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"Multiplicity, Instability and Sunspots in Games"

by

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MULTIPLICITY, INSTABILITY AND SUNSPOTS IN GAMES

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Abstract. This paper considers a class of two players games in the unit square for which a similar and high enough responsiveness of each player's strategy to the other player's strategy around a Nash equilibrium in pure strategies implies (*i*) the existence of at least two other Nash equilibria in pure strategies; (*ii*) the non local uniqueness of the strategies of this Nash equilibrium in the sets of rationalizable strategies; and (*iii*) the existence of nontrivial correlated equilibria arbitrarily close to this Nash equilibrium. Although a similar result can be shown to follow from Milgrom and Roberts' (1990) results for supermodular games, the games considered here are not necessarily supermodular, which makes clear that supermodularity is not necessary to obtain it. The simultaneous emergence of phenomena of multiplicity, instability and vulnerability to sunspots studied in this paper parallels similar patterns observed in other frameworks (e.g. overlapping generations economies and finite economies with asymmetric information), and thus hints at the existence of an underlying relation between different avatars of the indeterminacy of the outcome of economies and games that goes beyond the boundaries of any specific framework and may be common to every decision-making problem faced of simultaneous, independent and interrelated optimizers.

1. Introduction

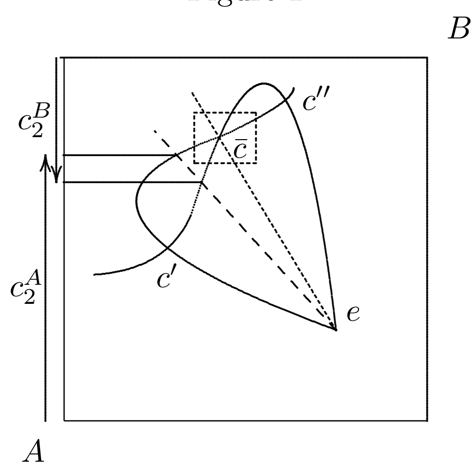
The simultaneous and independent decision-making of several agents may lead to a multiplicity of different outcomes. This is the case in, for instance, a competitive exchange economy with two agents with standard preferences[1] and two commodities

I want to thank David Cass, Jan Eeckhout, Armando Gomes, George Mailath, José Víctor Ríos and Antonio Villanacci for helpful discussions on the results presented in this paper. I also thank Roger Guesnerie for detailed comments on a previous version of this paper that helped to improve it and for useful discussions on this research.

[1]That is to say, representable by a continuous utility function u defined in $\mathbb{R}^2_+$, differentiable in $\mathbb{R}^2_{++}$, and such that, for all $c \in \mathbb{R}^2_{++}$, $Du(c) \in \mathbb{R}^2_{++}$ and $D^2u(c)$ is negative definite in the space orthogonal to $Du(c)$, and moreover is "well-behaved" in the boundary, in the sense that for all $i = 1, 2$, $Du(c) \cdot e_i \to 0$ as $c \to \bar{c}$, where $\bar{c}$ is any point of the axis of the i-th good distinct from the origin and e_i is the i-th vector of the canonical basis.

if it has a Walrasian equilibrium allocation with the adequate crossing of the two agents' offer curves (see Figure 1).[2] Under these circumstances, firstly, at least two other Walrasian equilibria do exist. Moreover, and secondly, the Walrasian equilibrium with the adequate crossing of the offer curves is unstable under the tâtonnement process; and finally, and thirdly, if an additional condition on the slopes of the offer curves at the crossing is satisfied,[3] then there exist arbitrarily close to this equilibrium other equilibria in which the agents make depend their choices on the realization of a sunspot on which they have asymmetric information[4] (in Figure 1, where agents A and B exchange commodities 1 and 2 in abscissae and ordinates respectively, it can be seen (*i*) a multiplicity of competitive equilibria $\bar{c}$, c', and c''; (*ii*) the instability of the tâtonnement around the equilibrium $\bar{c}$; and (*iii*) the support of a correlated equilibrium –constituted by the four corners of the small box in dashes– in which agent A (resp. B) supplies to the market two possible amounts of good 1 (resp. 2) according to the uncertain realization of a private sunspot correlated with the other agent's equally private sunspot). Thus multiplicity (of Walrasian equilibria), instability (of the the tâtonnement around a Walrasian equilibrium) and vulnerability to sunspots (of nearby equilibria) come hand in hand in this framework.

Figure 1



Also in the simplest overlapping generations economy,[5] whenever the agents

[2]Specifically, the determinant of the matrix formed by the gradients of the consumers' offer curves equations at the equilibrium allocation $\bar{c}$, with that of the consumer whose excess demand for good 1 is positive in the first row, must be negative, i.e.

$$\begin{vmatrix} Du^A(\bar{c}^A)^t + (\bar{c}^A - e^A)^t D^2 u^A(\bar{c}^A) \\ Du^B(\bar{c}^B)^t + (\bar{c}^B - e^B)^t D^2 u^B(\bar{c}^B) \end{vmatrix} < 0$$

if A is the consumer such that $e_1^A < \bar{c}_1^A$.

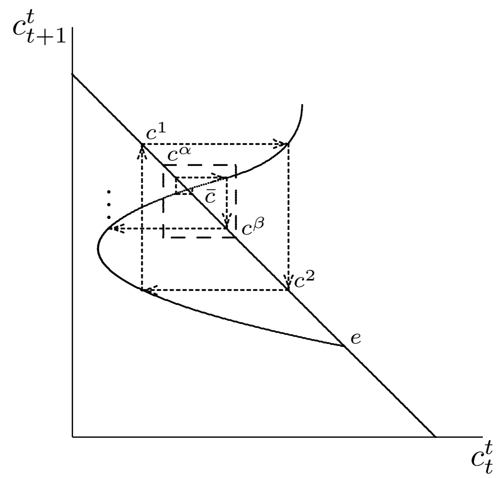
[3]Namely, the positive cone spanned by the gradient of agent A's offer curve at the Walrasian equilibrium allocation $\bar{c}$, $Du^A(\bar{c}^A)^t + (\bar{c}^A - e^A)^t D^2 u^A(\bar{c}^A)$, and its symmetric image across the axis of abscissae contains the opposite of the gradient of agent B's offer curve at the same point, $-(Du^B(\bar{c}^B)^t + (\bar{c}^B - e^B)^t D^2 u^B(\bar{c}^B))$.

[4]By a sunspot is meant a signal conveying no information about the fundamentals of the economy or, equivalently, a source of extrinsic uncertainty, i.e. uncertainty about a state of the world with respect to which the fundamentals of the economy are constant. See Maskin and Tirole (1987) for the existence of such an equilibrium in this setup. On this issue see also Dávila (1999). For the seminal papers on the concept of sunspot and sunspot equilibrium, see Shell (1977) and Cass and Shell (1982).

[5]A never-ending sequence of identical overlapping generations living for two periods with stan-

want to trade in consumption today for consumption tomorrow,[6] the steady state is, firstly, not the only non-autarkic stationary perfect foresight equilibrium if the slope of the offer curve of the representative agent at the steady state is smaller than 1 in absolute value (in effect, at least another stationary perfect foresight equilibrium exists in that case: a cycle of period 2).[7] Moreover, a steady state satisfying this condition is, secondly, indeterminate (i.e. unstable in the backward perfect foresight dynamics); and thirdly, there exist local sunspot equilibria around it[8] (see Figure 2, where it can be seen (*i*) a cycle of period 2 in which the global resources $e_1 + e_2$ of the economy are split between young and old agents at either c^1 or c^2 every other period; (*ii*) the instability of the steady state in the backward perfect foresight dynamics; and (*iii*) the support of a sunspot equilibrium in which the global resources are split at either c^α or c^β randomly according to a first order Markov chain).

Figure 2



Again multiplicity (of stationary equilibria), instability (of the perfect foresight dynamics around the steady state) and vulnerability to sunspots (of nearby equilibria) come hand in hand in this framework as well.

Another instance of this connection between the issues of uniqueness versus multiplicity, stability versus instability, and vulnerability to sunspots versus sunspot-proofness is provided in this paper. In the case presented here, two players play a game in which a similar and high enough responsiveness of each player's best reply to the other player's strategy around a Nash equilibrium in pure strategies

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dard preferences over the only existing good per period. For the seminal paper, see Samuelson (1956).

[6]The so-called Samuelsonian case in Gale (1973).

[7]See Azariadis and Guesnerie (1986). The case in which every agent wants to consume more today than what he is endowed with, borrowing against his future endowment (the so-called classical case in Gale (1973)), leads to a different but symmetric implication: multiplicity is guaranteed if the slope of the offer curve at the steady state is *bigger* than 1 in absolute value.

[8]See Guesnerie (1986), Azariadis and Guesnerie (1986) and Woodford (1986). For the existence of local sunspot equilibria in the classical case the condition to obtain local sunspot equilibria is still that the slope of the offer curve at the steady state is in absolute value smaller than 1, driving a wedge in that case between multiplicity on the one hand (which requires this slope to be bigger than 1 in absolute value), and indeterminacy and vulnerability to sunspots locally around the steady state on the other hand. As a matter of fact, in the classical case there are always sunspot equilibria, although maybe not arbitrarily close to the steady state (see Dávila (1994)).

(in a way such that a relatively low responsiveness on the part of one player can be offset by a high responsiveness on the part of the other, and necessarily with both players' choices moving in the same direction) implies (*i*) the existence of at least two other Nash equilibria (Theorem 1); (*ii*) the existence of (non-trivial) local correlated equilibria around the Nash equilibrium (Theorem 2); and (*iii*) the instability of the "virtual" eductive dynamics of beliefs that would make the players converge on it as the only (at least locally) rationalizable outcome (Theorem 3).

All the three previous examples hint at the existence of an underlying relation between the different manifestations of the problem of indeterminacy that plagues economies and games alike, and that seems to go well beyond the borders of any specific framework in which it has been exhibited until now. This paper intends to be another contribution at unveiling this relation.

Two issues are closely related to the results presented in this paper, and I now proceed to comment briefly on them. On the one hand, Milgrom and Roberts (1990) established that in supermodular games there is a largest and a smallest profiles of serially undominated strategies that, moreover, are Nash equilibria in pure strategies. Since the set of profiles of serially undominated strategies contains the supports of the Nash equilibria in pure strategies, the correlated equilibria (either subjective or objective), and the profiles of rationalizable strategies, then the uniqueness of a pure Nash equilibrium would imply the coincidence of the largest and the smallest profiles of serially undominated strategies and, hence, the non-existence of nontrivial correlated equilibria, as well as the local uniqueness of the Nash equilibrium as profile of rationalizable strategies. Therefore, Milgrom and Roberts' result points implicitly also to the fact that multiplicity, vulnerability to sunspots and lack of convergence of the iterative elimination of dominated strategies must come hand in hand in some games, namely in supermodular games. Notwithstanding, the games that I consider in this paper are not necessarily supermodular (more specifically, the increasing differences hypothesis will not necessarily be satisfied), and in this sense Milgrom and Roberts' and this paper can be considered as being, so to speak, linearly independent. On the negative side, while Milgrom and Roberts' result holds for strategic sets that are intervals in any finite dimensional real vector space and, more generally, complete lattices, mine are intervals in the real line. Still the results presented in this paper are enough to clarify that the connection at hand between the different forms of indeterminacy has nothing to do with supermodularity.

On the other hand, in the application of Guesnerie's investigation on the eductive justification of the rational expectations equilibrium of Guesnerie (1992) to simple overlapping generations economies in Guesnerie (1993), he finds an equivalence between the determinacy of the steady state, the non-existence of local sunspot equilibria around it, and the steady state being strongly rational (i.e. a locally unique rationalizable equilibrium).[9] This paper, can be seen as an extension of that result to a simple class of games.

The structure of the remainder paper is as follows. Section 2 presents the class of games considered. Section 3 defines the Nash equilibria in pure strategies of these games, establishes the existence of at least one and provides a sufficient condition for the existence of multiple Nash equilibria in pure strategies (Theorem 1). Sec-

[9] As well as equivalent to the convergence of any "reasonable" learning process. See also Guesnerie (1999).

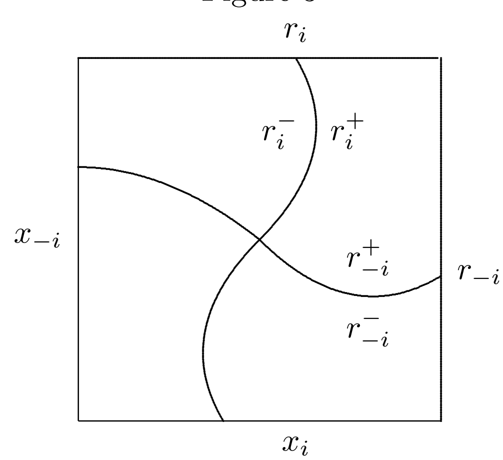
tion 4 defines a type of simple nontrivial correlated equilibria of these games (they have a finite support of profiles of strategies) and shows that the same sufficient condition on a Nash equilibrium in pure strategies that guarantees the existence of a multiplicity of other Nash equilibria, guarantees also the existence of nontrivial correlated equilibria in every neighborhood of the former (Theorem 2). Section 5 establishes that the same sufficient condition on a Nash equilibrium in pure strategies that guarantees the existence of other Nash equilibria and of nontrivial correlated equilibria around it, suffices to establish the lack of convergence of the iterated elimination of dominated strategies to a single profile, i.e. the non local uniqueness of the Nash equilibrium as a profile of rationalizable strategies (Theorem 3). Section 6 presents other results relating the three previous issues from a global viewpoint, as opposed to the local nature of the previous theorems. Section 7 concludes and the Appendix collects proofs and lemmas.

2. The game

Consider a class of games $\Gamma = \{I, \{X_i, f_i\}_{i \in I}\}$, whose set of players is $I = \{-1, 1\}$, and each player's $i \in I$ strategy set X_i and payoff function f_i are, respectively, the interval $[0, 1]$ of the real line without loss of generality, and a C^2 function $f_i\colon X_i \times X_{-i} \to \mathbb{R}$ that is strictly unimodal in its first argument, for every value of the second argument.[10]

For every player $i \in I$, let $r_i(x_{-i})$ denote the set of best replies of i to $-i$'s strategy x_{-i}, i.e. $\{x_i \in X_i | \forall x_i' \in X_i, f_i(x_i', x_{-i}) \leq f_i(x_i, x_{-i})\}$. Under the assumption made on the unimodality of the payoff functions, the set $r_i(x_{-i})$ is always a singleton whose only element will be denoted by $r_i(x_{-i})$ as well, abusing the notation slightly, and is in the interval $(0, 1)$. Hence $r_i = \{(x_{-i}, x_i) \in X_{-i} \times X_i | x_i = r_i(x_{-i})\}$ is a function that takes values in $(0, 1)$ and, moreover, is continuous because of the theorem of the maximum. We will use r_i^+ and r_i^- to denote the epigraph and hypograph, respectively, of r_i, i.e. $r_i^+ = \{(x_{-i}, x_i) \in X_{-i} \times X_i | x_i \geq r_i(x_{-i})\}$ and $r_i^- = \{(x_{-i}, x_i) \in X_{-i} \times X_i | x_i \leq r_i(x_{-i})\}$ (see Figure 3).

Figure 3



[10] More precisely, for each player $i \in I$ and all $x_{-i} \in [0, 1]$, there exists an $\bar{x}_i \in (0, 1)$ such that $f_i(\cdot, x_{-i})$ is strictly increasing in the interval $(0, \bar{x}_i)$ and strictly decreasing in the interval $(\bar{x}_i, 1)$.

3. Multiplicity of Nash equilibria

A *Nash equilibrium* in pure strategies of the game Γ is a $\hat{x} \in \times_{i\in I} X_i$ such that, for all $i \in I$, and all $x_i \in X_i$,

$$f_i(x_i, \hat{x}_{-i}) \leq f_i(\hat{x}_i, \hat{x}_{-i}). \tag{1}$$

Since, for all $i \in I$, X_i is a nonempty, compact, convex set in the real line, and f_i is continuous and quasi-concave in x_i, Nash's theorem guarantees the existence of a Nash equilibrium of this game.[11] Two examples of a game fitting in this framework with multiple Nash equilibria are characterized by the best reply functions $r_i(x_{-i}) = \frac{1}{2} + \frac{1}{\pi}\sin\big(2\pi(x_{-i} - \frac{1}{2})\big)$, for all $i \in I$, as well as $r_i(x_{-i}) = 5(x - \frac{1}{2})(x^2 - x) + \frac{1}{2}$, for all $i \in I$. In particular, the last example has three Nash equilibria in pure strategies: namely $(\frac{1}{2} - \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10})$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2} + \frac{\sqrt{5}}{10}, \frac{1}{2} - \frac{\sqrt{5}}{10})$.

The next proposition establishes a sufficient condition for the existence of multiple Nash equilibria in pure strategies. Namely, if the responsiveness of the players' best replies to the other player's strategy is similar and high enough around a Nash equilibrium in pure strategies,[12] then there exist at least two other Nash equilibria in pure strategies.

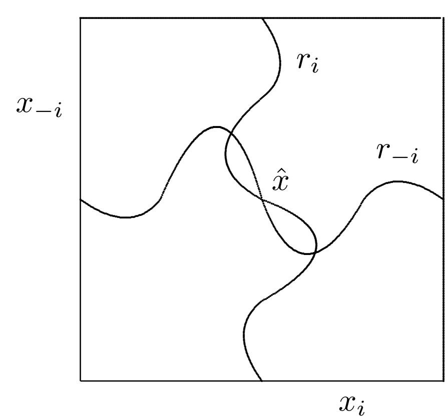
Theorem 1. *If $\hat{x} \in \times_{i\in I} X_i$ is a Nash equilibrium in pure strategies of Γ and*

$$\prod_{i\in I} r_i'(\hat{x}_{-i}) > 1, \tag{2}$$

then there exist at least two other Nash equilibria in pure strategies.

In effect, intuitively in order to have the right crossing of the best reply curves for a multiplicity of crossings to appear, the slope of r_i, for any given i, at the Nash equilibrium in pure strategies $\hat{x}$ must be bigger in absolute value to the reciprocal of the slope of r_{-i} at the same point, and both must have the same sign (see Figure 4).

Figure 4



[11] Notwithstanding, this result is proved in Proposition 1 in the appendix by means of an argument that requires little more than the intermediate value theorem and, more interestingly, will reappear in the proof of Proposition 5 in the appendix, which in turn will be needed in Section 6. Hence we state it explicitly there. More specifically, a Nash equilibrium of this game is shown in Proposition 1 to be associated to a zero of an adequately defined real-valued function on the real line. Note incidentally that the argument does not imply the uniqueness of the Nash equilibrium in pure strategies, since the function may have several zeros.

[12] Actually, it is the joint responsiveness of both players' best replies that matters, in such a way that if one player's best reply is not responsive enough, then the other's must be responsive enough to compensate for it

This intuitive proposition is a straightforward consequence of an also intuitive lemma that can be found along with its proof and that of Theorem 1 in the appendix (see Lemma 1). As a matter of fact, a much more simple proof of Theorem 1 can be produced resorting to more sophisticated mathematics, i.e. by means of the Poincaré-Hopf index theorem. In effect, the Nash equilibria in pure strategies of Γ happen to be zeros of an outward-pointing vector field $F(x_i, x_{-i}) = (x_i - r_i(x_{-i}), x_{-i} - r_{-i}(x_i))$ defined on the unit square by the equilibrium equations. The sufficient condition for the existence of another Nash equilibrium stated in Theorem 1, is nothing else than the condition on the index of the Nash equilibrium $\hat{x}$ to be distinct from the Euler characteristic of the unit square. Since the sum of the indices of all the zeros of the vector field must coincide with the Euler characteristic, hence the necessary existence of at least two other zeros of the vector field. i.e. two other Nash equilibria in pure strategies.

4. Non-trivial Correlated Equilibria around a Nash Equilibrium

I consider next a particularly simple class of correlated equilibria of this game, namely those with a finite support.

A *finite support correlated equilibrium* $\{p_i, S_i, x_i^*\}_{i\in I}$ of the game Γ consists of (i) a joint probability distribution over two privately observable random signals s_i (one for each player $i \in I$, taking values in $S_i = \{1, 2, \ldots, |S_i|\}$ with $|S_i| \geq 2$, and denoted p_i whenever defined over $S_i \times S_{-i}$),[13] and (ii) two increasing[14] functions $x_i^* \in X_i^{S_i}$,[15] for all $i \in I$, such that, for all $i \in I$ and all $\xi_i \in X_i^{S_i}$,

$$\sum_{s_i=1}^{|S_i|} \sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) f_i(\xi_i(s_i), x_{-i}^*(s_{-i})) \leq \sum_{s_i=1}^{|S_i|} \sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) f_i(x_i^*(s_i), x_{-i}^*(s_{-i})).^{16} \tag{3}$$

[13] That is to say, for each $i \in I$, $p_i \colon S_i \times S_{-i} \to [0,1]$ is a function such that $p_i(s_i, s_{-i}) = p_{-i}(s_{-i}, s_i)$ holds for all $i \in I$ and all $s_i \in S_i$, $s_{-i} \in S_{-i}$, in such a way that they denote actually the same joint distribution. Therefore these are *objective* correlated equilibria.

[14] That is to say, such that if $s_i < s_i'$, then $x_i^*(s_i) < x_i^*(s_i')$. Note that, therefore, by correlated equilibria I actually mean only the non-trivial ones, i.e. those for which the players truly randomize their choices.

[15] Throughout the paper any function $x_i \in X_i^{S_i}$ is, as usual, trivially supposed to have the entire S_i as domain, i.e. $x_i^{-1}(X_i) = S_i$.

[16] Note that whenever, for all $i \in I$, $|S_i| = 2$, then the support $\{x_i^*\}_{i\in I}$ of a correlated equilibrium $\{p_i, S_i, x_i^*\}_{i\in I}$ must be such that,

$$\begin{pmatrix} D_1^{11} & D_1^{12} & 0 & 0 \\ 0 & 0 & D_1^{21} & D_1^{22} \\ D_{-1}^{11} & 0 & D_{-1}^{12} & 0 \\ 0 & D_{-1}^{21} & 0 & D_{-1}^{22} \end{pmatrix}$$

with $D_i^{s_i s_{-i}} = D_1 f_i(x_i^*(s_i), x_{-i}^*(s_{-i}))$, for all $i \in I$, $s_i \in S_i$, and $s_{-i} \in S_{-i}$, is singular; otherwise the system of equations formed by the necessary and sufficient first order condition for the payoff maximization in mathematical expectation for each player and each realization of the uncertainty

The next proposition provides a sufficient condition on the relative slopes of the players' best reply functions at a Nash equilibrium in pure strategies for the existence of finite support non trivial correlated equilibria arbitrarily close to the Nash equilibrium strategy profile. I refer to them as *local* correlated equilibria, following the analogy with the local sunspot equilibria in the literature on overlapping generations economies. The proof of Theorem 2 can be found in the appendix.

Theorem 2. *If $\hat{x} \in \times_{i \in I} X_i$ is a Nash equilibrium in pure strategies of Γ and*

$$\prod_{i \in I} r_i'(\hat{x}_{-i}) > 1, \tag{4}$$

then there exist nontrivial correlated equilibria of Γ with support arbitrarily close to $\hat{x}$.

Thus, according to Theorems 1 and 2, the same condition guarantees the existence of local correlated equilibria around a Nash equilibrium in pure strategies and the existence of at least two other Nash equilibria.

5. Nash Equilibria with non locally unique rationalizable profiles of strategies

As in the overlapping generations framework and the finite economy with asymmetric information, an additional stability test can be carried out on an equilibrium to check whether a spontaneous coordination on it can be claimed on the grounds of such stability. The stability criterion used here to single out a Nash equilibrium in pure strategies is that of its local uniqueness as a profile of rationalizable strategies, i.e. as the result of iterative elimination of strategies that are not going to be best responses to any strategy once the common knowledge of this fact is taken into account.

More specifically, given the best reply functions r_i, for all $i \in I$, x_i is a rationalizable strategy for player i if, and only if, for all $n \in \mathbb{N}$,

$$x_i \in (r_i \circ r_{-i})^n \circ r_i(X_{-i}). \tag{5}$$

Accordingly, let R_i be the set $\cap_{n \in \mathbb{N}} (r_i \circ r_{-i})^n \circ r_i(X_{-i})$ of rationalizable strategies

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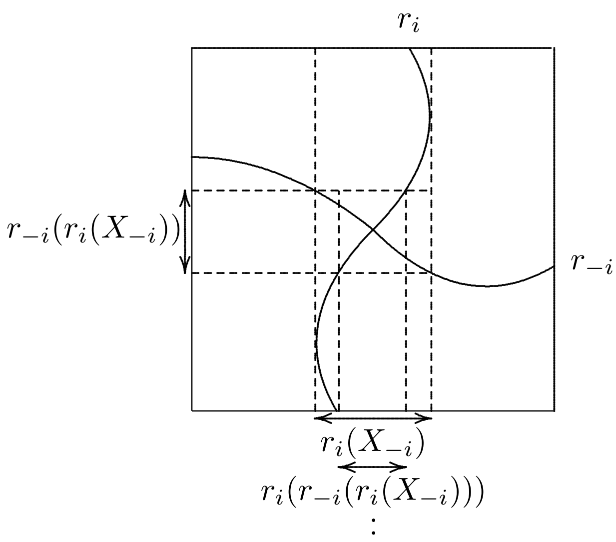
would be determinate and, hence, that system of equations along with the equation

$$p_i(1,1) + p_i(1,2) + p_i(2,1) + p_i(2,2) = 1$$

with the probabilities $p_i(s_i, s_{-i})$ as unknowns would be overdeterminate and, thus, would have no solution, contradicting the fact that $\{x_i^*\}_{i \in I}$ was the support of a correlated equilibrium. As long as $|S_i| > 2$ for some $i \in I$, the system has non-negative degrees of freedom $|S_i||S_{-i}| - (|S_i| + |S_{-i}| + 1)$, which are strictly positive indeed but for the case $(|S_i|, |S_{-i}|) = (2, 3)$ for any $i \in I$.

of player i (see Figure 5 for a few iterations).

Figure 5



Not surprisingly, every Nash equilibrium is a profile of rationalizable strategies.[16] The nonemptiness of the set of rationalizable strategies then follows immediately, while its convexity follows from the intermediate value theorem.[17] Thus from the non-emptiness and convexity of the sets of rationalizable strategies one can conclude the following sufficient condition for a Nash equilibrium being a profile of non isolated rationalizable strategies. Interestingly enough, this condition happens to be the same one as the one that guaranteed the existence of local correlated equilibria around the Nash equilibrium and the existence of other Nash equilibria in pure strategies as well. Its proof is provided in the appendix.

Theorem 3. *If $\hat{x} \in \times_{i \in I} X_i$ is a Nash equilibrium in pure strategies of Γ and*

$$\prod_{i \in I} r_i'(\hat{x}_{-i}) > 1, \tag{6}$$

then $\hat{x}$ is not an isolated point of the set of profiles of rationalizable strategies.

It may be worth to mention that although in general, if a game Γ has multiple Nash equilibria, then no profile of strategies is an isolated point of the set of profiles of rationalizable strategies (that is to say, the iterative elimination of dominated strategies will not converge to a single profile), nonetheless there still may be a Nash equilibrium for which the iterative elimination of dominated strategies, when constrained to start in a small enough neighborhood of it, does succeed to converge to it. In the context of his investigation about an eductive justification of some rational expectations equilibria as sensible outcomes of an economy, Guesnerie (1992) names such equilibria *locally strongly rational*. In effect, if the game has a finite number of multiple Nash equilibria in pure strategies, then according to the Poincaré-Hopf theorem there must be an odd number of them for which $\prod_{i \in I} r_i'(\hat{x}_{-i}) < 1$. If moreover for some of them $0 < \prod_{i \in I} r_i'(\hat{x}_{-i})$, then such equilibria are locally strongly rational in the sense defined above, as the next theorem shows, whose proof can be found in the appendix.

[16] A proof of this fact in this setup can be found in the appendix (see Proposition 2).

[17] See Proposition 3 in the appendix.

Theorem 4. *If $\hat{x}$ is a Nash equilibrium of Γ such that*

$$0 < \prod_{i \in I} r_i'(\hat{x}_{-i}) < 1, \tag{7}$$

then $\hat{x}$ is locally strongly rational.

6. Global results

The links connecting correlated and Nash equilibria in pure strategies are not constrained to be characterized by local conditions only. As a matter of fact, the mere existence of a multiplicity of Nash equilibria in pure strategy has immediate consequence on the existence of correlated equilibria and the the of rationalizable profiles of strategies, as the next theorems establish.

Theorem 5. *If there exist multiple Nash equilibria in pure strategies of the game Γ, then there exist non trivial correlated equilibria also.*

Similarly, multiplicity of Nash equilibria in pure strategies implies almost trivially that the set of rationalizable profiles is an interval of $\mathbb{R}^2$ with non empty interior and, hence, the lack of convergence of the iterated elimination of dominated strategies, as the next theorem states.

Theorem 6. *If there exist multiple Nash equilibria in pure strategies of the game Γ, then the set of rationalizable profiles of strategies has no isolated point.*

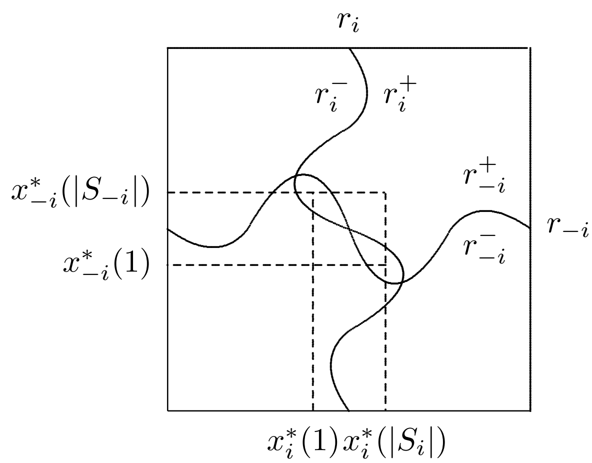
The converse statements are far from being true. For instance, if for all $i \in I$, $r_i(x_{-i}) = \frac{1}{2} + \frac{1}{4} i \sin 2\pi(x_{-i} - \frac{1}{2})$, then the only Nash equilibrium in pure strategies of this game is $x_i = \frac{1}{2}$ for all $i \in I$, while every strategy $x_i \in [\frac{1}{4}, \frac{3}{4}]$ is rationalizable, for all $i \in I$. Also, if for all $i \in I$, $r_i(x_{-i}) = \frac{1}{2} + \frac{1}{2} i (x_{-i} - \frac{1}{2})^{\frac{1}{3}}$, then the only Nash equilibrium in pure strategies of this game is $x_i = \frac{1}{2}$ for all $i \in I$, while the game can be shown to have correlated equilibria (by the same argument exposed in the proof of Theorem 2).

Nevertheless, a weaker version of the converse statement linking correlated equilibria to multiple Nash equilibria can be obtained on the grounds of global properties of the best reply functions that are a consequence of the two following properties of the supports of both correlated and Nash equilibria in pure strategies. On the one hand, at a correlated equilibrium no player will play with positive probability a strategy that may lead only to outcomes, so to speak, laying on the same side of his or her best reply function.[18] A consequence of this is that the best reply curve of every agent has to enter and exit the convex hull of the support of any correlated equilibrium at points where he plays either the minimum or the maximum of his

[18] The rationale for this is quite intuitive: should a player play with some probability a pure strategy such that, no matter what the other player does, the outcome is for sure in the, say, hypograph of the first player's best reply, then his payoff from every outcome would be bigger for a slightly higher strategy, and hence his expected payoff as well. See Proposition 4 in the appendix.

strategies played with positive probability (see Figure 6).

Figure 6



On the other hand, it can be shown that there exists a Nash equilibrium in pure strategies within the convex hull of the support of any non trivial correlated equilibrium with finite support.[19]

The two previous properties allow to establish the existence of multiple Nash equilibria whenever there exists a correlated equilibrium (not necessarily local to any Nash equilibrium in pure strategies) at the cost of giving conditions guaranteeing the existence of a Nash equilibrium not contained in the convex hull of the support of the correlated equilibrium. This is precisely what the next theorem does.

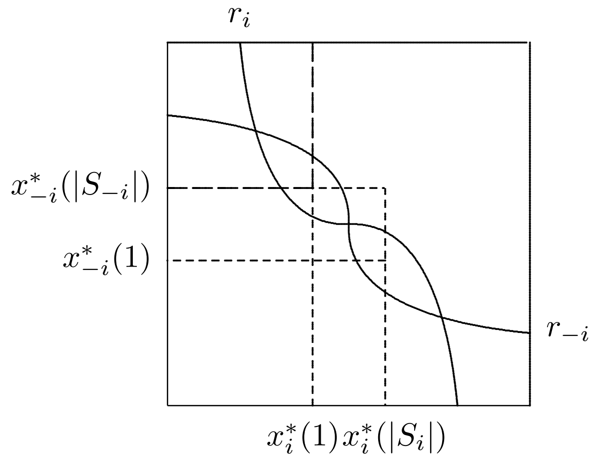
Theorem 7. *If there exists a non trivial correlated equilibrium of Γ and the best reply functions r_i are both non-decreasing or both non-increasing, then there exist multiple Nash equilibria in pure strategies.*

In effect, if $x_i^* \in X_i^{S_i}$ is player i's strategy in the correlated equilibrium, and the best replies are, say, both non-increasing (see Figure 7), then $r_i(x^*_{-i}(|S_{-i}|)) < x_i^*(1)$ and $x^*_{-i}(|S_i|) < r_{-i}(x_i^*(1))$ necessarily . Therefore, since both best replies are non-increasing, the restriction of the continuous vector field of equilibrium equations $F(x_i, x_{-i}) = (x_i - r_i(x_{-i}), x_{-i} - r_{-i}(x_i))$ to the subset $[0, x_i^*(1)] \times [x^*_{-i}(|S_{-i}|), 1]$ is still outward-pointing and hence must have at least one zero according to Poincaré-Hopf theorem, i.e. a Nash equilibrium in pure strategies of the game Γ. Since, according to Proposition 5 in the appendix, there is at least another Nash equilibrium in pure strategies in the convex hull of the support of the correlated equilibrium, then there is a multiplicity of these equilibria (a similar argument can be developed for the case of non-decreasing best replies).[20]

[19] See Proposition 5 in the appendix.

[20] In the appendix can be found an alternative proof that does not resort to the use of the Poincaré-Hopf theorem. Moreover, Proposition 6 gives a version of this result that does not require the monotonicity of the best reply functions.

Figure 7



Finally, as for the global relations between correlated equilibria and rationalizability in this setup, the existence of non-trivial correlated equilibria implies trivially that the set of profiles of rationalizable strategies is an interval of $\mathbb{R}^2$ with nonempty interior,[21] while the converse does not hold in general, as the following example shows. Let, for all $i \in I$,

$$r_{-i}(x_i) = \begin{cases} x_i + \frac{1}{2} & x_i < \frac{1}{4} \\ -x_i + 1 & \frac{1}{4} \leq x_i < \frac{3}{4} \\ x_i - \frac{1}{2} & \frac{3}{4} \leq x_i. \end{cases} \tag{8}$$

The set of rationalizable strategies of player i, for all $i \in I$, is in this game $[\frac{1}{4}, \frac{3}{4}]$, while there is no nontrivial correlated equilibrium.[22] Nonetheless, the very knife-edge nature of this example conveys the intuition that a generic converse may very likely still hold.

7. Conclusion

The previous sections have exhibited, in a specific class of games, a close connection between the issues of the existence of a multiplicity of Nash equilibria in pure strategies, the existence of local correlated equilibria around a Nash equilibrium, and the lack of local uniqueness of a Nash equilibrium as a profile of rationalizable strategies, i.e. the instability of the process of iterative elimination of dominated strategies around it. This connection parallels similar ones between the notions of multiplicity, instability, and vulnerability to sunspots in other seemingly completely unrelated frameworks as, for instance, the overlapping generations economies. Such a pervasive link between different forms of indeterminacy across economies and games, hints at a general phenomenon that may be common to every setup consisting of several optimizers that must, simultaneously and independently, make a decision as in the economies and games do, independently of whether they behave strategically or not.

[21] Since the set of profiles of rationalizable strategies contains the support of every correlated equilibrium and, moreover, the set of rationalizable strategies of each player is convex (see Proposition 3 in the appendix).

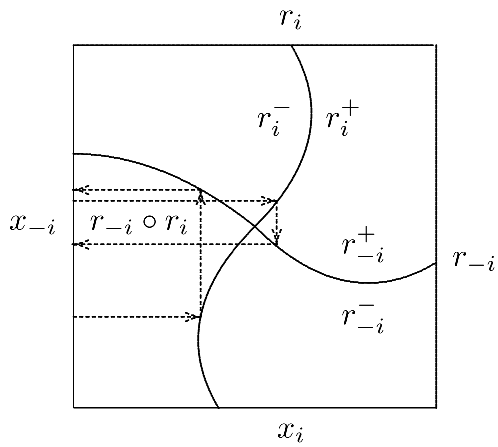
[22] There is no candidate to support that satisfies the necessary condition for it to be that of a correlated equilibrium established in Proposition 5 in the appendix.

Appendix

Proposition 1. *There exists at least one Nash equilibrium in pure strategies of the game* Γ.

Proof. For any $i \in I$, let ϕ_i be the function from $[0,1]$ to itself mapping each $x_{-i} \in X_{-i}$ to $\phi_i(x_{-i}) = x_{-i} - r_{-i}(r_i(x_{-i}))$. The function ϕ_i is continuous and takes positive and negative values, specifically $\phi_i(0) = -r_{-i}(r_i(0)) \in (-1, 0)$ while $\phi_i(1) = 1 - r_{-i}(r_i(1)) \in (0,1)$ (see figure 1A).

Figure 1A



Therefore, necessarily there exists an $\hat{x}_{-i} \in X_{-i}$ such that $\phi_i(\hat{x}_{-i}) = 0$, i.e. such that $\hat{x}_{-i} = r_{-i}(r_i(\hat{x}_{-i}))$ or, equivalently, $(r_i(\hat{x}_{-i}), \hat{x}_{-i}) \in r_{-i}$. Letting $\hat{x}_i$ be $r_i(\hat{x}_{-i})$, then $(\hat{x}_{-i}, \hat{x}_i) \in r_i \cap r_{-i}^{-1}$. Q.E.D.

Lemma 1. *Any two continuous curves in an open ball of the real plane whose endpoints alternate at the boundary of the ball, cross within it.*

Proof. Without loss of generality let the ball be $B_1(0)$ and the curves f and g be the ranges of two continuous functions from $[0,1]$ to the closure of $B_1(0)$ such that, in polar coordinates, for all $t \in (0,1)$, $\rho_f(t), \rho_g(t) \in (0,1)$, and

$$\rho_f(0) = \rho_f(1) = \rho_g(0) = \rho_g(1) = 1 \tag{9}$$

$$\theta_f(0) < \theta_g(1) < \theta_f(1) < \theta_g(0). \tag{10}$$

Moreover, without loss of generality as well, let g be such that $\min_{[0,1]} \rho_g \leq \min_{[0,1]} \rho_f$.

Then there exists a continuous mapping ϕ from $[0,1]$ to itself such that, for all $t \in [0,1]$, $\rho_g(t) \leq \rho_f(\phi(t))$ (e.g. $\phi(t) = \frac{1}{2}(t - \min_{[0,1]} \rho_f)^{\frac{1}{2n+1}} + \frac{1}{2}$, for $n \in \mathbb{N}$ big enough). Let $\tilde{f} = (\rho_f \circ \phi, \theta_f \circ \phi)$ and note that f is the range of $\tilde{f}$ as well. Also, for $\alpha > 1$ big enough, a zero of the equation

$$\rho_g(t^{\frac{1}{\alpha}}) = \rho_{\tilde{f}}(t) \tag{11}$$

bifurcates in two branches that converge to the zeros 0 and 1 each. On the other hand, $\min \theta_{\tilde{f}}^{-1}(\theta_g(1)) \in (0,1)$ is an asymptote to a zero of

$$\theta_g(t^{\frac{1}{\alpha}}) = \theta_{\tilde{f}}(t) \tag{12}$$

as α goes to infinity. Therefore, there exists some $\alpha > 1$ and some $t \in (0, 1)$ such that both equations hold. Let ψ be such that $\psi(t) = t^{\frac{1}{\alpha}}$. Since the ranges of $(\rho_{\tilde{f}}, \theta_{\tilde{f}})$ and $(\rho_g \circ \psi, \theta_g \circ \psi)$ are still f and g, then there exists $t \in (0, 1)$ such that

$$(\rho_{\tilde{f}}(t), \theta_{\tilde{f}}(t)) = (\rho_g \circ \psi(t), \theta_g \circ \psi(t)) \in f \cap g. \tag{13}$$

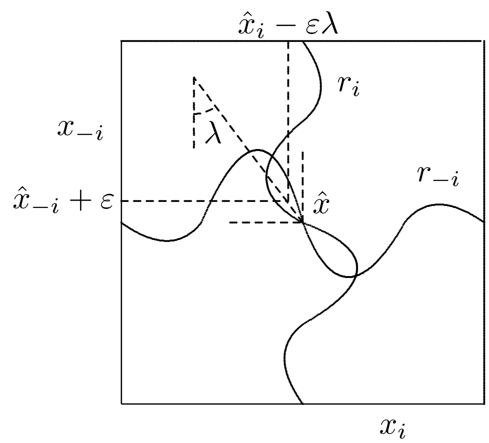
Q.E.D.

Proof of Theorem 1. Assume, without loss of generality, that $r_i'(\hat{x}_{-i}), r_{-i}'(\hat{x}_i) < 0$. Then there exists λ such that

$$r_i'(\hat{x}_{-i}) < \lambda < \frac{1}{r_{-i}'(\hat{x}_i)}. \tag{14}$$

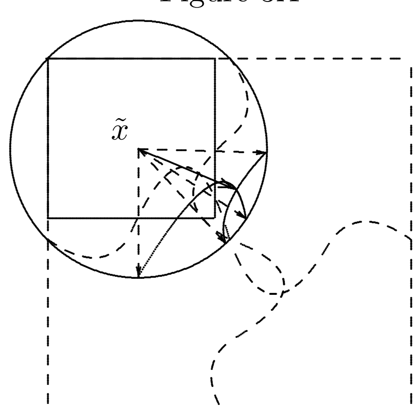
For $\varepsilon > 0$ small enough, consider the restriction of the best reply functions to the square formed by the segments joining consecutively $(0, \hat{x}_{-i} + \varepsilon)$, $(\hat{x}_i - \varepsilon\lambda, \hat{x}_{-i} + \varepsilon)$, $(\hat{x}_i - \varepsilon\lambda, 1)$, $(0, 1)$, and $(0, \hat{x}_{-i} + \varepsilon)$ again (see Figure 2A).

Figure 2A



Since Lemma 1 applies to the continuous transformation of this restriction that, for a given point $\tilde{x}$ within the restriction, maps every point x of the restriction to its homothecy by the reciprocal of the norm of the longest segment within the restricted set starting at $\tilde{x}$ and going through x (see Figure 3A),

Figure 3A



then the crossing of the transformed best replies guaranteed by Lemma 1 corresponds to a crossing of the best reply curves in the restricted set, i.e. to a Nash equilibrium $\hat{x}'$ distinct from $\hat{x}$. A similar argument shows the existence of a third Nash equilibrium $\hat{x}''$ with $\hat{x}''_i > \hat{x}_i$ and $\hat{x}''_{-i} < \hat{x}_{-i}$. Q.E.D.

Proof of Theorem 2. In effect, if $\prod_{i \in I} r'_i(x_{-i}) > 1$, that is to say $\prod_{i \in I} D_{12} f_i(\hat{x}_i, \hat{x}_{-i}) > \prod_{i \in I} D_{11} f_i(\hat{x}_i, \hat{x}_{-i})$, then, since $\prod_{i \in I} D_{11} f_i(\hat{x}_i, \hat{x}_{-i}) > 0$,[23] necessarily $\prod_{i \in I} D_{12} f_i(\hat{x}_i, \hat{x}_{-i}) > 0$ holds as well. Therefore, either, for all $i \in I$, $D_{12} f_i(\hat{x}_i, \hat{x}_{-i}) > 0$, or for all $i \in I$, $D_{12} f_i(\hat{x}_i, \hat{x}_{-i}) < 0$.

If, for all $i \in I$, $D_{12} f_i(\hat{x}_i, \hat{x}_{-i}) > 0$, then

$$0 < -\frac{D_{11} f_1(\hat{x}_1, \hat{x}_{-1})}{D_{12} f_1(\hat{x}_1, \hat{x}_{-1})} < -\frac{D_{12} f_{-1}(\hat{x}_{-1}, \hat{x}_1)}{D_{11} f_{-1}(\hat{x}_{-1}, \hat{x}_1)}, \tag{15}$$

and, for all $i \in I$, there exist $0 < \xi_i$ such that

$$-\frac{D_{11} f_1(\hat{x}_1, \hat{x}_{-1})}{D_{12} f_1(\hat{x}_1, \hat{x}_{-1})} < \frac{\xi_{-1}}{\xi_1} < -\frac{D_{12} f_{-1}(\hat{x}_{-1}, \hat{x}_1)}{D_{11} f_{-1}(\hat{x}_{-1}, \hat{x}_1)} \tag{16}$$

and $\lambda > 0$ small enough, such that for all $i \in I$, $(\hat{x}_{-i} - \lambda\xi_{-i}, \hat{x}_i - \lambda\xi_i), (\hat{x}_{-i} - \lambda\xi_{-i}, \hat{x}_i + \lambda\xi_i) \in r_i^+$ while $(\hat{x}_{-i} + \lambda\xi_{-i}, \hat{x}_i - \lambda\xi_i), (\hat{x}_{-i} + \lambda\xi_{-i}, \hat{x}_i + \lambda\xi_i) \in r_i^-$. Thus letting, for all $i \in I$,

$$\begin{aligned} x_i^*(1) &= \hat{x}_i - \lambda\xi_i \\ x_i^*(2) &\in r_{-i}^{-1}(x_i^*(1)) \cap (x_i^*(1), x_i^*(3)) \\ x_i^*(3) &= \hat{x}_i + \lambda\xi_i, \end{aligned} \tag{17}$$

for it to be the support of a correlated equilibrium with a joint distribution p_i, it should satisfy the equations

$$\sum_{s_{-i}=1}^{3} p_i(s_i, s_{-i}) D_1 f_i(x_i^*(s_i), x_{-i}^*(s_{-i})) = 0, \tag{18}$$

for all $i \in I$, and all $s_i \in \{1, 2, 3\}$. The existence of such a joint distribution p_i is guaranteed for this support. In effect, the matrix of coefficients of the previous system of linear equations is

$$\begin{pmatrix} D_1^{11} & D_1^{12} & D_1^{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1^{21} & D_1^{22} & D_1^{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_1^{31} & D_1^{32} & D_1^{33} \\ D_{-1}^{11} & 0 & 0 & D_{-1}^{12} & 0 & 0 & D_{-1}^{12} & 0 & 0 \\ 0 & D_{-1}^{21} & 0 & 0 & D_{-1}^{22} & 0 & 0 & D_{-1}^{23} & 0 \\ 0 & 0 & D_{-1}^{31} & 0 & 0 & D_{-1}^{32} & 0 & 0 & D_{-1}^{33} \end{pmatrix} \tag{19}$$

where $D_i^{s_i, s_{-i}} = D_1 f_i(x_i^*(s_i), x_{-i}^*(s_{-i}))$, for all $i \in I$, $s_i, s_{-i} \in \{1, 2, 3\}$. Therefore, for given probabilities $p_1(1,1)$, $p_1(1,2)$, $p_1(2,1)$, and $p_1(2,2)$, the two first equations determine $p_1(1,3)$ and $p_1(2,3)$, which along with the last three equations determine

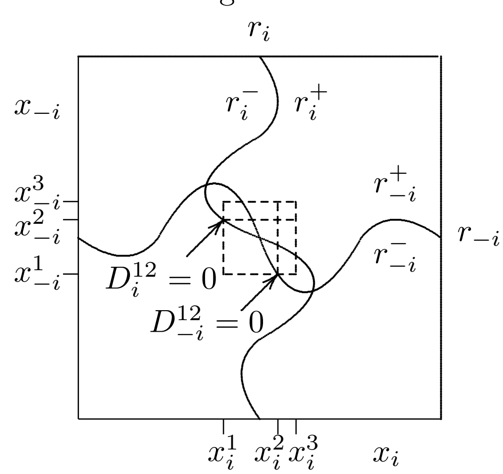
[23]Recall that, for all $i \in I$, $\hat{x}_i$ maximizes $f_i(x_i, \hat{x}_{-i})$, and f_i is differentiable and strictly quasi-concave. Hence $D_{11} f_i(\hat{x}_i, \hat{x}_{-i}) < 0$ for all $i \in I$.

$p_1(3,1)$, $p_1(3,2)$, and $p_1(3,3)$. But all these probabilities must satisfy as well the third equation, and hence the following consistency condition

$$\left(\frac{D^{11}_{-1}}{D^{13}_{-1}} - \frac{D^{33}_{1}}{D^{31}_{1}}\frac{D^{31}_{-1}}{D^{33}_{-1}}\frac{D^{11}_{1}}{D^{13}_{1}}\right) p_1^{11} + \left(\frac{D^{32}_{1}}{D^{31}_{1}}\frac{D^{21}_{-1}}{D^{23}_{-1}} - \frac{D^{33}_{1}}{D^{31}_{1}}\frac{D^{31}_{-1}}{D^{33}_{-1}}\frac{D^{12}_{1}}{D^{13}_{1}}\right) p_1^{12} + \left(\frac{D^{12}_{-1}}{D^{13}_{-1}} - \frac{D^{33}_{1}}{D^{31}_{1}}\frac{D^{32}_{-1}}{D^{33}_{-1}}\frac{D^{21}_{1}}{D^{23}_{1}}\right) p_1^{21} + \left(\frac{D^{32}_{1}}{D^{31}_{1}}\frac{D^{22}_{-1}}{D^{23}_{-1}} - \frac{D^{33}_{1}}{D^{31}_{1}}\frac{D^{32}_{-1}}{D^{33}_{-1}}\frac{D^{22}_{1}}{D^{13}_{1}}\right) p_1^{22} = 0. \tag{20}$$

For this equation to be satisfied by positive probabilities, there must be to coefficients with opposite signs, which is what happens to the coefficient of p_1^{12} (which becomes positive) and the coefficient of p_1^{21} (which becomes negative) if, for all $i \in I$, $x_i^*(2)$ is such that $D_i^{12} = 0$. This is indeed the case given that $x_i^*(2) \in r_{-i}^{-1}(x_i^*(1)) \cap (x_i^*(1), x_i^*(3))$ (see Figure 4A). Such a support is then the support of a continuum of correlated equilibria. Q.E.D.

Figure 4A



Proposition 2. *The strategies of every Nash equilibrium of the game* Γ *are rationalizable, i.e. for all* $i \in I$*,*

$$r_i \cap r_{-i}^{-1} \subset R_{-i} \times R_i. \tag{21}$$

Proof. Assume that there exists $i \in I$ such that $r_i \cap r_{-i}^{-1} \not\subset R_{-i} \times R_i$. Then there exists $(\hat{x}_{-i}, \hat{x}_i) \in r_i \cap r_{-i}^{-1}$ such that $(\hat{x}_{-i}, \hat{x}_i) \notin R_{-i} \times R_i$. Thus there exists $i \in I$ such that $\hat{x}_i \notin R_i$, i.e. such that there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} \hat{x}_i &\notin (r_i \circ r_{-i})^n \circ r_i(X_{-i}) \\ &= (r_i \circ r_{-i}) \circ (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i}). \end{aligned} \tag{22}$$

Hence there does not exist $x_i' \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$ such that $(x_i', \hat{x}_i) \in r_i \circ r_{-i}$ or, equivalently, for all $x_i' \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$, $(x_i', \hat{x}_i) \notin r_i \circ r_{-i}$. Therefore, for all $x_{-i}' \in X_{-i}$, either $(x_i', x_{-i}') \notin r_{-i}$ or $(x_{-i}', \hat{x}_i) \notin r_i$. Since the domain of r_{-i} is X_i and $(r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i}) \subset X_i$, it cannot be true that, for all $x_i' \in (r_i \circ r_{-i})^{n-1} \circ r_i(X_{-i})$ and all $x_{-i}' \in X_{-i}$, $(x_i', x_{-i}') \notin r_{-i}$. Thus necessarily, for all $x_{-i}' \in X_{-i}$, $(x_{-i}', \hat{x}_i) \notin r_i$ must hold. But $\hat{x}_{-i} \in X_{-i}$ and $(\hat{x}_{-i}, \hat{x}_i) \in r_i$! Therefore it must be true that, for all $i \in I$, $r_i \cap r_{-i}^{-1} \subset R_{-i} \times R_i$. Q.E.D.

Proposition 3. *For each player $i \in I$ of the game Γ, the set R_i of rationalizable strategies in nonempty and convex.*

Proof. Since, for all $i \in I$, r_i is continuous and X_i is a closed interval, then $r_i(X_i)$ is a closed interval, because of the intermediate value theorem, and hence convex. Moreover, $r_i \circ r_{-i}$ is continuous as well, and hence, for all $N \in \mathbb{N}$, $\cap_{n=1}^{N}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is convex. Assume $\cap_{n\in\mathbb{N}}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is not convex. Then there exist $x, x' \in \cap_{n\in\mathbb{N}}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$ and $x'' \in (x, x')$ such that $x'' \notin \cap_{n\in\mathbb{N}}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$. Then there exists $N \in \mathbb{N}$ such that $x'' \notin (r_i \circ r_{-i})^N \circ r_i(X_{-i})$ while $x, x' \in (r_i \circ r_{-i})^N \circ r_i(X_{-i})$, and hence $x'' \notin \cap_{n=1}^{N}(r_i \circ r_{-i})^N \circ r_i(X_{-i})$ while $x, x' \in \cap_{n=1}^{N}(r_i \circ r_{-i})^N \circ r_i(X_{-i})$, which contradicts that, for all $N \in \mathbb{N}$, $\cap_{n=1}^{N}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$ is convex. Therefore, $\cap_{n\in\mathbb{N}}(r_i \circ r_{-i})^n \circ r_i(X_{-i})$, i.e. R_i is convex. Q.E.D.

Proof of Theorem 3. Since $\prod_{i\in I} r_i'(\hat{x}_{-i}) > 1$, then by Theorem 1 there exist two more Nash equilibria whose strategies are one at each side of the original Nash equilibrium strategies. Since every Nash equilibrium is rationalizable and the set of rationalizable strategies of each player is convex, then it is an interval with the original Nash equilibrium in its interior. Q.E.D.

Proof of Theorem 4. If $0 < \prod_{i\in I} r_i'(\hat{x}'_{-i}) < 1$, assume with no loss of generality that

$$0 < r_{-i}'(\hat{x}_i') < \frac{1}{r_i'(\hat{x}'_{-i})}. \tag{23}$$

Then, for all $i \in I$, there exist $\lambda_i > 0$ small enough such that,

$$r_{-i}'(\hat{x}_i') < \frac{\lambda_{-i}}{\lambda_i} < \frac{1}{r_i'(\hat{x}'_{-i})} \tag{24}$$

and hence, for all $i \in I$, the image of $[\hat{x}_{-i} - \lambda_{-i}, \hat{x}_{-i} + \lambda_{-i}]$ by r_i is a proper subset of $[\hat{x}_i - \lambda_i, \hat{x}_i + \lambda_i]$. The convergence to $\hat{x}_i$ of the iterative elimination of dominated strategies, if constrained to the interval $[\hat{x}_i - \lambda_i, \hat{x}_i + \lambda_i] \times [\hat{x}_{-i} - \lambda_{-i}, \hat{x}_{-i} + \lambda_{-i}]$, follows immediately. Q.E.D.

Proof of Theorem 5. Assume there exist two distinct Nash equilibria in pure strategies $\hat{x}$ and $\hat{x}'$, and assume both $\prod_{i\in I} r_i'(\hat{x}_{-i}) < 1$ and $\prod_{i\in I} r_i'(\hat{x}'_{-i}) < 1$ (otherwise, Theorem 2 guarantees the existence of correlated equilibria already). Applying the Poincaré-Hopf theorem again, there must exist another zero of the vector field determined by the equilibrium equations with and index opposite to those of $\hat{x}$ and $\hat{x}'$, i.e. another Nash equilibrium in pure strategies $\hat{x}''$ such that $\prod_{i\in I} r_i'(\hat{x}''_{-i}) > 1$. Therefore, according to Theorem 2, there must exist non trivial correlated equilibria in every neighborhood of $\hat{x}''$. Q.E.D.

Proof of Theorem 6. It follows immediately from the facts that, on the one hand, every Nash equilibrium is a profile of rationalizable strategies (see Proposition 2) and, on the other hand, the set of rationalizable strategies of every agent is convex (see Proposition 3). Therefore, the multiplicity of Nash equilibria in pure strategies implies immediately that the set of rationalizable profiles is a cartesian product of intervals with nonempty interior and, hence, has no isolated point. Q.E.D.

Proposition 4. *If* $\{p_i, S_i, x_i^*\}_{i\in I}$ *is a finite support correlated equilibrium of the game* Γ*, then, for all* $i \in I$ *and all* $s_i \in S_i$*,*

$$\begin{aligned} &\{(x_{-i}^*(s_{-i}), x_i^*(s_i))\}_{s_{-i}\in S_{-i}} \not\subset r_i^+ \text{ and} \\ &\{(x_{-i}^*(s_{-i}), x_i^*(s_i))\}_{s_{-i}\in S_{-i}} \not\subset r_i^- . \end{aligned} \tag{25}$$

Proof. Since $\{p_i, S_i, x_i^*\}_{i\in I}$ is a finite support correlated equilibrium of Γ, then, for all $i \in I$, $x_i^* \in X_i^{S_i}$ solves

$$\max_{x_i\in X_i^{S_i}} \sum_{s_i=1}^{|S_i|} \sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) f_i(x_i(s_i), x_{-i}^*(s_{-i})), \tag{26}$$

and hence, for all $s_i = 1, 2, \ldots, |S_i|$, $x_i^*(s_i)$ solves

$$\max_{x_i(s_i)\in(0,1)} \sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) f_i(x_i(s_i), x_{-i}^*(s_{-i})). \tag{27}$$

Therefore, since f_i is differentiable and strictly unimodal, $x_i^*(s_i)$ is a critical point of the maximand $\sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) f_i(x_i^*(s_i), x_{-i}^*(s_{-i}))$, i.e.

$$\sum_{s_{-i}=1}^{|S_{-i}|} p_i(s_i, s_{-i}) D_1 f_i(x_i(s_i), x_{-i}^*(s_{-i})) = 0. \tag{28}$$

Assume, without loss of generality, that $\{(x_{-i}^*(s_{-i}), x_i^*(s_i))\}_{s_{-i}\in S_{-i}} \subset r_i^+$, then, for all $s_i \in S_i$,

$$D_1 f_i(x_i^*(s_i), x_{-i}^*(s_{-i})) \le 0, \tag{29}$$

and hence, for all $s_i \in S_i$,

$$D_1 f_i(x_i^*(s_i), x_{-i}^*(s_{-i})) = 0. \tag{30}$$

Therefore, because of the strict unimodality of f_i, for all $s_i, s_i' \in S_i$, $x^*(s_i) = x^*(s_i')$, i.e. x^* would be a Nash equilibrium in pure strategies actually. Therefore, if $\{p_i, S_i, x_i^*\}_{i\in I}$ is a non-trivial correlated equilibrium, then

$$\{(x_{-i}^*(s_{-i}), x_i^*(s_i))\}_{s_{-i}\in S_{-i}} \not\subset r_i^+. \tag{31}$$

Similarly, it can easily be established that $\{(x_{-i}^*(s_{-i}), x_i^*(s_i))\}_{s_{-i}\in S_{-i}} \not\subset r_i^-$. Q.E.D.

Proposition 5. *If there exists a finite support correlated equilibrium of the game* Γ*, then there exists a Nash equilibrium in pure strategies whose profile of strategies is in the convex hull of the support of the correlated equilibrium.*

Proof. Let $\{(p_i, x_i^*)\}_{i\in I}$ be a finite support correlated equilibrium. Assume, without loss of generality, that $x_i^*(1) < \cdots < x_i^*(|S_i|)$, for each $i \in I$. Consider, for each $i \in I$ as well, the function

$$\tilde{r}_i(x_{-i}) = \min\{\max\{r_i(x_{-i}), x_i^*(1)\}, x_i^*(|S_i|)\} \tag{32}$$

for all $x_{-i} \in [x^*_{-i}(1), x^*_{-i}(|S_{-i}|)]$, and for any $i \in I$ consider the function $\tilde{\phi}_i$ from $[x^*_{-i}(1), x^*_{-i}(|S_{-i}|)]$ to itself mapping each x_{-i} to $\tilde{\phi}_i(x_{-i}) = x_{-i} - \tilde{r}_{-i}(\tilde{r}_i(x_{-i}))$.

According to Proposition 4, for each $i \in I$, either $\tilde{r}_i(x^*_{-i}(1)) = x^*_i(1)$ or $\tilde{r}_i(x^*_{-i}(1)) = x^*_i(|S_i|)$ holds. If $\tilde{r}_i(x^*_{-i}(1)) = x^*_i(1)$ holds for both $i \in I$,[24] then for any $x'_{-i} > x^*_{-i}(1)$ but close enough to $x^*_{-i}(1)$, it still holds $\tilde{r}_i(x_{-i}) = x^*_i(1)$ and hence $\tilde{\phi}_i(x'_{-i}) > 0$. Moreover, there exists some x''_{-i} in the interval $[x^*_{-i}(1), x^*_{-i}(|S_{-i}|)]$ such that $\tilde{\phi}_i(x''_{-i}) < 0$, i.e. such that $x''_{-i} < \tilde{r}_{-i}(\tilde{r}_i(x''_{-i}))$. In effect, on the one hand, such an x''_{-i} exists whenever there is some x_i in $[x^*_i(1), x^*_i(|S_i|)]$ such that $\tilde{r}_{-i}(x_i) > \inf \tilde{r}_i^{-1}(x_i)$, since then there exists some $x_{-i} \in \tilde{r}_i^{-1}(x_i)$, satisfying $\tilde{r}_{-i}(x_i) > x_{-i}$ and $\tilde{r}_i(x_{-i}) = x_i$,[25] i.e. such that $\tilde{r}_{-i}(\tilde{r}_i(x_{-i})) > x_{-i}$. On the other hand, should there be no x_i in $[x^*_i(1), x^*_i(|S_i|)]$ such that $\tilde{r}_{-i}(x_i) > \inf \tilde{r}_i^{-1}(x_i)$, then it would hold that, for all x_i in $[x^*_i(1), x^*_i(|S_i|)]$, $\tilde{r}_{-i}(x_i) \leq \inf \tilde{r}_i^{-1}(x_i)$, but since $\tilde{r}_i^{-1}(x_i) \subset [x^*_{-i}(1), x^*_{-i}(|S_{-i}|)]$, certainly $\inf \tilde{r}_i^{-1}(x_i) \leq x^*_{-i}(|S_{-i}|)$ would hold as well, and hence so would $\tilde{r}_{-i}(x_i) \leq x^*_{-i}(|S_{-i}|)$ for all x_i in the interval $[x^*_i(1), x^*_i(|S_i|)]$. Thus, $\{(x^*_i(s_i), x^*_{-i}(|S_{-i}|))\}_{s_i \in S_i} \subset r^+_{-i}$ would have to be true, which contradicts Proposition 4.

Therefore, for some $\hat{x}_{-i}$ in (x'_{-i}, x''_{-i}), $\tilde{\phi}_i(\hat{x}_{-i}) = 0$ holds, that is to say $\hat{x}_{-i} = \tilde{r}_{-i}(\tilde{r}_i(\hat{x}_{-i}))$, and letting $\hat{x}_i$ be $\tilde{r}_i(\hat{x}_{-i})$, then $(\hat{x}_{-i}, \hat{x}_i) \in \tilde{r}_i$ for all $i \in I$. Now, should it be that case that $\hat{x}_i \in \{x^*_i(1), x^*_i(|S_i|)\}$, then for $(\hat{x}_{-i}, \hat{x}_i) \in \tilde{r}_i$ to hold for all $i \in I$, necessarily it would have to be true that $\hat{x}_{-i} \in \{x^*_{-i}(1), x^*_{-i}(|S_{-i}|)\}$ as well. Therefore, since $\hat{x}_{-i} \in (x'_{-i}, x''_{-i})$, with $x'_{-i} \in (x^*_{-i}(1), x^*_{-i}(|S_{-i}|))$ and $x''_{-i} \in [x^*_{-i}(1), x^*_{-i}(|S_{-i}|)]$, then $\hat{x}_{-i} \notin \{x^*_{-i}(1), x^*_{-i}(|S_{-i}|)\}$, and hence $\hat{x}_i \notin \{x^*_i(1), x^*_i(|S_i|)\}$ neither. Thus, $\hat{x}_i \in (x^*_i(1), x^*_i(|S_i|))$ for all $i \in I$ and hence $\tilde{r}_i(\hat{x}_{-i}) = r_i(\hat{x}_{-i})$ and $\tilde{r}_{-i}(r_i(\hat{x}_{-i})) = r_{-i}(r_i(\hat{x}_{-i}))$ for any $i \in I$. As a consequence, $\hat{x}_{-i} = r_{-i}(r_i(\hat{x}_{-i}))$ as well, and then $(\hat{x}_{-i}, \hat{x}_i) \in r_i$ for all $i \in I$, i.e. $\{\hat{x}_i\}_{i \in I}$ is a Nash equilibrium. Q.E.D.

In what follows a proof of Theorem 7 is provided without resorting to the Poincaré-Hopf Theorem.

Proof of Theorem 7. Let $\{(p_i, x^*_i)\}_{i \in I}$ be a finite support correlated equilibrium. Then Proposition 3 guarantees that, for all $i \in I$, there exists a $x_{-i}(1) \in X_{-i}$ such that $x^*_i(1)$ is the best reply to $x_{-i}(1)$, i.e. $r_i(x_{-i}(1)) = x^*_i(1)$, and $x_{-i}(1)$ is in the interior of the convex hull of the range of x^*_{-i}; there exists also $x_{-i}(|S_i|) \in X_{-i}$ such that $x^*_i(|S_i|)$ is the best reply to $x_{-i}(|S_i|)$, i.e. $r_i(x_{-i}(|S_i|)) = x^*_i(|S_i|)$, and $x_{-i}(|S_i|)$ is in the interior of the convex hull of the range of x^*_{-i}. For each $i \in I$, let $r^\wedge_i$ be such that $r^\wedge_i(x_{-i}) = \min\{r_i(x_{-i}), r_i(x_{-i}(1))\}$, and $r^\vee_i$ be such that $r^\vee_i(x_{-i}) = \max\{r_i(x_{-i}), r_i(x_{-i}(|S_i|))\}$, that is to say, in the case both r_i are strictly decreasing functions

$$r^\wedge_i(x_{-i}) = \begin{cases} r_i(x_{-i}(1)) = x^*_i(1) & \forall x_{-i} \in [0, x_{-i}(1)] \\ r_i(x_{-i}) & \forall x_{-i} \in [x_{-i}(1), 1] \end{cases} \tag{33}$$

and

$$r^\vee_i(x_{-i}) = \begin{cases} r_i(x_{-i}) & \forall x_{-i} \in [0, x_{-i}(|S_i|)] \\ r_i(x_{-i}(|S_i|)) = x^*_i(|S_i|) & \forall x_{-i} \in [x_{-i}(|S_i|), 1]. \end{cases} \tag{34}$$

Consider the continuous function $\phi^\wedge_i(x_{-i}) = x_{-i} - r^\vee_{-i}(r^\wedge_i(x_{-i}))$ from $[0,1]$ to itself. Then $\phi^\wedge_i(0) = -r^\vee_{-i}(r^\wedge_i(0)) \in (-1, 0)$ while $\phi^\wedge_i(1) = 1 - r^\vee_{-i}(r^\wedge_i(1)) \in (0, 1)$. Thus

[24] A similar argument can be easily developed for each of the three other possible cases.

[25] For all x_i in $[x^*_i(1), x^*_i(|S_i|)]$, $\tilde{r}_i^{-1}(x_i)$ contains its greatest lower bound because of the continuity of $\tilde{r}_i$

there exists $\hat{x}_{-i} \in X_{-i}$ such that $\hat{x}_{-i} = r^{\vee}_{-i}(r^{\wedge}_i(\hat{x}_{-i}))$, i.e. letting $\hat{x}_i$ be $r^{\wedge}_i(\hat{x}_{-i})$, it holds that $(\hat{x}_{-i}, \hat{x}_i) \in r^{\wedge}_i$ and $(\hat{x}_i, \hat{x}_{-i}) \in r^{\vee}_{-i}$. Now, since both r_i and r_{-i} are strictly decreasing, $r^{\wedge}_i$ does not meet $[0, x^*_{-i}(|S_{-i}|)) \times (x^*_i(1), 1]$, nor does $r^{\vee}_{-i}$ meet $(x^*_i(1), 1] \times [0, x^*_{-i}(|S_{-i}|))$. Then, necessarily, $\hat{x}_{-i} \in [x^*_{-i}(|S_{-i}|), 1] \subset [x_{-i}(1), 1]$ and $\hat{x}_i \in [0, x^*_i(1)] \subset [0, x_i(|S_{-i}|)]$, and therefore $r^{\wedge}_i(\hat{x}_{-i}) = r_i(\hat{x}_{-i})$ and $r^{\vee}_{-i}(\hat{x}_i) = r_{-i}(\hat{x}_i)$, that is to say, $(\hat{x}_{-i}, \hat{x}_i) \in r_i$ and $(\hat{x}_i, \hat{x}_{-i}) \in r_{-i}$. Thus $\{\hat{x}_i\}_{i \in I}$ is a Nash equilibrium of the game not contained in the convex hull of the support of the correlated equilibrium, and hence distinct from the one contained in it.

Similarly it can be proved that there is another Nash equilibrium $\hat{x}'_i \in [x^*_i(|S_i|), 1]$ and $\hat{x}'_{-i} \in [0, x^*_{-i}(1)]$ by means of the function $\phi^{\vee}_i(x_{-i}) = x_{-i} - r^{\wedge}_{-i}(r^{\vee}_i(x_{-i}))$. Finally, an analogous argument shows the existence of two Nash equilibria outside the convex hull of the support of the correlated equilibrium in the case that both best reply function are strictly increasing. Q.E.D.

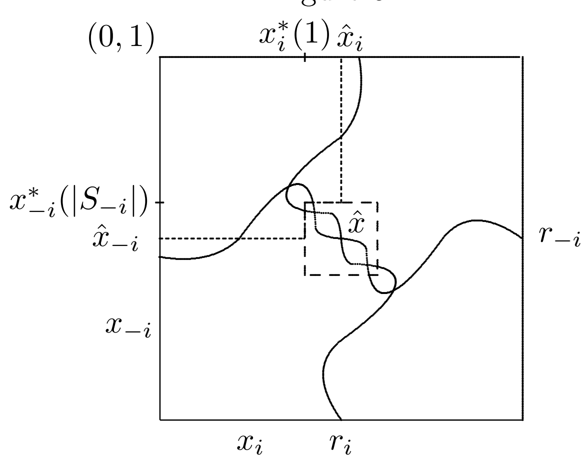
Proposition 6. *If there exists a finite support correlated equilibrium of the game* Γ *and either,*

(1) *for all* $i \in I$, $x_i(|S_i|) < r_i(x_{-i}(1))$ *and* $r_i(x_{-i}(|S_{-i}|)) < x_i(1)$ *or*
(2) *for all* $i \in I$, $x_i(|S_i|) < r_i(x_{-i}(|S_{-i}|))$ *and* $r_i(x_{-i}(1)) < x_i(1)$,

then there exist at least three Nash equilibria.

Proof. In effect, the existence of a correlated equilibrium guarantees the existence of a Nash equilibrium in pure strategies $(\hat{x}_i, \hat{x}_{-i})$ in the convex hull of its support. Moreover, Lemma 1 applies necessarily to a continuous deformation of a set of profiles of strategies, e.g. in the case (1) the set circumscribed by the polygon formed by the segments joining the points $(0,1)$, $(0, \hat{x}_{-i})$, $(x^*_i(1), \hat{x}_{-i})$, $(x^*_i(1), x^*_{-i}(|S_{-i}|))$, $(\hat{x}_i, x^*_{-i}(|S_{-i}|))$, $(\hat{x}_i, 1)$, and $(0,1)$ again (see Figure 5A).[26] Clearly, using the same argument, another Nash equilibrium must exist also in this case to the southeast of $\hat{x}$.

Figure 5A



[26] Consider the same continuous transformation is in the the proof of Theorem 1. Notice also that the call for Lemma 1 is essential here: a restriction of the vector field defined by the best reply functions that includes the Nash equilibrium within the convex hull of the support of the correlated equilibrium, needs not be outward-pointing, and thus no general argument based on the Poincaré-Hopf theorem can be made.

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