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“Some Results on the Solution of the Neoclassical Growth Model”

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# Some Results on the Solution of the Neoclassical Growth Model\*

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## Abstract

This paper presents some new results on the solution of the stochastic neoclassical growth model with leisure. We use the method of Judd (2003) to explore how to change variables in the computed policy functions that characterize the behavior of the economy. We find a simple close-form relation between the parameters of the linear and the loglinear solution of the model. We extend this approach to a general class of changes of variables and show how to find the optimal transformation. We report how in this way we reduce the average absolute Euler equation errors of the solution of the model by a factor of three. We also demonstrate how changes of variables correct for variations in the volatility of the economy even if we work with first order policy functions and how we can keep a linear representation of the laws of motion of the model if we use a nearly optimal transformation. We finish discussing how to apply our results to estimate dynamic equilibrium economies.

*Key words:* Dynamic Equilibrium Economies, Computational Methods, Changes of Variables, Linear and Nonlinear Solution Methods.

*JEL classifications:* C63, C68, E37.

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# 1. Introduction

This paper presents some new results on the solution of the stochastic neoclassical growth model with leisure. In an important recent contribution Judd (2003) has provided formulae to apply changes of variables to the solutions of dynamic equilibrium economies obtained through the use of perturbation techniques. Standard perturbation methods provide a Taylor expansion of the policy functions that characterize the equilibrium of the economy in terms of the state variables of the model and a perturbation parameter. Judd's derivations allow moving from this Taylor expansion to any other series in terms of nonlinear transformations of the state variables without the need to recompute the whole solution. This second approximation can be more accurate than the first because the change of variables induces nonlinearities that can help to track the true but unknown policy functions.

Judd's results are important for several reasons. First, we often want to solve problems with a large number of state variables. Solving these models is difficult and costly. For instance if we use perturbation methods, the number of derivatives required to compute the parameters of the Taylor expansion of the solution quickly explodes as we increase the order of the approximation (see Judd and Guu, 1997). One alternative strategy is to obtain a low order expansion (even just a linear one) of the solution and move to a more accurate representation using a change of variables. Computing this low order expansion and the change of variables is relatively inexpensive. Consequently, if we are able to select an adequate transformation, we can increase the accuracy of the solution enough to use our computation for quantitative analysis despite the high dimensionality of the state space.

Second, researchers frequently exploit approximated solutions to understand the analytics of dynamic equilibrium models. See for example the classical treatment of Campbell (1994) with the neoclassical growth model or the recent work of Woodford (2003) with monetary economies. Because of tractability considerations those analyses limit themselves to first order expansions. Clearly the usefulness of this approach depends on the quality of the approximation. Linear solutions, however, may perform poorly outside a small region around which we linearize. More importantly, as pointed out by Benhabib, Schmitt-Grohé and Uribe (2001), linear solutions may even lead to incorrect findings regarding the existence and

uniqueness of equilibrium. An optimal change of variables can address both issues increasing the accuracy of the solution and avoiding misleading local results while maintaining analytic tractability.

Third, a change of variables can be useful to estimate dynamic equilibrium economies using a likelihood approach. In general we do not know how to directly evaluate the likelihood function of these economies and we need to either linearize the model and use the Kalman filter (Schorfheide, 1999) or to resort to simulation methods that are expensive to implement (see Fernández-Villaverde and Rubio-Ramírez, 2002)). However a change of variables may deliver a representation of the economy suitable for efficient estimation while capturing some of the non-linearities of the model.

Motivated by the arguments above, this paper applies Judd’s methodology to the stochastic neoclassical growth model with leisure, the workhorse of dynamic macroeconomics. First, we derive a simple close-form relation between the parameters of the linear and the loglinear solution of the model. We extend this approach to a more general class of changes of variables: those that generate a policy function with a power function structure.

Second, we study the effects of that last particular class of changes of variable on the size of the Euler equation errors, i.e. those errors that appear in the optimality conditions of the agents because of the use of approximated solutions instead of the exact ones. We search for the optimal change of variable inside the class of power functions and report results for a benchmark calibration of the model and for alternative parametrizations. In that way we study the performance of the procedure both for a nearly linear case (the benchmark calibration) and for more nonlinear cases (for example those with higher variance of the productivity shock). We find that for the benchmark calibration we reduce the average absolute Euler equation errors by a factor of three. This reduction makes the new approximated solution of the model competitive to much more involved nonlinear methods, such as Finite Elements or Value Function Iteration, and to second order perturbations.

Third, sensitivity analysis reveals how the change of variables corrects for movements in the exogenous variance of the economy through changes in the optimal values of the parameters of the transformation. This is true even if we confine ourselves to a first order policy function. In comparison a standard linear (or loglinear) solution can only provide

a certainty equivalent approximation. We view this property as a key advantage of our approach.

Fourth, sensitivity analysis also suggests that a particular change of variables that respects the linearity of the solution is roughly optimal. We propose the use of this approximation because of two reasons. First, because as explained before, such linearity allows researchers to use the well understood toolbox of linear systems while achieving a level of accuracy not possible with the standard linearization. Second because it facilitates taking the model to the data. In addition to being linear, we show how this quasi-optimal approximated solution also implies that the disturbances to the model are normally distributed. Consequently we can write the economy in a state-space form and use the Kalman filter to perform likelihood based inference.

The rest of the paper is organized as follows. Section 2 presents the canonical stochastic neoclassical growth model. Section 3 outlines how we solve for the standard linear representation of the model policy functions in levels. Section 4 discusses changes of variables in general terms. Section 5 derives the relation between the linear and loglinear solution of the model. Section 6 explores the optimal change of variables within a flexible class of functions and reports a sensitivity analysis exercise. Section 7 discusses the use of the change of variables for estimation and section 8 offers some concluding remarks.

## **2. The Stochastic Neoclassical Growth Model**

As mentioned above we want to explore how the approximated solutions of the stochastic neoclassical growth model with leisure respond to nonlinear changes of variables. Three reasons lead us to use this model. First, its popularity (directly or with small changes) to address a large number of questions (see Cooley, 1995) makes it a natural laboratory to explore the potential of Judd's contribution. Second, because of this importance any analytical result regarding the (approximated) solution of the model is of interest in itself. Finally, since this model is nearly linear for a benchmark calibration, its use provides a particularly difficult laboratory for the change of variables approach: it bounds tightly the improvements we can obtain. We argue that even for this case we find important advantages of the technique and

that those improvements are likely to be higher for more nonlinear economies.<sup>1</sup>

Since the model is well known (see Cooley and Prescott, 1995) we provide only the minimum exposition required to fix notation. There is a representative agent in the economy, whose preferences over stochastic sequences of consumption  $c_t$  and leisure  $1 - l_t$  are representable by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \quad (1)$$

where  $\beta \in (0, 1)$  is the discount factor,  $E_0$  is the conditional expectation operator and  $U(\cdot, \cdot)$  satisfies the usual technical conditions.

There is one good in the economy, produced according to the aggregate production function  $y_t = e^{z_t} k_t^\alpha l_t^{1-\alpha}$  where  $k_t$  is the aggregate capital stock,  $l_t$  is the aggregate labor input and  $z_t$  is a stochastic process representing random technological progress. The technology follows a first order process  $z_t = \rho z_{t-1} + \epsilon_t$  with  $|\rho| < 1$  and  $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ . Capital evolves according to the law of motion  $k_{t+1} = (1 - \delta)k_t + i_t$  where  $\delta$  is the depreciation rate and the economy must satisfy the resource constraint  $y_t = c_t + i_t$ .

Since both welfare theorems hold in this economy, in order to determine the competitive equilibrium allocations and prices we can solve directly for the social planner's problem. We maximize the utility of the household subject to the production function, the evolution of the stochastic process, the law of motion for capital, the resource constraint and some initial conditions for capital and the stochastic process.

The solution to this problem is fully characterized by the equilibrium conditions:

$$U_c(t) = \beta E_t \{ U_c(t+1) (1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta) \} \quad (2)$$

$$U_c(t) = U_l(t) (1 - \alpha) e^{z_t} k_t^\alpha l_t^{-\alpha} \quad (3)$$

$$c_t + k_{t+1} = e^{z_t} k_t^\alpha l_t^{1-\alpha} + (1 - \delta) k_t \quad (4)$$

$$z_t = \rho z_{t-1} + \epsilon_t \quad (5)$$

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<sup>1</sup>Note that the special case of the model with log utility function, no leisure and total depreciation is not very informative regarding the usefulness of a change of variables: we already know that the exact solution of that case is loglinear. What we want to do is to evaluate the performance of those changes of variables in the general situation where we do not have an analytical solution.

given some initial capital  $k_0$ . The first equation is the standard Euler equation that relates current and future marginal utilities from consumption, the second one is the static first order condition between labor and consumption and the last two equations are the resource constraint of the economy and the law of motion of technology.

Solving for the equilibrium of this economy amounts to finding two policy functions for next period's capital  $k'(\cdot, \cdot, \cdot)$ , and labor  $l(\cdot, \cdot, \cdot)$  that deliver the optimal choice of these controls as functions of the two state variables, capital and technology level, and the standard deviation of the innovation to the productivity level  $\sigma$ .<sup>2</sup> The only point to notice is that we include the perturbation parameter as one explicit variable in the policy functions. That notation is useful in the very next section.

### 3. Solving the Model using a Perturbation Approach

The system of equations listed above does not have a known analytical solution and we need to use a numerical method to solve it. In a series of seminal papers Judd and coauthors (see Judd and Guu, 1992, Judd and Guu, 1997 and the textbook exposition in Judd, 1998, among others) have proposed to build a Taylor series expansion of the policy functions  $k'(\cdot, \cdot, \cdot)$ , and  $l(\cdot, \cdot, \cdot)$  around the deterministic steady state where  $z = 0$  and  $\sigma = 0$ .

If  $k_0$  is equal to the value of the capital stock in the deterministic steady state and the policy function for labor is smooth, we can use a Taylor approximation to approximate this policy function around  $(k_0, 0, 0)$  with the form

$$l_p(k, z, \sigma) \simeq \sum_{i,j,m} \frac{\partial^{i+j+m} l_p(k, z, \sigma)}{\partial k^i \partial z^j \partial \sigma^m} \Big|_{k_0, 0, 0} (k - k_0)^i z^j \sigma^m. \quad (6)$$

where the subscript  $p$  stands for perturbation approximation. The policy function for next period capital will have an analogous representation.

The key idea of the perturbation methods is to vary the parameter  $\sigma$  to find a case where the model can be solved analytically and to exploit implicit-function theorems to pin down the unknown coefficients  $\frac{\partial^{i+j+m} l_p(k, z, \sigma)}{\partial k^i \partial z^j \partial \sigma^m} \Big|_{k_0, 0, 0}$  in a recursive fashion.

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<sup>2</sup>Using the budget constraint  $c(\cdot, \cdot, \cdot)$  is a function of  $k(\cdot, \cdot, \cdot)$  and  $l(\cdot, \cdot, \cdot)$ .

The do so the first step is to linearize the equilibrium conditions of the model around the deterministic steady state, i.e. when  $\sigma = 0$ . Then we substitute capital and labor by the values given by the first order expansion of the policy function and we solve for the unknown coefficients. The next step is to find the second order expansion of the equilibrium conditions, again around the deterministic steady state, plug in the quadratic approximation of the policy function (evaluated with the first order coefficients found before) and solve for the unknown second order coefficients. We can then iterate on this procedure as many times as desired to get an approximation of arbitrary order. For a more detailed explanation of these steps we refer the reader to Judd and Guu (1992) and Aruoba, Fernández-Villaverde and Rubio-Ramírez (2003).

Since perturbations only deliver an asymptotically locally correct expression for the policy functions, the accuracy achieved by the method may be poor either away from the deterministic steady state or when the order of the expansion is low.

Several routes have been proposed to correct for these problems. One, by Collard and Juillard (2000), uses bias correction to find the approximation of the solution around a more suitable point than the deterministic steady state. The second one, suggested by Judd (2003), changes the variables in terms of which we express the computed solution of the model. We review Judd's proposal in next section.

## 4. The Change of Variables

The first order perturbation solution to the stochastic neoclassical growth model can be written as

$$f(x) \simeq f(a) + (x - a) f'(a)$$

where  $x = (k, z, \sigma)$  are the variables of the expansion,  $a = (k_0, 0, 0)$  is the deterministic steady state value of those variables and  $f(x) = (k'_p(k, z, \sigma), l_p(k, z, \sigma))$  is the unknown policy function of the model.<sup>3</sup>

Let us now transform both the domain and the range of  $f(x)$ . Thus, to express some

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<sup>3</sup>Higher order perturbation solutions could easily be considered but we concentrate in the first order approximations because of exposition reasons.



nonlinear function of  $f(x)$ ,  $h(f(x)) : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$  as a polynomial in some transformation of  $x$ ,  $Y(x) : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$ , we can use the Taylor series of  $g(y) = h(f(X(y)))$  around  $b = Y(a)$ , where  $X(y)$  is the inverse of  $Y(x)$ .

Using tensor notation Judd (2003) shows that

$$g(y) = h(f(X(y))) = g(b) + g_\alpha(b) (Y^\alpha(x) - b^\alpha) \quad (7)$$

where  $g_\alpha = h_A f_i^A X_\alpha^i$  comes from the application of the chain rule. From this expression it is easy to see that if we have computed the values of  $f_i^A$ , then it is straightforward to find the value of  $g_\alpha$ .

In the next two sections we obtain particular values for this general transformation, one for moving between a solution in levels and in logs and a second for a more general class of power functions.

## 5. A Particular Case: The Loglinearization

Since the exact solution of the stochastic neoclassical growth model in the case of log utility, total depreciation and no leisure choice is loglinear, a large share of practitioners have favored the loglinearization of the equilibrium conditions of the model over linearization in levels.<sup>4</sup> This practice generates the natural question of finding the relation between the coefficients on both representations. We use (7) to get a simple closed-form answer to this question.

A first order perturbation produces an approximated policy function in levels of the form:<sup>5</sup>

$$\begin{aligned} (k' - k_0) &= a_1 (k - k_0) + b_1 z \\ (l - l_0) &= c_1 (k - k_0) + d_1 z \end{aligned}$$

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<sup>4</sup>The wisdom of this practice is disputable. Some evidence in Christiano (1990) and Den Haan and Marcet (1994) suggest that this is the right choice in simpler version of the model. Aruoba, Fernández-Villaverde and Rubio-Ramírez (2003) find the opposite result using a model with leisure. They document that a linear solution in levels delivers systematically lower Euler equation errors.

<sup>5</sup>See Uhlig (1999) for details. Remember that this solution is the same as the one generated by a Linear Quadratic approximation of the utility function (Kydland and Prescott, 1982), the Eigenvalue Decomposition (Blanchard and Kahn, 1980 and King, Plosser and Rebelo, 2002), Generalized Schur Decomposition (Klein, 2000) or the QZ decomposition (Sims, 2002b) among others. Subject to applicability, all methods need to find the same policy functions since the linear space approximating a nonlinear space is unique.

where  $k$  and  $z$  are the current states of the economy,  $l_0$  is steady state value for labor and where for convenience we have dropped the subscript  $p$  where no ambiguity exists.<sup>6</sup> Analogously a loglinear approximation of the policy function will take the form:

$$\begin{aligned}\log k' - \log k_0 &= a_2 (\log k - \log k_0) + b_2 z \\ \log l - \log l_0 &= c_2 (\log k - \log k_0) + d_2 z\end{aligned}$$

or in equivalent notation:

$$\begin{aligned}\widehat{k}' &= a_2 \widehat{k} + b_2 z \\ \widehat{l} &= c_2 \widehat{k} + d_2 z\end{aligned}$$

where  $\widehat{x} = \log x - \log x_0$  is the percentage deviation of the variable  $x$  with respect to its steady state.

How do we go from one approximation to the second one? First we follow Judd's (2003) notation and write the linear system in levels as:

$$\begin{aligned}k'_p(k, z, \sigma) &= f^1(k, z, \sigma) = f^1(k_0, 0, 0) + f_1^1(k_0, 0, 0)(k - k_0) + f_2^1(k_0, 0, 0)z \\ l_p(k, z, \sigma) &= f^2(k, z, \sigma) = f^2(k_0, 0, 0) + f_1^2(k_0, 0, 0)(k - k_0) + f_2^2(k_0, 0, 0)z\end{aligned}$$

where:

$f^1(k_0, 0, 0) = k_0$	$f_1^1(k_0, 0, 0) = a_1$	$f_2^1(k_0, 0, 0) = b_1$
$f^2(k_0, 0, 0) = l_0$	$f_1^2(k_0, 0, 0) = c_1$	$f_2^2(k_0, 0, 0) = d_1$

Second we propose the changes of variables:

$h^1 = \log f^1$	$Y^1(x) = \log x^1$	$X^1 = \exp y^1$
$h^2 = \log f^2$	$Y^2 = x^2$	$X^2 = y^2$

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<sup>6</sup>It can be shown that the coefficients on  $\sigma$  are zero in the first order perturbation.

Judd's (2003) formulae for this particular example imply:

$$\begin{pmatrix} \log k'(\log k, z) \\ \log l(\log k, z) \end{pmatrix} = g(\log k, z) = \begin{pmatrix} \log f^1(k_0, 0, 0) \\ \log f^2(k_0, 0, 0) \end{pmatrix} + \begin{pmatrix} \log k - \log k_0 & z - z_0 \\ \log k - \log k_0 & z - z_0 \end{pmatrix} \begin{pmatrix} \frac{1}{k_0} \begin{pmatrix} f_1^1(k_0, 0, 0) & f_2^1(k_0, 0, 0) \end{pmatrix} \begin{pmatrix} k_0 \\ 1 \end{pmatrix} \\ \frac{1}{l_0} \begin{pmatrix} f_1^2(k_0, 0, 0) & f_2^2(k_0, 0, 0) \end{pmatrix} \begin{pmatrix} k_0 \\ 1 \end{pmatrix} \end{pmatrix},$$

and thus:

$$\begin{aligned} \log k' - \log k_0 &= f_1^1(k_0, 0, 0)(\log k - \log k_0) + \frac{1}{k_0} f_2^1(k_0, 0, 0) z \\ \log l - \log l_0 &= \frac{k_0}{l_0} f_1^2(k_0, 0, 0)(\log k - \log k_0) + \frac{1}{l_0} f_2^2(k_0, 0, 0) z \end{aligned}$$

We equating coefficients we obtain a nice and simple closed-form relation between the parameters of both representations:<sup>7</sup>

$a_2 = a_1$	$b_2 = \frac{1}{k_0} b_1$
$c_2 = \frac{k_0}{l_0} c_1$	$d_2 = \frac{1}{l_0} d_1$

Note that we have not used any assumption on the utility or production functions except that they satisfy the general technical conditions of the neoclassical growth model. Also moving from one coefficient set to the other one is an operation that only involves  $k_0$  and  $l_0$ ,

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<sup>7</sup>An alternative heuristic argument that delivers the same result is as follows. Take the system

$$\begin{aligned} (k' - k_0) &= a_1(k - k_0) + b_1 z \\ (l - l_0) &= c_1(k - k_0) + d_1 z \end{aligned}$$

and divide on both sides by the steady state value of the control variable:

$$\begin{aligned} \frac{k' - k_0}{k_0} &= a_1 \frac{k - k_0}{k_0} + \frac{1}{k_0} b_1 z \\ \frac{l - l_0}{l_0} &= c_1 \frac{k - k_0}{l_0} + \frac{1}{l_0} d_1 z \end{aligned}$$

and noticing that  $\frac{x' - x_0}{x_0} \simeq \log x - \log x_0$  we get back the same relation that the one presented in the paper. Of course our argument is more general and does not depend on an additional approximation.

values that we need to find anyway to compute the linearized version in levels. Therefore, once you have the linear solution, obtaining the loglinear one is immediate.

## 6. The Optimal Change of Variables

In the last section we showed how to find a loglinear approximation to the solution of the neoclassical growth model directly from its linear representation. Now we use the same approach to generalize our result to encompass the relationship between any power function approximation and the linear coefficients of the policy function. Also we search for the optimal change of variable inside this class of power functions and we report how the Euler equation errors improve with respect to the linear representation.

### 6.1. A Power Function Transformation

Before we argued that some practitioners have defended the use of loglinearizations to capture some of the nonlinearities in the data. This practice can be push one step ahead. We can generalize the log function into a general class of power function of the form:

$$\begin{aligned} k'_p(k, z; \gamma, \zeta, \mu, \varphi)^\gamma - k_0^\gamma &= a_3 \left( k^\zeta - k_0^\zeta \right) + b_3 z^\varphi \\ l_p(k, z; \gamma, \zeta, \mu, \varphi)^\mu - l_0^\mu &= c_3 \left( k^\zeta - k_0^\zeta \right) + d_3 z^\varphi \end{aligned}$$

with  $\varphi \geq 1$ . The last constraint assures that we will have real values for the power  $z^\varphi$ .

This class of functions is attractive because it provides a lot of flexibility in shapes with few free parameters while including the log transformation as the limit case when the coefficients  $\gamma$ ,  $\zeta$  and  $\mu$  tend to zero and  $\varphi$  is equal to 1. Also a similar power function with only two parameters is proposed by Judd (2003) in a simple optimal growth model without leisure and stochastic perturbations. His finding of notable improvements in the accuracy of the solution when he optimally selects the value of these parameters is suggestive of the advantages of using this parametric family.

The changes of variables for this family of functions are given by :

$h^1 = (f^1)^\gamma$	$Y^1 = (x^1)^\zeta$	$X_1 = (y^1)^{\frac{1}{\zeta}}$
$h^2 = (f^1)^\mu$	$Y_2 = (x^2)^\varphi$	$X_2 = (y^2)^{\frac{1}{\varphi}}$

Following the same reasoning than in the previous section we derive a form for the system in term of the original coefficients:

$$\begin{aligned}
k'_p(k, z; \gamma, \zeta, \mu, \varphi)^\gamma - k_0^\gamma &= \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1 (k^\zeta - k_0^\zeta) + \frac{\gamma}{\varphi} k_0^{\gamma-1} b_1 z^\varphi \\
l'_p(k, z; \gamma, \zeta, \mu, \varphi)^\mu - l_0^\mu &= \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1 (k^\zeta - k_0^\zeta) + \frac{\mu}{\varphi} l_0^{\mu-1} d_1 z^\varphi
\end{aligned}$$

Therefore, the relation of between the new and the old coefficients is again very simple to compute:

$a_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1$	$b_3 = \frac{\gamma}{\varphi} k_0^{\gamma-1} b_1$
$c_3 = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1$	$d_3 = \frac{\mu}{\varphi} l_0^{\mu-1} d_1$

As we pointed out before when  $\gamma$ ,  $\zeta$  and  $\mu$  tend to zero and  $\varphi$  is equal to 1 we get back the transformation derived in the previous section to move from the linear into the loglinear solution of the model.

## 6.2. Searching for the Optimal Transformation

One difference between the transformation from linear into a loglinear solution and the current transformation into a power law is that we have four free parameters  $\gamma$ ,  $\zeta$ ,  $\mu$  and  $\varphi$ . How do we select optimal values for those?

A reasonable criterion (and indeed part of the motivation for this whole exercise) is to select them in order to improve the accuracy of the solution of the model. This desideratum raises a question. How do we measure this accuracy, conditional on the fact that we do not know the true solution of the model?

Judd (1992) solves this problem evaluating the normalized Euler equation errors.<sup>8</sup> Note that in our model intertemporal optimality implies that:

$$U_c(t) = \beta E_t \{ U_c(t+1) (1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta) \}. \quad (8)$$

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<sup>8</sup>See also the discussion in Santos (2000) and the very similar approach in an statistical framework of Den Hann and Marcet (1994).

Since the solutions are not exact, condition (8) will not hold exactly when evaluated using the computed decision rules. Instead, for any choice of  $\gamma$ ,  $\zeta$ ,  $\mu$ , and  $\varphi$ , and its associated policy rules  $k'_p(\cdot, \cdot; \gamma, \zeta, \mu, \varphi)$  and  $l_p(\cdot, \cdot; \gamma, \zeta, \mu, \varphi)$  we can define the normalized absolute value Euler equation errors as:

$$EE(k, z; \Xi) = \left| 1 - \frac{(U_c^{-1}(E_t U_c(t+1)\beta R'(k, z; \Xi), l_p(k, z; \Xi)))}{c_p(k, z; \Xi)} \right| \quad (9)$$

where  $\Xi = (\gamma, \zeta, \mu, \varphi)$ ,  $R'(k, z; \Xi) = (1 + \alpha e^{z_{t+1}} k'(k, z; \Xi)^{\alpha-1} l'(k, z; \Xi)^{1-\alpha} - \delta)$  is the gross return on capital next period and where we have used the static optimality condition to substitute labor by its optimal choice.

This expression evaluates the (unit free) error in the Euler equation measured as a fraction of  $c_p(\cdot, \cdot; \Xi)$  as a function of the current states  $k$ , and  $z$  and the change of variables defined by the four parameters  $\gamma$ ,  $\zeta$ ,  $\mu$ , and  $\varphi$ . Judd and Guu (1997) interpret this function as the mistake, in dollars, incurred by each dollar spent. For example,  $EE(k, z; \Xi) = 0.01$  means that the consumer makes 1 dollar mistake for each 100 dollar spent.

A reasonable criterion is then to select values of  $\Xi$  to minimize the Euler error function. But here we face another problem. This function depends not only on the parameters  $\Xi$  but also on the values of the state variables. How do we eliminate that dependency? Do we minimize the Euler equation error at one particular point of the state space like the deterministic steady state? Do we better minimize some weighted mean of it?

A first choice could be the criterion:

$$\min_{\Xi} \int EE(k, z; \Xi) d\Phi$$

where  $\Phi$  is the stationary distribution of  $k$  and  $z$ . This choice is intuitive. We would weight each Euler equation error by the percentage of the time that the economy spends in that particular point: we would want to nail down the Euler equation errors for those parts of the stationary distribution where most of the action would happen while we would care less about accuracy in those points not frequently visited. The difficulty of the criterion is that we do not know that true stationary distribution (at least for capital) before we find the policy

functions of the model.

Of course we could solve a fixed-point problem where we get some approximation of the model, compute a stationary distribution, resolve the minimization problem, find the new stationary distribution and continue until convergence. This approach faces two problems. First it would be very expensive in terms of computing time. Second there is no known theory result that guaranties the convergence of such iteration to the right set of values for the parameters of the transformation.

Given the difficulties of the previous approach we do not follow that path. Instead we use a simpler strategy. Inspired by collocation schemes in projection methods we minimize the Euler equation errors over a grid of points of  $k$  and  $z$ :<sup>9</sup>

$$\min_{\Xi} SEE(\Xi) = \min_{\Xi} \sum_{k,z} EE(k, z; \Xi). \tag{10}$$

Setting this grid around the deterministic steady state (big enough to be representative of the stationary distribution of the economy but not too wide to avoid minimization over regions not very frequently visited) may achieve our desire goal of improving the accuracy of the solution.<sup>10</sup>

Inspection of (10) reveals that this problem depends, in general, on the values of the structural parameters of the model, i.e. those describing preferences and technology. As a consequence we need to take a stand on those before we can report any result. Our approach will be to calibrate of the model to match basic observations of the U.S. economy following the common strategy in macroeconomics and later to perform sensitivity analysis. That choice

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<sup>9</sup>As an alternative we also tried to solve the following problem:

$$\min_{\Xi} \sum_{k,z} \pi(z) EE(k, z; \Xi)$$

where  $\pi(z)$  is the ergodic distribution of shocks. The results, not reported in the paper, are very similar.

<sup>10</sup>When judging this scheme we need to remember that we do not look for the absolute best change of variables (although certainly finding it would be nice) but just for an improvement in the accuracy of the solution that can be easily found. This point can be important in models with dozens of state variables, where a multivariate maximization may be very costly. We can use a simulation procedure like a Markov chain Monte Carlo method to find an improvement “sufficiently good”. Those approaches are easy to code and get close to a global maximum (although not exactly at it) fairly quickly. In this paper however we can solve the problem using a simple Newton algorithm implemented in a *Mathematica* notebook that is available online at the following URL <http://www.econ.upenn.edu/~jesusfv>

allows us to evaluate how much accuracy we win in a “real life” situation by applying our change of variables and how confident we are in our results.

First we select as our utility function the CRRA form

$$\frac{\left(c_t^\theta (1 - l_t)^{1-\theta}\right)^{1-\tau}}{1 - \tau} \tag{11}$$

where  $\tau$  determines the elasticity of intertemporal substitution and  $\theta$  controls labor supply. Then we pick the benchmark calibration values as follows. The discount factor  $\beta = 0.9896$  matches an annual interest rate of 4% (see McGrattan and Prescott (2000) for a justification of this number based on their measure of the return on capital and on the risk-free rate of inflation-protected U.S. Treasury bonds). The parameter that governs risk aversion  $\tau = 2$  is a common choice in the literature.  $\theta = 0.357$  matches the microeconomic evidence of labor supply to 0.31 of available time in the deterministic steady state. We set  $\alpha = 0.4$  to match labor share of national income (after the adjustments to National Income and Product Accounts suggested by Cooley and Prescott (1995)). The depreciation rate  $\delta = 0.0196$  fixes the investment/output ratio and  $\rho = 0.95$  and  $\sigma = 0.007$  follow the stochastic properties of the Solow residual of the U.S. economy. Table 6.2.1 summarizes the discussion

6.2.1: Calibrated Parameters

Parameter	$\beta$	$\tau$	$\theta$	$\alpha$	$\delta$	$\rho$	$\sigma$
Value	0.9896	2.0	0.357	0.4	0.0196	0.95	0.007

Also we need to define the set of points for  $k$  and  $z$  over which we are going to sum the normalized absolute value Euler equation errors. We define a grid of 21 capital points covering  $\pm 30\%$  above and below the steady state capital and a grid of 21 perturbation points using Tauchen (1986) procedure.<sup>11</sup>

We report our main findings in Table 6.2.2 below. The last entry of the second row is the value of  $SEE(1, 1, 1, 1)$ , i.e. for the linear approximation for the Benchmark calibration. The third row is the solution to (10).

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<sup>11</sup>Aruoba, Fernández-Villaverde and Rubio-Ramírez (2003) report that the stationary distribution of the model, computed using highly accurate nonlinear methods, spends nearly all its time fluctuating between values of capital of 21 and 25. Our grid covers over 99% of the stationary distribution.



Table 6.2.2: Euler Equation Errors

$\gamma$	$\zeta$	$\mu$	$\varphi$	$SEE(\Xi)$
1	1	1	1	0.0856279
0.986534	0.991673	2.47856	1	0.0279944

We highlight several points from our optimal choice of parameter values. First, the parameters  $\gamma$  and  $\zeta$  are both very close to one suggesting that the nonlinearities in capital are of little importance (although not totally absent, since  $SEE(\Xi)$  suffers a bit when we impose that both parameters are equal to one). That result is confirmed by inspecting the nearly linear policy functions for capital found by very accurate but expensive nonlinear methods (see the results reported in Aruoba, Fernández-Villaverde and Rubio-Ramírez (2003)). Second,  $\mu$  is very far away from zero. Again the interpretation is that the labor supply function is much more nonlinear in capital and allowing the policy function to capture that behavior increases the accuracy of the solution. Third the coefficient on the technology shock,  $\varphi$ , stays at the constrained value of 1.<sup>12</sup>

The last column of the Table 6.2.2 shows that the optimal change of variables improves the average absolute value Euler equation error by a factor of around three. This means that for each mistake of three dollars made using the linear approximation the consumer would only have made a one dollar mistake using the optimal change of variable, a sizeable (but not dramatic) improvement in accuracy. This same result is shown in Figure 6.2.1 where we plot the decimal log of the absolute Euler equation errors at  $z = 0$  for the ordinary linear solution and the optimal change of variable.<sup>13</sup> In this graph we observe that when only one dimension is considered, the optimal change of variable can improve Euler equation errors from around 1 dollar every 10.000 dollars to 1 dollar every 1.000.000 dollars (an even more in some points).

We explore how the optimal change of variable compares with other more conventional nonlinear methods in terms of accuracy. In Figure 6.2.2 we replicate Figure 5.4.8. of Aruoba,

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<sup>12</sup>We checked that this finding was not a product of a particular initial guess. For very different initial guesses  $\varphi$  the minimization routine always pushed us back to a value of one.

<sup>13</sup>The use of decimal logs eases the reading of the graph. A value of -3 in the vertical axis represents an error of 1 dollar out of each 1000 dollars, a value of -4 an error of 1 dollar out of each 10000 dollars and so on.

Fernández-Villaverde, and Rubio-Ramírez (2003) adding the optimal change of variable solution as reported in Table 6.2.2. This figure plots the Euler equation errors for several linear and nonlinear solution methods applied to the stochastic neoclassical growth model with the same calibration than the one presented in this paper.

In this graph we can see how the optimal change of variable solution pushes a first order approximation to the accuracy level delivered by Finite Elements or Value Function Iteration for most of the interval. If we consider the cost of implementing the Finite Elements method or Value Function iteration, this result provides strong evidence in favor of using a combination of linear solution and optimal change of variable as a solution method for macroeconomic models.

We finish the discussion by pointing out how the new approximation is roughly comparable in terms of accuracy with a second order approximation of the policy function, a correction by variance and quadratic terms defended by Sims (2002a) and Schmitt-Grohé and Uribe (2002).<sup>14</sup>

### 6.3. Sensitivity Analysis

How does the solution to (10) depend on the chosen calibration? Are the optimal parameters values in  $\Xi$  robust to changes in the values of the structural parameters of the economy?

We analyzed in detail how the optimal change of variables depends on the different parameter values. We report only a sample of our findings with the most interesting results. We discuss the effect of changes in the share of capital  $\alpha$  (table 6.3.1), in the parameter governing the intertemporal elasticity of substitution  $\tau$  (table 6.3.2) and in the standard deviation of the technology shock  $\sigma$  (table 6.3.3). Note that since our minimization problem always found  $\varphi = 1$  (regardless of the initial guess for that parameter) we omit this last parameter.<sup>15</sup>

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<sup>14</sup>In addition the change of variables has the advantage of suffering much less from the problem of explosive behavior of simulations present in the second order approximation. See the explanation for this explosive behavior and a possible solution in Kim *et al.* (2003).

<sup>15</sup>Although we do not have a good intuition for this result we conjecture that it is a consequence of constant returns of scale in the production function.

Table 6.3.1: Optimal Parameters for different  $\alpha$ 's

$\alpha$	$\gamma$	$\zeta$	$\mu$
0.2	1.05064	1.06714	0.90284
0.4	0.98653	0.99167	2.47856
0.6	0.97704	0.97734	2.47796
0.8	0.94991	0.94889	2.47811

In Table 6.3.1 we see how  $\gamma$  and  $\zeta$  stay close to 1 although some variation in their values is induced by the changes in the capital share. On the other hand  $\mu$  changes quite substantially from 0.9 to 2.47. Interestingly all of this change happens when we move from  $\alpha = 0.2$  to  $\alpha = 0.4$ . This drastic change is due to the fact that the steepness of the marginal productivity of capital changes rapidly for values of  $\alpha$  around this range.

Table 6.3.2: Optimal Parameters for different  $\tau$ 's

$\tau$	$\gamma$	$\zeta$	$\mu$
4	1.02752	1.03275	2.47922
10	1.04207	1.04671	2.47830
50	1.01138	1.01638	2.47820

Table 6.3.2 shows the results are much more robust to changes in  $\tau$ . This result is important because it documents that the change of variables has a difficult time to capture the effects of large risk aversion.

Table 6.3.3: Optimal Parameters for different  $\sigma$ 's

$\sigma$	$\gamma$	$\zeta$	$\mu$
0.014	0.98140	0.98766	2.47753
0.028	1.04804	1.05265	1.73209
0.056	1.23753	1.22394	0.77869

Table 6.3.3 shows the effects of changing  $\sigma$  from its benchmark case of 0.07 to 0.014, 0.028 and 0.056. Here we observe more action:  $\gamma$  and  $\zeta$  move a bit and  $\mu$  goes all the way down from 2.48 to 0.78. Higher variance of the productivity shock increases the precautionary

motive for consumers and induces strong nonlinearities in the policy functions of the model (see for details, Aruoba, Fernández-Villaverde, and Rubio-Ramírez, 2003).

The variation in optimal parameter values is a very significant result. It shows how the change of variables allows for a correction based on the level of uncertainty existing in the economy. This correction cannot be achieved by the basic linear approach since that one produces a certainty equivalent approximation. We find this ability to adapt to different volatilities while keeping a first order structure in the solution a key advantage of the change of variables.

We conclude from our sensitivity analysis that the choices of optimal parameter values of  $\gamma$ ,  $\zeta$  and  $\varphi$  are stable across very different parametrizations and that the only relevant change is for the value of  $\mu$  when we vary capital share of the variance of the technology shock.

#### 6.4. A Linear Representation of the Solution

Our previous findings motivate the following observation. Since  $\gamma$  and  $\zeta$  are roughly equal and  $\varphi = 1$  across different parametrizations, instead of working with a four parameter change of variables, we can confine ourselves to the two parameter transformation:

$$\begin{aligned} k'^{\gamma} - k_0^{\gamma} &= a_3 (k^{\gamma} - k_0^{\gamma}) + b_3 z \\ l^{\mu} - l_0^{\mu} &= c_3 (k^{\gamma} - k_0^{\gamma}) + d_3 z \end{aligned}$$

while keeping nearly all the same level of accuracy obtained by the more general case.

To illustrate our point we plot the Euler Equation error for this restricted optimal case in figure 6.2.3. with  $\gamma = 1.11498$  and  $\mu = 0.948448$ . Comparing with the optimal change we can appreciate how we keep nearly all the increment in accuracy (more formally the  $SEE$  ( $\Xi$ ) moves now to 0.0420616).

The proposed two parameter transformation is very convenient because if we define  $\widehat{k} = k^{\gamma} - k_0^{\gamma}$  and  $\widehat{l} = l^{\mu} - l_0^{\mu}$  we can rewrite the equations as:

$$\begin{aligned} \widehat{k}' &= a_3 \widehat{k} + b_3 z \\ \widehat{l} &= c_3 \widehat{k} + d_3 z \end{aligned}$$

producing a linear system of difference equations.

This implies that we can study the neoclassical growth model (or a similar dynamic equilibrium economy) in the following way. First we find the equilibrium conditions of the economy. Second we linearize them (or loglinearize if it is easier) and solve the resulting problem using standard methods. Third we transform the variables (leaving the parameters of the transformation underdetermined until the final numerical analysis) and study the qualitative properties of the new system using standard techniques. If the transformation captures some part of the nonlinearities of the problem we may be able to avoid the problems singled out by Benhabib, Schmitt-Grohé and Uribe (2001). Fourth, if we want to study the quantitative behavior of the economy we pick values for the parameters according to (10) and simulate the economy using the transformed system.

## 7. Changes of Variables for Estimation

Often dynamic equilibrium models can be written in a state-space representation, with a transition equation that determines the law of motion for the states and a measurement equation that relates states and observables. It is well known that models with this representation are easily estimated using the Kalman Filter (see Harvey, 1989).

As a consequence a common practice to take the stochastic neoclassical growth model to the data has been to linearize the equilibrium conditions in levels or in logs to get a transition and a measurement equation, use the Kalman Formulae to evaluate the implied likelihood and then either to maximize it (in a classical perspective) or to draw from the posterior (from a Bayesian approach).

The variation that we propose is to use  $\widehat{k}' = a_3\widehat{k} + b_3z$  in the transition equation and (possibly)  $\widehat{l} = c_3\widehat{k} + d_3z$  in the measurement equation instead of the usual linear or loglinear relations. Since both equations are still linear in the transformed variables, the conditional distributions of the variables is normal and we can use the Kalman Filter to evaluate the likelihood. The only difference now will be that, in addition to depend on the structural parameters of the model, the likelihood will also be a function of the parameters  $\gamma$  and  $\zeta$ .

This approach allows us to keep a linear representation suitable for efficient estimation

while capturing an important part of the non-linearities of the model and avoiding the use of expensive simulation methods like the ones described in Fernández-Villaverde and Rubio-Ramírez (2002).

## 8. Concluding Remarks

This paper has explored the effects of changes of variables in the policy function of the stochastic neoclassical growth model as first proposed by Judd (2003). We have shown how this change of variables helps to obtain a more accurate solution to the model both for analytical and empirical applications.

The procedure proposed is conceptually straightforward and simple to implement yet powerful enough to substantially increase the quality of our solution to the neoclassical growth model. For our benchmark calibration the average Euler equation error is divided by three and the new policy function has a performance comparable with the policy functions generated by fully nonlinear methods. In addition, within the class of power functions considered in the paper, the optimal change of variables allows us to keep a linear structure of the model. This is useful for analytical and estimation purposes.

Several questions remain to be explored further. What is the optimal class of parametric families to use in the changes of variables? Are those optimal families robust across different dynamic equilibrium models? How big is the increment in accuracy in other types of models of interest to macroeconomist? How much do we gain in accuracy of our estimates by using a transformed linear state space representation of the model? We plan to address these issues in our future research.

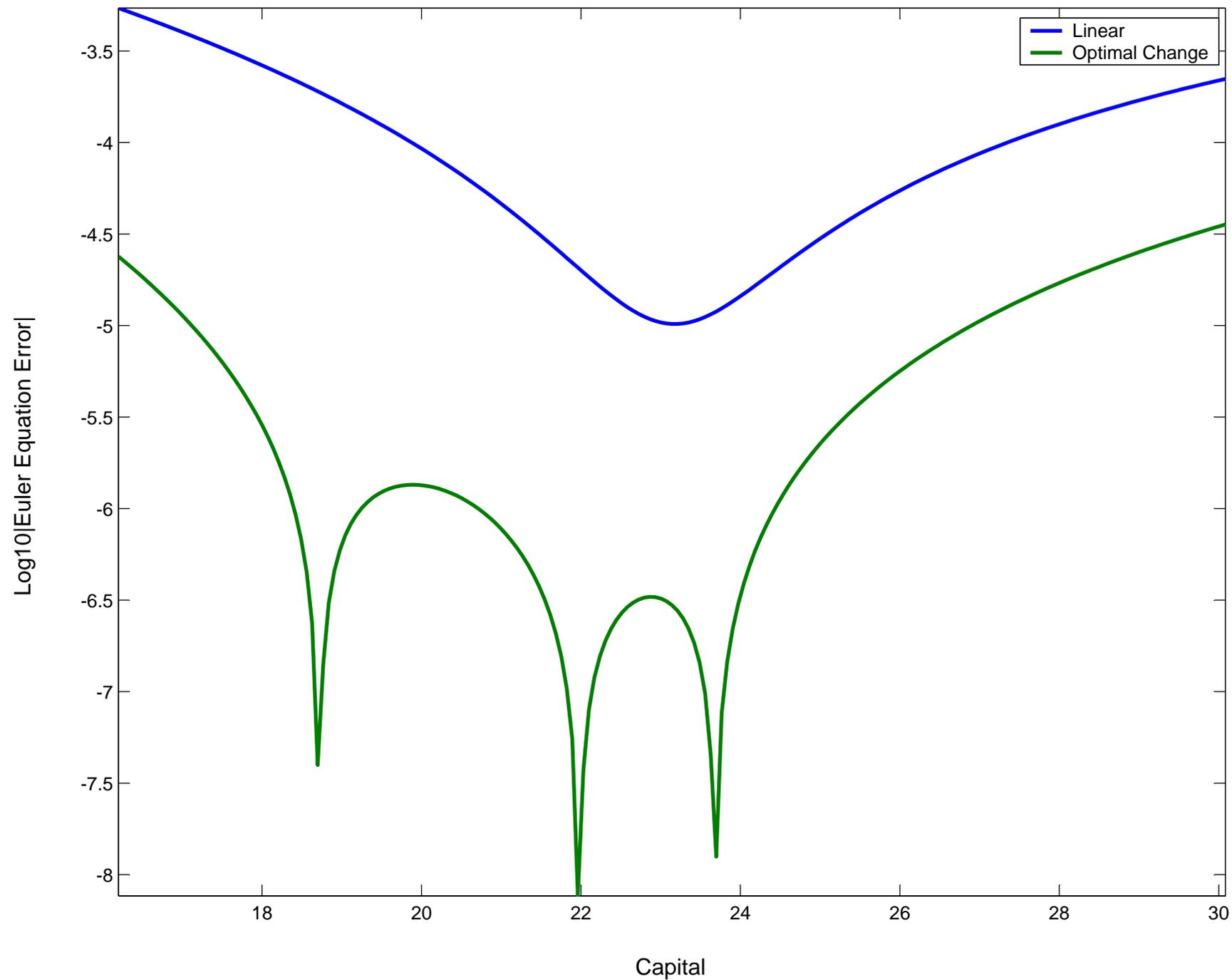
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**Figure 6.2.1 : Euler Equation Errors at  $z = 0$ ,  $\tau = 2 / \sigma = 0.007$**



**Figure 6.2.2 : Euler Equation Errors at  $z = 0$ ,  $\tau = 2 / \sigma = 0.007$**

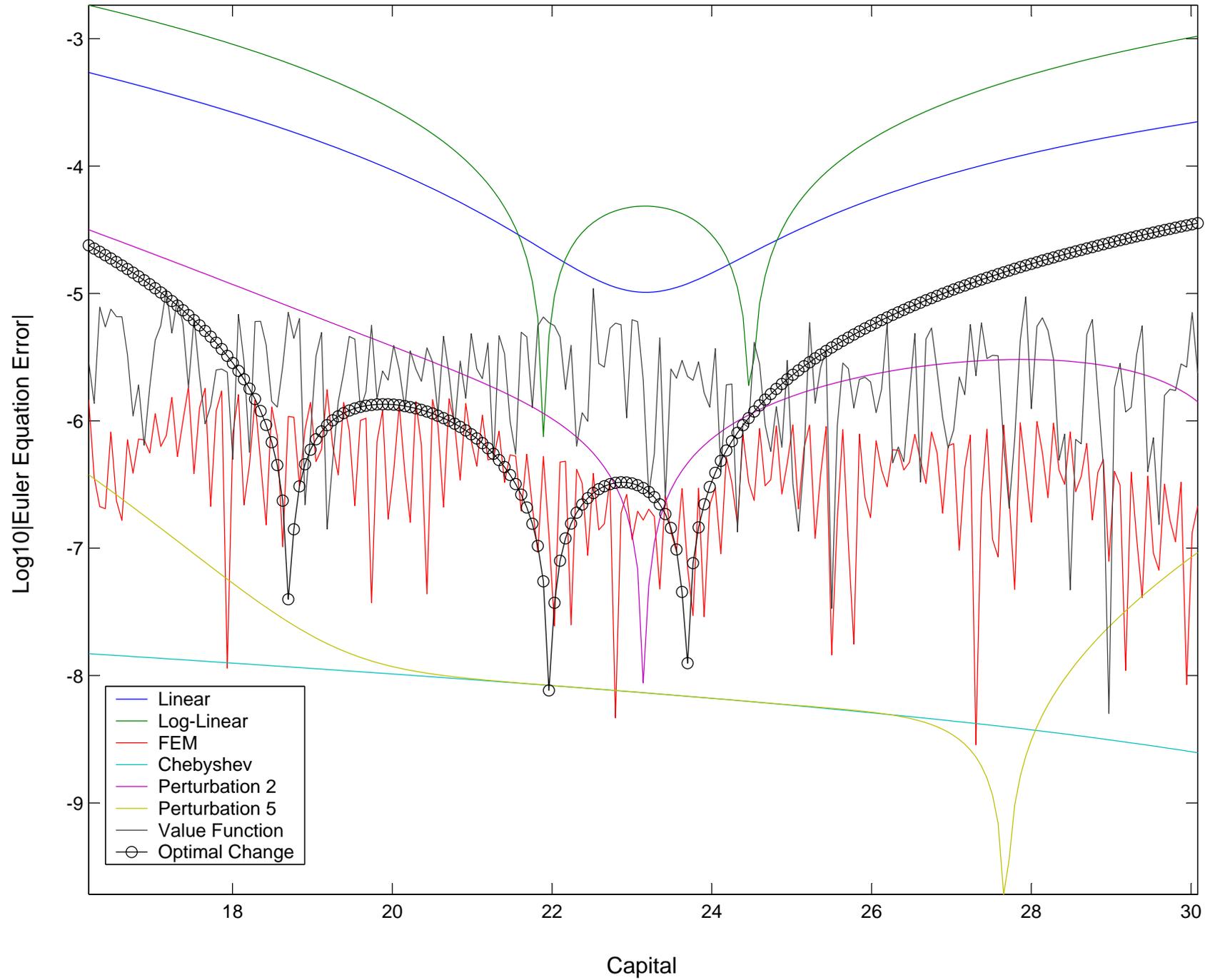


Figure 6.2.3. : Euler Equation Errors at  $z = 0$ ,  $\tau = 2 / \sigma = 0.007$

