Department of Economics
University of Pennsylvania

Preliminary Examination
Microeconomic Theory I – Suggested Solutions

Exam Date: August 7, 2017

Today's Date: August 9, 2017

Instructions

This exam has four questions and is worth 100 points.

Answer each question in a **SEPARATE** exam book.

If you need to make additional assumptions, state them clearly.

Be concise.

Write clearly.

Use WORDS – NOT math only!

Good luck!

1. (25 pts) Suppose a locally nonsatiated utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), \ldots, x_n(\mathbf{p}, y))$ defined on $\mathbb{R}^{n+1}_{++}$ ($\mathbf{p}$ is the price vector, y is income). Assume it and any other functions you use to answer this question are twice continuously differentiable.

   (a) (8 pts) State four properties this demand function must satisfy.

   Soln:

   i. It must be homogeneous of degree zero:

$$\mathbf{x}(t\mathbf{p}, ty) = \mathbf{x}(\mathbf{p}, y) \; \forall t > 0, \; (\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}.$$

   ii. It must satisfy the budget constraint with equality:

$$\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y \; \forall (\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}.$$

   iii. The Slutsky matrix $\mathbf{S} = [s_{ij}(\mathbf{p}, y)]$ must be negative semidefinite and symmetric at any $(\mathbf{p}, y)$, where

$$s_{ij}(\mathbf{p}, y) := \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}.$$

   (This can count as two properties, NSD and symmetry.)

   iv. The kernel (null space) of the Slutsky matrix must contain the price vector, i.e., the Slutsky matrix maps the price vector into the origin: $\mathbf{S}(\mathbf{p},y)\mathbf{p} = \mathbf{0}$.

$\blacksquare$

   (b) (17 pts) For each property you listed in (a), sketch a proof of why it must be satisfied.

   Soln:

   i. Multiplying all prices and income by the same positive constant does not change the constraint set in the consumer problem, and even more obviously does not change its objective function, u. Thus, $\mathbf{x}$ is a solution for the budget $(\mathbf{p}, y)$ iff it is a solution for the budget $(t\mathbf{p}, ty)$ for any $t > 0$.

   ii. Fix $(\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}$, and let $\mathbf{x} = \mathbf{x}(\mathbf{p}, y)$. Since $\mathbf{x}$ solves the consumer problem, it satisfies its constraint, i.e., $\mathbf{p} \cdot \mathbf{x} \leq y$. Suppose this inequality holds strictly. Then a neighborhood N of $\mathbf{x}$ exists such that $\mathbf{p} \cdot \mathbf{x}' < y$ for all $\mathbf{x}' \in N$. Because u is locally nonsatiated, $\mathbf{x}' \in N \cap \mathbb{R}^n_+$ exists such that $u(\mathbf{x}') > u(\mathbf{x})$. Thus, $\mathbf{x}'$ is feasible and affordable at $(\mathbf{p}, y)$, and so the fact that it is strictly preferred to $\mathbf{x}$ implies the contradiction that $\mathbf{x}$ does not solve the consumer problem at $(\mathbf{p}, y)$.

   iii. Recall the expenditure function,

$$e(\mathbf{p}, \bar{u}) := \min_{\mathbf{x} \geq 0} \mathbf{p} \cdot \mathbf{x} \text{ such that } u(\mathbf{x}) \geq \bar{u}.$$

   The solution to this minimization problem is the Hicksian demand function, $\mathbf{h}(\mathbf{p}, \bar{u})$. Since $e(\cdot, \bar{u})$ is the lower envelope of a bunch of affine functions of $\mathbf{p}$, it is a concave function of $\mathbf{p}$. Hence, since we've been told all functions in this problem are C^2, the matrix of cross-partials,

$$\left[\frac{\partial^2 e(\mathbf{p}, \bar{u})}{\partial p_i \partial p_j}\right],$$

exists and is negative semidefinite by the concavity of e in $\mathbf{p}$, and symmetric by Young's theorem. By the envelope theorem, $\partial e(\mathbf{p}, \bar{u})/\partial p_i = h_i(\mathbf{p}, \bar{u})$. Hence,

$$\left[\frac{\partial h_i(\mathbf{p}, \bar{u})}{\partial p_j}\right] = \left[\frac{\partial^2 e(\mathbf{p}, \bar{u})}{\partial p_i \partial p_j}\right],$$

which tells us that $\left[\frac{\partial h_i(\mathbf{p}, \bar{u})}{\partial p_j}\right]$ is negative semidefinite and symmetric at any $(\mathbf{p}, \bar{u})$. Since the Slutsky equation is in fact the equation

$$s_{ij}(\mathbf{p}, y) = \frac{\partial h_i(\mathbf{p}, u^*)}{\partial p_j},$$

where $u^* = u(\mathbf{x}(\mathbf{p}, y))$, we conclude that $[s_{ij}(\mathbf{p}, y)]$ is indeed negative semidefinite and symmetric.

iv. Since $h_i(\mathbf{p}, \bar{u})$ is homogeneous of degree zero in $\mathbf{p}$, it satisfies Euler's formula. Hence, for each i we have

$$\sum_j s_{ij} p_j = \sum_j \frac{\partial h_i}{\partial p_j} p_j = 0.$$

■

2. (25 pts) Mr. 1 has a complete and transitive preference ordering $\succeq_1$ over monetary lotteries that is monotone in the following sense: $\delta_x \succ_1 \delta_y$ for all $x > y$, where δ_x and δ_y are the degenerate lotteries that put probability one on the amounts x and y, respectively. Assume Mr. 1 is strictly risk averse, and let $\tilde{x}$ be a given non-degenerate monetary lottery.

(a) (5 pts) Let c_1 be Mr. 1's certainty equivalent for $\tilde{x}$. What is the relationship between c_1 and $\mathbb{E}\tilde{x}$? Prove your answer.

Soln: $\mathbb{E}\tilde{x} > c_1$.

Proof. By definition of strict risk aversion, we have

$$\delta_{\mathbb{E}\tilde{x}} \succ_1 \tilde{x}. \tag{1}$$

By definition of certainty equivalent, we have

$$\tilde{x} \sim_1 \delta_{c_1}. \tag{2}$$

The transitivity of $\succeq_1$, together with (1) and (2), implies $\delta_{\mathbb{E}\tilde{x}} \succ_1 \delta_{c_1}$. Hence, by the monotonicity of $\succeq_1$ we have $\mathbb{E}\tilde{x} > c_1$. ■

Now assume Mr. 1 satisfies the Expected Utility Hypothesis, and his Bernoulli utility function is u_1. Ms. 2 similarly has a Bernoulli utility function u_2. Both functions are twice differentiable, with $u_1'(x) > 0$ and $u_2'(x) > 0$ for all $x \in \mathbb{R}$. Let $A_i(x)$ denote the coefficient of absolute risk aversion of u_i, and assume $A_1(x) > A_2(x)$ for all $x \in \mathbb{R}$.

(b) (10 pts) Show that there exists a strictly concave increasing function $h : u_2(\mathbb{R}) \to \mathbb{R}$ such that $u_1 = h \circ u_2$.

Soln: An amount x gives utility $u_2(x)$ to 2 and utility $u_1(x)$ to 1. The function h that we seek maps the former into the latter:

$$h(u_2(x)) = u_1(x). \tag{3}$$

Now change variables from x to $v = u_2(x)$. As u_2 is strictly increasing it has a well-defined inverse, u_2^{-1}. Hence, this change of variables transforms (3) to

$$h(v) := u_1(u_2^{-1}(v)),$$

which is the definition of h on the domain $u_2(\mathbb{R})$.

To show that h is strictly increasing and strictly concave, first differentiate (3) with respect to x to obtain

$$h' u_2' = u_1'. \tag{4}$$

Since $u_1' > 0$ and $u_2' > 0$, this implies $h' > 0$, and so h is strictly increasing. Differentiating again yields

$$h''(u_2')^2 + h' u_2'' = u_1'',$$

which in light of (4) is

$$h''(u_2')^2 + \frac{u_1'}{u_2'} u_2'' = u_1''.$$

Substitute $-A_i u_i'$ for each u_i'' and rearrange to obtain

$$h'' = \frac{u_1'}{(u_2')^2}(A_2 - A_1) < 0,$$

which proves h is strictly concave. ∎

(c) (10 pts) Show that $c_1 < c_2$, where c_1 and c_2 are their certainty equivalents for $\tilde{x}$.

Soln: We have $u_1(c_1) = \mathbb{E}u_1(\tilde{x})$. This and $u_1 = h \circ u_2$ (from (b)) yield

$$h(u_2(c_1)) = \mathbb{E}h(u_2(\tilde{x})).$$

The strict concavity of h, the non-degeneracy of $\tilde{x}$, the strict monotonicity of u_2, and Jensen's inequality imply

$$\mathbb{E}h(u_2(\tilde{x})) < h(\mathbb{E}u_2(\tilde{x})).$$

By the definition of c_2,

$$h(\mathbb{E}u_2(\tilde{x})) = h(u_2(c_2))$$

These three displays yield $h(u_2(c_1)) < h(u_2(c_2))$. Hence, as $h \circ u_2$ is a strictly increasing function, we have $c_1 < c_2$. ∎

3. (25 pts) Three farmers, $i = 1, 2, 3$, grow corn along a river that is subject to flooding and is protected by a dyke that protects the adjacent land from flooding when the river gets too high. In the absence of flooding, each farmer's crop will be 100 units of corn. There has been very heavy rain and it is known that the dyke will be breached tomorrow (date $t = 1$), flooding exactly one of the farms and ruining its crop. There are thus three states of the world tomorrow, $s = 1, 2, 3$: in state s the farm of farmer $i = s$ is flooded. For each farmer $i = 1, 2, 3$, let ω^i denote his initial endowment vector of state-contingent crop, so that

$$\omega^1 = (0, 100, 100), \quad \omega^2 = (100, 0, 100), \quad \omega^3 = (100.100, 0).$$

Today (date $t = 0$) the farmers arrange for how the the corn that is harvested tomorrow (date $t = 1$) will be shared in each state. The utility function of farmer i is

$$U^i(x^i) = \Sigma_{s=1}^3 \pi_s^i u^i(x_s^i),$$

where x_s^i is his consumption of corn in state s, and π_s^i is his belief probability that state s will occur. Assume u^i is continuously differentiable, strictly concave, and strictly increasing.

(a) (8 pts) Suppose the farmers agree that the state probabilities are $(\pi_1, \pi_2, \pi_3) \gg \mathbf{0}$. Show that in any interior Pareto efficient allocation, farmer 1 will consume the same amount of corn regardless of whose farm is flooded.

Soln: Let x be any interior allocation. Suppose $x_s^1 > x_{s'}^1$ for some $s \neq s'$. Then, since the total amount of corn in each state is the same, for some $i \neq 1$ we have $x_s^i < x_{s'}^i$. The strict concavity of u^1 and u^i now implies that

$$\frac{\pi_s u^{1\prime}(x_s^1)}{\pi_{s'} u^{1\prime}(x_{s'}^1)} < \frac{\pi_s}{\pi_{s'}} < \frac{\pi_s u^{i\prime}(x_s^i)}{\pi_{s'} u^{i\prime}(x_{s'}^i)}.$$

Thus, the marginal rates of substitution of farmers 1 and i between corn in states s and s' are not equal. Since they must be equal if x were to be Pareto efficient, this proves x is not Pareto efficient. ■

For the remaining parts, assume instead that each farmer is sure that his farm will not be flooded, and believes that it is equally likely that the other two farms will be flooded: $\pi_i^i = 0$ for each i and $\pi_s^i = 1/2$ for $s \neq i$.

(b) (8 pts) Prove that if (x^{1*}, x^{2*}, x^{3*}) is a Pareto efficient allocation, then $x_i^{i*} = 0$ for all i.

Soln: Farmer i is positive that state i will not arise. Hence, reducing his corn consumption in state i does not decrease his utility, but giving it to one of the other two farmers in state i increases that farmer's utility since he puts positive probability on state i. Hence any allocation that gives farmer i corn contingent on state i is Pareto dominated. ■

(c) (9 pts) Find a competitive equilibrium price vector (p_1, p_2, p_3), where p_s is the price at date 0 for contingent corn to be consumed at date 1 in state s.

Soln: $(p_1, p_2, p_3) = (p, p, p)$ for any $p > 0$.

Proof. The logic of part (a) and the answer to part (b) indicate that the initial endowment is Pareto efficient. In addition, it is easily shown that there is no other feasible allocation that does not make at least one farmer worse off than the endowment allocation does. Hence, the endowment is the only possible equilibrium allocation. Assuming it is, the equilibrium price ratios must be determined by the corresponding MRS's. For example, as farmer 1 consumes his endowment in states 2 and 3, we have $x_2^1 = x_3^1 = 100 > 0$, and so

$$\frac{p_2}{p_3} = \frac{\frac{1}{2}u^{1\prime}(100)}{\frac{1}{2}u^{1\prime}(100)} = 1.$$

The analogous calculation for farmer 2 and states 1 and 3 shows that $p_1 = p_3$. Hence, $(p_1, p_2, p_3) = (p, p, p)$ for any $p > 0$. To verify that any such price vector is an equilibrium price vector, simply note (show) that ω^i solves farmer i's consumer problem given these prices. For example, farmer 1's problem reduces (since obviously he sets $x_1^1 = 0$) to

$$\max_{x_2^1, x_3^1} \tfrac{1}{2}u^1(x_2^1) + \tfrac{1}{2}u^1(x_3^1) \text{ s.t. } px_2^1 + px_3^1 \leq 200p.$$

The solution to this, given that u^1 is strictly concave and increasing, is $x_2^1 = x_3^1 = 100$, his endowment quantities. ■

4. (25 points) Walrasian equilibrium with production.

   (a) (8 pts) State precisely the definition of a Walrasian equilibrium for an economy with production.

   (b) (8 pts) Given standard assumptions on preferences and interior endowments, what conditions on the production technology are sufficient for a Walrasian equilibrium with production to exist? (Little if any credit will be given for trivial conditions such as "the production set is empty".)

   Soln: The production set Y must be closed, convex, and satisfy $Y \cap -Y = 0$. ■

   (c) (9 pts) Give an example in which one of the conditions on the technology you gave in (b) is not satisfied, and a Walrasian equilibrium nonetheless exists. A graphic example carefully done is sufficient.

   Soln: The easiest examples would have nonconvex production sets. A simple example would be to take a two good economy with a Walrasian equilibrium and make the production set nonconvex in an irrelevant region. ■