Microeconomic Theory II
Preliminary Examination Solutions

Exam date: August 7, 2017

1. Sheila moves first and chooses either H or L. Bruce receives a signal, h or l, about Sheila's behavior. The distribution of the signal is $\Pr(h|H) = p$, $\Pr(l|H) = 1-p$, $\Pr(h|L) = q$, $\Pr(l|L) = 1-q$, with $p > \frac{1}{2} > q$. After observing the signal, Bruce chooses either A or B. Payoffs are:

Action profile	Sheila's payoff	Bruce's payoff
HA	5	2
HB	2	1
LA	6	1
LB	4	2

(a) Suppose $p = 1$ and $q = 0$. Note that for these values of p and q, the signals are *perfectly* informative about Sheila's actions (i.e., this is equivalent to Burce observing Sheila's actions). What is the normal form of this game? What are the pure strategy Nash equilibria? **[10 points]**

Solution: Bruce's strategy space is $\{AA, AB, BA, BB\}$, where the first letter is Bruce's action choice if he observes h and the second is his choice if he observes l. Note that it is not enough just to list the pairs, you must also specify what the pairs mean. The normal form is

Bruce

Sheila		AA	AB	BA	BB
	H	$5,2$	$5,2$	$2,1$	$2,1$
	L	$6,1$	$4,2$	$6,1$	$4,2$

The pure strategy equilibria are (H, AB) and (L, BB). Note that in this case, observing the signal is equivalent to observing Sheila's action choice. ∎

(b) Again for the parameter values $p = 1$ and $q = 0$, what is the result of deleting weakly dominated strategies? Describe the extensive form game of perfect information that has the normal form obtained in part (a), and apply backward induction. **[5 points]**

Solution: First we can eliminate all of Bruce's strategies other than AB, since AB weakly dominates the other three strategies. Since Bruce is left with one strategy, AB, we delete L for Sheila, and the resulting profile is (H, AB). Note that when $p = 1$ and $q = 0$, observing the signal is equivalent to observing Sheila's action, yielding a game of perfect information, drawn in Figure 1.

The backward induction solution is (H, AB). ∎

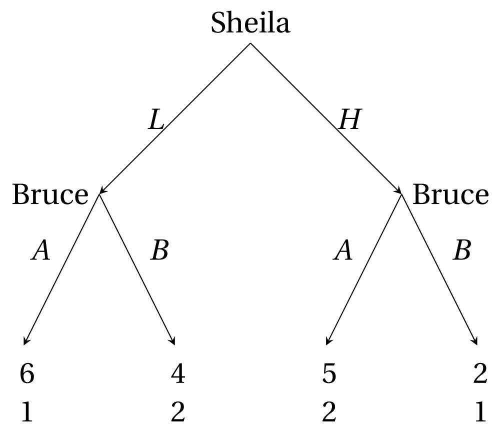

Figure 1: The extensive form for Question (b).

(c) Suppose $p, q \in (0,1)$ (that is, $0 < p, q < 1$). What is the normal form of this game? What are the pure strategy Nash equilibria? What happens to the pure strategy equilibria as $p \to 1$ and $q \to 0$? Explain why backward induction cannot be applied to the game when both p and q are strictly between 0 and 1. **[15 points]**

Solution: The strategy spaces for Sheila and Bruce are unchanged. It is still $\{H, L\}$ for Sheila, and $\{AA, AB, BA, BB\}$ for Bruce, where the first letter is Bruce's action choice if he observes h and the second is his choice if he observes l. The normal form is

		Bruce			
		AA	AB	BA	BB
Sheila	H	$5,2$	$2+3p, 1+p$	$5-3p, 2-p$	$2,1$
	L	$6,1$	$4+2q, 2-q$	$6-2q, 1+q$	$4,2$

The only pure strategy Nash equilibrium is (L, BB): first observe that the unique best reply for Bruce to H is AA, while the unique best reply to L is BB, and on $\{AA, BB\}$, Sheila has a strictly dominant choice L.

As $p \to 1$ and $q \to 0$, the uniqueness of the pure strategy equilibrium (L, BB) is unaffected.

Backward induction cannot be applied in this case, because there are no proper subgames: when Bruce sees, for example a signal of h, he does not have a singleton information set. This means that he cannot conclude that Sheila had chosen H (or L, for that matter). Given any signal, there is still uncertainly as to what Sheila had done. In the case where $p = 1$ and $q = 0$, there is no uncertainty. ■

2. A firm with a *single* vacancy is considering hiring one of two identically productive workers. Suppose the value of the output produced by either worker is s.

(a) Suppose that in the first period, the firm chooses a worker to bargain with, and then play proceeds as an alternating offer bargaining game with the firm making the initial offer (the other worker is excluded from the negotiations, and so receives a payoff of 0). The workers and firm discount the future with possibly different discount factors, δ_1, δ_2, and $\delta_F \in (0,1)$. Assume $\delta_1 > \delta_2$. Describe the subgame perfect equilibria of the game. **[20 points]**

Solution: If the firm chooses to bargain with worker i, the resulting subgame is the standard Rubinstein alternating offer bargaining game, with a unique equilibrium. In this unique equilibrium on the subgame, the firm offers a wage of

$$\frac{\delta_i(1-\delta_F)}{1-\delta_i\delta_F}s$$

whenever it is the firm's turn to make an offer, and accepts any wage demand from the worker that is no higher than

$$\frac{1-\delta_F}{1-\delta_i\delta_F}s,$$

and worker i asks for a wage of

$$\frac{1-\delta_F}{1-\delta_i\delta_F}s$$

whenever it is the worker's turn to make a wage demand, and accepts any wage offer from the firm that is at least as high as

$$\frac{\delta_i(1-\delta_F)}{1-\delta_i\delta_F}s.$$

We have now described the behavior of the players on the two bargaining subgames, one in which the firm bargains with worker 1 and the other in which the firm bargains with worker 2.

The firm bargains with the worker yielding the higher payoff, and so bargains with the worker i satisfying

$$s-\frac{\delta_i(1-\delta_F)}{1-\delta_i\delta_F}s > s-\frac{\delta_j(1-\delta_F)}{1-\delta_j\delta_F}s \iff \frac{\delta_j}{1-\delta_j\delta_F} > \frac{\delta_i}{1-\delta_i\delta_F}$$
$$\iff (1-\delta_i\delta_F)\delta_j > (1-\delta_j\delta_F)\delta_i \iff \delta_j > \delta_i.$$

Since $\delta_1 > \delta_2$, the firm prefers to bargain with worker 2, and so the firm's continuation payoff is

$$\frac{1-\delta_2}{1-\delta_2\delta_F}s,$$

worker 2's payoff is

$$\frac{\delta_2(1-\delta_F)}{1-\delta_2\delta_F}s,$$

and worker 1's payoff is 0. ■

(b) Suppose now that the government, concerned that a firm can exclude a worker by simply choosing to negotiate with the other worker, passes a law that gives the workers "equal access" to the firm as follows: Each worker i simultaneously announces an opening offer of a wage $w_i \le s$. The firm, on the basis of the offered wages, can decide to accept one of the offers, in which case that worker receives the wage, the firm receives the remaining surplus, and the other worker receives zero. Only if the firm rejects both offers, can the firm then chose a worker to bargain with (the other worker is then permanently excluded from further negotiations, and receives 0). The game now proceeds (as in part (a)) to an alternating offer bargaining game with the firm making the initial offer in the period after the rejection. Describe the subgame perfect equilibria of the game. Do the workers benefit from this law? **[10 points]**

Solution: First consider the subgames reached by the simultaneous announcement of the pair of wages (w_1, w_2). If the firm rejects both wages, the continuation game is in part (a), the firm chooses to bargain with worker 2, with that worker receiving a strictly positive payoff and worker 1 a zero payoff.

Since the firm makes the first wage offer in the period after rejecting the original bids, if

$$\min\{w_1, w_2\} < \frac{\delta_F \delta_2 (1-\delta_F)}{1-\delta_2\delta_F} s,$$

the firm accepts the smaller w_i; if the two wages are equal, accepting either worker is consistent with equilibrium. If the inequality is reversed, the firm rejects both offers and bargains with worker 2. If equality, accepting and rejecting the low wage is consistent with equilibrium.

Finally, both workers must offer a wage of 0 to the firm: Any positive wage offer is undercut by the other worker.

There are multiple equilibria due to indifferences (since the firm is indifferent between the workers when they make their equilibrium wage offers), but equilibrium payoffs are unique: *both* worker receive a payoff of 0 and the firm receives a payoff of s. In other words, the firm is in favor of this law, worker 1 is indifferent, and worker 2 is made worse off. ∎

3. An entrepreneur is contemplating selling all or part of his startup to outside investors. The profits from the startup are risky and the entrepreneur is risk averse. The entrepreneur's preferences over $x \in [0,1]$, the fraction of the startup the entrepreneur retains, and p, the price "per share" paid by the outside investors, are given by

$$u(x, \theta, p) = \theta x - x^2 + p(1-x),$$

where $\theta > 1$ is the value of the startup (i.e., expected profits). The quadratic term reflects the entrepreneur 's risk aversion. The outside investors are risk neutral, and so the payoff to an outside investor of paying p per share for $1-x$ of the startup is then

$$\theta(1-x) - p(1-x).$$

There are at least two outside investors, and the price is determined by a first price sealed bid auction: The entrepreneur first chooses the fraction of the firm to sell, $1-x$; the outside investors then bid, with the $1-x$ fraction going to the highest bidder (ties are broken with a coin flip). **Important convention:** The outside investors submit bids in "price per share" p, so the amount paid is $p(1-x)$.

(a) Suppose θ is public information. What fraction of the startup will the entrepreneur sell, and how much will he receive for it? **[5 points]**

Solution: When θ is public, the outside investors are bidding in a common value auction of known value, they are each indifferent between winning and losing, and so bid $p=\theta$. The entrepreneur sells all of the startup ($x=0$) for a price of $p=\theta$, giving a payoff of θ to the entrepreneur. All of the risk has been shifted from the risk averse entrepreneur to a risk neutral investor. ■

(b) Suppose now θ is privately known to the entrepreneur. The outside investors have common beliefs, assigning probability $\alpha \in (0,1)$ to $\theta = \theta_1 > 1$ and probability $1-\alpha$ to $\theta = \theta_2 > \theta_1$. Suppose $\theta_2 - \theta_1 > 2$. Characterize the separating perfect Bayesian equilibria. Are there any other perfect Bayesian equilibria? **[15 points]**

Solution: Let x_i be the share of the startup put up for sale by the entrepreneur of type θ_i. In a perfect Bayesian equilibrium, the investors have common beliefs about θ; let $p(x)$ be the common belief if x is put up for sale. Then, in a separating equilibrium, $p(x) \in [\theta_1, \theta_2]$ for all $x \in [0,1]$ and

$$p(x) = \theta_i, \qquad \text{if } x = x_i.$$

Since θ_1 is the lowest price the entrepreneur can receive, θ_1 cannot be deterred from setting $x_1 = 0$. In order for θ_1 to not find it optimal to choose x_2, incentive compatibility for θ_1 (IC(θ_1)) should be satisfied:

$$\begin{aligned} \theta_1 &\geq \theta_1 x_2 - x_2^2 + \theta_2(1-x_2) \\ &\iff x_2^2 \geq (\theta_2 - \theta_1)(1-x_2). \end{aligned}$$

Note that this is trivially satisfies at $x_2 = 1$, and violated at $x_2 = 0$. Thus IC(θ_1) requires the high type entrepreneur to retain at least a fraction $\bar{x} > 0$ of the startup, where $\bar{x}$ is the positive solution to $x_2^2 = (\theta_2 - \theta_1)(1-x_2)$.

We also need x_2 to not be so large that θ_2 does not prefer x_1, this is IC(θ_2):

$$\begin{aligned} \theta_2 x_2 - x_2^2 + \theta_2(1-x_2) = \theta_2 - x_2^2 &\geq \theta_2 x_1 - x_1^2 + \theta_1(1-x_1) = \theta_1 \qquad \text{(since } x_1 = 0\text{)} \\ &\iff x_2^2 \leq (\theta_2 - \theta_1). \end{aligned}$$

Clearly, $\bar{x}$ strictly satisfies this constraint.

For $x \neq 0$ or x_2 (i.e, out-of-equilibrium quantities), specifying $p(x) = \theta_1$, the worst possible price, yields a separating perfect Bayesian equilibrium: This is not trivial, since θ_2's optimal response to a constant price $p(x) = \theta_1$ is *not* to choose $x = 0$, but instead to choose $\max\{1, (\theta_2 - \theta_1)/2\}$.

Since $\theta_2 - \theta_1 \geq 2$, type θ_2's most profitable deviation is to $x = 1$, i.e., retaining the entire firm, leading to profits of $\theta_2 - 1$, which is clearly (weakly) smaller than $\theta_2 - x_2^2$, the payoff from choosing x_2. Thus, any $x_2 \in [\bar{x}, 1]$ is consistent with some separating perfect Bayesian equilibrium.

There are also pooling equilibria, where both types choose the same x_p, with $p(x_p) = \alpha\theta_1 + (1-\alpha)\theta_2 \equiv \bar{\theta}$, and $p(x) = \theta_1$ for $x \neq x_p$. Any x_p satisfying

$$\begin{aligned} \theta_1 x_p - x_p^2 + \bar{\theta}(1 - x_p) &\geq \theta_1 \\ \iff (1-\alpha)(\theta_2 - \theta_1)(1 - x_p) &\geq x_p^2 \end{aligned}$$

gives a pooling equilibrium. ■

(c) Maintaining the assumption that θ is privately known to the entrepreneur, suppose now that the outside investors' beliefs over θ have support $[\theta_1, \theta_2]$, so that there is a continuum of possible values for θ. What is the initial value problem (differential equation plus initial condition) characterizing separating perfect Bayesian equilibria? **DO NOT ATTEMPT TO SOLVE IT.** **[10 points]**

Solution: The payoff to the entrepreneur from choosing x when it leads to a belief $\hat{\theta}$ by the outside investors is

$$U(x, \theta, \hat{\theta}) = \theta x - x^2 + \hat{\theta}(1 - x).$$

Let $\tau : [\theta_1, \theta_2] \to [0, 1]$ be a separating strategy. Then, $\hat{\theta} = \tau^{-1}(x)$, and the entrepreneur solves

$$\max_x U(x, \theta, \tau^{-1}(x)) = \max_x \theta x - x^2 + \tau^{-1}(x)(1 - x).$$

The first order condition is

$$\begin{aligned} 0 &= \theta - 2x - \tau^{-1}(x) + \frac{d\tau^{-1}(x)}{dx}(1 - x) \\ &= \theta - 2x - \tau^{-1}(x) + (\tau'(\tau^{-1}(x)))^{-1}(1 - x). \end{aligned}$$

In equilibrium, $x = \tau(\theta)$ satisfies this first order condition, and so

$$\begin{aligned} 0 &= \theta - 2\tau(\theta) - \theta + (\tau'(\theta))^{-1}(1 - \tau(\theta)) \\ \Rightarrow \tau'(\theta) &= \frac{1 - \tau(\theta)}{2\tau(\theta)}. \end{aligned}$$

The initial condition is $\tau(\theta_1) = 0$. ■

4. (A mechanism design perspective on startup funding.) As in Question 3, there is an entrepreneur with a start-up who would like to obtain external funding. Unlike the previous question, though, we now consider a monopoly provider of investment funds (call it the bank). The bank proposes a mechanism to the entrepreneur to determine the terms of any funding (p, x), $x \in [0, 1]$, provided.

(a) We begin, as in Question 3(a) by assuming the parameter $\theta > 1$ is public information. What is the optimal take-it-or-leave-it offer from the bank? **[5 points]**

Solution: The bank chooses (p, x) to maximize

$$(\theta - p)(1 - x)$$

subject to the entrepreneur's participation (IR) constraint:

$$\theta x - x^2 + p(1 - x) \geq \theta - 1.$$

The participation constraint can be written as

$$(p - \theta)(1 - x) \geq x^2 - 1,$$

which will hold with equality. Thus, the bank maximizes

$$1 - x^2,$$

and so chooses $x = 0$, which implies $p = \theta - 1$. ■

As in Question 3(b), we now assume the parameter θ is privately known to the entrepreneur, and so not known by the bank. The bank assigns probability $\alpha \in (0, 1)$ to $\theta = \theta_1 > 1$ and probability $1 - \alpha$ to $\theta = \theta_2 > \theta_1$. Suppose $\theta_2 - \theta_1 > 2$.

(b) State the revelation principle and explain its importance in the current setting. **[10 points]**

Solution: The revelation principle states that any equilibrium of any mechanism the bank could propose is the truthtelling equilibrium of a direct mechanism (i.e., a mechanism in which the terms offered by the bank (p, x) are determined by the entrepreneur's report of its private information). This dramatically simplifies the search for the bank's optimal mechanism, since we need only optimize over the set of direct mechanisms. ■

(c) Describe the optimization problem the bank must solve in order to determine its optimal direct mechanism (in other words, state the objective function and all the constraints) **[5 points]**

Solution: In a direct mechanism, the entrepreneur reports its private information $\hat{\theta}$, and the terms offered by the bank (p, x) are specified as a function of $\hat{\theta}$. Denote (p_i, x_i) as the terms given to the report $\hat{\theta}_i$.

The bank maximizes

$$\alpha(\theta_1 - p_1)(1 - x_1) + (1 - \alpha)(\theta_2 - p_2)(1 - x_2).$$

The constraints are incentive compatibility for θ_1,

$$\theta_1 x_1 - x_1^2 + p_1(1 - x_1) \geq \theta_1 x_2 - x_2^2 + p_2(1 - x_2), \tag{IC1}$$

incentive compatibility for θ_2,

$$\theta_2 x_2 - x_2^2 + p_2(1-x_2) \geq \theta_2 x_1 - x_1^2 + p_1(1-x_1), \tag{IC2}$$

individual rationality for θ_1,

$$\theta_1 x_1 - x_1^2 + p_1(1-x_1) \geq \theta_1 - 1, \tag{IR1}$$

and individual rationality for θ_2,

$$\theta_2 x_2 - x_2^2 + p_2(1-x_2) \geq \theta_2 - 1. \tag{IR2}$$

■

(d) Prove that in any direct mechanism satisfying the constraints from part (c), the share retained by θ_2 is at least as large as that retained by θ_1. **[5 points]**

Solution: Summing IC1 and IC2 gives

$$\theta_1 x_1 + \theta_2 x_2 \geq \theta_1 x_2 + \theta_2 x_1,$$

so after rearranging

$$(\theta_2 - \theta_1)(x_2 - x_1) \geq 0,$$

and since $\theta_2 > \theta_1$, we have $x_2 \geq x_1$. ■

(e) One of the constraints in part (c) is redundant. Which one and why? **[5 points]**

Solution: Observe the IR constraints can be written as

$$(p_i - \theta_i)(1-x_i) - x_i^2 + 1 \geq 0.$$

Then, IR2 and IC1 implies IR1:

$$\begin{aligned}
(p_1 - \theta_1)(1-x_1) - x_1^2 + 1 &\geq (p_2 - \theta_1)(1-x_2) - x_2^2 + 1 && \text{(by IC1)} \\
&= (p_2 - \theta_2)(1-x_2) - x_2^2 + 1 + (\theta_2 - \theta_1)(1-x_2) \\
&\geq (\theta_2 - \theta_1)(1-x_2) && \text{(by IR2)} \\
&\geq 0. && \text{(since } x_2 \leq 1\text{)}
\end{aligned}$$

■