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" Local Sunspot Equilibria Reconsidered"

by

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LOCAL SUNSPOT EQUILIBRIA RECONSIDERED

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Abstract. This paper re-examines the conditions for the existence of local stationary sunspot equilibria (SSE) in the standard OLG model from a broader perspective than before. We say that local SSE exist around a steady state of a given OLG economy if, in any arbitrarily small neighborhood of the steady state, we can find a SSE of a "nearby" economy. We show that when the domain where "nearby" economies may lie is defined by agents' endowments and probabilities, the indeterminacy of the steady state remains both necessary and sufficient for the existence of local SSE. On the other hand, when the domain of economies is defined by by agents' preferences and probabilities, local SSE may exist even around determinate steady states.

We also show that if a slightly weaker notion of distance is used to identify "nearby" economies, SSE in the vicinity of a steady state equilibrium generically exist.

1. Introduction

It is by now a well established fact that the volatility in prices and allocation of resources in a market economy can be generated by the agents' self-fulfilling price

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expectations even when the fundamentals of the economy do not fluctuate. This is formalized by the notion of sunspot equilibrium.[1]

The existence of sunspot equilibria has been shown in various frameworks, both in economies with finite horizon[2] and with infinite horizon. In this paper we analyze sunspot equilibria in overlapping generations (OLG) economies.[3] In particular, we are primarily concerned with sunspot equilibria that are both *stationary*, so that the fluctuations induced by agents' beliefs are persistent, and *local,* i.e. such that fluctuations occur arbitrarily close to a deterministic steady state. These sunspot equilibria are especially interesting for two reasons: first, their stationarity makes them likelier to be the outcome of a learning process and hence gives a well founded rationale for the use of the rational expectations hypothesis; secondly, their local character addresses well the typical pattern exhibited by many economic time series that depart recurrently by only slightly from their trends. This explains why local stationary sunspot equilibria (SSE) have received considerable attention in the literature. Another important reason is that the conditions for the existence of local SSE are easier to characterize and can be stated in terms of properties of the underlying certainty economy, the economy where sunspot uncertainty is ignored by agents.

In particular, there is a close connection between the existence of local sunspot equilibria and the *indeterminacy* of the steady state equilibrium (that is, any perfect foresight equilibrium path locally converges to it, and thus there is a continuum of equilibrium paths around the steady state). The sufficiency of this condition for the existence of local 2-SSE (i.e., finite support with two sunspot states each period), under some regularity condition which excludes knife-edge cases a priori, was shown by Azariadis and Guesnerie (1986) for one-commodity overlapping generations economies and by Guesnerie (1986) for n-dimensional one-step forward-looking dynamical systems, where the necessity of the condition was also shown. Guesnerie's result was then generalized to k-SSE by Chiappori, Geoffard and Guesnerie (1992) within the same setup of Guesnerie (1986). A more general form of the result was proved by Woodford (1986), thus validating the conjecture in Woodford (1984); within a more general framework of dynamic economies, Woodford showed, under a similar regularity condition, that the indeterminacy of the steady state is

[1]For the seminal papers on the concept of sunspot equilibrium see Shell (1977) and Cass and Shell (1983).

[2]Cass (1992), Gottardi and Kajii (1999), among others.

[3]For the first studies of sunspot equilibria in the framework of overlapping generations economies see Shell (1977), Azariadis (1981) and Azariadis and Guesnerie (1986). From here an extensive literature on sunspot equilibria in overlapping generations economies and dynamical economies was developed in the 80's and 90's; see Chiappori and Guesnerie (1991) and Guesnerie and Woodford (1996) for surveys of this literature.

both necessary and sufficient for the existence of local SSE (not restricted to have a finite support).[4] But, to the best of our knowledge, the possibility of exhibiting local sunspot equilibria around a *determinate* steady state was excluded by all the previous results in the literature. If there is no local sunspot equilibria around a determinate steady state, an analyst can be assured again that sunspots do not matter - in the vicinity of the monetary steady state - if s/he studies the Pareto efficient steady state with a "normal" offer curve, for instance.

In this paper, we reconsider this possibility again, namely the existence of local stationary sunspot equilibria in the neighborhood of a determinate steady state, and we do it from a broader perspective than before. More precisely, local SSE exist around a steady state of a given economy if, in any arbitrarily small neighborhood of the steady state we can find a SSE of an economy "close" to the original economy.

In this set-up, an important role is played by the domain of economies where we can look for 'nearby' economies exhibiting sunspot equilibria in the vicinity of the steady state. In the literature on local sunspot equilibria referred to above, the domain of economies considered was defined by the probabilities describing the beliefs of the agents over the sunspot states. In other words, economies are close if the fundamentals, i.e. the agents' preferences and endowments, are the same and they differ by agents' beliefs only. We explore here the conditions for the existence of local SSE within a larger domain of economies, where we allow for "small" variations both in the fundamentals of the economy and agents' beliefs. We believe that such a more liberal interpretation of closeness conforms with the spirit of stationary local sunspot equilibria explained above.

To avoid the emergence of any other possible 'pathology', the analysis is carried out in the framework of the standard OLG model, with one commodity, one agent per generation and fiat money as the only asset. We examine first the case where the domain of economies is defined by agents' endowments as well as probabilities. Thus economies are considered to be close if the agents' endowments and probabilities over sunspot states are close, while preferences are the same. We show that in this case the indeterminacy of the steady state remains a necessary and sufficient condition for the existence of local SSE in its neighborhood; thus the aforementioned results extend to this more general framework.

On the other hand, when the domain of economies is defined by agents' prefer-

[4] On the other hand, in this more general set-up the indeterminacy condition is not always sufficient if the SSE considered are still restricted to have a finite support, as pointed out in Dávila (1997). In effect, Dávila (1997) showed that, in the presence of predetermined variables the indeterminacy of the steady state, while continuing to be a necessary condition for the existence of local sunspot equilibria, is no longer sufficient for the existence of local SSE with finite support whenever the dynamics is linear.

ences and probabilities, we show that local SSE may exist even around determinate steady states. To be more precise, we can construct a convergent sequence of economies, where there is a determinate steady state and a sunspot equilibrium around it for every economy. In fact, the steady state consumption is constant along the sequence. The sunspot equilibrium consumption allocation converges to the (deterministic) steady state of the limit economy. It turns out that whether or not the steady state is also determinate in the limit economy depends on the topology for which the convergence of utility functions takes place. The precise formulation of the existence result therefore depends on the strength of the notion of convergence (i.e. of distance) which is used in the space of utility functions describing preferences. In any case, our results show that local SSE prove to be quite pervasive, as it is hard to identify robust conditions on the fundamentals of the economies which can rule out their existence.

We also consider a slightly weaker notion of closeness, whereby an economy is viewed as “close” to another if it has the same preferences, probabilities and aggregate endowment, though its distribution among different generations (and/or heterogeneous households) may be different; the aggregate endowment is observable, but not its distribution. We show that in this case, even when the domain of economies considered is defined by endowments and probabilities, SSE ‘near’ a steady state equilibrium exist generically: for almost any choice of the agents’ beliefs, preferences and aggregate endowments, there is a way to redistribute the aggregate endowments so that the resulting economy has a SSE around the steady state equilibrium of the given economy. Our analysis provides a general and robust method for constructing SSE in the vicinity of any stationary equilibrium.

The results shown in this paper contribute to clarify the precise nature of the link between local SSE around a steady state and the indeterminacy of the latter. In this sense, they help completing the understanding of this links we already had from Guesnerie (1986) and Woodford (1986); at the same time they show that some qualifications to Woodford’s conjecture is needed.

Our analysis also establishes some results which are of somewhat independent interest: first, the singularity of non-sunspot stationary equilibria is shown to be necessary for the existence of local SSE around a steady state, even for the larger domain of economies we consider, and the conditions for the singularity of such equilibria are determined (Lemmas 1 and 2); secondly, any increasing function defined in a neighborhood of a point can be rationalized as the offer curve of a consumer with additively separable preferences (Lemma 4); and finally, we provide a method of constructing SSE around a steady state when the offer curve has a certain property (the last part of the proof of Proposition 4).

The rest of the paper is organized as follows. Section 2 presents the model. Local stationary sunspot equilibria are then defined in section 3. The conditions for the existence of local SSE are then characterized in Section 4 for the case in which the domain of economies is defined by endowments as well as probabilities and in Section 5 for the case in which economies are parameterized by preferences and probabilities.

2. The Model

We consider a class of simple, stationary overlapping generations economies. There is a single perishable commodity. Each period $t = 1, ...$ a new generation is born, composed of one (type of) agent, living two consecutive periods. Each generation is identical, and is characterized as follows, unless specified otherwise:

(1) a utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$, exhibiting standard properties: u is differentiable on R^2_{++}, monotone (i.e. $Du(c)$ is in R^2_{++} for every c in R^2_{++}), concave, differentiably strictly quasi concave[5] and well-behaved at the boundary (i.e. such that strictly positive prices imply interior solutions, which is guaranteed for instance, although not exclusively, by indifference curves included in R^2_{++}).

(2) an endowment vector $e \in R^2_{++}$.

Fiat money is the only asset in the economy. Each period the consumption good can be exchanged with money at the price p. Even though the 'fundamentals' of the economy (e, u) are deterministic, consumers may still face some uncertainty about the price level that will prevail in the market next period. Uncertainty is thus purely extrinsic, i.e., it is generated by sunspots.

Since our interest is to investigate the conditions for the existence of sunspot equilibria, it suffices to consider the case in which there are two possible realizations of the uncertainty each period. The choice problem of an agent who faces a price p_1 when young and a probability distribution of prices $\{p_2^1, p_2^2; \pi^1, \pi^2\}$ when old,[6] is

$$\begin{gathered} \max \sum_{j=1}^{2} \pi^j u(c_1, c_2^j) \\ p_1(c_1 - e_1) + p_2^j(c_2^j - e_2) = 0, \; j = 1, 2, \end{gathered} \tag{2.1}$$

where c_1 is the consumption when young and $\left(c_2^1, c_2^2\right)$ the plan of contingent consumptions when old. The expression of the budget constraints has been simplified

[5] That is to say, $D^2u(c)$ is negative definite in the orthogonal space of $Du(c)$, for every c in R^2_{++}.

[6] We denote by π^j the probability that the price when old will be p_2^j.

by substituting out the agent's money holdings. Under the above assumptions, the vector $\left(c_1; c_2^1, c_2^2\right)$ constitutes a solution to problem (2.1) if and only if it satisfies, together with a pair of positive numbers λ^1, λ^2, the following equations:

$$\begin{aligned} \sum_{j=1}^{2} \pi^j D_1 u(c_1, c_2^j) - \sum_{j=1}^{2} \lambda^j p_1 = 0, \\ \pi^j D_2 u(c_1, c_2^j) - \lambda^j p_2 = 0, \text{ for } j = 1, 2, \\ p_1(c_1 - e_1) + p_2^j(c_2^j - e_2) = 0, \text{ for } j = 1, 2. \end{aligned} \tag{2.2}$$

In line with the stationarity of the economy, we will consider the case where uncertainty is described by a first order Markov chain with two states, with transition matrix $\Pi = \left\{\pi^{i,j}\right\}_{i,j=1,2}$. A *sunspot economy* is then identified by the parameters describing its 'fundamentals' as well as the structure of the (sunspot) uncertainty, $\{(e,u), \Pi\}$. Note that if the distinction among sunspot states is ignored (i.e., agents have deterministic expectations over the level of future prices), we have a deterministic economy (e, u). We refer to this as the *certainty economy* associated to $\{(e,u), \Pi\}$.

3. Local Sunspot Equilibria

In this paper we will limit our attention to the stationary equilibria of the model described above.

Definition 1. *A stationary sunspot equilibrium (SSE)*[7] *for the economy* $\{(e,u), \Pi\}$ *is given by a collection of consumption levels in each possible state when young and old* $(c_1^i)_{i=1}^2$, $(c_2^i)_{i=1}^2$, *and prices* $\left(p^1, p^2\right)$, *such that* $c_1^1 \neq c_1^2$ *and*

(1) *for all* $i = 1, 2$, *the consumption when young and the plan of sunspot contingent consumptions when old* $(c_1^i; c_2^1, c_2^2)$ *constitutes the solution to the problem of the agent* (2.1) *when facing* $(p^i, \{p^1, p^2; \pi^{i1}, \pi^{i2}\})$,

(2) *the allocation of resources is feasible:*

$$c_1^j - e_1 + c_2^j - e_2 = 0, \qquad \textit{for all } j = 1, 2.$$

Therefore, in a SSE both the distribution of the global resources between contemporary young and old, and the prices fluctuate randomly following a 2-state Markov chain.

[7]Following the terminology of Guesnerie (1986) and Chiappori-Guesnerie (1989), we can also refer to such equilibrium as a 2-SSE.

A *non-sunspot stationary equilibrium* for the economy $\{(e,u),\Pi\}$ is then given by a set of consumption levels and prices, $(\bar{c}_1,\bar{c}_2)$, $\bar{p}$, such that the values $c_1^i = \bar{c}_1, c_2^i = \bar{c}_2, p^i = \bar{p}$, $i = 1,2$, constant across the 2 states, satisfy the same individual optimality and feasibility conditions of Definition 1.

We can similarly define stationary equilibria of a certainty economy (e,u). A *steady state equilibrium* of (e,u) is a specification of the agents' consumption when young and when old, together with a constant price level $(\bar{c}_1,\bar{c}_2,\bar{p})$ such that:

(1) $(\bar{c}_1,\bar{c}_2) \in \arg\max u(c_1,c_2)$ s.t. $\bar{p}(c_1 - e_1) + \bar{p}(c_2 - e_2) = 0$;

(2) $\bar{c}_1 - e_1 + \bar{c}_2 - e_2 = 0$.

It is immediate to see that any set of values $(\bar{c}_1,\bar{c}_2,\bar{p})$ defining a steady state equilibrium of the certainty economy (e,u) also constitutes a non-sunspot equilibrium $\{(\bar{c}_1,\bar{c}_2),\bar{p}\}$ of the sunspot economy $\{(e,u),\Pi\}$. Since steady state equilibria always exist, under our assumptions, so do non-sunspot stationary equilibria.

An important role in the analysis of the conditions for the existence of stationary sunspot equilibria of $\{(e,u),\Pi\}$ is played by the stability properties of the steady states of the associated certainty economy (e,u). Consider the equation

$$\Gamma(c_1,c_2) \equiv D_1u(c_1,c_2)(c_1 - e_1) + D_2u(c_1,c_2)(c_2 - e_2) = 0, \tag{3.1}$$

defining the representative agent's offer curve. Differentiating Γ with respect to c_i, $i = 1,2$, we obtain

$$D_i\Gamma(c_1,c_2) = D_iu(c_1,c_2)+D_{ii}u(c_1,c_2)(c_i-e_i)+D_{ji}u(c_1,c_2)(c_j-e_j),\ \ i \neq j \in \{1,2\}\,. \tag{3.2}$$

The vector $(D_1\Gamma(\bar{c}_1,\bar{c}_2), D_2\Gamma(\bar{c}_1,\bar{c}_2))$ is then the gradient, at the steady state $(\bar{c}_1,\bar{c}_2)$, of the representative agent's offer curve. It is said that the steady state $(\bar{c}_1,\bar{c}_2,\bar{p})$ is *indeterminate* whenever there is a neighborhood of the steady state where the perfect foresight forward dynamics converges to it, otherwise it is said to be *determinate*. In our framework the indeterminacy of the steady state requires the slope of the offer curve at it to be not bigger than 1 in absolute value. On the other hand, a sufficient condition for the indeterminacy of the steady state is that this same slope be strictly smaller than 1 in absolute value, i.e.

$$\left|\frac{dc_2}{dc_1}\right| = \left|\frac{D_1\Gamma(\bar{c}_1,\bar{c}_2)}{D_2\Gamma(\bar{c}_1,\bar{c}_2)}\right| < 1. \tag{3.3}$$

So the case where slope is equal to one constitutes a knife edge case in our analysis, which does not have any clear implication for determinacy. To avoid this unclear case, we will follow Woodford (1986) (as well as Guesnerie (1986), Chiappori, Geoffard, Guesnerie (1992)) in imposing the following condition in various parts of the analysis:

Condition R. *At any steady state,* $\left|\frac{dc_2}{dc_1}\right| \neq 1$.

It can be formally shown that the condition R is generically satisfied.[8]

From the characterization of the solutions of the individual optimization problem in (2.2), it follows that stationary equilibria of the sunspot economy can be obtained as solutions of the following system of equations:

$$\begin{aligned} \sum_{j=1}^{2} \pi^{ij} D_1 u(c_1^i, c_2^j) - \sum_{j=1}^{2} \lambda^{ij} p^i &= 0, \text{ for all } i, \\ \pi^{ij} D_2 u(c_1^i, c_2^j) - \lambda^{ij} p^j &= 0, \text{ for all } i, j, \\ p^i (c_1^i - e_1) + p^j (c_2^j - e_2) &= 0, \text{ for all } i, j, \\ (c_1^j - e_1 + c_2^j - e_2) &= 0, \text{ for all } j, \end{aligned} \tag{3.4}$$

for some positive vectors $\left(\lambda^{i1}, \lambda^{i2}\right)$, $i = 1, 2$. Use the second set of equations to substitute for λ^{ij} in the first set of equations. Notice, furthermore that, as long as $c_1^i \neq e_1$, we can substitute the budget constraints for $\left(p^1, p^2\right)$ and plug the solution in the remaining equations of (3.4). Then, we obtain the following, simpler system of equations characterizing a stationary equilibrium:

$$\begin{aligned} &\pi^{11}(D_1 u(c_1^1, c_2^1)(c_1^1 - e_1) + D_2 u(c_1^1, c_2^1)(c_2^1 - e_2)) + \\ &\quad \pi^{12}(D_1 u(c_1^1, c_2^2)(c_1^1 - e_1) + D_2 u(c_1^1, c_2^2)(c_2^2 - e_2)) = 0, \\ &\pi^{21}(D_1 u(c_1^2, c_2^1)(c_1^2 - e_1) + D_2 u(c_1^2, c_2^1)(c_2^1 - e_2)) + \\ &\quad \pi^{22}(D_1 u(c_1^2, c_2^2)(c_1^2 - e_1) + D_2 u(c_1^2, c_2^2)(c_2^2 - e_2)) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} c_1^1 + c_2^1 - e_1 - e_2 &= 0, \\ c_1^2 + c_2^2 - e_1 - e_2 &= 0. \end{aligned} \tag{3.6}$$

Conversely, if the system above has a solution c with $c_t^i \neq e_t$, for $i = 1, 2$ and $t = 1, 2$, (that is, c is not autarky) then there are p and λ such that c, p, and λ satisfy equations (3.4). To see this, pick any prices with $p^i/p^j = -\left(c_2^j - e_2\right) / \left(c_1^i - e_2\right)$. Then $p^i/p^j = -\left(c_1^j - e_1\right) / \left(c_2^i - e_2\right)$ also holds by the resource feasibility constraints (3.6), thus the budget constraint in (3.4) holds. So set $p^2 = 1$, and $p^1 = -\left(c_2^j - e_2\right) / \left(c_1^i - e_2\right)$, and set λ^{ij} so that the second equation in (3.4) holds. Then (3.5) implies the first equation of (3.4) holds as well.

[8] See Kehoe and Levine (1984).

Denote the terms on the left hand side of the equations (3.5), (3.6) as $\Phi(c; e, u, \Pi)$, where $c \equiv (c_1^1, c_1^2, c_2^1, c_2^2)$. The argument above has shown that a non-autarky consumption allocation c obtains at a stationary equilibrium of the economy (e, u, Π) if and only if $\Phi(c; e, u, \Pi) = 0$. The system of equations $\Phi(c; e, u, \Pi) = 0$ has the same number of variables c and equations, which motivates the following definition.

Definition 2. *Let c be a non-autarkic equilibrium consumption allocation of the economy (e, u, Π). An equilibrium c is regular if $D_c\Phi$ is invertible when evaluated at $(c; e, u, \Pi)$. On the other hand, when $D_c\Phi$ is not invertible, the equilibrium c is said to be singular.*[9]

Of all the stationary sunspot equilibria we will be primarily interested in local sunspot equilibria, i.e. in those that lie 'close' to a non-sunspot stationary equilibrium (and hence also, given the equivalence established above, to a steady state equilibrium of the associated certainty economy). More precisely, we will say that an overlapping generations economy has local sunspot equilibria whenever, in any arbitrarily small neighborhood of a non sunspot stationary equilibrium of the economy, there exist SSE for some nearby economy. Note that by definition the notion of local sunspot equilibria of a given economy depends on the way an economy is regarded to be close to the original one, and so does the existence of them.

If one takes the view of the literature on local sunspot equilibria, according to which two economies are considered to be close only if they share exactly the same fundamentals, and can differ only in the signals triggering the expectations-driven fluctuations, then there is a close link between the indeterminacy of the steady state and the existence of local sunspot equilibria around it. In our setup, it has been shown that, under condition R, there exist local sunspot equilibria around a steady state if, and only if, that steady state is indeterminate. As a result, the natural interest in local sunspot equilibria as a depiction of the fluctuations of the economic activity became twofold. Firstly, from a positive viewpoint, they addressed the typical pattern of many economic time series exhibiting fluctuations that depart only slightly from a steady state. Secondly, for practical purposes, their intimate connection with the asymptotic behavior of the perfect foresight dynamics around the steady state provided an immediate means to detect their existence.

Nonetheless, we contend that it is unduly restrictive to keep the focus of the analysis only on the differences in the signals as has been done in the literature. So we propose a broader concept of local sunspot equilibria. A formal definition requires the consideration of a sequence of economies, converging to some limit economy, and a sequence of associated SSE converging to a non-sunspot steady

[9]See also Kehoe - Levine (1984) .

state of the limit economy:

Definition 3. *Local stationary sunspot equilibria exist around a non-sunspot stationary equilibrium* $(\bar{c}_1, \bar{c}_2)$*,* $\bar{p}$ *of the economy* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$*, provided there is a sequence of economies* $\{(e^n, u^n), \Pi^n\}_n$*, and an associated sequence of stationary sunspot equilibria* $\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\}_n$*, which converge, respectively to* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ *and to* $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ *as* $n \to \infty$*.*

The above definition leaves various possibilities open both with regard to the space where 'nearby' economies may lie (or equivalently the parameters of the economies allowed to vary along the sequence) and the notion of 'closeness' which is used. The latter issue is particularly relevant when agents' preferences also vary along the sequence of economies considered, in which case different notions of convergence for a sequence of functions (describing the agents' preferences) can be utilized.

If we restrict our attention to the domain of economies defined by the set of Markov matrices Π (i.e.,when 'nearby' economies can only differ by probabilities, while only Π^n can vary with n), an immediate application of Guesnerie (1986) (also Woodford (1986b)) yields the following result, which establishes a clear relationship between existence of local SSE and stability properties of the steady state of the limit economy:

Proposition 1. *Let* $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ *be a steady state of* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ *and suppose condition R holds for* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$*. Then, there is a sequence of economies* $\{(e, u), \Pi^n\}_n$*, where only* Π^n *varies with* n*, and associated stationary sunspot equilibria* $\left\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\right\}$*, which converge, respectively to* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ *and to* $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ *as* $n \to \infty$*, if and only if* $(\bar{c}_1; \bar{c}_2)$ *is an indeterminate steady state of* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$*.*

Such result provides the strongest basis to the claim that the indeterminacy of the steady state is needed for the existence of local sunspot equilibria. In what follows, we will re-examine the validity of this claim when a larger domain of economies is considered. We will show that such relationship is no longer valid in this set-up. In particular when 'nearby' economies can also differ with regard to the agents' preferences, local SSE may also be found around determinate steady states. This shows that the relationship between existence of local sunspot equilibria and indeterminacy of the steady state is not as strong as previously thought.

4. Existence of Local Sunspot Equilibria I: letting endowments vary

In this section, we investigate the conditions for the existence of local stationary sunspot equilibria around a non-sunspot stationary equilibrium when 'nearby' economies can differ by endowments as well as probabilities. In this case, the existence of local SSE means that, in any arbitrarily small neighborhood of the steady state of a given economy, we can find a SSE for an economy characterized by an arbitrarily close level of endowments and probabilities (but the same preferences).

We will show that the indeterminacy of the steady state is again a necessary condition for the existence of local stationary sunspot equilibria in its vicinity. Thus Woodford and Guesnerie's result generalizes to such extended domain of economies.

The analysis builds on two intermediate results, which are of independent interest. We will argue first that, when the domain of economies considered is parameterized by endowments and probabilities, the existence of local SSE around a non-sunspot stationary equilibrium requires the latter to be a singular equilibrium:[10]

Lemma 1. *Suppose that there is a sequence of economies* $\{(e^n, \bar{u}), \Pi^n\}_n$*, where both* e^n *and* Π^n *can vary with* n*, and an associated sequence of stationary sunspot equilibria* $\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\}_n$*, which converge, respectively to* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ *and to* $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ *as* $n \to \infty$*. Then* $\{\bar{c}_1, \bar{c}_2, \bar{p}\}$ *is a singular equilibrium of the limit economy* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$.

Proof. Fix the utility function u at $\bar{u}$ throughout. From the hypothesis of the statement, it follows that both $\left\{(\overline{c}_1, \overline{c}_1; \overline{c}_2, \overline{c}_2), \bar{e}, \bar{u}, \bar{\Pi}\right\}$ and $\left\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), e^n, \bar{u}, \Pi^n\right\}$ for each n must be solutions of (3.5), (3.6)). Note that each certainty economy $(e^n, \bar{u})$ associated to $\{(e^n, \bar{u}), \Pi^n\}$ also has a steady state equilibrium $(\overline{c}_1^n, \overline{c}_2^n)$ and hence $\{(e^n, \bar{u}), \Pi^n\}$ has a non sunspot stationary equilibrium $(\overline{c}_1^n, \overline{c}_2^n, \bar{p}^n)$. Moreover, by the continuity of demand it follows that $(\overline{c}_1^n, \overline{c}_2^n)$ must lie close to $(\overline{c}_1, \overline{c}_2)$ and that $(\overline{c}_1^n, \overline{c}_2^n) \to (\overline{c}_1, \overline{c}_2)$. Hence the existence of local SSE implies that in any arbitrarily small neighborhood of $\left\{(\overline{c}_1, \overline{c}_1; \overline{c}_2, \overline{c}_2), \bar{e}, \bar{\Pi}\right\}$, the system given by (3.5), (3.6) has two solutions (a sunspot one and a non sunspot one); as a consequence, such equilibrium must be singular. Q.E.D.

To investigate the conditions under which a local SSE exists around a non-sunspot stationary equilibrium, we can then study the conditions under which the non sunspot stationary equilibrium fails to be regular. In the following result, we characterize the conditions for the singularity of non-sunspot stationary equilibria which are different from autarky.

Lemma 2. *A non-sunspot stationary equilibrium* $(\overline{c}_1, \overline{c}_2, \bar{p})$*, with* $\overline{c}_1 \neq e_1$*, of an economy* (e, u, Π) *is singular if and only if the slope of the offer curve at* $(\overline{c}_1, \overline{c}_2)$,

[10] Azariadis and Guesnerie (1986), and Guesnerie (1986) had earlier pointed out, using bifurcation arguments, that the singularity of a non-sunspot stationary equilibrium is closely related (in fact sufficient) for the existence of SSE around it.

$-\frac{D_1\Gamma}{D_2\Gamma}$ *is equal to* -1 *or* $-|\Pi|$.

Proof. Recall that the singularity of $(\bar{c}_1, \bar{c}_2, \bar{p})$ means that $\Phi(c; e, u, \Pi)$ has a singularity at $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), \bar{e}, \bar{\Pi}\}$.

The Jacobian $D_c\Phi$ evaluated at $(\bar{c}_1, \bar{c}_2, \bar{c}_1, \bar{c}_2, \bar{e}_1, \bar{e}_2)$ is given by the 4×4 matrix

$$\begin{pmatrix} D_1\Gamma & \pi^{11}D_2\Gamma & 0 & \pi^{12}D_2\Gamma \\ 0 & \pi^{21}D_2\Gamma & D_1\Gamma & \pi^{22}D_2\Gamma \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

where we used $D_i\Gamma$ to denote, for simplicity, $D_i\Gamma(\bar{c}_1, \bar{c}_2)$, and the columns correspond, respectively, to the variables c_1^1, c_2^1, c_1^2, c_2^2. Computing the determinant of this matrix we obtain,[11]

$$-(D_1\Gamma)^2 + \left(\pi^{11} + \pi^{22}\right) D_1\Gamma D_2\Gamma - (D_2\Gamma)^2 \left(\pi^{11} + \pi^{22} - 1\right).$$

Thus we see that the Jacobian is singular if and only if one of the following conditions is satisfied:

(1) $D_1\Gamma = D_2\Gamma = 0$;
(2) $D_1\Gamma \neq 0$ and $\frac{D_2\Gamma}{D_1\Gamma}$ is a root of

$$P_1(\lambda) \equiv (\pi^{11} + \pi^{22} - 1)\lambda^2 - (\pi^{11} + \pi^{22})\lambda + 1 = 0 \tag{4.1}$$

or $D_2\Gamma \neq 0$ and $\frac{D_1\Gamma}{D_2\Gamma}$ is a root of the polynomial

$$P_2(\lambda) \equiv \lambda^2 - (\pi^{11} + \pi^{22})\lambda + (\pi^{11} + \pi^{22} - 1) = 0. \tag{4.2}$$

It is immediate to see that one of the roots of these polynomials is 1 and that the other roots are the determinant of the Markov matrix $|\Pi| = \pi^{11} + \pi^{22} - 1$ for P_2 and its reciprocal $|\Pi|^{-1}$ for P_1.

We observe first that (1) can never occur at a steady state different from autarky. For any point (c_1, c_2) lying on the offer curve, using (3.1) and (3.2) we have in fact,

$$(D\Gamma_{c_1}(c_1, c_2), D\Gamma_{c_2}(c_1, c_2)) \cdot (c_1 - e_1, c_2 - e_2) = \sum_{i,j=1}^{2} (c_j - e_j) D_{ij}u(c_1, c_2)(c_i - e_i)$$

By the strict quasi concavity of u this expression is always negative, if $c_i \neq e_i$. Hence we cannot have $D\Gamma_1 = D\Gamma_2 = 0$.

[11] We used here the facts that $\pi^{21} = (1 - \pi^{22})$, $\pi^{12} = (1 - \pi^{11})$.

We are thus left with case (2), which requires that the slope of the offer curve is either -1 or $-|\Pi|$. Q.E.D.

On the basis of Lemma 1 we can say that the conditions identified in Lemma 2 are necessary conditions for the existence of local SSE around a non-sunspot stationary equilibrium $(\bar{c}_1, \bar{c}_2, \bar{p})$, different from autarky. Since the determinant $|\Pi|$ always lie in the interval $[-1, 1]$ (and, except for the cases in which the Markov matrix is the identity or a purely cyclical matrix, $|\Pi| \in (-1, 1)$), these conditions require the slope of the offer curve at $(\bar{c}_1, \bar{c}_2)$, $\left|\frac{D_1\Gamma}{D_2\Gamma}\right|$ at $(\bar{c}_1, \bar{c}_2)$ to be smaller than or equal to 1. We have so proved that the fact that the slope of the offer curve in absolute value at the steady state is not bigger than 1 is still a necessary condition for the existence of local SSE in the extended domain of economies parameterized by endowments as well as by probabilities. Moreover, the condition that this same slope is strictly smaller than 1 in absolute value is also sufficient for the existence of local stationary sunspot equilibria.[12] Note that these are respectively also the necessary and sufficient conditions for the indeterminacy of the steady state. Thus it follows that, under condition R, local stationary sunspot equilibria exist if, and only if, the steady state is indeterminate in the case of economies parameterized by endowments as well as by probabilities.

Proposition 2. *Assume that the condition R holds for the economy* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$. *Then, there is a sequence of economies* $\{(e^n, \bar{u}), \Pi^n\}_n$, *where both* e^n *and* Π^n *can vary with* n, *and an associated sequence of stationary sunspot equilibria* $\left\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\right\}_n$, *which converge, respectively to* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ *and to* $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ *as* $n \to \infty$, *for* $\bar{c}_1 \neq e_1$, *if and only if* $(\bar{c}_1; \bar{c}_2)$ *is an indeterminate steady state of* $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$.

Remark 1. *The results in Lemmas 1 and 2 also allow us to get an interesting characterization of other properties, beyond the absence of determinacy, of the economies exhibiting non-sunspot stationary equilibria around which local SSE exist. The slope of the offer curve (at the associated steady state) must equal -1, a condition which holds only for a negligible subset economies as parameterized by* (e, u) *while* Π *can be arbitrary. Alternatively, the slope of the offer curve can lie anywhere in the interval* $(-1, 1)$, *but the determinant of the Markov matrix* Π *has to equal the opposite of the slope of the offer curve. This imposes no condition on* (e, u) *other than the fact that the non autarchic steady state is necessarily indeterminate, but it restricts* Π *to lie in a manifold of dimension 1 in the 2 dimensional space of* 2×2 *Markov matrices which defines the domain of* Π.

[12] By the same argument than the one used at the end of the proof of Proposition 1.

To conclude this section, we show next that if a different, slightly weaker notion of "vicinity" is used to identify economies lying nearby a given economy, SSE arbitrarily close to a non-sunspot stationary equilibrium of a given economy can be found under very general - in fact generic - conditions. We will consider in particular the case where, in any arbitrarily small neighborhood of a non-sunspot stationary equilibrium $(\bar{c}_1, \bar{c}_2, \bar{p})$ of an economy $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$, we can find stationary sunspot equilibria of economies $\left\{(e, \bar{u}), \bar{\Pi}\right\}$, characterized by the same preferences and probabilities as $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ as well as the same aggregate endowment: $e_1 + e_2 = \bar{e}_1 + \bar{e}_2$. We will show that this is always possible provided $(\bar{c}_1, \bar{c}_2, \bar{p})$ constitutes a regular non sunspot stationary equilibrium of $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$, a condition which - as we saw above (see also remark 1) - is generically satisfied. Thus no condition is needed here with regard to the stability properties of the steady state for the existence of SSE close to it.[13]

Note that even though the economies $\left\{(e, \bar{u}), \bar{\Pi}\right\}$ have the same preferences, probabilities and aggregate endowments as $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$, the distribution of the aggregate endowments between young and old may be significantly different. Thus, we cannot say that such economies lie 'nearby' the economy $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ in the same sense of the definition of local sunspot equilibria (i.e., according to some distance in the space of economies as parameterized by e u, Π). We will use then a different term to refer to such stationary sunspot equilibria: rather than local SSE, we will speak of *SSE "near" a steady state.*

Proposition 3. *Consider an economy $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ with a regular non sunspot stationary equilibrium $\{(\bar{c}_1, \bar{c}_2), \bar{p}\}$. Then there are stationary sunspot equilibria 'near' it; that is, for any $\varepsilon > 0$ fixed arbitrarily, we can find an economy $\left\{(e, \bar{u}), \bar{\Pi}\right\}$,[14] such that $e_1 + e_2 = \bar{e}_1 + \bar{e}_2$, with a SSE $\left(c_1^1, c_1^2; c_2^1, c_2^2; p^1, p^2\right)$ lying within an ε neighborhood of $(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2; \bar{p}, \bar{p})$.*

As shown in the Appendix, the result easily extends to OLG economies with heterogeneous agents in each generation.

Proof. The proof of the proposition relies on the consideration of a class of auxiliary, stochastic economies, used to construct SSE arbitrarily near the non-sunspot equilibrium.[15]

Consider the class of stochastic economies $\{(\hat{e}, u), \Pi\}$, where the only difference from the sunspot economies considered so far is that the agent's endowment when

[13] It is in fact interesting to observe that the conditions for the existence of a SSE "near" a non-sunspot stationary equilibrium are the perfect complement of the conditions for the existence of local SSE near it.

[14] Here, $e >> 0$ may not hold.

[15] The logic of the construction used here is reminiscent of the one in Gottardi-Kajii (1999).

young and old is described by the vector $\hat{e} = (\hat{e}_1^1, \hat{e}_2^1, \hat{e}_1^2, \hat{e}_2^2)$, i.e. may also vary with the realization of the uncertainty. Let us denote by $\hat{\Phi}(c; \hat{e}, u, \Pi)$ the system of equations in (3.5), (3.6) when e is replaced by $\hat{e}$. It is immediate to see that c is a stationary equilibrium[16] of $(\hat{e}, u, \Pi)$ if $\hat{\Phi}(c; \hat{e}, u, \Pi) = 0$. Let $\widehat{\bar{e}} \equiv (\bar{e}_1, \bar{e}_1, \bar{e}_2, \bar{e}_2)$; since, by assumption $(\bar{c}; \bar{e}, \bar{u}, \bar{\Pi})$ is a regular solution of $\Phi(c; e, u, \Pi) = 0$, evidently so is also $\left(\bar{c}; \widehat{\bar{e}}, \bar{u}, \bar{\Pi}\right)$ of $\hat{\Phi}(c; \hat{e}, u, \Pi) = 0$. Thus by the implicit function theorem, for any stochastic endowment vector $\hat{e}$ that is close enough to $\widehat{\bar{e}}$, there is a stationary equilibrium for which the consumption level $(c_1^1, c_1^2; c_2^1, c_2^2)$ is close to the non-sunspot equilibrium level $(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2)$. It can immediately be verified that, as long as the endowment $(\hat{e}_1^1, \hat{e}_2^1, \hat{e}_1^2, \hat{e}_2^2)$ is stochastic, i.e. $\hat{e}_i^1 \neq \hat{e}_i^2$, for $i = 1$ and/or $i = 2$, the associated stationary equilibrium will also be stochastic $c_i^1 \neq c_i^2$, for $i = 1$ and/or $i = 2$.

In particular, consider the case where $\hat{e}_1^1 + \hat{e}_2^1 = \hat{e}_1^2 + \hat{e}_2^2$, i.e. where the aggregate endowment is constant in the two states, only its distribution between young and old agents vary with the realization of the uncertainty. To complete the proof we will show that in this case it is always possible to find a deterministic endowment vector (e_1, e_1, e_2, e_2), such that $(c_1^1, c_1^2; c_2^1, c_2^2)$ is also a stationary (now sunspot) equilibrium of the deterministic economy with endowment (e_1, e_2).

Lemma 3. *Let $\left\{\left(c_1^1, c_1^2; c_2^1, c_2^2\right), p^1, p^2\right\}$ be a stationary equilibrium of a stochastic economy with preferences u, probabilities Π and endowments $(\hat{e}_1^1, \hat{e}_2^1, \hat{e}_1^2, \hat{e}_2^2)$ such that $\hat{e}_1^1 + \hat{e}_2^1 = \hat{e}_1^2 + \hat{e}_2^2$. Then $\left\{(c_1^1, c_1^2; c_2^1, c_2^2), p^1, p^2\right\}$ also constitutes a stationary sunspot equilibrium of a sunspot economy, with the same preferences u and probabilities Π, and deterministic endowments $\tilde{e}_1, \tilde{e}_2$ obtained as solution to*

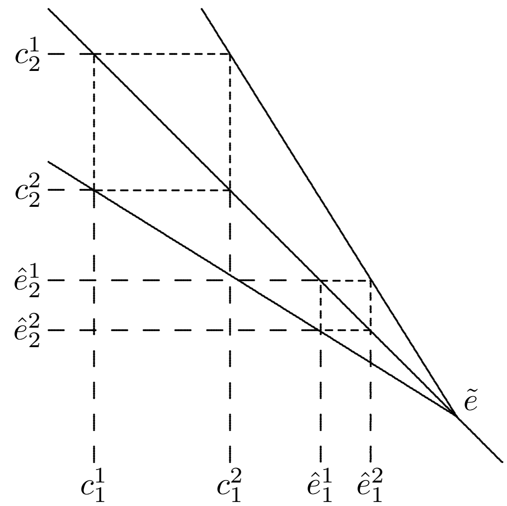
$$\begin{aligned} \tilde{e}_1 + \tilde{e}_2 &= \hat{e}_1^1 + \hat{e}_2^1 \\ p^1 \tilde{e}_1 + p^2 \tilde{e}_2 &= p^1 \hat{e}_1^1 + p^2 \hat{e}_2^2. \end{aligned}$$

We prove a more general version of the Lemma in Appendix. This establishes the claim of the Proposition. Q.E.D.

A simple geometric illustration of the argument of the proof of Lemma 3 is given in Figure 1.

[16]The definition of stationary equilibria for the stochastic economy $\{(\hat{e}, u), \Pi\}$ is perfectly analogous to the one of SSE given in Definition 1.

Figure 1



Lemma 3 provides a method for transforming any stationary equilibrium of a stochastic overlapping generations economy (with no aggregate risk) into a stationary sunspot equilibrium of a related sunspot economy. On this basis, the argument of the proof of Proposition 3 indicates then a recipe for constructing sunspot stationary equilibria around non-sunspot (or in fact any) stationary equilibrium. Notice that this argument requires nothing on the determinacy or indeterminacy properties of the steady state. On the other hand, the deterministic endowments $\tilde{e}_1, \tilde{e}_2$ found in Lemma 3 are not necessarily strictly positive.

5. Existence of Local Sunspot Equilibria II: letting utilities vary

We study in this section the conditions for the existence of local stationary sunspot equilibria when the space where 'nearby' economies may lie is parameterized by probabilities and utilities instead of endowments. Fix e at some level $\bar{e}$. We say in this case that there are local SSE around a non-sunspot stationary equilibrium if there is a sequence of utility functions $\{u^n : n = 1, 2, ...\}$, which converges to a function $\bar{u}$ in the C^k uniform convergence topology for some $k \geq 1$, a sequence of Markov matrices Π_n converging to $\bar{\Pi}$, and a sequence of associated SSE which converges to a non-sunspot stationary equilibrium of $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$.

We want to argue that local sunspot stationary equilibria exist, in this domain of economies, irrespective of the stability property of the steady state as well as of the sign of the equilibrium price of money (which may be positive or negative). Thus the existence of sunspot equilibria cannot be ruled out, in the vicinity of a steady state equilibrium, under quite general conditions, and in particular not even when the steady state is determinate.

The precise formulation of our claim will depend on the strength of the notion of convergence which is considered. When a strong notion is considered, i.e. we require the sequence $\{u^n\}_n$ to converge to $\bar{u}$ in the C^k uniform convergence topology for k arbitrarily large, we will show that local sunspot equilibria around a non-sunspot stationary equilibrium exist also when, for all the economies in the sequence, the steady state lying near the non-sunspot equilibrium is determinate and, in addition, the price of money is positive:

Proposition 4. *For any $k > 1$, we can find sequences of utility functions $\{u^n\}_n$, converging to $\bar{u}$ in the C^k uniform convergence topology, and sequences of probabilities $\{\Pi^n\}_n$ converging to $\bar{\Pi}$, such that*
(i) for each n, economy $\{(\bar{e}, u^n), \Pi^n\}$ has a stationary sunspot equilibrium and the sequence $\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\}_n$ converges to $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ as $n \to \infty$;
(ii) for each n, certainty economy $\{\bar{e}, u^n\}$ associated to $\{(\bar{e}, u^n), \Pi^n\}$ has a steady state equilibrium $\{\bar{c}_1^n, \bar{c}_2^n, \bar{p}^n\}$ around $\{\bar{c}_1, \bar{c}_2, \bar{p}\}$ which is determinate and such that $\bar{c}_1^n < \bar{e}_1$ (i.e., the price of money is positive).

Proof. Let $\bar{e}$ be such that $\bar{e}_1 > \bar{e}_2$. We will prove the result by constructing a sequence of utility functions $\{u^n\}_n$, converging to $\bar{u}$ in the C^k uniform convergence topology for some $k > 1$ such that all the certainty economies in the sequence $\{\bar{e}, u^n\}$ as well as the limit economy $\{\bar{e}, u^n\}$ have the same monetary steady state $(\bar{c}_1, \bar{c}_2)$, with $\bar{c}_1 < \bar{e}_1$.

We establish first a preliminary result which is of some independent interest: any given upward sloping differentiable function defined in a neighborhood of $\bar{c}_1, \bar{c}_2$ can be rationalized as the offer curve of a consumer with an additively separable, concave utility function (and endowment $\bar{e}$) :

Lemma 4. *Let f be a strictly increasing, C^∞ function defined on an open interval (a, b) around $\bar{c}_1$, with $b < e_1$, $f(a) > e_2$ and $f(\bar{c}_1) = \bar{c}_2$. There is an additively separable, concave utility function u such that f is (a part of) of the offer curve of an agent with preferences u and endowments e.*

Proof. Consider the following differential equation

$$u_2'(z) = \frac{e_1 - f^{-1}(z)}{y - e_2},$$

where $z \in (f(a), f(b))$. The right hand side is positive and C^∞; hence there exists a strictly increasing, C^∞ function u_2 which solves this equation. Moreover, u_2 is concave since the right hand side of the equation is decreasing in z. Since u_2' is bounded on $(f(a), f(b))$, there exists a monotonic, concave C^∞ extension of u_2 on

$[0, \infty)$, which we denote by u_2 abusing notation. Let $u(x_1, x_2) = x_1 + u_2(x_2)$ for $(x_1, x_2) \in \mathbb{R}^2_+$.

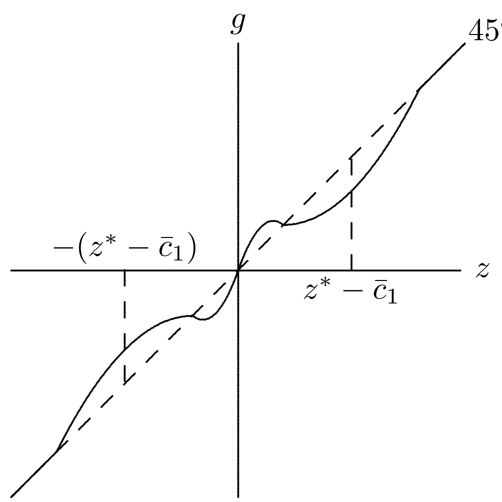
Then by construction,

$$\frac{D_2 u(x_1, x_2)}{D_1 u(x_1, x_2)} = u_2'(f(x_1)) = \frac{e_1 - x_1}{f(x_1) - e_2},$$

which implies that the gradient of u at $(x_1, f(x_1))$ is perpendicular to the line connecting $(x_1, f(x_1))$ and (e_1, e_2). That is, the first order condition for utility maximization is satisfied. Since u is concave, this shows that the offer curve generated by u coincides with the curve $\{(z, f(z)) : z \in (a, b)\}$ around $(\bar{c}_1, \bar{c}_2)$. Q.E.D.

Let g be a C^∞ function with domain $\mathbb{R}$, such that $g(0) = 0$, $g'(0) > 1$, $g(z) = z$ for $|z|$ large; moreover, there exists $z^* > \bar{c}_1$ such that $g(z^* - \bar{c}_1) < z^* - \bar{c}_1$ and $g[-(z^* - \bar{c}_1)] > -(z^* - \bar{c}_1)$. Thus the function g is "S-shaped" around 0, as shown in Figure 2. Since g is C^∞ and differs from the identity function only on a compact set, for each k $\frac{d^k}{dz^k} g(z)$ is bounded in z.

Figure 2



For every integer n, let f^n be a C^∞ function defined by the rule

$$f^n(z) = \frac{1}{n}\left[g\left(n^q(z - \bar{c}_1)\right) - n^q(z - \bar{c}_1)\right] + (z - \bar{c}_1) + \bar{c}_2,$$

where $q \in (0, 1)$ is a fixed constant. So for each n, the curve $\{(z, f^n(z)) : z \in (a, b)\}$ is also "S-shaped" as the curve $\{(z, g(z)) : z \in (a - \bar{c}_1, b - \bar{c}_1)\}$, with $f^n(z^{*n}) < (z^{*n} - \bar{c}_1) + \bar{c}_2$ and $f^n(\bar{c}_1 - (z^{*n} - \bar{c}_1)) > -(z^{*n} - \bar{c}_1) + \bar{c}_2$, where $z^{*n} - \bar{c}_1 = (z^* - \bar{c}_1)/n^q$, but the size of the "S-shape" is smaller and decreasing with n. Note that, since $g(0) = 0$ and $g\left(n^q\left(z - \bar{c}_1\right)\right) = n^q\left(z - \bar{c}_1\right)$ when $\left|n^q\left(z - \bar{c}_1\right)\right|$ is large, the term $\left[g\left(n^q\left(z - \bar{c}_1\right)\right) - n^q\left(z - \bar{c}_1\right)\right]$ is bounded; hence the function $f^n(z)$ converges uniformly to the function $f^*(z) := (z - \bar{c}_1) + \bar{c}_2$ for any q. By direct computation, $\frac{d}{dz}f^n(z) = n^{q-1}\left[g'\left(n^q\left(z - \bar{c}_1\right)\right) - 1\right] + 1$, which also converges uniformly to 1 ($= \frac{d}{dz}f^*(z)$) since $q < 1$. For $k \geq 2$, $\frac{d^k}{dz^k}f^n(z) = n^{kq-1}\frac{d^k}{dz^k}g\left(n^q\left(z - \bar{c}_1\right)\right)$, so f^n is C^k uniformly convergent to f^* if $q < \frac{1}{k}$. Thus by choosing q arbitrarily small we can ensure an arbitrarily strong form of convergence. Notice, furthermore, that for every n, we have $f^n(\bar{c}_1) = \bar{c}_2$ and $\frac{d}{dz}f^n(\bar{c}_1) > 1$.

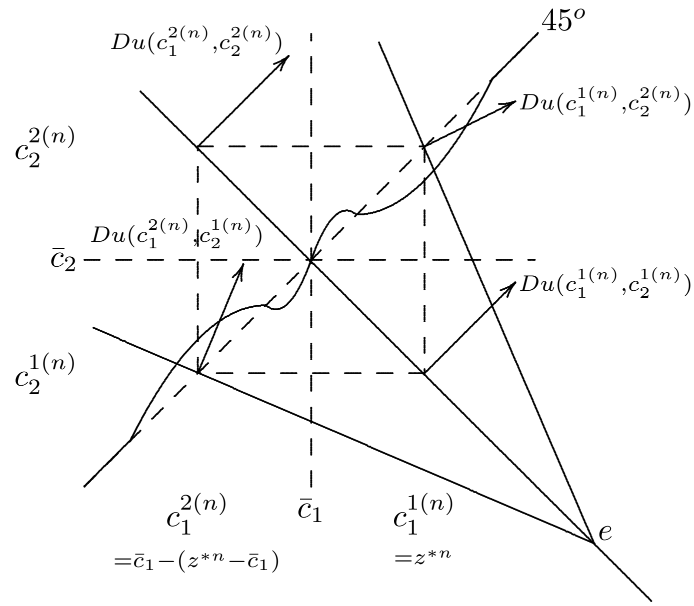
Now for each n, let u^n be the utility function, constructed using the argument in the proof of Lemma 4, which rationalizes the function f^n as the offer curve around $(\bar{c}_1, \bar{c}_2)$. Also let $\bar{u}$ be the utility function rationalizing f^* as the offer curve around $(\bar{c}_1, \bar{c}_2)$. By construction, since for each n $f^n(\bar{c}_1) = \bar{c}_2$, $f^*(\bar{c}_1) = \bar{c}_2$ and $\bar{c}_1 < e_1$, the consumption bundle $(\bar{c}_1, \bar{c}_2)$ is a steady state equilibrium with positive value of money of the certainty economy $(u^n, (\bar{e}_1, \bar{e}_2))$, for all n, as well as for the economy $(\bar{u}, \bar{e}_1, \bar{e}_2))$. Moreover, since $\frac{d}{dz}f^n(\bar{c}_1) = n^{q-1}\left[g'(0) - 1\right] + 1 > 1$ for all n, the steady state $(\bar{c}_1, \bar{c}_2)$ is determinate for each economy $(u^n, (\bar{e}_1, \bar{e}_2))$. On the other hand, as we already noticed, the slope of offer curve at the steady state is one for the limit economy $(\bar{u}, (e_1, e_2))$. Since f^n is convergent to f^* in the C^k uniform topology, $k < \frac{1}{q}$, so is Du^n to Du, and thus u^n converges to u in C^{k+1} uniform topology for $k < \frac{1}{q}$, by construction. This establishes the claim in part (ii) of the Proposition.

We show next that for each n, we can find a Markov matrix Π^n such that the stochastic economy $\{(\bar{e}, u^n), \Pi^n\}$ has a stationary sunspot equilibrium $\left\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\right\}$; moreover, the sequence $\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right)$ converges to the non-sunspot stationary equilibrium allocation $(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2)$.[17]

Let $c_1^{1(n)} = z^{*n}$, $c_2^{1(n)} = (\bar{c}_1 - z^{*n}) + \bar{c}_2$, $c_1^{2(n)} = \bar{c}_1 - (z^{*n} - \bar{c}_1)$, $c_2^{2(n)} = \bar{c}_2 - (\bar{c}_1 - z^{*n})$. The four points $\left(c_1^{i(n)}, c_2^{j(n)}\right)$, $i, j = 1, 2$, form a square box around $(\bar{c}_1, \bar{c}_2)$ as in Figure 3; by the property of z^{*n}, $\left(c_1^{1(n)}, c_2^{2(n)}\right)$ and $\left(c_1^{2(n)}, c_2^{2(n)}\right)$ (respectively $\left(c_1^{2(n)}, c_2^{1(n)}\right)$ and $\left(c_1^{2(n)}, c_2^{1(n)}\right)$) lie above (below) the offer curve. These points will describe the consumption allocation at a SSE, for some Markov matrix Π^n.

[17] Since the elements of $\{\Pi^n\}_n$ lie in a compact set, this sequence always admits a convergent subsequence.

Figure 3



Notice that for each $j = 1, 2$, $c_1^{j(n)} - e_1 + c_2^{j(n)} - e_2 = 0$, thus the market clearing condition (3.6) is satisfied for both states $j = 1, 2$. Moreover, since each pair of points $\left(c_1^{i(n)}, c_2^{1(n)}\right)$ and $\left(c_1^{i(n)}, c_2^{2(n)}\right)$, $i = 1, 2$, is such that one point lies above and the other lies below the offer curve, the inner product of the gradient of the utility function at $(c_1^{i(n)}, c_2^{j(n)})$ (see (3.2)) and the excess demand vector $(c_1^{i(n)} - e_1, c_2^{j(n)} - e_2)$ has opposite signs at $j = 1$ and $j = 2$. Thus for each i, we can find positive values π^{ij}, $j = 1, 2$ with $\pi^{i1} + \pi^{i2} = 1$ such that

$$\sum_{j=1}^{2} \pi^{ij} \left(D_1 u(c_1^{i(n)}, c_2^{j(n)})(c_1^{i(n)} - e_1) + D_2 u(c_1^{i(n)}, c_2^{j(2)})(c_2^{j(2)} - e_2) \right) = 0. \qquad (5.1)$$

Recall that equations (3.5), (3.6) characterize consumption allocations at a SSE. We have thus shown that the four points $\left(c_1^{i(n)}, c_2^{j(n)}\right)$, $i, j = 1, 2$, constructed as above constitute a sunspot stationary equilibrium for the economy $\{(\bar{e}, u^n), \Pi^n\}$. Notice that as n increases, the four points become arbitrarily close to $(\bar{c}_1, \bar{c}_2)$ since the "S-shape" of f^n gets arbitrarily small as n increases. This establishes claim (i) and completes then the proof of the Proposition. Q.E.D.

The argument of the proof of Proposition 4 shows how to find sequences of sunspot economies $\{(\bar{e}, u^n), \Pi^n\}_n$ exhibiting SSE and converging to $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ in the C^k uniform convergence, for any k arbitrarily large (it suffices to pick $q < \frac{1}{k}$ in the construction described above). The consumption allocation $(\bar{c}_1, \bar{c}_2)$ constitutes a determinate steady state for every certainty economy $(u^n, (\bar{e}_1, \bar{e}_2))$. But for the limit economy $(\bar{u}, (\bar{e}_1, \bar{e}_2))$ the slope of the offer curve at $(\bar{c}_1, \bar{c}_2)$ is one, as we already argued; thus the steady state may be either determinate or indeterminate.

We show next that if a weaker form of convergence is considered, in particular if we require the sequence $\{u^n\}_n$ to converge to $\bar{u}$ only in the C^1 convergence topology, we can exhibit local sunspot equilibria around a steady state which is determinate also for the limit economy:

Proposition 5. *There are sequences of economies $\{(\bar{e}, u^n), \Pi^n\}_n$, where $\{u^n\}_n$ converges to $\bar{u}$ in the C^1 uniform convergence topology and $\{\Pi^n\}_n \to \bar{\Pi}$ as $n \to \infty$, and associated stationary sunspot equilibria $\left\{\left(c_1^{1(n)}, c_1^{2(n)}; c_2^{1(n)}, c_2^{2(n)}\right), \left(p^{1(n)}, p^{2(n)}\right)\right\}_n$ which converge to $\{(\bar{c}_1, \bar{c}_1; \bar{c}_2, \bar{c}_2), (\bar{p}, \bar{p})\}$ even when $(\bar{c}_1, \bar{c}_2, \bar{p})$ is a determinate steady state equilibrium of the limit certainty economy $\{\bar{e}, \bar{u}\}$ and $\bar{c}_1 < \bar{e}_1$ (i.e., the price of money is positive).*

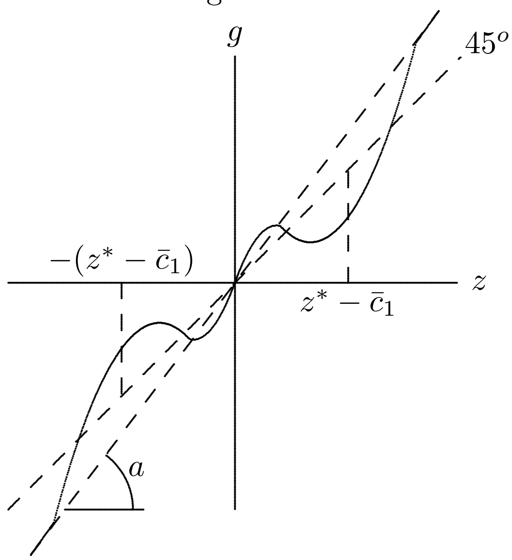
Proof. The proof relies on a slight modification of the construction described in the proof of Proposition 4 above. We will only outline here the main points of departure from the earlier construction.

Let g be a C^∞ function with domain $\mathbb{R}$, such that $g(0) = 0$, $g'(0) > a > 1$, where a is a fixed constant, and $g(z) = az$ for $|z|$ large. The function is again "S-shaped" around 0 and there is $z^* > \bar{c}_1$ such that $g(z^* - \bar{c}_1) < z^* - \bar{c}_1$ and $g[-(z^* - \bar{c}_1)] > -(z^* - \bar{c}_1)$, as shown in Figure 4. For each n, let f^n be a C^∞ function defined by the rule

$$f^n(z) = \frac{1}{n}\left[g(n(z - \bar{c}_1)) - n(a(z - \bar{c}_1))\right] + a(z - \bar{c}_1) + \bar{c}_2.$$

So the differences from the previous construction are that here q is set to be equal to one, and that f^n is uniformly convergent to the linear function $f^*(z) := a(z - \bar{c}_1) + \bar{c}_2$ with $a > 1$.

Figure 4



For each n, the curve $\{(z, f^n(z)) : z \in (a, b)\}$ is also "S-shaped" as the curve $\{(z, g(z)) : z \in (a - \bar{c}_1, b - \bar{c}_1)\}$. It is less obvious to see that we can find $z^{*n} > \bar{c}_1$ such that $f^n(z^{*n}) < (z^{*n} - \bar{c}_1) + \bar{c}_2$ and $f^n(\bar{c}_1 - (z^{*n} - \bar{c}_1)) > -(z^{*n} - \bar{c}_1) + \bar{c}_2$, but this can be verified as follows. Set $z^{*n} - \bar{c}_1 = (z^* - \bar{c}_1)/n$. Now,

$$\begin{aligned}
(z^{*n} - \bar{c}_1) + \bar{c}_2 - f^n(z^{*n}) &= \frac{z^* - \bar{c}_1}{n} + \bar{c}_2 - \\
&\quad - \left[\frac{1}{n}\left(g(z^* - \bar{c}_1) - a(z^* - \bar{c}_1)\right) + a(z^* - \bar{c}_1)/n + \bar{c}_2\right] \\
&= (z^* - \bar{c}_1)/n - \frac{1}{n} g(z^* - \bar{c}_1) \\
&= (z^* - \bar{c}_1)\left[\frac{1}{n}(1 - a) - \frac{1}{n}\left\{\frac{g(z^* - \bar{c}_1)}{(z^* - \bar{c}_1)} - a\right\}\right] \\
&= (z^* - \bar{c}_1)\left[\frac{1}{n}\left(1 - \frac{g(z^* - \bar{c}_1)}{(z^* - \bar{c}_1)}\right)\right] > 0,
\end{aligned}$$

since $z^* > \bar{c}_1$ and $1 - \frac{g(z^* - \bar{c}_1)}{(z^* - \bar{c}_1)} > 0$. A similar argument then shows that

$$f^n(\bar{c}_1 - (z^{*n} - \bar{c}_1)) > -(z^{*n} - \bar{c}_1) + \bar{c}_2.$$

By Lemma 4, the function $f^n(z)$, $z \in (a, b)$, can be rationalized as the offer curve of a consumer with preferences u^n and endowments $\bar{e}$. Applying then the same argument as in the proof of Proposition 4, for each n we can find a Markov matrix Π^n and a SSE of the economy $\{(\bar{e}, u^n), \Pi^n\}$.

Notice that in this case f^n is not C^1 convergent: $\frac{d}{dz} f^n(z) \to 1$ as $n \to \infty$ if $z \neq \bar{c}_1$ but $\frac{d}{dz} f^n(\bar{c}_1) = g'(0) > 1$, so f^n is C^0 uniformly convergent to f^* but not C^1 uniformly convergent. Thus the corresponding sequence of utility functions $\{u^n\}_n$ converges only in the C^1 uniform topology to the limit utility function $\bar{u}$, which rationalizes f^* as its offer curve. On the other hand, in this case the limit offer curve f^* has slope $a > 1$ around the steady state $(\bar{c}_1, \bar{c}_2)$. Thus $(\bar{c}_1, \bar{c}_2)$ is a determinate steady state allocation for all the certainty economies in the sequence $\{\bar{e}, u^n\}_n$ as well as for the limit economy $\{\bar{e}, \bar{u}\}$. Q.E.D.

To conclude the paper, let us comment on the relationship between the existence results obtained in Propositions 4 and 5, and the necessary condition for the existence of local SSE identified in Lemmas 1 and 2. The reader may wonder if these results are consistent with each other. Proposition 4 can be readily shown to be consistent with the lemmas. To see this, notice that the slope of the offer curve at the steady state $\{\bar{c}_1, \bar{c}_2\}$ of the limit economy $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ is in fact equal to one. According to Lemma 2, the non-sunspot stationary equilibrium $\{\bar{c}_1, \bar{c}_2, \bar{p}\}$ is a singular equilibrium of the limit economy $\left\{(\bar{e}, \bar{u}), \bar{\Pi}\right\}$ when the slope of the offer curve equals $-|\Pi|$ and this is in turn equal to 1 when $\pi^{11} = 0 = \pi^{22}$; thus for a Markov matrix exhibiting this property, the Jacobian $D_c\Phi$ is singular.

On the other hand, the existence result of Proposition 5, where the steady state $\{\bar{c}_1, \bar{c}_2\}$ is determinate also for the limit economy, seems to constitute an apparent contradiction to Lemmas 1 and 2. This is not so. To understand this relationship, we should observe that the result obtained in Lemma 1 relies on the application of the Implicit Function Theorem to the system $\Phi(c; e, u, \Pi)$, and hence requires the differentiability of Φ. Since the utility gradient Du appears in $\Phi(c; e, u, \Pi)$, the differentiability of Φ requires the uniform convergence of D^2U, i.e. we need the sequence $\{u^n\}_n$ to be at least C^2 convergent, a property which we saw does not hold for the local SSE constructed in the proof of Proposition 5. Thus the condition of Lemma 1 is not applicable to the framework of Proposition 5.

APPENDIX

We show here that the validity of the claims in Proposition 3 and Lemma 3 extends to overlapping generation economies with heterogeneous agents. These results provide then a general and robust method for constructing stationary sunspot equilibria in the vicinity of any stationary equilibrium.

Consider the following generalization of the model considered in the previous sections. In each generation there are $H > 1$ distinct agents. Each agent $h \in \{1, .., H\}$ is characterized by preferences u^h (with the same properties as before) and endowments $\hat{e}^h = (\hat{e}_1^{h,1}, \hat{e}_2^{h,1}, \hat{e}_1^{h,2}, \hat{e}_2^{h,2})$, which can possibly vary with the realization of the uncertainty.

Stationary equilibria are described in this case by a collection of vectors of consumption levels in all states $(c_1^{h,i}; c_2^{h,i1}, c_2^{h,i2})_{i=1}^{2}$[18] for each agent h and prices (p^1, p^2), which are obtained as solutions of the following system of equations:

$$\begin{aligned}
\sum_{j=1}^{k} \pi^{ij} D_1 u^h(c_1^{h,i}, c_2^{h,ij}) - \sum_{j=1}^{k} \lambda^{hij} p^i &= 0, && \text{for all } i, h \\
\pi^{ij} D_2 u^h(c_1^{h,i}, c_2^{h,ij}) - \lambda^{hij} p^j &= 0, && \text{for all } i, j, h \\
p^i(c_1^{h,i} - \hat{e}_1^{h,i}) + p^j(c_2^{h,ij} - \hat{e}_2^{h,j}) &= 0, && \text{for all } i, j, h \\
\sum_{h=1}^{H} (c_1^{h,j} - \hat{e}_1^{h,j} + c_2^{h,ij} - \hat{e}_2^{h,j}) &= 0, && \text{for all } i, j
\end{aligned} \tag{A.1}$$

for some, strictly positive $\left(\lambda^{h,i1}, ..., \lambda^{h,ik}\right)_{i=1}^{k}$.

Lemma A. *Let* $\left\{(c_1^{h1}, c_1^{h2}; c_2^{h1}, c_2^{h2})_{h=1}^H, p^1, p^2\right\}$ *be a stationary equilibrium of a stochastic economy with preferences* $(u^h)_{h=1}^H$*, probabilities described by the Markov matrix* Π *and endowments* $(\hat{e}^h)_{h=1}^H$*, such that, for all* $h = 1, \ldots, H$, $\hat{e}_1^{h1} + \hat{e}_2^{h1} = \hat{e}_1^{h2} + \hat{e}_2^{h2}$ *(i.e., the sum of the endowments is the same in each state). Then the same consumption levels and prices constitute also a stationary sunspot equilibrium of a sunspot economy, with the same preferences* $(u^h)_{h=1}^H$ *and probabilities* Π*, and deterministic endowments* $(\tilde{e}_1^h, \tilde{e}_2^h)_{h=1}^H$ *such that, for each* $h = 1, ..., H$,

$$\tilde{e}_1^h + \tilde{e}_2^h = \hat{e}_1^{h1} + \hat{e}_2^{h1} \tag{A.2}$$

$$p^1 \tilde{e}_1^h + p^2 \tilde{e}_2^h = p^1 \hat{e}_1^{h1} + p^2 \hat{e}_2^{h2}. \tag{A.3}$$

[18] Note that, with heterogeneous agents in each generation, agents' consumption when old typically depends on the realization of the uncertainty in both periods of the agent's lifetime. Feasibility then implies that aggregate consumption when old depends only on the state when old.

Proof. The inspection of the system of equations (A.1) characterizing stationary equilibria reveals that it suffices to show that the endowment values $(\tilde{e}_1^h, \tilde{e}_2^h)_{h=1}^H$ obtained as a solution of (A.2), (A.3) satisfy the last two sets of equations of (A.1), describing the agents' budget constraints and the feasibility conditions, i.e. that

$$p^i c_1^{hi} + p^j c_2^{hj} = p^i \tilde{e}_1^h + p^j \tilde{e}_2^h \qquad \text{for all } i, j, h \tag{A.4}$$

and

$$\sum_{h=1}^{H} (c_1^{hi} + c_2^{hi}) = \sum_{h=1}^{H} (\tilde{e}_1^h + \tilde{e}_2^h) \qquad \text{for all } i, j. \tag{A.5}$$

Evidently, (A.2) implies that (A.5) is always satisfied.

Consider next (A.4):

(1) for all $i = j \in \{1, 2\}$, then

$$\begin{aligned} p^i c_1^{hi} + p^i c_2^{hi} &= p^i \hat{e}_1^{hi} + p^i \hat{e}_2^{hi} \\ &= p^i (\hat{e}_1^{hi} + \hat{e}_2^{hi}) \\ &= p^i (\tilde{e}_1^h + \tilde{e}_2^h) \\ &= p^i \tilde{e}_1^h + p^i \tilde{e}_2^h, \end{aligned} \tag{A.6}$$

where the third equation follows by (A.2).

(2) if $i = 1$ and $j = 2$, then

$$\begin{aligned} p^1 c_1^{h1} + p^2 c_2^{h2} &= p^1 \hat{e}_1^{h1} + p^2 \hat{e}_2^{h2} \\ &= p^1 \tilde{e}_1^h + p^2 \tilde{e}_2^h \end{aligned} \tag{A.7-8}$$

(3) Finally, if $i = 2$ and $j = 1$, we have

$$p^2 c_1^{h2} + p^1 c_2^{h1} = p^2 \hat{e}_1^{h2} + p^1 \hat{e}_2^{h1}.$$

From (A.2), (A.5) it follows that

$$\begin{aligned} p^1 \hat{e}_1^{h1} + p^2 \hat{e}_2^{h2} &= p^1 \tilde{e}_1^h + p^2 (\hat{e}_1^{h1} + \hat{e}_2^{h1} - \tilde{e}_1^h) \\ &= p^1 \tilde{e}_1^h + p^2 (\hat{e}_1^{h2} + \hat{e}_2^{h2} - \tilde{e}_1^h). \end{aligned}$$

where the property $\hat{e}_1^{h1} + \hat{e}_2^{h1} = \hat{e}_1^{h2} + \hat{e}_2^{h2}$ was used to derive the last equality. Rearranging terms and simplifying we get

$$\begin{aligned} p^2 (\hat{e}_1^{h2} - \tilde{e}_1^h) &= -p^1 (\tilde{e}_1^h - \hat{e}_1^{h1}) \\ &= -p^1 (\hat{e}_2^{h1} - \tilde{e}_2^h), \end{aligned} \tag{A.9}$$

where the second equality follows from (A.2). Thus

$$p^2\hat{e}_1^{h2} + p^1\hat{e}_2^{h1} = p^1\tilde{e}_2^h + p^2\tilde{e}_1^h.$$

Hence we have shown that

$$p^2c_1^{h2} + p^1c_2^{h1} = p^2\tilde{e}_1^h + p^1\tilde{e}_2^h$$

as required. Q.E.D.

This Lemma implies then that if $\{(c_1^{h1}, c_1^{h2}; c_2^{h1}, c_2^{h2})_{h=1}^H, p^1, p^2\}$ is a regular, stationary equilibrium of an economy $\{(\hat{e}^h)_{h=1}^H, (u^h)_{h=1}^H, \Pi\}$, such that, for all $h = 1, \ldots, H$, $\hat{e}_1^{h1} + \hat{e}_2^{h1} = \hat{e}_1^{h2} + \hat{e}_2^{h2}$ and, for some h, $\hat{e}_1^{h1} \neq \hat{e}_1^{h2}$, by the Implicit Function Theorem there is a sunspot economy with stationary sunspot equilibria arbitrarily near $\{(c_1^{h1}, c_1^{h2}; c_2^{h1}, c_2^{h2})_{h=1}^H, p^1, p^2\}$.

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