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"Informational Size and Incentive Compatibility without Negligible Aggregate Uncertainty"

by

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Informational Size and Incentive Compatibility without Negligible Aggregate Uncertainty*

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Abstract

In McLean and Postlewaite (2001), we analyzed a general equilibrium model with asymetrically informed agents. We presented a notion of informational size and showed (among other things) that when agents' information as a whole resolved nearly all the uncertainty, the conflict between incentive compatibility and (ex post) efficiency can be made arbitrarily small if agents are sufficiently small informationally. This paper extends the analysis of the relationship between informational size and efficiency to the case in which there is nontrivial aggregate uncertainty, that is, when there is significant uncertainty about the world even when the information of all agents is known. We further show that the conflict between incentive compatibility and efficiency asymptotically vanishes when an economy is replicated.

1 Introduction

It is well understood that, in the presence of asymmetric information, incentive compatibility and Pareto efficiency often conflict: agents may benefit from misrepresenting their private information when that information is to be used in making decisions that affect them. In McLean and Postlewaite (2001), we addressed certain continuity issues when agents have small amounts of information that is not common knowledge. Much of the research in economics ignores the issue of asymmetry of information, implicitly or explicitly assuming that the characteristics of the economic environment are common knowledge among agents even though the assumption that all agents are *identically* informed is implausible. This working assumption greatly simplifies the analysis and is based on a belief that behavior should be continuous with respect to

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the information structure. When the asymmetry of information is "small enough," predicted behavior when asymmetries are ignored should be close to predicted behavior when these asymmetries are properly modeled.

McLean and Postlewaite (2001) formally address this question by introducing a measure of an agent's informational size and show that, in some cases, the unavoidable inefficiencies caused by incentive problems can be made small when agents are sufficiently informationally small. In that paper, two aspects of the informational structure are important. The first aspect is *aggregate uncertainty*: to what extent does the agents' information *in toto* resolve nearly all the uncertainty in the economy? The second is *informational variability* which, very roughly, measures the degree to which an agent's information is correlated with the information of others.[1] McLean and Postlewaite (2001) restrict attention to the case of *negligible aggregate uncertainty* (in which the agents' information resolves nearly all uncertainty) and show that incentive compatible, individually rational and nearly ex post efficient allocations will exist when each agent is sufficiently informationally small relative to his informational variability.[2]

This paper extends the analysis of the relationship between informational size and efficiency in two ways. First, we are able to prove the existence of incentive compatible, individually rational and nearly ex post efficient allocations without assuming negligible aggregate uncertainty (though we do maintain the assumption that agents are small informationally relative to informational variability). We should note that, while this weakens the conditions that are sufficient for the existence of such allocations, it does not extend our earlier results regarding the approximation of certain individually rational Pareto efficient allocations. Second, we prove that, for general exchange economies with asymmetric information, the conflict between incentive compatibility and efficiency asymptotically vanishes when an economy is replicated.

Both this paper and McLean and Postlewaite (2001) are related to Cremer and McLean (1985,1989), and to subsequent work by McAfee and Reny (1992). The possibility of full extraction in mechanism design problems also depends on the correlation of agents' information in a way that is related to, but not precisely the same, as our concept of informational variability. This is discussed in detail when that concept is formally introduced below.

[1] This is defined formally below.

[2] This is actually a corollary of a stronger result shown there that any individually rational Pareto efficient allocation can be approximated by an incentive compatible allocation.

2 Private Information Economies:

The model is similar, but not identical, to that in McLean and Postlewaite (2001). The interested reader will find a detailed discussion of the model and many of the assumptions there.

Throughout the paper, $||\cdot||$ will denote the 1-norm unless specified otherwise. Let $N = \{1, 2, ..., n\}$ denote the set of *economic agents*. Let $\Theta = \{\theta_1, .., \theta_m\}$ denote the (finite) *state space* and let $T_1, T_2, ..., T_n$ be finite sets where T_i represents the set of possible *signals* that agent i might receive. Let $T \equiv T_1 \times \cdots \times T_n$ and $T_{-i} \equiv \times_{j \neq i} T_i$. If $t \in T$, we will often write $t = (t_{-i}, t_i)$. If X is a finite set, define

$$\Delta_X := \{\rho \in \Re^{|X|} | \rho(x) \geq 0, \sum_{x \in X} \rho(x) = 1\}.$$

In our model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable but each agent i receives a "signal" t_i that may be correlated with nature's choice of θ. More formally, let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ with associated distribution $P \in \Delta_{\Theta \times T}$ where

$$P(\theta, t_1, .., t_n) = \text{Prob}\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

Without loss of generality, we will make the following "full support" assumptions regarding the marginal distributions: $\text{supp}(\widetilde{\theta}) = \Theta$ i.e. for each $\theta \in \Theta$,

$$P(\theta) = \text{Prob}\{\widetilde{\theta} = \theta\} > 0$$

and for each $i \in N$, $\text{supp}(\widetilde{t}_i) = T_i$ i.e. for each $t_i \in T_i$,

$$P(t_i) = \text{Prob}\{\widetilde{t}_i = t_i\} > 0.$$

Let $T^* = \text{supp}(\widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n) = \{t \in T | P(t) > 0\}$. We note that T^* need *not* be equal to T. If $t \in T^*$, let $P_\Theta(\cdot|t) \in \Delta_\Theta$ denote the induced conditional probability measure on Θ. Let $\chi_\theta \in \Delta_\Theta$ denote the degenerate measure concentrated on state θ.

The *consumption set* of each agent is $\Re^\ell_+$ and $w_i \in \Re^\ell_+$ denotes the *initial endowment* of agent i (an agent's initial endowment is independent of the state θ). For each $\theta \in \Theta$, let $u_i(\cdot, \theta) : \Re^\ell_+ \to \Re$ be the utility function of agent i in state θ. We note that this formulation differs from the more standard formulation that specifies utility functions of the form $\tilde{u}_i(\cdot, t)$, where t is the profile of agents' types. Clearly, our formulation is without loss of generality, since one can always define the space $\Theta \equiv T$. The formulation $u_i(\cdot, \theta)$ that we have chosen is advantageous for our purposes in that it focuses attention on the manner in which other agents' types affect agent i's utility. The utility of i for any bundle of goods x depends only on the state of nature θ, and the information of other agents affects i only through the way that information changes the likelihood of different states of nature. This focus simplifies our formulation of informational size below.

Throughout the paper, we make the following assumptions regarding utilities and endowments for each agent $i \in N$:

Continuity: For each $\theta \in \Theta$, $u_i(\cdot, \theta)$ is continuous.

Monotonicity: if $x, y \in \Re^{\ell}_{+}$, $x \geq y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.

Normalization: $u_i(0, \theta) = 0$

Nonzero endowment: $w_i \neq 0$.

Each $\pi \in \Delta_{\Theta}$ can be associated with a pure exchange economy in which each agent's utility for any bundle x is the expected utility of that bundle given the distribution π on Θ. More formally, the *expected economy corresponding to* π (expected economy for short) is the pure exchange economy in which agent i has endowment w_i and utility

$$v_i(x, \pi) := \sum_{\theta \in \Theta} u_i(x, \theta)\pi(\theta).$$

The expected economy corresponding to π will be denoted $e(\pi) = \{w_i, v_i(\cdot, \pi)\}_{i \in N}$ and we will define $e(\chi_{\theta}) := e(\theta)$. For each $\pi \in \Delta_{\Theta}$, an allocation for $e(\pi)$ is a collection $\{x_i(\pi)\}_{i \in N}$ satisfying $x_i(\pi) \in \Re^{\ell}_{+}$ for each i and $\sum_{i \in N}(x_i(\pi) - w_i) \leq 0$. For each $\pi \in \Delta_{\Theta}$, an allocation $\{x_i(\pi)\}_{i \in N}$ for the expected economy $e(\pi)$ is *efficient* if there is no other allocation $\{y_i(\pi)\}_{i \in N}$ for $e(\pi)$ such that

$$v_i(y_i(\pi), \pi) > v_i(x_i(\pi), \pi)$$

for each $i \in N$. For each $\varepsilon \geq 0$, an allocation $\{x_i(\pi)\}_{i \in N}$ for the expected economy $e(\pi)$ is $\varepsilon-$*efficient* if there is no other allocation $\{y_i(\pi)\}_{i \in N}$ for $e(\pi)$ such that

$$v_i(y_i(\pi), \pi) > v_i(x_i(\pi), \pi) + \varepsilon$$

for each $i \in N$.

The collection $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ will be called a *private information economy* (PIE for short). An PIE allocation $z = (z_1, z_2, ..., z_n)$ for the PIE is a collection of functions z_i: $T \rightarrow \Re^{\ell}_{+}$ satisfying $\sum_{i \in N}(z_i(t) - w_i) \leq 0$ for all $t \in T$. We will not distinguish between $w_i \in \Re^{\ell}_{+} \backslash \{0\}$ and the constant allocation that assigns the bundle w_i to agent i for all $t \in T$.

Recall that $P_{\Theta}(\cdot \mid t) \in \Delta_{\Theta}$ denotes the conditional distribution on Θ given $t \in T^{*}$. Given a PIE $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$, we can define a natural expected economy $e(t) := e(P_{\Theta}(\cdot \mid t))$ for each $t \in T^{*}$. In this notation, every PIE allocation z for the PIE $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ induces an allocation $z(t)$ in the expected economy $e(t)$ for each $t \in T^{*}$. Note that $e(t)$ depends on P and we are suppressing this dependence.

A PIE allocation z for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ is:

(*incentive compatible)* (IC) if

$$\sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} u_i(z_i(t_{-i},t_i),\theta)P(\theta,t_{-i}\mid t_i) \geq \sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} u_i(z_i(t_{-i},t_i'),\theta)P(\theta,t_{-i}\mid t_i)$$

for all $i\in N$, and all $t_i, t_i' \in T_i$.

(*ex post individually rational)* (XIR) if $z(t)$ is individually rational in $e(t)$ for all $t\in T^*$.

(*ex post* $\varepsilon-$*efficient*)($X_\varepsilon E$) if $z(t)$ is $\varepsilon-$efficient in $e(t)$ for all $t\in T^*$.

Note that allocations can depend on agents' types (their information) but not on θ, which is assumed to be unobservable. Hence, our use of the term "ex post" refers to events that occur *after* the realization of the signal vector t but *before* the realization of the state θ.

3 Nonexclusive Information, Incentive Compatibility and Ex Post Efficiency

In this section, we adress the following question: when can we find a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying IC, XIR and X_0E? To illustrate the ideas, we begin with two examples.

Example 1: **Independence**

Suppose that $\tilde{\theta}$ and $\tilde{t}$ are independent. Let P_Θ denote the marginal of P on Θ. Choose an allocation $\{\overline{x}_i\}_{i\in N}$ that is individually rational and Pareto efficient for the expected economy $e(P_\Theta)$ with utilities

$$v_i(x_i;P_\Theta) = \sum_{\theta\in\Theta} u_i(x_i,\theta)P_\Theta(\theta)$$

and endowments w_i. If we define an allocation $z(\cdot)$ for the PIE as

$$\begin{aligned} z_i(t) &= \overline{x}_i \text{ if } t\in T^* \\ &= w_i \text{ if } t\notin T^* \end{aligned}$$

then $z(\cdot)$ is XIR, X_0E and IC. Note that IC is trivial in this case; when all agents are announcing their types truthfully, the only possible effect that an agent can have on the outcome by misreporting his type is to change the resulting allocation from $\overline{x}$ to w. This cannot increase any agent's utility since $\overline{x}$ is individually rational.

Example 2: **Perfect Correlation**

Suppose that

$$T_i = \Theta = \{\theta_1, .., \theta_m\}$$

for each i and that, for each k,

$$\begin{aligned} P(\theta_k, t) &= P_\Theta(\theta_k) \text{ if } t = (\theta_k, .., \theta_k) \\ P(\theta_k, t) &= 0 \text{ if } t \neq (\theta_k, .., \theta_k). \end{aligned}$$

In words, each agent learns the state of nature θ precisely. Hence,

$$T^* = \{(\theta_k, .., \theta_k)\}_{k=1}^m.$$

and

$$P_\Theta(\cdot|t) = \chi_{\theta_k} \text{ if } t = (\theta_k, .., \theta_k).$$

For each k, choose an efficient, individually rational allocation $\{x_i(\theta_k)\}_{i\in N}$ for the (degenerate) expected economy $e(\theta_k)$. If we define an allocation $z(\cdot)$ for the PIE as

$$\begin{aligned} z_i(t) &= x_i(\theta_k) \text{ if } t = (\theta_k, .., \theta_k) \\ &= w_i \text{ if } t \notin T^* = \{(\theta_k, .., \theta_k)\}_{k=1}^m \end{aligned}$$

then $z(\cdot)$ is XIR, X_0E and IC. As in the first example, incentive compatibility follows from the fact that, whenever all other agents are announcing truthfully, the only possible effect that an agent can have on the outcome by misreporting his type is to change the resulting allocation from $\overline{x}$ to w.

The two examples presented above are special cases of the more general concept of *nonexclusive information* (Postlewaite and Schmeidler (1986)).

Definition: A measure $P \in \Delta_{\Theta\times T}$ satisfies *nonexclusive information* (NEI) if

$$t \in T^* \Rightarrow P_\Theta(\cdot|t) = P_\Theta(\cdot|t_{-i}) \text{ for all } i \in N.$$

The following (easily proved) lemma provides a simple but useful characterization of NEI.

Lemma 1: : The following are equivalent:
$P \in \Delta_{\Theta\times T}$ satisfies NEI if and only if, for each $i \in N$ and for all $t_i, t_i' \in T_i$,

$$[(t_{-i}, t_i) \in T^* \text{ and } (t_{-i}, t_i') \in T^*] \Rightarrow P_\Theta(\cdot|t_{-i}, t_i) = P_\Theta(\cdot|t_{-i}, t_i').$$

Examples 1 and 2 above demonstrate that for two particular instances in which NEI holds, there exist incentive compatible, ex post individually rational and ex post efficient mechanisms. The logic of those examples can be generalized, and we show next that if P satisfies NEI, then we can always find incentive compatible, ex post individually rational, ex post efficient mechanisms.

Proposition 1: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies and suppose that $P \in \Delta_{\Theta\times T}$ satisfies NEI. Then there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC and X_0E.

Proof: Let $z(\cdot)$ be a PIE allocation for $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying (i) $z(t)$ is an efficient, individually rational allocation for the expected economy $e(t)$ for each $t \in T^*$, (ii) $z(t) = z(\hat{t})$ if $t, \hat{t} \in T^*$ and $P_\Theta(\cdot|t) = P_\Theta(\cdot|\hat{t})$, and (iii) $z(t) = (w_1, .., w_n)$ if $t \notin T^*$.

Clearly, the PIE allocation $z(\cdot)$ is XIR and X_0E. To prove incentive compatibility, note that

$$\sum_{\theta\in\Theta} \sum_{t_{-i}\in T_{-i}} [u_i(z_i(t_{-i}, t_i), \theta) - u_i(z_i(t_{-i}, t_i'), \theta)]\, P(\theta, t_{-i} \mid t_i)$$

$$= \sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \sum_{\theta\in\Theta} [v_i(z_i(t_{-i}, t_i), P_\Theta(\cdot|t)) - v_i(z_i(t_{-i}, t_i'), P_\Theta(\cdot|t))]\, P(t_{-i} \mid t_i)$$

$$+ \sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \sum_{\theta\in\Theta} [v_i(z_i(t_{-i}, t_i), P_\Theta(\cdot|t)) - v_i(z_i(t_{-i}, t_i'), P_\Theta(\cdot|t))]\, P(t_{-i} \mid t_i)$$

$$= \sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \sum_{\theta\in\Theta} [v_i(z_i(t_{-i}, t_i), P_\Theta(\cdot|t)) - v_i(w_i, P_\Theta(\cdot|t))]\, P(t_{-i} \mid t_i)$$

$$\geq 0.$$

4 Informational Size and Variability of Beliefs

4.1 Informational Size

In the mechanism of Proposition 1, agents reveal their types and the announced type profile t is used to construct an updated probability $P_\Theta(\cdot|t)$ distribution on Θ. The mechanism then specifies an efficient, individually rational allocation for the economy $e(P_\Theta(\cdot|t))$.The mechanism is incentive compatible because each agent i is "informationally small" in the following sense: when agents other than i announce truthfully, there is no residual uncertainty about the state that can be resolved using i's announcement. In other words, i's information is irrelevant if all other agents are announcing truthfully.

To investigate these issues in a more general framework, we need to formalize the idea of *informational size*. If $t \in T^*$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ and $\chi_\theta \in \Delta_\Theta$ denotes the degenerate measure concentrated on θ. A natural notion of an agent's informational size is the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t_i', this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t_i'. We formalize this in the following definition.

Definition: Let

$$I_\varepsilon^i(t_i', t_i) = \{t_{-i} \in T_{-i} | (t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^* \text{ and } ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| > \varepsilon\}$$

The *informational size* of agent i is defined as

$$\nu_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \inf\{\varepsilon > 0 | \text{ Prob}\{\tilde{t}_{-i} \in I_\varepsilon^i(t_i', t_i) | \tilde{t}_i = t_i\} \leq \varepsilon\}$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν_i^P is "small." If agent i receives signal t_i but reports $t_i' \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t_i')$. If $t_{-i} \in I_\varepsilon(t_i', t_i)$, then this change is "large" in the sense that $||P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t_i')|| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon(t_i', t_i) | \tilde{t}_i = t_i\}$ is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t_i' instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type.

If all agents have zero informational size, then P must satisfy NEI. In fact, we have the following result which follows easily from Lemma 1.

Proposition 2: $P \in \Delta_{\Theta \times T}$ satisfies NEI if and only if $\nu_i^P = 0$ for each $i \in N$.

We conclude this section with the observation that informational size is not related to the "quality" of an agent's information regarding the state of nature. In example 1, an agent's private signal provides no information regarding the realization of $\tilde{\theta}$ (since $P_\Theta(\cdot|t_i) = P_\Theta(\cdot)$ for each $t_i \in T_i$) while in example 2, an agent's private signal provides perfect information regarding the realization of $\tilde{\theta}$ (since $P_\Theta(\cdot|t_i) = \chi_{t_i}$ for each $t_i \in T_i$). Hence, agents may have very good estimates of the true state conditional on their own types, yet each agent is informationally small.

4.2 Variability of Agents' Beliefs

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot|t_i)$ and $P_{T_{-i}}(\cdot|t_i')$, the conditional distributions on T_{-i} given different types t_i and t'_i for agent i. Formally, we define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t'_i \in T_i \setminus t_i} ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t'_i)||^2$$

and we will refer to this informally as the *variability of agents' beliefs.*

As mentioned in the introduction, our work is related to that of Cremer and McLean (1985,1989). Those papers, and subsequent work by McAfee and Reny (1992), demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems. The key ingredient there is the assumption that the collection of conditional distributions $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is a linearly independent set for each i (where $P_{T_{-i}}(\cdot|t_i)$ is the conditional distribution on T_{-i} given t_i). This of course, implies that $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t'_i)$ if $t_i \neq t'_i$ and, therefore, that $\Lambda_i^P > 0$. While linear independence implies that $\Lambda_i^P > 0$, the actual (positive) size of Λ_i^P is not relevant in the Cremer-McLean constructions, and full extraction will be possible. In the present work, we do not require that the collection $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ be linearly independent (or satisfy the weaker cone condition in Cremer and McLean (1988)). However, the "closeness" of the members of $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $z_i(t)$ satisfying $0 \leq z_i(t) \leq 1$ and

$$\sum_{t_{-i}} [z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i)] \, P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t'_i \in T_i$ if and only if $\Lambda_i^P > 0$. This means that, if the posteriors $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then the "incentive compatibility" inequalities above are strict. That is, the difference in the expected reward from a truthful report and from a false report will be positive. However, this difference decreases as $\Lambda_i^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Our results require that informational size and aggregate uncertainty be small relative to the variation in these posteriors.

Small incentives for truthful reporting (i.e., small values of Λ_i^P) are not a serious problem in the surplus extraction problem studied by Cremer and McLean since the rewards and punishments can be rescaled so that a false report results in a large negative expected payment. Of course, the punishments themselves may then become very large. However, such rescaling is not possible in our framework for two reasons. First, we deal with pure exchange economies where the feasibility requirement limits the size of punishments. Second, we do not restrict attention to quasilinear preferences. Since agents may be risk averse, punishments and rewards that have small (or zero) expected value can have large negative welfare effects.

4.3 An Example

We will illustrate the ideas of variability and informational size with a simple example. The example will be parametrized by m, a positive integer, and for each m we will completely specify the information structure. This information structure will have the property that variability will be independent of m, and agents' informational size will be arbitrarily small for sufficiently large m.

There are 3 agents and two possible states of the world, θ_1 and θ_2. Fix a positive integer $m > 3$ and define each agent's type set as $T_i = \{\frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}\}$. Let $Q := \{\frac{1}{m}, \frac{2}{m}, ..., \frac{m-3}{m}\}$ and for each $q \in Q$, let $R(q)$ be the set of the six possible orderings of the numbers $\{q, q + \frac{1}{m}, q + \frac{2}{m}\}$. Next, define $P \in \Delta_{\Theta \times T}$ as follows:

$$\begin{aligned} P(\theta_1, t_1, t_2, t_3) &= \frac{q}{6(m-3)} \text{ if } (t_1, t_2, t_3) \in R(q) \\ P(\theta_2, t_1, t_2, t_3) &= \frac{1-q}{6(m-3)} \text{ if } (t_1, t_2, t_3) \in R(q) \\ P(\theta, t_1, t_2, t_3) &= 0 \text{ if } (t_1, t_2, t_3) \notin \cup_{q \in Q} R(q). \end{aligned}$$

Therefore,

$$T^* = \cup_{q \in Q} R(q),$$

so

$$\begin{aligned} P(t_1, t_2, t_3) &= \frac{1}{6(m-3)} \text{ if } (t_1, t_2, t_3) \in \cup_{q \in Q} R(q) \\ &= 0 \text{ if } (t_1, t_2, t_3) \notin \cup_{q \in Q} R(q). \end{aligned}$$

and

$$P_\Theta(\theta_1 | t_1, t_2, t_3) = q \text{ if } (t_1, t_2, t_3) \in R(q).$$

Finally,

$$\begin{aligned} P(t_i) &= \frac{1}{3(m-3)} \text{ if } t_i \in \{\frac{1}{m}, \frac{m-1}{m}\} \\ P(t_i) &= \frac{2}{3(m-3)} \text{ if } t_i \in \{\frac{2}{m}, \frac{m-2}{m}\} \\ P(t_i) &= \frac{1}{(m-3)} \text{ if } t_i \in \{\frac{3}{m}, .., \frac{m-3}{m}\}. \end{aligned}$$

Defining $\text{Per}(x, y) = \{(x, y), (y, x)\}$, it follows that

$$\begin{aligned} P(t_2, t_3 | t_1 &= \frac{1}{m}) = \frac{1}{2} \text{ if } (t_2, t_3) \in \text{Per}(\frac{2}{m}, \frac{3}{m}) \\ &= 0 \text{ otherwise} \end{aligned}$$

$$
\begin{aligned}
P(t_2, t_3|t_1 &= \frac{2}{m}) = \frac{1}{4} \text{ if } (t_2, t_3) \in \text{ Per}(\frac{1}{m}, \frac{3}{m}) \cup \text{Per}(\frac{3}{m}, \frac{4}{m}) \\
&= 0 \text{ otherwise}
\end{aligned}
$$

$$
\begin{aligned}
P(t_2, t_3|t_1 &= \frac{3}{m}) = \frac{1}{6} \text{ if } (t_2, t_3) \in \text{ Per}(\frac{1}{m}, \frac{2}{m}) \cup \text{Per}(\frac{2}{m}, \frac{4}{m}) \cup \text{Per}(\frac{4}{m}, \frac{5}{m}) \\
&= 0 \text{ otherwise}
\end{aligned}
$$

and so on.

We claim that agents are informationally small if m is large. To see this, suppose that $(t_{-i}, t_i) \in T^*$. Then (t_{-i}, t_i) is a permutation of $\{\frac{j}{m}, \frac{j+1}{m}, \frac{j+2}{m}\}$ for some $j \in \{1, .., m-3\}$. Suppose that $t_i' \neq t_i$ and $(t_{-i}, t_i') \in T^*$. This means that $t_i = \frac{j}{m}$ or $\frac{j+2}{m}$. If $t_i = \frac{j}{m}$, then (t_{-i}, t_i') must be a permutation of $\{\frac{j+1}{m}, \frac{j+2}{m}, \frac{j+3}{m}\}$. If $t_i = \frac{j+2}{m}$, then (t_{-i}, t_i') must be a permutation of $\{\frac{j-1}{m}, \frac{j}{m}, \frac{j+1}{m}\}$. (The first cannot be a deviation in the case that $j = m-3$ and the second cannot be a deviation when $j = 1$). Therefore,

$$
(t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^* \Rightarrow ||P(\cdot|t_{-i}, t_i) - P(\cdot|t_{-i}, t_i')|| \leq \frac{1}{m}
$$

and it follows that $\nu_i^P = \frac{1}{m}$.

We also note that there exists a positive number λ such that $\Lambda_i^P > \lambda$ for each i since $P_{T_{-i}}(\cdot|t_i)$ is independent of m and $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t_i')$ whenever $t_i \neq t_i'$.

Summarizing, the example has the following properties:

1. For sufficiently large m, agents will be arbitrarily small informationally.
2. Informational variability is uniformly bounded away from 0 for all m.
3. The distribution P does not exhibit negligible aggregate uncertainty, even as $m \to \infty$.

5 Informational Size, Incentive Compatibility and Approximate Ex Post Efficiency

5.1 Preliminaries and the Main Approximation Lemma

Let $\{e(\theta)\}_{\theta \in \Theta}$ be a collection of (degenerate) expected economies. For each $\pi \in \Delta_\Theta$, let

$$
\Phi(\pi) = \{(v_1(x_1; \pi), .., v_n(x_n; \pi)) | (x_1, .., x_n) \text{ is feasible for } e(\pi)\}.
$$

That is, $\Phi(\pi)$ is the set of feasible utility vectors for the agents in the expected economy $e(\pi)$. Before proceeding, we will introduce several definitions. A map $f : \Delta_\Theta \to \Re^n$ is a *Lipschitz selection* for Φ if $f(\pi) \in \Phi(\pi)$ for all $\pi \in \Delta_\Theta$ and each f_i is uniformly Lipschitz on Δ_Θ. That is, for each i there exists a $K_i > 0$ such that

$|f_i(\pi) - f_i(\pi')| \leq K_i||\pi - \pi'||$ for each $\pi, \pi' \in \Delta_\Theta$. A selection f for Φ is *positive* if $f_i(\pi) > 0$ for all $\pi \in \Delta_\Theta$ and all $i \in N$.

We will show that certain feasible utilities can be approximated by incentive compatible PIE allocations when agents are informationally small relative to variability. The idea is as follows. Choose a Lipschitz selection $f : \Delta_\Theta \to \Re^n$. Since $f(\pi) = (f_1(\pi), .., f_n(\pi))$ is a feasible utility profile for $e(\pi)$, there exists an allocation $x(\pi) = (x_1(\pi), .., x_n(\pi))$ such that $f(\pi) = (v_1(x_1(\pi); \pi), .., v_n(x_n(\pi); \pi))$. Next, define a PIE allocation $y(\cdot)$ where $y(t) = x(P_\Theta(\cdot|t))$ for each $t \in T^*$ and where $P_\Theta(\cdot|t)$ is the posterior distribution on Θ given t. That is, $y(t)$ is an allocation that generates the desired utility for the distribution $P_\Theta(\cdot|t)$. Of course, $y(\cdot)$ need not be incentive compatible. When agents are informationally small, however, any agent who unilaterally misreports his type can change the posterior by only a small amount. Since the selection f is Lipschitz, the utility change resulting from that agent's misreport will be small. When the variability condition is satisfied, agents' types are correlated in a way that allows us to construct small rewards and punishments for the agents. These rewards have the property that, by truthfully announcing his type, an agent maximizes his utility - including his reward - if other agents are announcing truthfully. Given these rewards and punishments, we can modify the allocation $y(\cdot)$ in a way that the modified allocation will be incentive compatible when informational size is small relative to variability for all agents. The next proposition formalizes this.

Proposition 3: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies and suppose that f is a positive Lipschitz selection for Φ. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying

$$f_i(P_\Theta(\cdot|t)) \geq v_i(z_i(t); P_\Theta(\cdot|t)) \geq f_i(P_\Theta(\cdot|t)) - \varepsilon$$

for each $t \in T^*$ and for all $i \in N$. Moreover,

$$v_i(\zeta_i(t); P_\Theta(\cdot|t)) \geq v_i(z_i(t); P_\Theta(\cdot|t)) \geq v_i(\zeta_i(t); P_\Theta(\cdot|t)) - \varepsilon$$

for any PIE allocation $\zeta(\cdot)$ for $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying $v_i(\zeta_i(t); P(\cdot|t)) = f_i(P_\Theta(\cdot|t))$ for all $t \in T^*$.

Proof: See appendix.

5.2 The Main Result for Economies of Fixed Size

Proposition 3 of the previous section provides conditions on the information structure under which the utilities associated with a Lipschitz selection can be approximated by an incentive compatible PIE allocation. We next prove the existence of a Lipschitz selection that gives rise to an approximately efficient, strictly individually rational, incentive compatible PIE allocation.

In the previous subsection, we defined Φ, the correspondence that associates with every $\pi \in \Delta_\Theta$ the set of feasible utility vectors for the agents in the expected economy $e(\pi)$, and considered Lipschitz selections from that correspondence. We will now be interested in selections f that have the property that $f(\pi)$ is on the frontier of the utility possibility set $\Phi(\pi)$ for the expected economy $e(\pi)$. Toward this end, define

$$\Phi^0(\pi) = \{(v_1(x_1;\pi),..,v_n(x_n;\pi))|(x_1,..,x_n) \text{ is efficient and IR for } e(\pi)\}.$$

The following defines the restriction to strictly individually rational outcomes.

Definition: A selection $f : \Delta_\Theta \to \Re^n$ is strictly individually rational if $f_i(\pi) > v_i(w_i;\pi)$ for all $\pi \in \Delta_\Theta$ and all $i \in N$.

The monotonicity assumption and the nonzero endowment assumption imply that every strictly individually rational selection is positive. We next show that there exist Lipschitz selections that associate with each expected economy $e(\pi)$ a utility vector on the frontier of the feasible set for $e(\pi)$.

Lemma 2: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies. The correspondence $\Phi^0 : \Delta_\Theta \to \Re^n$ admits a positive Lipschitz selection. If the economy $e(\pi)$ has at least one strictly individually rational allocation for each $\pi \in \Delta_\Theta$, then the correspondence $\Phi^0 : \Delta_\Theta \to \Re^n$ admits a strictly individually rational Lipschitz selection.

Proof: See appendix.

Using Lemma 2 and Proposition 3 of the previous section, we can prove the following theorem.

Theorem 1: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies and suppose that $e(\pi)$ has at least one strictly individually rational allocation for each $\pi \in \Delta_\Theta$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC and $\mathrm{X}_\varepsilon E$.

Proof: Applying Lemma 2, let f be a strictly individually rational Lipschitz selection for Φ^0. Defining $g_i(\pi) = f_i(\pi) - v_i(w_i;\pi)$, it follows that there exists a positive number λ such that $f_i(\pi) - v_i(w_i;\pi) = g_i(\pi) \geq \lambda > 0$ for all i and π since each $g_i(\cdot)$ is positive and continuous on the compact set Δ_Θ. Choose $\varepsilon > 0$ and let

$0 < \eta < \min\{\varepsilon, \lambda\}$. Applying Proposition 3, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying

$$f_i(P_\Theta(\cdot|t)) \geq v_i(z_i(t); P_\Theta(\cdot|t)) \geq f_i(P_\Theta(\cdot|t)) - \eta$$

for each $t \in T^*$ and for all $i \in N$. Let $\zeta(\cdot)$ be a PIE allocation for $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying $v_i(\zeta_i(t); P(\cdot|t)) = f_i(P_\Theta(\cdot|t))$ for all $t \in T^*$. Obviously, $\zeta(t)$ is efficient and strictly individually rational in $e(t)$ for each $t \in T^*$. In addition,

$$v_i(\zeta_i(t); P(\cdot|t)) \geq v_i(z_i(t); P(\cdot|t)) \geq v_i(\zeta_i(t); P(\cdot|t)) - \eta$$

for all $t \in T^*$.

Since $v_i(\zeta_i(t); P(\cdot|t)) - v_i(w_i; P(\cdot|t)) = f(P(\cdot|t)) - v_i(w_i; P(\cdot|t)) \geq \lambda$ for each $t \in T^*$, it follows that

$$\begin{aligned} v_i(z_i(t); P(\cdot|t)) - v_i(w_i; P(\cdot|t)) &= v_i(z_i(t); P(\cdot|t)) - v_i(\zeta_i(t); P(\cdot|t)) \\ &\quad + v_i(\zeta_i(t); P(\cdot|t)) - v_i(w_i; P(\cdot|t)) \\ &\geq \lambda - \eta \\ &> 0 \end{aligned}$$

for each i so $z(t)$ is individually rational in $e(t)$.

To show that $z(t)$ is $\varepsilon-$efficient in $e(t)$, suppose instead that there exists an allocation y for $e(t)$ such that $v_i(y_i; P(\cdot|t)) > v_i(z_i(t); P(\cdot|t)) + \varepsilon$ for each i. Then

$$v_i(y_i; P(\cdot|t)) > v_i(z_i(t); P(\cdot|t)) + \varepsilon \geq v_i(\zeta_i(t); P(\cdot|t)) - \eta + \varepsilon > v_i(\zeta_i(t); P(\cdot|t))$$

for all i contradicting the efficiency of $\zeta(t)$ in the expected economy $e(t)$. This completes the proof of Theorem 1.

6 The Replica Problem

In the presence of a large number of agents, we might expect any single agent to be informationally small, and replica economies are a natural framework in which to investigate this conjecture.

6.1 Notation and Definitions:

Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of complete information economies and let $J_r = \{1, 2, ...r\}$. For each positive integer r and each θ, let $e^r(\theta) = \{w_{is}, u_{is}(\cdot, \theta)\}_{(i,s)\in N\times J_r}$ denote the r replication of $e(\theta)$ corresponding to state θ satisfying:

(1) $w_{is} = w_i$ for all $s \in J_r$

(2) $u_{is}(z,\theta) = u_i(z,\theta)$ for all $z \in \Re^{\ell}_{+}, i \in N$ and $s \in J_r$.

For any positive integer r, let $T^r = T \times \cdots \times T$ denote the r-fold Cartesian product and let $t^r = (t^r(1), .., t^r(r))$ denote a generic element of T^r where $t^r(s) = (t^r_1(s).., t^r_n(s)) \in T$. If $P^r \in \Delta_{\Theta \times T^r}$, then $e^r = (\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ is a PIE with nr agents.

6.2 Replica Economies and the Replica Theorem

Definition: A sequence of replica economies $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ is a conditionally independent sequence if there exists a $P \in \Delta_{\Theta\times T}$ such that

(a) For each r, each $s \in J_r$ and each $(\theta, t) \in \Theta \times T$,

$$\text{Prob}\{\widetilde{\theta} = \theta, \tilde{t}^r(s) = t\} = P(\theta, t_1, t_2, ..., t_n)$$

(b) For each r and each θ, the random vectors

$$\tilde{t}^r(1), \tilde{t}^r(2), ..., \tilde{t}^r(r)$$

are independent given $\widetilde{\theta} = \theta$.

(c) For every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in T$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

(d) For each i and each $t_i, t'_i \in T_i$, $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t'_i)$

Thus a conditionally independent sequence is a sequence of PIE's with nr agents containing r "copies" of each agent $i \in N$. Each copy of an agent i is identical, i.e., has the same endowment and the same utility function. Furthermore, the realizations of type profiles across cohorts are independent given the true value of $\widetilde{\theta}$. As r increases each agent is becoming "small" in the economy in terms of endowment, and we will show that each agent is also becoming informationally small. Note that, for large r, an agent may have a small amount of private information regarding the preferences of everyone through his information about $\widetilde{\theta}$.

Theorem 2: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of (degenerate) expected economies and suppose that for each θ, there exists a Pareto efficient, strictly individually rational allocation for the degenerate expected economy $e(\theta)$. Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence and suppose that each $u_i(\cdot;\theta)$ is concave. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$, there exists an allocation z^r for the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ which satisfies IC, XIR and X$_\varepsilon$E.

7 Discussion

7.1 Relation to earlier results

As noted in the introduction, our aim in this paper is to present results that complement those in McLean and Postlewaite (2001). We will now describe in somewhat more detail how the results in this paper relate those in our previous work. Throughout this discussion, we will assume that $T = T^*$.

In McLean and Postlewaite (2001), three concepts play an important role in constructing individually rational, approximately efficient allocations for private information economies: informational size, variability of beliefs and negligible aggregate uncertainty. The definition of informational size in McLean and Postlewaite (2001) is precisely the same as that presented in this paper. However, the definition of variability is different. In McLean and Postlewaite (2001), we defined

$$\Omega_i^P = \min_{t_i \in T_i} \min_{t_i' \in T_i \setminus t_i} ||P_\Theta(\cdot|t_i) - P_\Theta(\cdot|t_i')||^2$$

as our measure of the variability of agents' beliefs. Here, $P_\Theta(\cdot|t_i)$ is the conditional distribution on the state space Θ given $\tilde{t}_i = t_i$.

Furthermore, we required that the information structure exhibit *negligible aggregate uncertainty.* Informally, we say that a probability measure P exhibits *negligible aggregate uncertainty* if, for a set of t's with high probability, $P_\Theta(\cdot|t) \approx \chi_\theta$ for some $\theta \in \Theta$. More formally, define

$$\mu_i^P = \max_{t_i \in T_i} \inf\{\varepsilon > 0 | \text{Prob}\{\tilde{t} \in T \text{ and } ||P_\Theta(\cdot|\tilde{t}) - \chi_\theta|| > \varepsilon \text{ for all } \theta \in \Theta | t_i\} \leq \varepsilon\}$$

We define the *aggregate uncertainty* as $\mu^P \equiv \max_i \mu_i^P$ and we will say that P exhibits negligible aggregate uncertainty if μ^P is small. In this case, each agent knows that, conditional on his own signal, the aggregate information of all agents will, with high probability, provide a good prediction of the true state. The main result of McLean and Postlewaite (2001) may be stated as follows:

Proposition 4: Let $\{e(\theta)\}_{\theta \in \Theta}$ be a collection of (degenerate) expected economies and suppose that there exists a strictly individually rational, efficient allocation for each $e(\theta)$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Omega_i^P \text{ and } \max_i \nu_i^P \leq \delta \min_i \Omega_i^P,$$

there exists an allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying XIR and IC. Furthermore, there exists a set $E \subseteq T^*$ such that $Prob\{\tilde{t} \in E\} \geq 1 - \varepsilon$ and $z(t)$ is $\varepsilon-$efficient in $e(t)$ for all $t \in E$.

Proposition 4 bears a certain similarity to Theorem 1 but differs in several important ways. First, Proposition 4 does not require the existence of an efficient strictly individually rational Lipschitz selection. Instead, Proposition 4 only requires the existence of a collection $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ where $x(\theta)$ is a strictly individually rational, Pareto efficient allocation for $e(\theta)$. Each result requires that informational size be sufficiently small relative to *some* notion of variability of beliefs, but the notions of variability are defined with respect to different objects. In Theorem 1, these are beliefs regarding the types of other agents, while in Proposition 4, these are beliefs regarding the state of nature. In addition, Proposition 4 requires that the measure of aggregate uncertainty be sufficiently small relative to variability while Theorem 1 does not involve aggregate uncertainty in any way. Finally, the conclusion of Theorem 1 is stronger. The incentive compatible, individually rational PIE z constructed in Theorem 1 is $\varepsilon-$efficient in $e(t)$ for **all** $t \in T^*$ while the incentive compatible, individually rational PIE z constructed in Proposition 4 is $\varepsilon-$efficient in $e(t)$ only for $t \in E$, where $E \subseteq T^*$ has probability close to one.

Since Proposition 4 uses negligible aggregate uncertainty but does not require the existence of a Lipschitz selection, it is useful to outline the method of proof for Proposition 4. Let $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ be a collection where $x(\theta)$ is a strictly individually rational, Pareto efficient allocation for $e(\theta)$. In the presence of negligible aggregate uncertainty, we can partition T into $m+1$ disjoint sets with $A_k = \{t \in T | P(\cdot|t) \approx \chi_{\theta_k}\}$ for $k = 1, ..., m$, and $A_0 = T\backslash[\cup_{k\geq 1} A_k]$. In words, A_k is the set of $t \in T$ for which the posterior distribution on Θ is close to the degenerate distribution χ_{θ_k} that puts probability 1 on θ_k. Therefore, A_0 is the set of $t \in T$ for which the posterior is not close to χ_θ for any θ.

We next choose a PIE allocation $y(\cdot)$ with $y(t) = x(\theta_k)$ for $t \in A_k$, $k = 1, ..., m$, and $y(t) = w$ (the initial endowment) for $t \in A_0$. When aggregate uncertainty is small, the information profile $t \in T$ will, with high probability, resolve most of the uncertainty regarding the state of nature θ. There are two consequences of small aggregate uncertainty: the event A_0 has small probability and for each $t \in A_k$, $P_\Theta(\theta_k|t)$ is close to 1. Since $y(t)$ is efficient for the economy $e(\theta_k)$, it follows that $\Sigma_\ell u_i(y_i(t); \theta_\ell)P_\Theta(\theta_\ell|t)$ is close to $u_i(x_i(\theta_k); \theta_k)$ whenever $t \in A_k$. As a result, the PIE allocation $y(\cdot)$ is approximately efficient for most realizations of the signal vector.

The PIE $y(\cdot)$ as constructed is not incentive compatible in general. Suppose that i receives signal t_i and the other agents truthfully report t_{-i}. It may be the case that $(t_{-i}, t_i) \in A_k$ while $(t_{-i}, t_i') \in A_j$ for $j \neq k$. Hence, i receives $x_i(\theta_k)$ if he reports t_i, while he receives $x_i(\theta_j)$ if he reports t_i'. If $x_i(\theta_j)$ results in higher utility than $x_i(\theta_k)$, agent i may have an incentive to misreport. To say that agent i is informationally small means that there is (at most) a low probability that the posteriors on Θ given (t_{-i}, t_i) and (t_{-i}, t_i') put probability close to 1 on θ_k and θ_j, respectively. Thus, if an agent's informational size is small, the expected gain to that agent from a misreported type is also small. In order to offset this (small) potential gain that i might receive from misreporting, we modify the bundle $x_i(\theta_k)$ that i receives when $t \in A_k$. If the difference between $P(\theta_k|t_i)$ and $P(\theta_k|t_i')$ is sufficiently large for different types t_i and

t_i' , relative to informational size and aggregate uncertainty, then we can construct an allocation $z(t)$ by slightly modifying each $y(t)$ to ensure incentive compatibility. The final allocation $z(\cdot)$ will be be incentive compatible and approximately efficient for nearly all t.

We follow a different approach in the current paper. Instead of choosing a collection $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ where $x(\theta)$ is a strictly individually rational, Pareto efficient allocation for $e(\theta)$,we begin with a strictly individually rational Lipschitz selection f from the correspondence Φ^0. We can "invert" this Lipschitz selection to generate a mapping ζ from Δ_Θ into allocations such that, for any $\pi \in \Delta_\Theta$, $\zeta(\pi)$ is a strictly individually rational, Pareto efficient for the expected economy $e(\pi)$. In the current paper, the mapping $\zeta(\cdot)$ plays a role analogous to that of the collection $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ in our previous work, and it is important to point out that $\zeta(\cdot)$ need not even be continuous on Δ_Θ. For any announced profile of types t, we begin with the allocation $\zeta(P_\Theta(\cdot|t))$, and then modify it. When agents are informationally small, any agent who unilaterally misreports his type will change the posterior distribution on Θ by only a small amount, and hence, change his resulting utility by only a small amount since his *utility* depends on $P_\Theta(\cdot|t)$ in a Lipschitzian manner. As long as variability is sufficiently large relative to informational size, the requisite modifications to $\zeta(P_\Theta(\cdot|t))$ can be made that ensure incentive compatibility.

We conclude with the following observation. If we make the Lipschitz assumption of Theorem 1, then we can prove a variant of Theorem 1 that uses the notion of variability found in Proposition 4. Formally, we have:

Proposition 5: Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of complete information economies and suppose that $e(\pi)$ has at least one strictly individually rational allocation for each $\pi \in \Delta_\Theta$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \nu_i^P \le \delta \min_i \Omega_i^P$$

there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC and X_εE.

Proof: See appendix.

7.2 Lipschitz selections

There set of Lipschitz selections of Φ^0 is large. In the proof of Lemma 2, we demonstrate the existence of one such selection by showing that the solution to a particular programming problem yields a selection that is both efficient for each probability distribution in Δ_Θ, and Lipschitz. In that problem, we essentially maximize the minimum utility increase any agent receives relative to his initial endowment, when agent i's utility is measured along the ray $w_i + e$ (e is the vector of ones in $\Re_+^\ell$). Had we measured utility along a different ray, say $w_i + p$ where $p_i > 0$ for each i, then the associated maximization problem would have generated a different selection, but one that was still efficient and Lipschitz. It would be interesting to characterize the set of all such "weighted" selections, but that is beyond the scope of the present paper.

8 Bibliography

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9 Appendix: Proofs

9.1 Proof of Proposition 3

Choose $\varepsilon > 0$. In the next three steps, we will construct a PIE allocation. Following the construction of the allocation, we will show that it has the desired properties.

Step 1: Let f be a positive Lipschitz selection for Φ and let $K_i > 0$ denote the modulus of continuity of f_i. Since each $f_i(\cdot)$ is continuous on Δ_Θ and positive on Δ_Θ, and since Δ_Θ is compact, it follows that there exists a $\lambda > 0$ such that $f_i(\pi) \geq \lambda$ for all i and π. For each $\pi \in \Delta_\Theta$, choose an allocation $\{\zeta_i(\pi)\}_{i \in N}$ for $e(\pi)$ satisfying

$$v_i(\zeta_i(\pi); \pi) = f_i(\pi)$$

for each i and note that each $\zeta_i(\pi) \neq 0$. Define

$$M_i = u_i(\sum_{j \in N} w_j; \theta).$$

Suppose that

$$0 < \eta < \min\{\lambda, \varepsilon\}.$$

Then for each i and π there exists $\beta_i(\pi)$ such that $0 < \beta_i(\pi) < 1$ and

$$v_i(\zeta_i(\pi); \pi) - v_i(\beta_i(\pi)\zeta_i(\pi); \pi) = \eta.$$

To see this, define

$$\psi(\beta) = v_i(\zeta_i(\pi); \pi) - v_i(\beta\zeta_i(\pi); \pi).$$

Note that

$$\psi(1) = 0 < \eta$$

and

$$\psi(0) = v_i(\zeta_i(\pi); \pi) = f_i(\pi) \geq \lambda > \eta.$$

The continuity of $v_i(\cdot; \pi)$ implies that $\psi(\cdot)$ is continuous and the result follows.

Step 2: Suppose that $P \in \Delta_{\Theta \times T}$ with conditionals $P_{T_{-i}}(\cdot|t_i) \in \Delta_{T_{-i}}$ for all i and $t_i \in T_i$. Next, define

$$\alpha_i(t_{-i}|t_i) = \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}$$

for each $t \in T$ and note that

$$0 \leq \alpha_i(t_{-i}|t_i) \leq 1.$$

For each π, i and t, there exists a number $\tau_i(t_{-i}, t_i) \geq 0$ such that

$$v_i((1 + \tau_i(t_{-i}, t_i))\beta_i(\pi)\zeta_i(\pi); \pi) - v_i(\beta_i(\pi)\zeta_i(\pi); \pi) = \eta\alpha_i(t_{-i}|t_i).$$

[This is possible because $\beta_i(\pi)\zeta_i(\pi) \neq 0$]. Furthermore, $(1+\tau_i(t_{-i},t_i))\beta_i(\pi) \leq 1$. [If $(1+\tau_i(t_{-i},t_i))\beta_i(\pi) > 1$, then monotonicity implies that

$$\begin{aligned} v_i((1+\tau_i(t_{-i},t_i))\beta_i(\pi)\zeta_i(\pi);\pi) - v_i(\beta_i(\pi)\zeta_i(\pi);\pi) &> v_i(\zeta_i(\pi);\pi) - v_i(\beta_i(\pi)\zeta_i(\pi);\pi) \\ &= \eta \\ &\geq \eta\alpha_i(t_{-i}|t_i) \end{aligned}$$

a contradiction.]

Defining

$$x_i(\pi|t) = (1+\tau_i(t))\beta_i(\pi)\zeta_i(\pi)$$

it follows that the collections $\{x_i(\pi|t)\}_{\pi\in\Delta_\Theta, t\in T}$ satisfy

$$x_i(\pi|t) \in \Re^\ell_+ \text{ for each } i \text{ and } \sum_{i\in N}(x_i(\pi|t) - w_i) \leq 0.$$

Furthermore,

$$v_i(x_i(\pi|t_{-i},t_i);\pi) - v_i(\beta_i(\pi)\zeta_i(\pi);\pi) = \eta\alpha_i(t_{-i}|t_i)$$

for all $t \in T$. Therefore,

$$v_i(x_i(\pi|t_{-i},t_i);\pi) = v_i(\zeta_i(\pi);\pi) + \eta\alpha_i(t_{-i}|t_i) - \eta$$

and

$$v_i(\zeta_i(\pi);\pi) \geq v_i(x_i(\pi|t_{-i},t_i);\pi) = v_i(\beta_i(\pi)\zeta_i(\pi);\pi) + \eta\alpha_i(t_{-i}|t_i) \geq v_i(\zeta_i(\pi);\pi) - \eta.$$

Step 3: For each $t \in T^*$, let $q(t) = P_\Theta(\cdot|t)$ and define a PIE alocation $z(\cdot)$ as follows:

$$\begin{aligned} z_i(t) &= x_i(q(t)|t) \text{ if } t \in T^* \\ z_i(t) &= 0 \text{ if } t \notin T^* \end{aligned}$$

We will now prove that $z(\cdot)$ has the desired properties.

Claim 1: $z(\cdot)$ is a PIE allocation.

Proof: This follows from the observation that

$$x_i(\pi|t) \in \Re^\ell_+ \text{ and } \sum_{i\in N}(x_i(\pi|t) - w_i) \leq 0$$

for every $\pi \in \Delta_\Theta$ and $t \in T$.

Claim 2: For each $t \in T^*$

$$f_i(P_\Theta(\cdot|t)) \geq v_i(z_i(t);P_\Theta(\cdot|t)) \geq f_i(P_\Theta(\cdot|t)) - \varepsilon.$$

Proof: This is an immediate consequence of the definition of $z_i(t)$ and the assumption that $\eta < \varepsilon$.

Claim 3: Let $M = \max_i M_i, K = \max_i K_i$ and $B = \frac{|T|^{-\frac{5}{2}}}{2}$ and choose δ so that

$$0 < \delta < \min\{\frac{B\eta}{3(K+M)}\}.$$

If

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

then $z(\cdot)$ satisfies IC.

Proof: *Part 1*: Since

$$v_i(x_i(\pi|t_{-i},t_i);\pi) = v_i(\zeta_i(\pi);\pi) + \eta\alpha_i(t_{-i}|t_i) - \eta$$

for each $\pi \in \Delta_\Theta$ and each $(t_{-i},t_i) \in T$, it follows that

$$v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) = v_i(\zeta_i(q(t_{-i},t_i));q(t_{-i},t_i)) + \eta\alpha_i(t_{-i}|t_i) - \eta$$

and

$$v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i');q(t_{-i},t_i')) = v_i(\zeta_i(q(t_{-i},t_i'));q(t_{-i},t_i')) + \eta\alpha_i(t_{-i}|t_i') - \eta$$

whenever $(t_{-i},t_i),(t_{-i},t_i') \in T^*$. Therefore,

$$\begin{aligned} & v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) - v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i'));q(t_{-i},t_i')) \\ = \; & v_i(\zeta_i(q(t_{-i},t_i));q(t_{-i},t_i)) - v_i(\zeta_i(q(t_{-i},t_i'));q(t_{-i},t_i')) \\ & +\eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) \\ \geq \; & \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) - K_i||P_\Theta(\cdot|t_{-i},t_i)) - P_\Theta(\cdot|t_{-i},t_i')|| \end{aligned}$$

Part 2: Applying the conclusion of Part 1, it follows that

$$\begin{aligned} & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \left[v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) - v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i');q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\ \geq & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \left[\eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) - K_i||P(\cdot|t_{-i},t_i)) - P(\cdot|t_{-i},t_i')||\right] P(t_{-i}|t_i) \\ \geq & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) - 3K_i\nu_i^P \end{aligned}$$

Part 3: The normalization assumption implies that $v_i(0; q(t)) = 0$ for each $t \in T^*$. Therefore,

$$\begin{aligned}
& \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \left[v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i); q(t_{-i},t_i)) - v_i(0; q(t_{-i},t_i)) \right] P(t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \left(v_i(\zeta_i(q(t_{-i},t_i)); q(t_{-i},t_i)) + \eta\alpha_i(t_{-i}|t_i) - \eta \right) P(t_{-i}|t_i) \\
\geq & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \left(\eta\alpha_i(t_{-i}|t_i) - \eta + \lambda \right) P(t_{-i}|t_i) \\
\geq & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i}|t_i) P(t_{-i}|t_i)
\end{aligned}$$

Part 4: Finally,

$$\begin{aligned}
& \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \left[v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i'); q(t_{-i},t_i')) - v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i'); q(t_{-i},t_i)) \right] P(t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \sum_{\theta} u_i(x_i(q(t_{-i},t_i')|t_{-i},t_i'); \theta) \left[P(\theta|t_{-i},t_i')) - P(\theta|t_{-i},t_i)) \right] P(t_{-i}|t_i) \\
\geq & -M_i \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} ||P(\cdot|t_{-i},t_i)) - P(\cdot|t_{-i},t_i')||P(t_{-i}|t_i) \\
\geq & -3M_i\nu_i^P.
\end{aligned}$$

Part 5: Let X be a finite set with cardinality k and let $p, q \in \Delta_X$. Then

$$\left[\frac{p}{||p||_2} - \frac{q}{||q||_2} \right] \cdot p \geq \frac{k^{-\frac{5}{2}}}{2} \left[||p-q|| \right]^2$$

where $||\cdot||_2$ denotes the 2-norm and $||\cdot||$ denotes the 1-norm. To see this, note that direct computation yields

$$\left[\frac{p}{||p||_2} - \frac{q}{||q||_2} \right] \cdot p = \frac{||p||_2}{2} \left\| \frac{p}{||p||_2} - \frac{q}{||q||_2} \right\|_2^2 .$$

The result now follows by combining the facts that $||p||_2 \geq 1/\sqrt{k}, k(||p-q||_2)^2 \geq ||p-q||^2$ and

$$\left[\left\|\frac{p}{||p||_2} - \frac{q}{||q||_2}\right\|_2\right]^2 \geq \frac{1}{k}\left[||p-q||_2\right]^2.$$

To complete the proof of Claim 3, we combine Parts 2,3,4 and 5 to obtain

$$\begin{aligned}
& \sum_{\theta}\sum_{t_{-i}} \left[u_i(z_i(t_{-i},t_i);\theta) - u_i(z_i(t_{-i},t_i');\theta)\right] P(\theta,t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^*}} \left[v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) - v_i(z_i(t_{-i},t_i');q(t_{-i},t_i))\right] P(t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \left[v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) - v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i');q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
& + \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \left[v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i');q(t_{-i},t_i')) - v_i(x_i(q(t_{-i},t_i')|t_{-i},t_i');q(t_{-i},t_i))\right] P(t_{-i}|t_i) \\
& + \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \left[v_i(x_i(q(t_{-i},t_i)|t_{-i},t_i);q(t_{-i},t_i)) - v_i(0;q(t_{-i},t_i))\right] P(t_{-i}|t_i) \\
\geq & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) + \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i}|t_i)P(t_{-i}|t_i) - 3(K_i+M_i)\nu_i^P
\end{aligned}$$

Since $\alpha_i(t_{-i}|t_i') = 0$ if $(t_{-i},t_i') \notin T^*$, it follows from part 5 that

$$\begin{aligned}
& \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) + \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i}|t_i)P(t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T^*}} \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) \\
= & \sum_{\substack{t_{-i} \\ :(t_{-i},t_i)\in T}} \eta\left(\alpha_i(t_{-i}|t_i) - \alpha_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) \\
\geq & \ \eta B\Lambda_i^P
\end{aligned}$$

Therefore,

$$\sum_{\theta}\sum_{t_{-i}} \left[u_i(z_i(t_{-i},t_i);\theta) - u_i(z_i(t_{-i},t_i');\theta)\right] P(\theta,t_{-i}|t_i) \geq \eta B\Lambda_i^P - 3(K_i+M_i)\nu_i^P \geq 0.$$

This completes the proof of Claim 3.

9.2 Proof of Lemma 2:

For each $\pi \in \Delta_\Theta$, let $v(w;\pi) := (v_1(w_1;\pi), .., v_n(w_n;\pi))$ and define

$$\mu(\pi) = \arg\max[\mu | v(w;\pi) + \mu e \in \Phi(\pi)].$$

Note that $\mu(\pi)$ is well defined since $\Phi(\pi)$ is compact and that $\mu(\pi) \geq 0$. For each $\pi \in \Delta_\Theta$, choose a feasible allocation $x(\pi)$ for $e(\pi)$ satisfying $v_i(x_i(\pi);\pi) = v_i(w_i;\pi) + \mu(\pi)e$ for each $i \in N$. Finally, let $f : \Delta_\Theta \to \Re^n$ be the map defined by $f_i(\pi) = v_i(x_i(\pi);\pi)$ for each i.

Claim 1: For each π, $f(\pi) \in \Phi^0(\pi)$. If the economy $e(\pi)$ has at least one strictly individually rational allocation for each $\pi \in \Delta_\Theta$, then f is strictly individually rational.

Proof: First we show that $x(\pi)$ is efficient in $e(\pi)$. Suppose that $x(\pi)$ is not efficient in $e(\pi)$. Then there exists an allocation $y = (y_1, .., y_n)$ satisfying $v_i(y_i;\pi) > v_i(x_i(\pi);\pi)$ for all i. Choose i_0 so that

$$v_{i_0}(y_{i_0};\pi) - v_{i_0}(w_{i_0};\pi) = \min_{i \in N} [v_i(y_i;\pi) - v_i(w_i;\pi)] := \sigma$$

and note that $\sigma > \mu(\pi)$. For each $i \neq i_0$, there exists a $\beta_i \in [0,1]$ such that $v_i(\beta_i y_i;\pi) - v_i(w_i;\pi) = \sigma$ (this follows from montonicity, continuity and normalization.) Defining $z_{i_0} = y_{i_0}$ and $z_i = \beta_i y_i$, otherwise, it follows that $(z_1, .., z_n)$ is a feasible allocation for $e(\pi)$ and that

$$(v_1(z_1;\pi), .., v_n(z_n;\pi)) = v(w;\pi) + \sigma e.$$

Since $\sigma > \mu(\pi)$, we arrive at a contradiction and, therefore, $x(\pi)$ is efficient in $e(\pi)$.

Individual rationality follows from the nonnegativity of $\mu(\pi)$. Suppose that, in addition, the economy $e(\pi)$ has at least one strictly individually rational allocation. Then using precisely the same argment used above in the proof of efficiency, we can construct a feasible allocation $(z_1, .., z_n)$ for $e(\pi)$ satisfying

$$(v_1(z_1;\pi), .., v_n(z_n;\pi)) = v(w;\pi) + \sigma e.$$

and $\sigma > 0$. Therefore, $\mu(\pi) > 0$ and $x(\pi)$ is strictly individually rational.

Claim 2: The selection f is positive.

Proof: Let $\lambda = \min_i \min_\theta u_i(w_i;\theta)$. Monotonicity, normalization and nonzero endowment imply that $\lambda > 0$. Since

$$f_i(\pi) = v_i(x_i(\pi);\pi) \geq v_i(w_i;\pi) = \sum_\theta u_i(w_i;\theta)\pi(\theta) \geq \min_\theta u_i(w_i;\theta) \geq \lambda$$

we conclude that f is positive.

Claim 3: Let

$$M = \max_i \max_\theta u_i(\sum_{j \in N} w_j; \theta)$$

and note that $M > 0$. For each i, the mapping f_i is uniformly Lipschitz on Δ_Θ with modulus $K_i = 3M$.

Proof: Choose $\pi, \pi' \in \Delta_\Theta$ and w.l.o.g., suppose that $\mu(\pi) \leq \mu(\pi')$. To prove the claim, it is enough to show that $\mu(\pi') \leq \mu(\pi) + 2M||\pi - \pi'||$ for each i since we would then conclude that

$$\begin{aligned} |f_i(\pi) - f_i(\pi')| &= |\,(v_i(w_i; \pi) + \mu(\pi)) - (v_i(w_i; \pi') + \mu(\pi'))\,|| \\ &\leq |v_i(w_i; \pi) - v_i(w_i; \pi')| + |\mu(\pi) - \mu(\pi')| \\ &\leq M||\pi - \pi'|| + 2M||\pi - \pi'|| \\ &= 3M||\pi - \pi'||. \end{aligned}$$

First we show that there exists a $y \in \Phi(\pi)$ such that $|y_i - f_i(\pi')|| \leq M||\pi - \pi'||$ for each i. To see this, simply let $y = (v_1(x_1(\pi'); \pi), .., v_n(x_n(\pi'); \pi))$ and observe that

$$|y_i - f_i(\pi')| = |v_i(x_i(\pi'); \pi) - v_i(x_i(\pi'); \pi')| \leq M||\pi - \pi'||.$$

To complete the proof, suppose that $\mu(\pi') > \mu(\pi) + 2M||\pi - \pi'||$ and choose $y \in \Phi(\pi)$ satisfying $|y_i - f_i(\pi')|| \leq M||\pi - \pi'||$ for each i. We claim that $y_i > v_i(x_i(\pi); \pi)$ for each i, contradicting the efficiency of the allocation $x(\pi)$ in $e(\pi)$. To see this, suppose that $y_i \leq v_i(x_i(\pi); \pi) = v_i(w_i; \pi) + \mu(\pi)$ for some i. Using the fact that

$$v_i(w_i; \pi') - v_i(w_i; \pi) \geq -|v_i(w_i; \pi') - v_i(w_i; \pi)| \geq -M||\pi - \pi'||$$

we deduce that

$$\begin{aligned} f_i(\pi') - y_i &= v_i(w_i; \pi') + \mu(\pi') - y_i \\ &> v_i(w_i; \pi') + \mu(\pi) + 2M||\pi - \pi'|| - y_i \\ &= (v_i(w_i; \pi) + \mu(\pi) - y_i) + v_i(w_i; \pi') - v_i(w_i; \pi) + 2M||\pi - \pi'|| \\ &\geq v_i(w_i; \pi') - v_i(w_i; \pi) + 2M||\pi - \pi'|| \\ &\geq 2M||\pi - \pi'|| - M||\pi - \pi'|| \\ &= M||\pi - \pi'||. \end{aligned}$$

This is impossible since $|y_i - f_i(\pi')| \leq M||\pi - \pi'||$. Therefore, there exists a $y \in \Phi(\pi)$ such that $y_i > v_i(x_i(\pi); \pi)$ for each i. Hence, the hypothesis that $\mu(\pi') > \mu(\pi) + 2M||\pi - \pi'||$ leads to a contradiction and the proof of Claim 3 is complete.

9.3 Proof of Theorem 2:

Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence and suppose that each $u_i(\cdot;\theta)$ is concave. Choose $\varepsilon > 0$.

Step 1:

For each $t^r \in T^r$, let $\varphi(t^r)$ denote the "empirical frequency distribution" that t^r induces on T. More formally, $\varphi(t^r)$ is a probability measure on T defined for each $\tau \in T$ as follows:

$$\varphi(t^r)(\tau) = \frac{|\{s \in J_r | t^r(s) = \tau\}|}{r}$$

(We suppress the dependence of φ on r for notational convenience.)

Claim: For every $\rho > 0$, there exists an integer $\hat{r}$ such that for all $r > \hat{r}$,

$$\nu_i^{P^r} \leq \rho\ .$$

Proof of Claim: Choose $\rho > 0$. Applying the argument in the appendix to Gul-Postlewaite(1992) (see the analysis of their equation (9)), together with the definition of φ and the law of large numbers, it follows that there exists $\lambda > 0$ and an integer $\hat{r}$ such that for all $r > \hat{r}$,

$$||\varphi(t^r) - P_T(\cdot|\theta_k)|| < \lambda \Rightarrow ||P^r_\Theta(\cdot|t^r) - \chi_{\theta_k}|| < \rho/2 \text{ for all } t^r \text{and } k \geq 1,$$

$$||\varphi(t^r_{-is}, t_i) - \varphi(t^r_{-is}, t'_i)|| < \lambda/2 \text{ for all } t_i, t'_i \in T_i \text{ and all } t^r \text{and all } i,$$

and

$$\text{Prob}\{||\varphi(\tilde{t}^r) - P_T(\cdot|\theta_k)|| < \lambda/2 | \tilde{t}^r_{is} = t_i, \tilde{\theta} = \theta_k\} > 1 - \rho \text{ for all } t_i, t'_i \in T_i \text{ and } k \geq 1.$$

Choose $t_i, t'_i \in T_i$, $k \geq 1$ and $r > \hat{r}$. Then

$$\text{Prob}\{||P^r_\Theta(\cdot|\tilde{t}^r_{-is}, t_i) - P^r_\Theta(\cdot|\tilde{t}^r_{-is}, t'_i)|| \leq \rho | \tilde{t}^r_{is} = t_i, \tilde{\theta} = \theta_k\}$$

$$\begin{aligned}
&\geq \text{Prob}\{||\varphi(\tilde{t}^r_{-is}, t_i) - P_T(\cdot|\theta_k)|| < \lambda/2 \text{ and } ||\varphi(\tilde{t}^r_{-is}, t'_i) - P_T(\cdot|\theta_k)|| < \lambda | \tilde{t}^r_{is} = t_i, \tilde{\theta} = \theta_k\} \\
&\geq \text{Prob}\{||\varphi(\tilde{t}^r_{-is}, t_i) - P_T(\cdot|\theta_k)|| < \lambda/2 \text{ and } ||\varphi(\tilde{t}^r_{-is}, t_i) - \varphi(\tilde{t}^r_{-is}, t'_i)|| < \lambda/2 | \tilde{t}^r_{is} = t_i, \tilde{\theta} = \theta_k\} \\
&= \text{Prob}\{||\varphi(\tilde{t}^r_{-is}, t_i) - P_T(\cdot|\theta_k)|| < \lambda/2\ | \tilde{t}^r_{is} = t_i, \tilde{\theta} = \theta_k\} \\
&> 1 - \rho
\end{aligned}$$

Hence,

$$\text{Prob}\{||P^r_\Theta(\cdot|\tilde{t}^r_{-is}, t_i) - P^r_\Theta(\cdot|\tilde{t}^r_{-is}, t'_i)|| \leq \rho | \tilde{t}^r_{is} = t_i\} \geq 1 - \rho$$

and we conclude that $\nu_i^{P^r} \leq \rho$.

Step 2:

For the "basic" PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ (i.e., the PIE with $r = 1$), we can apply Lemma 2 and conclude that there exists a strictly individually rational Lipschitz selection f for Φ^0 with $f_i(\pi) - v_i(w_i;\pi) \geq \lambda > 0$ for all i and π. Let $K_i > 0$ denote the modulus of continuity of f_i and define

$$M_i = u_i(\sum_{j\in N} w_j; \theta).$$

For each $\pi \in \Delta_\Theta$, choose an allocation $\{\zeta_i(\pi)\}_{i\in N}$ for $e(\pi)$ satisfying

$$v_i(\zeta_i(\pi);\pi) = f_i(\pi)$$

for each i and note that each $\zeta_i(\pi) \neq 0$. Finally, choose η such that $0 < \eta < \min\{\varepsilon, \lambda\}$.

Suppose that, for each $\pi \in \Delta_\Theta$, the allocation $\{\zeta_i(\pi)\}_{i\in N}$ is an allocation for $e(\pi)$ satisfying

$$(v_1(\zeta_1(\pi);\pi), .., v_n(\zeta_n(\pi);\pi)) = f(\pi).$$

Duplicating verbatim the steps 1 and 2 in the proof of Proposition 3, we construct

$$x_i(\pi|t) = (1 + \tau_i(t))\beta_i(\pi)\zeta_i(\pi)$$

for each π and each $t \in T^*$. From the construction, it follows that the collections $\{x_i(\pi|t)\}_{\pi\in\Delta_\Theta, t\in T}$ satisfy (i)

$$x_i(\pi|t) \in \Re^\ell_+ \text{ and } \sum_{i\in N}(x_i(\pi|t) - w_i) \leq 0.$$

(ii)

$$v_i(\zeta_i(\pi);\pi) \geq v_i(x_i(\pi|t_{-i}, t_i);\pi) \geq v_i(\zeta_i(\pi);\pi) - \eta$$

and

(iii)

$$v_i(x_i(\pi|t_{-i}, t_i);\pi) = v_i(\zeta_i(\pi);\pi) + \eta\alpha_i(t_{-i}|t_i) - \eta.$$

where

$$\alpha_i(t_{-i}|t_i) = \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}.$$

Step 3:

We now use this construction for the "basic" PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ to define a mechanism for the replica PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$. For each $r \geq 1$ and each $t \in T^{r*}$, let $q(t^r) = P^r_\Theta(\cdot|t^r)$ and define a collection

$$\begin{aligned} z^r_{i,s}(t^r) &= x_i(q(t^r)|t^r(s)) \text{ if } t^r \in T^{r*} \\ z^r_{i,s}(t^r) &= 0 \text{ if } t^r \notin T^{r*}. \end{aligned}$$

It follows from the construction of $x_i(\pi|t)$ in Step 2 above that

$$v_i(\zeta_i(q(t^r)); q(t^r)) \geq v_i(x_i(q(t^r)|t^r(s)); q(t^r)) \geq v_i(\zeta_i(q(t^r)); q(t^r)) - \eta$$

To complete the proof, we will show that mechanism $z^r(\cdot)$ is an individually rational, incentive compatible, ε-efficient PIE allocation for $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ whenever r is sufficiently large.

Claim 1: For each positive integer r, the mechanism $z^r(\cdot)$ is a PIE allocation for $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$.

Proof: Since

$$x_i(\pi|t) \in \Re^\ell_+ \text{ and } \sum_{i\in N}(x_i(\pi|t) - w_i) \leq 0$$

for every $\pi \in \Delta_\Theta$ and $t \in T$, it follows that

$$\sum_{s=1}^{r}\sum_{i\in N}(z^r_{i,s}(t^r) - w_i) = \sum_{s=1}^{r}\sum_{i\in N}(x_i(q(t^r)|t^r(s)) - w_i) \leq 0$$

whenever $t^r \in T^{r*}$ and

$$\sum_{s=1}^{r}\sum_{i\in N}(z^r_{i,s}(t^r) - w_i) = \sum_{s=1}^{r}\sum_{i\in N}(0 - w_i) \leq 0$$

whenever $t^r \notin T^{r*}$.

Claim 2: For each positive integer r, the mechanism $z^r(\cdot)$ is ex post IR and ex post ε-efficient for the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$.

Proof: Since $v_i(\zeta_i(q(t^r)); q(t^r)) - v_i(w_i; q(t^r)) = f(q(t^r)) - v_i(w_i; q(t^r)) \geq \rho$ for each $t^r \in T^{r*}$, it follows that

$$\begin{aligned} v_i(z^r_{i,s}(t^r); q(t^r)) - v_i(w_i; q(t^r)) &= v_i(z_{i,s}(t^r); q(t^r)) - v_i(\zeta_i(q(t^r)); q(t^r)) \\ &\quad + v_i(\zeta_i(q(t^r)); q(t^r)) - v_i(w_i; q(t^r)) \\ &\geq \rho - \eta \\ &> 0 \end{aligned}$$

for each i so $z^r(t^r)$ is individually rational for each $t^r \in T^{r*}$. That is, $z^r(\cdot)$ satisfies XIR for the replica PIE e^r.

To show that $z^r(\cdot)$ satisfies $\text{X}_\varepsilon\text{E}$ in the the replica PIE e^r, suppose that $t^r \in T^{r*}$ and that $y^r(\cdot)$ is a PIE allocation for e^r satisfying

$$v_i(y^r_{is}(t^r); q(t^r)) > v_i(z^r_{is}(t^r); q(t^r)) + \varepsilon$$

for each (i, s). For each i, let

$$\overline{y}_i = \frac{1}{r}\sum_{s=1}^{r} y^r_{is}(t^r)$$

and therefore,

$$\sum_{i=1}^{n} \overline{y}_i = \frac{1}{r} \sum_{i=1}^{n} \sum_{s=1}^{r} y_{is}^r(t^r) \leq \sum_{i=1}^{n} w_i.$$

Since each $v_i(\cdot; t^r)$ is concave and

$$v_i(z_{is}^r(t^r); q(t^r)) \geq v_i(\zeta_i(q(t^r)); q(t^r)) - \eta,$$

it follows that

$$\begin{aligned} v_i(\overline{y}_i; q(t^r)) &\geq \frac{1}{r} \sum_s v_i(y_{is}^r(t^r); q(t^r)) \\ &> \frac{1}{r} \sum_s v_i(z_{is}^r(t^r); q(t^r)) + \varepsilon \\ &\geq v_i(\zeta_i(q(t^r)); q(t^r)) - \eta + \varepsilon \\ &> v_i(\zeta_i(q(t^r)); q(t^r)) \end{aligned}$$

for each i and we conclude that $\{\zeta_i(q(t^r))\}_{i \in N}$ is not Pareto optimal in $e(q(t^r))$, a contradiction.

Claim 3: There exists a positive integer $\hat{r}$ such that, whenever $r > \hat{r}$, the mechanism $z^r(\cdot)$ satisfies IC.

Proof: Let $K = \max_i K_i$, $M = \max_i M_i$ and define $B = \frac{|T|^{-\frac{5}{2}}}{2}$. Assumption (d) of the Theorem implies that $\Lambda_i^P > 0$. Hence, we can apply the result of step 1 and conclude that there exists a positive integer $\hat{r}$ such that $\eta B \Lambda_i^P - 3(K + M)\nu_i^{P^r} > 0$ whenever $r > \hat{r}$. We will show that the mechanism $z^r(\cdot)$ satisfies IC whenever $r > \hat{r}$. Choose $(i, s) \in N \times J_r$. Since

$$v_i(x_i(\pi|t_{-i}, t_i); \pi) = v_i(\zeta_i(\pi); \pi) + \eta\alpha_i(t_{-i}|t_i) - \eta$$

for each $\pi \in \Delta_\Theta$ and each $(t_{-i}, t_i) \in T$, it follows that

$$v_i(x_i(q(t_{-i,s}^r, t_i)|t_{-i}^r(s), t_i); q(t_{-i,s}^r, t_i)) = v_i(\zeta_i(q(t_{-i,s}^r, t_i)); q(t_{-i,s}^r, t_i)) + \eta\alpha_i(t_{-i}^r(s)|t_i) - \eta$$

and

$$v_i(x_i(q(t_{-i,s}^r, t_i')|t_{-i}^r(s), t_i'); q(t_{-i,s}^r, t_i')) = v_i(\zeta_i(q(t_{-i,s}^r, t_i')); q(t_{-i,s}^r, t_i')) + \eta\alpha_i(t_{-i}^r(s)|t_i') - \eta$$

whenever $(t_{-i,s}^r, t_i), (t_{-i,s}^r, t_i') \in T^{r*}$. Therefore,

$$\begin{aligned} & v_i(x_i(q(t_{-i,s}^r, t_i)|t_{-i}^r(s), t_i); q(t_{-i,s}^r, t_i)) - v_i(x_i(q(t_{-i,s}^r, t_i')|t_{-i}^r(s), t_i'); q(t_{-i,s}^r, t_i')) \\ \geq\ & \eta\left(\alpha_i(t_{-i}^r(s)|t_i) - \alpha_i(t_{-i}^r(s)|t_i')\right) - K_i||P_\Theta^r(\cdot|t_{-i,s}^r, t_i)) - P_\Theta^r(\cdot|t_{-i,s}^r, t_i')|| \end{aligned}$$

Duplicating the arguments in parts 2,3 and 4 of Claim 3 in the proof of Proposition 3, we have the following inequalities:

$$\sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\in T^{r*}}} \left[v_i(x_i(q(t^r_{-i,s},t_i)|t^r_{-i}(s),t_i);q(t^r_{-i,s},t_i)) - v_i(x_i(q(t^r_{-i,s},t'_i)|t^r_{-i}(s),t'_i);q(t^r_{-i,s},t'_i))\right] P^r(t^r_{-i,s}|$$

$$\geq \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\in T^{r*}}} \eta\left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r_{-i,s}|t_i) - 2K_i\nu_i^{P^r}$$

$$\sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\notin T^{r*}}} \left[v_i(x_i(q(t^r_{-i,s},t_i)|t^r_{-i}(s),t_i);q(t^r_{-i,s},t_i)) - v_i(0;q(t^r_{-i,s},t_i))\right] P^r(t^r_{-i,s}|t_i)$$

$$\geq \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\notin T^{r*}}} \eta\alpha_i(t_{-i}(s)|t_i)P^r(t_{-i,s}|t_i)$$

and

$$\sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\in T^{r*}}} \left[v_i(x_i(q(t^r_{-i,s},t'_i)|t^r_{-i}(s),t'_i);q(t_{-i,s},t'_i)) - v_i(x_i(q(t^r_{-i,s},t'_i)|t^r_{-i}(s),t'_i);q(t_{-i,s},t_i))\right] P^r(t_{-i,s}|$$

$$\geq -3M_i\nu_i^{P^r}.$$

To complete the proof of Claim 2, we combine these inequalities to obtain

$$\sum_{\theta}\sum_{t^r_{-i,s}} \left[u_i(z^r_{i,s}(t_{-i,s},t_i);\theta) - u_i(z^r_{i,s}(t_{-i,s},t'_i);\theta)\right] P^r(\theta,t^r_{-i,s}|t_i)$$

$$\geq \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\in T^{r*}}} \eta\left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r_{-i,s}|t_i) - 3K_i\nu_i^{P^r}$$

$$+ \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\notin T^{r*}}} \eta\alpha_i(t_{-i}(s)|t_i)P^r(t_{-i,s}|t_i) - 3M_i\nu_i^{P^r}.$$

Since $\alpha_i(t_{-i}(s)|t'_i) \geq 0$, it follows from part (5) of Claim 3 in the proof of Proposition 3 that

$$\sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\in T^{r*}}} \eta\left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r_{-i,s}|t_i) + \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*} \\ (t_{-i,s},t'_i)\notin T^{r*}}} \eta\alpha_i(t_{-i}(s)|t_i)P^r(t_{-i,s}|t_i)$$

$$
\begin{aligned}
&\geq \sum_{\substack{t^r_{-i,s} \\ :(t_{-i,s},t_i)\in T^{r*}}} \eta\left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r_{-i,s}|t_i) \\
&= \sum_{t^r_{-i,s}} \eta\left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r_{-i,s}|t_i) \\
&= \eta \sum_{t^r(1)\in T} \cdots \sum_{t^r_{-i}(s)\in T_{-i}} \cdots \sum_{t^r(r)\in T} \left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P^r(t^r(1),..,t^r_{-i}(s),..,t^r(r)|t_i) \\
&= \eta \sum_{t^r_{-i}(s)\in T_{-i}} \left(\alpha_i(t^r_{-i}(s)|t_i) - \alpha_i(t^r_{-i}(s)|t'_i)\right) P(t_{-i}(s)|t_i) \\
&\geq \eta B \Lambda^P_i
\end{aligned}
$$

Therefore

$$
\sum_\theta \sum_{t^r_{-i,s}} \left[u_i(z^r_{i,s}(t_{-i,s},t_i);\theta) - u_i(z^r_{i,s}(t_{-i,s},t'_i);\theta)\right] P^r(\theta, t^r_{-i,s}|t_i) \geq \eta B\Lambda^P_i - 3(K+M)\nu^{P^r}_i > 0
$$

and the the proof of incentive comapatibility is complete.

9.4 Proof of Proposition 5

We begin with the following claim.

Claim: Suppose that $P \in \Delta_{\Theta\times T}$. Then for each $t_i, t'_i \in T_i$,

$$
||P_\Theta(\cdot|t_i) - P_\Theta(\cdot|t'_i)|| \leq 3\nu^P_i + ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t'_i)||.
$$

Proof: Choose $P \in \Delta_{\Theta\times T}$. Then

$$
\begin{aligned}
P(\theta|t_i) - P(\theta|t'_i) &= \sum_{t_{-i}} P(\theta|t_{-i},t_i)P(t_{-i}|t_i) - P(\theta|t_{-i},t'_i)P(t_{-i}|t'_i) \\
&= \sum_{t_{-i}} P(\theta|t_{-i},t_i)P(t_{-i}|t_i) - \sum_{t_{-i}} P(\theta|t_{-i},t'_i)P(t_{-i}|t_i) \\
&\quad + \sum_{t_{-i}} P(\theta|t_{-i},t'_i)P(t_{-i}|t_i) - P(\theta|t_{-i},t'_i)P(t_{-i}|t'_i) \\
&= \sum_{t_{-i}} \left[P(\theta|t_{-i},t_i) - P(\theta|t_{-i},t'_i)\right] P(t_{-i}|t_i) + \sum_{t_{-i}} P(\theta|t_{-i},t'_i)\left[P(t_{-i}|t_i) - P(t_{-i}|t'_i)\right]
\end{aligned}
$$

Therefore,

$$
|P(\theta|t_i) - P(\theta|t'_i)| \leq \sum_{t_{-i}} |P(\theta|t_{-i},t_i) - P(\theta|t_{-i},t'_i)|P(t_{-i}|t_i) + \sum_{t_{-i}} P(\theta|t_{-i},t'_i)|P(t_{-i}|t_i) - P(t_{-i}|t'_i)|
$$

and we conclude that

$$\begin{aligned}
||P_\Theta(\cdot|t_i) - P_\Theta(\cdot|t_i')|| &= \sum_\theta |P(\theta|t_i) - P(\theta|t_i')| \\
&\leq \sum_{t_{-i}} \sum_\theta |P(\theta|t_{-i}, t_i) - P(\theta|t_{-i}, t_i')| P(t_{-i}|t_i) \\
&\quad + \sum_\theta \sum_{t_{-i}} P(\theta|t_{-i}, t_i')|P(t_{-i}|t_i) - P(t_{-i}|t_i')| \\
&= \sum_{t_{-i}} ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| P(t_{-i}|t_i) + ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')||.
\end{aligned}$$

Since $||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')|| \leq 2$, it follows that

$$\sum_{t_{-i}} ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| P(t_{-i}|t_i) \leq 3\nu_i^P$$

and the proof of the claim is complete.

Let $\nu = \max_i \nu_i^P, \Omega = \min_i \Omega_i^P$, and $\Lambda = \min_i \Lambda_i^P$. Since $\nu \leq 1$ and $||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')|| \leq 2$, we conclude that

$$\begin{aligned}
||P_\Theta(\cdot|t_i) - P_\Theta(\cdot|t_i')||^2 &\leq \left(3\nu_i^P\right)^2 + 6\nu_i^P ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')|| + ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')||^2 \\
&\leq 9\nu_i^P + 12\nu_i^P + ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')||^2 \\
&= 21\nu_i^P + ||P_{T_{-i}}(\cdot|t_i) - P_{T_{-i}}(\cdot|t_i')||^2.
\end{aligned}$$

Therefore,

$$\Omega \leq 21\nu + \Lambda.$$

Now suppose that the hypotheses of the Proposition are satisfied and choose $\varepsilon > 0$. Applying Theorem 1, there exists $\hat{\delta} > 0$ such that whenever $P \in \Delta_{\Theta \times T}$ and satisfies

$$\max_i \nu_i^P \leq \hat{\delta} \min_i \Lambda_i^P$$

there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC and X$_\varepsilon$E. Let

$$\delta = \min\{\frac{1}{42}, \frac{\hat{\delta}}{2}\}$$

Then

$$\nu \leq \delta\Omega \leq \delta\,(21\nu + \Lambda) = 21\delta\nu + \delta\Lambda \leq (1/2)\nu + (\hat{\delta}/2)\Lambda$$

from which it follows that $\nu \leq \hat{\delta}\Lambda$. This completes the proof.