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PIER Working Paper 01-013

"Majority Rule in a Stochastic Model of Bargaining"

by

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http://papers.ssrn.com/sol3/papers.cfm?abstract_id=267780

Majority Rule in a Stochastic Model of Bargaining*

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Running Title: Majority Rule

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*We thank two anonymous referees for their useful suggestions. Ed Green, George Mailath, Andy McLennan and seminar and conference participants at several institutions provided useful comments. Merlo acknowledges the financial support of the National Science Foundation and the C.V. Starr Center for Applied Economics at New York University.

Abstract: In this paper we consider multilateral stochastic bargaining models with general agreement rules. For n-player games where in each period a player is randomly selected to allocate a stochastic level of surplus and $q \leq n$ players have to agree on a proposal to induce its acceptance, we characterize the set of stationary subgame perfect equilibrium payoffs and establish their existence. We show that for agreement rules other than the unanimity rule, the equilibrium payoffs need not be unique. Furthermore, even when the equilibrium is unique, it need not be efficient. *Journal of Economic Literature* Classification Numbers: C73, C78, D70.

Keywords: Noncooperative bargaining, voting rules, stochastic games.

1. Introduction

Many real bargaining situations involve negotiations among more than two players over the allocation of some surplus. In some contexts, agreement entails the unanimous consent of all negotiating parties. For example, in a Chapter 11 corporate bankruptcy negotiation, all classes of creditors have to agree on a plan to restructure the bankrupt firm. In other contexts, agreement among only a subset of the players is sufficient to implement a particular allocation. For example, when political parties in a parliamentary democracy bargain over the formation of a new government, a simple majority is typically enough to determine the allocation of cabinet ministries among the parties represented in the parliament.[1]

Most of the existing theoretical literature on multilateral bargaining restricts attention to unanimity games (see, e.g., the survey in Osborne and Rubinstein [15]). Notable exceptions are the papers by Baron and Ferejohn [3], Harrington [10], Baron and Kalai [4], Eraslan [7], and Banks and Duggan [2]. These papers study (complete information) multilateral sequential bargaining games with random proposers and general agreement rules which build on Rubinstein's [17] "divide-the-dollar" game. In these n-player games ($n \geq 3$), at least q players ($q \in \{1, ..., n\}$) have to agree on how to allocate a certain amount of surplus, and the discounted value of the surplus decreases in a deterministic fashion with each rejected offer.[2]

[1] In other political negotiations, qualified majorities are sometime required to implement specific agreements. For example, a 5/7 (super)majority rule is used in the European Council.

[2] Banks and Duggan [2] consider a more general framework by allowing the object of the negotiation to be multidimensional. Even in their model, however, the set of feasible agreements is fixed.

Merlo and Wilson [12, 13] consider a general class of multilateral bargaining games where the surplus to be allocated follows a stochastic process. In many negotiations, the terms of an agreement may depend on aspects of the environment which change during the negotiating process. In such cases, the surplus to be allocated may evolve over time according to a stochastic process.[3] To provide an analog to the paradigm of the "divide-the-dollar" game in deterministic environments, in a stochastic environment we may think of the "divide-the-yen" game, where the dollar value of the surplus to be allocated oscillates stochastically over time.[4] In their analysis of this class of games, Merlo and Wilson [12, 13] restrict attention to games with unanimity rule.

In this paper, we combine the two literatures and consider multilateral stochastic bargaining models with general agreement rules. The game we study is an n-player game with a q-quota agreement rule. In each period a state is realized which determines the total utility to be allocated if an agreement is reached in that period, and a player is randomly selected to make a proposal. The selected player may either propose an allocation or pass. If he proposes an allocation, each of the remaining players in turn accepts or rejects the proposal. If more than $n - q$ players reject the proposal or the proposer passes, a new state is realized, a new proposer is selected,

[3] We have in mind a situation where agents bargain over complex agreements and a complete set of contingent contracts is not available. For a detailed description of the environment see Merlo and Wilson [12].

[4] For an application of a stochastic bargaining model to the process of government formation in a parliamentary democracy, see Merlo [11]. Eraslan [8] uses a stochastic model to analyze the negotiations behind bankruptcy reorganizations of large, publicly held corporations. For a discussion of other possible applications see Merlo and Wilson [13].

and the process is repeated until q players agree upon some proposed allocation. We characterize the set of stationary subgame perfect equilibrium payoffs and establish their existence.[5]

For the class of games we consider, if the surplus to be allocated is restricted to be constant over time, Baron and Ferejohn [3] and Eraslan [7] have shown that for any agreement rule there exists a unique stationary subgame perfect equilibrium payoff. Furthermore, the equilibrium is efficient and an agreement is reached in the first period. When the surplus to be allocated is allowed to evolve stochastically over time, Merlo and Wilson [13] have shown that under unanimity rule there exists a unique stationary subgame perfect equilibrium payoff. Furthermore, the equilibrium is efficient, even though it may involve delays.[6] In this paper, we show that when the surplus to be allocated is allowed to evolve stochastically over time, for general agreement rules the stationary subgame perfect equilibrium payoff need not be unique. Furthermore, even when the equilibrium is unique, for any agreement rule other than unanimity it need not be efficient. In particular, the kind of inefficiency that may emerge in equilibrium is induced by the fact that agreement may be reached "too soon."

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[5]It is well know that in (deterministic and stochastic) multilateral bargaining games like the ones considered here, if players are sufficiently patient, then any allocation of the surplus can be sustained as a subgame perfect equilibrium payoff regardless of the agreement rule (see, e.g., Baron and Ferejohn [3] and Merlo and Wilson [12]). Stationarity, on the other hand, typically selects a unique equilibrium (see, e.g., Baron and Ferejohn [3], Eraslan [7], and Merlo and Wilson [13]). Like these papers, we restrict attention to stationary subgame perfect equilibria.

[6]Once we suppose that the discounted size of the surplus need not decline in every period, a temporary delay in agreement should not be unexpected as a possible equilibrium outcome. As shown by Merlo and Wilson [13], delays are, however, efficient.

The intuition for our results is as follows. Whenever agreement entails less than unanimous approval, there exists a differential treatment between the players who are included in a proposal (i.e., those who are allocated a positive share of the surplus) and the players who are excluded (i.e., those who are allocated a zero share). In a stochastic environment, there may be incentives for the players to delay agreement until a larger level of surplus is realized. If all players have veto power (i.e., in the unanimity game), then the interests of all players are aligned in their pursuit of the optimal time to agree (see Merlo and Wilson [13]). If, on the other hand, agreement among only a subset of the players is sufficient to implement an allocation, then while all players may gain in expected terms by waiting, the actual gains from waiting will be captured only by those players who will be included in the proposal that ultimately will be agreed upon. This tension generates the possibility of inefficient agreements where players fail to realize the gains from waiting. It may also generate multiplicity of equilibrium payoffs. Players who are offered a positive payoff in a state where the level of surplus is relatively small may be induced to accept it if they expect to be excluded from future agreements when the level of surplus is relatively large. This in turn may induce the proposer to make such a proposal in that state even though there exists other equilibria where no proposal is made in that state and payoffs are higher.

Before turning our attention to the analysis of the game, a few remarks are in order. Our paper is related to the literature on noncooperative coalitional bargaining games. In fact, q-quota games are a special class of n-person games in coalitional form. In the context of these models, Chatterjee et al. [5] and Okada [14] show that when the surplus to be divided is non stochastic but depends on a coalition, the conflict inherent in coalition formation may induce an inefficient stationary subgame perfect equilibrium payoff.

Our results are also relevant for the literature on the relative desirability of alternative voting rules. For instance, in the context of models of collective choice under uncertainty, Feddersen and Pesendorfer [9] illustrate the inferiority of the unanimity rule.[7] In the context of multilateral stochastic bargaining games with complete information, our analysis indicates that the unanimity rule dominates all other (q-quota) voting rules.

The two key features of the bargaining games we consider are a stochastic surplus and a q-quota agreement rule. To further motivate our analysis, consider the following concrete bargaining situation that can be analyzed using our theoretical framework. The process of government formation in a parliamentary democracy entails bargaining among the set of parties represented in parliament over the allocation of cabinet ministries. Approval of a government proposal requires the support of a simple parliamentary majority. Furthermore, bargaining over governments takes time and a stochastic environment best describes the changing political and economic situation while parties bargain over the formation of a new government.[8] Since political

[7]See also Austen-Smith and Banks [1] and Persico [16].

[8]For example, Merlo [11] reports that the average duration of a negotiation over the formation of a new government in Italy is 5 weeks and the maximum duration is 18 weeks.

and economic variables affect government stability and a more durable government implies a larger level of surplus, these considerations lead one to consider a bargaining environment where the surplus to be divided follows a stochastic process.[9]

2. The Game

Consider the following class of stochastic bargaining games with complete information. Let $\{\sigma_t\}_{t=0}^{\infty}$ denote an independently and identically distributed stochastic process with state space S and distribution function F, where S is a closed Borel subset of $[0, \bar{s}]$, with $0 < \bar{s} < \infty$.[10] Let $N = \{1, ..., n\}$ denote a set of players, where $n \geq 2$. We refer to an element $s \in S$ as a *state*, and an element $i \in N$ as a *player*. A state $s \in S$ denotes the size of the surplus to be divided among the players, if they agree in that state. For $t = 0, 1, 2...$, let $\sigma^t \equiv (\sigma_0, \sigma_1, ..., \sigma_t)$ denote the t-period state-history, with typical realization $(s_0, s_1, ..., s_t)$.

Players have an identical single date payoff function which is linear in their surplus share and discount the future at a common discount factor $\beta \in (0, 1)$.[11] For any state $s \in S$, let $X(s) \equiv \{x \in I\!R^n_+ : \sum_{i=1}^n x_i \leq s\}$ denote the set of feasible payoff vectors to be allocated in state s. For an allocation $x \in X(s)$, x_i denotes the amount of surplus

[9]For a more detailed explanation of the stochastic nature of the surplus in negotiations over the formation of a new government see Merlo [11]. Notice that in the context of this application, the assumption of random proposers–the third main feature of the bargaining game we consider–can also be justified on empirical grounds (see Diermeier and Merlo [6]).

[10]Merlo and Wilson [12, 13] consider a general Markov process. While our characterization extends to the more general case, the analysis becomes significantly more complex. We therefore restrict attention to the i.i.d. case.

[11]Like Baron and Ferejohn [3] and Merlo and Wilson [13], we consider bargaining games with transferable utility.

awarded to player i.

For any player $i \in N$, let p_i denote the time and state invariant probability player i is selected as proposer in each period, where $p_i \geq 0$, $\sum_{i \in N} p_i = 1$.

Let $q \in \{1, ..., n\}$ denote the number of players who have to agree on a proposal to induce its acceptance. We refer to the following game as a q-quota game. In period zero, after state s is realized, a player is selected as proposer. The proposer can pass or propose an allocation in $X(s)$. If he proposes an allocation, all other players respond sequentially (in some fixed order) by either accepting or rejecting the proposal.[12] An agreement is reached if q players (including the proposer) accept the proposal. Otherwise a new state s' is realized and a new proposer is selected in the next period. This process continues until an allocation is proposed and accepted.

When $q = n$, the agreement rule is unanimity and the game is a special case of Merlo and Wilson [13]. When $q = 1$, the proposer is a one period dictator. If n is odd, when $q = (n+1)/2$, the agreement rule is majority rule. For a general q, we refer to the agreement rule as a q-quota rule.[13]

An outcome for the q-quota game is a random vector (η_q, τ_q) where τ_q is a stopping time and η_q is measurable with respect to σ_{τ_q} and satisfies $\eta_q \in X(\sigma_{\tau_q})$ if τ_q is finite and $\eta_q = 0$, otherwise. Given a realization of σ, τ_q denotes the period in which a proposal is accepted, and η_q denotes the proposed allocation which is accepted in

[12] The actual order in which players respond to a proposal does not affect the results of the model. Therefore, we leave the order unspecified.

[13] When the distribution F is degenerate, the game reduces to the one studied by Baron and Ferejohn [3] and Eraslan [7].

state σ_{τ_q}. Define $\beta^\infty = 0$. Then in the game outcome (η_q, τ_q), the von Neumann-Morgenstern payoff vector is given by $E[\beta^{\tau_q}\eta_q]$. For notational simplicity, in the rest of the paper we suppress the subscript q unless required by the context.

A history at date t, h^t, is a sequence of realized states, proposers, and actions taken up to date t. A (behavior) strategy for player i, ψ_i, is a probability distribution over feasible actions for each date t and history at date t. A strategy profile ψ is a n-tuple of strategies, one for each player. Let $G(h^t)$ denote the game from date t on with history h^t. Let $\psi|h^t$ denote the restriction of ψ to the histories consistent with h^t. Then $\psi|h^t$ is a strategy profile on $G(h^t)$. A strategy profile ψ is subgame perfect (SP) if, for every history h^t, $\psi|h^t$ is a Nash equilibrium of $G(h^t)$. A strategy profile is stationary if the actions prescribed at any history depend only on the current level of surplus, proposer and offer. A stationary, subgame perfect (SSP) outcome and payoff are the outcome and payoff generated by an SSP strategy profile.

3. Characterization of SSP Payoffs

In this section, we characterize the set of SSP payoffs and establish their existence. In our characterization we focus on the SSP continuation payoffs.[14] Let v denote the von Neumann-Morgenstern continuation payoff vector generated by an SSP strategy profile. In the remainder of the paper, we simply refer to an SSP continuation payoff

[14]Unlike the actual (realized) SSP payoffs, the SSP continuation payoffs do not depend on the identity of the proposer, whether or not the proposer makes a proposal, and, when a proposal is made, the identity of the players included in the proposal. This simplifies the characterization of the equilibrium payoffs. Note that, given an SSP continuation payoff, the actual SSP payoff can be easily computed.

vector as an SSP payoff vector. Let M^i denote the set of n-dimensional real vectors such that the ith component is zero and let $e \in I\!R^n$ denote the n-dimensional unit vector.

Theorem 1. *v is an SSP payoff vector for the q-quota game if and only if*

$$\begin{aligned} v_i &= \beta \int \Bigg\{ p_i[\alpha_i(s)(s - \sum_{j \neq i} r_{ij} v_j) + (1 - \alpha_i(s)) v_i] \\ &\quad + \sum_{j \neq i} p_j [\alpha_j(s) r_{ji} v_i + (1 - \alpha_j(s)) v_i] \Bigg\} dF(s), \end{aligned} \tag{1}$$

where for all $i \in N$, $(r_{ij})_{j=1}^n$ is a minimizer for the program

$$\begin{aligned} \min_z \quad & z'v \\ \textit{subject to} \quad & z'e = q - 1, \\ & z \in [0,1]^n \cap M^i \end{aligned} \tag{2}$$

and for all $i \in N$ and $s \in S$, $\alpha_i(s) \in [0,1]$ satisfies

$$\alpha_i(s) = \begin{cases} 1 & \textit{if } s - \sum_{j \neq i} r_{ij} v_j > v_i \\ 0 & \textit{if } s - \sum_{j \neq i} r_{ij} v_j < v_i \end{cases} \tag{3}$$

Proof: Suppose the vector of SSP continuation payoffs is given by v. Fix s and let i denote the proposer. Consider an SSP response to a proposal $x \in X(s)$. Player

j accepts the proposal if $x_j \geq v_j$ and rejects it if $x_j < v_j$. As it is common in the literature on sequential bargaining, we impose the (mild) restriction on a player's strategy that a player always accepts a proposal that gives him his continuation payoff (even though he is indifferent between accepting and rejecting the proposal).

Note that the proposer needs only $q-1$ votes in addition to his vote for a proposal to be accepted. Then, if the proposer decides to make an offer that will be accepted, he will solve the program

$$\begin{aligned} \min_{z} \quad & z'v \\ \text{subject to} \quad & z'e = q-1, \\ & z \in \{0,1\}^n \cap M^i \end{aligned} \tag{4}$$

Let Γ_i denote the set of minimizers of (4). Note that an SSP proposal in pure strategies by player i can be identified by the $(n-1)$-dimensional vector which specifies the players to whom player i offers their continuation payoff. Thus, each $\gamma_i = (\gamma_{ij})_{j=1}^n \in \Gamma_i$ corresponds to a pure proposal. A minimizer of (2), however, does not necessarily correspond to a pure proposal. Rather, it corresponds to a mixed proposal, where player i randomizes over the proposals corresponding to the elements in Γ_i (possibly with degenerate probabilities).

Notice that any proposal corresponding to an element in Γ_i yields the same payoff to player i. Hence, in equilibrium, player i randomizes over such proposals. It can be verified that r_{ij} is a minimizer of (2) if and only if there exists a probability distribution $\pi_i(.)$ over Γ_i such that

$$r_{ij} = \sum_{\gamma_i \in \Gamma_i} \gamma_{ij} \pi_i(\gamma_i).$$

In other words, randomizing over pure proposals is payoff equivalent to offering mixed proposals. Intuitively, r_{ij} denotes the probability that player j is offered his continuation payoff when player i is the proposer who proposes an allocation that will be accepted.

If player i offers an allocation in state s that is accepted, this allocation yields the payoff $s - \sum_{j \neq i} r_{ij} v_j$ to the proposer and it yields the expected payoff $r_{ij} v_j$ to player j. If no proposal is accepted in state s, then all the players receive their continuation payoffs.

Given these restrictions on the SSP strategies, a payoff maximizing proposer obtains a payoff of $s - \sum_{j \neq i} r_{ij} v_j$ from any SSP proposal that is accepted. But the proposer can also guarantee himself v_i by passing. Then the proposer offers an allocation that will be accepted if

$$s - \sum_{j \neq i} r_{ij} v_j > v_i,$$

passes if

$$s - \sum_{j \neq i} r_{ij} v_j < v_i,$$

and is indifferent between proposing an allocation that will be accepted and passing if

$$s - \sum_{j \neq i} r_{ij} v_j = v_i.$$

Let $\alpha_i(s)$ denote the probability that player i proposes an allocation that will be accepted in state s. Then $\alpha_i(s)$ must satisfy the restrictions imposed in equation (3). Note that while it is sensible to assume that a player accepts a proposal that makes him indifferent between accepting and rejecting, there is no natural argument to break the proposer's indifference between proposing and passing.

In equilibrium, the offer probabilities r_{ij} and proposal probabilities α_i must induce the continuation payoffs v, that is, $v = \beta E[v]$. Note that in this expression the expectation is taken over the recognition probabilities as well as next period's state. Next we show that this is satisfied by equation (1).

Let s denote the next period's state. With probability p_i, player i is the proposer next period. With probability $\alpha_i(s)$ player i proposes an allocation that will be accepted in which case his payoff is $s - \sum_{j \neq i} r_{ij} v_j$. With probability $1 - \alpha_i(s)$ player i passes and receives his continuation payoff v_i. Thus, conditional on being the proposer, next period's expected payoff for player i discounted back to the current period is

$$\beta p_i[\alpha_i(s)(s - \sum_{j \neq i} r_{ij} v_j) + (1 - \alpha_i(s))v_i] \tag{5}$$

Now consider the case when player i is not the proposer next period. With probability p_j, player $j \neq i$ is the proposer. Player j proposes an allocation that will be accepted with probability $\alpha_j(s)$ in which case the expected payoff to player i is $r_{ji}v_i$. With probability $1-\alpha_j(s)$ player j passes in which case player i receives his continuation payoff v_i. Thus, conditional on not being the proposer, next period's expected payoff for player i discounted back to the current period is

$$\beta \sum_{j \neq i} p_j[\alpha_j(s) r_{ji} v_i + (1-\alpha_j(s))v_i] \tag{6}$$

By (5) and (6) the continuation payoff for player i is given by equation (1).

To complete the proof consider the following strategy. When player i is not the proposer, he accepts a proposal if and only if the proposal gives him at least v_i. When player i is the proposer in state s, he proposes an allocation with probability $\alpha_i(s)$ and passes with probability $1-\alpha_i(s)$. If he proposes an allocation, the allocation he proposes is (x_i, x_j) with probability $\pi(\gamma_i)$, where

$$x_i = s - \sum_{j \neq i} \gamma_{ij} v_j$$

and for all $j \neq i$

$$x_j = \gamma_{ij} v_j,$$

and $\pi_i(.)$ is the probability distribution on Γ_i that induces the offer probabilities r_{ij}. Clearly, this strategy implements the payoffs given by (1) and no player has an incentive to unilaterally deviate from it. ■

Our characterization of the SSP payoffs is based on the observation that, if agreement is reached in any period of the q-quota game, the proposer may extract any surplus over what the "cheapest" $q-1$ other players obtain by delaying agreement until the next period.[15] Note that for any player $i \in N$ and for any state $s \in S$, $\alpha_i(s)$ is the probability player i makes a proposal in state s, and $(r_{ij})_{j=1}^{n}$ is the vector of probabilities player i offers their continuation payoff to any other player. If a proposal is made in any period, it is accepted. If no proposal is made in a period, then all the players receive their continuation payoff.

For any $v \in I\!\!R^n$ define $r(v;q) = r_1(v;q) \times \ldots \times r_n(v;q)$ where $r_i(v;q)$ is the set of minimizers to the program defined in (2). Let $\alpha(r,v) = \alpha_1(r,v) \times \ldots \times \alpha_n(r,v)$ where $\alpha_i(r,v)$ is the set of proposal probabilities that satisfy equation (3). Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $r = [r_{ij}]$, define the mapping $A(.;\alpha,r) : I\!\!R^n \to I\!\!R^n$ as

$$\begin{aligned} A_i(v;\alpha,r) \;=\; \beta \int \Bigg\{ & p_i[\alpha_i(s)(s - \sum_{j \neq i} r_{ij} v_j) + (1-\alpha_i(s))v_i] \\ & + \sum_{j \neq i} p_j[\alpha_j(s) r_{ji} v_i + (1-\alpha_j(s))v_i] \Bigg\} dF(s), \end{aligned} \tag{7}$$

for all $i \in N$.

[15]In the proof, we impose the restriction that a player votes in favor of a proposal whenever he is indifferent between accepting the proposal and rejecting it. This restriction guarantees that the proposer's maximization problem is always well defined. Note, however, that the restriction binds only in those states where the proposer is indifferent between proposing an allocation and passing.

Let $C = \{v \in I\!R^n_+ : \sum_{i=1}^n v_i \leq \overline{s}\}$ and define the set-valued mapping $T(.;q)$ on C as

$$\begin{aligned} T(v;q) &= \{g \in I\!R^n : \exists r \in r(v;q), \exists \alpha \in \alpha(r,v) \\ &\quad \textit{such that } g = A(v;\alpha,r)\}. \end{aligned} \tag{8}$$

Note that the operator T is indexed by q since $r(.)$ is indexed by q while $\alpha(.)$ and $A(.)$ are independent of q (although, in equilibrium, they depend on q through r). The next theorem provides an alternative characterization of the SSP payoffs that will be useful in the rest of the paper.

Theorem 2. *v is an SSP payoff vector for the q-quota game if and only if it is a fixed point of the set-valued mapping $T(v;q)$, that is $v \in T(v;q)$.*

Proof: The result follows immediately from Theorem 1 and the definition of $T(.;q)$. ■

We can now establish the existence of SSP payoffs.

Theorem 3. *There exists an SSP payoff vector.*

Proof: First note that $T(.;q)$ maps C to non-empty subsets of C. It is easily seen that $T(v;q)$ is convex for all v since $r(.;q)$ and $\alpha(.;q)$ are convex valued. Furthermore, $T(.;q)$ is upper semi-continuous since $r(.;q)$ and $\alpha(.;q)$ are upper semi-continuous and

A is continuous in v, α and r. Finally, for all $v \in C$, $T(v; q)$ is a closed subset of the compact set C and hence, $T(v; q)$ is compact. Thus, the result follows from Kakutani Fixed Point Theorem. ■

Theorem 3 proves the existence of an SSP payoff vector in mixed strategies where the proposer is allowed to randomize over the selection of his coalition partners. Unlike under the unanimity rule (Merlo and Wilson [13]), under general q-quota agreement rules an SSP payoff in pure strategies does not necessarily exist. To illustrate this point consider the following example. There are 3 players, a deterministic surplus of size 1, the discount factor is $\beta = 0.95$, the proposer selection probabilities are $p_1 = 0.2$, $p_2 = 0.3$ and $p_3 = 0.5$, and the agreement rule is majority rule. In this example, the unique SSP payoff vector is $v = (0.328, 0.328, 0.344)$. In equilibrium, player 1 always offers to player 2, player 2 always offers to player 1, and player 3 randomizes (he offers to player 1 with probability 0.62 and to player 2 with probability 0.38). It is easy to verify that no equilibrium in pure strategy exists in this example.

4. Non Uniqueness of SSP Payoffs

In this section, we address the issue of uniqueness of the SSP payoffs. For the class of games we consider here, when $q = n$ (i.e., in the unanimity game), Merlo and Wilson [13] show that the SSP payoff is unique. The same result holds for any q-quota game where the surplus to be divided is constant over time (Baron and Ferejohn [3] and Eraslan [7]). We show that in general, for stochastic q-quota bargaining games the SSP payoff need not be unique. To illustrate this point we present an example where in a game with majority rule there are multiple equilibrium payoffs even when the players are symmetric, that is, $p_i = 1/n$ for all $i \in N$ (the case considered by Baron and Ferejohn [3]).

Before presenting the example, we first provide a simpler characterization of the set of SSP payoffs for the case where players are symmetric. It is easy to show that when all players are equally likely to be selected as proposer they have the same SSP payoff.[16] Let v denote the (common) SSP payoff to a player. For any $v \in [0, \bar{s}/n]$, let α_v denote the set of proposal probabilities that satisfy

$$\alpha(s) = \begin{cases} 1 & \text{if } s > qv \\ 0 & \text{if } s < qv \end{cases} \tag{9}$$

Define the mapping $\bar{T}(.;q) : [0, \bar{s}/n] \to [0, \bar{s}/n]$ as

$$\begin{aligned} \bar{T}(v;q) &= \Bigg\{ g \in [0, \bar{s}/n] : \exists \alpha \in \alpha_v \textit{ such that} \\ & \qquad g = \beta \int [\alpha(s)\frac{s}{n} + (1-\alpha(s))v] dF(s) \Bigg\}. \end{aligned}$$

Lemma 1. *When the players are symmetric, v is an SSP payoff for the q-quota game if and only if $v \in \bar{T}(v;q)$.*

[16] This result follows from the fact that for any two players i and j, $p_i \leq p_j$ implies that $v_i \leq v_j$. The proof of this intuitive result is omitted and is available from the authors upon request.

Proof: First suppose that v is an SSP payoff for the q-quota game. Observe that when the players are symmetric, the optimized value of the objective function of program (2) for player i (that is, the sum of the shares of surplus player i has to give to his coalition partners in order to induce acceptance of his proposal) is equal to $(q-1)v$ for all $i \in N$ regardless of the offer probabilities. Thus, the proposal probabilities $\alpha_i(s)$ do not depend on the proposer and can be written as in (9). In other words, even if the players do not employ symmetric strategies, the proposal probabilities satisfy (9) and the right hand side of (9) does not depend on the identity of the proposer. For any v, let $\bar{\alpha}(s) = \sum_{i=1}^n p_i \alpha_i(s)$. Then $\bar{\alpha}(s)$ also satisfies (9). To see that $v \in \bar{T}(v;q)$ note that since v is an SSP payoff for the q-quota game, by Theorem 2, $v \in T_i(ve;q)$ for all i, where e is the n-dimensional unit vector and $T_i(.;.)$ is the ith component of the set-valued mapping T defined in (8) above. Thus, $v \in (\sum_{i=1}^n T_i(ve;q))/n$. But $(\sum_{i=1}^n T_i(ve;q))/n = \bar{T}(v;q)$. Thus, $v \in \bar{T}(v;q)$.

Next suppose $v \in \bar{T}(v;q)$. Then there exists α satisfying (9) such that

$$v = \beta \int [\alpha(s)\frac{s}{n} + (1-\alpha(s))v]dF(s).$$

Let $\underline{\alpha} = (\alpha, \ldots, \alpha)$. It suffices to find SSP offer probabilities $r \in r(v;q)$ such that $v = A_i(ve;\underline{\alpha}, r)$. Let $r_{ij} = (q-1)/(n-1)$ for all i and for all $j \neq i$. It is straightforward to verify that $r \in r(v;q)$ and $v = A_i(ve;\underline{\alpha}, r)$. Since, as explained above, the proposal probabilities do not depend on r it is clear that $v \in T_i(ve;q)$ for all i. Thus, v is an SSP payoff for the q-quota game by Theorem 2. ■

Consider the following example. There are 3 players with equal proposer selection probabilities $p_i = 1/3$, for all $i \in N$. The common discount factor β is equal to 0.99. There are two possible sizes of the surplus, $S = \{1, 2\}$. Each state is realized with equal probability, $\Pr[s = 1] = \Pr[s = 2] = 0.5$.

Note that in equilibrium agreement always occurs when the large level of surplus is realized. Then, there are three possible outcomes to consider: (i) agreement occurs on the large surplus level only; (ii) agreement occurs on both surplus levels; or (iii) when the small level of surplus is realized agreement occurs with some positive probability (not equal to one). Let v', v'', and v''' denote the payoffs corresponding to each of these three outcomes, respectively. We show that when the agreement rule is majority rule (i.e., $q = 2$) all three outcomes can occur in equilibrium.

Consider first v'. By Lemma 1, for any $q \in \{1, 2, 3\}$, v' is an SSP payoff for the q-quota game, if and only if

$$v' = 0.99[\frac{1}{2}v' + \frac{1}{2}\frac{2}{3}], \tag{10}$$

$$qv' < 2, \tag{11}$$

and

$$qv' > 1. \tag{12}$$

From (10) we obtain $v' = 0.653$. It is easy to see that the inequality in (11) is satisfied for all q while the inequality in (12) is violated for $q = 1$. Hence, v' is an SSP payoff for the q-quota game if and only if $q \geq 2$. The proposer's SSP strategy that induces this payoff is such that $\alpha(1) = 0$, $\alpha(2) = 1$, and, for example, $r_{ij} = 0.5$ for all $i, j = 1, 2, 3$ and $i \neq j$.

Next consider v''. By Lemma 1, v'' is an SSP payoff for the q-quota game, if and only if

$$v'' = 0.99[\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{2}{3}], \tag{13}$$

$$qv'' < 2, \tag{14}$$

and

$$qv'' < 1. \tag{15}$$

Solving (13) we obtain $v'' = 0.495$, which implies that the inequality in (14) is satisfied for all q, while the inequality in (15) is satisfied only if $q \leq 2$. Hence, v'' is an SSP payoff for the q-quota game if and only if $q \leq 2$. The proposer's SSP strategy that induces this payoff is such that $\alpha(1) = 1$, $\alpha(2) = 1$, and, for example, $r_{ij} = 0.5$ for all $i, j = 1, 2, 3$ and $i \neq j$.

Finally, observe that, by Lemma 1, v''' is an SSP payoff for the q-quota game, if and only if there exists an $\alpha \in [0, 1]$ such that

$$v''' = 0.99[\frac{1}{2}(\alpha\frac{1}{3} + (1-\alpha)v''') + \frac{1}{2}\frac{2}{3}], \tag{16}$$

$$qv''' < 2, \tag{17}$$

and

$$qv''' = 1. \tag{18}$$

From (18) we obtain $v''' = 1/q$, which implies that the inequality in (17) is satisfied for all q. Also, it is easy to see that the value of α such that (16) holds is in the unit interval only if $q = 2$. Hence, v''' is an SSP equilibrium payoff for the q-quota game if and only if $q = 2$. In this equilibrium, $v''' = 0.5$. The proposer's SSP strategy that induces this payoff is such that $\alpha(1) = 0.94$, $\alpha(2) = 1$, and, for example, $r_{ij} = 0.5$ for all $i, j = 1, 2, 3$ and $i \neq j$.

This example illustrates that while in the unanimity game (i.e., when $q = 3$) there is a unique equilibrium payoff, in the game where agreement requires approval of a simple majority (i.e., when $q = 2$) there are three equilibrium payoffs. Similar examples can be constructed where multiple SSP payoffs arise for any q-quota game (with $q < n$).

The intuition for this result is as follows. In the unanimity game, agreement occurs only when the large level of surplus is realized. In this game, since all players have veto power, they all have a common interest to maximize the expected surplus to be allocated and hence delay agreement until the large level of surplus is realized. Under other agreement rules, when agreement only requires approval of $q < n$ players, in any equilibrium only q players receive a positive share of the surplus. This creates an asymmetry between the players who are included in a proposal (i.e., those who are allocated a positive share) and the players who are excluded (i.e., those who are allocated a zero share). This asymmetry generates an incentive for players who are offered a positive share of the surplus today to agree even if the current level of surplus is small, since some of them (perhaps all) may be excluded from future agreements. Knowing that his proposal would be accepted, the same argument may then induce a proposer to make a proposal when the small level of surplus is realized.

Note that in the previous example two of the equilibria with majority rule are *ex-ante* inefficient. In fact, when $q = 1$ the unique equilibrium in a q-quota game is *ex-ante* inefficient. We turn our attention to efficiency and to comparing the equilibrium outcomes of games with different agreement rules next.

5. Comparing Agreement Rules

When players are asymmetric with respect to their probability of being selected as proposer, the comparison of individual equilibrium payoffs across games with different agreement rules is uninformative. This point can easily be illustrated through an example given in Baron and Ferejohn [3], where the level of surplus is fixed and equal to one, and there are three players with discount factor equal to 0.8 and with proposer selection probabilities equal to 0.2, 0.35, and 0.45, respectively. In this example, the unique SSP payoff vector of the q-quota game is $v = (0.2, 0.35, 0.45)$ when $q = 1$ or $q = 3$, and is $v = (1/3, 1/3, 1/3)$ when $q = 2$. Moreover, as illustrated in the previous section, when the level of surplus is stochastic there can be multiple equilibria depending on the agreement rule even when the proposer selection probabilities are the same for all players. This makes the comparison of equilibrium payoffs across games with different agreement rules problematic.

In this section, we show that the SSP total payoff (defined as the sum of the individual SSP payoffs) in any equilibrium of a q-quota game where $q < n$ is never larger than the SSP total payoff in the unique equilibrium of the unanimity game. This inequality is strict if the level of surplus in any period is a random draw from a continuous density over the support $S = [0, \bar{s}]$.

Let $w \in [0, \bar{s}]$ denote an SSP total payoff for the q-quota game if there is an SSP payoff v for the q-quota game such that $w = \sum_{i=1}^{n} v_i$. Recall that $C = \{v \in I\!R_+^n : \sum_{i=1}^{n} v_i \leq \bar{s}\}$. Define the set-valued mapping $H(.;q)$ on $[0, \bar{s}]$ as

$$\begin{aligned} H(w;q) \;\; = \;\; & \Bigg\{ g \in [0,\bar{s}] : \exists v \in C \;\; \textit{such that} \;\; w = \sum_{i=1}^{n} v_i \;\; \textit{and} \\ & \exists r \in r(v;q), \exists \alpha \in \alpha(r,v) \quad \textit{such that} \\ & g = \beta \int [s \sum_{i=1}^{n} \alpha_i(s) p_i + w(1 - \sum_{i=1}^{n} \alpha_i(s) p_i)] dF(s) \Bigg\}. \end{aligned} \tag{19}$$

Theorem 4. *If w is an SSP total payoff for the q-quota game then $w \in H(w;q)$.*

Proof: Suppose w is an SSP total payoff for the q-quota game. Then by definition there exists an SSP payoff v for the q-quota game such that $w = \sum_{i=1}^{n} v_i$. By Theorem 2, $v \in T(v;q)$. Thus, there exists $r \in r(v;q)$ and $\alpha \in \alpha(r,v)$ such that $v_i = A_i(v;\alpha,r)$. Hence, $w = \sum_{i=1}^{n} A_i(v;\alpha,r)$ which in turn implies that $w \in H(w;q)$. ■

Note that when $q = n$ (i.e., the agreement rule is unanimity), the equilibrium offer probabilities must satisfy $r_{ij} = 1$ for all i and for all $j \neq i$. This implies that for any $v \in C$, $r(v; n)$ is a singleton. Hence, for all $i \in N$, the set of SSP proposal probabilities must satisfy

$$\alpha_i(s) = \begin{cases} 1 & \text{if } s > \sum_{j=1}^{n} v_j \\ 0 & \text{if } s < \sum_{j=1}^{n} v_j \end{cases} \tag{20}$$

This implies that the set of agreement states does not depend on the proposer except when $s = \sum_{j=1}^{n} v_j$. In this case, however, the total payoff does not depend on whether the proposer passes or proposes an allocation that will be accepted. Thus, abusing notation, we can write $H(w; n)$ as

$$H(w; n) = \beta \int \max\{s, w\} dF(s), \tag{21}$$

which is the same operator used by Merlo and Wilson [13] to characterize the SSP total payoffs of stochastic games with unanimity rule.

Theorem 5. *If $w \in H(w; n)$ then w is the unique SSP total payoff for the unanimity game.*

Proof: It can be verified that $H(.; n)$ is a contraction mapping (see Merlo and Wilson [13]). Thus, $H(.; n)$ has a unique fixed point, say w. If w is not an SSP total payoff for the unanimity game, there exists an SSP total payoff w' for the unanimity game such that $w \neq w'$. By Theorem 4, $w' = H(w'; n)$ which is a contradiction. ■

Note that for the unanimity game the operator $H(.;n)$ fully characterizes the SSP total payoff. For a game with a general q-quota agreement rule this is not the case. In fact, it is possible to have $w \in H(w;q)$ even though w is not an SSP total payoff for the q-quota game.

Theorem 6. *Let w^n be the SSP total payoff for the unanimity game and let w^q be an SSP total payoff for the q-quota game. Then $w^n \geq w^q$. Furthermore, if the level of surplus in any period is a random draw from a continuous density over the support $S = [0, \bar{s}]$, then $w^n > w^q$.*

Proof: First note that, for any $f \in [w^q, \bar{s}]$,

$$H(f;n) = \beta \int \max\{s, f\} dF(s) \geq \beta \int \max\{s, w^q\} dF(s) \geq w^q.$$

To see the last inequality is true observe the following. Since w^q is an SSP total payoff for the q-quota game, there exists SSP proposal probabilities α_i such that

$$w^q = \beta \int [s \sum_{i=1}^{n} \alpha_i(s) p_i + w^q(1 - \sum_{i=1}^{n} \alpha_i(s) p_i)] dF(s).$$

Note that $s \geq w^q$ implies $\alpha_i(s) = 1$ for all i while the converse is not true. In particular, it is possible to have $\max\{s, w^q\} = w^q > s$ while $\sum_{i=1}^{n} \alpha_i(s) p_i > 0$ by the definition of SSP proposal probabilities. Hence, $\max\{s, w^q\} \geq s \sum_{i=1}^{n} \alpha_i(s) p_i + w^q(1 - \sum_{i=1}^{n} \alpha_i(s) p_i)$.

Next note that $H(.;n)$ is a contraction that maps $[w^q, \bar{s}]$ to itself. Thus $H(.;n)$ has a fixed point in this set. But w^n is the unique fixed point of $H(.;n)$ and hence $w^n \geq w^q$. When the level of surplus in any period is a random draw from a continuous density over the support $S = [0, \bar{s}]$, it can be verified that $H(w^q;n) > w^q$. Since $w^n \geq w^q$, and w^q is not a fixed point of $H(.;n)$ it must be the case that $w^n > w^q$. ■

For the class of games we consider, Merlo and Wilson [13] show that the unique stationary subgame perfect equilibrium of the unanimity game is *ex-ante* efficient. Theorem 6 implies that the equilibrium outcomes of a stochastic bargaining game with a q-quota rule may be *ex-ante* inefficient. An interesting feature of this result is that the inefficiency arises because players may reach an agreement "too soon". This phenomenon is due to the fact that, *ceteris paribus*, it is "cheaper" for a proposer to obtain the votes of q players (including himself) rather than n players. Furthermore, *ceteris paribus*, each player has a lower continuation payoff when agreement requires less than unanimous approval, because of the possibility of being excluded from future agreements. These two factors may induce agreements to occur on "sub-optimal" levels of surplus.

6. Concluding Remarks

It should be clear from our analysis that the source of multiplicity and inefficiency of equilibria in q-quota stochastic bargaining games derives from a limitation on the set of contracts agents are allowed to sign. If agents have a complete set of contingent contracts at their disposal, then none of these issues would arise (see Merlo and

Wilson [12]). In particular, in the specific context we are considering, to guarantee uniqueness and efficiency of the equilibrium it would be sufficient to allow agents to sign binding contracts specifying the composition of the coalition they restrict themselves to bargain with over time. While this option is certainly interesting, we believe there are many real bargaining situations where such contracts are not available.

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