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"Multilateral Negotiations and Formation of Coalitions"

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Multilateral Negotiations and Formation of Coalitions

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Abstract

This paper studies multilateral negotiations among n players in an environment where there are externalities and where contracts forming coalitions can be written and renegotiated. The negotiation process is modeled as a sequential game of offers and counteroffers, and we focus on the stationary subgame perfect equilibria, which jointly determine both the expected value of players and the Markov state transition probability that encodes the path of coalition formation. The existence of equilibria is established, and Pareto efficiency is guaranteed if the grand coalition is efficient, despite the existence of externalities. Also, for almost all games (except in a set of measure zero) the equilibrium is locally unique and stable, and the number of equilibria is finite and odd. Global uniqueness does not hold in general (a public good provision example has seven equilibria), but a sufficient condition for global uniqueness is derived. Using this sufficient condition, we show that there is a globally unique equilibrium in three-player superadditive games. Comparative statics analysis can be easily carried out using standard calculus tools, and some new insights emerge from the investigation of the classic apex and quota games.

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1. Introduction

In this paper we study multilateral negotiations among n players where contracts forming coalitions can be written and renegotiated, and where the formation of coalitions may impose externalities on other players. What is the path of coalition formation? What is the value of players? This paper develops a new non-cooperative model of coalitional bargaining that provides answers to these two fundamental questions in economics.

The externalities present in the environment are described by a set of exogenous parameters, conveniently expressed using a partition function form. The partition function form assigns a worth to each coalition depending on the coalition structure (or collection of coalitions) formed by the remaining players. This general formulation is valuable because it can address situations in which the formation of coalitions impose positive or negative externalities (see also Ray and Vohra (1999), Bloch (1996), and Jehiel and Moldovanu (1995)).

Our multilateral negotiation procedure follows the traditional approach of using a dynamic game with complete information where at each stage a player becomes the proposer (such as Selten (1981), Chatterjee et al. (1993), Hart and Mas-Colell (1996), and Ray and Vohra (1999), among many others). Proposers make offers to form coalitions with a certain sharing of the coalitional value, followed by players who have received offers making their response whether or not to accept the offer. However, in contrast with most of the bargaining literature, here contracts forming coalitions can be renegotiated, so a coalition may continue actively seeking profitable deals with other players or coalitions after forming (a few other models also consider renegotiations: Gul (1989), Seidmann and Winter (1998), and Jehiel and Moldovanu (1999)).

The equilibrium concept used is stationary subgame perfect Nash equilibrium or Markov perfect equilibrium (MPE) where the set of states are all possible coalition structures. The MPE solutions characterize, jointly, both the expected equilibrium value of coalitions, and the Markov state transition probability that describes the path of coalition formation. Thus, the equilibria provide answers to both questions we posed, and our goal is to develop a thorough analysis of the equilibrium properties.

We begin by establishing the existence of equilibria for all games. The equilibria exhibit desirable efficiency properties: the equilibrium is Pareto efficient if the grand coalition is efficient and frictions (delay between offers) are insignificant. The Coasian conjecture that an efficient outcome should eventually emerge thus holds in our model, despite the fact that there can be widespread externalities in the economy (generalizing Seidmann and Winter's (1998) result for economies with externalities). In contrast, Jehiel and Moldovanu (1995, 1999) find that the Coasian conjecture fails to hold in the case of sale of an indivisible good where buyers impose negative externalities on other buyers (note that this problem can also be represented using partition function forms); this paper points out that the reason such failure occurs in Jehiel and Moldovanu is the lack of ability to write an efficient grand coalition contract. Furthermore, it is important to note that the ability of coalitions to renegotiate is critical for the Pareto efficiency result. For this reason, the equilibrium in non-cooperative models of coalitional bargaining in which coalitions are forced to leave the game is not Pareto efficient (even if grand coalition is efficient and frictions are insignificant).

The practical interest and applicability of our multilateral negotiations model hinges upon, to a great extent, whether or not one can compute the equilibria using efficient numerical methods. We show that the problem of finding equilibria is equivalent to finding solutions of a mixed nonlinear complementarity problem (*MNCP*). Such problems have been extensively studied in the mathematical programing literature (see Harker and Pang (1990) and Cottle, Pang, and Stone (1992)), and several numerical algorithms have been developed. Hence, the computation of equilibria is a task that can be undertaken using several proven numerical algorithms.

We proceed by showing that for almost all games (except in a closed set of measure zero) the equilibria are locally unique and locally stable. These properties imply that the predictions of the model about both the expected payoffs of players and the path of coalition formation are sharp, in the sense that, at least locally, they are unique and robust to small perturbations of the exogenous parameters of the game. Moreover, the number of equilibrium solutions is finite and odd for almost all games.

Thus we extend to multilateral bargaining models similar results that hold for other well-known economic models such as Walrasian equilibrium of competitive economies (Debreu (1970)) and Nash equilibrium of n-person strategic form games (Wilson (1971) and Harsanyi (1973)).

Interestingly, global uniqueness of equilibria does not hold, and we provide an example of a game with multiple (seven) equilibrium solutions. Nonetheless, we derive a sufficient condition for the global uniqueness, and argue that this sufficient condition is weak and is likely to be satisfied by a large class of games. Along these lines, we prove that the sufficient condition holds for three-player coalitional bargaining games if the grand coalition is efficient (which includes the class of superadditive games), and thus there is a globally unique equilibrium for a general class of three-player games (see also Gomes (2000)).

We demonstrate the applicability of the model analyzing two classic games–apex and quota games (see Shapley (1953), Davis and Maschler (1965), and Maschler (1992))–games for which there are also results from experimental studies. Comparison of the equilibrium payoffs predicted by our model with established solution concepts from cooperative game theory, such as the nucleolus, bargaining set, kernel, core, and Shapley value, shows that our predictions are different than all other cooperative solution concepts. Our cursory look at the experimental results provides support for the predictions of our model with respect to both payoffs and coalition formation. A point worth noting is that our model allows for a more comprehensive empirical analysis than cooperative models, because it predicts not only the payoffs but also the dynamics of coalition formation.

How do the equilibrium value of players and the path of coalition formation change as a result of changes in exogenous parameters such as the partition function form and the probability of being the proposer? Knowing how to address these questions is of considerable practical interest to negotiators, as they, for example, may be able to invest in changing the likelihood of being proposers in negotiations. We show how to answer these questions using standard calculus results (the implicit function theorem), which provides a powerful tool for quickly answering comparative statics questions by simply evaluating Jacobian matrices at the solution. We illustrate the applications of the technique using the apex and quota games, and some interesting insights emerge. Surprisingly, a player sometimes may not benefit by investing in obtaining more initiative to propose in negotiations, because other players may adjust their strategies in such a way that lead the proposer to be worse off. The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase as well.

Finally, we also explore the role of agents' ability to credibly commit to leave the game. As Ray and Vohra (1999, 2000) point out, this commitment is economically relevant in situations where there are externalities. Ray and Vohra show that in public good provision problems inefficiencies may arise as a result of players' commitment ability: the first player who is given a chance to leave the game may want to do so when it is in the best interest of the remaining players to provide the public good after a player has irrevocably left the game. Such type of behavior is not possible in our model, as agents lack the commitment ability to leave the game.

The remainder of the paper is organized as follows: section 2 presents the negotiation model; section 3 addresses the existence and efficiency of equilibria; section 4 shows how to compute the equilibria; section 5 develops the local uniqueness, stability, and genericity properties of the equilibria; section 6 includes the examples; section 7 addresses the number of equilibrium solutions; and section 8 concludes.

2. The Model

Let $N = \{1, 2, \cdots, n\}$ be a set with n players. Negotiations are modeled as an infinite horizon complete information game. All players have the same expected intertemporal utility function and are risk-neutral and have common discount factor $\delta \in (0,1)$: thus their utility over a stream of random payoffs $(x_t)_{t=0}^{\infty}$ is $\sum_{t=0}^{\infty} \delta^t E(x_t)$.

Contractual agreements can be written among any subset S of players, creating a *coalition* S. A *coalition structure (c.s.)* $\pi = \{S_1, ..., S_m\}$ is a *partition* of the set of players N into disjoint coalitions $S_k \in \pi$, and Π is the set of all coalition structures. The underlying economic opportunities are given by a *partition function* *form* $v = (v_i(\pi))_{\substack{\pi \in \Pi \\ i \in \pi}}$, where $v_i(\pi) \in R$.[1] The partition function represents the exogenous parameters, concisely described in vector form, and details the aggregate utility flow received by coalition i belonging to coalition structure π during one period of the game. Specifically, given that the prevailing c.s. is π, then coalition $i \in \pi$ receives a utility flow equal to $(1-\delta)\, v_i(\pi)$ (we multiply by $(1-\delta)$ to normalize the payoffs, so that, if the game stays at c.s. π forever, the worth of coalition i is $v_i(\pi)$).

Observe that the partition function form allow us to capture the existence of externalities associated with the formation of coalitions, as the value $v_i(\pi)$ of coalition i depends on the c.s. formed by players $N \backslash i$. The traditional *characteristic function form* corresponds to the special case of the partition function where $v_i = v_i(\pi) = v_i(\pi')$ for all partitions π and π' with $i \in \pi \cap \pi'$.

Once a coalition S forms all players $i \in S$ give up all their decision making to coalition S, who acts as a player (or principal) whose objective is to maximize the aggregate expected utility of the coalitional members. Moreover, after coalition S forms we allow for this contract to be renegotiated, so that further coalitions among S and other players (or coalitions) can also be written. Of course, an integral part of the contractual agreement creating coalition S is a rule specifying how the value created by the coalition is to be shared among the coalitional members for every possible state of the world in the future. This sharing rule can be very complex, especially given the fact that the coalitional value is contingent on the c.s. formed by players in $N \backslash S$. However, any sharing rule, regardless of how complicated it is, ends up giving each player $j \in S$ a share α_j of the expected value of coalition S, where $\sum_{j \in S} \alpha_j = 1$ (see Ray and Vohra (1999) for an alternative approach to deal with games with externalities). Therefore, without any loss of generality, it is enough to concentrate on the simple sharing rule that gives each player $j \in S$ a fraction α_j of the value created by coalition S in all future states of the world.[2] In our model, players are farsighted and, when forming a coalition, take into account both the sharing rule *and* the coalition's expected value.

[1] For notational simplicity, we typically denote coalitions (or players) by the symbol i.

[2] Naturally, the players j in coalition S receive a fraction α_j of any payoffs received by coalition S, including payoffs received by coalition S if it later expands.

The game starts at the node where the n players have not yet formed any coalitions (c.s. $\{\{1\},...,\{n\}\}$). The *coalition bargaining game* is the dynamic game with the following extensive form: Say that at the beginning of a certain period of the game the c.s. is π. One of the coalitions (or players) $i \in \pi$ is randomly chosen with probability $p_i(\pi)$ to be the proposer (the only restriction is that $p_i(\pi) \geq 0$ and $\sum_{i\in\pi} p_i(\pi) = 1$).[3] Player i then proposes to form coalition $S \subset \pi$ including himself (i.e., $i \in S$) and offer a sharing rule $\alpha = (\alpha_j)_{j\in S}$, where $\sum_{j\in S} \alpha_j = 1$. Players in S respond (the order of response is not important, and may be either simultaneous or sequential) by either accepting or rejecting the offer. If the offer is accepted by all players in S then coalition S forms and the new c.s. is πS where

$$\pi S = \{\cup_{j\in S} j\} \cup (\pi \backslash S), \tag{1}$$

which is a coarser partition of N than π (for example, if $\pi = \{\{1,2\},\{3,4\},\{5\}\}$ and coalition $i = \{1,2\}$ proposes to form coalition $S = \{\{1,2\},\{3,4\}\} \subset \pi$, then $\pi S = \{\{1,2,3,4\},\{5\}\}$). Otherwise, if any of the players in S rejects the offer then no coalition is formed and the c.s. remains equal to π. After a lapse of one period of time, the game is repeated starting with the prevailing c.s. with a new proposer being randomly chosen as just described.

Our notion of equilibrium is *stationary subgame perfect Nash equilibrium* or *Markov perfect equilibrium (MPE)*, where the set of states is all the coalition structures. A strategy profile σ is an *MPE* if it is a subgame perfect equilibrium and the strategies are such that they depend only on the current coalition structure and the current proposer, but neither on the history of the game nor on calendar time.

So far we have described the time discounting version of the negotiation model. Generalizing the two-person bargaining of Binmore, Rubinstein, and Wolinsky (1986) to multilateral negotiations, we derive a variation of the model in which players are indifferent to the passage of time but face the probability of exogenous breakdown of the negotiation process. Say that during every period of the game there is a

[3] Three specifications for the proposer probabilities that have been used in the literature are: (i) $p_i(\pi) = \frac{1}{|\pi|}$ (proposers are chosen with equal probabilities), (ii) $p_i(\pi) = \frac{|i|}{n}$ (proposers are chosen with probabilities proportional to the coalitional size), and (iii) $p_i(\pi) = 1$ for some $i \in \pi$.

probability $(1-\delta)$ that the negotiation process stops at the current period with the prevailing c.s. (and a probability δ that negotiations continue). In addition, say that if negotiations stop with a c.s. π the value of coalition $i \in \pi$ is $v_i(\pi)$ (the utility $v_i(\pi)$ is interpreted as a fixed payoff rather than a flow of utility). The game with this specification and the same extensive form as described above is the exogenous risk of breakdown version of the coalitional bargaining game.

The coalitional bargaining game (v, p, δ) is the dynamic game with either the time discounting or exogenous risk of breakdown versions, where (v, p, δ) are the exogenous parameters, respectively, the partition function form, the proposer probability, and the discount rate or the probability of exogenous breakdown of negotiations. Even though having both the time discounting model and the exogenous risk of breakdown model available is useful in applications, as one of the interpretations may be more appropriate for a particular situation, it is easy to show that both versions have exactly the same set of equilibria (see section 3.1). For this reason, we focus throughout the paper on the time discounting version of the model.

3. Existence and Efficiency of Equilibria

3.1. *Existence*

Before establishing the existence of equilibria, we derive some general properties that must be satisfied by any Markov perfect equilibrium. Suppose that we are given an *MPE*, and let the variables $\phi_i(\pi)$ represent the expected equilibrium outcome of coalition (or player) $i \in \pi$ when the c.s. is π. What is the expected equilibrium payoff of player i in the subgame starting in the rejection node with c.s. equal to π? Player i's utility in the current period is the flow $(1-\delta)\, v_i(\pi)$; his utility in the subgame starting next period with the proposal stage is $\phi_i(\pi)$ (because this subgame is just like the original game and the equilibrium is stationary), which has a present value equal to $\delta\phi_i(\pi)$. This implies that player i's expected utility is equal to

$$x_i(\pi) = \delta\phi_i(\pi) + (1-\delta)\, v_i(\pi)\,. \tag{2}$$

By the same reasoning, the value of a coalition S that has just formed is $x_S(\pi S)$.[4]

In the interpretation with exogenous risk of breakdown, player i's expected utility in case of rejection is also $x_i(\pi)$, and the value of coalition S is also $x_S(\pi S)$. For this reason, and due to the results of lemma 1, the *MPE* solutions of the model with time discounting and exogenous risk of breakdown are exactly the same.

Now let us consider what is the best response strategy of a player receiving an offer. Say that player $i \in \pi$ receives an offer $\left(S, (\alpha_i)_{i \in S}\right)$ where $S \subset \pi$ and $i \in S$. Player i's best response is simply to accept the offer if and only if the value of the share α_i of the coalition S, which is worth $t_i = \alpha_i x_S(\pi S)$, is greater than or equal to the value $x_i(\pi)$ that player i can obtain if he rejects the offer. Thus player i accepts the offer if and only if $t_i = \alpha_i x_S(\pi S) \geq x_i(\pi)$, and the best response strategy of a player proposing an acceptable offer to player i is to offer him a share worth $t_i = x_i(\pi)$.

In addition, the best response strategy of a proposer j is to offer to form coalition S, where $j \in S$, that maximizes

$$x_S(\pi S) - \sum_{i \in S \backslash j} x_i(\pi) = x_S(\pi S) - \sum_{i \in S} x_i(\pi) + x_j(\pi),$$

because we have just argued that the offer to players in $S \backslash j$ are worth $x_i(\pi)$. Therefore the objective function of proposer j is to maximize the excess function, defined by

$$e(\pi)(S)(x) = x_S(\pi S) - \sum_{i \in S} x_i(\pi), \tag{3}$$

over all possible $S \subset \pi$ with $j \in S$. For simplicity, we refer to the behavioral strategy $\sigma_i(\pi)$ of proposer i as a probability distribution over $\Sigma_i(\pi) = \{S \subset \pi : i \in S\}$. Also, we define $\Sigma(\pi) = \underset{i \in \pi}{\times} \Delta^{\Sigma_i(\pi)}$ as the set of behavioral strategy profiles when the c.s. is π and let $\Sigma = \underset{\pi \in \Pi}{\times} \Sigma(\pi)$ be the set of behavioral strategy profiles.[5]

[4]In order to simplify the notation, we often refer to $S \subset \pi$ also as $S = \cup_{j \in S} j \subset N$.

[5]We use the following standard notation: $\times$ is the Cartesian product, $|A|$ is the cardinality of set A, and $\mathbb{I}_{[A]}$ is the indicator function that is equal to one or zero, respectively, if statement A is true or false.

The necessary part of the following lemma follows directly from the above discussion and the definition of *MPE*.

LEMMA 1: *A payoff structure $\phi_i(\pi)$ and a strategy profile $\sigma_i(\pi)$ is an MPE of the coalitional bargaining game (v, p, δ) if and only if the following system of equations is satisfied, where $x_i(\pi) = \delta\phi_i(\pi) + (1-\delta)\, v_i(\pi)$:*
1) the support of $\sigma_i(\pi) \in \Delta^{\Sigma_i(\pi)}$ is

$$supp\,(\sigma_i(\pi)) \subset \arg\max_{S \ni i} \{e(\pi)(S)(x)\}, \tag{4}$$

2) the expected equilibrium outcome of player i conditional on player j being chosen to be the proposer $\phi_i^j(\pi)$ is equal to

$$\phi_i^j(\pi) = \begin{cases} \max\limits_{S \ni i} \{e(\pi)(S)(x)\} + x_i(\pi) & j = i \\ \sum_{S \subset \pi} \sigma_j(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S)\right) & j \neq i \end{cases}, \tag{5}$$

3) the following system of equations holds

$$\phi_i(\pi) = \left(\sum_{j \in \pi} p_j(\pi)\, \phi_i^j(\pi)\right), \tag{6}$$

for all $\pi \in \Pi$, and $i, j \in \pi$.

PROOF: See appendix.

There is a one-to-one relation between $\phi_i(\pi)$ and $x_i(\pi)$ given by equation (2). For convenience we will be solving for the vectors $x_i(\pi)$ instead of $\phi_i(\pi)$ from now on. The vector x, as well as the partition function form v, belongs to the Euclidean space R^d and the proposer probability parameter p belongs to the space Δ^d, where $\Delta^d = \{p \in R^d : p_i(\pi) \geq 0$ and $\sum_{i \in \pi} p_i(\pi) = 1$ for all $\pi \in \Pi\}$, and the dimension $d = \sum_{\pi \in \Pi} |\pi|$.

Any payoff x candidate for equilibria must satisfy some obvious restrictions. For example, the payoff x satisfies $x_i(\pi) \geq \underline{v}_i$, where the lower bound is $\underline{v}_i = \min\limits_{\pi \ni i} \{v_i(\pi)\}$, because player i can get at least $\underline{v}_i$ by refusing to participate in any coalitions. It must also be the case, due to a feasibility constraint, that $\sum_{i\in\pi} x_i(\pi) \leq \overline{v}$, where the upper bound is $\overline{v} = \max_{\pi\in\Pi}\left\{\sum_{i\in\pi} v_i(\pi)\right\}$. Therefore any *MPE* payoff x must belong to the convex and compact set $X \subset R^d$ defined by $X = \underset{\pi\in\Pi}{\times} X(\pi)$, where

$$X(\pi) = \{x(\pi) \in R^{|\pi|} \text{ such that } \sum_{i\in\pi} x_i(\pi) \leq \overline{v} \text{ and } x_i(\pi) \geq \underline{v}_i\}.$$

Now consider the correspondence $\mathcal{F} : X \to R^d$, where $\mathcal{F}(x)$ is the set of payoffs $y = (y_i(\pi)) \in R^d$,

$$\mathcal{F}(x) = \left\{ y \in R^d : \begin{array}{l} \text{where } y_i(\pi) = \delta p_i(\pi) \max_{S\ni i}\{e(\pi)(S)(x)\} + (1-\delta)v_i(\pi) \\ +\delta\left(\sum_{S\subset\pi}\sum_{j\in\pi} p_j(\pi)\sigma_j(\pi)(S)\left(\mathbb{I}_{[i\in S]}x_i(\pi) + \mathbb{I}_{[i\notin S]}x_i(\pi S)\right)\right), \\ \text{and } supp(\sigma_i(\pi)) \subset \arg\max_{S\ni i}\{e(\pi)(S)(x)\} \end{array} \right\}. \tag{7}$$

It follows immediately from lemma 1 that the fixed points of $\mathcal{F}$ are the *MPE* payoffs of the game. Using this property, the next theorem shows the existence of equilibria for all games.

THEOREM 1: *There exist equilibria for all coalitional bargaining games* (v, p, δ).

We show in the appendix that the theorem follows from the Kakutani fixed point theorem because $\mathcal{F}(X) \subset X$, $\mathcal{F}(x)$ is convex and non-empty for all $x \in X$, and $\mathcal{F}$ is an upper hemicontinuous correspondence.

3.2. *Efficiency*

We now analyze the efficiency of equilibria. We will see that it is possible for the equilibria to be inefficient. There are two types of inefficiency in coalitional bargaining models: (1) delays in reaching an efficient state (a state π is efficient if $\sum_{i\in\pi} v_i(\pi) = \max_{\pi\in\Pi}\left\{\sum_{i\in\pi} v_i(\pi)\right\}$), and (2) asymptotic inefficiency, or inefficiency in the limit when the time between offers vanishes (or $\delta \to 1$).

As in other models in the literature, except for very special types of games (see Chatterjee et al. (1993)), equilibria in our model typically involve delays in the formation of an efficient c.s. Roughly, delay occurs in equilibrium, because proposers often find more profitable to form smaller coalitions, sharing the gains among few players, rather than forming a larger and perhaps more efficient coalition, in which the gains have to be shared among a larger number of players (i.e., the excess $e(\pi)(S)(x) = x_S(\pi S) - \sum_{i \in S} x_i(\pi)$ may be maximized at a coalition S where πS is an inefficient state). We will see several examples of games in which delay occurs in our model (see Gomes (2000) for a model of coalitional bargaining where there are no delays).

In contrast, we will show that our game is asymptotically efficient if the grand coalition is efficient (a weak condition that, for example, is satisfied by superadditive games). The equilibria converge to an efficient c.s. after a finite number of periods, and thus the equilibria is Pareto efficient when the time between offers shrinks to zero. Thus the Coasian conjecture that an efficient outcome eventually should arise holds when the grand coalition is efficient, despite the fact that there can be widespread externalities in the game (Seidmann and Winter (1998) show a similar result for games without externalities).

This efficiency result is opposed to the results of Chatterjee et al. (1993), Ray and Vohra (1999), and Okada (1996), where, even in strictly superadditive games (except for games satisfying the restrictive condition $v(N)/|N| \geq v(S)/|S|$), the equilibrium is asymptotically Pareto inefficient. The main reason for the difference is that we allow for coalitions to renegotiate agreements, while in their models, once a coalition reaches an agreement, the coalition leaves the game.

Additionally, we show that if the grand coalition is efficient, all offers that are proposed when the current c.s. is inefficient are accepted with no delays and the negotiations move to a new state with probability one–a generalization of Okada's (1996) main result to games with externalities and renegotiations.

THEOREM 2: *Consider a partition function form v where the grand coalition is efficient. Then all equilibria of the coalitional bargaining game (v, δ) converge to an* *efficient c.s. after a finite number of periods (bounded by $|\Pi|$). So the equilibria are Pareto efficient when the time between offers shrinks to zero (or $\delta \to 1$). Furthermore, for all δ, if π is an inefficient c.s. then negotiations move to a new state with probability one, and thus delays in the formation of coalitions never happen.*

Proof: Let $v_N(\{N\}) = V$, and let x be any *MPE* payoff. We first prove the last part of the theorem. Consider any c.s. π that is inefficient (i.e., $\sum_{i\in\pi} v_i(\pi) < V$). This implies that $\sum_{i\in\pi} x_i(\pi) = \sum_{i\in\pi} \delta\phi_i(\pi) + (1-\delta)\, v_i(\pi) < V$, because $\sum_{i\in\pi} \phi_i(\pi) \leq V$. Now assume, by contradiction, that there is a player, say j, who proposes, in equilibrium, an offer that is rejected with positive probability. The maximum excess of player j then must be zero, because that is the excess j gets if the offer is rejected, and j's equilibrium offer maximizes his excess. But if j proposes $S = N$, j can get an excess of at least $V - \sum_{i\in\pi} x_i(\pi) > 0$, which is a contradiction. This proves the last part of the theorem.

We now prove the first part of the theorem. We have just seen that if negotiations are in an inefficient c.s. π then the negotiations move to a new c.s. with probability one in the next period of the game. But there are only a finite number of coalition structures, and it is impossible to go back to a c.s. that has been already played before. Therefore, inefficient c.s. are played only a finite number of times. Thus the minimum level of efficiency of any *MPE* equilibrium is $\delta^{|\Pi|}V + (1-\delta)\left(1 + ... + \delta^{|\Pi|-1}\right)\underline{V}$ where $\underline{V} = \min_{\pi\in\Pi}\left\{\sum_{i\in\pi} v_i(\pi)\right\}$. But this minimum converges to V when $\delta \to 1$, proving that Pareto efficiency is reached in the limit. Q.E.D.

Theorem 2, despite being more general than Okada's (1996) and Seidmann and Winter's (1998) results, has a much simpler proof. We will provide in section 6 examples showing that if the grand coalition is not efficient then both (a) the equilibrium may not be Pareto efficient in the limit, and (b) players may propose to remain in an inefficient state (or pass up their opportunities to propose).

4. Computing the Equilibria

We now show that the problem of finding equilibria can be restated as the solution of a certain mixed nonlinear complementarity problem (see Harker and Pang (1990) for a survey about complementarity problems). Mixed nonlinear complementarity problems (*MNCP*) are the subject of numerous studies in the mathematical programming literature (see Harker and Pang (1990)), and a large number of algorithms have been proposed for solving *MNCP* problems.

We now introduce the *MNCP* problem associated with multilateral bargaining games. First define the maps $f(x,\sigma,\lambda)$, $h(\sigma)$, and $g(\lambda,x)$ by

$$\begin{aligned}
f_i(\pi)(x,\sigma,\lambda) &= x_i(\pi) - \delta p_i(\pi)\lambda_i(\pi) - (1-\delta)v_i(\pi) \\
&\quad -\delta\left(\sum_{S\subset\pi}\sum_{j\in\pi} p_j(\pi)\sigma_j(\pi)(S)\left(\mathbb{I}_{[i\in S]}x_i(\pi) + \mathbb{I}_{[i\notin S]}x_i(\pi S)\right)\right), \\
h_i(\pi)(\sigma) &= \sum_{S\subset\pi : i\in S}\sigma_i(\pi)(S) - 1, \qquad (8) \\
g_i(\pi)(S)(\lambda,x) &= \lambda_i(\pi) - e(\pi)(S)(x),
\end{aligned}$$

for all i,π,S satisfying $\pi\in\Pi$, $i\in\pi$, and $i\in S\subset\pi$.

The mixed nonlinear complementarity problem is the problem of finding triples (x,σ,λ) that satisfy all conditions

$$\begin{aligned}
f(x,\sigma,\lambda) &= 0, \\
h(\sigma) &= 0, \qquad (9) \\
g(\lambda,x) &\geq 0, \\
\sigma &\geq 0 \text{ and } \sigma^T g(\lambda,x) = 0.
\end{aligned}$$

Note that $\sigma^T g(\lambda,x) = 0$ is equivalent to $\sigma_i(\pi)(S) . g_i(\pi)(S)(\lambda,x) = 0$ for all i,π,S satisfying $\pi\in\Pi$, $i\in\pi$, and $i\in S\subset\pi$.

THEOREM 3: *If (x,σ) is an equilibrium then (x,σ,λ) is a solution of the mixed nonlinear complementarity problem (9), where $\lambda_i(\pi) = \max_{S\ni i}\{e(\pi)(S)x\}$. Reciprocally, if (x,σ,λ) is a solution of MNCP (9) then (x,σ) is an equilibrium.*

PROOF: Consider the necessary part of the theorem, and say that (x,σ) is an *MPE*. Then all the conditions in items 1, 2, and 3 of theorem 1 hold. Replacing expression (5) of $\phi_i^j(\pi)$ into equation (6), and considering that, by definition, $x_i(\pi) =$

$\delta\phi_i(\pi) + (1-\delta)\, v_i\,(\pi)$, we obtain the system of equations

$$\begin{aligned} x_i\,(\pi) \;&=\; \delta p_i(\pi) \max_{S \ni i} \{e\,(\pi)\,(S)\,(x)\} + (1-\delta) v_i\,(\pi) \\ &+\delta\left(\sum_{S\subset\pi}\left(\sum_{j\in\pi} p_j(\pi)\sigma_j\,(\pi)\,(S)\right)\left(\mathbb{I}_{[i\in S]}x_i(\pi) + \mathbb{I}_{[i\notin S]}x_i(\pi S)\right)\right), \end{aligned}$$

after noting that $\sum_{S\subset\pi} \sigma_i\,(\pi)\,(S)\left(\mathbb{I}_{[i\in S]}x_i(\pi) + \mathbb{I}_{[i\notin S]}x_i(\pi S)\right) = x_i(\pi)$.
Let $\lambda_i(\pi) = \max_{S \ni i} \{e\,(\pi)\,(S)\,x\}$ and consider the triple (x, σ, λ). We then have that (x, σ, λ) satisfies the equation $f(x, \sigma, \lambda) = 0$. Since σ is a probability distribution then $h\,(\sigma) = 0$ and $\sigma \geq 0$ are automatically satisfied. Also, by definition of λ, $\lambda_i(\pi) - e\,(\pi)\,(S)\,(x) \geq 0$, so that $g\,(\lambda, x) \geq 0$. Finally, $\sigma_i\,(\pi)\,(S)\,g_i(\pi)\,(S)\,(\lambda, x) = 0$ follows from definition of λ and the support restriction of σ in (4). The reciprocal follows using the same arguments. Q.E.D.

Theorem 3 is useful because there are several numerical algorithms for solving *MNCP* problems. These algorithms, as we now briefly describe, typically, are based on both Newton's method for solving a system of equations and on methods for solving linear complementarity problems (*LCP*), such as Lemke-Howson's algorithm (see Cottle et al. (1992) for a comprehensive treatment of the *LCP*): When solving a system of nonlinear equations, starting from a given initial condition, Newton's algorithm solves the linear approximation of the system of equations to obtain the updating direction. In the case of *MNCP* problems, instead of just solving for a linear system of equations, one solves a linear complementarity problem to obtain the updating direction, so as to satisfy the inequalities in the system.

The results of this section thus imply that the computation of equilibrium points can be accomplished using very efficient numerical methods.

5. Comparative Statics Analysis, Local Uniqueness and Stability of Equilibria

In this section we show that almost all games (except in a set of measure zero) have equilibria that are locally unique and locally stable, and satisfy a natural regularity condition. These properties are very useful. For example, they imply that almost all games have only a finite number of equilibria, and provide powerful tools for comparative statics analysis in multilateral bargaining games.

We now argue that instead of focusing on the strategy profile $\sigma \in \Sigma$, it is more convenient to focus on the associated Markov transition probability $\mu = \mu(\sigma)$, which is defined as

$$\mu(\sigma)(\pi)(S) = \sum_{j \in \pi} p_j \sigma_j(\pi)(S), \tag{10}$$

where $\mu(\pi)(S)$ represents the probability of moving from state π to state πS.

The following result shows that uniqueness of strategy profiles do not hold in general and thus the best we can hope is to have uniqueness with respect to expected payoffs and the Markov transition probabilities.

LEMMA 2: *If* (x, σ) *is an MPE then* (x, σ') *is an MPE for any* $\sigma' \in \Sigma$, *with* $\mu(\sigma) = \mu(\sigma')$, *and supp* $(\sigma'_i) \subset \arg\max_{S \ni i} \{e(S)(x)\}$.

PROOF: By theorem 3, (x, σ, λ) solves the *MNCP* (9), and thus $f(x, \sigma, \lambda) = 0$. Inverting the order of summation in the expression of $f(x, \sigma, \lambda)$ (see definition 8) we get

$$\begin{aligned} f_i(\pi)(x, \sigma, \lambda) &= x_i(\pi) - \delta p_i(\pi)\lambda_i(\pi) - (1-\delta)v_i(\pi) \\ &\quad -\delta\left(\sum_{S \subset \pi} \mu(\sigma)(\pi)(S)\left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S)\right)\right) \end{aligned}$$

where $\mu(\sigma)$ is as in equation (10). Therefore, for any σ' as above, $f(x, \sigma', \lambda) = 0$, and by the reciprocal of theorem 3 this implies that (x, σ') is an *MPE*. Q.E.D.

Several examples in which such lack of uniqueness in terms of strategy profile occurs are given in section 6. Our efforts from now on are concentrated on proving uniqueness in terms the of pair (x, μ). Observe that this captures the essential aspects of the game: x is the expected payoffs of all players for all states (c.s.), and μ is the transition probability that contains all information needed to determine the dynamics of formation of coalitions. From now on, two equilibrium points (x, σ) and (x', σ') are equivalent if and only if $x = x'$ and $\mu(\sigma) = \mu(\sigma')$.

The strategy profile σ belongs to a space with much higher dimensionality than the transition probabilities μ, and when passing from σ to μ some important information is lost. For example, suppose that $\mu(S) > 0$ for some $S \subset \pi$. Who are the players that choose coalition S with positive probability? It is certainly possible that the best strategy for player $j \in S$ is to choose coalition S, but that another player $i \in S$ is strictly better off choosing a different coalition.

We now introduce the useful concept of coalitional dynamic structure (*CDS*) that allow us to recover all the essential information about the strategy profile σ, that is not recorded in μ.

5.1. *Coalitional dynamic structures*

We start by defining an equivalence relation on the set π induced by the strategy profile σ. Given any two players $i, j \in \pi$, say that $i \rightarrow j$ if and only if there exists a coalition $S \subset \pi$ with $i, j \in S$ such that $\sigma_i(\pi)(S) > 0$. Also, say that there is a *path* from i to j if there exists a sequence of players $i_1, ..., i_k$ belonging to π such that $i \rightarrow i_1 \rightarrow ... \rightarrow i_k \rightarrow j$. Finally, we say that i and j is *connected,* $i \longleftrightarrow j$, if there is a path from i to j and a path from j to i. It is straightforward to verify that connection is an equivalence relation (transitivity, symmetry, and reflexivity hold). Let the equivalence classes of this relation (the maximal connected components) be denoted as $P_r(\pi)$, where $q(\pi)$ is the number of equivalence classes and $r = 1, ..., q(\pi)$. Also, let $C_r(\pi) = \cup_{i \in P_r(\pi)} supp(\sigma_i(\pi))$ be the union of the offers in the support of the strategy profile of players in $P_r(\pi)$.[6] The coalitional dynamic structure associated with σ is a partition of the set of players and $supp(\sigma)$ into the equivalence classes induced by the connection relation.

DEFINITION 1: *The coalitional dynamic structure (CDS) associated with* σ *is* $\mathcal{C}(\sigma) = (C(\pi), P(\pi))_{\pi \in \Pi}$ *where, for each c.s.* π*:*

[6]Note that $C_r(\pi) \cap C_{r'}(\pi) = \emptyset$ if $r \neq r'$. Otherwise, there exist $i \in P_r(\pi)$, $j \in P_{r'}(\pi)$, and $S \in C_r(\pi) \cap C_{r'}(\pi)$ such that $\sigma_i(\pi)(S) > 0$ and $\sigma_j(\pi)(S) > 0$. But this implies $i \longleftrightarrow j$ (contradiction).

i) $P(\pi) = (P_1(\pi), ..., P_{q(\pi)}(\pi))$ *is a partition of* π*, and* $P_r(\pi)$ *are the equivalence classes of the connection relation;*
ii) $C(\pi) = (C_1(\pi), ..., C_{q(\pi)}(\pi))$ *is a partition of* $supp(\sigma(\pi)) = \cup_{i\in\pi} supp(\sigma_i(\pi))$ *and* $C_r(\pi) = \cup_{i\in P_r(\pi)} supp(\sigma_i(\pi))$.
The set of coalitional dynamic structures is $CDS = \{\mathcal{C}(\sigma) : \sigma \in \Sigma\}$.

An example may help clarify the definition of *CDS*: Say that $\pi = \{1,2,3\}$ and that $supp(\sigma_1) = \{\{1,2\}\}$, $supp(\sigma_2) = \{\{1,2\}\}$, and $supp(\sigma_3) = \{\{1,2,3\}\}$ then the *CDS* $\mathcal{C}(\sigma) = (C, P)$, where $C = (\{\{1,2\}\}, \{\{1,2,3\}\})$ and $P = (\{1,2\}, \{3\})$. Now if $supp(\sigma_1) = \{\{1,2\}, \{1,3\}\}$, $supp(\sigma_2) = \{\{1,2\}, \{2,3\}\}$, and $supp(\sigma_3) = \{\{1,3\}, \{2,3\}\}$ then the *CDS* $\mathcal{C}(\sigma) = (C, P)$, where $C = (\{\{1,2\}, \{1,3\}, \{2,3\}\})$ and $P = (\{1,2,3\})$.

We are interested in analyzing the problem of finding an equilibrium point (x, σ) with a given *CDS* $\mathcal{C}$ with $C(\pi) = (C_1(\pi), ..., C_{q(\pi)}(\pi))$ and $P(\pi) = (P_1(\pi), ..., P_{q(\pi)}(\pi))$. By definition of *CDS*, this implies that the excesses of all coalitions belonging to the same equivalence class $C_r(\pi)$ are equal, that is,

$$\lambda_r(\pi) = x_S(\pi S) - \sum_{i\in S} x_i(\pi), \tag{11}$$

for all $S \in C_r(\pi)$ and $r = 1, ..., q(\pi)$ (see proof of lemma 3). In addition, the associated Markov transition probability $\mu = \mu(\sigma)$ satisfies

$$\sum_{S\in C_r(\pi)} \mu(\pi)(S) = \sum_{j\in P_r(\pi)} p_j(\pi),$$

because $supp(\sigma_j(\pi)) \subset C_r(\pi)$ for all $j \in P_r(\pi)$.

Therefore, if (x, σ) is an equilibrium then (x, μ, λ) solves the following systems of equations, or problem $F(\mathcal{C})$

$$F_{\mathcal{C}}(x, \mu, \lambda) = \begin{pmatrix} f_{\mathcal{C}}(x, \mu, \lambda) \\ E_{\mathcal{C}}(x, \lambda) \\ M_{\mathcal{C}}(\mu) \end{pmatrix} = 0 \tag{12}$$

where the maps $f_{\mathcal{C}}(x,\mu,\lambda)$, $E_{\mathcal{C}}(x,\lambda)$, and $M_{\mathcal{C}}(\mu)$ associated with *CDS* $\mathcal{C}$ are defined by

$$\begin{aligned}
(f_{\mathcal{C}})_i(\pi)(x,\mu,\lambda) &= x_i(\pi) - \delta p_i(\pi)\lambda_r(\pi) - (1-\delta)v_i(\pi) \\
&\quad - \delta\left(\sum_S \mu(\pi)(S)\left(\mathbb{I}_{[i\in S]}x_i(\pi) + \mathbb{I}_{[i\notin S]}x_i(\pi S)\right)\right), \\
E_{\mathcal{C}}(\pi)(S)(x,\lambda) &= \sum_{i\in S} x_i(\pi) + \lambda_r(\pi) - x_S(\pi S), \\
M_{\mathcal{C}}(\pi)(r)(\mu) &= \sum_{j\in P_r(\pi)} p_j(\pi) - \sum_{S\in C_r(\pi)} \mu(\pi)(S),
\end{aligned} \tag{13}$$

for all r,i, and S satisfying $r = 1,...,q(\pi)$, $i \in P_r(\pi)$, $S \in C_r(\pi)$, and all $\pi \in \Pi$.

The reciprocal result also holds if we impose some additional restrictions on the solutions of $F(\mathcal{C})$. Any set of payoffs x that are candidates for equilibrium with an associated *CDS* $\mathcal{C}$ must satisfy

$$e(\pi)(S)(x) \geq e(\pi)(T)(x) \text{ for all } S \in C_r(\pi) \text{ and } T \notin C_r(\pi) \text{ with } T \cap P_r(\pi) \neq \emptyset, \tag{14}$$

because of equalities (11) and inequalities (4). Thus, the set of payoffs $\mathcal{E}_{\mathcal{C}}$ consistent with $\mathcal{C}$ is

$$\mathcal{E}_{\mathcal{C}} = \left\{x \in R^d : \text{ such that all inequalities (14) hold}\right\}.$$

Moreover, any transition probability μ that is consistent with a *CDS* $\mathcal{C}$ satisfies $\mu = \mu(\sigma)$ where σ is a strategy profile with a *CDS* $\mathcal{C}$ (i.e., $\mathcal{C}(\sigma) = \mathcal{C}$).[7] Thus, the set of transition probabilities $\mathcal{M}_{\mathcal{C}}$ consistent with $\mathcal{C}$ is

$$\mathcal{M}_{\mathcal{C}} = \{\mu = \mu(\sigma) : \text{ where } \sigma \in \Sigma \text{ and } \mathcal{C}(\sigma) = \mathcal{C}\}. \tag{15}$$

[7]Observe that if the strategy profile σ is such that for all r and $i \in P_r(\pi)$, $supp(\sigma_i(\pi)) = C_r(\pi) \cap \{S \subset \pi : i \in S\}$ then $\mathcal{C}(\sigma) = \mathcal{C}$.

The following lemma, proved in the appendix, provides yet another useful characterization of *MPE*.

Lemma 3: *If* (x, σ) *is an MPE of the bargaining game then* (x, λ, μ) *is a solution of problem* $F(\mathcal{C}(\sigma))$, *where* $\mu = \mu(\sigma) \in \mathcal{M}_{\mathcal{C}}$, $x \in \mathcal{E}_{\mathcal{C}}$, *and* $\lambda_r(\pi) = e(\pi)(S)(x)$ *for any* $S \in C_r(\pi)$. *Reciprocally, if* (x, λ, μ) *is a solution of problem* $F(\mathcal{C})$ *satisfying* $\mu \in \mathcal{M}_{\mathcal{C}}$ *and* $x \in \mathcal{E}_{\mathcal{C}}$ *then there exists an MPE* (x, σ) *of the bargaining game with* $\mu = \mu(\sigma)$ *and* $\mathcal{C} = \mathcal{C}(\sigma)$.

This result allows us to transform the problem of finding equilibria into a lower-dimensional equivalent problem of finding solutions of the system of equations $F(\mathcal{C})$.

5.2. *Strongly Regular Games and Generic Local Uniqueness*

We now introduce the concepts of regularity and strong regularity. We show that solutions of strongly regular games are locally unique and locally stable. It immediately follows, due to the compactness of the solution space, that all strongly regular games have only a finite number of solutions. We also show that almost all games $(v, p) \in R^d \times \Delta^d$ are strongly regular.[8] In other words, except in a set of measure zero, all games are strongly regular.

Consider the following definition of regularity: a solution $z = (x, \mu, \lambda)$ of problem $F(\mathcal{C})$ is a *regular solution* if the Jacobian $d_z F_{\mathcal{C}}$ is nonsingular; a coalitional dynamic structure $\mathcal{C}$ is *regular* if all the solutions of problem $F(\mathcal{C})$ are regular. Finally, a *regular game* is a game where all *CDS*s are regular. Note that, by definition, if there exists no triple (x, μ, λ) solution of $F(\mathcal{C})$ then $\mathcal{C}$ is regular.

In addition, we say that a solution (x, μ, λ) of problem $F(\mathcal{C})$ is *strong* if all the inequalities in (14) are strict. Similar concepts are extended to *CDS*s and games in the natural way.

The Jacobian matrix associated with a *CDS* $\mathcal{C}$ with $C(\pi) = (C_1(\pi), ..., C_{q(\pi)}(\pi))$

[8] We remind that $\Delta^d = \{p \in R^d : p_i(\pi) \geq 0$ and $\sum_{i \in \pi} p_i(\pi) = 1$ for all $\pi \in \Pi\}$.

is a matrix of order $d+m+q$ that has the following special structure

$$d_z F_{\mathcal{C}} = \begin{bmatrix} d_x f_{\mathcal{C}} & d_\lambda f_{\mathcal{C}} & d_\mu f_{\mathcal{C}} \\ & E_{\mathcal{C}} & 0 \\ & 0 & M_{\mathcal{C}} \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} m &= \sum_{\pi \in \Pi} \sum_{r=1}^{q(\pi)} m_r(\pi) \text{ and } m_r(\pi) = |C_r(\pi)|, \\ q &= \sum_{\pi \in \Pi} q(\pi) \text{ and } d = \sum_{\pi \in \Pi} |\pi|. \end{aligned}$$

Observe that $\mu \in R^m$, $\lambda \in R^q$, and $x \in R^d$, and matrix $E_{\mathcal{C}}$, the incidence matrix associated with $\mathcal{C}$, has m rows and $d+q$ columns, and matrix $M_{\mathcal{C}}$ has q rows and m columns.

Also, whenever $\mathcal{C}$ is a regular *CDS and* there exists a solution of problem $F(\mathcal{C})$, the incidence matrix $E_{\mathcal{C}}$ has rank m. This is so because $d_z F_{\mathcal{C}}$ is nonsingular only if $rank(E_{\mathcal{C}}) = m$.

Moreover, note that the solution at a given c.s. π only depends on what happen at coalition structures that are coarser than π. This implies that the Jacobian matrix $d_z F_{\mathcal{C}}$ can be partitioned into an upper block triangular structure with diagonal blocks equal to $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ where $z(\pi) = (x(\pi), \mu(\pi), \lambda(\pi))$ for all $\pi \in \Pi$, where all entries to the left of the diagonal blocks are zero. Therefore, the Jacobian matrix $d_z F_{\mathcal{C}}$ is nonsingular if and only if all the diagonal blocks $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ are nonsingular.

We now show that the implicit function theorem implies that strongly regular *MPE* solutions are locally unique and locally stable.

THEOREM 4: *(Local uniqueness and stability) Strongly regular coalitional bargaining games have equilibria that are locally unique and locally stable.*

PROOF: See appendix.

Specifically, we show that for any game (v^*, p^*) and strongly regular solution (x^*, μ^*), associated with a *CDS* $\mathcal{C}$, there exists an open neighborhood $B \subset R^d \times \Delta^d$

of (v^*, p^*), an open neighborhood $W \subset R^d \times R^m$ of (x^*, μ^*) and a local mapping $(x(v,p), \mu(v,p)) \in R^d \times R^m$ such that $(x(v,p), \mu(v,p))$ is the *only MPE* solution for all $(v,p) \in B$ in the neighborhood W.

We have just argued that the set of strongly regular games is an open set. We now show that except in a set of measure zero, all games are strongly regular. Therefore the set of games that are not strongly regular is closed and has measure zero, and strong regularity is a generic property.

This genericity result is established using the transversality theorem from differential topology (Guillemin and Pollack (1974) and Hirsch (1976)).

THEOREM 5: *(Genericity) Almost all coalitional bargaining games (v,p) in $R^d \times \Delta^d$ are strongly regular.*

PROOF: We first show that if the incidence matrix $E_{\mathcal{C}}$ of the *CDS* $\mathcal{C}$ has rank m then, except in a closed set of measure zero of R^d, all the solutions of $F(\mathcal{C})$ are strongly regular. The solutions of problem $F(\mathcal{C})$ can be represented as the zeros of $F_{\mathcal{C}}(z,v) = 0$, where $z = (x, \lambda, \mu)$ and we take into account the dependency with respect to the game. The Jacobian of this mapping is

$$d_{(z,v)}F_{\mathcal{C}} = \begin{bmatrix} d_x f_{\mathcal{C}} & d_\lambda f_{\mathcal{C}} & d_\mu f_{\mathcal{C}} & -(1-\delta)I \\ & E_{\mathcal{C}} & 0 & 0 \\ & 0 & M_{\mathcal{C}} & 0 \end{bmatrix}.$$

The Jacobian is a surjective matrix (with rank equal to the number of rows) because all blocks $E_{\mathcal{C}}$, $M_{\mathcal{C}}$, and $-(1-\delta)I$ have rank equal to the number of rows, and because of the disposition of zeros in the Jacobian. Thus $F_{\mathcal{C}}$ is transversal to zero, $F_{\mathcal{C}} \pitchfork 0$. By the transversality theorem, for almost every v, $F_{\mathcal{C}}(v)$ is also transversal to zero, $F_{\mathcal{C}}(v) \pitchfork 0$. Thus the square Jacobian matrix $d_z F_{\mathcal{C}}(v)$ is surjective at all solutions of $F(\mathcal{C})$, and thus nonsingular. Therefore, $\mathcal{C}$ is a regular *CDS* for almost all games.

Now consider an hyperplane H in the space R^{d+m+q} obtained by replacing any one of the inequality signs in (14) by an equality sign. Consider the problem $F_{\mathcal{C}}(z,v) = 0$ restricted to the domain $H \times R^d$. Applying the transversality theorem to this new problem implies that for almost all v there exists no solution, because the codimension of H in the space R^{d+m+q} is 1. Using the fact that a finite union of sets of measure zero is a set of measure zero, we conclude that there exists a set $\widetilde{R}_{\mathcal{C}}$ of games, with complement of measure zero, where $F(\mathcal{C})$ is strongly regular. Note that all games in $R_{\mathcal{C}} = \widetilde{R}_{\mathcal{C}} \times \Delta^d$ are strongly regular, because the argument holds for all $p \in \Delta^d$, and the Cartesian product of a set of measure zero and Δ^d has measure zero.

The case where the incidence matrix $E_{\mathcal{C}}$ has rank smaller than m is more difficult to address. We show that, except in a closed set of measure zero, $F(\mathcal{C})$ has no solutions. The proof proceeds using induction on the number of players and is included in the appendix. Q.E.D.

5.3. *Comparative Statics Analysis*

So far we have focused on obtaining the solution of the coalitional bargaining games for fixed exogenous parameters v (value of coalitions) and p (proposers' probabilities). An important issue is to understand how the value of players and the path of coalition formation changes in response to changes in these exogenous parameters. Strongly regular games are very convenient because they allow us to perform comparative statics analysis using standard calculus tools.

The following corollary is an immediate application of the implicit function theorem and theorem 4.

Corollary 1: *(Comparative Statics) Let (v, p) be a strongly regular game and $z = (x, \mu, \lambda)$ be an equilibrium with CDS $\mathcal{C}$. The first-order effects of a change in the exogenous parameters (v, p) on the solution z is given by the sensitivity matrix $\mathcal{S}_{\mathcal{C}} = -\left[d_z F_{\mathcal{C}}\right]^{-1} d_{(v,p)} F_{\mathcal{C}}$ (i.e., $\Delta z = \mathcal{S}_{\mathcal{C}}\left(\Delta v, \Delta p\right)$). In particular, the effect of a local change Δv of coalitional values are given by $\Delta z = \left(\left[d_z F_{\mathcal{C}}\right]^{-1}\right)_{\cdot x}(1-\delta)\,\Delta v$, where $\left(\left[d_z F_{\mathcal{C}}\right]^{-1}\right)_{\cdot x}$ denotes the submatrix with the first d columns of the inverse Jacobian.*

The first-order effects with respect to changes in value Δv are given by the sensitivity matrix $-\left[d_z F_{\mathcal{C}}\right]^{-1} d_v F_{\mathcal{C}}$. But since

$$d_v F_{\mathcal{C}} = \begin{pmatrix} d_v f_{\mathcal{C}} \\ d_v E_{\mathcal{C}} \\ d_v M_{\mathcal{C}} \end{pmatrix} = -\begin{pmatrix} (1-\delta) I \\ 0 \\ 0 \end{pmatrix}, \tag{17}$$

the sensitivity matrix $-\left[d_z F_{\mathcal{C}}\right]^{-1} d_v F_{\mathcal{C}}$ simplifies to $\left(\left[d_z F_{\mathcal{C}}\right]^{-1}\right)_{\cdot x} (1-\delta)$. Therefore, once we have obtained an equilibrium point, all that is needed to determine first-order effects of changes in value is to evaluate the inverse of the Jacobian matrix at the solution.

The ability to conduct comparative statics analysis in multilateral bargaining games using calculus is a very powerful tool. Some of its applications are illustrated in the next section.

6. Examples

We start by analyzing two classic games: apex and quota games. These examples serve to illustrate several points. First, we demonstrate how to apply the ideas introduced in the paper, such as coalitional dynamic structures and comparative statics analysis. Second, we compare our solution concept with well-established solution concepts such as the nucleolus, bargaining set, kernel, core, and Shapley value. Moreover, since results from experimental studies are available for the examples we choose, we can evaluate whether the solution, and in particular the structure of coalition formation predicted by the model, makes sensible economic predictions.

In a third example, we consider two different public good provision problems. The purpose of this example is to illustrate that it is possible to have multiple (seven) equilibria, and to exemplify the important economic role played by agents' ability (or lack of ability) to commit to leave the game.

6.1. *Quota Games*

Quota games have been studied by Shapley (1953) and Maschler (1992) (who also report some experimental results).[9] Consider a four-player quota game, where each pairwise coalition gets $v_{\{i,j\}} = \omega_i + \omega_j$ for all distinct pairs $i, j \in N$, where the quotas of the four players are $(\omega_1, \omega_2, \omega_3, \omega_4) = (10, 20, 30, 40)$, and all remaining coalitions get $v_S = 0$ for all $S \subset N$, $S \neq \{i, j\}$. Players are very patient (i.e., we are interested in the limit when δ converges to 1), and they all have an equal chance to be proposers ($p_i(\pi) = \frac{1}{|\pi|}$).

The solution is depicted in Figure 1, where we describe the expected value $\phi_i(\pi)$ of each player for all c.s. in the equilibrium path, and the equilibrium transition probabilities $\mu(\pi)(S)$ (for simplicity, we omitted from the figure off-the-equilibrium strategies). The *CDS* at the initial state is $\mathcal{C} = (\{\{2,3\},\{2,4\},\{3,4\}\},\{\{1\}\})$, and the excesses are $e(S) = 4.938$ for $S = \{2,3\},\{2,4\}$, and $\{3,4\}$, and it can also be easily verified that this solution is a strong regular solution.[10]

The solution $(17.41, 17.53, 27.53, 37.53)$ is different from the nucleolus (Schmeidler (1969)) and the core (both of which coincide with the quota $(10, 20, 30, 40)$), the kernel, the bargaining set (Maschler (1992)), and the Shapley value (which is equal to $(17.5, 20, 28.33, 34.16)$).[11] In our solution, player 1 gets 7.41 more than his quota and players 2, 3 and 4 get each 2.47 less than their quota values. This example illustrates that the solution proposed in the paper is different from all the other major existing solution concepts.

In addition to predicting the values, the solution also predicts the structure of coalition formation. Interestingly, the equilibrium strategy of player 1 is to wait for

[9] I thank Prof. Maschler for suggesting this example.

[10] The equilibrium strategies at the initial state are not unique, though (see section 4). An example of an equilibrium strategy profile is $\sigma_1(\{1\}) = 1$, $\sigma_2(\{2,3\}) = 0.344$, $\sigma_2(\{2,4\}) = 0.656$, $\sigma_3(\{3,4\}) = 0.951$, $\sigma_3(\{2,3\}) = 0.049$, and $\sigma_4(\{3,4\}) = 1$.

[11] Since the bargaining set contains the kernel, and the kernel contains the nucleolus, this implies that our solution is different from the kernel and bargaining set (see Maschler 1992). Also, because the game is not superadditive, we compared our solution with the Shapley value of the superadditive cover of the game.

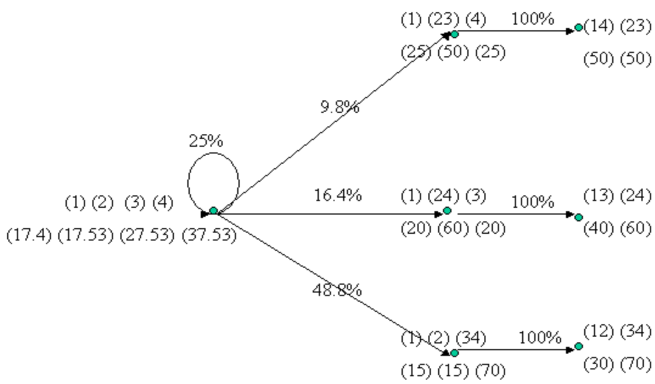

Figure 1: MPE solution of the quota game $(10, 20, 30, 40)$.

a pairwise coalition to form, an strategy that allows player 1 to get significantly more than his quota. The solution thus makes predictions that are consistent with experimental results reported in Maschler (1992), where player 1 realized that he was weak and that his condition would improve if he waited until a pairwise coalition formed, and captures an important strategic element of the game. Indeed, player 1 is better off if the coalition $\{2,3\}$ forms, rather than $\{2,4\}$ or $\{3,4\}$, because in the ensuing pairwise bargaining with 4, player 1 can get a payoff equal to 25.[12] The solution also predicts that players 2, 3, and 4 get approximately their quotas, which is also consistent with the results reported by Maschler. Interestingly, this example also serves to illustrate that Okada's (1996) result that–delays in coalition formation never happen–does not hold for games that are not superadditive.

Note that the quota game has three different absorbing states or c.s.[13]: $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, and $\{\{1,4\},\{2,3\}\}$. Given the stationary transition

[12]However, strategies considered in this paper rule out the possibility that player 1 makes side payments to players 2 and/or 3 in order to encourage them to form coalition $\{2,3\}$.

[13]An absorbing state is a c.s. π such that $\mu(\pi,\pi)=1$.

probability μ, one can easily compute the probability of reaching each of the absorbing states, from any given starting state. In particular, starting at the initial state, the probabilities of reaching each of the three absorbing states above are 65.06%, 21.86%, and 13.06%, respectively. This empirically testable prediction of the model is novel, and has many potential economic applications: existing cooperative solution concepts do not make such types of predictions about the formation of coalitions.

How do players' value change with changes in quotas and proposers' probabilities? This comparative statics exercise can be readily answered by evaluating the sensitivity matrices at the solution, and it produces some surprising results. Evaluating the value-sensitivity matrix with respect to changes in quotas, as we have seen in section 5.3, yields

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0.366 & 0.549 & 0.062 & 0.022 \\ 0.211 & 0.816 & -0.020 & -0.007 \\ 0.211 & -0.183 & 0.979 & -0.007 \\ 0.211 & -0.183 & -0.020 & 0.992 \end{bmatrix} \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \\ \Delta\omega_4 \end{bmatrix},$$

and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial\omega_i} > 0$ and $\frac{\partial\mu(\{j,k\})}{\partial\omega_i} < 0$ for all distinct i, j, and k in {2,3,4} (for the sake of space we report only the signs of entries).[14]

The information contained in the value sensitivity matrix provides some interesting insights: Increases in the quota of player 1 are shared by all players. However, increases in the quotas of either player 2, 3 or 4 are almost completely appropriated by them (in fact, the other two players distinct from player 1 suffer a loss). Other predictions are expected: When a player's quota goes up, all coalitions including this player become more likely to form (and coalitions not including this player are less likely to form).

The comparative statics with respect to changes in proposers' probability is de-

[14]Note that the main diagonal is positive, so the value of all players increases when their quotas increase, and the sum of the entries in each column is equal to one, as the value of coalitions that forms increases by the amount of the increase in the quota.

scribed by the value-sensitivity matrix

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0 & -5.42 & 1.96 & 3.45 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \end{bmatrix} \begin{bmatrix} \Delta p'_1 \\ \Delta p'_2 \\ \Delta p'_3 \\ \Delta p'_4 \end{bmatrix},$$

where, in order to preserve the sum of probabilities equal to one, we consider $p_i = p'_i/(\sum_{j=1}^{4} p'_j)$, and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial p'_i} < 0$ and $\frac{\partial\mu(\{j,k\})}{\partial p'_i} > 0$ for all distinct i, j, and k in $\{2, 3, 4\}$.

This comparative statics analysis also reveals some surprising results: When player 2 has more initiative to propose, he benefits and player 1 loses from it. Interestingly, though, the opposite happens when players 3 and 4 have more initiative. Their equilibrium payoffs decrease when they have more initiative to propose![15]

6.2. *Apex Games*

Apex games, introduced by Davis and Maschler (1965), are another interesting class of n-person games that have received considerable attention. In this game, only two types of coalitions create non-zero value: any coalition with the Apex player (player 1), or the coalition with all the $n-1$ remaining players (the Base players). For concreteness, consider the 5-player game $N = \{1, 2, 3, 4, 5\}$, where $v_{\{1,j\}} = 100$ for $j = 2, ..., 5$, $v_{\{2,3,4,5\}} = 100$, and $v_S = 0$ otherwise. Players are very patient (δ is infinitesimally close to 1), and all players have equal chance to be proposers.

The solution is depicted in Figure 2. The *CDS* at the initial state is $\mathcal{C} = (\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3, 4, 5\}\})$, and the excesses are $e(S) = \frac{300}{7}$ for $S \in \mathcal{C}$.[16]

[15] This result can be rationalized as follows: when p_2 increases, coalitions {2,3} and {2,4} are less likely to form and coalition {3,4} more likely; since player 1's gains are lowest when coalition {3,4} forms he indirectly suffers when p_2 increases. By similar reasoning, when p_4 increases, coalitions {2,4} and {3,4} are less likely to form and coalition {2,3} more likely, which benefits player 1 and hurts the other players.

[16] The equilibrium strategies for the initial state, which are not unique (see section 4), are given by $\sigma_1(\{1, j\}) = 0.25$, $\sigma_j(\{2, 3, 4, 5\}) = 0.25$, and $\sigma_j(\{1, j\}) = 0.75$, for all $j = 2, 3, 4, 5$.

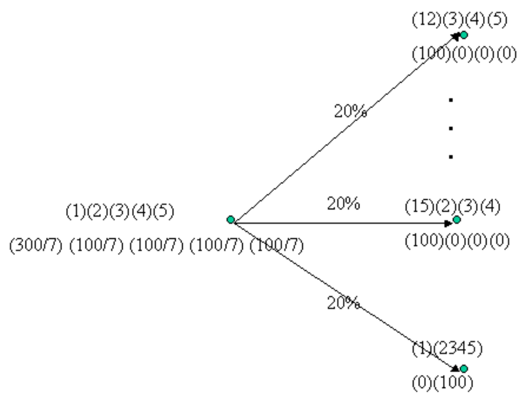

Figure 2: MPE solution of the apex game.

The solution for the game is $(42.9, 14.3, 14.3, 14.3, 14.3)$. This solution coincides with the kernel of the game and the nucleolus. However, it is different from the bargaining set, the core (which is empty), and the Shapley value (which is equal to $(60, 10, 10, 10, 10)$).

Moreover, the model also predicts that any of the four apex coalitions $\{1, j\}$, $j = 2, ..., 5$, can form with 20% probability (thus there is an 80% chance that an apex coalition forms), and the base coalition $\{2, 3, 4, 5\}$ can form with 20% probability. Several experimental tests of the Apex game with 4 and 5 players have been conducted (see Kahan and Rapoport (1984) for a survey of these studies). Particular attention has been given in these studies to the frequencies of formation of apex coalitions versus the base coalition. Two of these studies (Selten and Schuster (1968) and Albers (1978)) considered 5-person apex games similar to the one considered above.[17]

[17]The games played were slightly different: $v_S = 100$ for all $S \subset N$ with $1 \in S$ and $|S| \geq 2$, $v_{\{2,3,4,5\}} = 100$, and $v_S = 0$ otherwise. For this modified game the prediction of our equilibrium concept is as described above; thus the difference is immaterial.

Albers (1978), based on 25 plays of the game, reports that the apex coalition, the base coalition, and other coalition structure forms with frequencies equal to 76%, 20%, and 4%, respectively. Similarly, Selten and Schuster (1968), based on 12 plays of the game, report frequencies equal to 67%, 17%, and 17%, respectively. Overall these results seem to provide support for the predictions of the model.

Comparative statics results for the apex game can also be easily obtained. The sensitivity matrix describing the changes in value is

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \\ \Delta\phi_5 \end{bmatrix} = \begin{bmatrix} 0.23 & -0.48 & 0 & 0 \\ 0.74 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \\ -0.26 & 0.17 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v(\{1,2\}) \\ \Delta v(\{2,3,4,5\}) \\ \Delta p_1' \\ \Delta p_2' \end{bmatrix},$$

where $p_i = p_i'/(\sum_{j=1}^{5} p_j')$, and the sensitivity matrix describing the changes in coalition formation is

$$\begin{bmatrix} \Delta\mu(\{1,2\} \\ \Delta\mu(\{1,3\} \\ \Delta\mu(\{1,4\} \\ \Delta\mu(\{1,5\} \\ \Delta\mu(\{2,3,4,5\} \end{bmatrix} = \begin{bmatrix} 0.0309 & -0.0009 & -0.2 & -2.2 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.01 & -0.0009 & -0.2 & 0.8 \\ -0.0009 & 0.0036 & 0.8 & -0.2 \end{bmatrix} \begin{bmatrix} \Delta v(\{1,2\}) \\ \Delta v(\{2,3,4,5\}) \\ \Delta p_1' \\ \Delta p_2' \end{bmatrix}.$$

Surprisingly, these results indicate that changes in proposer probabilities have no effect on the value of players. Also, as was the case with the previous example, whenever the value of a coalition increases then both the equilibrium value of all coalitional members and the probability that this coalition forms increase as well.

6.3. *Public Good Provision: Multiple Equilibria Example*

We analyze separately two related public good provision problems among three symmetric firms. In the first example, if firms stay independent they each make zero profit. If any two firms merge they make a profit of 1, and the firm that remains independent receives a positive externality and profits 3. Regulators prevent full consolidation (a three-way merger) from happening. The partition function that describes this game is thus $v_i(\{\{1\},\{2\},\{3\}\} = 0$, $v_{\{i,j\}}(\{\{i,j\},\{k\}\}) = 1$, and $v_{\{k\}}(\{\{i,j\},\{k\}\}\}) = 3$. The grand coalition is ruled out (or has a low value). Assume that proposers are chosen with equal probabilities and $\delta \in (0.5, 1)$. Note that Pareto efficient equilibria have a total value of 4.

There are seven *MPE* solutions for this game of three different types. In many respects this game loosely resembles a three-player war of attrition.

One equilibrium is the symmetric solution in which the expected equilibrium payoffs are $x = (0.5, 0.5, 0.5)$; the transition probabilities are $\mu(\{i,j\}) = \frac{1-\delta}{5\delta}$, for all pairs $\{i,j\}$, and $\mu(\emptyset) = \frac{8\delta-3}{5\delta}$, where $\emptyset$ represents no proposal (or remaining at the initial state); and the *CDS* is $\mathcal{C} = (\{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}\})$. In this equilibrium, each of the three firms refrains from proposing with high probability, and only proposes with a small probability to the other two firms. They all reject any proposals below 0.5, and thus firms are indifferent between proposing or not. This symmetric equilibrium is the most inefficient.

In the second type of equilibrium (there are three symmetric cases), the expected equilibrium payoffs are $x = (0,1,1)$; the transition probabilities are $\mu(\{1,2\}) = \mu(\{1,3\}) = \frac{1-\delta}{2\delta}$, and $\mu(\emptyset) = \frac{2\delta-1}{\delta}$; and the *CDS* is $\mathcal{C} = (\{\emptyset, \{1,2\}, \{1,3\}\})$. In this equilibrium, firms 2 and 3 reject any proposals lower than 1, make no proposal with high probability, and, when proposing, choose to form a coalition with firm 1. Firm 1 cannot afford to pay more than 1 to form a coalition and thus it makes no proposals with probability one. This equilibrium is more efficient than the first one, but is still inefficient.

In the third type of equilibrium (again there are three symmetric cases), the expected equilibrium payoffs are $x = \left(\frac{6\delta}{3-\delta}, \frac{\delta}{3-\delta}, \frac{\delta}{3-\delta}\right)$, which converges to $x = (3, 0.5, 0.5)$ when $\delta \to 1$; the transition probabilities are $\mu(\emptyset) = \frac{1}{3}$, $\mu(\{2,3\}) = \frac{2}{3}$; and the *CDS* is $\mathcal{C} = (\{\emptyset, \{2,3\}\})$. In this equilibrium, firm 1's strategy is to refrain from proposing and reject any proposal worth less than 3, and firm 2's and firm 3's strategies are to always "give in" and propose to form the coalition $\{2,3\}$. This equilibrium is, in the limit, Pareto efficient.

The second public good provision problem is similar to an example considered by Ray and Vohra (1999, 2000), and has very different equilibria than the first one. Maintain everything as fixed in the previous example with the exception of the following aspect: Now, regulators allow for full consolidation of the industry, and the total value of the three-way merger is 6. In terms of partition functions the game is then: $v_i(\{\{1\},\{2\},\{3\}\} = 0$, $v_{\{i,j\}}(\{\{i,j\},\{k\}\}) = 1$, $v_{\{k\}}(\{\{i,j\},\{k\}\}\}) = 3$, and $v_{\{3\}}(\{\{1,2,3\}\}) = 6$.

The unique equilibrium of this game in our model is $\phi = (2,2,2)$ and $\mu(\{1,2,3\}) = 1$ (a three-way merger with an equal split of gains). Note that this is a Pareto efficient equilibrium. In our setting a player cannot credibly commit to leave the game, while in Ray and Vohra this is possible. This assumption can lead to very different economic implications. The equilibrium, in Ray and Vohra, is for a player to leave the game, because it is in the best interest for the remaining players to provide the public good, after the first player has left, and thus the first player to leave gets 3. Thus the commitment ability may lead to inefficiencies.

Summarizing, these two examples serve to illustrate the distinctions between our model and the model of Ray and Vohra. Without the commitment ability, in the first public good example, many equilibria exist, including inefficient equilibria; while in the second public good example the commitment ability leads to a unique and efficient equilibrium. With the commitment ability, players commit to exit in the first opportunity, leaving the other players to provide the public good; this is efficient in the first example but inefficient in the second one.

In the next section we see that the odd number of equilibria in the examples of this section is not just coincidental, but holds in general. The analysis of the global number of equilibria is the subject of the next section.

7. The Global Number of Equilibria

We show in this section that almost all games have an odd number of *MPE* equilibria. Similar results hold for other well-known economic models such as competitive economies (Debreu (1970)) and n-person normal form games (Wilson (1971) and Harsanyi (1973)).

Additionally, we derive a sufficient condition for the global uniqueness of equilibria. The result states that if the index of each equilibrium solution is non-negative, where the index is shown to be equal to the sign of the determinant of the Jacobian matrix $dF_{\mathcal{C}}$, then there is a globally unique equilibrium. We argue that this sufficient condition is very weak and likely to be satisfied by a large class of games. Along these lines, we prove that the sufficient condition holds for three-player superadditive games, and thus there is only one equilibrium in these games.

Our first result is an application of the Lefschetz fixed point theorem ($LFPT$) for correspondences (see McLennan (1989)), which is restated in the appendix for completeness.

Theorem 6: *Almost all games (all strongly regular games) have a finite and odd number of MPE equilibria. Moreover,* $\sum_{\mathcal{C}\in CDS} \sum_{y\in MPE(\mathcal{C})} sgn \det(dF_{\mathcal{C}}(y)) = +1$, *where the summation is over all CDSs* $\mathcal{C}$ *and all MPEs* y *with CDS* $\mathcal{C}$.

Proof: Let v be any strongly regular game. The set of fixed points of $\mathcal{F}$ (see definition (7)), $\mathcal{F}^* = \{x \in X : x \in \mathcal{F}(x)\}$, corresponds to the equilibria points of the game by theorem 1. The set $\mathcal{F}^*$ is finite: all the equilibrium points are, by lemma 3, solutions of $F_{\mathcal{C}}(x, \lambda, \mu) = 0$ for some CDS $\mathcal{C}$. But since the game is regular the solutions are locally isolated (theorem 4), and since the solution belongs to the compact X then there is only a finite number of solutions.

By theorem 1, $\mathcal{F} : X \to X$ is an upper hemicontinuous convex-valued correspondence (thus $\mathcal{F}(x)$ is contractible for all $x \in X$). The set $X \subset R^d$, Cartesian product of simplexes, is a simplicial complex and thus $\mathcal{F}$ satisfies the conditions of the $LFPT$.

Let U_{x^*} be an open neighborhood around each $x^* \in \mathcal{F}^*$, so that x^* is the only fixed point in $\overline{U}_{x^*}$. The Additivity Axiom of the Lefschetz index implies

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}). \tag{18}$$

In addition, the Lefschetz index is

$$\Lambda(\mathcal{F}, X) = 1. \tag{19}$$

But $\mathcal{F}$ can be approximated by a continuous map $f' : X \to X$ such that $\Lambda(\mathcal{F}, X) = \Lambda(f', X)$ (Continuity Axiom), and X is a contractible set, and thus there is an homotopy $\varphi : X \times [0,1] \to X$ where $\varphi_1 = I_X$ and $\varphi_0 = z_0 \in X$. Therefore, any continuous map $f' : X \to X$ is homotopic to the constant map so, by the Weak Normalization and Homotopy Axioms, $\Lambda(\mathcal{F}, X) = \Lambda(f', X) = 1$. Therefore, equations (18) and (19) imply,

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}) = 1.$$

The next lemma, proved in the appendix, establishes the formula for $\Lambda(\mathcal{F}, U_{x^*})$, which completes the proof.

Lemma 4: *The Lefschetz index of a strongly regular MPE solution (x^*, λ^*, μ^*) is equal to $\Lambda(\mathcal{F}, U_{x^*}) = sgn \det(dF_{\mathcal{C}}(x^*, \lambda^*, \mu^*))$, and is equal to either $+1$ or -1.*

Q.E.D.

Theorem 6 implies that a sufficient condition for global uniqueness of equilibria is that $\det(dF_{\mathcal{C}}) \geq 0$, at all solutions of problem $F(\mathcal{C})$ and for all *CDS*s $\mathcal{C}$.

Corollary 2: *All coalitional bargaining games have a globally unique equilibrium if $\det(dF_{\mathcal{C}}) \geq 0$ where the Jacobian is evaluated at any solution of problem $F(\mathcal{C})$ for all CDSs $\mathcal{C}$.*

Because of the special structure of the Jacobian $dF_{\mathcal{C}}$, we conjecture that a sufficient condition for $\det(dF_{\mathcal{C}}) \geq 0$ at all solutions of problem $F(\mathcal{C})$ is that the inequalities $x_i(\pi) - x_i(\pi S) \geq 0$, for all $S \in C_r(\pi)$ and $i \notin S$, hold. Direct computation of determinants reveals, for all games with three players and all *CDS*s $\mathcal{C}$, that the inequalities imply that $\det(dF_{\mathcal{C}}) \geq 0$ (see corollary 3).

The inequality $x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, is a weak condition that has a natural economic interpretation: player i is excluded from the offer S if and only if moving from c.s. π to πS imposes a negative externality on player i (i.e., $x_i(\pi S) \leq x_i(\pi)$). We show in the next corollary that these inequalities hold for all three-player games where the grand coalition is efficient.

COROLLARY 3: *Almost all three-player games where the grand coalition is efficient, in particular superadditive game, have a globally unique equilibrium.*

PROOF: See appendix.

We remark that in the public good provision example of section 6.3 (a strongly regular game with seven equilibria) the grand coalition was not efficient, and therefore it is not in contradiction to corollary 3.

Interestingly, one can easily verify that in three-player games where the grand coalition is efficient, proposers are chosen with equal probabilities, and the discount rate δ converges to 1, the unique *MPE* solution converges to the coalitional bargaining value solution proposed by Gomes (2000). The coalitional bargaining value solution is a piecewise linear in the partition function form v (like the nucleolus) and has eight regions of linearity: the regions of linearity correspond to solutions with *CDS*s equal to $(\{\{1,2,3\}\})$, $(\{\{i,j\}\},\{\{1,2,3\}\})$, $(\{\{i,j\}\},\{\{i,k\}\})$ or $(\{\{i,j\},\{i,k\}\})$, and $(\{\{1,2\},\{1,3\},\{2,3\}\})$ (where i, j, and k are distinct elements of $N = \{1,2,3\}$). Moreover, as shown by Gomes (2000), this limit *MPE* solution coincides either with the Shapley value (for the case where the solution is associated with the *CDS* $(\{\{1,2\},\{1,3\},\{2,3\}\})$), or with the nucleolus for the other *CDS*s.

8. Conclusion

This paper studied n-player coalitional bargaining games in an environment with widespread externalities (where the exogenous parameters are expressed in a partition function form). The coalitional bargaining problem is modeled as a dynamic non-cooperative game in which contracts forming coalitions may be renegotiated. The equilibrium concept used is stationary subgame or Markov perfect equilibrium, where the set of states is all possible coalition structures. The equilibrium characterizes, simultaneously, both the expected value of coalitions, and the state transition probability that describes the path of coalition formation. A comprehensive analysis of the equilibrium properties is developed.

The existence of equilibria is established, and Pareto efficiency is guaranteed if the grand coalition is efficient and if frictions (delay between offers) are insignificant. Also, for almost all games (except in a closed set of measure zero) the equilibrium is locally unique and stable to small perturbations of the exogenous parameters, and the number of equilibria is finite and odd. Global uniqueness does not hold in general (a public good provision example with multiple (seven) equilibria is provided), but a sufficient condition for global uniqueness is derived, and this sufficient condition is shown to prevail in three-player superadditive games.

Comparative statics analysis can be easily performed using standard calculus tools, allowing us to understand how the value of players and the path of coalition formation changes in response to changes in the exogenous parameters. Being able to answer comparative statics questions is valuable to negotiators, because they may be able, for example, to invest in changing the likelihood of being proposers. Applications of the technique are illustrated using the apex and quota games, and some interesting insights emerge: surprisingly, a player may not benefit from having more initiative to propose (other players may adjust their strategies in such a way that lead the proposer to be worse off). The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase.

Appendix

Proof of Lemma 1: The necessary part follows directly from the discussion before the statement of the result and the definition of *MPE* solution. Let us prove the sufficient part of the theorem. Suppose that we are given payoffs and strategy profiles $\left(\phi, \phi^j, \sigma\right)$ satisfying all the conditions of the lemma. We use the one-stage deviation principle for infinite-horizon games. This result states that in any infinite-horizon game with observed actions that is continuous at infinity, a strategy profile σ is subgame perfect if and only if there is no player i and strategy σ_i' that agrees with σ_i except at a single stage t of the game and history h^t, such that σ_i' is a better response to σ_{-i} than σ_i conditional on history h^t being reached (see Fudenberg and Tirole (1991)).

Note first that the game is continuous at infinity: for each player i his utility function is such that, for any two histories h and h' such that the restrictions of the histories to the first t periods coincides, then the payoff of player i, $\left|u_i(h) - u_i(h')\right|$, converges to zero as t converge to infinity. It is immediately clear that the negotiation game is continuous at infinity because $\left|u_i(h) - u_i(h')\right| \leq M\left(\delta^{t+1} + \delta^{t+2} + \cdots\right) = \frac{M}{1-\delta}\delta^{t+1}$, for M large enough. The strategy profile σ_i is such that, by construction, no single deviation σ_i' at both the proposal and response stage can lead to a better response than σ_i. Therefore, by the one-stage deviation principle, the stationary strategy profile σ is a subgame perfect Nash equilibrium. Q.E.D.

Proof of Theorem 1: We show that the correspondence $\mathcal{F}$ satisfies all the conditions of the Kakutani fixed point theorem.

We find convenient to use the map $y_i^j(\pi)(x, \sigma_j)$ and $y_i(\pi)(x, \sigma)$ where

$$y_i^j(\pi)(x, \sigma_j) = \begin{cases} \max\limits_{S \ni i} \left\{x_S(\pi S) - \sum_{j \in S \setminus i} x_j(\pi)\right\} & j = i \\ \sum_{S \subset \pi} \sigma_j(\pi)(S)\left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S)\right) & j \neq i \end{cases}.$$

and,

$$y_i(\pi)(x,\sigma) = \delta\left(\sum_{j\in\pi} p_j(\pi)\, y_i^j(\pi)(x,\sigma(j))\right) + (1-\delta)v_i(\pi).$$

Note that $z \in \mathcal{F}(x)$ if and only if $z = y(x,\sigma)$ and $\sigma \in \Sigma(x)$.

(1) $\mathcal{F}(X) \subset X$, where X is non-empty subset that is compact and convex. Take any $x \in X$ and $z = y(x,\sigma)$ with $\sigma \in \Sigma(x)$. We first show that $\sum_{i\in\pi} z_i(\pi) \leq \overline{v}$ and then $z_i(\pi) \geq \underline{v}_i$ which implies that $z \in X$. First,

$$\begin{aligned}\sum_{i\in\pi} z_i(\pi) &= \sum_{i\in\pi} y_i(\pi)(x,\sigma) = \\ &= \delta\left(\sum_{j\in\pi} p_j(\pi) \sum_{i\in\pi} y_i^j(\pi)(x,\sigma_j)\right) + (1-\delta)\sum_{i\in\pi} v_i(\pi)\end{aligned}$$

where the order of the first summation has been inverted. But

$$\sum_{i\in\pi} y_i^j(\pi)(x,\sigma) = \sum_{S\subset\pi} \sigma_j(\pi)(S)\left(x_S(\pi S) + \sum_{i\notin S} x_i(\pi S)\right) \leq \overline{v}$$

because $\sum_S \sigma_j(\pi)(S) = 1$, $\sigma_j(\pi)(S) \geq 0$, and $x_S(\pi S) + \sum_{i\notin S} x_i(\pi S) \leq \overline{v}$, for $x \in X$. Therefore

$$\sum_{i\in\pi} z_i(\pi) \leq \delta \sum_{j\in\pi} p_j(\pi)\,\overline{v}(\pi) + (1-\delta)\overline{v}(\pi) = \overline{v}.$$

Also, we have that $z_i(\pi) \geq \underline{v}_i$. First note that $y_i^j(\pi)(x,\sigma) \geq \underline{v}_i$ for all j, because $y_i^j(\pi)(x,\sigma) \geq x_i(\pi) \geq \underline{v}_i$ as player i can always choose not to make any offer ($S = \{i\}$) and $x \in X$. Also, $y_i^j(\pi)(x,\sigma) = \sum_{S\subset\pi} \sigma_j(\pi)(S)\left(\mathbb{I}_{[i\in S]} x_i(\pi) + \mathbb{I}_{[i\notin S]} x_i(\pi S)\right) \geq \underline{v}_i$ because $x \in X$. Therefore,

$$y_i(\pi)(x,\sigma) = \delta\left(\sum_{j\in\pi} p_j(\pi)\, y_i^j(\pi)(x,\sigma_j)\right) + (1-\delta)v_i(\pi) \geq \underline{v}_i.$$

(2) $\mathcal{F}(x)$ is a convex (and non-empty) set for all $x \in X$: Say that $z, z' \in \mathcal{F}(x)$ with $z = y(x,\sigma)$ and $z' = y(x,\sigma')$ where $\sigma, \sigma' \in \Sigma(x)$. Then, for any $\lambda \in [0,1]$, $\lambda z + (1-\lambda) z' = y(x, \lambda\sigma + (1-\lambda)\sigma') \in \mathcal{F}(x)$ because $\lambda\sigma + (1-\lambda)\sigma' \in \Sigma(x)$ ($\Sigma(x)$ is convex).

(3) $\mathcal{F}$ is u.h.c., that is, for any sequence $(x^n, y(x^n,\sigma^n)) \to (x,z)$ with $\sigma^n \in \Sigma(x^n)$ then $z \in \mathcal{F}(x)$ (i.e., there exists an $\sigma \in \Sigma(x)$ such that $y(x,\sigma) = z$). The sequence (σ^n) belongs to Σ a compact subset of a finite-dimension Euclidean space. Therefore, there exists a subsequence of (σ^{n_k}) that converges to $\sigma \in \Sigma$. Rename this subsequence as (σ^n) for notational simplicity. We have that $\sigma_i^n(\pi)(S) \to \sigma_i(\pi)(S)$ for all $S \subset \pi$ and $i \in \pi \in \Pi$, and that $y(x^n,\sigma^n) \to y(x,\sigma)$, due to the continuity of y, and thus $z = y(x,\sigma)$.

It is sufficient to show that $\sigma \in \Sigma(x)$. By the definition of $\Sigma(x)$, $\sigma \in \Sigma(x)$ if and only if $\sigma \in \Sigma$ and $\sigma_i(\pi)(S) = 0$ for all $S \subset \pi$ and $i \in \pi \in \Pi$ such that $x_S(\pi S) - \sum_{j\in S} x_j(\pi) < \max\limits_{S \ni i} \left(x_S(\pi S) - \sum_{j\in S} x_j(\pi) \right)$. Consider any $S \subset \pi$ for which the inequality above holds. By continuity, we have that there exists a large enough n_0 such that for all $n \geq n_0$, $x_S^n(\pi S) - \sum_{j\in S} x_j^n(\pi) < \max\limits_{S \in \pi_i} \left(x_S^n(\pi S) - \sum_{j\in S} x_j^n(\pi) \right)$. But since $\sigma^n \in \Sigma(x^n)$, this implies that $\sigma_i^n(S) = 0$, and $\sigma_i(S) = 0$. Q.E.D.

Proof of lemma 3: For all σ satisfying $\mathcal{C} = \mathcal{C}(\sigma)$ then $\mu = \mu(\sigma)$ satisfy

$$\begin{aligned} \sum_{S \in C_r(\pi)} \mu(\pi)(S) &= \sum_{S \in C_r(\pi)} \sum_{j \in \pi} p_j(\pi)\, \sigma_j(\pi)(S) = \\ &= \sum_{j\in\pi} p_j(\pi) \sum_{S \in C_r(\pi)} \sigma_j(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi) \text{ for all } r, \end{aligned}$$

because if $j \in P_r(\pi)$ then $supp(\sigma_j(\pi)) \subset C_r(\pi)$, which corresponds to the last set of equations in $F(\mathcal{C})$.

Now, if $i \to j$ then there exist a coalition S such that $i, j \in S$ and $\sigma_i(\pi)(S) > 0$. But because $supp(\sigma_i(\pi)) \subset \arg\max\limits_{\{S \subset \pi : i \in S\}} \{e(\pi)(S)\, x\}$ then

$$e_i := \max_{\{S\subset\pi : i \in S\}} \left\{ x_S(\pi S) - \sum_{j \in S} x_j(\pi) \right\} \leq e_j.$$

Repeating the same argument, if there is a path from i to j then $e_i \leq e_j$, and if i is connected to j then both have the same excess $e_i = e_j$. Thus, $\lambda_r(\pi) = e_i = x_S(\pi S) - \sum_{j \in S} x_j$, for all $S \in \mathcal{C}_r$ and all $i \in P_r(\pi)$. Substituting the expressions for the excesses into equation (8) finishes the if part of the proof. The reciprocal follows directly from the construction of the polyhedral sets $\mathcal{M}_{\mathcal{C}}$ and $\mathcal{E}_{\mathcal{C}}$. Q.E.D.

Proof of Theorem 4: The implicit function theorem immediately implies that, for any game (v^*, p^*) and regular solution (x^*, μ^*, λ^*) there exists an open neighborhood $B \subset R^d \times R^d$ of (v^*, p^*), an open neighborhood $\widetilde{W} \subset R^d \times R^m \times R^q$ of (x^*, μ^*, λ^*), and a mapping $(x(v,p), \lambda(v,p), \mu(v,p)) \in R^n \times R^m \times R^q$ such that $z(v,p) = (x(v,p), \lambda(v,p), \mu(v,p))$ is the only solution of problem $F(\mathcal{C})$ in $\widetilde{W}$ for all games $(v,p) \in B$. Note that since $\lambda(v,p)$ is a function of $x(v,p)$ (see second set of equations in (13)) then $z(v,p)$ is the only solution in a cylinder $W \times R^q$ for W an open neighborhood of (x^*, μ^*).

It remains to show that $z(v,p)$ is an *MPE* solution. By lemma 3, $z(v,p)$ is an *MPE* solution if $x(v,p) \in \mathcal{E}_{\mathcal{C}}$ and $\mu(v,p) \in \mathcal{M}_{\mathcal{C}}$. Choosing the open neighborhood W small enough, we get that $x(v,p) \in \mathcal{E}_{\mathcal{C}}$ because x^* is a strong solution ($x^* \in int(\mathcal{E}_{\mathcal{C}})$). It remains to show that $\mu(v,p) \in \mathcal{M}_{\mathcal{C}}$. Consider the following lemma.

Lemma 5: *The dimension of the affine hull of* $\mathcal{M}_{\mathcal{C}}$ *is* $\dim(aff(\mathcal{M}_{\mathcal{C}})) = m - q$, *and*

$$aff(\mathcal{M}_{\mathcal{C}}) = L_{\mathcal{C}} = \left\{ \mu \in R^m : \sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi) \right\}.$$

Furthermore, $\mathcal{M}_{\mathcal{C}}$ *is an open set relative to its affine subspace, that is,* $ri(\mathcal{M}_{\mathcal{C}}) = \mathcal{M}_{\mathcal{C}}$.

Proof of lemma 5: First note that $\mathcal{M}_{\mathcal{C}}$ is a convex set. We have seen that $\mathcal{M}_{\mathcal{C}} \subset L_{\mathcal{C}}$, then $aff(\mathcal{M}_{\mathcal{C}}) \subset L_{\mathcal{C}}$. But $\dim(L_{\mathcal{C}}) = m - q$: the matrix $M_{\mathcal{C}}$ of order $q \times m$ has rank q. Thus it is enough to show that $\dim aff(\mathcal{M}_{\mathcal{C}}) = m - q$ to conclude that $aff(\mathcal{M}_{\mathcal{C}}) = L_{\mathcal{C}}$.

The affine space $aff(\mathcal{M}_{\mathcal{C}}) = aff(\mathcal{M}_{\mathcal{C}_1}) \oplus ... \oplus aff(\mathcal{M}_{\mathcal{C}_q})$ ($\oplus$ is the direct sum) where $\mathcal{M}_{\mathcal{C}_r} := \left\{ (\mu(S))_{S \in \mathcal{C}_r} : \mu \in \mathcal{M}_{\mathcal{C}} \right\}$. Therefore it is enough to prove that

$\dim(aff\,(\mathcal{M}_{\mathcal{C}_r})) = m_r - 1$ since then $\dim(aff\,(\mathcal{M}_{\mathcal{C}})) = \sum_{r=1}^{q} \dim(aff\,(\mathcal{M}_{\mathcal{C}_r})) = m - q$. In the sequel, we drop the subscript r from $\mathcal{C}_r$ and P_r.

In order to complete the proof of the lemma it is sufficient to prove the following claim: Given any vector $(\mu(S))_{S\in\mathcal{C}}$ such that $\sum_{S\in\mathcal{C}} \mu(S) = 0$ then there exist vectors $(\sigma_i(S))_{S\ni i}$ for all $i \in P$, such that $\sum_{S\in\mathcal{C}} \sigma_i(S)\mathbb{I}_{[i\in S]} = 0$ and $\sum_{i\in P} \sigma_i(S)\mathbb{I}_{[i\in S]} = \mu(S)$.

We prove the claim by induction. The induction hypothesis is: Let P_1 and P_2 be two disjoint subsets of P and let $C_k = \{S \in C : S \cap P_k \neq \emptyset\}$ for $k = 1, 2$ and suppose that each P_k is connected. That is, for any i and j in P_k there exists a path linking i and j (i and j are directly linked if there exists a subset $S \subset C_k$ with $i, j \in S$). Suppose that the claim hold for each pair (P_k, C_k). We now show that the claim also holds for $P' = P_1 \cup P_2$ and $C' = C_1 \cup C_2$ if this pair is connected (thus $C_1 \cap C_2 \neq \emptyset$). Take any $(\mu(S))_{S\in C'}$ such that $\sum_{S\in C'} \mu(S) = 0$. Construct $\mu_k(S) = \mu(S)$ if $S \in C' \backslash C_{-k}$ and $\mu_k(S) = \alpha_k \mu(S)$ if $S \in C_1 \cap C_2$ where α_k is such that $\sum_{S\in C'\backslash C_{-k}} \mu(S) + \alpha_k \sum_{S\in C_1\cap C_2} \mu(S) = 0$. Note that there always exist a solution, because $C_1 \cap C_2 \neq \emptyset$ and the solution satisfies $\mu_1(S) + \mu_2(S) = \mu(S)$. Since $\sum_{S\in C_k} \mu_k(S) = 0$ there exists $\left(\sigma_i^k(S)\right)_{S\ni i}$ for all $i \in P_k$, such that $\sum_{S\in C} \sigma_i^k(S)\mathbb{I}_{[i\in S]} = 0$ and $\sum_{i\in P} \sigma_i^k(S)\mathbb{I}_{[i\in S]} = \mu_k(S)$. The sum $\sigma_i(S) = \sigma_i^1(S) + \sigma_i^2(S)$ is such that $\sum_{S\in C'} \sigma_i(S)\mathbb{I}_{[i\in S]} = 0$ and $\sum_{i\in P'} \sigma_i(S)\mathbb{I}_{[i\in S]} = \mu(S)$ as required by the claim. In order to complete the proof start with $P_1 = \{i\}$ and $P_2 = \{j\}$ such that there exists a $S \cap \{i, j\} \neq \emptyset$ and $S \in C$. After applying the claim we obtain a set P'. We can proceed iterative by letting $P_1 = P'$ and choosing a new $P_2 = \{k\}$ that is connected to P_1 (it always exist because of the connection assumption) until we obtain $P' = P$. Q.E.D.

Lemma 5 shows that the dimension of the affine hull of $\mathcal{M}_{\mathcal{C}}$ is $m - q$ and that $\mathcal{M}_{\mathcal{C}}$ is open relative to the affine hull. Thus, there exists an open neighborhood W of $(x^*, \mu^*) \in \mathcal{E}_{\mathcal{C}} \times \mathcal{M}_{\mathcal{C}}$ such that $(x(v,p), \mu(v,p)) \in W \subset \mathcal{E}_{\mathcal{C}} \times \mathcal{M}_{\mathcal{C}}$. Q.E.D.

Proof of Theorem 5 (continuation): Consider the combined induction hypothesis: Almost all games with less than n players are strongly regular *and* all such games have local solution mappings $x(v,p)$ that are surjective.

All games with 1 player are strongly regular and the solution mapping $x(v) = v$ is surjective. The only *CDS* is $\mathcal{C} = \{\{1\}\}$ and obviously the Jacobian matrix of problem

$F(\mathcal{C})$ is nonsingular.

Let π be a c.s. with n players, and let us represent by a subscript 0 the references to the c.s. π and by the subscript -0 the references to all its proper subgames. Let $V_0 \times \Delta_0$ represent the set of all $(v_i(\pi), p_i(\pi))$ and $V_{-0} \times \Delta_{-0}$ the set of all $(v_i(\pi'), p_i(\pi'))_{\substack{\pi' \in \Pi \\ \pi' \neq \pi}}$.

Let $R_{-0} \subset V_{-0} \times \Delta_{-0}$ be the set of games that are strongly regular and the local mappings $x_{-0}(v, p)$ are surjective. According to the induction hypothesis almost all games of $V_{-0} \times \Delta_{-0}$ belong to R_{-0}. Consider the solutions of $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) = 0$ where $z_0 = (x_0, \lambda_0, \mu_0)$ and the Jacobian matrix at the solution,

$$d_{(z_0,v_0,p_0,v_{-0},p_{-0})} F_{\mathcal{C}_0} = \begin{bmatrix} d_{x_0} f_{\mathcal{C}} \; d_{\lambda_0} f_{\mathcal{C}} & d_{\mu_0} f_{\mathcal{C}} & -(1-\delta) I_0 & 0 & 0 \\ E_{\mathcal{C}_0} & 0 & 0 & 0 & -\left(d_{(v_{-0},p_{-0})} g\,(x_{-0})\right) \\ 0 & M_{\mathcal{C}_0} & 0 & -M'_{\mathcal{C}_0} & 0 \end{bmatrix}, \tag{20}$$

where $g : V_{-0} \to R^m$ is the linear map $g\,(x_{-0})\,(S) = x_S(\pi S)$ for all the m sets in the *CDS* $\mathcal{C}$. Note that the linear map g is surjective, and thus the composition $g \circ x_{-0}\,(v_{-0,}p_{-0})$ is surjective (the composition of surjective maps is surjective). But then we have that $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) \overline{\pitchfork} 0$. Therefore, by the transversality theorem, for almost every $(v, p) \in V \times \Delta$, $F_{\mathcal{C}_0}(z_0) \overline{\pitchfork} 0$.

If the rank of $E_{\mathcal{C}}$ is smaller than m then the Jacobian matrix $d_z F_{\mathcal{C}_0}$ where $z = (x, \lambda, \mu)$ cannot have full rank (see expression (16)), and thus it must be the case that for almost every game, say $R_{\mathcal{C}} \subset V \times \Delta$, there exists no solution to problem $F(\mathcal{C})$. If the rank of $E_{\mathcal{C}}$ is equal to m we have already proven that for almost every game the solution of $F(\mathcal{C})$ are strongly regular.

It remains to show that the local solution mappings $x\,(v, p)$ of problem $F(\mathcal{C})$ are surjective. But

$$d_{(v,p)} x = \begin{bmatrix} d_{(v_0,p_0)} x_0 & d_{(v_{-0},p_{-0})} x_0 \\ 0 & d_{(v_{-0},p_{-0})} x_{-0} \end{bmatrix},$$

because x_{-0} does not depend on $(v_{0,}p_0)$, and it is thus enough to prove that $x_0\,(v, p)$ is surjective (by the induction hypothesis $d_{(v_{-0},p_{-0})}x_{-0}$ is surjective). The implicit function theorem gives us the expression of the derivative of the local mappings (refer to (13)) as, $d_{(v,p)}x_0(v,p) = [d_z F_{\mathcal{C}}]_n^{-1} d_{(v,p)}b_{\mathcal{C}_0}$, where $[d_z F_{\mathcal{C}}]_n^{-1}$ is the submatrix of $[d_z F_{\mathcal{C}}]^{-1}$ restricted to the first n rows, and $b_{\mathcal{C}_0}(v,p) = ((1-\delta)v_0, g \circ x_{-0}(v_{-0},p_{-0}), h_0(p))$ where $h_0(p)(r) = \sum_{j \in P_r(\pi)} p_j(\pi)$. But the product of matrix $d_{(v,p)}b_{\mathcal{C}_0}$, which is surjective, and matrix $[d_z F_{\mathcal{C}}]_n^{-1}$, which is also surjective, is a surjective matrix. Thus we conclude that $x_0(v,p)$ is surjective.

So all games in $R = \cap_{\mathcal{C} \in CDS} R_{\mathcal{C}}$ are strongly regular (and satisfy the surjectivity induction hypothesis), and the complement of R, $(R^d \times \Delta^d) \backslash R$, has measure zero because there are only a finite number of *CDS*s. Q.E.D.

Lefschetz Fixed Point Theorem (LFPT) (McLennan 1989): Let $\mathcal{T}$ be the collection of admissible triples (X, F, U) where $X \subset R^m$ is a finite simplicial complex, $F : X \to X$ is a upper hemicontinuous contractible valued correspondence (u.h.c.c.v.), $U \subset X$ is open, and there are no fixed points of F in $\overline{U} - U$. Then there is a unique Lefschetz fixed point index $\Lambda(X, F, U)$ that satisfying the following axioms (when X is implicitly given we just say $\Lambda(F, U)$):

(Localization axiom): If $F_0, F_1 : X \to X$ are u.h.c.c.v. correspondences that agree on $\overline{U}$, and (X, F_1, U), $(X, F_0, U) \in \mathcal{T}$, then $\Lambda(X, F_1, U) = \Lambda(X, F_0, U)$.

(Continuity axiom): If $(X, F, U) \in \mathcal{T}$, then there is a neighborhood W of $Gr(F)$ such that $\Lambda(X, F', U) = \Lambda(X, F, U)$ for all u.h.c.c.v. correspondences $F' : X \to X$ with $Gr(F') \in W$.

(Homotopy axiom): If $h : [0,1] \times X \to X$ is a homotopy with $(X, h_t, U) \in \mathcal{T}$, for all t, then $\Lambda(X, h_0, U) = \Lambda(X, h_1, U)$.

(Additivity axiom): If $(X, F, U) \in \mathcal{T}$ and $U_1, ..., U_r$ is a collection of pairwise disjoint open subsets of U such that there are no fixed points of F in $U - (\cup_{k=1}^r U_k)$ then $\Lambda(X, F, U) = \sum_{k=1}^r \Lambda(X, F, U_k)$.

(Weak Normalization axiom): For $y \in X$, let c_y be the constant correspondence $c_y(x) = \{y\}$. If $y \in U$ then $\Lambda(X, c_y, U) = 1$.

(Commutativity axiom): If $X \subset R^m$ and $Y \subset R^n$ are finite simplicial complexes, $f : X \to Y$ and $g : Y \to X$ are continuous functions, and $\Lambda(X, g \circ f, U) = \Lambda(X, f \circ g, g^{-1}(U))$.

PROOF OF LEMMA 4: Define the correspondence $\mathbf{F}(x) = x - \mathcal{F}(x)$, where $\mathcal{F}(x)$ is the correspondence defined in (7). The Lefschetz index of $\mathcal{F}$ and the degree of $\mathbf{F}$ are related by $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}, U, 0)$ (see McLennan (1989)), and, for convenience, we work in the remainder of the proof with the concept of degree.

For each point x consider the mixed linear complementarity problem, or $MLCP(0)$

$$\begin{aligned} h(\sigma) &= 0, \\ g(\lambda, x) &\geq 0, \\ \lambda \text{ free variable}, \sigma &\geq 0 \text{ and } \sigma^T g(\lambda, x) = 0, \end{aligned} \tag{21}$$

where the functions h and g were defined in (8). Let $z(x) = (\lambda(x), \sigma(x))$ be a solution of the $MLCP(0)$ (there can be multiple solutions). Note that $\mathbf{F}(x) = \{f(x, z(x)) : \ z(x) \text{ is a solution of } MLCP(0)\}$, where the function f has also been defined in (8).

Let (x^*, λ^*, μ^*) be any strongly regular *MPE* with an associated *CDS* $\mathcal{C} = (C, P)$, with $C = (C_1, ..., C_q)$ and $P = (P_1, ..., P_q)$. By lemma 3, there exists $\sigma^* \in \Sigma_C$ such that $\mu^* = \mu^*(\sigma^*)$, and (x^*, σ^*) is *MPE*. Furthermore, because all points in P_r are connected, we can choose a strategy profile σ^* satisfying $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$ for all $i \in P_r$.

Consider now the perturbed mixed linear complementarity problem, or $MLCP(\varepsilon)$

$$\begin{gathered} h(\varepsilon)(\lambda, \sigma) = h(\sigma) + \varepsilon(\lambda - \lambda^*) = 0, \\ g(\varepsilon)(x, \sigma, \lambda) = g(x, \lambda) + \varepsilon(\sigma - \sigma^*) \geq 0, \\ \lambda \text{ free variable}, \sigma \geq 0, \ \sigma^T g(\varepsilon) = 0, \end{gathered} \tag{22}$$

where $\varepsilon > 0$. The Jacobian matrix $M(\varepsilon)$ of $MLCP(\varepsilon)$ is a P-matrix (i.e., a matrix with all its principal minors positive). This is so because (see Cottle et al. (1992, pg. 154)), $M(\varepsilon) = M + \varepsilon I$, where M is the Jacobian of $MLCP(0)$, is a P_0-matrix (i.e., a matrix with all its principal minors nonnegative). Let us prove that M is a P_0-matrix: Consider the principal matrix $M_{\beta\beta}$ associated with a subset β of lines (or columns).[18] We now show that either $\det(M_{\beta\beta})$ is equal to zero or one. Note first

[18]We refer to the lines corresponding to ∂h_i and $\partial g_i(S)$ as lines λ_i and $\sigma_i(S)$, and the columns

that $\det(M_{\beta\beta}) = \prod_{i\in\pi}\det\left(M_{\beta_i\beta_i}\right)$ where $\beta = \cup_i\beta_i$ and β_i are the elements of β with entry i (either λ_i or $\sigma_i(S)$ for some $S \ni i$). But $\det\left(M_{\beta_i\beta_i}\right) = 1$ if $\beta_i = \{\lambda_i, \sigma_i(S)\}$ and is zero otherwise. Therefore, we conclude that all principal minors of M are nonnegative, and thus M is a P_0-matrix.

Given that $MLCP(\varepsilon)$ has a P-matrix then there is a unique solution $z_\varepsilon(x)$ (Cottle et al. (1992, pg. 150)) for all x: $MLCP(\varepsilon)$ can be transformed into a standard LCP eliminating the variable λ and the equation $h(\varepsilon) = 0$ (this is possible because $M_{\lambda\lambda}(\varepsilon) = \varepsilon I$ is nonsingular), and the transformed LCP also has a P-matrix (the Schur complement of $M_{\lambda\lambda}(\varepsilon)$ in $M(\varepsilon)$). Note that, in addition, we have that $z_\varepsilon(x^*) = (\lambda^*, \sigma^*)$, and that $z_\varepsilon(x)$ converge to a solution of $MLCP(0)$ when $\varepsilon \to 0$ (Cottle et al. (1992, pg. 442)), and that $z_\varepsilon(x)$ is piecewise linear in x.

We now show that, because x^* is a strong solution, there exists an $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exists an open neighborhood U_ε of x^* such that $z_\varepsilon(x)$ is smooth in U_ε. Moreover, if we let α represent the index set

$$\alpha = \{\sigma_i(S) : \text{for all } S \in \mathcal{C}_r \text{ and } i \in S\}, \tag{23}$$

then all $\sigma_i(S)$-coordinates of the solution $z_\varepsilon(x)$ that do not belong to α are zero, and $z_\varepsilon(x)$ are explicitly given by $(M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$, where $M_{\alpha\alpha}(\varepsilon)$ is

$$M_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} \varepsilon I_{\alpha\alpha} & (d_\lambda g)_\alpha \\ d_\alpha h & \varepsilon I_{\lambda\lambda} \end{bmatrix},$$

and the vector $q_\alpha(x)$ has λ_i-coordinate equal to $(\varepsilon\lambda_i^* - 1)$, and $\sigma_i(S)$-coordinate in α equal to $\varepsilon\sigma_i(S)^* + e(S)(x)$, for all $0 < \varepsilon < \overline{\varepsilon}$ and $x \in U_\varepsilon$.

In order to prove the above claim consider the function

$$\varphi(x) = \min \cup_{r=1}^{q} \{e(S)(x) - e(T)(x) : S \in C_r, T \cap P_r \neq \emptyset, \text{ and } T \notin C_r\}.$$

Naturally, the function φ is continuous in x and, because x^* is a strong solution,

corresponding to $\frac{\partial \cdot}{\partial \lambda_i}$ and $\frac{\partial \cdot}{\partial \sigma_i(S)}$ as columns λ_i and $\sigma_i(S)$. Also, we use the standard notation that $A_{\alpha\alpha}$, $A_{\cdot\alpha}$, and $A_{\alpha\cdot}$ represent the submatrix of A with, respectively, rows and columns, columns, and rows extracted from the index set α. Also, $\overline{\alpha}$ denotes the complementary set of α.

$\varphi(x^*) > 0$. Therefore, there exists an $\overline{\varepsilon} > 0$ and an open neighborhood $U \subset U_{x^*}$ of x^*, such that all $x \in U$ satisfy $\varphi(x) > 2\overline{\varepsilon}$. Now suppose that the solution $z_\varepsilon(x)$ for $x \in U$ is such that a $\sigma_i(T)$-coordinate is non-zero for $T \notin C_r$ and $i \in P_r$. Then $g_i(\varepsilon)(T) = 0$ which is equivalent to $\lambda_i + \varepsilon(\sigma_i(T) - \sigma_i^*(T)) - e(T)(x) = 0$, and implies $e(T)(x) \geq \lambda_i - \varepsilon$. Also, $g_i(\varepsilon)(S) \geq 0$ for all S, and thus $\lambda_i + \varepsilon(\sigma_i(S) - \sigma_i^*(S)) - e(S)(x) \geq 0$, which implies that $e(S)(x) \leq \lambda_i + \varepsilon$. Therefore, $e(S)(x) - e(T)(x) \leq 2\varepsilon \leq 2\overline{\varepsilon}$ for $x \in U$, in contradiction with $\varphi(x) > 2\overline{\varepsilon}$ for all $x \in U$. Now, since $z_\varepsilon(x^*) = (\lambda^*, \sigma^*)$, and $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$, and $z_\varepsilon(x)$ is continuous, then there exists an open neighborhood $U_\varepsilon \subset U_{x^*}$ of x^* where all $\sigma_i(S)$-coordinates of the solution belonging to α are non-zero. This implies that $g_i(\varepsilon)(S) = 0$ holds for all $\sigma_i(S)$ in α, and thus $z_\varepsilon(x) = (M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$.

Define the mapping $\mathbf{F}_\varepsilon(x) = f(x, z_\varepsilon(x))$ (this mapping is well-defined due to the uniqueness of $z_\varepsilon(x)$), where $\mathbf{F}_\varepsilon(x^*) = 0$. Since f is smooth and $z_\varepsilon(x) \to z(x)$ then $\mathbf{F}_\varepsilon(x) \to \mathbf{F}(x)$. Therefore, for every $\delta > 0$ there exists $\overline{\varepsilon}$ such that $\underset{x \in \overline{U}}{dist}(\mathbf{F}_\varepsilon(x), \mathbf{F}(x)) < \delta$, for all $0 < \varepsilon \leq \overline{\varepsilon}$. But since $\mathbf{F}(x)$ has no zeros in the boundary of ∂U then $\mathbf{F}_\varepsilon(x)$ also does not have any zeros in ∂U. By the homotopy and continuity property of the degree, $\deg(\mathbf{F}, U, 0) = \deg(\mathbf{F}_\varepsilon, U, 0)$, for ε close to zero.

Therefore, it only remains to show that $\deg(\mathbf{F}_\varepsilon, U, 0) = sgn(\det(dF_{\mathcal{C}}(z^*)))$ for ε close to zero, where $z^* = (x^*, \lambda^*, \mu^*)$. This result follows from $sgn(\det(d_x\mathbf{F}_\varepsilon(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*))) \neq 0$, as we will show. Indeed, this implies that $\mathbf{F}_\varepsilon$ is nonsingular at x^*, and thus there exists an open neighborhood $V \subset U$ of x^* where x^* is the only zero of $\mathbf{F}_\varepsilon$. But since the point x^* is the only zero of $\mathbf{F}(x)$ in $U \subset U_{x^*}$, and $\mathbf{F}_\varepsilon(x) \to \mathbf{F}(x)$ then there are no zeros of $\mathbf{F}_\varepsilon$ in the compact region $\overline{U} \backslash V$, for ε small enough, and thus x^* is the only zero of $\mathbf{F}_\varepsilon$ in U. A well-known property of the degree then implies that $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}_\varepsilon, U, 0) = sgn(\det(d_x\mathbf{F}_\varepsilon(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$.

We now show that $sgn(\det(d_x\mathbf{F}_\varepsilon(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$, for ε small enough. Consider $F(x, \sigma, \lambda)(\varepsilon)$,

$$F(x, \sigma, \lambda)(\varepsilon) = \begin{pmatrix} f(x, \sigma, \lambda) \\ h(\sigma) + \varepsilon(\lambda - \lambda^*) \\ g(\lambda, x) + \varepsilon(\sigma - \sigma^*) \end{pmatrix}.$$

Simple linear algebra shows that the Jacobian $d_x\mathbf{F}_\varepsilon(x^*)$ is the Schur complement of $M_{\alpha\alpha}(\varepsilon)$ in $dF_{\alpha\alpha}(\varepsilon)$ $(d_x\mathbf{F}_\varepsilon(x^*) = dF_{\alpha\alpha}(\varepsilon)/M_{\alpha\alpha})$, where

$$dF_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} (d_x f) & d_\alpha f & d_\lambda f \\ (d_x g)_\alpha & \varepsilon I_{\alpha\alpha} & (d_\lambda g)_\alpha \\ 0 & d_\alpha h & \varepsilon I_{\lambda\lambda} \end{bmatrix}, \tag{24}$$

is evaluated at point $(x^*, \lambda^*, \sigma^*)$. Therefore, $\det(d_x\mathbf{F}_\varepsilon(x^*)) = \det(dF_{\alpha\alpha}(\varepsilon))/\det(M_{\alpha\alpha})$ (see Cottle et al. (1992, pg. 75)). But since $\det(M_{\alpha\alpha}) > 0$ (M is a P-matrix) then $sgn(\det(d_x\mathbf{F}_\varepsilon(x^*))) = sgn(\det(dF_{\alpha\alpha}(\varepsilon)))$.

We claim that $sgn(\det(dF_{\alpha\alpha}(\varepsilon))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$. In order to prove the claim we use the following formula for the determinant (Cottle et al. (1992), pg. 60): for an arbitrary diagonal matrix D, $\det(A+D) = \sum_\gamma \det D_{\overline{\gamma\gamma}} \det A_{\gamma\gamma}$ where the summation ranges over all subsets γ of lines. Observe that matrix $dF_{\alpha\alpha}(\varepsilon) = A + D$, where $A = dF_{\alpha\alpha}(0)$ and D is the diagonal matrix,

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon I_{\alpha\alpha} & 0 \\ 0 & 0 & \varepsilon I_{\lambda\lambda} \end{bmatrix}.$$

Developing the expression for $\det(dF_{\alpha\alpha}(\varepsilon))$ using the formula above we get a polynomial in ε ($\det D_{\overline{\gamma\gamma}}$ is a power of ε). We are only interested in the non-zero coefficient with lowest order because, when ε converges to zero, this is the coefficient that determines the sign of $\det(dF_{\alpha\alpha}(\varepsilon))$.

The rows and columns of matrix $A = dF_{\alpha\alpha}(0)$ corresponding to $\sigma_i(S)$ and λ_i are

$$
\begin{aligned}
R(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\in S]} e(x_j) + e(\lambda_i), \\
R(\lambda_i) &= -\sum_{S\in C_r} e(\sigma_i(S)), \\
C(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\notin S]} x_j(S) e(x_j) - e(\lambda_i), \\
C(\lambda_i) &= -p_i e(x_i) + \sum_{S\in C_r} e(\sigma_i(S)),
\end{aligned}
$$

where vectors $e(x_i)$, $e(\lambda_i)$, and $e(\sigma_i(S))$ are the unit vectors at, respectively, coordinates x_i, λ_i, and $\sigma_i(S)$.

Consider $A_{\alpha\cdot}$ the submatrix of A corresponding to the rows α of A. Let β be a maximal subset of α such that $rank(A_{\beta\cdot})$ is different from zero ($|\beta| = rank(A_{\alpha\cdot})$ and $rank(A_{\beta\cdot}) = rank(A_{\alpha\cdot})$). Note that $A_{\gamma\gamma}$ where γ is the set of lines $\gamma = \beta \cup \{\lambda_i : i\in\pi\} \cup \{x_i : i \in \pi\}$ is equal to $A_{\gamma\gamma} = dF_{\beta\beta}(0)$, according to the definition (24). Also, $\det A_{\gamma'\gamma'} = 0$ for set of lines γ' that strictly contains γ because β is a maximal subset of α such that $rank(A_{\beta\cdot}) \neq 0$. We now show that $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*)) \neq 0$, which proves that the lowest-order non-zero coefficient is equal to a positive integer (the number of maximal subsets $\beta \subset \alpha$) multiplied by $\det(dF_{\mathcal{C}}(z^*))$, and thus $sgn(\det(dF_{\alpha\alpha}(\varepsilon))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$, for ε small enough.

We now propose an algorithm replaces all rows and columns $\sigma_i(S)$'s with the same S by only one row and column $\sigma_i(S)$ for all $S\in C_r$, and also replaces all rows and columns λ_i for all $i \in P_r$ by only one row and column λ_r for each $r = 1, ..., q$.

Algorithm: Start with matrix $A = dF_{\beta\beta}(0)$.

Step 1: Choose an element r, that have not yet been chosen, from the set $\{1, 2, ..., q\}$ and proceed to the next step, or else, stop if the choice is not possible.

Step 2: Choose two distinct rows $\sigma_i(S)$ and $\sigma_j(S)$ of A with $j \neq i$ and $S \in C_r$ and proceed to the next step, or else return to step 1 if the choice is not possible.

Step 3: Subtract row $\sigma_i(S)$ from row $\sigma_j(S)$ (i.e., $R(\sigma_j(S)) = R(\sigma_j(S)) - R(\sigma_i(S))$), and add column λ_j to column λ_i (i.e., $C(\lambda_i) = C(\lambda_i) + C(\lambda_j)$). The matrix that is obtained after the two operations have the same determinant as matrix A. Let this matrix be the new matrix A. After these two operations, row $\sigma_j(S)$ of A has only one non-zero entry at column λ_j, with a value equal to 1. The determinant of A can be computed by a co-factor expansion along row $\sigma_j(S)$, and $|A| = (-1)^{(\#\sigma_j(S)+\#\lambda_j)}|A'|$, where A' is the submatrix obtained after deleting row $\sigma_j(S)$ and column λ_j of matrix A.

Now, perform the following symmetric transformations on the submatrix A': Subtract column $\sigma_i(S)$ from column $\sigma_j(S)$ (i.e., $C(\sigma_j(S)) = C(\sigma_j(S)) - C(\sigma_i(S))$) and add row λ_j to row λ_i (i.e., $R(\lambda_i) = R(\lambda_i) + R(\lambda_j)$). The matrix that is obtained after the two operations have the same determinant as A'. Let this matrix be the new matrix A'. After these two operations, column $\sigma_j(S)$ of A' has only one non-zero entry at row λ_j, with a value equal to -1. The determinant of A' can be computed by a co-factor expansion along column $\sigma_j(S)$, and $|A'| = (-1) \times (-1)^{(\#\sigma_j(S)+\#\lambda_j-1)}|A''|$, where A'' is the submatrix of A' obtained after deleting column $\sigma_j(S)$ and row λ_j: observe that the column $\sigma_j(S)$ of A' is in the same location as row $\sigma_j(S)$ of A', but row λ_j appears one entry before column λ_j of A (because the row $\sigma_j(S)$ that has been removed appears before row λ_j). Putting together the expressions for the determinant yields $|A| = |A''|$. Let matrix A'' be the new matrix A, and return to step 2.

Because β is a maximal subset of α with $rank(A_{\beta\cdot}) \neq 0$ and $rank(E_C) \neq 0$, the algorithm starts with matrix $A = dF_{\beta\beta}(0)$ and ends with matrix $A = \det(dF_{\mathcal{C}}(z^*))$ (maintaining the same determinant in all steps).

Therefore, $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*))$, as we claimed. Q.E.D.

PROOF OF COROLLARY 3: Let (x, σ) be any *MPE*, and say that $\mu = \mu(\sigma)$. We only need to analyze the c.s. $\pi = \{\{1\}, \{2\}, \{3\}\}$ because we already know that two-player games have a unique equilibrium (Rubinstein (1982)).

We first show that $X(i, S) = x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, if there is a positive probability that S is chosen in equilibrium. Say that $S = \{j, k\}$ (if $S = \emptyset$ (no proposal case) then $x_i(\pi S) = x_i$ and if $S = N = \{1, 2, 3\}$ then there are no elements $i \notin S$). In order to simplify the notation, let $x_i = x_i(\pi)$, $x_i(\pi S) = x_i(jk)$, $x_S(\pi S) = x_{jk}(jk)$, and $V = v_N(\{N\})$. Suppose that S is chosen in equilibrium with positive probability.

Then $e(S)(x) \geq e(N)(x)$, which is equivalent to,

$$x_{jk}(jk) - x_j - x_k \geq V - x_i - x_j - x_k, \tag{25}$$

and

$$x_{jk}(jk) + x_i(jk) + x_i - x_i(jk) \geq V. \tag{26}$$

But since there is no delay in the formation of the grand coalition when the game is at the c.s. $\{\{jk\}, \{i\}\}$, we have that

$$x_{jk}(jk) + x_i(jk) = \delta V + (1-\delta)(v_{jk}(jk) + v_i(jk)).$$

Replacing this expression into (26) yields

$$X(i, jk) = x_i - x_i(jk) \geq (1-\delta)(V - (v_{jk}(jk) + v_i(jk))) \geq 0.$$

We now compute $\det(dF_{\mathcal{C}})$ for all admissible *CDS* $\mathcal{C} = (C, P)$, and show that $\det(dF_{\mathcal{C}}) \geq 0$. From the definition of *CDS*s it follows that $P = (P_1, ..., P_q)$ is a partition of N and $C = (C_1, ..., C_q)$ is an ordered disjoint collection of subsets $S \subset N$ satisfying: for all $S \in C_r$ then $S \cap P_r \neq \emptyset$ and $S \subset \cup_{s=1}^{r} P_s$, and also $\cup_{S \in C_r} S \supset P_r$. Moreover, theorem 2 implies that there is no $S = \{i\}$ that is chosen in equilibrium, and thus $C_r \subset \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. A list of all admissible *CDS*s (except for permutations of the players) follows with the corresponding value for $\det(dF_{\mathcal{C}})$ (i, j, and k are distinct elements of N, and $d_i = 1 - \delta \sum_S \mu(S) \mathbb{I}_{[i \in S]}$, $z(i, jk) = \delta X(i, jk)$, and $w_i = \delta p_i$):

CDS $\mathcal{C}$	$\det(dF_{\mathcal{C}})$
$(\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\})$	$z(2,13)z(1,23)z(3,12)$
$(\{\{1,2\},\{1,3\},\{1,2,3\}\})$	$z(3,12)z(2,13)(w_1+d_1)$
$(\{\{1,2\},\{1,2,3\}\})$	$z(3,12)(d_2w_1+d_1d_2+w_2d_1)$
$(\{\{1,2,3\}\})$	$d_1w_3d_2+d_1d_3d_2+d_1d_3w_2+d_2d_3w_1$
$(\{\{1,2\},\{2,3\},\{1,3\}\})$	$\sum_{i,j,k}(d_i+2w_i)\,z(k,ij)z(j,ik)$
$(\{\{1,2\},\{1,3\}\})$	$\sum_{i,j\neq 1}(d_iw_1+d_id_1+w_id_1)z(j,1i)$
$(\{\{1,2\}\},\{\{1,2,3\},\{1,3\},\{2,3\}\})$	$z(2,13)z(1,23)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,2,3\},\{1,3\}\})$	$z(2,13)(w_1+d_1)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,2,3\}\})$	$(d_2w_1+d_2d_1+w_2d_1)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,3\},\{2,3\}\})$	$\left(\sum_{i,j\neq 3}(2w_i+d_i)z(j,i3)\right)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,3\}\})$	$(d_2w_1+d_2d_1+w_2d_1)(w_3+d_3)$

Note that the first 6 entries of the table corresponds to *CDS*s with $P=(\{1,2,3\})$ and the last entries to *CDS*s with $P=(\{1,2\},\{3\})$.

The determinant for all *CDS*s are nonnegative because it is a sum of nonnegative terms. Corollary 2 implies that there is a unique global *MPE* solution. Q.E.D.

References

Albers, W. (1978), "Bloc Forming Tendencies as Characteristics of the Bargaining Behavior in Different Version of Apex Games," in *Beitrage zur experimentellen Wirtschaftsforschung, vol VIII: Coalition Forming Behavior,* ed. H. Sauermann. Tubingen: J.C.B. Mohr.

Binmore, K., Rubinstein, A., and Wolinsky, A. (1986), "The Nash Bargaining Solution in Economic Modelling," *Rand Journal of Economics*, 17, 176-188.

Bloch, F. (1996), "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division," *Games and Economic Behavior*, 14, 90-123.

Chatterjee, K., Dutta, B., Ray, D., and Sengupta, K. (1993), "A Non-cooperative Theory of Coalitional Bargaining," *Review of Economic Studies*, 60, 463-477.

Cottle, R., Pang, J., and Stone, R. (1992), *The Linear Complementarity Problem.* Academic Press London.

Davis, M., and maschler, M. (1965), "The Kernel of a Cooperative Game," *Naval Research Logistics Quarterly*, 12, 223-259.

Debreu, G. (1970), "Economies with a Finite Set of Equilibria," *Econometrica*, 38, 387-92.

Fudenberg, D., and Tirole, J. (1991), *Game Theory.* Cambridge, Massachusetts: The MIT Press.

Gomes, A. (2000), "Externalities and Renegotiations in Three-Player Coalitional Bargaining," University of Pennsylvania, *CARESS* working paper 01-07.

Gomes, A., Hart, S., and Mas-Colell, A. (1999), "Finite Horizon Bargaining and the Consistent Field," *Games and Economic Behavior*, 27, 204-228.

GUL, F. (1989), "Bargaining Foundations of Shapley Value," *Econometrica*, 57, 81-95.

GUILLEMIN, V., AND POLLACK, A. (1974), *Differential Topology.* Englewood Cliffs, New Jersey: Prentice-Hall.

HARKER, P., AND PANG, J. (1990), "Finite-Dimensional Variation Inequality and Nonlinear Complementarity Problems: A Survey of Theory," *Math. Programming,* 37, 293-308.

HARSANYI, J. (1973), "Oddness of the Number of Equilibria Points: A New Proof," *International Journal of Game Theory*, 2, 235-250.

HART, S., AND MAS-COLELL, A. (1996), "Bargaining and Value," *Econometrica*, 64, 357-380.

HIRSCH, M. (1976), *Differential Topology.* Springer-Verlag New York.

JEHIEL, P., AND MOLDOVANU, B. (1995), "Negative Externalities May Cause Delay," *Econometrica*, 63, 1321-1335.

JEHIEL, P., AND MOLDOVANU, B. (1995), "Cyclical Delay in Bargaining with Externalities," *Review of Economic Studies*, 62, 619-637.

JEHIEL, P., AND MOLDOVANU, B. (1999), "Resale Markets and the Assignment of Property Rights," *Review of Economic Studies*, 66, 971-991.

KAHAN, J., AND RAPOPORT, A. (1984), *Theories of Coalition Formation.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

MASCHLER, M. (1992), "The Bargaining Set, Kernel, and Nucleolus," in *Handbook of Game Theory*, ed. by R. Aumann and S. Hart. Elsevier Science Publishers B.V., 592-667.

MCLENNAN, A. (1989), "Fixed Points of Contractible Valued Correspondences," in *International Journal of Game Theory*, 18, 175-184.

Okada, A. (1996), "A Noncooperative Coalitional Bargaining Game with Random Proposers," *Games and Economic Behavior*, 16, 97-108.

Ray, D., and Vohra, R. (1999), "A Theory of Endogenous Coalition Structures," *Games and Economic Behavior*, 26, 268-336.

Ray, D., and Vohra, R. (2000), "Coalitional Power and Public Goods," working paper.

Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97-109.

Seidmann, D., and Winter, E. (1998), "A Theory of Gradual Coalition Formation," *Review of Economic Studies, 65, 793-815.*

Selten, R. (1981), "A Non-Cooperative Model of Characteristic Function Bargaining," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern,* ed. by V. Bohn,and Nachtkamp. Mannheim: Bibliographisches Institut Mannheim.

Selten, R., and Schuster, K. (1968), "Psychological Variables and the Coalition Formation Behavior," in *Risk and Uncertainty*, eds. by K. Borch and J. Mossin. London: Macmillan.

Shapley, L. (1953), "Quota Solutions of n-Person Games," in *Contributions to the Theory of Games II*, ed. by H.W. Kuhn and A.W. Tucker. Princeton University Press.

Schmeidler, D. (1969), "The Nucleolus of a Characteristic Function game," *SIAM Journal of Applied Mathematics*, 17, 1163-1170.

Wilson, R. (1971), "Computing Equilibria of n-person Games," *SIAM Journal of Applied Mathematics*, 21, 80-87.