Prelim Examination

Friday August 11, 2017. Time limit: 150 minutes

Instructions:

(i) The exam consists of two parts. The total number of points for each part is 40. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

Part I

Question 1: Concepts (9 Points).

(i) (3 Points) What does it mean for an estimator to be *admissible*?

(ii) (3 Points) What is *Neyman-Pearson lemma*?

(iii) (3 Points) What is the *Cramer-Rao lower bound*?

Question 2: Asymptotics in a Location-Shift Model (12 Points).

Consider the following location-shift model

$$Y_i = \theta_0 + U_i, \quad U_i \sim (0,1), \quad \mathbb{E}[|U_i|^4] < \infty, \quad i = 1, \ldots, n \tag{1}$$

(i) (2 Points) Derive a quasi-maximum likelihood estimate (QMLE) of θ_0 under the assumption that the U_i's are normally distributed. Denote the quasi-likelihood function by $p(Y_{1:n}|\theta)$ (Note: It's called a "quasi" MLE because according to the DGP in (1) the U_i's do not have to be normally distributed.)

(ii) (2 Points) Derive the probability limit and the asymptotic distribution of the QMLE $\hat{\theta}$.

(iii) (3 Points) Now derive a posterior distribution $p(\theta|Y_{1:n})$ for θ under the assumption that the U_i's are normally distributed. Use the prior $\theta \sim N(0,1)$.

(iv) (5 Points) Show that the posterior distribution is consistent in the following sense: for any $\delta > 0$

$$\mathbb{P}\big[|\theta - \theta_0| > \delta \,\big|\, Y_{1:n}\big] \longrightarrow 0$$

as $n \longrightarrow \infty$. In plain English, what does this result imply?

Question 3: Stopping Rules (19 Points)

Suppose that $\{Y_i\}_{i=1}^3$ is a sequence of $iidN(\theta, 1)$ random variables. Define the sample mean for a sample of n observations as $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$. Here the sample size n is either 1, 2, or 3. For the Bayesian calculations below, use the improper prior $p(\theta) \propto 1$.

(i) (1 Point) Consider the following experiment, $\mathcal{E}_1$. Generate a draw of Y_1 and Y_2. The realizations are $y_1 = 1$ and $y_2 = -1.5$. What is the likelihood function for θ based on this experiment?

(ii) (3 Points) Derive a 95% Bayesian credible interval and a 95% frequentist confidence interval based on the outcome of $\mathcal{E}_1$.

(iii) (6 Points) Now consider the following alternative experiment, $\mathcal{E}_2$: sample sequentially, starting with Y_1, and terminate either if $\bar{Y}_n < 0$ or if $n = 3$. Thus, depending on the y_i values, the sample size is either $n(Y) = 1$, $n(Y) = 2$, or $n(Y) = 3$. What is the likelihood function for θ based on this experiment? (Hint: use $\phi(x)$ and $\Phi(x)$ to denote the pdf and cdf of a $N(0, 1)$ random variable.

(iv) (2 Points) Assume that you observe $y_1 = 1$ and $y_2 = -1.5$ in $\mathcal{E}_2$. Compare the likelihood functions conditional on these two observations from experiment $\mathcal{E}_2$ and $\mathcal{E}_1$. What is the 95% Bayesian credible interval for θ in $\mathcal{E}_2$. Does the stopping rule matter?

(v) (7 Points) Compute the frequentist coverage probability for the following interval:

$$\begin{array}{ll} Y_1 \pm 1.96 & \text{if } n = 1 \\ (Y_1 + Y_2)/2 \pm 1.96/\sqrt{2} & \text{if } n = 2 \\ (Y_1 + Y_2 + Y_3)/3 \pm 1.96/\sqrt{3} & \text{if } n = 3 \end{array}$$

From a pre-experimental perspective, does the Bayesian credible interval in (iv) have the correct frequentist coverage probability? (Hint: some parts of the coverage probability formula are difficult to evaluate. Feel free to use graphical illustrations.)

Part II

Question 4: (12') Asymptotic Covariance Estimation

Take a regression model with i.i.d. data $\{(Y_i, X_i) : i = 1, ..., n\}$ and scalar X_i,

$$\begin{aligned} Y_i &= X_i\theta + U_i, \\ E(U_i|X_i) &= 0, \\ \Omega &= E(X_i^2 U_i^2). \end{aligned}$$

Let $\widehat{\theta}$ be the OLS estimator of θ with residual

$$\widehat{U}_i = Y_i - X_i\widehat{\theta}.$$

(i) (6') Provide an estimator of Ω, denoted by $\widehat{\Omega}$, and show its consistency.

(ii) (6') Derive the asymptotic distribution of $\sqrt{n}(\widehat{\Omega} - \Omega)$.

Question 5: (28') GMM

Take the model

$$\begin{aligned} Y_i &= X_i'\theta + U_i, \\ E(Z_iU_i) &= 0, \end{aligned}$$

where $Y_i \in R$, $X_i \in R^k$, and $Z_i \in R^l$, with $l \geq k$. Assume the observations are i.i.d. Consider the following statistic

$$\begin{aligned} J_n(\theta) &= n\overline{m}_n(\theta)'W_n\overline{m}_n(\theta) \\ \overline{m}_n(\theta) &= n^{-1}\sum_{i=1}^{n} Z_i(Y_i - X_i'\theta) \end{aligned}$$

for some weighting matrix $W_n \to_p W$.

(i) (4') Consider the hypothesis

$$H_0 : \theta = \theta_0.$$

Derive the asymptotic distribution of $J_n(\theta_0)$ under H_0 as $n \to \infty$.

(ii) (2') Write down an appropriate weighting matrix W_n which takes advantage of H_0 and yields a known asymptotic distribution in part (i).

(iii) (2') What is the asymptotic limit of $J_n(\theta_0)$ under the alternative $H_1 : \theta \neq \theta_0$?

(iv) (2') Describe a test of H_0 against H_1 based on this statistic at the significance level α.

(v) (2') Use the result in part (iv) to construct a confidence set for θ. The asymptotic coverage of this confidence set is $1 - \alpha$.

(vi) (4') Define an estimator of θ and derive its asymptotic distribution.

(vii) (2') What is the optimal choice of W_n for the estimation of θ?

(viii) (4') How do you define the exogeneity condition and the relevance condition of the instruments in this problem?

(ix) (4') If the relationship between Y_i and X_i is nonlinear, i.e., $Y_i = g(X_i, \theta) + U_i$ for some known nonlinear function $g(x, \theta)$. How to define $J_n(\theta)$ and W_n for the estimation of θ? Suppose you have instruments Z_i that satisfy all requirements.

(x) (2') How to define the relevance condition with this nonlinear specification?

END OF EXAM