Microeconomic Theory I
Preliminary Examination
Suggested Solutions

University of Pennsylvania

June 5, 2017

Instructions

This exam has four questions and is worth 100 points.

Answer each question in a **SEPARATE** exam book.

If you need to make additional assumptions, state them clearly.

Be concise.

Write clearly.

Use WORDS to explain your reasoning.

Good luck!

1. (25 pts) *Classical production and demand theory*

   (a) (15 pts) A competitive firm produces one output, q, using two inputs, x_1 and x_2, which have prices $w = (w_1, w_2)$. The firm has a continuous increasing production function f on $\mathbb{R}^2_+$. Find f, given that the firm's cost function is

$$c(q,w) = \begin{cases} 2\sqrt{w_1 w_2 q} - w_2 & \text{if } \dfrac{w_1}{w_2} \geq \dfrac{1}{q} \\ \\ w_1 q & \text{if } \dfrac{w_1}{w_2} \leq \dfrac{1}{q} \end{cases}.$$

Soln: The production function is $f(x) = x_1(x_2 + 1)$.
Derivation. At (q, w) for which $w_1/w_2 > 1/q$, the conditional factor demands are

$$x_1(q,w) = \frac{\partial c}{\partial w_1} = \sqrt{\frac{w_2 q}{w_1}},$$

$$x_2(q,w) = \frac{\partial c}{\partial w_2} = \sqrt{\frac{w_1 q}{w_2}} - 1.$$

Now, fix an arbitrary $(x_1, x_2) \in \mathbb{R}^2_{++}$, and consider the two equation system

$$x_1 = \sqrt{\frac{w_2 q}{w_1}}, \quad x_2 = \sqrt{\frac{w_1 q}{w_2}} - 1$$

in the two unknowns w_2/w_1 and q. Rearrange it to

$$\frac{x_1}{\sqrt{q}} = \sqrt{\frac{w_2}{w_1}} \text{ and } \frac{\sqrt{q}}{x_2 + 1} = \sqrt{\frac{w_2}{w_1}},$$

to obtain

$$\frac{x_1}{\sqrt{q}} = \frac{\sqrt{q}}{x_2 + 1}.$$

From this we obtain $q = x_1(x_2 + 1)$. As this is the case for any $x \gg 0$, and we know the production function is continuous on $\mathbb{R}^2_+$, it must be

$$f(x) = x_1(x_2 + 1)$$

on $\mathbb{R}^2_+$. (This production function is easily verified to be correct by solving the cost minimization problem for it to show that the resulting cost function is indeed the one given in the problem.) ■

   (b) (10 pts) In a three-good world, suppose a consumer's demands for goods 1 and 2 are given by

$$x_1(p,y) = \frac{p_2}{p_3} \quad \text{and} \quad x_2(p,y) = \frac{p_1}{p_3}$$

on some open subset of $\mathbb{R}^4_{++}$. Can these demands arise from the maximization of a continuous utility function representing locally nonsatiated strictly convex preferences on $\mathbb{R}^3_+$? Prove your answer.

Soln: The answer to the question is "No."

Proof 1. For, suppose they do. Then the Slutsky matrix $S = [s_{ij}]$ is symmetric and negative semidefinite at any (p, y) in the given neighborhood. Hence, the determinants of the leading principal submatrices alternate in sign (weakly), and in particular,

$$\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} = s_{11}s_{22} - s_{12}s_{21} \geq 0.$$

However, recalling that

$$s_{ij} = \frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial y},$$

we obtain a contradiction using the given demands for goods 1 and 2:

$$s_{11}s_{22} - s_{12}s_{21} = 0 \cdot 0 - 2p_3^{-1} < 0.$$

Proof 2. For, suppose they do. Then Walras' Law holds, and so the demand for good 3 in the given open set must be

$$x_3(p, y) = \frac{y - p_1 x_1(p, y) - p_2 x_2(p, y)}{p_3} = \frac{y}{p_3} - \frac{2p_1p_2}{p_3^2}.$$

Thus, the bottom right element of the Slutsky matrix at such (p, y) is

$$\begin{aligned} s_{33}(p, y) &= \frac{\partial x_3}{\partial p_3} + x_3 \frac{\partial x_3}{\partial y} \\ &= \left(-\frac{y}{p_3^2} + \frac{4p_1p_2}{p_3^3}\right) + \left(\frac{y}{p_3} - \frac{2p_1p_2}{p_3^2}\right)\left(\frac{1}{p_3}\right) \\ &= \frac{2p_1p_2}{p_3^3}. \end{aligned}$$

As this term is positive, the Slutsky matrix is not negative semidefinite. This is the desired contradiction, as we know the Slutsky matrix must be negative semidefinite if these demand functions arise from a continuous utility function representing locally nonsatiated strictly convex preferences. ■

2. (25 pts) *Risk aversion and comparative statics*

There are two states of the world, ℓ and n, which will occur with probabilities $\pi \in (0, 1)$ and $1 - \pi$, respectively. In state ℓ, Alex loses $L > 0$ dollars. In state n, he loses 0 dollars. His initial wealth (income) is $w > 0$. Before the state is realized, Alex can purchase insurance: he pays px to obtain a policy that pays x dollars if state ℓ occurs. His state-contingent incomes given x are thus $y_\ell = w - L + x - px$ and $y_n = w - px$. Alex chooses x to maximize his expected utility of income subject to the constraint $x \geq 0$. His Bernoulli utility function u is C^2 and satisfies $u' > 0$ and $u'' < 0$.

Lastly, assume the price of insurance exceeds the actuarily fair rate: $p > \pi$.

(a) (5 pts) Show that Alex's optimal coverage, x_a, is less than L.

Soln: x_a solves the program

$$\max_{x \geq 0} \pi u(w - L + x(1 - p)) + (1 - \pi)u(w - px),$$

If $x_a = 0$, then $x_a < L$ is immediate. If $x_a > 0$, it satisfies with equality the FOC for this program, which yields

$$\frac{u'(y_n^a)}{u'(y_\ell^a)} = \frac{\pi}{p}\frac{1-p}{1-\pi} < 1, \tag{1}$$

where $y_\ell^a := w - L + x_a(1-p)$ and $y_n^a := w - px_a$. It follows from this and $u'' < 0$ that $y_n^a > y_\ell^a$. This in turn implies $x_a < L$. ■

(b) (10 pts) Assuming u exhibits decreasing absolute risk aversion (DARA), determine whether insurance coverage is a normal or inferior good for Alex.

Soln: It is an inferior good for Alex. Let k denote the RHS of (1), and rearrange the equation to

$$u'(w - px_a) - ku'(w - L + x_a(1-p)) = 0.$$

This equation determines x_a as a function of the parameters, in particular w. Differentiating it totally with respect to w and rearranging yields

$$x_a'(w) = \left[-u''(y_n^a) + ku''(y_\ell^a)\right]/D,$$

where

$$D := -pu''(y_n^a) - k(1-p)u''(y_\ell^a) > 0.$$

Letting $A = -u''/u'$, we have

$$\begin{aligned} x_a'(w)D &= -u''(y_n^a) + ku''(y_\ell^a) \\ &= A(y_n^a)u'(y_n^a) - kA(y_\ell^a)u'(y_\ell^a) \\ &= \left[A(y_n^a) - A(y_\ell^a)\right]u'(y_n^a) \\ &< 0, \end{aligned}$$

where the third equality follows from (1), and the inequality comes from DARA and $y_n^a > y_\ell^a$. We conclude that $x_a'(w) < 0$. ■

(c) (10 pts) Barb is in the same insurance market, and her Bernoulli utility function, v, satisfies the same assumptions as does u. However, Barb is strictly more risk averse than Alex. Determine whether Barb purchases more or less insurance than does Alex. (Assume they both purchase positive amounts.)

Soln: Barb purchases more.

Let x_b be Barb's optimal x, and let the consumers' corresponding optimal state-contingent consumptions be y_l^i and y_n^i for $i = a, b$. By the argument in (a), we have $y_n^i > y_\ell^i$ for both $i = a, b$, Alex's FOC is (1), and Barb's FOC is

$$\frac{v'(y_n^b)}{v'(y_\ell^b)} = \frac{\pi}{p}\frac{1-p}{1-\pi}. \tag{2}$$

Since Barb is strictly more risk averse than Alex, there exists a strictly concave increasing function h such that $v = h \circ u$. Hence, $v'(z) = h'(u(z))u'(z)$ for any z. From this and (1) we obtain

$$\begin{aligned} \frac{v'(y_n^a)}{v'(y_\ell^a)} &= \frac{h'(u(y_n^a))u'(y_n^a)}{h'(u(y_\ell^a))u'(y_\ell^a)} \\ &= \frac{h'(u(y_n^a))}{h'(u(y_\ell^a))}\frac{\pi}{p}\frac{1-p}{1-\pi} \\ &< \frac{\pi}{p}\frac{1-p}{1-\pi}, \end{aligned}$$

since $y_n^a > y_\ell^a$ implies $u_a(y_n^a) > u_a(y_\ell^a)$ implies $h'(u_a(y_n^a)) < h'(u_a(y_\ell^a))$. Hence, from (2) we now have

$$\frac{v'(y_n^a)}{v'(y_\ell^a)} < \frac{v'(y_n^b)}{v'(y_\ell^b)},$$

or rather,

$$\frac{v'(w - px^a)}{v'(w - L + x^a(1-p))} < \frac{v'(w - px^b)}{v'(w - L + x^b(1-p))}.$$

This implies, since $v'' < 0$, that $x_b > x_a$. ■

3. (25 pts)

   (a) (5 pts) Define the core of a pure exchange economy with l goods and n agents.

   (b) (5 pts) State the core convergence theorem.

   Soln: Assuming strict convexity of preferences, we know that the "equal treatment" property holds for core allocations. Using this fact, the core convergence theorem states that if an allocation is in the core for all replicas it must be a Walrasian allocation. Note particularly that it is NOT true that for a sufficiently large number of replications all allocations in the core are Walrasian. ■

   (c) (5 pts) Give a graphic example of a two-person, two-good economy in which there does not exist a competitive equilibrium, but there exists an allocation in the core.

   Soln: The simplest example would have the initial endowment Pareto efficient with a unique supporting hyperplane that was not a Walrasian price. There are two things that must be clear. First, the proposed core allocation must be individually rational. Second, it must be clear that not only is the proposed allocation not competitive, but there is no other competitive equilibrium. ■

   (d) (10 pts) Consider a two person, two good exchange economy for which agents have utility functions that are continuous and increasing, but not necessarily concave. Prove the core is nonempty. State clearly any additional assumptions you make.

   Soln: Proof: Let $(\bar{x}_i, \bar{y}_i)$ be i's initial endowment, $i = 1, 2$, and let $x_1, y_1; x_2, y_2 \in \arg\max u_2(x_2, y_2)$ s.t. $u(x_1, y_1) \geq u(\bar{x}_1, \bar{y}_1)$ and $(x_1, y_1) + (x_2, y_2) \leq (\bar{x}_1, \bar{y}_1) + (\bar{x}_2, \bar{y}_2)$. Solution exists and is in the core by a simple argument. ■

4. (25 pts) Consider a standard two-period economy, dated $t = 0$ and $t = 1$. Agents consume in both periods. There are three states of nature in the second period. There is a single consumption good, and it is used as a numeraire; hence, the spot price of a unit of consumption at either date is 1. At date 0 agents can trade in two primary securities. Security 1 has the second-period payoff vector $r_1 = (1, 0, 0)$, and security 2 has the second-period payoff vector $r_2 = (1, 2, 3)$. The prices of these securities at date 0 are $q_1 = 0.1$ and $q_2 = 1.1$.

   (a) (10 pts) Suppose there is a *derivative* security denoted security 3. Security 3 is a call option on security 2 with strike price of 1. What are the minimum and maximum possible prices for this security that are consistent with no arbitrage?

   Soln: Let (v_1, v_2, v_3) be the state prices. Since $q_1 = 0.1$, $v_1 = .1$. $q_2 = 1v_1 + 2v_2 + 3 \cdot v_3$; since $q_2 = 1.1$ we have $2v_2 + 3v_3 = 1$. Hence $v_2 = \frac{1-3v_3}{2}$. $v_2 \geq 0$ so $v_3 \leq 1/3$. The call option with strike price 1 will be exercised in states 2 and 3 and gives return vector $(0, 1, 2)$. The price of this return is then $v_2 + 2v_3 = \frac{1-3v_3}{2} + 2v_3 = \frac{1+v_3}{2}$. Since $v_2 \geq 0$ $v_3 \leq 1/3$ The is lowest when $v_3 = 0:\ 1/2$; the highest price is when $v_3 = 1/3 : 2/3$. ■

(b) (5 pts) Suppose that the price of security 3 in part (a) is 1. Show that the system is arbitrage free.

Soln: There was a typo in the question – the price of the security was supposed to have been .6 not 1. As a result of the typo the system is not arbitrage free. The simplest answer is to note that in part (a) it was shown that the highest price of the asset consistent with no arbitrage is 2/3. ■

(c) (10 pts) Assume these security prices, $q = (.1, 1.1, .6)$, arise in an incomplete markets equilibrium with the specified three securities.

i. What would be the market price of a put option on asset 2 with a strike price of 3?

Soln: The easiest way to show this is to compute the state price vector, which is the solution to the three price equations: $q_1 = v_1$; $q_2 = v_1 + 2v_2 + 3v_3$; $q_3 = 2v_2 + 3$; the solution is $v = (.1, .2, .2)$. A put option with strike price 3 will be exercised whenever the value is less than 3, that is in states 1 and 2; the return is then $(2, 1, 0)$; using the state price vector, we see that $q_3 = 2v_1 + v_2 = .4$. ■

i. What would be the risk-free interest rate on a loan taken at date $t = 0$?

Soln: The cost of getting 1 for sure next period is the price of the security with return $(1, 1, 1)$. This price is $v_1 + v_2 + v_3 = .5$. Hence an investment of .5 will give a risk free return of 1. Hence the interest rate on a risk free loan would be 100%. ■