_{i_1} - B_{i_3}), (B_{i_2} - B_{i_3}))$ in conjunction with the characteristic function of Y_2 (identified in (a)) allows identification of the joint characteristic function of $((A_{i_1} - A_{i_3}), (A_{i_2} - A_{i_3}))$, which according to the Kotlarski argument in turn implies that the probability density functions of A_{i_1} and A_{i_2} are identified.

(c) The probability density functions $g_{A_{i_1}}$, h_{Y_2} uniquely determine the probability distribution and thus the characteristic function of $Y_2 * A_{i_1}$, which allows unique identification of the probability distribution of Y_1 from the characteristic function of B_{i_1}.

(d) The argument developed in Laffont and Vuong (1996) can be applied to establish identification of the probability density functions from the probability distribution of A_{i_1} and A_{i_2}.

Thus I have established that h_{Y_1}, h_{Y_2}, h_{X_1}, h_{X_2} are identified from the joint distribution of four arbitrary bids.

Similar to the one-dimensional case, the exact realizations of y_{1j}, y_{2j} and $\{x_{ij}\}$ are not uniquely identified.

Details of the estimation procedure

Step 1.

1. The log transformation of bid data is performed to obtain $LB_{i_l} = \log(B_{i_l})$ and $LB_{i_p} = \log(B_{i_p})$, where $i_l = 1, .., m_{01}$ and $i_p = m_{01} + 1, .., m_0$.

2. The joint characteristic function of an arbitrary pair (LB_{i_l}, LB_{i_p}) is estimated by

$$\widehat{\Psi}(t_1,t_2) = \frac{1}{m_{01}m_{02}} \sum_{1\le l\le m_{01}, m_{01}+1\le p\le m_0} \frac{1}{n_0}\sum_{j=1}^{n_0} \exp(it_1 \cdot B_{i_l j} + it_2 \cdot B_{i_p j})$$

and the derivative of $\Psi(.,.)$ with respect to the first argument, $\Psi_1(.,.)$, by

$$\widehat{\Psi_1}(t_1,t_2) = \frac{1}{m_{01}m_{02}} \sum_{1\le l\le m_{01},\ m_{01}+1\le p\le m_0} \frac{1}{n_0}\sum_{j=1}^{n_0} iB_{i_l j}\exp(it_1 \cdot B_{i_l j} + it_2 \cdot B_{l_p j})$$

I average over all possible pairs to enhance efficiency.

3. The characteristic functions of the log of individual bid components LA_k, $k = 1, 2$, and the log of the common cost component LY are estimated as

$$\begin{aligned}
\widehat{\Phi}_{LY}(t) &= \exp\left(\int_0^t \frac{\widehat{\Psi}_1(0,u_2)}{\widehat{\Psi}(0,u_2)} du_2\right), \\
\widehat{\Phi}_{LA_1}(t) &= \frac{\widehat{\Psi}(t,0)}{\widehat{\Phi}_{\log Y}(t)}, \\
\widehat{\Phi}_{LA_2}(t) &= \frac{\widehat{\Psi}(0,t)}{\widehat{\Phi}_{\log Y}(t)}.
\end{aligned}$$

4. The inversion formula is used to estimate densities g_{LA_k}, $k = 1, 2$, and g_{LY}.

$$\begin{aligned}
\widehat{g}_{LA_k}(u_1) &= \frac{1}{2\pi}\int_{-T}^{T} \exp(-itu_1)\widehat{\Phi}_{LA_k}(t)dt, \\
\widehat{h}_{LY}(u_2) &= \frac{1}{2\pi}\int_{-T}^{T} \exp(-itu_2)\widehat{\Phi}_{LY}(t)dt
\end{aligned}$$

for $u_1 \in [\log(\underline{a_k}), \log(\overline{a_k})]$, and $u_2 \in [\log(\underline{y}), \log(\overline{y})]$, where T is a smoothing parameter.

5. The densities of A_k and Y are obtained as

$$\begin{aligned}\widetilde{g}_{A_k}(a) &= \frac{\widehat{g}_{LA_k}(\log(a))}{a}, \\ \widetilde{h}_Y(y) &= \frac{\widehat{h}_{LY}(\log(y))}{y},\end{aligned}$$

for $a \in [\underline{a_k}, \overline{a_k}]$, and $y \in [\underline{y}, \overline{y}]$.

Step 2

1. The estimate of the individual bid component density, $\widetilde{g}_{A_k}(.)$, $k = 1, 2$, is used to generate a sample of pseudo-bids $\{\widetilde{a}_{kj}\}$, $j = 1, ..., L$.

2. The sample of pseudo bids is used to generate a sample of pseudo-costs as

$$\begin{aligned}\widetilde{x}_{1j} &= \widetilde{a}_{1j} + \frac{(1-\widetilde{G}_{A_1}(a_{1j}))\cdot(1-\widetilde{G}_{A_2}(a_{1j}))}{(m_1-1)\cdot\widetilde{g}_{A_1}(a_{1j})\cdot(1-\widetilde{G}_{A_2}(a_{1j}))+m_2\cdot\widetilde{g}_{A_2}(a_{1j})\cdot(1-\widetilde{G}_{A_1}(a_{1j}))}, \\ \widetilde{x}_{2j} &= \widetilde{a}_{2j} + \frac{(1-\widetilde{G}_{A_1}(a_{2j}))\cdot(1-\widetilde{G}_{A_2}(a_{2j}))}{m_1\cdot\widetilde{g}_{A_1}(a_{2j})\cdot(1-\widetilde{G}_{A_2}(a_{2j}))+(m_2-1)\cdot\widetilde{g}_{A_2}(a_{2j})\cdot(1-\widetilde{G}_{A_1}(a_{2j}))},\end{aligned}$$

where

$$\widetilde{G}_{A_k}(a) = \int_{\underline{\widehat{a}}_k^1}^{a} \widetilde{g}_{A_k}(z)dz$$

and $\underline{\widehat{a}}_k^1$ is an estimate of the lower bound of the support of $g_{A_k}(.)$ (see part A of the Appendix for the discussion of the support estimation).

3. The density of the individual cost component is non-parametrically estimated from the sample of pseudo-cost

$$\widetilde{h}(x) = \frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} K_h\left(\frac{x-\widetilde{x}_{kj}}{\delta_k}\right),$$

where $K_h(.)$ is a kernel function, and δ_{h_k} is the bandwidth.

3a. The estimation procedure described in Step 1 leads to a zero-mean distribution of $\log(A_1)$, which does not necessarily correspond to the random variable X_1 such that $EX_1 = 1$. To arrive at the final estimates of the distributions in question we have to perform an adjustment. Let e denote the mean of the estimated distribution of random variable X_1. Then $\widehat{h}_{X_k}(x) = \frac{\widetilde{h}_{X_k}(ex)}{e}$, $\widehat{h}_Y(y) = e\widetilde{h}_Y(\frac{y}{e})$ are the final estimates of the individual and common cost component probability density functions.

4. I have also constructed an estimate of the cost density function

$$\widehat{h}_{C_k}(c) = \int_{\underline{y}}^{\overline{y}} \widehat{h}_{X_k}(\frac{c}{y})\widehat{h}_Y(y)dy,$$

for $c \in [\underline{x} \cdot \underline{y}, \overline{x} \cdot \overline{y}]$.

5. An average inverse bid function was estimated as

$$\widehat{\vartheta}_k(b) = \int_{\underline{y}}^{\overline{y}} y \cdot \widehat{\phi}_k(\frac{b}{y})\widehat{h}_Y(y)dy,$$

where $\widehat{\phi}_k(.)$ is an estimate of the individual inverse bid function given by

$$\widehat{\phi}_1(a) = a + \frac{(1-\widetilde{G}_{A_1}(a)) \cdot (1-\widetilde{G}_{A_2}(a))}{(m_1-1) \cdot \widetilde{g}_{A_1}(a) \cdot (1-\widetilde{G}_{A_2}(a)) + m_2 \cdot \widetilde{g}_{A_2}(a) \cdot (1-\widetilde{G}_{A_1}(a))},$$

$$\widehat{\phi}_2(a) = a + \frac{(1-\widetilde{G}_{A_1}(a)) \cdot (1-\widetilde{G}_{A_2}(a))}{m_1 \cdot \widetilde{g}_{A_1}(a) \cdot (1-\widetilde{G}_{A_2}(a)) + (m_2-1)\widetilde{g}_{A_2}(a) \cdot (1-\widetilde{G}_{A_1}(a))}.$$

Both integrals were computed using Monte-Carlo integration with respect to $\widehat{h}_Y(.)$.

Properties of the Estimator (Proposition 5)

I start by describing how the supports of distributions of the individual bid and the common cost components can be estimated. Then I proceed to the proof of proposition 5.

Estimation of the Support Bounds

Strictly speaking bounds of the support are recovered during the inversion procedure when the density function of the distribution in question is computed. According to the inversion formula, the density function recovered from the theoretical characteristic function should approach zero as smoothing parameter T approaches infinity at every point outside of the support. Therefore, the upper and lower bounds of the support are respectively defined as lower and upper limits of the points where density function is equal to zero. In estimation, the density function recovered from the estimated characteristic function does not, in general, equal zero outside of the support. An econometrician, therefore, has to choose cut-off points that correspond to sufficiently low values of the estimated density function. Unfortunately, econometric theory does not provide us with guidelines on how to choose such cut-off points. That is why I use a different approach in this paper. I estimate bounds of the supports for the distributions of interest using restrictions imposed by the model with unobserved auction heterogeneity. If data are generated by the model with unobserved auction heterogeneity then this approach leads to consistent estimators of the support bounds. The proof of this statement and the derivation of the rate of convergence are given together with the proof of proposition 5. Below I describe a procedure to estimate support bounds of the distributions of the individual bid and the common cost components.

Notice that the distributions of the components are identified up to the location only. So, I start with arbitrary choice of supports, then estimate the shift in supports that accompanies the first-stage estimation. Finally, after the second-stage I adjust supports, so that estimated distributions satisfy assumption (D_3).

Initially, I ignore the assumption (D_3), $EX_{i1} = 1$. Instead, I assume that there are no restrictions on the means of the distributions. To fix the supports of the distributions in question, I assume that support of LY is symmetric around zero. I denote the support of the log of the common component by $[-y^0, y^0]$ and the support of the log of the individual bid component of the type 1 bidder by $[\underline{a_1}^0, \overline{a_1}^0]$. Then the support of the log of bids for type 1 is given by $[\underline{a}_1^0 - y^0, \overline{a}_1^0 + y^0]$, and the support of the differences in logs of bids is given by $[\underline{a}_1^0 - \overline{a}_1^0, \overline{a}_1^0 - \underline{a}_1^0]$. Since the bounds of these supports can be estimated as $[min(\log(b_{1lj})), max(\log(b_{1lj}))]$ and $[min(\log(b_{1lj}) - \log(b_{1pj})), max(\log(b_{1lj}) - \log(b_{1pj}))]$, we arrive at the system of equations

$$\begin{aligned} min(\log(b_{1lj})) &= \widehat{\underline{a}}_1^0 - \widehat{y}^0, \\ max(\log(b_{1lj})) &= \widehat{\overline{a}}_1^0 + \widehat{y}^0, \\ max(\log(b_{1lj}) - \log(b_{1pj})) &= \widehat{\overline{a}}_1^0 - \widehat{\underline{a}}_1^0, \end{aligned}$$

which can be solved to get

$$\begin{aligned} \widehat{y}^0 &= \frac{max(\log(b_{1lj})) - min(\log(b_{1lj})) - max(\log(b_{1lj}) - \log(b_{1pj}))}{2}, \\ \widehat{\underline{a}}_1^0 &= \frac{min(\log(b_{1lj})) + max(\log(b_{1lj})) - max(\log(b_{1lj}) - \log(b_{1pj}))}{2}, \\ \widehat{\overline{a}}_1^0 &= \frac{min(\log(b_{1lj})) + max(\log(b_{1lj})) + max(\log(b_{1lj}) - \log(b_{1pj}))}{2}. \end{aligned}$$

Formulas for the estimation of the characteristic function of the common cost component in (2) and (4) have been derived under the assumption that $E(LA_1) = 0$. Hence, the mean of the common component equals the bids' mean. Thus, the probability density functions $\widehat{h}_Y(.)$ and $\widehat{h}_{LA_1}(.)$ are shifted so as to achieve a zero mean for the distribution of the log of the individual bid component LA_1. If the symmetrization of the common component support initially assigned a mean of e_1 to the individual bid component of type 1, then Step 1 is going to produce density $\widehat{h}_{LY}(.)$ with the support $[-\widehat{y}^0+e_1, \widehat{y}^0+e_1]$ and $\widehat{g}_{LA_1}(.)$ with the support $[\widehat{\underline{a}}_1^0-e_1, \widehat{\underline{a}}_1^0-e_1]$, where the shift factor e_1 is given by the solution to the equation

$$\int_{\widehat{\underline{a}}_1^0-e_1}^{\widehat{\underline{a}}_1^0-e_1} a\widehat{g}_{LA_1}(a)da = 0.$$

I use this equation to estimate e_1 through a line search method.

The procedure described above produces estimates for the supports of Y and A_k

$$\begin{aligned}
[\widehat{\underline{y}}_1, \widehat{\overline{y}}_1] &= [\exp(-\widehat{y}^0+\widehat{e}_1), \exp(\widehat{y}^0+\widehat{e}_1)], \\
[\widehat{\underline{a}}_1^1, \widehat{\overline{a}}_1^1] &= [\exp(\widehat{\underline{a}}_1^0-\widehat{e}_1), \exp(\widehat{\overline{a}}_1^0-\widehat{e}_1)], \\
[\widehat{\underline{a}}_2^1, \widehat{\overline{a}}_2^1] &= [\exp(min(\log(b_{2lj}))+\widehat{y}^0-\widehat{e}_1), \exp(max(\log(b_{2lj}))-\widehat{y}^0+\widehat{e}_1)].
\end{aligned}$$

Proof of proposition 5 :

First, I describe a set of technical assumptions needed to establish the rate of convergence for the estimators of density functions. Assumption 1 is a more technical version of assumptions D_2,which assumes $R=1$. The proof is given for a more general case of $R \geq 1$. Assumption 2 concerns the properties of the kernel used in the second-stage estimation. Finally, assumption 3 describes the choice of the bandwidth in the second-stage estimation.

Assumption 1:

(i) The supports of H_{X_k} are given by $S(H_k) = [\underline{x}, \overline{x}]$, $x \geq 0$, $x \leq x$, for $k \in \{1, 2\}$;

(ii) $h_k(.)$ are bounded away from zero on every closed subset of the interior of $S(H_k)$;

(iii) $H_k(.)$ admit up to $R+1$ continuous bounded derivatives, with $R \geq 1$.

Assumption 2:

(i) The kernel $K_h(.)$ is symmetric with bounded hypercube support and twice continuous bounded derivatives;

(ii) $\int K_h(x)dx = 1$

(iii) $K_h(.)$ is of order $R+1$. Thus, moments of order strictly smaller than the given order vanish.

Assumption 3:

(i) The bandwidth δ_{h_k} is of the form

$$\delta_{h_k} = \lambda_{h_k} (\frac{\log(L)}{L})^{R/(2R+1)},$$

where λ_{h_k} is a strictly positive constant and L is the number of pseudo-bid draws in the second-stage estimation.

The proof consists of several steps.

(1) First, I establish that the distribution function and the probability density functions of the individual bid components inherit properties of the distribution function and the probability density functions of the individual cost component. Namely,

Lemma 2

Given Assumption 1, the distribution functions $G_{A_k}(.)$ satisfy:

(i) its supports $S(G_{A_k})$ are given by $[\underline{a}_k, \overline{a}_k]$ with $\overline{a}_k = \overline{x}$;

(ii) G_{A_k} admit up to $R+1$ continuous bounded derivatives on every closed subset of the interior of $S(G_{A_k})$.

(iii) For every closed subset of the interior of $S(G_{A_k})$, there exists $c_g > 0$ such that $|G^{(r)}_{A_k}(a)| \geq c_g > 0$ on this subset.

Proof

The point (i) is established in section 2. To show that the points (ii) and (iii) holds, I use the relationship between the distribution functions of the individual bid components and the distribution functions of the individual cost components. Namely,

$$G_{A_k}(a) = H_{X_k}(\phi_k(a)),$$

where $\phi_k(.)$ is the inverse individual bid function of the bidder of type k. Then, $G^{(r)}_{A_k}(a)$ is a sum of terms like $H^{(u)}_{X_k}(\phi_k(a))\ \phi^{(v)}_k(a)$ for $u, v = \{1, .., r\}$. By the equilibrium characterization $\phi_k(.)$ is continuously differentiable and monotonically increasing. This implies that $\phi_k(.)$ is bounded and $a > \phi_k(a)$ on every closed subset of the interior of $S(G_{A_k})$. The necessary first order conditions for the game with $Y = 1$ can be re-written as follows

$$\phi^{'}_k(a) = \frac{(1 - H_{X_k}(\phi(a)))}{(m-1)h_{X_k}(\phi(a))}\{\frac{m_{-k} - 1}{a - \phi_k(a)} - \frac{m_{-k}}{a - \phi_{-k}(a)}\}.$$

This representation can further be used to show that $\{\phi^{(u)}_k(.)\}_{u \leq R+1}$ are well defined, continuos and therefore bounded on every closed subset of the interior of $S(G_{A_k})$. Hence $\{G^{(r)}_{A_k}(.)\}_{r\leq R+1}$ are well defined, continuos and bounded on every closed subset of the interior of $S(G_{A_k})$. Point (iii) then follows from the properties of $\{H^{(u)}_{X_k}(.)\}_{u\leq R+1}$ and properties of the equilibrium bid function.

(2) If probability density functions of cost components are ordinarily smooth of order $\varkappa > 1$, then Theorems 3.1- 3.2 in Li and Vuong (1998) apply that establish uniform consistency of the first stage estimators. In particular, they establish that

$$\begin{aligned} \sup_{y\in S(H_{LY})} |\widehat{h}_{LY}(y) - h_{LY}(y)| &= O(\frac{n}{\log\log n})^{\frac{-(2\varkappa-1)}{2(2+5\varkappa)}} \\ \sup_{a\in S(G_{LA_k})} |\widehat{g}_{LA_k}(a) - g_{LA_k}(a)| &= O(\frac{n}{\log\log n})^{\frac{-(2\varkappa-1)}{2(2+6L)}}. \end{aligned}$$

Since,

$$\begin{aligned} h_Y(y) &= \frac{h_{LY}(\log(y))}{\log(y)} \\ h_{A_k}(a) &= \frac{h_{LA_k}(\log(a))}{\log(a)} \end{aligned}$$

and $a \in [\underline{a}_k, \overline{a}_k]$, $\underline{a}_k > \underline{x}_k > 0$, then

$$\begin{aligned} \sup_{y\in S(H_Y)} |\widehat{h}_Y(y) - h_Y(y)| &= O(\frac{n}{\log\log n})^{\frac{-(2\varkappa-1)}{2(2+5\varkappa)}} \\ \sup_{a\in S(G_{A_k})} |\widehat{g}_{A_k}(a) - g_{A_k}(a)| &= O(\frac{n}{\log\log n})^{\frac{-(2\varkappa-1)}{2(2+6L)}}. \end{aligned}$$

(3) Uniform consistency of the estimators for the individual inverse bid function and the probability density function of the individual component follows the logic of Proposition 3 and Theorem 3 of Guerre, Perrigne, Vuong (2000).

(a) First, we derive the rate of convergence for the support bounds, $\underline{a}_k$ and $\overline{a}_k$. Recall that bounds of supports have been derived in several steps. First, supports of the distributions of LB_{1i} and $(LB_{1i_1} - LB_{1i_2})$ have been estimated as

$$[min(\log(b_{1lj})), max(\log(b_{1lj}))]$$
$$[min(\log(b_{1lj}) - \log(b_{1pj})), max(\log(b_{1lj}) - \log(b_{1pj}))].$$

These are maximum likelihood estimators for the support bounds of corresponding densities. (They are well defined due to (v) of Lemma 2.) We know that they converge to the true value of the support bounds at the rate of n. The preliminary estimates for the bounds of LA_k supports, $\underline{\widehat{a}}_k^0$ and $\widehat{\overline{a}}_k^0$, are obtained as linear functions of the support bounds for LB_{1i} and $(LB_{1i_1} - LB_{1i_2})$. Therefore, they also converge to the true support bounds at the rate of n. Next stage obtains intermediate estimates of the support bounds, $\underline{\widehat{a}}_k^1$ and $\widehat{\overline{a}}_k^1$. They are obtained from $\underline{\widehat{a}}_k^0$ and $\widehat{\overline{a}}_k^0$ through a shift by an adjustment factor e_1. An extremum estimator for e_1 is obtained by minimizing

$$\widehat{Q}_n = (\int_{\underline{\widehat{a}}_1^0 - \widehat{e}_1}^{\widehat{\overline{a}}_1^0 - \widehat{e}_1} a\widehat{g}_{LA_1}(a)da)^2,$$

or

$$\widehat{Q}_n = (\int_{\underline{\widehat{a}}_1^0}^{\widehat{\overline{a}}_1^0} a\widehat{g}_{LA_1}(a + \widehat{e}_1)da - \widehat{e}_1)^2.$$

The usual results for extremum estimators apply. Notice that $\widehat{Q}_n \to (\int_{\underline{a}_1^0}^{\overline{a}_1^0} ag_{LA_1}(a+e_1)da - e_1)^2$ at the same rate as $\widehat{g}_{LA_1}$ converges to g_{LA_1} (see Li and Vuong (1998) for an appropriate rate of convergence). Let us denote this rate by d_n. It can be shown that all standard conditions for the convergence of extremum estimators hold and $\widehat{e}_1$ converges to e_1 at the rate d_n. Thus, intermediate estimators of the support bounds of LA_k, $\underline{\widehat{a}}_k^1$ and $\widehat{\overline{a}}_k^1$, converge to the corresponding true values at the rate d_n. The bounds of supports for A_k are estimated as $\underline{\widehat{a}}_k = exp(\underline{\widehat{a}}_k^1)$ and $\widehat{\overline{a}}_k = exp(\widehat{\overline{a}}_k^1)$, respectively. The smoothness of the exponential function ensures consistency of these estimators. The delta method can be used to show that the rate of convergence remains equal to d_n.

(b) The rate of convergence for $\widehat{g}_{A_k}(.)$ is established in Li and Vuong (1998). Recall that here we denote it d_n. Now we derive a rate of convergence for $\widehat{G}_{A_k}$. The estimator for G_{A_k} is defined as

$$\widehat{G}_{A_k}(a) = \int_{\underline{\widehat{a}}_k}^{a} \widehat{g}_{A_k}(a)da.$$

To establish consistency we consider

$$\left|\widehat{G}_{A_k}(a) - G_{A_k}(a)\right| \leq \left|\int\limits_{\widehat{\underline{a}}_k}^{\underline{a}_k} \widehat{g}_{A_k}(a)da\right| + \left|\int\limits_{\underline{a}_k}^{a} (\widehat{g}_{A_k}(a) - g_{A_k}(a))da\right|.$$

Since g_{A_k} is a continuous function with bounded support, (D_9), then g_{A_k} is a bounded function. For large enough n, $\widehat{g}_{A_k}$ is also bounded a.s. due to uniform convergence of $\widehat{g}_{A_k}$ to g_{A_k}. Then, part (b) implies that the first summand converges to zero at the rate d_n. The second summand also converges to zero at the rate d_n since support of g_{A_k} is bounded. Therefore, $\widehat{G}_{A_k}$ converges to G_{A_k} at the rate d_n.

(d) Next, we prove uniform consistency of the estimator for the individual cost component. Recall that the individual cost components corresponding to the individual bid components a_k are estimated as

$$\begin{aligned}
\widetilde{x}_1 &= \widetilde{a}_1 + \frac{(1-\widetilde{G}_{A_1}(a_1))\cdot(1-\widetilde{G}_{A_2}(a_1))}{(m_1-1)\cdot\widetilde{g}_{A_1}(a_1)\cdot(1-\widetilde{G}_{A_2}(a_1)) + m_2\cdot\widetilde{g}_{A_2}(a_1)\cdot(1-\widetilde{G}_{A_1}(a_1))}, \\
\widetilde{x}_{2j} &= \widetilde{a}_{2j} + \frac{(1-\widetilde{G}_{A_1}(a_2))\cdot(1-\widetilde{G}_{A_2}(a_2))}{m_1\cdot\widetilde{g}_{A_1}(a_2)\cdot(1-\widetilde{G}_{A_2}(a_2)) + (m_2-1)\cdot\widetilde{g}_{A_2}(a_2)\cdot(1-\widetilde{G}_{A_1}(a_2))}.
\end{aligned}$$

Similar to Guerre, Perrigne and Vuong (2000) I restrict my attention to the subset of the support

$$V(G_{A_k}) = \{a \in [\underline{a}_k, \overline{a}_k] \text{ such that } (a \pm 2\delta_k) \in S(H_k).$$

Notice that for every $a_{1j} \in V(G_{A_k})$ corresponding x_{1j} is finite. For every $a \in V(G_{A_k})$, $\widehat{g}_{A_k}(a) \geq c_g > 0$ and $(1-\widehat{G}_{A_k}(a)) \geq c_G > 0$ for some c_g and c_G, since $\widehat{g}_{A_k}$ and $\widehat{G}_{A_k}$ uniformly converge to g_{A_k} and G_{A_k}, respectively, and (ii) of Lemma 2.

Below I sketch the argument that establishes uniform convergence of $\widetilde{x}_{1j}$ to x_{1j}.

Let us denote

$$\begin{aligned}
\xi_1(a_1) &= \frac{(1-G_{A_1}(a_1))\cdot(1-G_{A_2}(a_1))}{(m_1-1)\cdot g_{A_1}(a_1)\cdot(1-G_{A_2}(a_1)) + m_2\cdot g_{A_2}(a_1)\cdot(1-G_{A_1}(a_1))}, \\
\widetilde{\xi}_1(a_1) &= \frac{(1-\widetilde{G}_{A_1}(a_1))\cdot(1-\widetilde{G}_{A_2}(a_1))}{(m_1-1)\cdot\widetilde{g}_{A_1}(a_1)\cdot(1-\widetilde{G}_{A_2}(a_1)) + m_2\cdot\widetilde{g}_{A_2}(a_1)\cdot(1-\widetilde{G}_{A_1}(a_1))},
\end{aligned}$$

$$
\begin{aligned}
\zeta_1(a_1) &= (m_1-1)\cdot g_{A_1}(a_1)\cdot(1-G_{A_2}(a_1)) + m_2\cdot g_{A_2}(a_1)\cdot(1-G_{A_1}(a_1)),\\
\widetilde{\zeta}_1(a_1) &= (m_1-1)\cdot \widetilde{g}_{A_1}(a_1)\cdot(1-\widetilde{G}_{A_2}(a_1)) + m_2\cdot \widetilde{g}_{A_2}(a_1)\cdot(1-\widetilde{G}_{A_1}(a_1)),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_1(a_1) &= (1-G_{A_1}(a_1))\cdot(1-G_{A_2}(a_1)),\\
\widetilde{\varepsilon}_1(a_1) &= (1-\widetilde{G}_{A_1}(a_1))\cdot(1-\widetilde{G}_{A_2}(a_1)).
\end{aligned}
$$

Then

$$
|\widetilde{x}_{1j} - x_{1j}| = \left|\widetilde{\xi}_1(a_1) - \xi_1(a_1)\right|,
$$

which in turn can be bounded by

$$
\left|\widetilde{\xi}_1(a_1) - \xi_1(a_1)\right| \leq \frac{1}{\widetilde{C}_1 C_1}\left|\widetilde{\varepsilon}_1(a_1)\zeta_1(a_1) - \varepsilon_1(a_1)\widetilde{\zeta}_1(a_1)\right|
$$

or

$$
\left|\widetilde{\xi}_1(a_1) - \xi_1(a_1)\right| \leq \frac{1}{\widetilde{C}_1 C_1}(|\widetilde{\varepsilon}_1(a_1) - \varepsilon_1(a_1)|\cdot|\zeta_1(a_1)| + \left|\widetilde{\zeta}_1(a_1) - \zeta_1(a_1)\right|\cdot|\varepsilon_1(a_1)|),
$$

or

$$
\left|\widetilde{\xi}_1(a_1) - \xi_1(a_1)\right| \leq \frac{1}{\widetilde{C}_1}|\widetilde{\varepsilon}_1(a_1) - \varepsilon_1(a_1)| + \frac{\widetilde{c}_G c_G}{\widetilde{C}_1 C_1}\left|\widetilde{\zeta}_1(a_1) - \zeta_1(a_1)\right|,
$$

where $C_1 = (m_1+m_2-1)c_g c_G$ and $\widetilde{C}_1 = (m_1+m_2-1)\widetilde{c}_g\widetilde{c}_G$.

Pointwise application of delta method allows us to conclude that

$$
\begin{gathered}
|\widetilde{\varepsilon}_1(a_1) - \varepsilon_1(a_1)| = O_p(d_n), \text{ a.s.}\\
\left|\widetilde{\zeta}_1(a_1) - \zeta_1(a_1)\right| = O_p(d_n), \text{ a.s.}
\end{gathered}
$$

Then

$$
\left|\widetilde{\xi}_1(a_1) - \xi_1(a_1)\right| = O_p(d_n), \text{ a.s.}
$$

To conclude the proof we note that δ_{h_k} converges to zero as n diverges to infinity (we choose L so that it diverges to infinity together with n) and thus the statement above holds everywhere on the interior of the support.

(d) Finally, we establish uniform convergence of the probability density function of the individual cost component. Here again we consider closed subsets of the support interior. Recall that

$$\widetilde{h}_{X_k}(x) = \frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} K_h(\frac{x - \widetilde{x}_{kj}}{\delta_{h_k}}).$$

here L is the size of the pseudo-sample. We will make it a function of n, i.e., $L = L(n)$. Let us denote

$$\widetilde{\widetilde{h}}_{X_k}(x) = \frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} K_h(\frac{x - x_{kj}}{\delta_{h_k}}).$$

Then

$$\left|\widetilde{h}_{X_k}(x) - h_{X_k}(x)\right| \leq \left|\widetilde{h}_{X_k}(x) - \widetilde{\widetilde{h}}_{X_k}(x)\right| + \left|\widetilde{\widetilde{h}}_{X_k}(x) - h_{X_k}(x)\right|.$$

The rate of convergence for the second term depends solely on L and is equal to $(\frac{\log(L)}{L})^{R/(2R+1)}$ (see Stone 1992). Next, we focus on the first term:

$$\left|\widetilde{h}_{X_k}(x) - \widetilde{\widetilde{h}}_{X_k}(x)\right| = \left|\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} (K_h(\frac{x - \widetilde{x}_{kj}}{\delta_{h_k}}) - K_h(\frac{x - x_{kj}}{\delta_{h_k}})\right|.$$

A second-order Taylor expansion gives

$$\left|\widetilde{h}_{X_k}(x) - \widetilde{\widetilde{h}}_{X_k}(x)\right| \leq \left|\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} \frac{1}{\delta_{h_k}} \frac{dK_h}{dx}(\frac{x - x_{kj}}{\delta_{h_k}}) \cdot (x - \widetilde{x}_{kj}) + \frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} \frac{1}{\delta_{h_k}^2} \cdot \frac{d^2K_h}{dx^2}(\frac{x - x_{kj}}{\delta_k^2}) \cdot (x - \widetilde{x}_{kj})^2\right|$$

or

$$\left|\widetilde{h}_{X_k}(x) - \widetilde{\widetilde{h}}_{X_k}(x)\right| \leq \left|\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} \frac{dK_h}{dx}(\frac{x - x_{kj}}{\delta_{h_k}})\right| \cdot \left|\frac{x - \widetilde{x}_{kj}}{\delta_{h_k}}\right| + \left|\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} (\frac{1}{\delta_{h_k}}) \frac{d^2K_h}{dx^2}(\frac{x - x_{kj}}{\delta_{h_k}})\right| \cdot \left|\frac{x - \widetilde{x}_{kj}}{\delta_{h_k}}\right|^2.$$

The terms

$$\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} \frac{dK_h}{dx}(\frac{x - x_{kj}}{\delta_{h_k}}),$$

$$\frac{1}{L\delta_{h_k}} \sum_{j=1}^{n} (\frac{1}{\delta_k}) \frac{d^2K_h}{dx^2}(\frac{x - x_{kj}}{\delta_{h_k}})$$

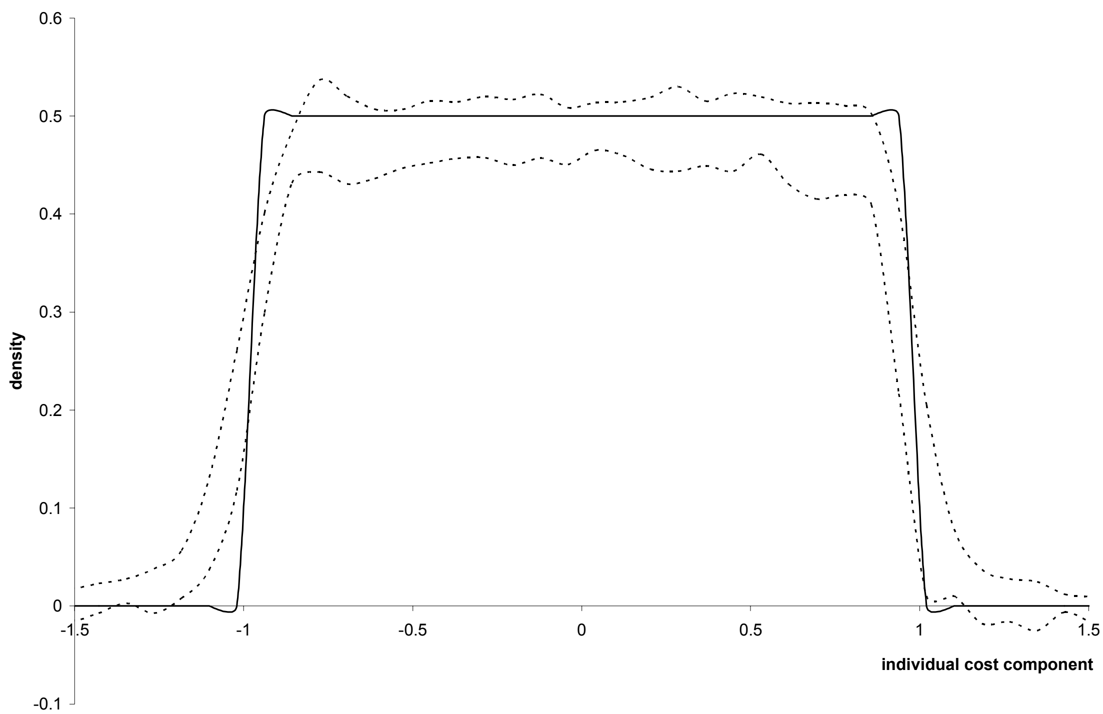
can be considered as kernel estimators. It can be shown that they converge to

$$h_{X_k}(x)\int \frac{dK_h}{dx}(x-z)dzdx,$$
$$h_{X_k}(x)\int \frac{d^2K_h}{dx^2}(x-z)dzdx$$

respective, which ensures that corresponding terms are bounded. Recall that the rate of convergence for $(x-x_{kj})$ is given by d_n. If δ_{h_k} is of the order r_{h_k}, then the rate of convergence for the first term is given by d_n/r_{h_k}. This is also the rate of convergence for $\widetilde{h}_{X_k}(x)-\widetilde{\widetilde{h}}_{X_k}(x)$, since the second term has a smaller order of magnitude. The bandwidth is chosen as a function of the number of random draws L, which, in turn, is a function of n, the number of auctions in the data set. The number of draws L can always be chosen so that $d_n/r_{h_k}\rightarrow 0$. The rate of convergence for $(\widetilde{h}_{X_k}(x)-h_{X_k}(x))$ is then given by $max\{d_n/r_{h_k},(\frac{\log(L)}{L})^{R/(2R+1)}\}$. Notice that the second term is equal to r_{h_k}. Therefore, the fastest achievable rate of convergence corresponds to $r_{h_k} \simeq \sqrt[2]{d_n}$. This determines the choice of L as a function of n.

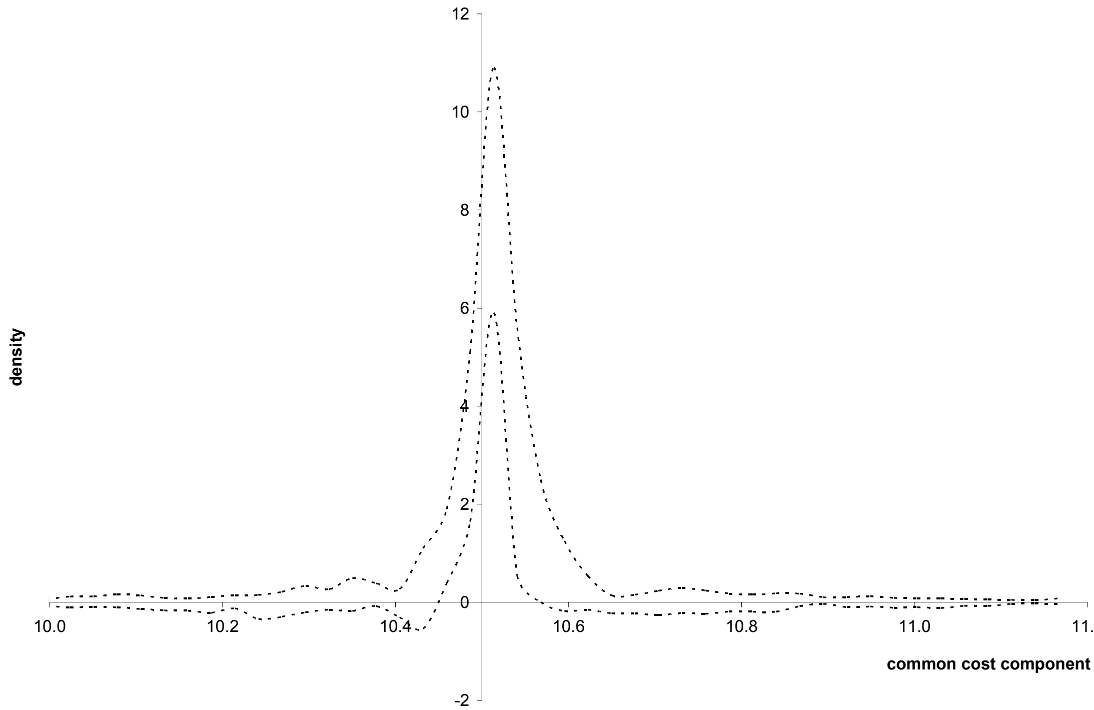
B Results of Monte Carlo Study

Figure 1b: The Probability Density Function of the Individual Cost Component (independent costs)



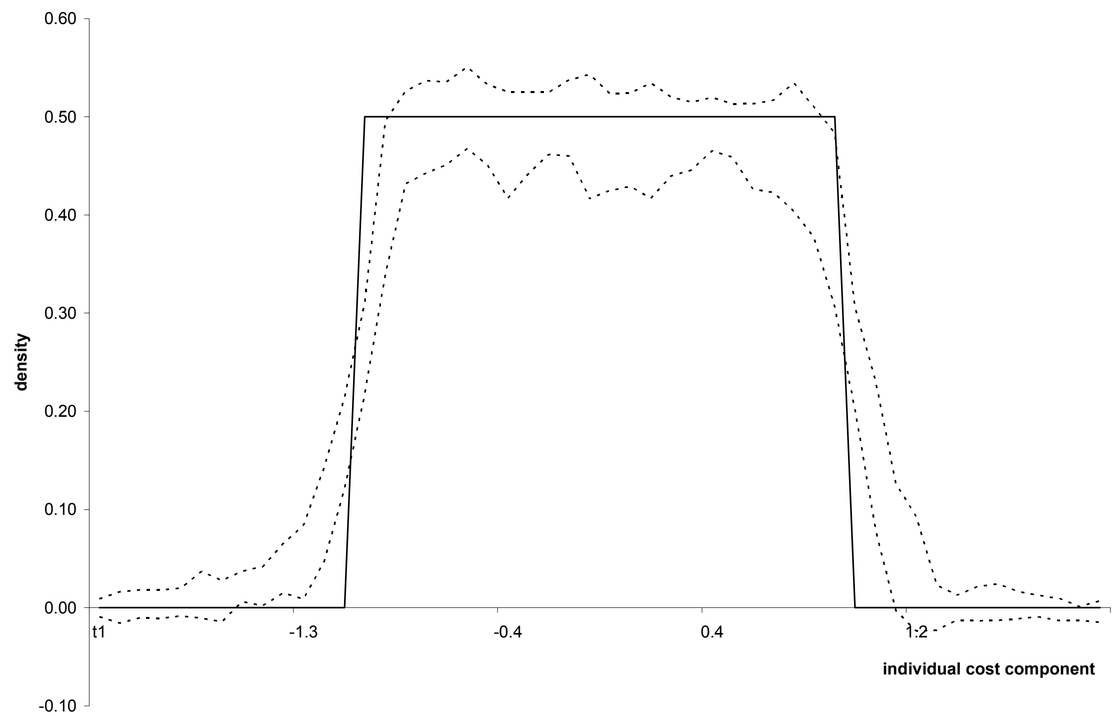
Solid line - true density function; Dotted lines - 5% and 95% pointwise quantiles of the density estimator

Figure 2b: The Probability Density Function of the Common Cost Component (independent costs)



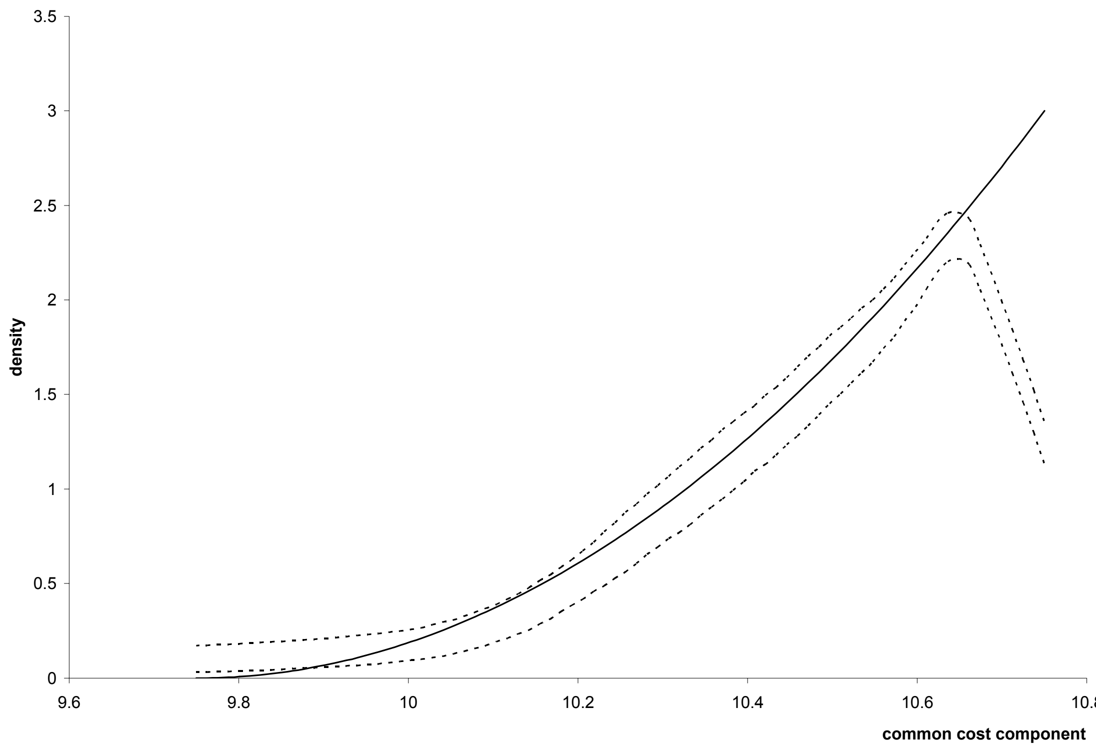
Dotted lines indicate 5% and 95% pointwise quantiles of the density estimator

Figure 3b: The Probability Density Function of the Individual Cost Component (unobserved auction heterogeneity)



Solid line - the true density function; Dotted lines - 5% and 95% pointwise quantiles of the density estimator

Figure 4b: The Probability Density Function of the Common Cost Component (unobserved auction heterogeneity)



Solid line - true density function; Dotted lines - 5% and 95% pointwise quantiles of the density estimator

Table 1b: Description of the Simulated Models

	Model						
	Cov (C1,C2)	Cov (X1,X2)	Cov (X,Y)	Var (X)	Var (Y)	Var (C)	E(Mark-up)
Case 1	0	0	0	0.33	0	0.33	0.38
Case 2a	0.04	0	0	0.33	0.04	0.37	0.09
Case 2b	0.34	0	0	0.33	0.34	0.67	0.09
Case3a	0.16	0	0	0.33	0.16	0.49	0.47
Case 3b	0.04	0	0	0.33	0.04	0.37	0.21
Case 4a	0.52	0.18	0	0.36	0.34	0.7	0.27
Case 4b	0.36	0.02	0	0.2	0.34	0.54	0.15
Case 5a	0.52	0	0.18	0.36	0.34	0.7	0.11
Case 5b	0.2	0	0.02	0.2	0.18	0.38	0.06

Case 1	Independent Costs
Case 2	Unobserved Auction Heterogeneity
Case 3	Affiliated Private Values
Case 4	Correlated Individual Components
Case 5	Individual Components Correlated with Common Component

Table 2b: Summary of Estimation Results

	Estimates			
	Var (X)	**Var(Y)**	**Var(C)**	**Mark-up**
Case 1	0.325	0.013	0.337	0.362
	[0.28, 0.34]	[0.01, 0.02]	[0.28, 0.35]	[0.357,0.393]
Case 2a	0.321	0.0362	0.364	0.082
	[0.27, 0.36]	[0.034, 0.045]	[0.312, 0.386]	[0.078, 0.095]
Case 2b	0.318	0.335	0.687	0.092
	[0.26, 0.37]	[0.27, 0.356]	[0.61, 0.72]	[0.082, 0.098]
Case 3a			0.33	0.18
			[0.266, 0.369]	[0.153, 0.21]
Case 3b			0.36	0.33
			[0.272, 0.385]	[0.284, 0.37]
Case 4a	0.176	0.36	0.525	0.137
	[0.149, 0.192]	[0.347, 0.401]	[0.501, 0.584]	[0.123, 0.146]
Case 4b	0.254	0.461	0.727	0.238
	[0.182, 0.291]	[0.416, 0.521]	[0.71, 0.773]	[0.201, 0.268]
Case 5a	0.187	0.21	0.36	0.052
	[0.174, 0.193]	[0.19, 0.217]	[0.352, 0.368]	[0.5, 0.61]
Case 5b	0.174	0.522	0.71	0.098
	[0.169, 0.179]	[0.512, 0.53]	[0.7, 0.722]	[0.093, 0.108]

Table 3b: Analysis of Misspecified Models

Var(Y)		E(C)	Var (C)	E(Mark-up)
0.04	model	10.5	0.37	0.09
0.04	UAH	[10.44, 10.59]	[0.32, 0.41]	[0.078, 0.095]
0.04	APV	[10.37, 10.48]	[0.35, 0.44]	[0.097, 0.12]
0.04	IPV	[10.41, 10.5]	[0.34, 0.44]	[0.95, 0.11]
0.34	model	10.5	0.67	0.03
0.34	UAH	[10.28, 10.64]	[0.63, 0.72]	[0.02, 0.038]
0.34	APV	[8.31, 9.57]	[0.89, 1.02]	[0.042, 0.55]
0.34	IPV	[8.85, 9.84]	[0.83,0.96]	[0.4, 0.52]
1	model	10.5	1.33	0.01
1	UAH	[10.15, 10.76]	[1.27, 1.43]	[0.008, 0.014]
1	APV	[7.6, 8.8]	[2.01, 2.27]	[0.033, 0.047]
1	IPV	[8.1, 9.2]	[1.86, 2.05]	[0.31, 0.044]

In the brackets - 5% and 95% pointwise quantiles of the estimator

Notations: UAH - estimation procedure that relies on the assumption of unobserved auction heterogeneity; APV - estimation procedure that relies on the assumption of affiliated private values; IPV - estimation procedure that relies on the assumption of independent private values

C Michigan Highway Procurement Auctions

Table 1c: Descriptive Statistics of Data

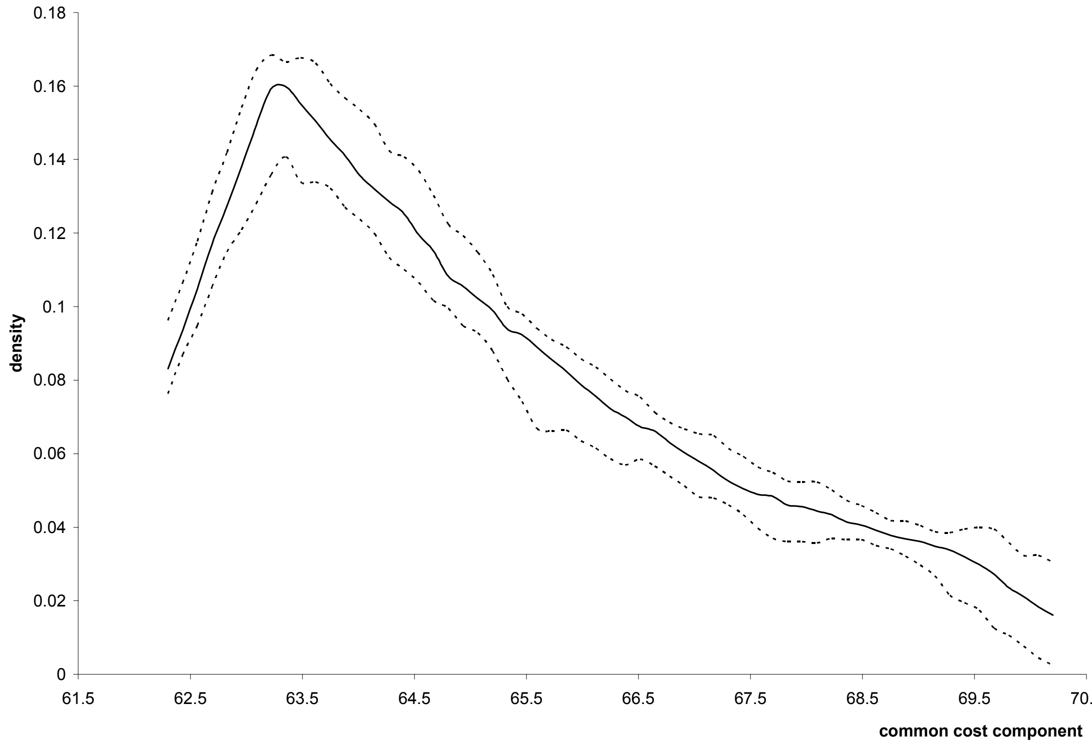
Number of bidders		**overall**	**1**	**2**	**3**	**4**	**5**	**6**
Number of observations		3,947	71	673	1126	1026	365	192
Engineers Estimate (hdrds. th.)	mean	12.80	13.34	10.27	12.60	13.90	12.90	16.40
	std.dev	2.35	2.88	1.41	3.02	2.26	1.79	3.39
Winning Bid (hdrds. th.)	mean	11.10	14.12	10.00	11.80	12.90	11.80	15.20
	std.dev	2.32	3.05	1.50	2.89	2.25	1.66	3.35
Money Left on the Table	mean	0.07	NA	0.11	0.08	0.07	0.05	0.04
	std.dev	0.05	NA	0.08	0.06	0.06	0.05	0.04
Number of Regular Bidders	mean	1.92	0.79	1.43	1.65	2.07	2.16	2.29
	std.dev	1.06	0.41	0.62	0.72	0.98	1.21	1.32

Table 2c: Bid Analysis (maintenance projects)

dependent variable	log (B_1)		
number of observations	947		
number of bidders	4		

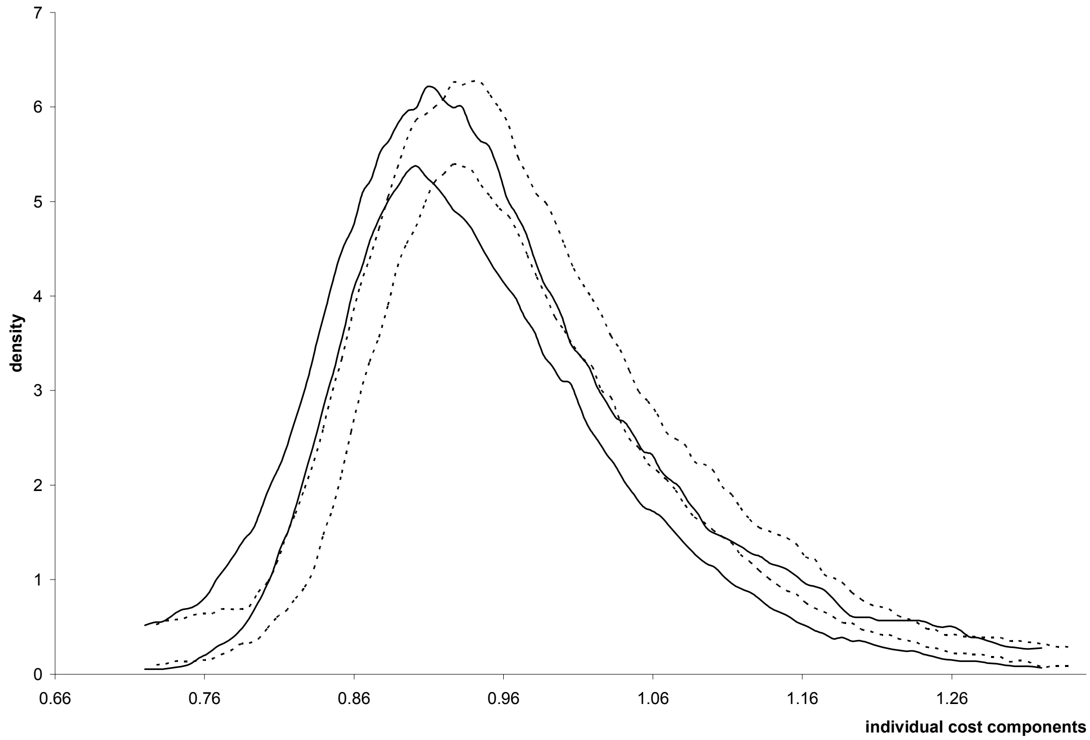
Variable	**OLS**	**Auction Dummies**	**Random Effects**
Log (Estimate)	0.9100	0.9300	0.8800
	(0.0530)	(0.1400)	(0.1300)
Duration	-0.0210	-0.0490	-0.0210
	(0.0500)	(0.0250)	(0.0500)
Distance	0.0312	0.0540	0.0270
	(0.0160)	(0.0120)	(0.0160)
Length	0.0700	0.0950	0.0650
	(0.4700)	(0.3900)	(0.4900)
Marking (dummy)	0.1000	0.1200	0.1000
	(0.0360)	(0.0470)	(0.0470)
Landscaping (dummy)	0.8700	0.8100	0.8600
	(0.0330)	(0.0330)	(0.0330)
Sign (dummy)	-0.0200	-0.0300	-0.0130
	(0.0160)	(0.0160)	(0.0160)
Log (Load Remaining)	0.0310	0.0392	0.0270
	(0.0157)	(0.0127)	(0.0144)
Number of Potential Bidders	-0.1100	-0.1330	-0.1100
	(0.0310)	(0.0230)	(0.0290)
Fringe (Dummy)	0.2500	0.2800	0.2200
	(0.1100)	(0.1200)	(0.1000)
….			
Constant	0.1750	0.1620	0.1680
	(0.0580)	(0.0520)	(0.0480)
R^2	0.857	0.914	R^2(within)=0.00 R^2(between)=0.934 R^2(overall)=0.917 σ_u^2=0.16 σ_ε^2=0.22 ρ=0.48
other variables:	district dummies, regular bidders dummies		

Figure 1c: The Probability Function of the Common Cost Component



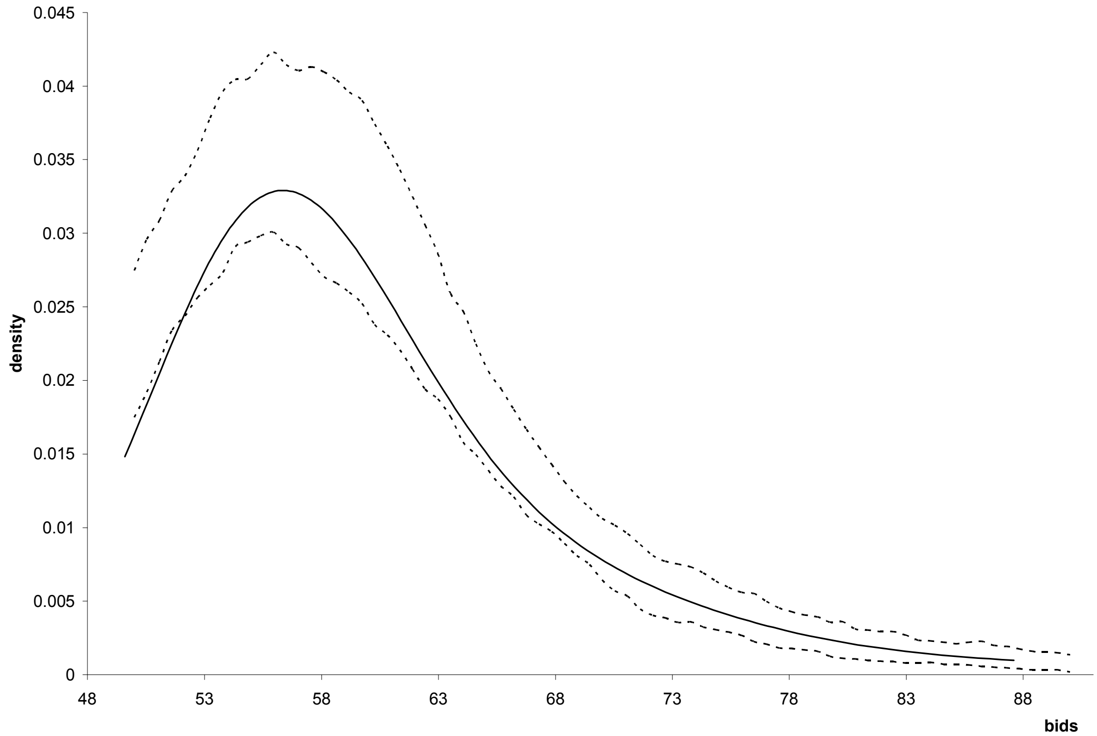
Solid line - the probability density function of common component; Dotted lines: 5% and 95% pointwise quantiles of the density estimator

Figure 2c: The Probability Density Functions of the Individual Cost Components



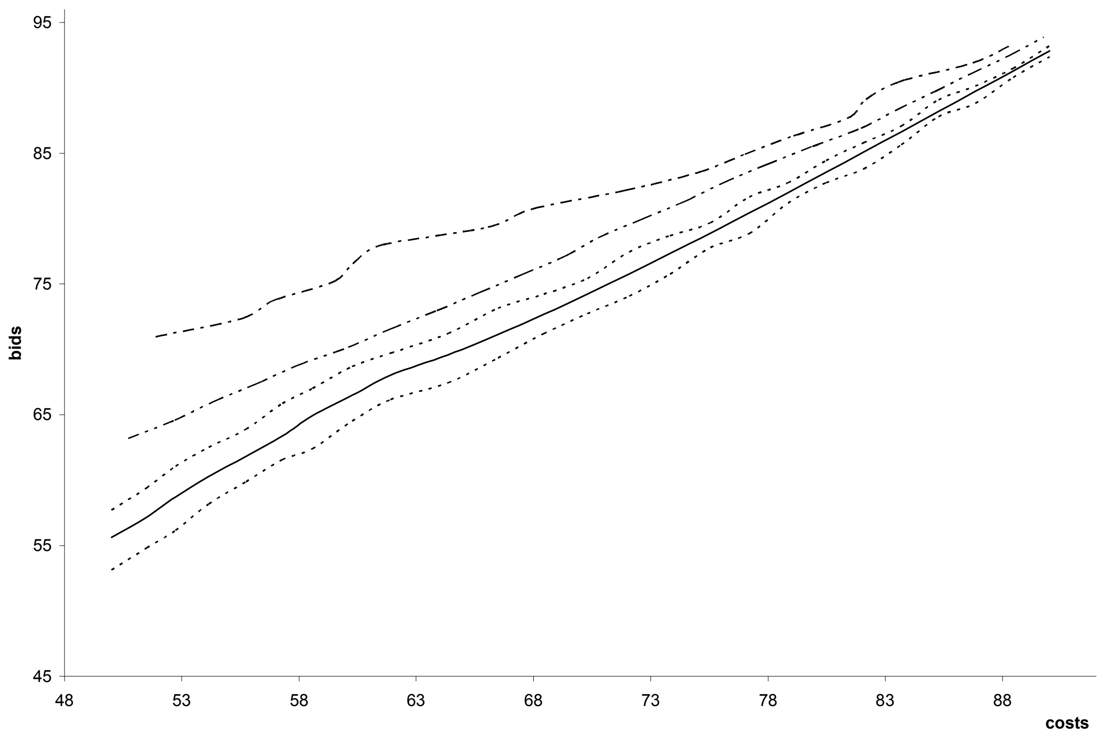
Solid lines - 5% and 95% pointwise quantiles of the density estimator for the regular type of bidders; Dotted lines - 5% and 95% pointwise quantiles of the density estimator for the "fringe" type

Figure 3c: Fit of the Model



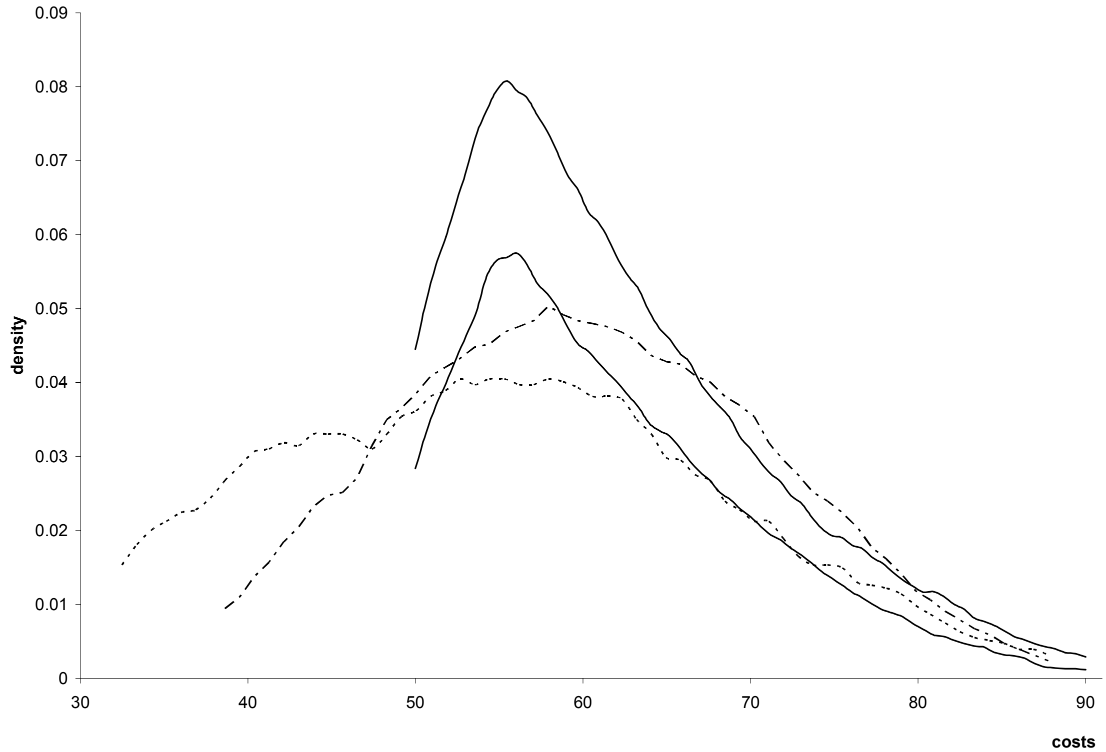
Solid line - the estimated density of bids for the regular type of bidders; Dotted lines - 5% and 95% quantiles of the estimated density for simulated bids

Figure 4c: Bidding Strategies



Solid line - expected bidding strategy estimated under the assumption of unobserved auction heterogeneity; Dotted lines - 5% and 95% pointwise quantiles for the estimator based on the assumption of unobserved auction heterogeneity; the first perforated line - the bidding strategy estimated under the assumption of affiliated private values; the second perforated line - the bidding strategy estimated under independent private values

Figure 5c: The Probability Density Functions



Solid lines - 5% and 95% pointwise quantiles of the expected probability density function estimated under the assumption of unobserved auction heterogeneity; dotted line - the probability density function estimated under the assumption of independent private values; perforated line - the probability density function estimated under the assumption of affiliated private values

Table 2c: Evaluating Validity of Independence Assumption

	Method 1	**Method 2**	
	($\log(B_1)$, $\log(B_2)$)	**($\log(B_1)$-$\log(B_3)$, $\log(B_2)$-$\log(B_3)$)**	
recovered as		common component	individual component
Var(X_{reg})	0.252	0.284	0.205
	(0.21, 0.3)	(0.2, 0.33)	(0.18, 0.29)
Var(X_{fringe})	0.27		0.217
	(0.22, 0.32)		(0.19, 0.31)