Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://www.econ.upenn.edu/pier

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"Efficient Auction Mechanisms with Interdependent Valuations and Multidimensional Signals"

by

Richard P. McLean and Andrew Postlewaite

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Efficient Auction Mechanisms with Interdependent Valuations and Multidimensional Signals*

Richard McLean
Rutgers University

Andrew Postlewaite
University of Pennsylvania

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Abstract

We develop an auction model for the case of interdependent values and multidimensional signals in which agents' information is not independent. We show that a modification of the Vickrey auction which includes payments to the bidders will result in an efficient outcome under very general conditions. Further, we provide a definition of informational size such that the necessary payments to bidders will be arbitrarily small if agents are sufficiently informationally small.

Keywords: Auctions, Incentive Compatibility, Mechanism Design, Interdependent Values.

JEL Classification: C70, D44, D60, D82

1 Introduction

The efficiency of market processes has been a central concern in economics since its inception. Auction mechanisms constitute a very important class of market processes, yet the analysis of auctions has typically focused on their revenue generating properties rather than their efficiency properties. This is partly due to the fact that, for many of the problems typically studied, efficiency is trivial. When bidders have private values, a standard Vickrey auction guarantees that the object will be sold to the buyer with the highest valuation for the object. In the case of pure common values - that is, when all buyers have the same value for the object - any outcome that with

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probability one assigns the object to *some* bidder will be efficient. The intermediate case in which bidders' values are interdependent but not identical is more problematic. When bidders' values are interdependent, any single bidder's value may depend on the information of other agents and, hence, he may not even know his own value. It is not clear what it would mean for an agent to bid his "true" value, even before we ask if it is optimal for him to do so.

Several recent papers have examined the possibility of efficient auctions. In the case of two bidders, Maskin (1992) extended the Vickrey auction to the case of interdependent values in a way that assures an efficient outcome. Dasgupta and Maskin (1998) and Perry and Reny (1998) use the basic idea in Maskin (1992) to construct auction mechanisms that guarantee efficient outcomes for the case in which there are multiple units to be sold. In these papers, an agent's information regarding the value of the object(s) to be sold must be represented by a one dimensional signal. Dasgupta and Maskin provide simple examples showing that, if agents' types are independent, there do not exist mechanisms that are efficient when signals are multidimensional. Jehiel and Moldovanu (1998) prove a general theorem about the impossibility of efficient mechanisms when bidders have independent types and multidimensional signals.

The exclusion of multidimensional types is quite restrictive. For many problems, individuals have private information of two very different kinds: information about the qualitative features of the object being sold and information about themselves that affects their personal valuation of an object with particular physical characteristics, but does not affect others' valuations. Potential bidders for an oil tract may have information about the size and nature of the oil field and, in addition, information regarding their own cost of retrieving and processing the oil. Agents bidding in a spectrum auction may have information about the number and characteristics of the individuals covered by the license being sold, as well as information about the value to their company from serving that population. In such problems, the information (type) of an agent is multidimensional and, hence, existing papers on efficient auctions provide no guidance.

The work described above on the possibility/impossibility of efficient auction mechanisms restricts attention to the case in which agents' types are independent. While this is a natural place to begin, the independence assumption is not compelling for all auctions. Many problems have the general structure of the problem described in the previous paragraph. The value to a given prospective buyer of the object(s) being sold depends on two qualitatively different things: objective characteristics of the object itself (the quantity and quality of oil in a tract to be auctioned off or the demographic characteristics of the consumers covered by a license in a spectrum auction), and idiosyncratic characteristics of the buyer (his cost of extracting the oil in the field or his cost of serving the customers covered by a given spectrum license). When bidders' types include information about objective characteristics of the object, it is plausible that their types are correlated.

It is well known that, if agents' types are correlated, then mechanisms can be designed to induce truthful revelation of private information, and that information can be used to ensure efficient outcomes. (See Cremer and McLean (1985,1988), and subsequent work by McAfee and Reny (1992).) Mechanisms that rely on correlation of types to induce truthful revelation are sometimes criticized on the grounds that in such mechanisms, the payments to and from agents can be very large. The use of very large payments makes it clear that such mechanisms will not be of use in the presence of limited liability or nonlinear preferences over money.

In this paper, we show that there exist efficient auction mechanisms for interdependent value auction problems when agents' types are correlated. These auction mechanisms are essentially Vickrey auctions augmented by payments to (not from) the agents. Most importantly, we link the payment made to an agent to that agent's "informational size". If all agents are receiving signals correlated with the common but unobservable value of the object, then any single agent's signal may add little to the information contained in the aggregate of the other agents' signals. Informally, we can think of an agent as being informationally small if it is unlikely that the probability distribution of the objective characteristics of the object is very sensitive to that agent's information, given the information of others. When agents are informationally small, the payments necessary for our augmented Vickrey auction will be small. Hence, agents' "informational rents" - as represented by the payments made to them - are linked to their informational size. However, we should emphasize that we are not proposing that agents are *necessarily* informationally small and, consequently, that efficient outcomes can always be assured with small augmented payments.

Our model is described in Section 2, and in Section 3 we present an example with a simple information structure in which agents receive conditionally independent signals of the state of nature. Section 4 provides an analysis for the more complicated problem with general information structures that include the conditionally independent structure of the example in section 3 as a special case. In section 5, we present the most general result in the paper in a framework that subsumes the model of section 4. Section 6 presents an example of a nonrevelation auction mechanism that, for some auction problems, achieves the same outcome as the revelation mechanisms. Some concluding comments are contained in Section 7 and the proofs are given in Section 8.

2 Auctions

Let $\Theta = \{\theta_1, .., \theta_m\}$ represent the finite set of states of nature. Each $\theta \in \Theta$ represents a complete physical description of the object being sold (e.g., the amount and quality of oil). Let T_i be a finite set of possible types of agent i. As stressed in the introduction, an agent's information may be of two qualitatively different kinds: information about the objective characteristics of the object being sold, and idiosyncratic information about the agent himself. The former is of interest to other agents - and consequently is the cause of the interdependence of agents' valuations - while the latter is irrelevant to other agents in calculating their valuations. The state of nature is unobservable but agent $i's$ information about the physical characteristics of the object to be sold will be captured by the correlation between his type t_i and nature's choice of θ. His type t_i will also capture any idiosyncratic information he may have. Agent i is characterized by a utility function $v_i : \Theta \times T_i \to R_+$. That is, agent $i's$ utility depends on the physical characteristics of the object, and his type t_i.

Let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ $(T \equiv T_1 \times \cdots \times T_n)$ with associated distribution P where

$$P(\theta, t_1, .., t_n) = Prob\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions : $P(\theta)$ =Prob$\{\widetilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t)$ =Prob$\{\widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\} > 0$ for each $t \in T$.

If X is a finite set, let Δ_X denote the set of probability measures on X. The set of probability measures on $\Theta \times T$ satisfying the full support conditions will be denoted $\Delta^*_{\Theta \times T}$

In problems with differential information, it is standard to assume that agents have utility functions $w_i : T \to R_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $w_i(t) = \sum_{\theta \in \Theta} [v_i(\theta, t_i) P(\theta|t)]$. Alternatively, given utility functions $w_i : T \to R_+$, we can define $\Theta \equiv T$ and define $v_i(t, t_i) = w_i(t)$. Our formulation will be useful in that it highlights the nature of the interdependence: agents care about other agents' types to the extent that they provide additional information about the physical characteristics of the object being sold.

An *auction problem* is a collection $(v_1, .., v_n, P)$ where $P \in \Delta^*_{\Theta \times T}$. An *auction mechanism* is a collection $\{q_i, x_i\}_{i \in N}$ where $q_i : T \to \mathbb{R}_+$ and $x_i : T \to \mathbb{R}$ are functions satisfying

$$\sum_{i \in N} q_i(t) \leq 1 \text{ for all } t \in T.$$

For any vector of types $t \in T$, let

$$\hat{v}_i(t) = \hat{v}_i(t_{-i}, t_i) = \sum_{\theta \in \Theta} v_i(\theta, t_i) P(\theta | t_{-i}, t_i).$$

Although $\hat{v}$ depends on P, we suppress this dependence for notational simplicity. The number $\hat{v}_i(t)$ represents $i's$ valuation for the object conditional on the informational state $t \in T$.

Definition: An auction mechanism $\{q_i, x_i\}_{i \in N}$ is:

incentive compatible (IC) if for each $i \in N$,

$$\sum_{t_{-i}} [q_i(t_{-i}, t_i)\hat{v}_i(t_{-i}, t_i) - x_i(t_{-i}, t_i)] P(t_{-i}|t_i) \geq \sum_{t_{-i}} [q_i(t_{-i}, t_i')\hat{v}_i(t_{-i}, t_i) - x_i(t_{-i}, t_i')] P(t_{-i}|t_i)$$

whenever $t_i, t_i' \in T_i$.

ex post individually rational (XIR) if

$$q_i(t)\hat{v}_i(t) - x_i(t) \geq 0 \text{ for all } i \text{ and all } t \in T.$$

ex post efficient (*XE*) if

$$\hat{v}_i(t) = \max_j \{\hat{v}_j(t)\} \text{ whenever } q_i(t) > 0.$$

For a given auction problem $(v_1, .., v_n, P)$, we will be interested in the second price auction using the conditional values $v_i(t)$. For each $t \in T$, let

$$I(t) = \{i \in N | \hat{v}_i(t) = \max_j \hat{v}_j(t)\}$$

and define

$$w_i(t) = \max_{j:j \neq i} \hat{v}_j(t).$$

Formally, we define a *Vickrey auction with conditional values* (Vickrey auction for short) to be the auction mechanism $\{q_i^*, x_i^*\}_{i \in N}$ defined as follows:

$$q_i^*(t) = \begin{cases} \frac{1}{\#I(t)} \text{ if } i \in I(t) \\ 0 \text{ if } i \notin I(t) \end{cases}$$

and

$$x_i^*(t) = q_i^*(t) w_i(t) \; .$$

It is straightforward to show that this Vickrey auction mechanism is ex post efficient and ex post individually rational. It will generally *not* be incentive compatible. However, as we will show below, it is often possible to modify the Vickrey auction payments so as to make truthful revelation an equilibrium when agents are informationally small in a sense to be defined below. .

Let $\{z_i\}_{i \in N}$ be an n-tuple of functions $z_i : T \to \mathbb{R}_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a "reward" to agent i. The associated *augmented Vickrey auction with conditional values* (augmented Vickrey auction for short) is the auction mechanism $\{q_i^*, x_i^* - z_i\}_{i \in N}$

We present an example in the next section that illustrates our notion of augmented Vickrey auctions and the relationship between informational size and the payments that agents receive. This example also illustrates the main ideas in the proofs of Theorems 1 and 2 discussed in sections 4 and 5 below.

3 Example

3.1 The Problem

Three wildcatters are competing for the right to drill for oil on a tract of land. It is common knowledge that the amount of oil is either 20 or 30, each equally likely. The state in which the quantity is 20 is denoted θ_L and the state in which the quantity is 30 is denoted θ_H; let $\Theta = \{\theta_L, \theta_H\}$. Each wildcatter i performs a private test that provides information in the form of a noisy signal of the state which we denote s_i. That is, agent i's private test yields a signal H (high) or L (low); for each i, let $S_i = \{H, L\}$. The distribution of the signal for agent i, conditional on the state, is given in the table below ($\rho > 1/2$).

	state	θ_L	θ_H
signal			
L		ρ	$1-\rho$
H		$1-\rho$	ρ

Agents' signals are independent, conditional on the state θ. For this information structure, it is relatively easy (though arithmetically tedious) to compute conditional probabilities like $P(\theta|s_1, s_2, s_3)$ and $P(s_1, s_2|s_3)$. For example, let $s_3 = H$. The conditional distribution $S_1 \times S_2$ is defined for each (s_1, s_2) as follows: $P(H, H|H) = \rho^3 + (1-\rho)^3$ and $P(H, L|H) = P(L, H|H) = P(L, L|H) = \rho(1-\rho)$. These specific computations will be used in the analysis that follows.

In addition to the signal regarding the amount of oil, each of the wildcatters has private information regarding his own cost of extraction. We assume that the extraction cost c_i of wildcatter i is drawn from a finite set. Hence, agent i's type t_i is the pair (s_i, c_i) comprising his privately observed extraction cost c_i and his privately observed signal s_i. We will assume that the vector of extraction costs (c_1, c_2, c_3) is independent of the state-signal vector (θ, s_1, s_2, s_3) although the $c_i's$ may be correlated. The price of oil is 1. Agent i's payoff v_i as a function of the state θ and his type depends only on θ and his private extraction cost. If $t_i = (c_i, s_i)$, then his payoff should he obtain the right to drill is given by:

$$\begin{aligned} v_i(\theta_L, t_i) &= 20 - c_i \\ v_i(\theta_H, t_i) &= 30 - c_i. \end{aligned}$$

Consider the following auction mechanism. Agents announce their types and the posterior distribution on θ given the agents' announcements of their signals is calculated. Let $P_\Theta(\cdot|s_1, s_2, s_3)$ denote this posterior distribution on Θ. Next, compute the agents' expected valuations $\hat{v}_i$ for the object, where

$$\hat{v}_i(t_1, t_2, t_3) = \hat{v}_i(s_1, s_2, s_3, c_i) = v_i(\theta_L, c_i) \cdot P_\Theta(\theta_L|s_1, s_2, s_3) + v_i(\theta_H, c_i) \cdot P_\Theta(\theta_H|s_1, s_2, s_3).$$

Let $\{q_i^*, x_i^*\}_{i\in\{1,2,3\}}$ be the associated Vickrey auction defined in section 2. The drilling rights are then given to an agent i for whom $\hat{v}_i(s_1, s_2, s_3, c_i)$ is highest and that agent pays a price equal to the higher of the other two agents' valuations.

3.2 An Incentive Compatible Augmented Vickrey Auction

Suppose that the triple of signals (s_1, s_2, s_3) is commonly known by the agents. Then the "reduced" Vickrey auction in which only the values of the $c_i's$ are private information is a pure private value auction in which bidder i's expected payoff depends only on c_i but not on c_j for $j \neq i$. The vector of signals affects the payoffs as well, but this is a vector of known parameters in the reduced auction. For each fixed triple of signals, the Vickrey mechanism $\{q_i^*, x_i^*\}_{i\in\{1,2,3\}}$ defined for the original auction induces a Vickrey mechanism in the resulting reduced auction in which only the extraction costs c_i are private information.

More formally, fix $i = 3$ and define

$$U_3(s_1, c_1, s_2, c_2, s_3', c_3'|s_3, c_3) := q_i^*(s_1, c_1, s_2, c_2, s_3', c_3')\hat{v}_i(s_1, s_2, s_3, c_3) - x_i^*(s_1, c_1, s_2, c_2, s_3', c_3').$$

¿From this definition, it follows (setting $s_3' = s_3$) that $U_3(s_1, c_1, s_2, c_2, s_3, c_3'|s_3, c_3)$ is precisely the expected payoff to bidder 3 in the reduced auction when the (commonly known) vector of signals is (s_1, s_2, s_3), the announced vector of costs is (c_1, c_2, c_3') and bidder 3 has true cost c_3. In the reduced auction, truthful reporting of c_3 is a dominant strategy in the following sense:

$$U_3(s_1, c_1, s_2, c_2, s_3, c_3|s_3, c_3) \geq U_3(s_1, c_1, s_2, c_2, s_3, c_3'|s_3, c_3)$$

for all signal triples (s_1, s_2, s_3), all pairs (c_1, c_2) and all costs c_3 and c_3' for bidder 3.

Truthful announcement of private costs is a dominant strategy in the reduced auction in which the true vector of signals is known. However, truthful announcement of i's type (s_i, c_i) in the actual problem is not a dominant strategy for the Vickrey mechanism; if bidder i announces L when he has in fact received signal H, he will lower the expected valuations of all agents. In the event that agent i wins the object, he will pay a lower price by doing so. We can, however, construct an augmented Vickrey mechanism that is incentive compatible if ρ is sufficiently close to 1. Let $z_i(s_1, s_2, s_3)$ denote the nonnegative reward to agent i when the announced vector of signals is (s_1, s_2, s_3). In general, the reward functions defining an augmented Vickrey auction will depend on the vector of announced types. In this example, the reward functions can be chosen to depend only on the announced signals and not on the announced private costs.

Now, fix $i = 3$ and suppose that his true type is $t_3 = (c_3, H)$ where $c_3 \in C_3$ (the argument for other agents is exactly the same.) To evaluate the gain from a false report of (c_3', L) in the augmented Vickrey mechanism $\{q_i^*, x_i^* - z_i\}_{i\in\{1,2,3\}}$, we must compute

$$[U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)+z_3(s_1,s_2,L)]-[U_3(s_1,c_1,s_2,c_2,H,c_3|H,c_3)+z_3(s_1,s_2,H)]$$

and then weight these gains with $P(s_1,c_1,s_2,c_2|H,c_3)$ in order to compute the total expected gain from a false report of (c_3',L). When ρ is close enough to one, we will show that the rewards $z_3(\cdot)$ can be chosen so that this expected gain is nonpositive. Hence, the resulting augmented Vickrey auction will be incentive compatible.

First, note that

$$\begin{aligned}
& U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)-U_3(s_1,c_1,s_2,c_2,H,c_3|H,c_3)\\
= & U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)-U_3(s_1,c_1,s_2,c_2,H,c_3'|H,c_3)\\
& +U_3(s_1,c_1,s_2,c_2,H,c_3'|H,c_3)-U_3(s_1,c_1,s_2,c_2,H,c_3|H,c_3)\\
\le & U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)-U_3(s_1,c_1,s_2,c_2,H,c_3'|H,c_3)
\end{aligned}$$

since truthful reporting of c_3 is a dominant strategy in the reduced auction. Since the independence assumption implies that

$$P(s_1,c_1,s_2,c_2|H,c_3)=P(s_1,s_2|H)P(c_1,c_2|c_3),$$

it suffices to show that $z_3(\cdot)$ can be chosen so that the sum of the terms

$$[U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)+z_3(s_1,s_2,L)]-[U_3(s_1,c_1,s_2,c_2,H,c_3'|H,c_3)+z_3(s_1,s_2,H)]$$

weighted by $P(s_1,s_2|H)$ will be nonpositive for each c_1,c_2,c_3 and c_3' when $\rho\approx 1$.

First, we estimate the gain from a misreport of L when bidders 1 and 2 receive different signals. In this case, $P(s_1,s_2|H)\approx 0$ if $s_1\neq s_2$ and $\rho\approx 1$. Therefore,

$$[U_3(s_1,c_1,s_2,c_2,L,c_3'|H,c_3)-U_3(s_1,c_1,s_2,c_2,H,c_3'|H,c_3)]\,P(s_1,s_2|H)$$

may be positive but will be close to zero since utilities are bounded. Hence, the contribution to the *total* expected gain from misreporting when other bidders receive different signals is close to

$$[z_3(L,H,L)-z_3(L,H,H)]\,P(L,H|H)+[z_3(H,L,L)-z_3(H,L,H)]\,P(H,L|H).$$

What is the estimated gain to bidder 3 from a misreport of L when the other two bidders receive the same signal? There are two possibilities. If the other two agents receive signal L, then a false report of L results in a gain of

$$U_3(L,c_1,L,c_2,L,c_3'|H,c_3)-U_3(L,c_1,L,c_2,H,c_3'|H,c_3)+z_3(L,L,L)-z_3(L,L,H).$$

If the other two agents receive signal H, then a false report of L results in a gain of

$$U_3(H,c_1H,c_2,L,c_3'|H,c_3)-U_3(H,c_1,H,c_2,H,c_3'|H,c_3)+z_3(H,H,L)-z_3(H,H,H).$$

When ρ is close to one, a bidder who observes H will believe it very likely that the other bidders' signals are both H and he will believe it very unlikely that the other bidders' signals are both L, i.e., $P(H,H|H) \approx 1$ while $P(L,L|H) \approx 0$.

Since utilities are bounded, the contribution to bidder 3's total expected gain from a misreport when the other two bidders receive the same signal is close to

$$[U_3(H,c_1H,c_2,L,c_3'|H,c_3) - U_3(H,c_1,H,c_2,H,c_3'|H,c_3)]\,P(H,H|H)$$
$$+\,[z_3(L,L,L) - z_3(L,L,H)]\,P(L,L|H) + [z_3(H,H,L) - z_3(H,H,H)]\,P(H,H|H).$$

Summarizing, the *total* expected gain from a false report of L is close to

$$[U_3(H,c_1H,c_2,L,c_3'|H,c_3) - U_3(H,c_1,H,c_2,H,c_3'|H,c_3)]\,P(H,H|H)$$
$$+\sum_{s_1\in S_1}\sum_{s_2\in S_2}[z_3(s_1,s_2,L) - z_3(s_1,s_2,H)]\,P(s_1,s_2|H)$$

when $\rho \approx 1$.

To complete the argument, we must show that this estimated total expected gain can be made nonpositive, when $\rho \approx 1$, by a judicious choice of the reward function $z_3(\cdot)$. Let $z_3(s_1,s_2,s_3) = \varepsilon$ if $s_1 = s_3$ or $s_2 = s_3$ (or both) and zero otherwise. Hence, a bidder will receive a reward of ε for announcing a signal that is in the majority, and nothing otherwise. When $\rho \approx 1$, the estimated total expected gain from a misreport of L is now

$$[U_3(H,c_1H,c_2,L,c_3'|H,c_3) - U_3(H,c_1,H,c_2,H,c_3'|H,c_3)]\,P(H,H|H) - \varepsilon\,[P(H,H|H) - P(L,L|H)]\,.$$

If ρ is sufficiently close to one, this estimate will be nonpositive. This hinges on two features of this example. First, one can verify that $P_\Theta(\cdot|H,H,H) \approx P_\Theta(\cdot|H,H,L)$ and $P(H,H|H) \approx 1$ when $\rho \approx 1$. Consequently[1], we can choose ρ sufficiently close to one so that

$$[U_3(H,c_1H,c_2,L,c_3'|H,c_3) - U_3(H,c_1,H,c_2,H,c_3'|H,c_3)]\,P(H,H|H) < \varepsilon/2.$$

Second, $P(H,H|H) - P(L,L|H) \approx 1$ when $\rho \approx 1$. Consequently, we can choose ρ sufficiently close to one so that

$$P(H,H|H) - P(L,L|H) > 1/2.$$

Combining these two observations, we see that for ρ sufficiently close to one, a misreport leads to a nonpositive expected "gain" in utility. The same argument holds for the case in which an agent observes L but falsely reports H. In particular, the estimated total expected gain from a misreport of H is close to

$$[U_3(L,c_1,L,c_2,H,c_3'|L,c_3) - U_3(L,c_1,L,c_2,L,c_3'|L,c_3)]\,P(L,L|L) - \varepsilon\,[P(L,L|L) - P(H,H|L)]\,.$$

When $\rho \approx 1$, this estimate is also nonpositive and it follows that the mechanism is incentive compatible.

[1] See Lemmas A.1 and A.2 in the appendix for a formal argument.

3.3 Discussion of the Example

In this example, we have shown the following: for every $\varepsilon > 0$, there exists a $\tilde{\rho} > 0$ such that, whenever $\tilde{\rho} < \rho < 1$, there exists an incentive compatible augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i\in\{1,2,3\}}$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for all t. These rewards are small when the agents' signals are accurate as a result of a subtle interplay of two ideas: *informational size* and the *variability of agent's beliefs*. We now illustrate these concepts in the example.

In the example, we have shown that agent 3's estimated total expected gain from announcing signal L when his true signal is H is given by

$$[U_3(H, c_1 H, c_2, L, c_3'|H, c_3) - U_3(H, c_1, H, c_2, H, c_3'|H, c_3)]\, P(H, H|H) \\ + \sum_{s_1\in S_1} \sum_{s_2\in S_2} [z_3(s_1, s_2, L) - z_3(s_1, s_2, H)]\, P(s_1, s_2|H).$$

Using a similar argument, agent 3's estimated total expected gain in utility from announcing signal H when his true signal is L is given by

$$[U_3(L, c_1, L, c_2, H, c_3'|L, c_3) - U_3(L, c_1, L, c_2, L, c_3'|L, c_3)]\, P(L, L|L) \\ + \sum_{s_1\in S_1} \sum_{s_2\in S_2} [z_3(s_1, s_2, H) - z_3(s_1, s_2, L)]\, P(s_1, s_2|L).$$

Consider the first terms in each of these estimates. These "utility" terms may be positive but they will be bounded from above by small positive numbers because agent 3 is "informationally small" in the following sense: the probability that agent 3 has a large effect on the conditional distribution on states $P_\Theta(\cdot|s_1, s_2, s_3)$ is small[2]. When $\rho \approx 1$ in the example, then $P(H, H|H) \approx 1$ and $P_\Theta(\cdot|H, H, H) - P_\Theta(\cdot|H, H, L) \approx 0$. Given a positive number ε, it follows that

$$[U_3(H, c_1 H, c_2, L, c_3'|H, c_3) - U_3(H, c_1, H, c_2, H, c_3'|H, c_3)]\, P(H, H|H) < \varepsilon/2$$

for ρ close enough to one. Similarly, $P(L, L|L) \approx 1$ and $P_\Theta(\cdot|L, L, H) - P_\Theta(\cdot|L, L, L) \approx 0$ imply that

$$[U_3(L, c_1, L, c_2, H, c_3'|L, c_3) - U_3(L, c_1, L, c_2, L, c_3'|L, c_3)]\, P(L, L|L) < \varepsilon/2.$$

Now consider the second term (the "reward" term) in each of these estimates. We constructed rewards $z_3(\cdot)$ that satisfy

$$\sum_{s_1\in S_1} \sum_{s_2\in S_2} [z_3(s_1, s_2, L) - z_3(s_1, s_2, H)]\, P(s_1, s_2|H) < -\varepsilon/2$$

[2]Again, this follows from Lemmas A.1 and A.2.

and

$$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [z_3(s_1, s_2, H) - z_3(s_1, s_2, L)] \, P(s_1, s_2|L) < -\varepsilon/2$$

for ρ close enough to one. Hence the (negative) expected gain in the reward "dominates" the (possibly positive) utility term in the expressions above. Combining these observations, it follows that agent 3 will truthfully reveal his signal. In a more general model in which the probabilistic structure is more complex than the conditionally independent noisy signal structure of the example, our ability to find $z_i's$ for which the expected gain in reward will dominate the expected gain in utility will depend on the difference in the conditional distributions $P(\cdot, \cdot|H)$ and $P(\cdot, \cdot|L)$ on $S_1 \times S_2$. If, for example, these conditional distributions were equal, then we cannot find a system of rewards satisfying the inequalities

$$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [z_3(s_1, s_2, L) - z_3(s_1, s_2, H)] \, P(s_1, s_2|H) < 0$$

and

$$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [z_3(s_1, s_2, H) - z_3(s_1, s_2, L)] \, P(s_1, s_2|L) < 0.$$

If the utility gain terms can be small positive numbers, then we will have difficulty constructing an incentive compatible mechanism. Hence, the closeness of 3's beliefs $P(\cdot, \cdot|H)$ and $P(\cdot, \cdot|L)$ on $S_1 \times S_2$ will play a role in our analysis.

In summary, agents must be informationally small and beliefs must be sufficiently variable in order to construct augmented Vickrey auctions that satisfy incentive compatibility. In the next section, we present a model that generalizes several features of this example and formalizes the concepts of informational size and variability.

4 Efficient Auction Mechanisms: Informational Independence

4.1 The Model

Let $S_1, ..., S_n$ and $C_1, ..., C_n$ be finite sets. An element $s_i \in S_i$ will be referred to as agent i's *signal*. An element $c_i \in C_i$ will be referred to as agent i's *personal* characteristic. Let $T_i = S_i \times C_i, T \equiv T_1 \times \cdots \times T_n$ and $T_{-i} \equiv \times_{j \neq i} T_j$. The product sets S, C, S_{-i}, C_{-i} are defined in a similar fashion. We will often write $t = (s, c)$ and $t_i = (s_i, c_i)$ where s and c (s_i and c_i) denote the respective projections of t (t_i) onto S and C (S_i and C_i). An agent's type is an element $t_i = (s_i, c_i) \in T_i$. Both the signal s_i and the personal characteristic c_i are private information to i with the following interpretations: s_i represents a signal that is correlated with nature's choice of θ and c_i represents a set of other idiosyncratic payoff relevant characteristics of agent i that provide no information about θ or s_{-i} beyond that contained in s_i. In our example, the extraction cost c_i of each wildcatter corresponds to the agent's personal characteristic and, since costs are assumed to be independent of the state and the agents' signals, it is certainly the case that c_i contains no information about θ or s_{-i} beyond that contained in s_i. The next definition formalizes this idea.

Definition: A probability measure $P \in \Delta^*_{\Theta\times S\times C}$ satisfies *Informational Independence* if for each $(\theta, s, c) \in \Theta \times S \times C$, (i) $P_\Theta(\theta|s, c_i) = P_\Theta(\theta|s)$ and (ii) $P_{S_{-i}}(s_{-i}|s_i, c_i) = P_{S_{-i}}(s_{-i}|s_i)$.

Let $\Delta^I_{\Theta\times T}$ denote the set of measures in $\Delta^*_{\Theta\times T}$ satisfying Informational Independence. If P satisfies Informational Independence, then for each s_i, (i) implies that the random variables $\tilde{\theta}$ and $\tilde{c}_i$ are independent conditional on $\tilde{s} = s$ while (ii) implies that the random variables $\tilde{s}_{-i}$ and $\tilde{c}_i$ are independent conditional on $\tilde{s}_i = s_i$. This accounts for the use of the term "independence" in the definition.

The first condition captures the notion that i's personal characteristic c_i contains no information beyond that contained in the signal profile signal s that is useful for predicting the state of nature. The second condition states that $i's$ personal characteristic c_i contains no information beyond that contained in his signal s_i that is useful predicting the signals of other agents.

It is easy to show that Informational Independence is satisfied when the random vectors $(\tilde{\theta}, \tilde{s})$ and $\tilde{c}$ are stochastically independent, i.e., when

$$P(\theta, t) \equiv P(\theta, s, c) = P(\theta, s)P(c).$$

The Informational Independence condition is weaker, however, than stochastic independence of the random vectors $(\tilde{\theta}, \tilde{s})$ and $\tilde{c}$. In the example of section 3, $(\tilde{\theta}, \tilde{s})$ and $\tilde{c}$ are stochastically independent.

4.2 Informational Size and Variability of Beliefs

We now formalize the idea of informational size discussed in section 3.3 above. Our example indicates that a natural notion of an agent's informational size is the degree to which he can alter this posterior distribution on Θ when other agents are announcing truthfully. Any vector of agents' signals $s = (s_{-i}, s_i) \in S$ induces a conditional distribution on $P_\Theta(\cdot|s_{-i}, s_i)$ on Θ and, if agent i unilaterally changes his announcement from s_i to s'_i, this conditional distribution will (in general) change. If i receives signal s_i but announces $s'_i \neq s_i$, the set

$$\{s_{-i} \in S_{-i} |\ ||P_\Theta(\cdot|s_{-i}, s_i) - P_\Theta(\cdot|s_{-i}, s'_i)|| > \varepsilon\}$$

consists of those s_{-i} for which agent i's misrepresentation will have (at least) an "$\varepsilon-$effect" on the conditional distribution. (Here and throughout the paper, $|| \cdot ||$ will denote the 1-norm.) Let $\nu_i^{P,S}(s_i, s_i')$ be defined as the smallest ε such that

$$\text{Prob}\{\ ||P_\Theta(\cdot|\tilde{s}_{-i}, s_i) - P_\Theta(\cdot|\tilde{s}_{-i}, s_i')|| > \varepsilon | \tilde{s}_i = s_i\} \leq \varepsilon$$

and define the *informational size* of agent i as

$$\nu_i^{P,S} = \max_{s_i, s_i' \in S_i} \nu_i^{P,S}(s_i, s_i').$$

Note that $\nu_i^P = 0$ for every i if and only if $P_\Theta(\cdot|s) = P_\Theta(\cdot|s_{-i})$ for every $s \in S$ and $i \in N$. [3]

There are two important features of this definition. First, an agent's informational size depends only on that part of his information that is useful in predicting θ. Hence, an agent's type $t_i = (s_i, c_i)$ can embody substantial amounts of information (in c_i) that is not known to other agents, yet agent i can still be informationally small. This is a feature of the example in section 3. Second, an informationally small agent may have very accurate information about the state θ. Informational size depends on how much the information contained in his signal adds to the information contained in the aggregate of the other agents' signals. When other agents also have very accurate information, an agent with accurate information may add little. This is also a feature of the example of section 3 where the conditional $P_\Theta(\cdot|s_i)$ is nearly degenerate for each i and each s_i when ρ is close to 1. Hence, agents have very good estimates of the true state conditional on their signals, yet each agent is informationally small.

In our discussion in section 3.3 above, we indicated that the ability to give agent i an incentive to reveal his information will depend on the magnitude of the difference between $P_{S_{-i}}(\cdot|s_i)$ and $P_{S_{-i}}(\cdot|s_i')$, the conditional distributions on S_{-i} given different signals for agent i. We will refer to this magnitude informally as the *variability of agents' beliefs.*

To define formally the measure of variability, we treat each conditional $P_{S_{-i}}(\cdot|s_i) \in \Delta_{S_{-i}}$ as point in a Euclidean space of dimension equal to the cardinality of S_{-i}. Our measure of variability is defined as

$$\Lambda_i^{P,S} = \min_{s_i \in S_i} \min_{s_i' \in S_i \setminus s_i} ||P_{S_{-i}}(\cdot|s_i) - P_{S_{-i}}(\cdot|s_i')||^2.$$

4.3 The Result

We now state our first result on the possibility of efficient mechanisms.

Theorem 1: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) If $P \in \Delta_{\Theta\times T}^I$ satisfies $\Lambda_i^{P,S} > 0$ for each i, then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i\in N}$ for the auction problem $(v_1, .., v_n, P)$.

[3] This is essentially the case of nonexclusive information introduced by Postlewaite and Schmeidler (1986) and is discussed further in the last section.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^I_{\Theta \times T}$ satisfies

$$\max_i \nu_i^{P,S} \leq \delta \min_i \Lambda_i^{P,S},$$

there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem $(v_1, .., v_n, P)$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t.

Part (i) of Theorem 1 states that, if $\Lambda_i^{P,S}$ is positive for each agent i, then there exists an incentive compatible augmented Vickrey mechanism for the auction problem $(v_1, .., v_n, P)$. The hypotheses of part (i) only requires that each $\Lambda_i^{P,S}$ be positive and places no lower bound on the magnitude of $\Lambda_i^{P,S}$. Furthermore, the informational size of the agents is not important. On the other hand, the conclusion of part (i) places no upper bound on the size of the reward z_i. These rewards can be quite large.

Part (ii) of the theorem states that there exists an incentive compatible augmented Vickrey mechanism with small payments if, for each i, $\Lambda_i^{P,S}$ is large enough relative to the informational size of agent i. To illustrate part (ii), consider again the example in section 3 where we showed the following: for every $\varepsilon > 0$, there exists a $\tilde{\rho} > 0$ such that, whenever $\tilde{\rho} < \rho < 1$, there exists an incentive compatible augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in \{1,2,3\}}$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for all t. This result can now be deduced as an application of (ii) since, in the example, each $\nu_i^{P,S} \to 0$ and each $\Lambda_i^{P,S} \to 1$ as $\rho \to 1$.

While the technical details of the proof are deferred until the last section, we can sketch the ideas here for the special case in which $T_i = S_i$ (i.e., each C_i is a singleton). There are two key steps. First, we show (see Lemmas A.1 and A.2) that for all i , all $s_i, s_i' \in S_i$ and all $s_{-i} \in S_{-i}$,

$$(q_i^*(s)\hat{v}_i(s) - x_i^*(s)) - (q_i^*(s_{-i}, s_i')\hat{v}_i(s) - x_i^*(s_{-i}, s_i')) \geq -M||P_\Theta(\cdot|s_{-i}, s_i) - P_\Theta(\cdot|s_{-i}, s_i')||$$

where

$$M = \max_\theta \max_i \max_{s_i} v_i(\theta, s_i).$$

This result is of some interest in its own right. If $||P_\Theta(\cdot|s_{-i}, s_i) - P_\Theta(\cdot|s_{-i}, s_i')||$ is "small" uniformly in s_i, s_i' and s_{-i}, then truthful reporting is an "approximate" dominant strategy in the (unaugmented) Vickrey mechanism $\{q_i^*, x_i^*\}$. For example, if $\tilde{\theta}$ and $\tilde{s}$ are independent, then $\hat{v}_i(s)$ depends only on s_i. In this case, $||P_\Theta(\cdot|s_{-i}, s_i) - P_\Theta(\cdot|s_{-i}, s_i')|| = 0$ for all i , all $s_i, s_i' \in T_i$ and all $s_{-i} \in S_{-i}$ and we deduce the classic result for Vickrey auctions: truthful reporting is a dominant strategy with pure private values.

Of course, $||P_\Theta(\cdot|s_{-i}, s_i) - P_\Theta(\cdot|s_{-i}, s_i')||$ is generally not uniformly small. However, we can use the concept of informational size to show that

$$\sum_{s_{-i}} \left[(q_i^*(s)\hat{v}_i(s) - x_i^*(s)) - (q_i^*(s_{-i}, s_i')\hat{v}_i(s) - x_i^*(s_{-i}, s_i'))\right] P(s_{-i}|s_i) \geq -3M\hat{\nu}_i^{P,S}.$$

If all agents are informationally small, then truthful reporting is "approximately" incentive compatible in the (unaugmented) Vickrey mechanism $\{q_i^*, x_i^*\}$. If $z_i(s)$ is the reward to i when the bidders announce s, then the associated augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}$ will be incentive compatible if and only if

$$\sum_{s_{-i}} [z_i(s_{-i}, s_i) - z_i(s_{-i}, s_i')] \, P(s_{-i}|s_i) - 3M\hat{\nu}_i^{P,S} \geq 0$$

for each $s_i, s_i' \in S_i$. This is the generalization of the analysis of the example in section 3.3.

It can be shown that there exists a collection of numbers $\zeta_i(s)$ satisfying $0 \leq \zeta_i(s) \leq 1$ and

$$\sum_{s_{-i}} [\zeta_i(s_{-i}, s_i) - \zeta_i(s_{-i}, s_i')] \, P(s_{-i}|s_i) > 0$$

for each $s_i, s_i' \in S_i$ if and only if $\Lambda_i^{P,S} > 0$. Part (i) of the theorem now follows: choose $\zeta_i(s)$ to satisfy these inequalities, define $z_i(s) = \alpha\zeta_i(s)$ and choose α large enough so that incentive compatibility is satisfied. Of course, as we mentioned above, the resulting $z_i' s$ can be large.

Part (ii) is more delicate. Unfortunately, the optimal value $val_i(P)$ of the linear program

$$\max_{\beta, \zeta_i(s)} \ \beta$$

$$s.t. \ \sum_{s_{-i}} [\zeta_i(s_{-i}, s_i) - \zeta_i(s_{-i}, s_i')] \, P(s_{-i}|s_i) \geq \beta \text{ for all } s_i, s_i'$$

$$0 \leq \zeta_i(s) \leq 1 \text{ for all } s$$

is not bounded from below by a positive number, uniformly in P. If this were the case, then the existence of an incentive compatible augmented Vickrey auction with small payments would depend *only* on informational size. Instead, $val_i(P) \to 0$ as $\Lambda_i^{P,S} \to 0$. In order to prove (ii), we require that each $\Lambda_i^{P,S}$ be large enough relative to the informational size of agent i.

5 Efficient Auction Mechanisms: The General Case

5.1 The model

The information structure in the previous section assumed that the set of types of an agent could be expressed as the Cartesian product of signals and personal characteristics and that the information structure satisfied our informational independence assumption. In this section, we show how the information structure for general incomplete information problems, even those without a product structure, can be represented in a way that decomposes agents' information into "signals" and "private characteristics."

Definition: For each i, let $\Pi_i = \{A_i^1, A_i^2, ..., A_i^{k_i}\}$ be a partition of T_i. The collection of partitions $\mathcal{C} = \{\Pi_1, .., \Pi_n\}$ is an *information decomposition* (ID) of $P \in \Delta^*_{\Theta \times T}$ if for each $(A_1, .., A_n) \in \Pi_1 \times \cdots \times \Pi_n$, the following hold:

(i) for all $t, \hat{t} \in A_1 \times \cdots \times A_n$,

$$P_\Theta(\cdot|t) = P_\Theta(\cdot|\hat{t}).$$

(ii) for all i and for all $t_i, t_i' \in A_i$,

$$\text{Prob}\{\tilde{t}_j \in A_j, j \neq i | \tilde{t}_i = t_i\} = \text{Prob}\{\tilde{t}_j \in A_j, j \neq i | \tilde{t}_i = t_i'\}.$$

Roughly speaking, one can identify the event $A_i \in \Pi_i$ with the signal s_i in the model of section 4 and t_i with the personal characteristic c_i. To see this, choose an "event" profile $(A_1, .., A_n)$ and note that condition (i) in the definition of ID is equivalent to

$$\text{Prob}\{\tilde{\theta} = \theta | \ \tilde{t} \in A_1 \times \cdots \times A_n\} = \text{Prob}\{\tilde{\theta} = \theta | \ \tilde{t} = t\}$$

for each $t \in A_1 \times \cdots \times A_n$. Therefore, condition (i) has an interpretation similar to that of condition (i) in the definition of informational independence: a specific type profile $t \in A_1 \times \cdots \times A_n$, contains no information beyond that contained in the event profile $(A_1, .., A_n)$ that is useful predicting the state of nature.

Condition (ii) is equivalent to

$$\text{Prob}\{\tilde{t}_j \in A_j, j \neq i | \tilde{t}_i \in A_i\} = \text{Prob}\{\tilde{t}_j \in A_j, j \neq i | \tilde{t}_i = t_i\}$$

for all $t_i \in A_i$. Hence, condition (ii) has an interpretation similar to that of condition (ii) in the definition of informational independence: a specific type $t_i \in A_i$ contains no information beyond that contained in the event A_i that is useful predicting the events of other agents.

Every measure P has at least one information decomposition: this is the trivial decomposition in which $\Pi_i = \{\{t_i\}\}_{t_i \in T_i}$. However, a measure P can have more than one ID. If each $T_i = S_i \times C_i$ as in section 4 and if P satisfies Informational Independence, then P has a second common value decomposition defined by

$$\Pi_i = \{\{s_i\} \times C_i\}_{s_i \in S_i}.$$

5.2 Informational Size and Variability of Beliefs

For each information decomposition $\mathcal{C}$ of a measure P, one can define informational size and variability. To see this, define a "natural" probability measure $P^{\mathcal{C}}$ on $\Theta \times \Pi_1 \times \cdots \times \Pi_n$ as follows: for each $(A_1, .., A_n) \in \Pi_1 \times \cdots \times \Pi_n$,

$$\begin{aligned} P^{\mathcal{C}}(\theta, A_1, .., A_n) &= \text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1 \in A_1, .., \tilde{t}_n \in A_n\} \\ &= \sum_{t_1 \in A_1} \cdots \sum_{t_n \in A_n} P(\theta, t_1, .., t_n). \end{aligned}$$

Let $P^{\mathcal{C}}_{\Theta}(\cdot|A_1, .., A_n) \in \Delta_{\Theta}$ denote the induced conditional distribution on Θ given the events $A_1, .., A_n$. Let $P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i) \in \Delta_{\Pi_{-i}}$ denote the induced conditional distribution on $\Pi_{-i} = \times_{j \neq i} \Pi_j$ given that i's event is A_i. These induced conditional probability measures can be used to define informational size and variability in ways analogous to the definitions of these concepts given in section 4. Let

$$\Phi^{\varepsilon}_i(A_i, A'_i) = \{A_{-i} \in \Pi_{-i} \,|\, ||P^{\mathcal{C}}_{\Theta}(\cdot|A_{-i}, A_i) - P^{\mathcal{C}}_{\Theta}(\cdot|A_{-i}, A'_i)|| > \varepsilon\}.$$

Let $\nu^{P,\mathcal{C}}_i(A_i, A'_i)$ be defined as the smallest ε such that

$$\sum_{A_{-i} \in \Phi^{\varepsilon}_i(A_i, A'_i)} P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) \leq \varepsilon$$

and define the informational size of agent i as

$$\nu^{P,\mathcal{C}}_i = \max_{A_i, A'_i \in \Pi_i} \nu^{P,\mathcal{C}}_i(A_i, A'_i).$$

Proposition 1: If $\mathcal{C}$ and $\mathcal{C}'$ are information decompositions of a measure $P \in \Delta^*_{\Theta \times T}$, then $\nu^{P,\mathcal{C}}_i = \nu^{P,\mathcal{C}'}_i$.

Proposition 1 states that informational size is invariant with respect to the choice of information decomposition. For example, when each $T_i = S_i \times C_i$ and the informational independence assumption is satisfied, then the ID with $\Pi_i = \{\{s_i\} \times C_i\}_{s_i \in S_i}$ for each i and the ID with $\Pi_i = \{\{t_i\}\}_{t_i \in T_i}$ for each i yield the same informational size for each agent.

The correct notion of distributional variability for an information decomposition should depend only on the partitions Π_i. The relevant notion of variability measures the difference between $P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i)$ and $P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A'_i)$ for $A_i, A'_i \in \Pi_i$.

Formally, our measure of variability is defined as

$$\Lambda^{P,\mathcal{C}}_i = \min_{A_i \in \Pi_i} \min_{A'_i \in \Pi_i \setminus A_i} ||P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i) - P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A'_i)||^2.$$

5.3 The Results

Using these definitions of informational size and variability of beliefs, we can generalize Theorem 1 as follows.

Theorem 2: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) Let $P \in \Delta^*_{\Theta \times T}$. If there exists an information decomposition $\mathcal{C} = \{\Pi_1, .., \Pi_n\}$ with $\Lambda_i^{P,\mathcal{C}} > 0$ for each i, then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem $(v_1, .., v_n, P)$.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^{P,\mathcal{C}} \leq \delta \min_i \Lambda_i^{P,\mathcal{C}}$$

for some information decomposition $\mathcal{C}$ of P, there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem $(v_1, .., v_n, P)$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t.

Theorem 1 is an immediate corollary of Theorem 2. To see this, let $\Pi_i = \{\{s_i\} \times C_i\}_{s_i \in S_i}$. If $\hat{s} \in S$ is a vector of signals and if $A_j = \{\hat{s}_j\} \times C_j$ for each j, then $P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i} | \{\hat{s}_i\} \times C_i) = P_{S_{-i}}(\hat{s}_{-i} | \hat{s}_i)$ and $P^{\mathcal{C}}_{\Theta}(\cdot | A_1, .., A_n) = P_{\Theta}(\cdot | \hat{s})$ for each i. Hence, $\nu_i^{P,\mathcal{C}} = \nu_i^{P,S}$, $\Lambda_i^{P,\mathcal{C}} = \Lambda_i^{P,S}$ and Theorem 1 follows.

It is possible that a measure P has only one ID, the trivial decomposition which we denote $\mathcal{C}^0$. For this decomposition, it is easy to verify that $\nu_i^{P,\mathcal{C}^0}(A_i, A_i')$ is the smallest positive ε such that

$$\text{Prob}\{\ ||P_{\Theta}(\cdot | \tilde{t}_{-i}, t_i) - P_{\Theta}(\cdot | \tilde{t}_{-i}, t_i')|| > \varepsilon | \tilde{t}_i = t_i\} \leq \varepsilon.$$

Furthermore,

$$\Lambda_i^{P,C^0} = \min_{t_i \in T_i} \min_{t_i' \in T_i \setminus t_i} ||P_{T_{-i}}(\cdot | t_i) - P_{T_{-i}}(\cdot | t_i')||^2$$

where $P_{T_{-i}}(\cdot | t_i)$ is the conditional on T_{-i} given $\tilde{t}_i = t_i$. For the trivial ID $\mathcal{C}^0$, we have the following corollary to Theorem 2.

Corollary 1: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) If $P \in \Delta_{\Theta \times T}$ satisfies $P_{T_{-i}}(\cdot | t_i) \neq P_{T_{-i}}(\cdot | t_i')$ for each $i = 1, ..., n$ and for each $t_i, t_i^{'} \in T_i$ with $t_i \neq t_i^{'}$, then there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem $(v_1, .., v_n, P)$.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}$ satisfies

$$\max_i \nu_i^{P,\mathcal{C}^0} \leq \delta \min_i \Lambda_i^{P,\mathcal{C}^0},$$

there exists an incentive compatible Augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i \in N}$ for the auction problem $(v_1, .., v_n, P)$ satisfying $0 \leq z_i(t) \leq \varepsilon$ for every i and t.

As a final remark on the relationship between our results, we note that Corollary 1 can also be deduced as a special case of Theorem 1 in which each C_i is a singleton and T_i is identified with S_i. If each C_i is a singleton, then informational independence is trivially satisfied and Corollary 1 follows from Theorem 1.

6 A Simple Nonrevelation Mechanism

We analyzed augmented Vickrey auctions above and demonstrated that they can achieve efficient outcomes with small augmenting payments when agents are informationally small. The general mechanism design approach that we use in this paper has been criticized on the grounds that revelation games are unrealistic for many problems. The examples used to illustrate mechanisms typically have simple information structures, as in our example in section 3, in which an agent's type is simply a pair of numbers - the quantity of oil and the cost of extracting it. In general, however, an agent's type encompasses all information he may have, including his beliefs about all relevant characteristics of the object, his beliefs about others' beliefs, etc. When types are realistically described, it seems unlikely that the revelation game could actually be played.

We are sympathetic to this argument; however, we want to stress that the underlying logic by which efficient outcomes are obtained in our model does not depend on the particular revelation game we used; the same outcome can be obtained through a non-revelation game. Consider first the following two-stage reformulation of our augmented Vickrey auction. In the first stage, agents announce their types; each bidder i receives a payment, $z_i(t)$, that depends on the vector of announcements. The announced types are made common knowledge among the bidders. In the second stage, the bidders engage in a standard Vickrey auction.

This reformulation is still essentially a revelation game, but it differs from the original form in that agents control their bids for the object rather than having those bids computed by the mechanism directly. However, if the announced vector in the first stage, t, is truthful, each agent i knows his value for the object given this information, $\hat{v}_i(t)$. Then, just as in a standard Vickrey auction, announcing this value in the second stage is a dominant strategy. The argument that there exist augmented payment functions $\{z_i(t)\}_{i\in N}$ that make truthful revelation in the first stage a Bayesian equilibrium is exactly as before.

This reformulation of our augmented Vickrey auction focuses attention on the way the revelation of agents' types affects the second stage: it transforms the initial asymmetric information problem into a symmetric information problem in which all agents have the aggregate information available to the set of agents. It also suggests nonrevelation games that might serve the same purpose as the revelation stage in the mechanism above. We will illustrate this two stage game in an example.

Consider a three bidder problem (similar to that of section 3) in which $\Theta =$

$\{\theta_L, \theta_H\}$ where $\theta_L = 20$ and $\theta_H = 30$. An agent's type is a noisy signal of the state denoted t_i. In particular, agent i's signal t_i will be H (high) or L (low), and the distribution of the signal for agent i, conditional on the state, is given in the table below.

	state	θ_L	θ_H
signal			
L		ρ	$1-\rho$
H		$1-\rho$	ρ

Agents' types are independent, conditional on the state θ, the two states are equally likely and $\rho > 1/2$. Suppose that

$$v_i(\theta, t_i) = \theta$$

for each i. As ρ gets closer to one, the informational size of each agent goes to zero. Furthermore, $P(H, H|H) = \rho^3 + (1-\rho)^3 = P(L, L|L)$, so both converge to one as ρ gets closer to one. For every $\varepsilon > 0$, we can apply Corollary 1 and deduce that, for all ρ close enough to 1, there exists an incentive compatible augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i\in\{1,2,3\}}$ with $0 \leq z_i \leq \varepsilon$ for every i.

Our goal here is to construct a nonrevelation game that proceeds in two stages. The second stage is a sealed bid second price auction. In the first stage, each bidder forecasts the highest bid, different from his own, that will be submitted in the second stage. At the end of stage 1, the forecasts are publicly revealed to all agents. Agents then submit their bids and the winner is determined. In addition, any agent whose forecast of the highest second stage bid other than his own is within 4 of that bid will receive a payment equal to 1. We will exhibit strategies for this two stage mechanism that are equilibrium strategies when agents are sufficiently small informationally, and which assure an efficient outcome.

A strategy for bidder i is a pair (α_i, β_i) where

$$\alpha_i : T_i \to \Re$$

and

$$\beta_i : T_i \times \Re^3 \to \Re$$

where $\alpha_i(t_i)$ represents i's forecast of the higher of the other two second stage bids as a function of his signal t_i, and $\beta_i(t_i, a_1, a_2, a_3)$ represents i's bid in the second stage Vickrey auction as a function of his signal t_i and the vector of forecasts (a_1, a_2, a_3). Consider strategies for which an agent's forecast function is the highest expected value of the object, conditional on t_i, to the players different from i. Formally,

$$\hat{\alpha}_i(t_i) = \max_{j:j\neq i} E[\hat{v}_j(\tilde{t})|\tilde{t}_i = t_i]$$

Because of the symmetry of the example, this expected value is the same for all other players. Therefore,

$$\hat{\alpha}_i(H) \equiv \hat{\alpha}(H) = E[\tilde{\theta}|\tilde{t}_i = H] = 30\rho + 20(1-\rho)$$

and

$$\hat{\alpha}_i(L) \equiv \hat{\alpha}(L) = E[\tilde{\theta}|\tilde{t}_i = L] = 20\rho + 30(1-\rho).$$

Since $\rho > 1/2$, it follows that $\hat{\alpha}(H) > \hat{\alpha}(L)$ so the function $\hat{\alpha}$ is "invertible." If each agent i uses $\hat{\alpha}_i(\cdot) = \hat{\alpha}(\cdot)$ at the first stage, then, at the second stage, each bidder can infer the signals of the other two agents from their forecasts. To construct the second stage strategies, define an "inference" function $\sigma : \Re \to \{H, L\}$ where $\sigma(a) = H$ if $a \geq 25$ and $\sigma(a) = L$ if $a < 25$. In the second stage, each bidder bids his expected value given his own type and the inferred signals:

$$\hat{\beta}_1(t_1, a_1, a_2, a_3) = \hat{v}_i(t_1, \sigma(a_2), \sigma(a_3))$$

$$\hat{\beta}_2(t_2, a_1, a_2, a_3) = \hat{v}_2(\sigma(a_1), t_2, \sigma(a_3))$$

$$\hat{\beta}_3(t_3, a_1, a_2, a_3) = \hat{v}_3(\sigma(a_1), \sigma(a_2), t_3).$$

To show that $(\hat{\alpha}_i, \hat{\beta}_i)_{i \in N}$ is a perfect Bayesian equilibrium, we need beliefs. Agent i will use the forecasts of other agents to form beliefs about the signals those agents have received. A belief function for i is a map that associates with each forecast vector a_{-i} a probability measure $\mu_i(\cdot|a_{-i}) \in \Delta_{T_{-i}}$. For any number a, define a probability measure $\delta(\cdot|a)$ on $\{H, L\}$ as follows

$$\begin{aligned} (\delta(H|a), \delta(L|a)) &= (1,0) \text{ if } a \geq 25 \\ (\delta(H|a)\delta(L|a)) &= (0,1) \text{ if } a < 25. \end{aligned}$$

Now define a belief function $\hat{\mu}_i$ for i as follows: for each $a_{-i} \in \Re^2$,

$$\hat{\mu}_i(t_{-i}|a_{-i}) = \prod_{j \neq i} \delta(t_j|a_j).$$

Note that these beliefs are consistent with Bayes rule given the forecasting functions $\hat{\alpha}_i$.

Proposition 2: The strategy profile $(\hat{\alpha}_i, \hat{\beta}_i)_{i=1,2,3}$, together with the belief functions $(\hat{\mu}_i)_{i=1,2,3}$, constitute a symmetric perfect Bayesian equilibrium of the two stage bidding problem.

As we noted above, when agents follow these strategies, all private information is revealed prior to the second stage Vickrey auction, since the first stage forecasts are completely revealing. Conditional on all private information being revealed prior to the second stage Vickrey auction, it is a dominant strategy to bid one's true value for the object in the Vickrey auction, and the second stage strategies $\hat{\beta}_i$ are optimal if the first stage strategies $\hat{\alpha}_i$ are followed. It remains to be shown that the first stage strategies $\hat{\alpha}_i$ are optimal.

When a bidder makes a forecast in the first stage, he faces the same problem described in the example in section 3: making an accurate forecast may reveal information to other bidders, and hence may affect the price that the bidder will pay should he win the object in the second stage. Since agents are using belief functions that convert forecasts into degenerate probabilities on signals, forecasting is equivalent to a revelation game. Making deliberately misleading forecasts is similar to misreporting one's type in the original revelation game. The same argument made there applies here. If ρ is close to one (so the informational size of each bidder is close to 0), then with very high probability, a misreported type by a single bidder will not greatly affect other bidders' expected values. Thus, the expected gain to any given bidder from misleading the other bidders, and possibly lowering the price he will pay should he win goes to 0 as ρ goes to 1. Furthermore, as ρ goes to 1, it is increasingly likely that all bidders receive a signal that is identical to the true state of nature. If bidder i sees signal H, then the highest of the other second stage bids will, with high probability, be close to 30 irrespective of bidder i's announcement. Forecasting 30 when the signal H is received will get the prize of 1 almost surely (similarly forecasting 20 when the signal L is received will almost surely get the prize). For ρ sufficiently high, the strategies $(\hat{\alpha}_i, \hat{\beta}_i)$ then constitute an equilibrium of the two stage mechanism.[4]

The mechanism described above appends a forecast stage to a standard Vickrey auction and bears some resemblance to problems in which a communication stage is added to a Bayesian game. While there is some similarity, there is an important difference. When a Bayesian game is expanded to include the possibility of cheap talk, the equilibrium set of the resulting communication game typically includes a "babbling equilibrium" in which all players send messages independently of their types (and, consequently, no information is revealed). In the second stage, player i ignores the messages of other players and chooses an action in the second stage as a functions his type only. More precisely, for any first stage message profile, the beliefs of a player i of type t_i regarding the types of other players are constant and equal to his prior given t_i and players choose second stage strategies that are identical to a Bayes-Nash equilibrium of the original game. In the communication game, it is optimal for players to babble in the first stage since players do not condition actions in the second stage on first stage messages. Since players' beliefs are simply their priors, irrespective of the first stage messages, any Bayes-Nash equilibrium of the original game can be used in the second stage to complete the description of the babbling

[4] As in previous sections, there may be other equilibria as well; we focussed only on the equilibrium that corresponds to the truthful equilibrium in the revelation game. The issue of multiple equilibria is discussed in more detail in the next section.

equilibrium. This is simply the standard argument used to show that any equilibrium outcome of the original Bayesian game corresponds to an equilibrium outcome of the communication game.

An augmented Vickrey auction is a Bayesian game and each Bayes-Nash equilibrium is naturally associated with a babbling equilibrium in the expanded communication game. However, the forecast stage of the game above is not a "cheap talk" extension of the Vickrey auction since the forecasts have payoff implications: players who forecast well will receive rewards. There are strategies in our forecast game with features similar to the babbling equilibrium in a game with communication. In the first stage, players make forecasts that are constant with respect to their signals (and, consequently, no information is revealed). In the second stage, players ignore the forecasts and choose bids in the second stage as functions of their personal signals only. More precisely, for any first stage *forecast* profile, the beliefs of a bidder i of type t_i regarding the signals of other bidders are constant and equal to his prior given t_i and bidders choose second stage strategies that are identical to a Bayes-Nash equilibrium, say $(\sigma_1, \sigma_2, \sigma_3)$, of the original augmented Vickrey auction. These strategies and beliefs, however, may not constitute a perfect Bayesian equilibrium in our game because of the payoff relevance of the forecasts. To get the prize in the example, an agent's forecast of the high bid other than his own must be within 4 of that bid. If ρ is close to 1, and bidder 1 (e.g.) receives the signal H, then bidder 1 believes it very likely (since $P(H, H|H) \approx 1$) that the other two bidders will be submitting Bayes-Nash equilibrium bids $\sigma_2(H)$ and $\sigma_3(H)$. If ρ is close to 1, and bidder 1 receives the signal L, then bidder 1 believes it very likely (since $P(L, L|L) \approx 1$) that the other two bidders will be submitting Bayes-Nash equilibrium bids $\sigma_2(L)$ and $\sigma_3(L)$. If

$$|\max\{\sigma_2(H), \sigma_3(H)\} - \max\{\sigma_2(L), \sigma_3(L)\}| \geq 8$$

then a first stage separating strategy will yield the prize with high probability irrespective of bidder 1's signal while a pooling strategy in the first stage will yield the prize for only one of these realized signals.

This example suggests how a pre-auction stage in which agents forecast some aspect of the auction can be of use. If agents forecast the winning bid, the median bid, the average bid, or some other statistic of the actual auction bids, when their valuation is correlated with other agents' valuations, their best forecasts will typically at least partially reveal their private information. Payments that are related to the accuracy of their forecasts can give them an incentive to improve their forecasts, and when bidders are informationally small, the prizes needed to provide incentives need not be large.

7 Discussion

1. As pointed out in the example, truthful revelation is an equilibrium for our augmented Vickrey auction mechanisms, but not the unique equilibrium. Furthermore, the additional equilibria will generally be inefficient. There are two relevant points in this regard. First, much of the work on auctions is in the tradition of "weak" implementation where the approach is to maximize an objective function subject to incentive constraints. The models of this paper give rise to the same concerns regarding additional equilibria as those in the previous literature. In fairness, however, the nontruthful equilibrium in the example in section 3 is potentially more problematic than the nontruthful equilibria in some models simply because agents can presumably identify the equilibrium strategies in the example more easily than in many models.

The second point is that, while truth is not a unique equilibrium, there is a vast literature on "exact" implementation (see, e.g., Postlewaite and Schmeidler (1986), or the surveys of Moore (1992) and Palfrey (1992)) that demonstrates how outcomes that are truthful equilibria of revelation games can be achieved as unique equilibrium outcomes through a clever choice of a nonrevelation game. There is no reason to believe that such techniques could not be used for the problem considered in this paper, but it is beyond the scope of this paper to do so.

2. In section 4, we assume that agents' type sets are finite. If the signals and personal characteristics of agents' information are separated, it is only the signal sets that need to be finite. The personal part can be finite, a continuum or some combination without affecting the possibility of efficient mechanisms.

3. In McLean and Postlewaite (2000) we used a notion similar to the idea of informational size above. That paper dealt with pure exchange economies with private information in which an agent's utility functions depended only on the realized state $\theta \in \Theta$. The preferences in the present paper are more general in the sense that agent i's utility may depend on depend on his type t_i as well as the state. The extension of the methods is possible because of the properties of the Vickrey auction for which there are no counterparts in a general equilibrium environment.

4. We treated the case of a single object to be sold. Our techniques can be extended to the problem of auctioning K identical objects when bidders' valuations exhibit "decreasing marginal utility." i.e., when $v_i(k+1,\theta,t_i) - v_i(k,\theta,t_i) \geq v_i(k+2,\theta,t_i) - v_i(k+1,\theta,t_i)$ where $v_i(k,\theta,t_i)$ is the payoff to bidder when the state is θ, his type is t_i and he is awarded k objects.

5. We now expand briefly on the relationship of our paper to those of Cremer and McLean on full surplus extraction (1985,1988). The main point of the Cremer-McLean papers is that correlation of agents' types allows full surplus extraction. In the models in those papers (as in the present paper), players' payoffs include payments that depend on other agents' types. In the Cremer-McLean setup, the type

of correlation (for example, the full rank condition in their 1985 paper) permits the construction of announcement dependent lotteries, where truthful revelation generates a lottery with zero conditional expected value while a lie generates negative conditional expected value. If the lotteries are appropriately rescaled, then the incentive for truthful reporting can be made arbitrarily large and an incentive compatible mechanism that extracts the full surplus can be found.

In part (i) of (for example) Corollary 1, we only require that the conditional distribution on T_{-i} be different for different $t_i's$. That is, we only require that $\Lambda_i^{P,\mathcal{C}_0}$ be positive. This is weaker than the full rank condition (and is also weaker than the cone condition in their 1988 paper) and the implication is concomitantly weaker. Our assumption only permits the construction of announcement dependent lotteries where truthful revelation generates a lottery whose conditional expected value exceeds the conditional expected value from a lie. Using the full rank condition and some additional assumptions on the conditional payoff $\hat{v}(t)$, Cremer-McLean construct a mechanism that extracts the full surplus from bidders (see Corollary 2 in Cremer-McLean, 1985). This mechanism is necessarily ex post efficient. Under the weaker conditions of this paper, we construct (in part (i)) a mechanism that is ex post efficient but which may not extract the full surplus. In addition, the payments in a Cremer-McLean mechanism can be positive or negative and they can be large in absolute value. Our paper differs in that we introduce only nonnegative payments. Hence, our techniques do not require unlimited liability on the part of buyers (although the seller may be constrained by the necessary payments that would induce incentive compatibility).

The more interesting part of our results - the ability to induce incentive compatibility with *small* payments when agents are informationally small - has no counterpart in the Cremer-McLean analysis.

6. We have focused on efficient auction mechanisms in this paper. As mentioned above, the problem is closely related to the problem of surplus extraction. The revenue raised in our auctions leaves bidders with some surplus. This is easy to see, since the outcome of the auction mechanism is the same as if all information were common knowledge and we used a standard Vickrey auction, which typically leaves bidders with positive surplus. The mechanism does extract most of the surplus associated with the signal part of agents' information. Agents get some surplus through the receipt of the payments used to induce incentive compatibility, but as the theorems demonstrate, the size of that surplus is related to agents' informational size.

We do not claim, however, that our mechanism maximizes the seller's surplus; it clearly does not in some cases. If the personal parts of bidders' types are correlated, some of the surplus that buyers get in the conditional Vickrey auction can be captured with a Cremer-McLean type scheme. It is easy to see that there sometimes exist such modifications that do not alter the efficiency of the outcome.

7. Many auction papers restrict attention to symmetric problems in which bidders'

types are drawn from the same distribution. It should be noted that we make no assumptions on the distribution of bidders' types. However, if agents' beliefs exhibit positive variability, then their types cannot be independent. Several papers analyzing interdependent value auction problems make assumptions regarding the impact of a bidder's information on his own valuation relative to other bidders' valuations (see, e.g., Maskin (1992), Dasgupta and Maskin (1998) and Perry and Reny (1998). We make no such assumptions.

8. The example in section 3 illustrates the way in which multidimensional signals naturally arise, where agents receive signals both about the quantity of oil in the tract and their cost of extracting the oil. Situations in which the signal space would be of even higher dimension would naturally arise if there is more than a single physical characteristic that describes the object being sold. For the example of the oil field, in addition to the amount of oil, it might be important to account for the depth of the oil, the kind of rock through which one must drill to reach the oil, the sulphur content of the oil, etc. Although signals sets are finite, we make no assumption regarding the dimension an agent's signal.[5] The earlier comments regarding the plausibility of a revelation mechanism, however, become more relevant when the signal space is substantially more complex. It should be noted, though, that the nonrevelation mechanism in which each agent forecasts the highest bid different from his own is no more complex in the case of more complicated signals, although the bidders' problem in deciphering the information contained in those forecasts presumably becomes more complex.

9. Our definition of informational size generalizes the concept of nonexclusive information introduced in Postlewaite and Schmeidler (1986). Nonexclusive information was introduced to characterize informational problems in which incentive compatibility would not be an issue. Heuristically, this would be the case when, for any agent and for any information he might have, the agent's information was redundant given all other agents' information. When nonexclusive information obtains, it is straightforward to induce truthful revelation. Roughly speaking this is because when a single agent misrepresents his information in this case, the agents' reports will be inconsistent, thus revealing that some agent misreported with probability one.

One can characterize this as an agent having no scope for altering the posterior distribution as he contemplates announcing various types he might be. Our measure of informational size extends this concept in the sense that, when an agent has positive informational size, the agent's different types (typically) result in different posterior distributions, given other agents' reported types. Being small informationally means that an agent is unlikely to alter much the posterior given other agents' reported types.

[5] Of course agents might have different informational sizes for the different characteristics, and hence, get different informational rents for them.

8 Proofs:

8.1 Proof of Proposition 1:

Let $P \in \Delta^*_{\Theta\times T}$ and suppose that $\mathcal{C}$ is an information decomposition of P. In addition, let $\mathcal{C}^0$ be the trivial ID in which $\Pi^0_i = \{\{t_i\}\}_{t_i\in T_i}$ for each i. To prove Proposition1, it is enough to show that $\nu_i^{P,\mathcal{C}_0} = \nu_i^{P,\mathcal{C}}$ for each i.

Choose $t_i, t'_i \in T_i$. Let A_i and A'_i denote the elements of Π_i with $t_i \in A_i$ and $t'_i \in A'_i$. Next, define

$$S^\varepsilon_i(t_i,t'_i) = \{t_{-i} \in T_{-i} | \; ||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i}t'_i)|| > \varepsilon\}$$

and

$$\Phi^\varepsilon_i(A_i,A'_i) = \{A_{-i} \in \Pi_{-i} \; |||P^{\mathcal{C}}_\Theta(\cdot|A_{-i},A_i) - P^{\mathcal{C}}_\Theta(\cdot|A_{-i},A'_i)|| > \varepsilon\}.$$

Since $P^{\mathcal{C}}_\Theta(\cdot|A_{-i},A_i) = P_\Theta(\cdot|t_{-i},t_i)$ and $P^{\mathcal{C}}_\Theta(\cdot|A_{-i},A'_i) = P_\Theta(\cdot|t_{-i},t'_i)$ whenever $t_{-i} \in A_{-i}$, it follows that

$$t_{-i} \in S^\varepsilon_i(t_i,t'_i) \Leftrightarrow t_{-i} \in A_{-i} \text{ for some } A_{-i} \in \Phi^\varepsilon_i(A_i,A'_i).$$

Therefore,

$$\sum_{A_{-i}\in\Phi^\varepsilon_i(A_i,A'_i)} P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) = \sum_{A_{-i}\in\Phi^\varepsilon_i(A_i,A'_i)} \sum_{t_{-i}\in A_{-i}} P(t_{-i}|t_i) = \sum_{t_{-i}\in S^\varepsilon_i(t_i,t'_i)} P(t_{-i}|t_i)$$

and we conclude that

$$\begin{aligned}
\nu_i^{P,\mathcal{C}_0}(\{t_i\},\{t'_i\}) &= \inf\{\varepsilon>0| \sum_{t_{-i}\in S^\varepsilon_i(t'_i,t_i)} P(t_{-i}|t_i) \le \varepsilon\} \\
&= \inf\{\varepsilon>0| \sum_{A_{-i}\in\Phi^\varepsilon_i(A_i,A'_i)} P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) \le \varepsilon\} \\
&= \nu_i^{P,\mathcal{C}}(A_i,A'_i).
\end{aligned}$$

This implies that $\nu_i^{P,\mathcal{C}_0} = \nu_i^{P,\mathcal{C}}$ and the proof of Proposition 1 is complete.

8.2 Preparations for the Proof of Theorem 2:

In this section, we begin with two lemmas that are of some independent interest.

Lemma A.1: Let $(v_1,..,v_n)$ be a collection of payoff functions and let $\{q^*_i,x^*_i\}_{i\in N}$ be the associated Vickrey auction mechanism. For every $i \in N$ and for each $t \in T$ and $t'_i \in T_i$,

$$(q^*_i(t)\hat{v}_i(t) - x^*_i(t)) - (q^*_i(t_{-i},t'_i)\hat{v}_i(t) - x^*_i(t_{-i},t'_i)) \ge -|w_i(t_{-i},t'_i) - w_i(t_{-i},t_i)|.$$

Proof: Choose $t \in T$ and $t_i' \in T_i$.

Case 1: Suppose that $\hat{v}_i(t_{-i}, t_i') < w_i(t_{-i}, t_i')$. Then

$$q_i^*(t_{-i}, t_i') = x_i^*(t_{-i}, t_i') = 0$$

so

$$\begin{aligned}
&(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i')) \\
&\quad = \quad q_i^*(t)\hat{v}_i(t) - x_i^*(t) \\
&\quad \geq \quad 0 \\
&\quad \geq \quad -|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)|.
\end{aligned}$$

Case 2: Suppose that $\hat{v}_i(t_{-i}, t_i') > w_i(t_{-i}, t_i')$. Then

$$q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i') = \hat{v}_i(t) - w_i(t_{-i}, t_i').$$

If $\hat{v}_i(t_{-i}, t_i) > w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = \hat{v}_i(t) - w_i(t_{-i}, t_i).$$

If $\hat{v}_i(t_{-i}, t_i) \leq w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = 0 \geq \hat{v}_i(t) - w_i(t_{-i}, t_i).$$

Therefore,

$$\begin{aligned}
&(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i')) \\
&\quad \geq \quad (\hat{v}_i(t) - w_i(t_{-i}, t_i)) - (\hat{v}_i(t) - w_i(t_{-i}, t_i')) \\
&\quad = \quad w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i) \\
&\quad \geq \quad -|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)|.
\end{aligned}$$

Case 3: Suppose that $\hat{v}_i(t_{-i}, t_i') = w_i(t_{-i}, t_i')$. Then

$$q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i') = \frac{1}{|I(t_{-i}, t_i')|}\left(\hat{v}_i(t) - w_i(t_{-i}, t_i')\right).$$

If $\hat{v}_i(t_{-i}, t_i) > w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = \hat{v}_i(t) - w_i(t_{-i}, t_i) \geq \frac{1}{|I(t_{-i}, t_i')|}\left(\hat{v}_i(t) - w_i(t_{-i}, t_i)\right).$$

If $\hat{v}_i(t_{-i}, t_i) \leq w_i(t_{-i}, t_i)$, then

$$q_i^*(t)\hat{v}_i(t) - x_i^*(t) = 0 \geq \frac{1}{|I(t_{-i}, t_i')|}\left(\hat{v}_i(t) - w_i(t_{-i}, t_i)\right).$$

Therefore,

$$\begin{aligned}
&(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i')) \\
&\quad \geq \frac{1}{|I(t_{-i}, t_i')|}(\hat{v}_i(t) - w_i(t_{-i}, t_i)) - \frac{1}{|I(t_{-i}, t_i')|}(\hat{v}_i(t) - w_i(t_{-i}, t_i')) \\
&\quad = \frac{1}{|I(t_{-i}, t_i')|}(w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)) \\
&\quad \geq -\frac{1}{|I(t_{-i}, t_i')|}|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)| \\
&\quad \geq -|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)|.
\end{aligned}$$

This completes the proof of Lemma 1.

If each $\hat{v}_i(t)$ is a function of t_i only, then $|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)| = 0$ and Lemma A.1 yields the familiar result for Vickrey auctions with pure private values: it is a dominant strategy to truthfully report one's type.

Lemma A.2: Let $(v_1, .., v_n)$ be a collection of payoff functions and let $\{q_i^*, x_i^*\}_{i\in N}$ be the associated Vickrey auction mechanism. Let

$$M = \max_{\theta} \max_{i} \max_{t_i} v_i(\theta, t_i)$$

and let $P \in \Delta^*_{\Theta\times T}$. For every $i \in N$ and for each $t_{-i} \in T_{-i}$, $t_i \in T_i$ and $t_i' \in T_i$,

$$|w_i(t_{-i}, t_i') - w_i(t_{-i}, t_i)| \leq M||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||.$$

Proof: Choose $t_{-i}, t_i, t_i', j \neq i$ and $j' \neq i$ so that

$$w_i(t_{-i}, t_i) = \max_{k\neq i} \sum_{\theta\in\Theta} [v_k(\theta, t_k)P_\Theta(\theta|t_{-i}, t_i)] = \sum_{\theta\in\Theta} [v_j(\theta, t_j)P_\Theta(\theta|t_{-i}, t_i)]$$

and

$$w_i(t_{-i}, t_i') = \max_{k\neq i} \sum_{\theta\in\Theta} [v_k(\theta, t_k)P_\Theta(\theta|t_{-i}, t_i')] = \sum_{\theta\in\Theta} [v_{j'}(\theta, t_{j'})P_\Theta(\theta|t_{-i}, t_i')].$$

Note that t_j and $t_{j'}$ are, respectively, the j and j' components of the vector t_{-i}. From the definitions of t_j and $t_{j'}$, it follows that

$$\sum_{\theta\in\Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_\Theta(\theta|t_{-i}, t_i) \geq 0$$

and

$$\sum_{\theta\in\Theta} [v_j(\theta, t_j) - v_{j'}(\theta, t_{j'})] P_\Theta(\theta|t_{-i}, t_i') \leq 0.$$

Therefore,

$$\begin{aligned}
&\sum_{\theta\in\Theta} v_{j'}(\theta,t_{j'})\left[P_\Theta(\theta|t_{-i},t_i)-P_\Theta(\theta|t_{-i},t_i')\right] \\
&\quad\le \sum_{\theta\in\Theta} v_{j'}(\theta,t_{j'})\left[P_\Theta(\theta|t_{-i},t_i)-P_\Theta(\theta|t_{-i},t_i')\right]+\sum_{\theta\in\Theta}\left[v_j(\theta,t_j)-v_{j'}(\theta,t_{j'})\right]P_\Theta(\theta|t_{-i},t_i) \\
&\quad= w_i(t_{-i},t_i)-w_i(t_{-i},t_i') \\
&\quad= \sum_{\theta\in\Theta} v_j(\theta,t_j)\left[P_\Theta(\theta|t_{-i},t_i)-P_\Theta(\theta|t_{-i},t_i')\right]+\sum_{\theta\in\Theta}\left[v_j(\theta,t_j)-v_{j'}(\theta,t_{j'})\right]P_\Theta(\theta|t_{-i},t_i') \\
&\quad\le \sum_{\theta\in\Theta} v_j(\theta,t_j)\left[P_\Theta(\theta|t_{-i},t_i)-P_\Theta(\theta|t_{-i},t_i')\right]
\end{aligned}$$

and we conclude that

$$|w_i(t_{-i},t_i)-w_i(t_{-i},t_i')|\le M||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||.$$

This completes the proof of Lemma 2.

We prove one final technical result.

Lemma A.3: Let X be a finite set with cardinality k and let $p,q\in\Delta_X$. Then

$$\left[\frac{p}{||p||_2}-\frac{q}{||q||_2}\right]\cdot p\ge\frac{k^{-\frac{5}{2}}}{2}\left[||p-q||\right]^2$$

where $||\cdot||_2$ denotes the 2-norm and $||\cdot||$ denotes the 1-norm.

Proof: Direct computation shows that

$$\left[\frac{p}{||p||_2}-\frac{q}{||q||_2}\right]\cdot p=\frac{||p||_2}{2}\left\|\frac{p}{||p||_2}-\frac{q}{||q||_2}\right\|_2^2$$

The result follows by combining the facts that $||p||_2\ge 1/\sqrt{k}, k(||p-q||_2)^2\ge||p-q||^2$ and

$$\left[\left\|\frac{p}{||p||_2}-\frac{q}{||q||_2}\right\|_2\right]^2\ge\frac{1}{k}\left[||p-q||_2\right]^2.$$

8.3 Proof of Theorem 2:

We prove part (ii) first. Choose $\varepsilon>0$. Let

$$M=\max_\theta\max_i\max_{t_i} v_i(\theta,t_i)$$

and let K be the cardinality of T. Choose δ so that

$$0<\delta<\frac{\varepsilon}{6MK^{\frac{5}{2}}}.$$

Suppose that $P \in \Delta^*_{\Theta\times T}$ has an information decomposition satisfying

$$\max_i \nu_i^{P,\mathcal{C}} \leq \delta \min_i \Lambda_i^{P,\mathcal{C}}.$$

Define $\hat{\nu}^{P,\mathcal{C}} = \max_i \nu_i^{P,\mathcal{C}}$and $\Lambda^{P,\mathcal{C}} = \min_i \Lambda_i^{P,\mathcal{C}}$. Therefore $\hat{\nu}^{P,\mathcal{C}} \leq \delta\Lambda^{P,\mathcal{C}}$.

Next, define

$$\zeta_i(A_{-i}, A_i) = \frac{P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i)}{||P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i)||_2}$$

for each $(A_1, .., A_n) \in \Pi_1 \times \cdots \times \Pi_n$ and note that

$$0 \leq \zeta_i(A_{-i}, A_i) \leq 1$$

for all i, A_{-i} and A_i. Now we define an augmented Vickrey auction mechanism. For each $t \in T$, let

$$z_i(t) := \varepsilon\zeta_i(A_1, .., A_n) \text{ if } t_i \in A_i \text{ for each i.}$$

The mechanism $\{q^*_i, x^*_i - z_i\}_{i\in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$q^*_i(t)\hat{v}_i(t) - x^*_i(t) \geq 0$$

and

$$z_i(t_{-i}, t_i) \geq 0.$$

To prove incentive compatibility, we consider two cases. First suppose that $t_i, t'_i \in A_i$ for some $A_i \in \Pi_i$. From part (i) of the definition of information decomposition, it follows that $|w_i(t_{-i}, t'_i) - w_i(t_{-i}, t_i)| = 0$ for all $t_{-i} \in T_{-i}$ and incentive compatibility is a consequence of Lemma A.1.

Now suppose that $t_i \in A_i$ and $t'_i \in A'_i$ with $A_i \neq A'_i$. The proof of incentive compatibility will follow from the next two claims.

Claim 1: For each i and for each $t_i, t'_i \in T_i$ with $t_i \neq t'_i$,

$$\sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i))\, P(t_{-i}|t_i) \geq \frac{K^{-\frac{5}{2}}}{2}\varepsilon\Lambda^{P,\mathcal{C}}.$$

Proof of Claim 1: The definition of common value decomposition implies that $\sum_{t_{-i}\in A_{--i}} P(t_{-i}|\hat{t}_i) = P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i)$ whenever $\hat{t}_i \in A_i$. Therefore,

$$\begin{aligned}&\sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t'_i))\, P(t_{-i}|t_i) \\ &= \varepsilon \sum_{A_{-i}} [\zeta_i(A_{-i}, A_i) - \zeta_i(A_{-i}, A'_i)] \left[\sum_{t_{-i}\in A_{--i}} P(t_{-i}|t_i)\right]\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \sum_{A_{-i}} \left[\zeta_i(A_{-i}, A_i) - \zeta_i(A_{-i}, A_i')\right] P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) \\
&= \varepsilon \sum_{A_{-i}} \left[\frac{P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i)}{||P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i)||_2} - \frac{P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i')}{||P^{\mathcal{C}}_{\Pi_{-i}}(\cdot|A_i')||_2}\right] P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) \\
&\geq \frac{\varepsilon K^{-\frac{5}{2}}}{2} \left[||P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i) - P^{\mathcal{C}}_{\Pi_{-i}}(A_{-i}|A_i')||\right]^2 \\
&\geq \frac{\varepsilon K^{-\frac{5}{2}}}{2} \Lambda_i^{P,\mathcal{C}}
\end{aligned}
$$

where the last inequality is an application of Lemma A.3.

Claim 2:

$$
\sum_{t_{-i}} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i'))\right] P(t_{-i}|t_i) \geq -3M\hat{\nu}^{P,\mathcal{C}}
$$

Proof of Claim 2: Choose $t_i, t_i' \in T_i$ and define

$$
S_i(t_i', t_i) = \{t_{-i} \in T_{-i}|\ ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}t_i')|| > \hat{\nu}^{P,\mathcal{C}}\}.
$$

If $\mathcal{C}^0$ denotes the trivial information decomposition of P in which $\Pi_i^0 = \{\{t_i\}\}_{t_i \in T_i}$ for each i, then $\nu_i^{P,\mathcal{C}_0} = \nu_i^{P,\mathcal{C}}$ as a consequence of Proposition 1. Since $\nu_i^{P,\mathcal{C}} \leq \hat{\nu}^{P,\mathcal{C}}$, we conclude that

$$
\text{Prob}\{\tilde{t}_{-i} \in S_i(t_i', t_i)|\tilde{t}_i = t_i\} \leq \nu_i^{P,\mathcal{C}_0} = \nu_i^{P,\mathcal{C}} \leq \hat{\nu}^{P,\mathcal{C}}.
$$

If $t_{-i} \notin S_i(t_i', t_i)$, then Lemmas A.1 and A.2 imply that

$$
\sum_{t_{-i} \notin S_i(t_i', t_i)} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i'))\right] P(t_{-i}|t_i) \geq -M\hat{\nu}^{P,\mathcal{C}}.
$$

Finally, note that

$$
|q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i')| \leq M
$$

for all i, t_i, t_i' and t_{-i}.

Combining these observations , we conclude that

$$
\begin{aligned}
&\sum_{t_{-i}} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i'))\right] P(t_{-i}|t_i) \\
&= \sum_{t_{-i} \in S_i(t_i', t_i)} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i'))\right] P(t_{-i}|t_i) \\
&\quad + \sum_{t_{-i} \notin S_i(t_i', t_i)} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i}, t_i')\hat{v}_i(t) - x_i^*(t_{-i}, t_i'))\right] P(t_{-i}|t_i) \\
&\geq -M\hat{\nu}^{P,\mathcal{C}} - 2M\hat{\nu}^{P,\mathcal{C}} \\
&= -3M\hat{\nu}^{P,\mathcal{C}}
\end{aligned}
$$

and the proof of claim 2 is complete.

Applying Claims 1 and 2, it follows that

$$\begin{aligned}
&\sum_{t_{-i}} \left[(q_i(t)\hat{v}_i(t) - x_i(t)) - (q_i(t_{-i},t_i')\hat{v}_i(t) - x_i(t_{-i},t_i')) \right] P(t_{-i}|t_i) \\
&\quad = \sum_{t_{-i}} \left[(q_i^*(t)\hat{v}_i(t) - x_i^*(t)) - (q_i^*(t_{-i},t_i')\hat{v}_i(t) - x_i^*(t_{-i},t_i')) \right] P(t_{-i}|t_i) \\
&\qquad\quad + \sum_{t_{-i}} \left(z_i(t_{-i},t_i) - z_i(t_{-i},t_i') \right) P(t_{-i}|t_i) \\
&\quad \geq \varepsilon \frac{k^{-\frac{5}{2}}}{2} \Lambda^{P,\mathcal{C}} - 3M\hat{\nu}^{P,\mathcal{C}} \\
&\quad \geq 0.
\end{aligned}$$

and the proof of part (ii) is complete.

Part (i) follows from the computations in part (ii). We have shown that, for any ID of P and for any positive number α, there exists an augmented Vickrey auction $\{q_i^*, x_i^* - z_i\}_{i\in N}$ satisfying

$$\sum_{t_{-i}} \left[(q_i(t)\hat{v}_i(t) - x_i(t)) - (q_i(t_{-i},t_i')\hat{v}_i(t) - x_i(t_{-i},t_i')) \right] P(t_{-i}|t_i) \geq \alpha \frac{k^{-\frac{5}{2}}}{2} \Lambda_i^{P,\mathcal{C}} - 3M\hat{\nu}^{P,\mathcal{C}}$$

for each i and each t_i, t_i'. If $\Lambda_i^{P,\mathcal{C}} > 0$ for each i, then α can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

8.4 Proof of Proposition 2:

Suppose that player 1 sees signal $t_1 = H$ and the players announce a_1, a_2, a_3. Let $t_2 = \sigma(a_2)$ and $t_3 = \sigma(a_3)$. We must consider bidder 1's best response for the case in which his first period forecast a_1 is constrained to satisfy $a_1 \geq 25$ and the case in which this forecast is constrained to satisfy $a_1 < 25$.

If 1 forecasts $a_1 \geq 25$, then 2 bids $\hat{v}(H, t_2, t_3)$ and 3 bids $\hat{v}(H, t_2, t_3)$. Player 1's best response in the second stage is a bid of $\hat{v}(H, t_2, t_3)$. Hence, 1 receives no surplus even if he wins the object. However, he will gain a reward of 1 if his forecast a_1 of the highest bid different from his own is close to $\hat{v}(H, t_2, t_3)$. Therefore, Player 1's payoff in the second stage is equal to

$$F(|\hat{v}(H, t_2, t_3) - a_1|)$$

where

$$\begin{aligned}
F(x) &= 1 \text{ if } x \leq 4 \\
&= 0 \text{ if } x > 4.
\end{aligned}$$

In order to determine his optimal first period forecast when that forecast is constrained to satisfy $a_1 \geq 25$, bidder 1 must solve the optimization problem

$$\max_{a_1} \sum_{t_2,t_3} F(|\hat{v}(H,t_2,t_3) - a_1|)P(t_2,t_3|H)$$

$$\text{subject to } a_1 \geq 25.$$

Note that

$$\sum_{t_2,t_3} F(|\hat{v}(H,t_2,t_3) - a_1|)P(t_2,t_3|H) \geq F(|\hat{v}(H,H,H) - a_1|)P(H,H|H)$$

and that $P(H,H|H) \to 1$, $\hat{v}(H,H,H) \to 30$ and $\hat{\alpha}(H) \to 0$ as $\rho \to 1$. Therefore, bidder 1 can guarantee himself an expected payoff close to 1 if $\rho \approx 1$ by choosing $a_1 = \hat{\alpha}(H)$.

Now suppose that bidder 1 forecasts $a_1 < 25$. In this case, 2 bids $\hat{v}(L,t_2,t_3)$ and 3 bids $\hat{v}(L,t_2,t_3)$. Bidder 1's second stage best response is still $\hat{v}(H,t_2,t_3)$. Since $\hat{v}(H,t_2,t_3) > \hat{v}(L,t_2,t_3)$ for each (t_2,t_3) pair (recall that $\rho > 1/2$), bidder 1 now wins the object and receives a surplus equal to $\hat{v}(H,t_2,t_3) - \hat{v}(L,t_2,t_3)$. In addition, he will gain a reward of 1 if his forecast a_1 is close enough to $\hat{v}(L,t_2,t_3)$. Thus, Player 1's payoff in the second stage is equal to

$$\hat{v}(H,t_2,t_3) - \hat{v}(L,t_2,t_3) + F(|\hat{v}(L,t_2,t_3) - a_1|).$$

As $\rho \to 1$,it follows that

$$\sum_{t_2,t_3} [\hat{v}(H,t_2,t_3) - \hat{v}(L,t_2,t_3)]\, P(t_2,t_3|H) \to 0$$

since $P(H,H|H) \to 1$ and

$$\hat{v}(H,H,H) - \hat{v}(L,H,H) \to 30 - 30 = 0.$$

Since $0 \leq F(\cdot) \leq 1$, it follows that

$$\begin{aligned}\sum_{t_2,t_3} & [F(|\hat{v}(L,t_2,t_3) - a_1|)]\, P(t_2,t_3|H) \\ & \leq \sum_{t_2,t_3} [F(|\hat{v}(L,H,H) - a_1|)]\, P(H,H|H) \\ & \quad +1 - P(H,H|H).\end{aligned}$$

Since $\hat{v}(L,H,H) \to 30$ as $\rho \to 1$, it follows that

$$\max_{a_1<25} \sum_{t_2,t_3} [F(|\hat{v}(L,t_2,t_3) - a_1|)]\, P(t_2,t_3|H) \leq 1 - P(H,H|H)$$

when ρ is close to one. Combining these observations, it follows that

$$\max_{a_1<25} \sum_{t_2,t_3} [\hat{v}(H,t_2,t_3) - \hat{v}(L,t_2,t_3) + F(|\hat{v}(L,t_2,t_3) - a_1|)] \to 0$$

as $\rho \to 1$.

¿From these computations, it follows that, when ρ is close enough to one, bidder 1's best response to $(\hat{\alpha}_2, \hat{\beta}_2), (\hat{\alpha}_3, \hat{\beta}_3)$ is a first stage announcement of $a_1 = \alpha(H)$ and a second stage bid of $\hat{v}(H, \sigma(a_2), \sigma(a_3))$. A similar argument applies if $s_1 = L$.

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