A NORMALIZED VALUE FOR INFORMATION PURCHASES

Antonio Cabrales
University College London
Olivier Gossner
École Polytechnique and LSE Math Dept
Roberto Serrano
Brown University

http://www.econ.brown.edu/faculty/serrano

Preliminaries

- When can one say that a new piece of information is more valuable to a d.m. than another?
- Difficulties:
- (i) The agent's priors matter
- (ii) The agent's preferences and/or wealth matter
- And (iii) the decision problem in which information will be applied matters

Blackwell (1953)

- Blackwell's (1953) ordering: an information structure (i.s.) α is more informative than β whenever β is a garbling of α.

- Or a d.m. with *any* utility function would prefer to use α over β in *any* decision problem.

- Can one complete this partial ordering on the basis of similar decision-theoretic considerations? E.g., can one find classes of preferences and problems such that "$\alpha \succeq_I \beta$ in terms of β being rejected at some price whenever α is" gives a complete ordering of i.s.'s?

Basic Notation

- Agent's initial wealth w,
- increasing and concave monetary and twice differentiable utility function $u\colon \mathbb{R} \to \mathbb{R}$.
- Coefficient of absolute risk aversion at wealth z:

$$\rho(z) = -\frac{u''(z)}{u'(z)}$$

- Coefficient of relative risk aversion at wealth z:

$$\rho_R(z) = -\frac{u''(z)z}{u'(z)}$$

Investments in Assets

- Let K be the finite set of states of nature.

- Agent's prior belief p with full support.

- Investment opportunity or asset: $x \in I\!R^K$, yielding wealth $w + x_k$ in state k.

- Opting out: $0_K \in B$.

- No-arbitrage asset $x \in I\!R^k$ (given p): $\sum_k p(k) x_k \leq 0$.

- B^*: set of all no-arbitrage assets.

Information Structures

- An i. s. α: finite set of signals $s \in S_\alpha$, and transition prob. $\alpha_k \in \Delta(S_\alpha)$ for every $k \in K$.

- $\alpha_k(s)$: prob. of signal s in state k.

- Repres. by a stochastic matrix: rows (states k), columns (signals s).

- Non-redundant signals:

$$\forall s, \exists k \quad \text{s.t.} \quad \alpha_k(s) > 0.$$

I.s. as a distribution over posteriors

- Total prob. of s:

$$p_\alpha(s) = \sum_k p(k)\alpha_k(s),$$

- posterior prob. on K given s: q^s_α, derived from Bayes' rule:

$$q^s_\alpha(k) = \frac{p(k)\alpha_k(s)}{p_\alpha(s)}$$

Examples of I.S.'s

- Most informative i. s. (according to Blackwell) $\overline{\alpha}$:

  for any s, there exists a unique k such that $q^s_{\overline{\alpha}}(k) = 1$.

- Excluding i. s. α:

  for any s, there exists a k such that $q^s_{\alpha}(k) = 0$.

- The least informative i.s. $\underline{\alpha}$:

  for any s and k, $q^s_{\underline{\alpha}}(k) = p(k) > 0$.

Valuable Information

- Given u, w, B and $q \in \Delta(K)$, the maximal expected utility that can be reached by choosing a $x \in B$:

$$v(u, w, B, q) = \sup_{x \in B} \sum_k q(k) u(w + x_k).$$

- The ex-ante expected payoff before receiving signal s from α:

$$\pi(\alpha, u, w, B) = \sum_s p_\alpha(s) v(u, w, B, q_\alpha^s).$$

  Opting out assures that both are at least $u(w)$.

Ruin-Averse Utility

- Ruin averse utility function u: $u(0) = -\infty$
- equivalent to $\rho_R(z) \geq 1$ for every $z > 0$.
- Let $\mathcal{U}^*$ be the set of ruin averse u.

Information Purchasing and Informativeness Ordering

The agent with utility function u and wealth w *purchases information* α at price μ given an investment set B when:

$$\pi(\alpha, u, w - \mu, B) \geq u(w).$$

Otherwise, he rejects α at price μ.

Definition 0: Information structure α *ruin-avoiding investment dominates* information structure β whenever , for every wealth w and price $\mu < w$ such that α is rejected by all agents with utility $u \in \mathcal{U}^*$ at wealth w for every opportunity set $B \subseteq B^*$, so is β.

A Key Lemma

Lemma 0: Given an information structure α, price μ and wealth level $w > \mu$, α is rejected by all agents with utility $u \in \mathcal{U}^*$ at wealth level w given every opportunity set $B \subseteq B^*$ if and only if α is rejected by an agent with ln utility at wealth w for the opportunity set B^*.

Intuition: the ln function majorizes all u in the class (the least risk averse, values information the most).

Entropy ordering

Following Shannon (1948), entropy of a prob. distribution $q \in \Delta(K)$:

$$H(q) = -\sum_{k \in K} q(k) \log_2 q(k)$$

where $0 \log_2(0) = 0$ by convention.

- $H(p)$: measure of the level of uncertainty of the investor with belief p.

- Always ≥ 0, and is equal to 0 only with certainty.

- Concave: distributions closer to the extreme points in $\Delta(K)$ have lower uncertainty; global maximum at the uniform.

Entropy Informativeness and the First Main Result

Recall: following α,

- prob. of s: $p_\alpha(s)$,

- posterior on K following s: q^s_α.

The *entropy informativeness* of i. s. α:

$$I^E(\alpha) = H(p) - \sum_s p_\alpha(s) H(q^s_\alpha).$$

Minimal at $\underline{\alpha}$; maximal at $\overline{\alpha}$; complete ordering.

Theorem 0: Information structure α ruin-avoiding investment dominates information structure β if and only if $I^E(\alpha) \geq I^E(\beta)$.

Information Purchases

- An information purchase (i.p.) is a pair $a = (\mu, \alpha)$, where α is an i.s. and $\mu > 0$ is a price.

- Can one rank "objectively" the value of any i.p., capturing the information-price trade-off?

- Back to class $\mathcal{U}$ of concave and strictly in creasing, twice differentiable $u: \mathbb{R} \to \mathbb{R}$: ruin is possible for sufficiently high prices μ.

- Recall B^*, the set of all non-arbitrage investments given prior p:

$$\{x \in \mathbb{R}^k : \sum_k p(k) x_k \leq 0\}.$$

Ordering Preferences for Information

Whenever agent 2 participates in the market for information, for sure so does agent 1:

Definition 1 Let $u_1, u_2 \in \mathcal{U}$. Agent u_1 uniformly likes (or likes, for short) information better than agent u_2 if for every pair of wealth levels w_1, w_2, and every information purchase a, if agent u_2 accepts a at wealth w_2, then so does agent u_1 at wealth w_1.

Preferences for Information and Risk Aversion

Given $u \in \mathcal{U}$ and wealth $z \in I\!R$, recall

$$\rho_u(z) = -\frac{u''(z)}{u'(z)}$$

be the Arrow-Pratt coefficient of absolute risk aversion.

Let $\overline{R}(u) = \sup_z \rho_u(z)$,
and $\underline{R}(u) = \inf_z \rho_u(z)$.

Theorem 1 Given $u_1, u_2 \in \mathcal{U}$, u_1 likes information better than u_2 if and only if $\overline{R}(u_1) \leq \underline{R}(u_2)$.

Ordering Information Purchases

"Duality" of value w.r.t. preferences for information roughly means that, if we are measuring the information/price tradeoff correctly, people who like information more should make more valuable purchases:

Definition 2 Let $a_1 = (\mu, \alpha)$ and $a_2 = (\nu, \beta)$ be two i.p.'s. We say that a_1 is more valuable than a_2 if, given two agents u_1, u_2 such that u_1 uniformly likes information better than u_2 and any two wealth levels w_1, w_2, whenever agent u_2 accepts a_2 at wealth level w_2, so does agent u_1 with a_1 at wealth level w_1.

Relative Entropy or Kullback-Leibler Divergence

Following Kulback and Leibler (1951), for two probability distributions p and q, relative entropy from p to q:

$$d(p||q) = \sum_k p_k \ln \frac{p_k}{q_k}.$$

- Always non-negative,
- equals 0 if and only if $p = q$,
- finite whenever the support of q contains that of p, and infinite otherwise.

Normalized Value of Information Purchases

Normalized value of an i.p. $a = (\mu, \alpha)$:

$$\mathcal{NV}(a) = -\frac{1}{\mu} \ln \left(\sum_s p_\alpha(s) \exp(-d(p||q_\alpha^s)) \right).$$

- Decreasing in the price μ,
- increasing in each relative entropy $d(p||q_\alpha^s)$,
- 0 for $a = (\mu, \underline{\alpha})$,
- $+\infty$ if for every signal s, there exists k such that $q_\alpha^s(k) = 0$ (excluding i.p.).
- ignoring μ, free energy or stochastic complexity.

Main Result

Theorem 2 Let a_1 and a_2 be two information purchases. Then, a_1 is more valuable than a_2 if and only if $\mathcal{NV}(a_1) \geq \mathcal{NV}(a_2)$.

CARA Agents

Given $r > 0$ let $u_C^r(w) = -\exp(-rw)$.

Recall: an i.p. a is excluding if for every signal s there exists a state k such that $q_s(k) = 0$; it is nonexcluding otherwise.

Lemma: If a is nonexcluding, there exists a unique number $\mathcal{NV}(a)$ such that for every w,

1. If $r > \mathcal{NV}(a)$, u_C^r rejects a at wealth w,

2. if $r \leq \mathcal{NV}(a)$, u_C^r accepts a at wealth w.

Sketch of Proof

- unique CARA indifferent between accepting and rejecting a,

- optimal investment for CARA with ARA r and belief q:

$$x_k = -\frac{1}{r}(-d(p||q) + \ln \frac{p_k}{q_k}).$$

- The rest of the proof of Theorem 2 uses Theorem 1 to "sandwich" a CARA agent between any two agents that are ordered according to "uniformly liking information."

Demand for Information

Theorem 3 Consider an information purchase a and $u \in \mathcal{U}$.

1. If $\underline{R}(u) > \mathcal{N}\mathcal{V}(a)$, then agent u rejects a at all wealth levels w.

2. If $\overline{R}(u) \leq \mathcal{N}\mathcal{V}(a)$, then agent u accepts a at all wealth levels w.

For DARA (decreasing ARA), one can say more:

Theorem 4 Consider an information purchase a and the class of utility functions $\mathcal{U}_{DA}$.

1. An agent $u \in \mathcal{U}_{DA}$ rejects a at all wealth levels if and only if $\underline{R}(u) > \mathcal{NV}(a)$.

2. An agent $u \in \mathcal{U}_{DA}$ accepts a at all wealth levels if and only if $\overline{R}(u) \leq \mathcal{NV}(a)$.

Properties of the Normalized Value

- Continuous in all variables.
- Monotonic with respect to the Blackwell ordering.
- Preserves value through mixtures of i.s.'s.
- For a fixed price, coincides with entropy informativeness for small information.