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“Informational Size and Efficient Auctions”

by

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Informational Size and Efficient Auctions*

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Abstract

We develop an auction model for the case of interdependent values and multidimensional signals in which agents' signals are correlated. We provide conditions under which a modification of the Vickrey auction which includes payments to the bidders will result in an ex post efficient outcome. Furthermore, we provide a definition of informational size such that the necessary payments to bidders will be arbitrarily small if agents are sufficiently informationally small.

Keywords: Auctions, Incentive Compatibility, Mechanism Design, Interdependent Values.

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1 Introduction

The efficiency of market processes has been a central concern in economics since its inception. Auction mechanisms constitute a very important class of market processes, yet the analysis of auctions has typically focused on their revenue generating properties rather than their efficiency properties. This is partly due to the fact that, for many of the problems typically studied, efficiency is trivial. When bidders have private values, a standard Vickrey auction guarantees that the object will be sold to the buyer with the highest value for the object. In the case of pure common values – that is, when all buyers have the same value for the object – any outcome that with probability one assigns the object to *some* bidder will be efficient. The intermediate case in which bidders’ values are not identical but may depend on other bidders’ signals is more problematic. When bidders’ values are interdependent in this way, any single bidder’s value may depend on the information of other agents and, hence, he may not even know his own value. It is not clear what it would mean for an agent to bid his “true” value, even before we ask if it is optimal for him to do so.

Several papers have studied efficient auctions with interdependent values and independent types.¹ While this is a natural place to begin, the independence assumption is not compelling for many problems. A prototypical problem is one in which an object is to be sold, and individual bidders have private information about the object (say the quantity or quality of oil in a tract to be sold) that affects other agents’ values for the object. When bidders’ types include information about objective characteristics of the object being sold, their types will typically not be independent. When agents’ types are statistically dependent, we show that there exist efficient auction mechanisms for interdependent value auction problems that are essentially Vickrey auctions augmented by payments to (not from) the agents. Most importantly, we link the payment made to an agent to that agent’s “informational size,” a concept formulated in McLean and Postlewaite (2002). To explain informational size, suppose that all agents are receiving signals correlated with the common but unobservable value of the object. Informally, we can think of an agent as being informationally small if that agent’s signal adds little to the information contained in the aggregate of the other agents’ signals. That is, an agent is informationally small if it is unlikely that the probability distribution of the objective characteristics of the object is very sensitive to that agent’s information, given the information of others. When agents are informationally small, the payments necessary for our augmented Vickrey auction will be small. Hence, agents’ “informational rents” – as represented by the payments

¹See, e.g., Maskin (1992), Dasgupta and Maskin (1998), and Perry and Reny (1998).

made to them – are linked to their informational size.²

In this paper, we use a technique related to that employed by Cremer and McLean (1985,1988) in that we construct systems of lotteries (i.e., “side bets”) that facilitate truthful revelation of types. Cremer and McLean showed that, when agents’ types exhibit a certain statistical dependence, then mechanisms can be designed to induce truthful revelation of private information, and that information can be used to ensure efficient outcomes.³ When agents’ types are not independent, the multidimensionality of information poses no problems for Bayes-Nash implementation. However, mechanisms that rely on statistical dependence of types to extract the full surplus are sometimes criticized on the grounds that in such mechanisms, the payments to and from agents can be very large. The use of very large payments makes their usefulness questionable in the presence of limited liability or nonlinear preferences over money. Our lotteries will generally not extract the full surplus from the agents but we do identify conditions under which efficient outcomes can always be assured with small augmented payments.

Our definition of informational size generalizes the concept of nonexclusive information introduced in Postlewaite and Schmeidler (1986). Nonexclusive information was introduced to characterize informational problems in which incentive compatibility would not be an issue. Heuristically, this would be the case when, for any agent and for any information he might have, that agent’s information is redundant given the combined information of all other agents’. In the presence of nonexclusive information, it is straightforward to induce truthful revelation. In this case, roughly speaking, the agents’ reports will be inconsistent when a single agent misrepresents his information, thus revealing that some agent misreported with probability one. One can characterize this situation as one in which an agent has no ability to alter the posterior distribution as he contemplates the type he will announce. Our measure of informational size extends this concept in the sense that, when an agent has positive informational size, the agent’s different types (typically) result in different posterior distributions, given other agents’ reported types. When an agent is informationally small, that agent is unlikely to have a large effect on the posterior given other agents’ reported types.

Our model is described in Section 2 and, in Section 3 we present an example with a simple information structure in which agents receive conditionally independent signals of the state of nature. Section 4 provides an analysis of a more general problem with information structures that include the conditionally independent structure of the example in section 3 as a special case. The analysis in section 4 assumes that agents’ types are exogenously specified in a form that separates the part of an agent’s

²However, we should emphasize that we are not proposing that agents are *necessarily* informationally small and, consequently, that efficient outcomes can always be assured with small augmented payments.

³See also McAfee and Reny (1992) for subsequent work.

information that affects other agents' values from the part of the agent's information that affects only his own value. In Section 5, we show how the information structure for general incomplete information problems can be represented in a way that decomposes agents' information into these two components. Since for many asymmetric information problems these two aspects of an agent's information are qualitatively different, this decomposition is of some independent interest. Some concluding comments are contained in Section 6 and the proofs are given in Section 7.

2 Auctions

Let $\Theta = \{\theta_1, \dots, \theta_m\}$ represent the finite set of states of nature. Each $\theta \in \Theta$ represents a complete physical description of the object being sold (e.g., the amount and quality of oil). Let T_i be a finite set of possible types of agent i . As stressed in the introduction, an agent's information may be of two qualitatively different kinds: information about the objective characteristics of the object being sold, and idiosyncratic information about the agent himself. The former is of interest to other agents - and consequently is the cause of the interdependence of agents' values - while the latter is irrelevant to other agents in calculating their values. The state of nature is unobservable but agent i 's information about the physical characteristics of the object to be sold will be captured by the correlation between his type t_i and nature's choice of θ . His type t_i will also capture any idiosyncratic information he may have. Agent i 's value is represented by a function $v_i : \Theta \times T_i \rightarrow R_+$. That is, agent i 's value for the object depends on the physical characteristics of the object θ , and his type t_i .

Let $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n)$ be an $(n+1)$ -dimensional random vector taking values in $\Theta \times T$ ($T \equiv T_1 \times \dots \times T_n$ and $T_{-i} \equiv \times_{j \neq i} T_j$) with associated distribution P where

$$P(\theta, t_1, \dots, t_n) = \text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1 = t_1, \dots, \tilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions : $P(\theta) = \text{Prob}\{\tilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t) = \text{Prob}\{\tilde{t}_1 = t_1, \dots, \tilde{t}_n = t_n\} > 0$ for each $t \in T$.

If X is a finite set, let Δ_X denote the set of probability measures on X . The set of probability measures on $\Theta \times T$ satisfying the full support conditions will be denoted $\Delta_{\Theta \times T}^*$.

In problems with differential information, it is standard to assume that agents have utility functions $w_i : T \rightarrow R_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $w_i(t) = \sum_{\theta \in \Theta} [v_i(\theta, t_i) P(\theta|t)]$. Alternatively, given utility functions $w_i : T \rightarrow R_+$, we can define $\Theta \equiv T$ and define $v_i(t, t_i) = w_i(t)$. Our formulation will be useful in that it highlights the nature of

the interdependence: agents care about other agents' types to the extent that they provide additional information about the physical characteristics of the object being sold.

An *auction problem* is a collection (v_1, \dots, v_n, P) where $P \in \Delta_{\Theta \times T}^*$. An *auction mechanism* is a collection $\{q_i, x_i\}_{i \in N}$ where $q_i : T \rightarrow \mathbb{R}_+$ is the probability that agent i is awarded the object given a vector of announced types, and each $x_i : T \rightarrow \mathbb{R}$ is a transfer function.

For any vector of types $t \in T$, let

$$\hat{v}_i(t) = \hat{v}_i(t_{-i}, t_i) = \sum_{\theta \in \Theta} v_i(\theta, t_i) P(\theta | t_{-i}, t_i).$$

Although \hat{v} depends on P , we suppress this dependence for notational simplicity. The number $\hat{v}_i(t)$ represents i 's value for the object conditional on the informational state $t \in T$.

Definition: An auction mechanism $\{q_i, x_i\}_{i \in N}$ is:

incentive compatible (IC) if for each $i \in N$,

$$\sum_{t_{-i}} [q_i(t_{-i}, t_i) \hat{v}_i(t_{-i}, t_i) - x_i(t_{-i}, t_i)] P(t_{-i} | t_i) \geq \sum_{t_{-i}} [q_i(t_{-i}, t'_i) \hat{v}_i(t_{-i}, t_i) - x_i(t_{-i}, t'_i)] P(t_{-i} | t_i)$$

whenever $t_i, t'_i \in T_i$.

ex post individually rational (XIR) if

$$q_i(t) \hat{v}_i(t) - x_i(t) \geq 0 \text{ for all } i \text{ and all } t \in T.$$

ex post efficient (XE) if

$$\hat{v}_i(t) = \max_j \{\hat{v}_j(t)\} \text{ whenever } q_i(t) > 0.$$

For a given auction problem (v_1, \dots, v_n, P) , we will be interested in the second price auction using the conditional values $\hat{v}_i(t)$. For each $t \in T$, let

$$I(t) = \{i \in N | \hat{v}_i(t) = \max_j \hat{v}_j(t)\}$$

and define

$$w_i(t) = \max_{j: j \neq i} \hat{v}_j(t).$$

Formally, we define a *Vickrey auction with conditional values* (Vickrey auction for short) to be the auction mechanism $\{q_i^*, x_i^*\}_{i \in N}$ defined as follows:

$$q_i^*(t) = \begin{cases} \frac{1}{|I(t)|} & \text{if } i \in I(t) \\ 0 & \text{if } i \notin I(t) \end{cases}$$

and

$$x_i^*(t) = q_i^*(t)w_i(t) .$$

It is straightforward to show that this Vickrey auction mechanism is ex post efficient and ex post individually rational. It will generally *not* be incentive compatible. However, as we will show below, it is often possible to modify the Vickrey auction payments so as to make truthful revelation an equilibrium when agents are informationally small in a sense to be defined below.

Let $\{z_i\}_{i \in N}$ be an n -tuple of functions $z_i : T \rightarrow \mathbb{R}_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a “reward” to agent i . The associated *augmented Vickrey auction with conditional values* (augmented Vickrey auction for short) is the auction mechanism $\{q_i^*, x_i^* - z_i\}_{i \in N}$.

We present an example in the next section that illustrates our notion of augmented Vickrey auctions and the relationship between informational size and the payments that agents receive. This example also illustrates the main ideas in the proofs of Theorems 1 and 2 discussed in sections 4 and 5 below.

3 Example

Three wildcatters are competing for the right to drill for oil on a tract of land. It is common knowledge that the amount of oil is either θ_L or θ_H , each equally likely with $\theta_L < \theta_H$. We will abuse notation and denote the states by θ_L and θ_H ; let $\Theta = \{\theta_L, \theta_H\}$. Each wildcatter i performs a private test that provides information in the form of a noisy signal of the state which we denote s_i . That is, agent i ’s private test yields a signal H (high) or L (low); for each i , let $S_i = \{H, L\}$. The distribution of the signal for agent i , conditional on the state, is given in the table below ($\rho > 1/2$).

	<i>state</i>	θ_L	θ_H
<i>signal</i>			
L		ρ	$1 - \rho$
H		$1 - \rho$	ρ

Agents’ signals are independent, conditional on the state θ .

In addition to the signal regarding the amount of oil, each of the wildcatters has private information regarding his own cost of extraction. We assume that the extraction cost c_i of wildcatter i is drawn from a finite set. Hence, agent i ’s type t_i is the pair (s_i, c_i) comprising his privately observed extraction cost c_i and his privately observed signal s_i . We will assume that the vector of extraction costs (c_1, c_2, c_3) is independent of the state-signal vector (θ, s_1, s_2, s_3) . The price of oil is 1. Agent i ’s payoff v_i as a function of the state θ and his type t_i depends only on θ and his private extraction cost c_i . If $t_i = (c_i, s_i)$, then his payoff should he obtain the right to drill is given by:

$$v_i(\theta_j, t_i) = \theta_j - c_i, \quad j = L, H$$

Consider the following auction mechanism. Agents announce their types and the posterior distribution on Θ given the agents' announcements of their signals, $P_\Theta(\cdot|s_1, s_2, s_3)$, is calculated. Next, compute the agents' expected values \hat{v}_i for the object, where

$$\hat{v}_i(t_1, t_2, t_3) = \hat{v}_i(s_1, s_2, s_3, c_i) = v_i(\theta_L, c_i) \cdot P_\Theta(\theta_L|s_1, s_2, s_3) + v_i(\theta_H, c_i) \cdot P_\Theta(\theta_H|s_1, s_2, s_3).$$

The drilling rights are awarded to the agent i for whom $\hat{v}_i(s_1, s_2, s_3, c_i)$ is highest and that agent pays a price equal to the higher of the other two agents' values. In addition, any agent who has announced a signal equal to that announced by the majority receives a (small) payment $\bar{z} > 0$.

Truth is generally not a dominant strategy for the unaugmented Vickrey mechanism. If agent 3 (for example) announces L when he has in fact received signal H , his announcement of L will lower the computed expected values of all agents. In the event that agent 3 wins the object, he will pay a lower price by announcing L . However, the introduction of the reward \bar{z} in the augmented mechanism will offset this possible gain in expected utility when ρ is close to 1. When $\rho \approx 1$, we make two important observations. First, an agent who observes signal H (resp. L) places probability close to one on the event that the other two agents have also seen H (resp. L). Second, the conditional distributions $P_\Theta(\cdot|H, H, H)$ and $P_\Theta(\cdot|H, H, L)$ are almost equal and the conditional distributions $P_\Theta(\cdot|L, L, L)$ and $P_\Theta(\cdot|L, L, H)$ are almost equal. Combining these observations, it follows that an agent who sees signal H will believe that, with high probability, an announcement of L will have little effect on the posterior distribution on Θ . Agent 3 affects the computations of other agents' values only through the posterior distribution on Θ . Since the values of agents are continuous in this posterior, it follows that any expected gain that he may hope to obtain by altering other agents' values through misreporting will be small.

On the other hand, a misreported signal will, with probability close to 1, put an agent in the minority and that agent will lose the reward \bar{z} paid to those in the majority. On balance, the incentive provided by the small payment \bar{z} will offset the small gain from misrepresenting when ρ is sufficiently close to 1. In summary, if ρ is sufficiently close to 1, truthful revelation will be an equilibrium.⁴

Incentive compatibility is obtained in the example when $\rho \approx 1$ as a result of an interplay of two ideas: *informational size* and the *variability of agent's beliefs*. In the example, each agent is *informationally small* when $\rho \approx 1$ in the sense that, upon observing the signal H , agents conclude that an announcement of L will with high

⁴It is not the unique equilibrium, however; all agents reporting signal L would be an equilibrium. We discuss this in the last section.

probability have only a small effect on the distribution over states conditioned on the information of all three agents. Although this means that the gains from lying are small, they are positive. That the reward \bar{z} for making an announcement that is in the majority offsets, at least partially, this gain is due to the *variability of agents' beliefs*, defined as the difference between the conditional distributions $P(\cdot, \cdot | H)$ and $P(\cdot, \cdot | L)$ on $S_1 \times S_2$. Because agent 3's probability distributions over the other agents' signals, $P(\cdot, \cdot | H)$ and $P(\cdot, \cdot | L)$, are different, agent 3's expected reward from truthfully announcing his signal is greater than the expected reward from lying. How *much* greater depends on the magnitude of the difference between $P(\cdot, \cdot | H)$ and $P(\cdot, \cdot | L)$. If, for example, these conditional distributions were equal, then we could not find a system of rewards that induced truthful revelation. To illustrate, if $\rho \approx 1/2$, agents would be informationally small, since their announcement would affect the posterior distribution on Θ very little. However, as ρ gets closer to $1/2$, the incentive to truthfully report is decreasing, since an agent is nearly as likely to be in the majority when he lies as when he announces truthfully. Whether a given reward will induce truthful announcement depends on the relationship between informational size and variability of beliefs.

The mechanism illustrates when agents' information can be truthfully elicited at low cost: when informational size is small relative to variability of beliefs. This is possible because (by construction) each agent's information is of little value given other agents' information, hence information rents are small. Once the information about the physical characteristics is elicited, there is no residual interdependence and a simple Vickrey auction can be used.

In the next section, we present a model that generalizes the insights of this example and formalizes the concepts of informational size and variability.

4 Efficient Auction Mechanisms

4.1 The Model

In this section we will assume that the set of types for agent i has the special product form $T_i = S_i \times C_i$ where S_1, \dots, S_n and C_1, \dots, C_n are finite sets. An element $s_i \in S_i$ will be referred to as agent i 's *signal*. An element $c_i \in C_i$ will be referred to as agent i 's *personal* characteristic. Let $S \equiv S_1 \times \dots \times S_n$ and $S_{-i} \equiv \times_{j \neq i} S_j$. The product sets C and C_{-i} are defined in a similar fashion. We will often write $t = (s, c)$ and $t_i = (s_i, c_i)$ where s and c (s_i and c_i) denote the respective projections of t (t_i) onto S and C (S_i and C_i). Both the signal s_i and the personal characteristic c_i are private information to i with the following interpretations: s_i represents a signal that is correlated with nature's choice of θ and c_i represents a set of other idiosyncratic payoff relevant characteristics of agent i that provide no information about θ or s_{-i} beyond that contained in s_i . In our example, the extraction cost c_i of each wildcatter

corresponds to the agent's personal characteristic and, since costs are assumed to be independent of the state and the agents' signals, it is certainly the case that c_i contains no information about θ or s_{-i} beyond that contained in s_i . We assume⁵ that the random vectors $(\tilde{\theta}, \tilde{s})$ and \tilde{c} are *stochastically independent*, i.e.,

$$P(\theta, t) \equiv P(\theta, s, c) = P(\theta, s)P(c).$$

We denote by $\Delta_{\Theta \times S \times C}^I$ the set of measures in $\Delta_{\Theta \times S \times C}^*$ satisfying this stochastic independence assumption.

4.2 Informational Size and Variability of Beliefs

We now formalize the idea of informational size discussed in section 3 above. Our example indicates that a natural notion of an agent's informational size is the degree to which he can alter this posterior distribution on Θ when other agents are announcing truthfully. Any vector of agents' signals $s = (s_{-i}, s_i) \in S$ induces a conditional distribution on $P_{\Theta}(\cdot | s_{-i}, s_i)$ on Θ and, if agent i unilaterally changes his announcement from s_i to s'_i , this conditional distribution will (in general) change. If i receives signal s_i but announces $s'_i \neq s_i$, the set

$$\{s_{-i} \in S_{-i} \mid \|P_{\Theta}(\cdot | s_{-i}, s_i) - P_{\Theta}(\cdot | s_{-i}, s'_i)\| > \varepsilon\}$$

consists of those s_{-i} for which agent i 's misrepresentation will have (at least) an " ε -effect" on the conditional distribution. (Here and throughout the paper, $\|\cdot\|$ will denote the 1-norm.) Let

$$\nu_i^P(s_i, s'_i) = \min\{\varepsilon \in [0, 1] \mid \text{Prob}\{\|P_{\Theta}(\cdot | \tilde{s}_{-i}, s_i) - P_{\Theta}(\cdot | \tilde{s}_{-i}, s'_i)\| > \varepsilon \mid \tilde{s}_i = s_i\} \leq \varepsilon\}.$$

To show that $\nu_i^P(s_i, s'_i)$ is well defined, let

$$F(\varepsilon) = \text{Prob}\{\|P_{\Theta}(\cdot | \tilde{s}_{-i}, s_i) - P_{\Theta}(\cdot | \tilde{s}_{-i}, s'_i)\| \leq \varepsilon \mid \tilde{s}_i = s_i\}.$$

Hence, the set $\{\varepsilon \in [0, 1] \mid 1 - F(\varepsilon) \leq \varepsilon\}$ is nonempty (since $1 - F(1) \leq 1$), bounded and closed (since F is right continuous with left hand limits.)

Finally, define the *informational size* of agent i as

$$\nu_i^P = \max_{s_i, s'_i \in S_i} \nu_i^P(s_i, s'_i).$$

Note that $\nu_i^P = 0$ for every i if and only if $P_{\Theta}(\cdot | s) = P_{\Theta}(\cdot | s_{-i})$ for every $s \in S$ and $i \in N$.⁶

⁵This assumption can be weakened. See point 9 in the discussion section.

⁶This is essentially the case of nonexclusive information introduced by Postlewaite and Schmeidler (1986) and is discussed further in the last section.

There are two important features of this definition. First, an agent's informational size depends only on that part of his information that is useful in predicting θ , and second, an informationally small agent may have very accurate information about the state θ .

In our discussion of the example in section 3 above, we indicated that the ability to give agent i an incentive to reveal his information will depend on the magnitude of the difference between $P_{S_{-i}}(\cdot|s_i)$ and $P_{S_{-i}}(\cdot|s'_i)$, the conditional distributions on S_{-i} given different signals for agent i . We will refer to this magnitude informally as the *variability of agents' beliefs*.

To define formally the measure of variability, we treat each conditional $P_{S_{-i}}(\cdot|s_i) \in \Delta_{S_{-i}}$ as a point in a Euclidean space of dimension equal to the cardinality of S_{-i} . Our measure of variability is defined as⁷

$$\Lambda_i^{P,S} = \min_{s_i \in S_i} \min_{s'_i \in S_i \setminus s_i} \|P_{S_{-i}}(\cdot|s_i) - P_{S_{-i}}(\cdot|s'_i)\|^2.$$

4.3 The Result

We now state our first result on the possibility of efficient mechanisms.

Theorem 1: Let (v_1, \dots, v_n) be a collection of payoff functions.

(i) If $P \in \Delta_{\Theta \times S \times C}^I$ satisfies $\Lambda_i^{P,S} > 0$ for each i , then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) .

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times S \times C}^I$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^{P,S},$$

there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t .

Part (i) of Theorem 1 states that, if $\Lambda_i^{P,S}$ is positive for each agent i , then there exists an incentive compatible augmented Vickrey mechanism for the auction problem (v_1, \dots, v_n, P) . The hypotheses of part (i) only require that each $\Lambda_i^{P,S}$ be positive and places no lower bound on the magnitude of $\Lambda_i^{P,S}$. Furthermore, the informational size of the agents is not important. On the other hand, the conclusion of part (i) places no upper bound on the size of the reward z_i . These rewards can be quite large.

Part (ii) of the theorem states that there exists an incentive compatible augmented Vickrey mechanism with small payments if, for each i , $\Lambda_i^{P,S}$ is large enough relative to the informational size of agent i . To illustrate part (ii), consider again the example in section 3 where we showed the following: for every $\varepsilon > 0$, there exists a $\tilde{\rho} > 0$ such

⁷See McLean and Postlewaite (2002) for further discussion of informational size and variability.

that, whenever $\tilde{\rho} < \rho < 1$, there exists an incentive compatible augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in \{1,2,3\}}$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for all t . This result can now be deduced as an application of (ii) since, in the example, each $\nu_i^P \rightarrow 0$ and each $\Lambda_i^{P,S} \rightarrow 1$ as $\rho \rightarrow 1$.

While the technical details of the proof are deferred until the last section, we can sketch the ideas here for the special case in which $T_i = S_i$ (i.e., each C_i is a singleton). Let

$$M = \max_{\theta} \max_i \max_{s_i} v_i(\theta, s_i)$$

and define

$$U_i^*(s_{-i}, s'_i) = q_i^*(s_{-i}, s'_i) \hat{v}_i(s) - x_i^*(s_{-i}, s'_i).$$

There are two key steps. First, we show (see Lemmas A.1 and A.2) that for all i , all $s_i, s'_i \in S_i$ and all $s_{-i} \in S_{-i}$,

$$U_i^*(s_{-i}, s_i) - U_i^*(s_{-i}, s'_i) \geq -M \|P_{\Theta}(\cdot | s_{-i}, s_i) - P_{\Theta}(\cdot | s_{-i}, s'_i)\|.$$

This Lipschitz-like property is of some interest in its own right. For example, if $\tilde{\theta}$ and \tilde{s} are independent, then $\hat{v}_i(s)$ depends only on s_i . In this case, $\|P_{\Theta}(\cdot | s_{-i}, s_i) - P_{\Theta}(\cdot | s_{-i}, s'_i)\| = 0$ for all i , all $s_i, s'_i \in T_i$ and all $s_{-i} \in S_{-i}$ and we deduce the classic result for Vickrey auctions: truthful reporting is a dominant strategy with pure private values.

Of course, $\|P_{\Theta}(\cdot | s_{-i}, s_i) - P_{\Theta}(\cdot | s_{-i}, s'_i)\|$ is generally not uniformly small. However, we can use the concept of informational size to show that

$$\sum_{s_{-i}} [U_i^*(s_{-i}, s_i) - U_i^*(s_{-i}, s'_i)] P(s_{-i} | s_i) \geq -3M \hat{\nu}_i^P.$$

If all agents are informationally small, then truthful reporting is “approximately” incentive compatible in the (unaugmented) Vickrey mechanism $\{q_i^*, x_i^*\}$. If $z_i(s)$ is the reward to i when the bidders announce s , then the associated augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}$ will be incentive compatible if

$$\sum_{s_{-i}} [z_i(s_{-i}, s_i) - z_i(s_{-i}, s'_i)] P(s_{-i} | s_i) - 3M \hat{\nu}_i^P \geq 0$$

for each $s_i, s'_i \in S_i$. Irrespective of an agent’s informational size, such a collection of z_i ’s can be found if $\Lambda_i^{P,S} > 0$. This is the content of Part (i) of the theorem. However, these z_i ’s can be large. If we desire small rewards (as in Part (ii)), then the situation is more delicate and we require that each $\Lambda_i^{P,S}$ be large enough relative to the informational size of agent i .

We now explain briefly the relationship of our paper to those of Cremer and McLean on full surplus extraction (1985, 1988). The main point of the Cremer-McLean papers is that correlation of agents’ types allows full surplus extraction.

In the models in those papers (as in the present paper), players' payoffs include payments that depend on other agents' types. In the Cremer-McLean setup, the type of correlation (for example, the full rank condition in their 1985 paper) permits the construction of announcement dependent lotteries, where truthful revelation generates a lottery with zero conditional expected value while a lie generates negative conditional expected value. If the lotteries are appropriately rescaled, then the incentive for truthful reporting can be made arbitrarily large and an incentive compatible mechanism that extracts the full surplus can be found.

We should note that we require a somewhat weaker condition than is used in those papers: that the conditional distribution on T_{-i} be different for different t'_i s. That is, we only require that $\Lambda_i^{P,S}$ be positive. This is weaker than the full rank condition (and is also weaker than the cone condition in their 1988 paper) and the implication is concomitantly weaker. Our assumption only permits the construction of announcement dependent lotteries where truthful revelation generates a lottery whose conditional expected value exceeds the conditional expected value from a lie. Using the full rank condition and some additional assumptions on the conditional payoff $\hat{v}(t)$, Cremer-McLean construct a mechanism that extracts the full surplus from bidders (see Corollary 2 in Cremer-McLean, 1985). This mechanism is necessarily ex post efficient. Under the weaker conditions of this paper, we construct a mechanism that is ex post efficient but which may not extract the full surplus. In addition, the payments in a Cremer-McLean mechanism can be positive or negative and they can be large in absolute value. Our paper differs in that we introduce only nonnegative payments. Hence, our techniques do not require unlimited liability on the part of buyers (although the seller may be constrained by the necessary payments that would induce incentive compatibility). Further, they allow us to induce incentive compatibility with *small* payments when agents are informationally small.

5 Efficient Auction Mechanisms: The General Case

The mechanism in the previous section is successful in achieving an efficient outcome because it deals differently with the component of an agent's information that affects other agents' values and with the component that affects only his own value. Since second-price auction techniques handle the latter, one need only extract the former to achieve efficient outcomes. The information structure in the previous section assumed that the set of types of an agent could be expressed as the Cartesian product of signals and personal characteristics and that the information structure satisfied stochastic independence. Stated differently, we assumed that agents' types were exogenously decomposed into "private" and "common" components. General information structures will typically not have this form, and consequently the result in the previous section may not apply. In this section, we show how the information structure for general incomplete information problems, even those without a product structure,

can be represented in a way that decomposes agents' information into "signals" and "private characteristics."

In order to extend the ideas of the special model of section 4 to the general problem defined in section 2, we need to define the appropriate generalizations of informational size and variability of beliefs. Let

$$\nu_i^P(t_i, t'_i) = \min\{\varepsilon \in [0, 1] | \text{Prob}\{ ||P_\Theta(\cdot | \tilde{t}_{-i}, t_i) - P_\Theta(\cdot | \tilde{t}_{-i}, t'_i)|| > \varepsilon | \tilde{t}_i = t_i \} \leq \varepsilon\}$$

and define the *informational size* of agent i as

$$\nu_i^P = \max_{t_i, t'_i \in T_i} \nu_i^P(t_i, t'_i).$$

This is the definition introduced in McLean and Postlewaite (2002). If $T_i = S_i \times C_i$ and if $P \in \Delta_{\Theta \times S \times C}^I$, then the definition of ν_i^P given above coincides with the definition of informational size given in section 4.

To extend the notion of variability of beliefs, we begin with the definition of information decomposition.

Definition: An *information decomposition* (ID) of $P \in \Delta_{\Theta \times T}^*$ is a collection \mathcal{D} consisting of sets R_1, \dots, R_n and surjections $g_i : T_i \rightarrow R_i$ satisfying:

(i) for all i , for all $t_i, t'_i \in T_i$ and for all $t_{-i} \in T_{-i}$,

$$g_i(t_i) = g_i(t'_i) \Rightarrow P_\Theta(\cdot | t_{-i}, t_i) = P_\Theta(\cdot | t_{-i}, t'_i).$$

(ii) for all i , for all $t_i, t'_i \in T_i$ and for all $r_{-i} \in R_{-i}$,

$$g_i(t_i) = g_i(t'_i) \Rightarrow \text{Prob}\{g_j(\tilde{t}_j) = r_j \forall j \neq i | \tilde{t}_i = t_i\} = \text{Prob}\{g_j(\tilde{t}_j) = r_j \forall j \neq i | \tilde{t}_i = t'_i\}.$$

We interpret $g_i(t_i)$ as that "part" of an agent's information that is "informationally relevant" for predicting the state of nature θ . Condition (i) has the following interpretation: given a type profile $t_{-i} \in T_{-i}$, a type $t_i \in T_i$ contains no information that is useful in predicting the state θ beyond that contained in the informationally relevant part $g_i(t_i)$. Condition (ii) states that a specific type $t_i \in T_i$ contains no information beyond that contained in $g_i(t_i)$ that is useful in predicting the informationally relevant profile of other agents.

Every measure P has at least one information decomposition: this is the trivial decomposition in which $T_i = R_i$ and $g_i = id$. It may be the case that for a given information structure, the trivial decomposition is the only decomposition; this would be the case, for example, when agents' private information consisted solely of noisy signals about the state θ , with no "private characteristic". However, a measure P can have in addition a nontrivial information decomposition. If each $T_i = S_i \times C_i$ as in section 4 and if $P \in \Delta_{\Theta \times S \times C}^I$, then P has a second information decomposition where $R_i = S_i$ and g_i is the projection of T_i onto S_i .

Given an information decomposition $\mathcal{D} = \{R_i, g_i\}_{i \in N}$ for $P \in \Delta_{\Theta \times T}^*$, we let $P^{\mathcal{D}}$ denote the distribution on $R = R_1 \times \dots \times R_n$ induced by the map $(t_1, \dots, t_n) \mapsto (g_1(t_1), \dots, g_n(t_n))$. That is, for each $(r_1, \dots, r_n) \in R$,

$$P^{\mathcal{D}}(r_1, \dots, r_n) = \text{Prob}\{\tilde{t}_i \in g_i^{-1}(r_i) \ \forall i \in N\}.$$

Given an information decomposition \mathcal{D} , let

$$\Lambda^{P, \mathcal{D}} = \min_{r_i \in R_i} \min_{r'_i \in R_i \setminus r_i} \|P_{R_{-i}}^{\mathcal{D}}(\cdot | r_i) - P_{R_{-i}}^{\mathcal{D}}(\cdot | r'_i)\|^2.$$

If each $T_i = S_i \times C_i$, $R_i = S_i$ and g_i is the projection of T_i onto S_i , then $\Lambda_i^{P, \mathcal{D}}$ coincides with $\Lambda_i^{P, S}$ as defined in section 4. Minimal information decompositions are important for representing an information structure since it will typically be the case our measure of variability will be 0 for information structures that are not minimal.

Using these definitions of informational size and variability of beliefs, we can generalize Theorem 1 as follows.

Theorem 2: Let (v_1, \dots, v_n) be a collection of payoff functions.

(i) Let $P \in \Delta_{\Theta \times T}^*$. If there exists an information decomposition \mathcal{D} for P with $\Lambda_i^{P, \mathcal{D}} > 0$ for each i , then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) .

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}^*$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^{P, \mathcal{D}}$$

for some information decomposition \mathcal{D} of P , there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t .

Theorem 1 is an immediate corollary of Theorem 2. It is possible that a measure P has only one ID, the trivial decomposition (denoted \mathcal{D}^0) where $T_i = R_i$ and $g_i = id$. For this decomposition, it follows from the definitions that

$$\Lambda_i^{P, \mathcal{D}^0} = \min_{t_i \in T_i} \min_{t'_i \in T_i \setminus t_i} \|P_{T_{-i}}(\cdot | t_i) - P_{T_{-i}}(\cdot | t'_i)\|^2$$

where $P_{T_{-i}}(\cdot | t_i)$ is the conditional on T_{-i} given $\tilde{t}_i = t_i$. For the trivial ID \mathcal{D}^0 , we have the following corollary to Theorem 2.

Corollary 1: Let (v_1, \dots, v_n) be a collection of payoff functions.

(i) If $P \in \Delta_{\Theta \times T}^*$ satisfies $P_{T_{-i}}(\cdot | t_i) \neq P_{T_{-i}}(\cdot | t'_i)$ for each $i = 1, \dots, n$ and for each $t_i, t'_i \in T_i$ with $t_i \neq t'_i$, then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) .

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}^*$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^{P, \mathcal{D}^0},$$

there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem (v_1, \dots, v_n, P) satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t .

As a final remark on the relationship between our results, we note that Corollary 1 can also be deduced as a special case of Theorem 1 in which each C_i is a singleton and T_i is identified with S_i . If each C_i is a singleton, then stochastic independence is trivially satisfied and Corollary 1 follows from Theorem 1.

6 Discussion

1. As pointed out in the example, truthful revelation is an equilibrium for our augmented Vickrey auction mechanisms, but not the unique equilibrium. One should be able to use the techniques in the literature on exact implementation to construct nonrevelation games that eliminate the multiplicity of equilibria.⁸

2. In this paper, we focus on the augmented Vickrey auction and show that an efficient outcome can be assured with payments to the agents that depend on the agents' informational size. The mechanism that we analyze will not, in general, maximize the net revenue to the seller. In proving our theorem, we demonstrate that for any limit on the total payments to the agents, we can guarantee a structure of payments depending on agents' announcements that will assure incentive compatibility if agents are sufficiently informationally small. Although the payments that we construct will not typically be the minimal payments that induce truthful announcement, it must be the case that any increase in expected net revenue to the seller that can be achieved through optimizing the structure of payments to agents is limited by the total payments identified in our result.

There is a second way the mechanism we analyze may be inefficient that may be more important, however. In our mechanism agents announce their types, and these types are used to calculate agents' conditional values. The agent with the highest conditional value obtains the object at the second highest conditional value, and the difference between the first and second highest values constitutes a rent to the winning bidder. Suppose agents' types consist of a signal about θ and a private characteristic. The winning bidder's rent will then depend on his private characteristic and the private characteristic of the agent with second highest conditional value. While the seller may not be able to eliminate this rent when the private characteristics are stochastically independent, we have not made any assumptions regarding such independence. If private characteristics are not independent, there may be scope for

⁸See, e.g., Postlewaite and Schmeidler (1986), or the surveys of Moore (1992) and Palfrey (1992).

extending our techniques to extract this rent. Of course, the possibility of extracting this rent has no bearing on whether the auction mechanism is efficient, which is the focus of this paper.

3. In section 4, we assume that agents' type sets are finite. If the signals and personal characteristics of agents' information are separated, it is only the signal sets that need to be finite. The set of personal characteristics can be finite, a continuum or some combination without affecting the possibility of efficient mechanisms.

4. As mentioned in the introduction, McLean and Postlewaite (2002) introduced a notion of informational size similar to that used in this paper. That paper deals with pure exchange economies with private information in which an agent's utility function depends only on the realized state $\theta \in \Theta$. The preferences in the present paper are more general in the sense that agent i 's utility may depend on his type t_i as well as the state θ . The extension of our methods to this case is possible because of the properties of the Vickrey auction for which there are no counterparts in a general equilibrium environment.

5. We treated the case of a single object to be sold. Our techniques can be extended to the problem of auctioning K identical objects when bidders' values exhibit "decreasing marginal utility," i.e., when $v_i(k+1, \theta, t_i) - v_i(k, \theta, t_i) \geq v_i(k+2, \theta, t_i) - v_i(k+1, \theta, t_i)$ where $v_i(k, \theta, t_i)$ is the payoff to bidder when the state is θ , his type is t_i and he is awarded k objects.

6. While many auction papers restrict attention to symmetric problems in which bidders' types are drawn from the same distribution, we do not make such assumptions. In our results, the distributional hypotheses relate an agent's informational size to the variability of his beliefs. Several papers analyzing interdependent value auction problems make assumptions regarding the impact of a bidder's information on his own value relative to other bidders' values (see, e.g., Maskin (1992), Dasgupta and Maskin (1998) and Perry and Reny (1998)). We make no such assumptions.

7. The general mechanism design approach that we use in this paper has been criticized on the grounds that revelation games are unrealistic for many problems. The examples used to illustrate mechanisms typically have simple information structures, as in our example in section 3, in which an agent's type is simply a pair of numbers - the quantity of oil and the cost of extracting it. In general, however, an agent's type encompasses all information he may have, including his beliefs about all relevant characteristics of the object, his beliefs about others' beliefs, etc. When types are realistically described, it seems unlikely that the revelation game could actually be played.

We are sympathetic to this argument, but we want to stress that the underlying logic by which efficient outcomes are obtained in our model does not depend on the particular revelation game we used; similar outcomes might be obtained through a non-revelation game. Consider first the following two-stage game. The second stage

is a standard Vickrey auction. In the first stage, agents forecast the highest bid in the second stage, excluding their own bid, and these forecasts are made common knowledge prior to the second stage. An agent is rewarded if the error is his forecast is smaller than some specified level.

Suppose that agents with favorable private information about the value of the object to others forecast high bids. When these forecasts are made public, each agent may be able to infer other agents' information from their forecasts. If they are able to do this, the asymmetry of information will have been eliminated, and the second stage Vickrey auction will assure an efficient outcome. Of course, agents might "manipulate" the system by making strategic rather than naive forecasts that will take into account the effects of their announcements in the second stage auction. However, the effect of strategic forecasting will be small if agents are informationally small. Hence, as in the case our mechanism, the reward for correct forecasting will dominate the potential benefits from strategic forecasting when bidders are informationally small.⁹

8. In this paper we investigated the general problem of the conflict between the extraction of information from agents and the use of that information to ensure efficient allocations. Pesendorfer and Swinkels (2000) analyze a model in which a number of objects are to be auctioned off to a number of bidders. They assume an informational structure that is similar to ours: each agent gets information about a personal taste parameter and a signal about a common value component. Pesendorfer and Swinkels study the problem when the number of agents increases and provide conditions under which the objects are allocated efficiently in the limit. It is easy to see that in their framework, each agent's informational size goes to zero as the number of agents goes to infinity.

9. Theorem 1 of Section 4 assumed that the random vectors $(\tilde{\theta}, \tilde{s})$ and \tilde{c} were stochastically independent. However, the conclusions of Theorem 1 will hold under a weaker condition that we call *Informational Independence*. Formally, a probability measure $P \in \Delta_{\Theta \times S \times C}^*$ satisfies *Informational Independence* if for each $(\theta, s, c) \in \Theta \times S \times C$, (i) $P_{\Theta}(\theta|s, c) = P_{\Theta}(\theta|s, c_{-i})$ and (ii) $P_{S_{-i}}(s_{-i}|s_i, c_i) = P_{S_{-i}}(s_{-i}|s_i)$. Obviously, informational independence is weaker than the stochastic independence assumption of section 4. Furthermore, it can be shown that, if $P \in \Delta_{\Theta \times S \times C}^*$ satisfies informational independence, then P admits an information decomposition $\mathcal{D} = \{g_i, R_i\}_{i \in N}$ where $R_i = S_i$ and g_i is the projection of T_i onto S_i . As a result, Theorem 1 will still hold under the assumption of Informational Independence.

10. It is straightforward to see that for many situations in which the number of agents becomes large, it is likely that each agent will become informationally small. Consequently, each agent's "informational rent" will become small. This does not

⁹See McLean and Postlewaite (2001) for an investigation of such a mechanism. Roust (2002) reports on experiments that are motivated by this mechanism.

imply, of course, that the agents' aggregate informational rents get small. McLean and Postlewaite (2003) investigate the conditions under which aggregate informational rents asymptotically go to zero as the economy is replicated.

7 Proofs:

7.1 Preparations for the Proof of Theorem 2:

In this section, we begin with two lemmas that are of some independent interest.

Lemma A.1: Let (v_1, \dots, v_n) be a collection of payoff functions and let $\{q_i^*, x_i^*\}_{i \in N}$ be the associated Vickrey auction mechanism. For every $i \in N$ and for each $t \in T$ and $t'_i \in T_i$,

$$(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i)) \geq -|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)|.$$

Proof: Choose $t \in T$ and $t'_i \in T_i$.

Case 1: Suppose that $\hat{v}_i(t_{-i}, t'_i) < w_i(t_{-i}, t'_i)$. Then

$$q_i^*(t_{-i}, t'_i) = x_i^*(t_{-i}, t'_i) = 0$$

so

$$\begin{aligned} & (q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i)) \\ &= q_i^*(t)\hat{v}_i(t) - x_i^*(t) \\ &\geq 0 \\ &\geq -|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)|. \end{aligned}$$

Case 2: Suppose that $\hat{v}_i(t_{-i}, t'_i) > w_i(t_{-i}, t'_i)$. Then

$$q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i) = \hat{v}_i(t) - w_i(t_{-i}, t'_i).$$

If $\hat{v}_i(t_{-i}, t_i) > w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = \hat{v}_i(t) - w_i(t_{-i}, t_i).$$

If $\hat{v}_i(t_{-i}, t_i) \leq w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = 0 \geq \hat{v}_i(t) - w_i(t_{-i}, t_i).$$

Therefore,

$$\begin{aligned} & (q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i)) \\ &\geq (\hat{v}_i(t) - w_i(t_{-i}, t_i)) - (\hat{v}_i(t) - w_i(t_{-i}, t'_i)) \\ &= w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i) \\ &\geq -|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)|. \end{aligned}$$

Case 3: Suppose that $\hat{v}_i(t_{-i}, t'_i) = w_i(t_{-i}, t'_i)$. Then

$$q_i^*(t_{-i}, t'_i) \hat{v}_i(t) - x_i^*(t_{-i}, t'_i) = \frac{1}{|I(t_{-i}, t'_i)|} (\hat{v}_i(t) - w_i(t_{-i}, t'_i)).$$

If $\hat{v}_i(t_{-i}, t_i) > w_i(t_{-i}, t_i)$, then

$$q_i^*(t) \hat{v}_i(t) - x_i^*(t) = \hat{v}_i(t) - w_i(t_{-i}, t_i) \geq \frac{1}{|I(t_{-i}, t'_i)|} (\hat{v}_i(t) - w_i(t_{-i}, t_i)).$$

If $\hat{v}_i(t_{-i}, t_i) \leq w_i(t_{-i}, t_i)$, then

$$q_i^*(t) \hat{v}_i(t) - x_i^*(t) = 0 \geq \frac{1}{|I(t_{-i}, t'_i)|} (\hat{v}_i(t) - w_i(t_{-i}, t_i)).$$

Therefore,

$$\begin{aligned} & (q_i^*(t) \hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i) \hat{v}_i(t) - x_i^*(t_{-i}, t'_i)) \\ & \geq \frac{1}{|I(t_{-i}, t'_i)|} (\hat{v}_i(t) - w_i(t_{-i}, t_i)) - \frac{1}{|I(t_{-i}, t'_i)|} (\hat{v}_i(t) - w_i(t_{-i}, t'_i)) \\ & = \frac{1}{|I(t_{-i}, t'_i)|} (w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)) \\ & \geq -\frac{1}{|I(t_{-i}, t'_i)|} |w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)| \\ & \geq -|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)|. \end{aligned}$$

This completes the proof of Lemma 1.

If each $\hat{v}_i(t)$ is a function of t_i only, then $|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)| = 0$ and Lemma A.1 yields the familiar result for Vickrey auctions with pure private values: it is a dominant strategy to truthfully report one's type.

Lemma A.2: Let (v_1, \dots, v_n) be a collection of payoff functions and let $\{q_i^*, x_i^*\}_{i \in N}$ be the associated Vickrey auction mechanism. Let

$$M = \max_{\theta} \max_i \max_{t_i} v_i(\theta, t_i)$$

and let $P \in \Delta_{\Theta \times T}^*$. For every $i \in N$ and for each $t_{-i} \in T_{-i}$, $t_i \in T_i$ and $t'_i \in T_i$,

$$|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)| \leq M \|P_{\Theta}(\cdot | t_{-i}, t_i) - P_{\Theta}(\cdot | t_{-i}, t'_i)\|.$$

Proof: Choose $t_{-i}, t_i, t'_i, j \neq i$ and $j' \neq i$ so that

$$w_i(t_{-i}, t_i) = \max_{k \neq i} \sum_{\theta \in \Theta} [v_k(\theta, t_k) P_{\Theta}(\theta | t_{-i}, t_i)] = \sum_{\theta \in \Theta} [v_j(\theta, t_j) P_{\Theta}(\theta | t_{-i}, t_i)]$$

and

$$w_i(t_{-i}, t'_i) = \max_{k \neq i} \sum_{\theta \in \Theta} [v_k(\theta, t_k) P_{\Theta}(\theta | t_{-i}, t'_i)] = \sum_{\theta \in \Theta} [v_{j'}(\theta, t_{j'}) P_{\Theta}(\theta | t_{-i}, t'_i)].$$

Note that t_j and $t_{j'}$ are, respectively, the j and j' components of the vector t_{-i} . From the definitions of t_j and $t_{j'}$, it follows that

$$\sum_{\theta \in \Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_{\Theta}(\theta | t_{-i}, t_i) \geq 0$$

and

$$\sum_{\theta \in \Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_{\Theta}(\theta | t_{-i}, t'_i) \leq 0.$$

Therefore,

$$\begin{aligned} & \sum_{\theta \in \Theta} v_{j'}(\theta, t_{j'}) [P_{\Theta}(\theta | t_{-i}, t_i) - P_{\Theta}(\theta | t_{-i}, t'_i)] \\ & \leq \sum_{\theta \in \Theta} v_{j'}(\theta, t_{j'}) [P_{\Theta}(\theta | t_{-i}, t_i) - P_{\Theta}(\theta | t_{-i}, t'_i)] + \sum_{\theta \in \Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_{\Theta}(\theta | t_{-i}, t_i) \\ & = w_i(t_{-i}, t_i) - w_i(t_{-i}, t'_i) \\ & = \sum_{\theta \in \Theta} v_j(\theta, t_j) [P_{\Theta}(\theta | t_{-i}, t_i) - P_{\Theta}(\theta | t_{-i}, t'_i)] + \sum_{\theta \in \Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_{\Theta}(\theta | t_{-i}, t'_i) \\ & \leq \sum_{\theta \in \Theta} v_j(\theta, t_j) [P_{\Theta}(\theta | t_{-i}, t_i) - P_{\Theta}(\theta | t_{-i}, t'_i)] \end{aligned}$$

and we conclude that

$$|w_i(t_{-i}, t_i) - w_i(t_{-i}, t'_i)| \leq M \|P_{\Theta}(\cdot | t_{-i}, t_i) - P_{\Theta}(\cdot | t_{-i}, t'_i)\|.$$

This completes the proof of Lemma 2.

We prove one final technical result.

Lemma A.3: Let X be a finite set with cardinality $k \geq 2$ and let $p, q \in \Delta_X$. Then

$$\left[\frac{p}{\|p\|_2} - \frac{q}{\|q\|_2} \right] \cdot p \geq \frac{k^{-\frac{5}{2}}}{4(k-1)} [\|p - q\|_1]^2$$

where $\|\cdot\|_2$ denotes the 2-norm and $\|\cdot\|$ denotes the 1-norm.

Proof: Let e denote the k -vector of ones and define $f_i(z) = z_i / e \cdot z$. If $z \neq 0$, then $\partial_j f(z) = -1 / (e \cdot z)^2$ if $j \neq i$ and $\partial_i f(z) = (e \cdot z - z_i) / (e \cdot z)^2$. Therefore,

$$\|\nabla f_i(z)\|_2 = \sqrt{\sum_{j=1}^k \partial_j f(z)^2} = \frac{1}{(e \cdot z)^2} \sqrt{k - 1 + (e \cdot z - z_i)^2}.$$

Let $C := \{z \in \mathfrak{R}_+^k | e \cdot z \geq 1, \|z\|_2 = 1\}$. Note that C is convex and that C is contained in an open set that does not include the origin. Choose $x \in \mathfrak{R}_+^k$ and $y \in \mathfrak{R}_+^k$ with $\|x\|_2 = \|y\|_2 = 1$. Applying the Mean Value Theorem, we conclude that there exists a $c^i \in C$ with $c^i \neq 0$ such that

$$f_i(x) - f_i(y) = \nabla f_i(c^i) \cdot (x - y).$$

Since $c^i \in C$, it follows that $1 \leq (e \cdot c^i)^2 \leq k$ and $(e \cdot c^i - c_i^i)^2 \leq k - 1$. Hence, we conclude that $\|\nabla f_i(c^i)\|_2 \leq \sqrt{2(k-1)}$. Combining these observations, it follows that

$$\left| \frac{x_i}{e \cdot x} - \frac{y_i}{e \cdot y} \right| \leq \|\nabla f_i(c^i)\|_2 \|x - y\|_2 \leq \sqrt{2(k-1)} \|x - y\|_2.$$

If $p, q \in \Delta_X$, then we can set $x = p/\|p\|$ and $y = q/\|q\|$ and conclude that

$$\|p - q\|_1 = \sum_{i=1}^k |p_i - q_i| \leq k \sqrt{2(k-1)} \left\| \frac{p}{\|p\|_2} - \frac{q}{\|q\|_2} \right\|_2.$$

To complete the proof, it is easy to verify the identity

$$\left\| \frac{p}{\|p\|_2} - \frac{q}{\|q\|_2} \right\|_2^2 = \frac{2}{\|p\|_2} \left[\frac{p}{\|p\|_2} - \frac{q}{\|q\|_2} \right] \cdot p.$$

Since $\|p\|_2 \geq 1/\sqrt{k}$, we conclude that

$$\|p - q\|_1^2 \leq [4k^{5/2}(k-1)] \left[\frac{p}{\|p\|_2} - \frac{q}{\|q\|_2} \right] \cdot p.$$

7.2 Proof of Theorem 2:

We prove part (ii) first. Choose $\varepsilon > 0$. Let

$$M = \max_{\theta} \max_i \max_{t_i} v_i(\theta, t_i),$$

let $|T|$ denote the cardinality of T and define

$$K = \frac{|T|^{-\frac{5}{2}}}{4(|T| - 1)}$$

Choose δ so that

$$0 < \delta < \frac{\varepsilon K}{3M}.$$

Suppose that $P \in \Delta_{\Theta \times T}^*$ has an information decomposition satisfying

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^{P, \mathcal{D}}.$$

Define $\hat{\nu}^P = \max_i \nu_i^P$ and $\Lambda^{P,\mathcal{D}} = \min_i \Lambda_i^{P,\mathcal{D}}$. Therefore $\hat{\nu}^P \leq \delta \Lambda^{P,\mathcal{D}}$.

Next, define

$$\zeta_i(r_{-i}, r_i) = \frac{P_{R_{-i}}^{\mathcal{D}}(r_{-i}|r_i)}{\|P_{R_{-i}}^{\mathcal{D}}(\cdot|r_i)\|_2}$$

for each $(r_1, \dots, r_n) \in R_1 \times \dots \times R_n$ and note that

$$0 \leq \zeta_i(r_{-i}, r_i) \leq 1$$

for all i , r_{-i} and r_i . Now we define an augmented Vickrey auction mechanism. For each $t \in T$, let

$$z_i(t) := \varepsilon \zeta_i(g_1(t_1), \dots, g_n(t_n)).$$

The mechanism $\{q_i^*, x_i^* - z_i\}_{i \in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$q_i^*(t) \hat{v}_i(t) - x_i^*(t) \geq 0$$

and

$$z_i(t_{-i}, t_i) \geq 0.$$

To prove incentive compatibility, we consider two cases. First suppose that $g_i(t_i) = g_i(t'_i)$. From part (i) of the definition of information decomposition, it follows that $|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)| = 0$ for all $t_{-i} \in T_{-i}$ and incentive compatibility is a consequence of Lemma A.1.

Now suppose that $g_i(t_i) = r_i$ and $g_i(t'_i) = r'_i$ with $r_i \neq r'_i$. The proof of incentive compatibility will follow from the next two claims.

Claim 1:

$$\sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i)) P(t_{-i}|t_i) \geq \varepsilon K \Lambda^{P,\mathcal{D}}.$$

Proof of Claim 1: Part (ii) of the definition of information decomposition implies that

$$\sum_{\substack{t_{-i} \in T_{-i} \\ :g_{-i}(t_{-i})=r_{-i}}} P(t_{-i}|\hat{t}_i) = P_{R_{-i}}^{\mathcal{D}}(r_{-i}|\hat{r}_i)$$

whenever $g_i(\hat{t}_i) = \hat{r}_i$. Therefore,

$$\begin{aligned} & \sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i)) P(t_{-i}|t_i) \\ &= \sum_{t_{-i}} (\zeta_i(g_{-i}(t_{-i}), g_i(t_i)) - \zeta_i(g_{-i}(t_{-i}), g_i(t'_i))) P(t_{-i}|t_i) \end{aligned}$$

$$\begin{aligned}
&= \varepsilon \sum_{r_{-i}} [\zeta_i(r_{-i}, r_i) - \zeta_i(r_{-i}, r'_i)] \left[\sum_{\substack{t_{-i} \in T_{-i} \\ :g_{-i}(t_{-i})=r_{-i}}} P(t_{-i}|t_i) \right] \\
&= \varepsilon \sum_{A_{-i}} [\zeta_i(r_{-i}, r_i) - \zeta_i(r_{-i}, r'_i)] P_{R_{-i}}^{\mathcal{D}}(r_{-i}|r_i) \\
&= \varepsilon \sum_{A_{-i}} \left[\frac{P_{R_{-i}}^{\mathcal{D}}(r_{-i}|r_i)}{\|P_{R_{-i}}^{\mathcal{D}}(\cdot|r_i)\|_2} - \frac{P_{R_{-i}}^{\mathcal{D}}(r_{-i}|r'_i)}{\|P_{R_{-i}}^{\mathcal{D}}(\cdot|r'_i)\|_2} \right] P_{R_{-i}}^{\mathcal{D}}(r_{-i}|r_i) \\
&\geq \varepsilon K \left[\|P_{R_{-i}}^{\mathcal{D}}(\cdot|r_i) - P_{R_{-i}}^{\mathcal{D}}(\cdot|r'_i)\| \right]^2 \\
&\geq \varepsilon K \Lambda_i^{P, \mathcal{D}}
\end{aligned}$$

where the last inequality is an application of Lemma A.3.

Claim 2:

$$\sum_{t_{-i}} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \geq -3M\hat{\nu}^P$$

Proof of Claim 2: Define

$$S_i(t'_i, t_i) = \{t_{-i} \in T_{-i} \mid \|P_{\Theta}(\cdot|t_{-i}, t_i) - P_{\Theta}(\cdot|t_{-i}, t'_i)\| > \hat{\nu}^P\}.$$

Since $\nu_i^P \leq \hat{\nu}^P$, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in S_i(t'_i, t_i) \mid \tilde{t}_i = t_i\} \leq \nu_i^P \leq \hat{\nu}^P.$$

If $t_{-i} \notin S_i(t'_i, t_i)$, then Lemmas A.1 and A.2 imply that

$$\sum_{t_{-i} \notin S_i(t'_i, t_i)} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \geq -M\hat{\nu}^P.$$

Finally, note that

$$|q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i)| \leq M$$

for all i, t_i, t'_i and t_{-i} .

Combining these observations, we conclude that

$$\begin{aligned}
&\sum_{t_{-i}} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \\
&= \sum_{t_{-i} \in S_i(t'_i, t_i)} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \\
&\quad + \sum_{t_{-i} \notin S_i(t'_i, t_i)} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \\
&\geq -M\hat{\nu}^P - 2M\hat{\nu}^P \\
&= -3M\hat{\nu}^P
\end{aligned}$$

and the proof of claim 2 is complete.

Applying Claims 1 and 2, it follows that

$$\begin{aligned}
& \sum_{t_{-i}} [(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t'_i)\hat{v}_i(t) - x_i^*(t_{-i}, t'_i))] P(t_{-i}|t_i) \\
& \quad + \sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i)) P(t_{-i}|t_i) \\
& \geq \varepsilon K \Lambda^{P, \mathcal{D}} - 3M\hat{\nu}^P \\
& \geq 0.
\end{aligned}$$

and the proof of part (ii) is complete.

Part (i) follows from the computations in part (ii). We have shown that, for any information decomposition \mathcal{D} of P and for any positive number α , there exists an augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ satisfying

$$\sum_{t_{-i}} [(q_i(t)\hat{v}_i(t) - x_i(t)) - (q_i(t_{-i}, t'_i)\hat{v}_i(t) - x_i(t_{-i}, t'_i))] P(t_{-i}|t_i) \geq \alpha K \Lambda_i^{P, \mathcal{D}} - 3M\hat{\nu}^P$$

for each i and each t_i, t'_i . If $\Lambda_i^{P, \mathcal{D}} > 0$ for each i , then α can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

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