Heterogeneity and Selection in Dynamic Panel Data

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Abstract

This paper shows nonparametric identification of dynamic panel data models with nonseparable heterogeneity and dynamic selection by nonparametrically differencing out these two sources of bias. For $T = 3$, the model is identified by using a proxy variable. For $T = 6$, the three additional periods construct the proxy to establish identification. As a consequence of these identification results, a constrained maximum likelihood criterion follows, which corrects for selection and allows for one-step estimation. Applying this method, I investigate whether SES affects adult mortality. The method circumvents the survivorship bias and accounts for unobserved heterogeneity. I find that employment has protective effects on survival for the male adults in the NLS Original Cohorts: Older Men.

Keywords: constrained maximum likelihood, dynamic panel, selection, nonparametric differencing, nonseparable heterogeneity

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1 Introduction

Dynamics, *nonseparable heterogeneity*, and *selection* have been separately treated in the panel data literature, in spite of their joint relevance to a wide array of applications. First, common economic variables of interest are modeled to follow dynamics, e.g., assets, income, physical capital and human capital. Second, many economic models entail nonseparable heterogeneity, i.e., an additively separable residual does not summarize abilities, preferences and technologies. Third, most empirical panel data are unbalanced by (self-) selection. Indeed, consideration of these three issues – dynamics, nonseparable heterogeneity, and selection – is essential, but existing econometric methods do not handle them at the same time.

To fill this gap, this paper proposes a set of conditions for identification of dynamic panel data models in the presence of both nonseparable heterogeneity and dynamic selection.[1] Nonparametric point identification is achieved by using information involving either a proxy variable or a slightly longer panel. Specifically, the model is point-identified using $T = 3$ periods of unbalanced panel data and a proxy variable. A special case of this identification result occurs by constructing the proxy variable from three additional periods, i.e., $T = 6$ in total.

For example, consider the dynamics of socio-economic status (SES) and its causal effects on adult mortality.[2] Many unobserved individual characteristics such as genetics, patience and innate abilities presumably affect both SES and survival in non-additive ways, which would incur a bias unless explicitly accounted for. Furthermore, a death outcome of the survival selection induces subsequently missing observations, which may result in a selection bias, often called survivorship bias. The following dynamic panel model with selection accommodates this example:

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T & \text{(Dynamic Panel Model)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T-1 & \text{(Selection Model)} \\ F_{Y_1 U} & & \text{(Initial Condition)} \end{cases}$$

The first equation models the dynamics of the observed state variable Y_t, such as SES, as a first-order Markov process with unobserved heterogeneity U. The second equation models a binary choice of the selection variable, D_t, such as survival, as a Markov decision process with unobserved heterogeneity U. The initial condition $F_{Y_1 U}$ models the dependence of the initial state Y_1 on unobserved heterogeneity U.[3] The period-specific shocks $(\mathcal{E}_t, V_t)$ are exogenous, leaving the fixed effect U as the only source of endogeneity. The economic agent drops out of

[1] In the introductory section of his monograph, Hsiao (2003) particularly picks up heterogeneity and selection as the two major sources of bias in panel data analysis, which motivates the goal of this paper.

[2] Among many biological and socioeconomic factors of mortality (Cutler, Deaton, and Lleras-Muney, 2006), the role of SES and economic environments has been investigated by a number of empirical researches (e.g., Ruhm, 2000; Deaton and Paxson, 2001; Snyder and Evans, 2006; Sullivan and von Wachter, 2009a,b)

[3] The distribution $F_{Y_1 U}$ features the *initial conditions problem* for dynamic panel data models. See Wooldridge (2005) and Honoré and Tamer (2006) for discussions on the initial conditions problem in the contexts of nonlinear and binary outcome models. Blundell and Bond (1998) and Hahn (1999) use semiparametric distributions to obtain identifying restrictions and efficiency gain. In applications, the initial condition $F_{Y_1 U}$ together with the function g are important to disentangle spurious state dependence of a long-run outcome (Heckman, 1981a,b).

the panel upon $D_t = 0$, as in the case of death. Consequently, data is observed in the following manner: (Y_2, D_2) is observed if $D_1 = 1$; (Y_3, D_3) is observed if $D_1 = D_2 = 1$; and so on. Heckman and Navarro (2007) introduced this formulation of dynamic selection.

How can we nonparametrically point identify the nonseparable functions (g, h) and the initial condition $F_{Y_1 U}$ under this setup of endogenously unbalanced panel data? Common ways to handle selection include matching and weighting. These approaches, however, presume selection on observables, parametric models, and additively separable models, none of which is assumed in this paper. Even without selection, the standard panel data techniques such as first differencing, demeaning, projection, and moment restrictions do not generally work for nonseparable and nonparametric models.

The literature on nonseparable cross section models proposes constructing auxiliary variables, such as a proxy variable or a control variable, to remove endogeneity (e.g., Garen, 1984; Imbens and Newey, 2009).[4] Likewise, Altonji and Matzkin (2005) show that a control variable can be also constructed from panel data for sibling and neighborhood panels. This paper complements Altonji and Matzkin along two dimensions. First, we show that a proxy variable can be constructed from dynamic panel data, similar to their construction of a control variable from sibling and neighborhood panel data. Second, the proxy variable, akin to a control variable,[5] handles not only nonseparable heterogeneity, but also dynamic selection. We propose a method of using the proxy variable to *nonparametrically difference out* both nonseparable heterogeneity and dynamic selection at the same time.

The nonparametric differencing relies on a nonclassical proxy variable, which we define as a noisy signal of true unobserved heterogeneity with a nonseparable noise. This definition is reminiscent of nonclassical measurement errors (ME).[6] A natural approach to identification, therefore, is to adapt the methods used in the nonclassical ME literature to the current context. This paper follows the spectral decomposition approach (e.g., Hu, 2008; Hu and Schennach, 2008) to nonparametrically identify mixture components.[7]

The identification procedure is outlined as follows. First, the method of nonparametric differencing removes the influence of nonseparable heterogeneity and selection. After removing these two sources of bias, the spectral decomposition identifies the mixture component $f_{Y_t | Y_{t-1} U}$, which in turn represents the observational equivalence class of the true nonseparable function g by normalizing the distribution of the exogenous error $\mathcal{E}_t$, following Matzkin (2003, 2007). This sequence yields nonparametric point identification of g from short unbalanced panel data. The selection function h can be similarly identified by a few additional steps of the spectral decomposition and solving integral equations[8] to identify representing mixture components.

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[4]Chesher (2003) can be also viewed as a control variable method, cf. Imbens and Newey (2009; Theorem 2).

[5]Proxy and control variables are similar in that both of them are correlated with unobserved factors. But they differ in terms of independence conditions: if X denotes an endogenous regressor and U denotes unobserved factors, then a proxy variable Z and a control variable Z' satisfy $Z \perp\!\!\!\perp X \mid U$ and $U \perp\!\!\!\perp X \mid Z'$, respectively.

[6]See Lewbel (2006), Mahajan (2006), Schennach (2007), Hu (2008), Hu and Schennach (2008), and Schennach, Song, and White (2011) for the literature on the nonclassical ME.

[7]See Henry, Kitamura, and Salanié (2010) for general identification results for mixture models.

[8]Precisely, they are the Fredholm equations of the first kind. See Carrasco, Florens, and Renault (2007).

2 Background

Selection is of natural interest in panel data analysis because attrition is an issue in most, if not all, panel data sets. While many applications focus on the dynamic model g as the object of primary interest, the selection function h also helps to explain important causal effects in a variety of economic problems. In the SES and mortality example, identification of the survival selection function h allows us to learn about the causal effects of SES on mortality. Generally, the selection function h can be used to model hazards of panel attrition. Examples include (i) school dropout (Cameron and Heckman, 1998; Eckstein and Wolpin, 1999; Belzil and Hansen, 2002; Heckman and Navarro, 2007); (ii) retirement from a job (Stock and Wise, 1990; Rust and Phelan, 1997; Karlstrom, Palme, and Svensson, 2004; French, 2005; Aguirregabiria, 2010; French and Jones, 2011); (iii) replacement of depreciated capital (Rust, 1987) and replacement of managers (Brown, Goetzmann, Ibbotson, and Ross, 1992); (iv) sterilization (Hotz and Miller, 1993); (v) exit from markets (Aguirregabiria and Mira, 2007; Pakes, Ostrovsky, and Berry, 2007); (vi) recovery from a disease (Crawford and Shum, 2005); and (vii) death (Contoyannis, Jones, and Rice, 2004; Halliday, 2008). Examples (i)–(v) are particularly motivated by rational hazards formulated in the following structural framework.

Example 1 (Optimal Stopping as a Rational Choice of Hazard)**.** Suppose that an economic agent knows her current utility or profit as a function π of state y_t and heterogeneity u. Let v_t^d denote a selection-specific private shock for each choice $d \in \{0, 1\}$, which is known to the agent. She also knows her exit value as a function $\overline{\nu}$ of state y_t and heterogeneity u. Using the dynamic function g, define the value function ν as the fixed point of the Bellman equation

$$\nu(y_t, u) = \mathrm{E}[\max\{\pi(y_t, u) + V_t^1 + \beta \mathrm{E}[\nu(g(y_t, u, \mathcal{E}_{t+1}), u)], \quad \pi(y_t, u) + V_t^0 + \beta\overline{\nu}(y_t, u)\}],$$

where β denotes the rate of time preference. The reduced-form self-selection function h is then defined by

$$h(y_t, u, v_t) := \mathbb{1}\{\underbrace{\beta \mathrm{E}[\nu(g(y_t, u, \mathcal{E}_{t+1}), u)]}_{\text{Continuation value}} - \underbrace{\beta\overline{\nu}(y_t, u)}_{\text{Exit Value}} \geqslant \underbrace{v_t^0 - v_t^1}_{\substack{\| \\ v_t}}\}.$$

The agent decides to exit at time t if $h(Y_t, U, V_t) = 0$. Identification of the reduced form h is important in many applications.[9] Moreover, the reduced form h also reveals the *heterogeneous* conditional choice probability (CCP), $f_{D_t|Y_tU}$, which in turn can be used to recover heterogeneous structural primitives by using the method of Hotz and Miller (1993).[10] □

[9] Counterfactual policy analysis is often possible with reduced-form selection function as a sufficient statistic; see the Marschak's (1953) maxim discussed by Heckman (2000) and Heckman and Vytlacil (2007).

[10] I keep identification of the primitives out of the scope of this paper. Primitives are known to be generally under-identified without additional restrictions (Rust, 1994; Magnac and Thesmar, 2002; Pesendorfer and Schmidt-Dengler, 2008). These features may be more generally treated in the literature of set identification and set inference, e.g., Bajari, Benkard, and Levin (2007) and the follow-up literature.

Identification of CCP under finite heterogeneous types has been discussed by Magnac and Thesmar (2002) and Kasahara and Shimotsu (2009). Aguirregabiria and Mira (2007) considered market-level unobserved heterogeneity as a variant of their main model. While we focus on identification of heterogeneous CCP, Arcidiacono and Miller (2011) suggested a method of estimating heterogeneous CCP.

As this example suggests, nonparametric identification of the heterogeneous CCP follows as a byproduct of our identification results,[11] showing a connection between this paper and the literature on structural dynamic discrete choice models. When attrition, $D_t = 0$, is associated with hazards or ends of some duration, our identification results also entail nonparametric identification of the mixed hazard model and the distribution of unobserved heterogeneity.[12] In this sense, our objective is also related to the literature on duration analysis (e.g., Lancaster, 1979; Elbers and Ridder, 1982; Heckman and Singer, 1984; Honoré, 1990; Ridder, 1990; Horowitz, 1999; Ridder and Woutersen, 2003).

The paper covers three econometric topics, (A) panel data, (B) selection/missing data, and (C) nonseparable models. To show the place of this paper, I briefly discuss these related branches of the literature. Because the field is extensive, the following list is not exhaustive.

(A) and (B): panel data with selection has been discussed from the perspective of (i) a selection model (Hausman and Wise, 1979; Das, 2004), (ii) variance adjustment (Baltagi, 1985; Baltagi and Chang, 1994), (iii) additional data such as refreshment samples (Ridder, 1992; Hirano, Imbens, Ridder, and Rubin, 2001; Bhattacharya, 2008), (iv) matching (Kyriazidou, 1997), (v) weighting (Hellerstein and Imbens, 1999; Moffitt, Fitzgerald, and Gottschalk, 1999; Wooldridge, 2002), and (vi) partial identification (Khan, Ponomareva, and Tamer, 2011). We contribute to this literature by allowing nonseparability in addition to selection/missing data.

(A) and (C): nonseparable panel models have been treated with (i) random coefficients and interactive fixed effects (Hsiao, 1975; Pesaran and Smith, 1995; Hsiao and Pesaran, 2004; Graham annd Powell, 2008; Arellano and Bonhomme, 2009; Bai, 2009). (ii) bias reduction (discussed in the extensive body of literature surveyed by Arellano and Hahn, 2005), (iii) identification of local partial effects (Altonji and Matzkin, 2005; Altonji, Ichimura, and Otsu, 2011; Graham and Powell, 2008; Arellano and Bonhomme, 2009; Bester and Hansen, 2009; Chernozhukov, Fernández-Val, Hahn, and Newey, 2009; Hoderlein and White, 2009), (iv) partial identification (Honoré and Tamer, 2006; Chernozhukov, Fernández-Val, Hahn, and Newey, 2010), (v) partial separability (Evdokimov, 2009), and (vi) assumptions of surjective and/or injective operators (Kasahara and Shimotsu, 2009; Bonhomme, 2010; Hu and Shum, 2010; Shiu and Hu, 2011). The paper contributes to this literature by introducing selection/missing data in addition to allowing nonseparability.

Identification of a nonseparable dynamic panel data model is studied by Shiu and Hu (2011) who use independently evolving covariates as auxiliary variables, similar to one of the two identification results of this paper using a proxy as an auxiliary variable. This paper complements Shiu and Hu along two dimensions. First, our identification result using $T = 6$ periods eliminates the need to assume the independently evolving covariates or any other auxiliary variable. Second, we can allow for selection/missing data in addition to dynamics and nonseparability.

[11] Taking the expectation of $h(y, u, \cdot)$ with respect to the distribution of the exogenous error V_t yields the heterogeneous CCP, $f_{D_t|Y_tU}(1 \mid y, u)$ for each (y, u). The heterogeneous CCP is also identified by Kasahara and Shimotsu (2009), which this paper complements by introducing missing observations in data.

[12] The nonparametric mixed hazard model and the marginal distribution F_U of unobserved heterogeneity follow from the identified survival selection function h and the initial condition F_{Y_1U}, respectively.

3 An Overview

We start out with an informal overview of the identification strategy in this section, followed by formal identification results summarized in Section 4.

Briefly described, the nonparametric differencing method works in the following manner. Let z denote a proxy variable. Observed data A_z, which are contaminated by mixed heterogeneity and selection, can be decomposed as $A_z = B_z C$, where B_z contains model information and C contains the two sources of bias, i.e., heterogeneity and selection. The contaminant holder, C, does not depend on z by an exclusion restriction. Thus, using two values of z, say $z = 0, 1$, selectively eliminates C by the operator composition $A_1 A_0^{-1} = B_1 C C^{-1} B_0^{-1} = B_1 B_0^{-1}$, without losing the model information B_z. This shows how heterogeneity and selection contained in C are nonparametrically differenced out, and is analogous to the familiar first differencing method which eliminates fixed effects by using two values of t instead of two values of z.

Section 3.1 sketches the identification strategy using $T = 3$ periods of panel data and a proxy variable. An intuition is the following. First, using variations in Y_1 in the equation $y_2 = g(Y_1, U, \mathcal{E}_2)$ involving the first two periods, $t = 1, 2$, we can retrieve information about $(U, \mathcal{E}_2)$ associated with $Y_2 = y_2$. This is comparable to the first stage in the cross section context except for the endogeneity of the first-stage regressor Y_1. The proxy variable, which is correlated with Y_1 only through U, disentangles U and Y_1 to fix the endogeneity. We then use this knowledge about U to identify the heterogeneous dynamic through the equation $Y_3 = g(y_2, U, \mathcal{E}_3)$ involving the latter two periods, $t = 2, 3$, which is comparable to the second stage.

Section 3.2 sketches the identification strategy using $T = 6$ periods without a proxy variable. With six periods, the three consecutive observations, Y_2, Y_3 and Y_4, together constitute a substitute for the proxy. Intuitively, controlling for the adjacent states, Y_2 and Y_4, the intermediate state Y_3 is correlated with (Y_1, Y_5, Y_6) only through the heterogeneity U. This allows Y_3 to serve as a proxy for U, conditionally on Y_2 and Y_4. The constructed proxy identifies both $F_{Y_6|Y_5U}$, which represents the dynamic function g, and the initial condition F_{Y_1U}.

3.1 A Sketch of the Identification Strategy

Consider the model $(g, h, F_{Y_1U}, \zeta, F_{\mathcal{E}_t}, F_{V_t}, F_W)$ where

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T & \text{(State Dynamics)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T-1 & \text{(Selection)} \\ F_{Y_1U} & & \text{(Initial Condition)} \\ Z = \zeta(U, W) & & \text{(Optional: Nonclassical Proxy)} \end{cases} \tag{3.1}$$

The observed state variable Y_t, such as SES, follows a first-order Markov process g with nonseparable heterogeneity U. The selection variable D_t, such as survival, follows a Markov decision process h with heterogeneity U. The outcome $D_t = 0$, such as death, indicates attrition after which the counterfactual state variable Y_t becomes unobservable. The distribution F_{Y_1U} of (Y_1, U) models dependence of the initial state Y_1 on unobserved heterogeneity U. The last optional equation models the proxy variable Z as a noisy signal of the true unobserved heterogeneity U with a nonseparable noise variable W.[13] This proxy equation is optional when $T \geqslant 6$, because the three additional periods construct the proxy. The functional relations in (3.1) together with the following exogeneity assumptions define the econometric model.

$$\begin{array}{lll} \text{(i)} & \text{Exogeneity of } \mathcal{E}_t\text{:} & \mathcal{E}_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s<t}, \{V_s\}_{s<t}, W) \text{ for all } t \geq 2. \\ \text{(ii)} & \text{Exogeneity of } V_t\text{:} & V_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s\leqslant t}, \{V_s\}_{s<t}) \text{ for all } t \geq 1. \\ \text{(iii)} & \text{Exogeneity of } W\text{:} & W \perp\!\!\!\perp (Y_1, \{\mathcal{E}_t\}_t, \{V_t\}_t). \end{array} \tag{3.2}$$

This construction of the model leaves the nonseparable fixed effect U as the only source of endogeneity, and is in accordance with the standard assumptions in the panel data literature. We assume that the exogenous shocks, $\mathcal{E}_t$, V_t, and the noise, W, are continuously distributed, and normalize the distributions $F_{\mathcal{E}_t}, F_{V_t}$, and F_W to Uniform$(0,1)$ so that the model consists of only the four elements $(g, h, F_{Y_1 U}, \zeta)$.

The nonseparable fixed effect U and the exogenous errors $\mathcal{E}_t$, V_t and W are unobservable by econometricians. Observation of the the state variable Y_t is contingent on self-selection by economic agents. For a three-period panel data, the states are observed according to the rule:

$$\begin{array}{ll} \text{Observe } Y_1. & \\ \text{Observe } Y_2 & \text{if } D_1 = 1. \\ \text{Observe } Y_3 & \text{if } D_1 = D_2 = 1. \end{array}$$

Consequently, panel data reveals the parts, $F_{Y_2 Y_1 Z D_1}(\,\cdot,\ \cdot,\ \cdot, 1)$ and $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot, 1, 1)$, of the joint distributions, but they will become unobservable once a '1' is replaced by a '0' in the slot of D_t.

In the current section, we consider a special case where Y_t, U, and Z are Bernoulli random variables for ease of exposition. This special case conveniently allows to describe the identification strategy by means of matrices instead of operators. The basic idea of the identification strategy for this special case extends to more general cases, as formally stated in Section 4. For this setting, the function g can be represented by a heterogeneous Markov transition probability, $f_{Y_{t+1}|Y_t U}$. Similarly, the selection function h can be represented by the heterogeneous conditional choice probability (heterogeneous CCP), $f_{D_t|Y_t U}$. In this way, the model $(g, h, F_{Y_1 U}, \zeta)$ in (3.1) can be represented by the quadruple $(f_{Y_{t+1}|Y_t U},\ f_{D_t|Y_t U},\ f_{Y_1 U},\ f_{Z|U})$ of conditional and joint mass functions. Given this statistical representation, identification amounts to that

$$f_{Y_2 Y_1 Z D_1}(\,\cdot,\ \cdot,\ \cdot, 1) \ \&\ f_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot, 1, 1) \quad \overset{\substack{\text{uniquely}\\ \text{determine}}}{\longmapsto} \quad (f_{Y_{t+1}|Y_t U},\ f_{D_t|Y_t U},\ f_{Y_1 U},\ f_{Z|U}).$$

The exogeneity in (3.2) implies the following two conditional independence restrictions:

$$\text{Exogeneity of } \mathcal{E}_3 \Rightarrow \text{ Markov Property:} \quad Y_3 \perp\!\!\!\perp (Y_1, D_1, D_2, Z) \mid (Y_2, U) \tag{3.3}$$

$$\text{Exogeneity of } W \Rightarrow \text{Redundant Proxy:} \quad Z \perp\!\!\!\perp (Y_2, Y_1, D_2, D_1) \mid U \tag{3.4}$$

[13] A standard proxy Z is an additively separable function of U and W (cf. Wooldridge, 2001; Ch. 4). Our proxy model ζ allows for nonseparability and nonlinearity to avoid a misspecification bias. One can think of the pair (U, W) as fixed unobserved characteristics, where U is the part that enters the economic model whereas W is the part excluded from these functions (i.e., exclusion restriction). Therefore, W is exogenous by construction.

See Lemma 3 in the appendix for a derivation of the above conditional independence restrictions. The Markov property (3.3) states that the current state Y_2 and the heterogeneity U are sufficient statistics for the distribution of the next state Y_3. The redundant proxy (3.4) states that, once the true heterogeneity U is controlled for, the proxy Z is redundant for the model.[14] These independence restrictions derive the following chain of equalities for each y_1, y, y_3, z:

$$
\begin{aligned}
&\underbrace{f_{Y_3Y_2Y_1ZD_2D_1}(y_3, y, y_1, z, 1, 1)}_{\text{Observed Data}} = \sum_u f_{Y_3Y_2Y_1ZUD_2D_1}(y_3, y, y_1, z, u, 1, 1) \\
= \;& \sum_u f_{Y_3|Y_2Y_1ZUD_2D_1}(y_3 \mid y, y_1, z, u, 1, 1) \cdot f_{Z|Y_2Y_1UD_2D_1}(z \mid y, y_1, u, 1, 1) \cdot f_{Y_2Y_1UD_2D_1}(y, y_1, u, 1, 1) \\
\overset{(*)}{=} \;& \sum_u \underbrace{f_{Y_3|Y_2U}(y_3 \mid y, u)}_{\text{Model } g} \cdot \underbrace{f_{Z|U}(z \mid u)}_{\text{Model } \zeta} \cdot \underbrace{f_{Y_2Y_1UD_2D_1}(y, y_1, u, 1, 1)}_{\substack{\text{Nonparametric Residual} \\ \text{Involving Selection } D_2 = D_1 = 1 \\ \text{\& Nonseparable Fixed Effect } U}}
\end{aligned}
\tag{3.5}
$$

where the last equality $(*)$ follows from (3.3) and (3.4). The object $f_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot, 1, 1)$ on the left-hand side can be observed from data because the slots of D_1 and D_2 contain '1.' The right-hand side consists of three factors, where $f_{Y_3|Y_2U}$ and $f_{Z|U}$ represent g and ζ, respectively. The last factor $f_{Y_2Y_1UD_2D_1}(\,\cdot,\ \cdot,\ \cdot, 1, 1)$ can be thought of as the *nonparametric residual* of the observed data after extracting the two preceding economic components, g and ζ. This nonparametric residual absorbs the selection, $D_2 = D_1 = 1$, which is a source of selection bias. Moreover, the nonparametric residual also absorbs the nonparametric distribution of the nonseparable fixed effect U, which is a source of endogeneity bias. In other words, the two sources of bias – nonseparable heterogeneity and selection – captured by the nonparametric residual are isolated from the economic models (g, ζ) in the decomposition (3.5).

For convenience of calculation, we rewrite the equality (3.5) in terms of matrices as

$$
L_{y,z} = P_y Q_z \tilde{L}_y \qquad \text{for each } y \in \mathcal{Y} \text{ and } z \in \mathcal{Z}, \tag{3.6}
$$

where $L_{y,z}$, P_y, Q_z, and $\tilde{L}_y$ are defined as[15]

$$
\begin{aligned}
L_{y,z} &:= \begin{bmatrix} f_{Y_3Y_2Y_1ZD_2D_1}(0, y, 0, z, 1, 1) & f_{Y_3Y_2Y_1ZD_2D_1}(0, y, 1, z, 1, 1) \\ f_{Y_3Y_2Y_1ZD_2D_1}(1, y, 0, z, 1, 1) & f_{Y_3Y_2Y_1ZD_2D_1}(1, y, 1, z, 1, 1) \end{bmatrix} && \rightarrow \text{Observed data} \\
P_y &:= \begin{bmatrix} f_{Y_3|Y_2U}(0 \mid y, 0) & f_{Y_3|Y_2U}(0 \mid y, 1) \\ f_{Y_3|Y_2U}(1 \mid y, 0) & f_{Y_3|Y_2U}(1 \mid y, 1) \end{bmatrix} && \rightarrow \text{Represents model } g \\
Q_z &:= \operatorname{diag}(f_{Z|U}(z \mid 0) \quad f_{Z|U}(z \mid 1))' && \rightarrow \text{Represents model } \zeta \\
\tilde{L}_y &:= \begin{bmatrix} f_{Y_2Y_1UD_2D_1}(y, 0, 0, 1, 1) & f_{Y_2Y_1UD_2D_1}(y, 1, 0, 1, 1) \\ f_{Y_2Y_1UD_2D_1}(y, 0, 1, 1, 1) & f_{Y_2Y_1UD_2D_1}(y, 1, 1, 1, 1) \end{bmatrix} && \rightarrow \text{Residual}
\end{aligned}
$$

In (3.6), the observed matrix $L_{y,z}$ is decomposed into three factors, P_y, Q_z, and $\tilde{L}_y$. The matrices P_y and Q_z represent the economic models g and ζ, respectively. The matrix $\tilde{L}_y$ contains the remainder as the nonparametric residual, and particularly contains the two sources of bias.

[14] The redundant proxy assumption is stated in terms of conditional moments in the context of linear additively separable models; see Wooldridge (2001), Ch. 4.

[15] These two-by-two matrices follow from the simplifying assumption of the current subsection that Y_t, U, and Z are Bernoulli random variables. In general cases, integral and multiplication operators replace these matrices.

Given the decomposition (3.6), the next step is to eliminate the nonparametric residual matrix $\tilde{L}_y$ in order to nonparametrically difference out the influence of selection and nonseparable heterogeneity, or to remove biases induced by them. Using the two values of z of the proxy variable, 0 and 1, we form the following composition:

$$\underbrace{L_{y,1}L_{y,0}^{-1}}_{\text{Observed Data}} = P_y Q_1 \tilde{L}_y \tilde{L}_y^{-1} Q_0^{-1} P_y^{-1} = \underbrace{P_y}_{g} \underbrace{Q_1 Q_0^{-1}}_{\zeta} \underbrace{P_y^{-1}}_{g}. \tag{3.7}$$

The nonparametric residual matrix $\tilde{L}_y$ has been eliminated as desired. Consequently, the observed data on the left hand side is now purely linked to a product of model components (g, ζ) without any influence of the selection or the nonseparable heterogeneity.

The composition (3.7) is valid provided that P_y, Q_z, and $\tilde{L}_y$ are all non-singular. The following rank restrictions guarantee that they are indeed non-singular under the current setting.

$$\text{Heterogeneous Dynamics:} \quad \mathrm{E}[g(y,0,\mathcal{E}_t)] \neq \mathrm{E}[g(y,1,\mathcal{E}_t)] \tag{3.8}$$

$$\text{Nondegenerate Proxy Model:} \quad 0 < \mathrm{E}[\zeta(u,W)] < 1 \text{ for each } u \in \{0,1\} \tag{3.9}$$

$$\text{No Extinction:} \quad \mathrm{E}[h(y,u,V_t)] > 0 \text{ for each } u \in \{0,1\} \tag{3.10}$$

$$\text{Initial Heterogeneity:} \quad \begin{array}{c} \mathrm{E}[U \mid Y_2 = y, Y_1 = 0, D_1 = 1] \neq \\ \mathrm{E}[U \mid Y_2 = y, Y_1 = 1, D_1 = 1] \end{array} \tag{3.11}$$

Restriction (3.8) requires that the dynamic model g is a nontrivial function of the unobserved heterogeneity, and implies that the matrix P_y is non-singular.[16] Restriction (3.9) requires that the proxy model (ζ, F_W) exhibits nondegeneracy, and implies that the matrix Q_0 is non-singular. Restriction (3.10) requires a positive survival probability for each heterogeneous type $u \in \{0,1\}$, and hence drives no type U into extinction. Restriction (3.11) requires that the unobserved heterogeneity is present at the initial observation. The last two restrictions (3.10) and (3.11) together imply that the nonparametric residual matrix $\tilde{L}_y$ is non-singular.

Now that the nonparametric residual $\tilde{L}_y$ containing the two sources of bias has gone, it remains to identify the elements of the matrices P_y and Q_z from equation (3.7). This can be accomplished by showing the uniqueness of eigenvalues and eigenvectors (e.g., Hu, 2008; Kasahara and Shimotsu, 2009). Because the matrix Q_z is diagonal, (3.7) forms a diagonalization of the observable matrix $L_{y,1}L_{y,0}^{-1}$. The diagonal elements of $Q_1 Q_0^{-1}$ and the columns of P_y are the eigenvalues and the eigenvectors of $L_{y,1}L_{y,0}^{-1}$, respectively. Therefore, $Q_1 Q_0^{-1}$ is identified by the eigenvalues of the observable matrix $L_{y,1}L_{y,0}^{-1}$ without an additional assumption.

On the other hand, identification of P_y follows from the following additional restriction:

$$\text{Relevant Proxy:} \quad \mathrm{E}[\zeta(0,W)] \neq \mathrm{E}[\zeta(1,W)]. \tag{3.12}$$

This restriction (3.12) requires that the proxy model ζ is a nontrivial function of the true unobserved heterogeneity on average. It characterizes the relevance of Z as a proxy of U,

[16] Under the current simplified setting with Bernoulli random variables, a violation of this assumption implies absence of endogeneity in the dynamic model, and thus the dynamic function g would still be identified. However, the other functions are not guaranteed to be identified without this assumption.

and implies that the elements of $Q_1 Q_0^{-1}$ (i.e., the eigenvalues of $L_{y,1} L_{y,0}^{-1}$) are distinct. The distinct eigenvalues uniquely determine the corresponding eigenvectors up to scale. Since an eigenvector $[f_{Y_{t+1}|Y_t U}(0 \mid y,u),\ f_{Y_{t+1}|Y_t U}(1 \mid y,u)]'$ is a vector of conditional densities which sum to one, the scale is also uniquely determined. Therefore, P_y and $Q_1 Q_0^{-1}$ are identified by the observed data $L_{y,1} L_{y,0}^{-1}$. The identified eigenvalues[17] take the form of the proxy odds $f_{Z|U}(1 \mid u)/(1 - f_{Z|U}(1 \mid u))$, which in turn uniquely determines the diagonal elements $f_{Z|U}(z \mid u)$ of Q_z for each z. This procedure heuristically shows how the the elements (g, ζ) of the model is identified from endogenously unbalanced panel data.

Remark 1. The general identification procedure consists of six steps. The current subsection presents a sketch of the first step to identify (g, ζ). Five additional steps show that the remaining two elements $(h, F_{Y_1 U})$ of the model are also identified. Figure 1 summarizes all the six steps. Section 4 presents a complete identification result.

Discussion 1. This sketch of the identification strategy demonstrates how the proxy handles both selection and nonseparable heterogeneity at the same time. The trick of Equation (3.5) or (3.6) is to isolate the selection ($D_1 = D_2 = 1$) and the nonparametric distribution of the nonseparable heterogeneity U into the nonparametric residual matrix $\tilde{L}_y$, which in turn is eliminated in Equation (3.7). Our method thus can be considered as a *nonparametric differencing* facilitated by a proxy variable, nonparametrically differencing out both nonseparable fixed effects and endogeneous selection. This process is analogous to the first differencing method which differences out fixed effects arithmetically. Our nonparametric differencing occurs in the non-commutative group of matrices (generally the group of linear operators), whereas the first differencing occurs in $(\mathbb{R}, +)$. In the non-commutative group, the proxy Z plays the role of selectively canceling out the nonparametric residual matrix $\tilde{L}_y$ while leaving the P_y and Q_z matrices intact. The use of a proxy parallels the classical idea of using instruments as means of removing endogeneity (Hausman and Taylor, 1981). Instrumental variables are useful for additive models because projection (moment restriction) of additive models on instruments removes fixed effects as in Hausman and Taylor. This projection method is not generally feasible for nonseparable and nonparametric models. Therefore, this paper uses a proxy variable, akin to a control variable,[18] to nonparametrically difference out the nonseparable fixed effect along with selection as argued above. This point is revisited in Section 3.2: Discussion 3.

3.2 A Sketch of the Identification Strategy for $T = 6$

When $T = 6$, we identify the model $(g, h, F_{Y_1 U})$ without using an outside proxy variable or the proxy model ζ. In the presence of an outside proxy, the main identification strategy was to

[17] Even though we obtain real eigenvalues in this spectral decomposition, $L_{y,1} L_{y,0}^{-1}$ need not be symmetric. Note that a Hermitian operator is sufficient for real spectrum, but not necessary. This identification result holds as far as the identifying restrictions are satisfied.

[18] If X denotes an endogenous regressor and U denotes unobserved factors, then a proxy, a control variable, and an instrument are characterized by $Z \perp\!\!\!\perp X \mid U$, $X \perp\!\!\!\perp U \mid Z$, and $Z \perp\!\!\!\perp U$, respectively. The conditional independence (3.4) thus characterizes Z as a proxy rather than a control variable or an instrument.

derive the decomposition $L_{y,z} = P_y Q_z \tilde{L}_y$ from which $\tilde{L}_y$ was eliminated (cf. Section 3.1). A similar idea applies to the case of $T = 6$ without an outside proxy.

Again, assume that Y_t, U, and Z follow the Bernoulli distribution for ease of exposition. Let $Z := Y_3$ for notational convenience. Using the exogeneity restriction (3.3) yields the following decomposition of the observed data $f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,1,1,1,1,1)$.

$$\underbrace{f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(y_6,y_5,y_4,\overbrace{z}^{y_3},y_2,y_1,1,1,1,1,1)}_{\text{Observed from Data } \to\ L_{y_5,y_4,z,y_2}} = \sum_u \underbrace{f_{Y_6|Y_5U}(y_6 \mid y_5,u)}_{\text{Model } g\ \to\ P_{y_5}}$$
$$\times \underbrace{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,z,1,1,1 \mid y_2,u)}_{\text{An Alternative to Proxy Model } \zeta\ \to\ Q_{y_4,z,y_2}} \cdot \underbrace{f_{Y_5|Y_4U}(y_5 \mid y_4,u) \cdot f_{Y_2Y_1UD_2D_1}(y_2,y_1,u,1,1)}_{\text{To Be Eliminated } \to\ \tilde{L}_{y_5,y_4,y_2}}$$

This equality can be equivalently written in terms of matrices as $L_{y_5,y_4,z,y_2} = P_{y_5} Q_{y_4,z,y_2} \tilde{L}_{y_5,y_4,y_2}$ for each (y_5,y_4,z,y_2), where the 2×2 matrices are defined as

$$\begin{aligned}
L_{y_5,y_4,z,y_2} &:= [f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(i,y_5,y_4,y_3,y_2,j,1,1,1,1,1)]_{(i,j)\in\{0,1\}\times\{0,1\}} \\
P_{y_5} &:= \left[f_{Y_6|Y_5U}(i \mid y_5,j)\right]_{(i,j)\in\{0,1\}\times\{0,1\}} \\
Q_{y_4,z,y_2} &:= \operatorname{diag}\left(f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,z,1,1,1 \mid y_2,0) \quad f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,z,1,1,1 \mid y_2,1)\right)' \\
\tilde{L}_{y_5y_4y_2} &:= \left[f_{Y_5|Y_4U}(y_5 \mid y_4,u) \cdot f_{Y_2Y_1UD_2D_1}(y_2,j,i,1,1)\right]_{(i,j)\in\{0,1\}\times\{0,1\}}
\end{aligned}$$

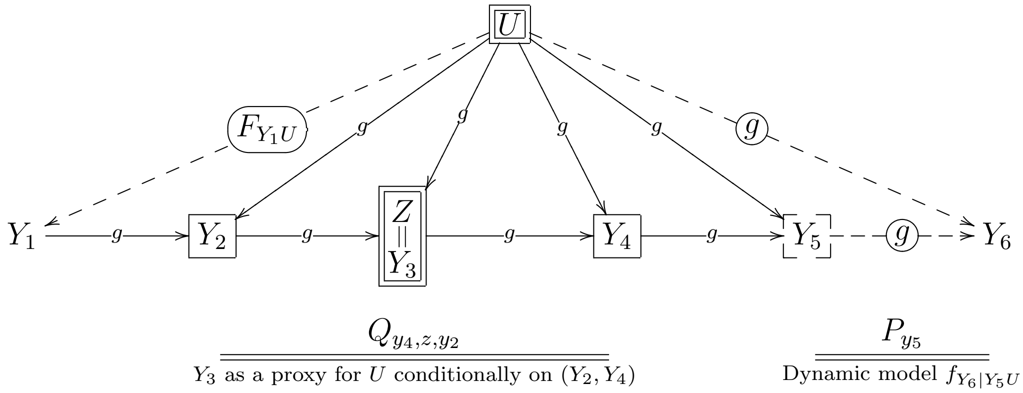
Similarly to the case with an outside proxy, varying $z = y_3$ while fixing (y_5,y_4,y_2) eliminates $\tilde{L}_{y_5,y_4,y_2}$ because it does not depend on $z = y_3$. Under rank restrictions, the composition

$$\underbrace{L_{y_5,y_4,1,y_2} L^{-1}_{y_5,y_4,0,y_2}}_{\text{Observed Data}} = P_{y_5} Q_{y_4,1,y_2} \tilde{L}_{y_5y_4y_2} \tilde{L}_{y_5y_4y_2} Q^{-1}_{y_4,0,y_2} P^{-1}_{y_5} = P_{y_5} \underbrace{Q_{y_4,1,y_2} Q^{-1}_{y_4,0,y_2}}_{\text{Diagonal}} P^{-1}_{y_5}$$

yields the eigenvalue-eigenvector decomposition to identify the dynamic model g represented by the matrix P_{y_5}. Five additional steps identify the rest (h, F_{Y_1U}) of the model.

Discussion 2. Why do we need $T = 6$? For convenience of illustration, ignore selection. The arrows in the diagram below indicate the directions of the causal effects. We are interested in g and F_{Y_1U} enclosed by round shapes. First, note that a variation in Y_6 in response to a variation in (Y_5, U) reveals g. Second, a variation in U in response to a variation in Y_1 reveals $F_{U|Y_1}$, hence F_{Y_1U}. We can see from the causal diagram that Y_2, Y_3, and Y_4 are correlated with U, and hence we may conjecture that they could serve as a proxy of U. However, any of them, say $Z := Y_3$, cannot be a genuine proxy because the redundant proxy assumption similar to (3.4), say $Z \perp\!\!\!\perp Y_5 \mid Z$, would be violated with this choice $Z = Y_3$. That is, even if we control for U, Y_3 is still correlated with Y_5 through the dynamic channel along the horizontal arrows. In order to shut out this dynamic channel, we control the intermediate state Y_4 between Y_3 and Y_5. Using the language of the causal inference literature, we say that (U, Y_4) "d-separates" Y_3 and Y_5 in the causal diagram below, and this d-separation implies the conditional independence restriction $Y_3 \perp\!\!\!\perp Y_5 \mid (U, Y_4)$; see Pearl (2000). Therefore Y_3 is now a genuine proxy of U conditionally on the fixed Y_4 to analyze the dynamic model g. Similarly, we control the intermediate state

Y_2 between Y_1 and Y_3 to analyze the initial condition F_{Y_1U}. The causal diagram indicates that (U, Y_2) "d-separates" Y_1 and Y_3, hence $Y_3 \perp\!\!\!\perp Y_1 \mid (U, Y_2)$. This makes Y_3 a genuine proxy of U conditionally on the fixed Y_2 to analyze the initial condition F_{Y_1U}. Controlling for the two adjacent states, Y_2 and Y_4, costs the consecutive three periods (Y_2, Y_3, Y_4) for the constructed proxy model Q_{y_4,z,y_2}. This is an intuition behind the requirement of three additional periods for identification without an outside proxy variable. See Appendix C.2 for a formal proof.



Discussion 3. The idea of using three additional time periods to construct a proxy variable parallels the well-known idea of using lags as instruments to form identifying restrictions for additively separable dynamic panel data models (e.g., Anderson and Hsiao, 1982; Arellano and Bond, 1991; Ahn and Schmidt, 1995; Arellano and Bover, 1995; Blundell and Bond, 1998; Hahn, 1999). Because projection or moment restriction on instruments is not generally a viable option for nonseparable and nonparametric models, the literature on nonseparable cross section models proposes constructing a proxy variable or a control variable from instruments (Garen, 1984; Florens, Heckman, Meghir, and Vytlacil, 2008; Imbens and Newey, 2009). Altonji and Matzkin (2005) show that a control variable can also be constructed from panel data for sibling and neighborhood panels. This paper proposes constructing a proxy variable from three additional observations of dynamic panel data, similar to Altonji and Matzkin's construction of a control variable from sibling and neighborhood panels. The constructed proxy variable turns out to account for not only nonseparable heterogeneity but also selection as argued above.

4 Identification

This section formalizes the identification result, a part of which is sketched in Section 3.

4.1 Identifying Restrictions

Identification is proved by showing the well-definition of the inverse DGP correspondence

$$(F_{Y_2Y_1ZD_1}(\,\cdot,\ \cdot,\ \cdot, 1),\ F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot, 1, 1)) \ \mapsto\ (g, h, F_{Y_1U}, \zeta, F_{\mathcal{E}_t}, F_{V_t}, F_W),$$

up to observational equivalence classes represented by a certain normalization of the error distributions $F_{\mathcal{E}_t}$, F_{V_t}, and F_W a la Matzkin (2003). To this end, we invoke the following four restrictions on the set of potential data-generating models.

Restriction 1 (Representation)**.** Each of the functions g, h, and ζ is non-decreasing and càglàd (left-continuous with right limit) in the last argument. The distributions of $\mathcal{E}_t$, V_t, and W are absolutely continuous with convex supports, and each of $\{\mathcal{E}_t\}_t$ and $\{V_t\}_t$ is identically distributed across t.

The weak – as opposed to strict – monotonicity of the functions with respect to idiosyncratic errors accommodates discrete outcomes Y_t, D_t, and Z under absolutely continuous distributions of errors $(\mathcal{E}_t, V_t, W)$. The purpose of Restriction 1 is to construct representations of the equivalence classes of nonseparable functions up to which g, h, and ζ are uniquely determined by the distributions $F_{Y_t|Y_{t-1}U}$, $F_{D_t|Y_tU}$, and $F_{Z|U}$, respectively. The independence restriction stated below in addition to Restriction 1 allows for their quantile representations in particular.

Restriction 2 (Independence)**.**
(i) Exogeneity of $\mathcal{E}_t$: $\mathcal{E}_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s<t}, \{V_s\}_{s<t}, W)$ for all $t \geq 2$.
(ii) Exogeneity of V_t: $V_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s\leqslant t}, \{V_s\}_{s<t})$ for all $t \geq 1$.
(iii) Exogeneity of W: $W \perp\!\!\!\perp (Y_1, \{\mathcal{E}_t\}_t, \{V_t\}_t)$.

In the context of Section 3.1, Restriction 2 (i) and (iii) imply the conditional independence restrictions (3.3) and (3.4), respectively. Parts (i) and (ii) impose exogeneity of the idiosyncratic errors $\mathcal{E}_t$ and V_t, thus leaving U as the only source of endogeneity. Part (iii) requires exogeneity of the noise W in the nonseparable proxy model ζ. This means that the unobserved characteristics consist of two parts (U, W) where U is the part that enters the functions g and h, whereas W is the part excluded from those functions (i.e., exclusion restriction), and hence is exogenous by construction. Part (iii) implies $Z \perp\!\!\!\perp (Y_1, \{\mathcal{E}_t\}_t, \{V_t\}_t) \mid U$, which is similar to the redundant proxy restriction in the classical sense as discussed in Section 3.1: once the true unobserved heterogeneity U is controlled for, the proxy Z is redundant for $(g, h, F_{Y_1,U})$. These independence conditions play the role of decomposing observed data into model components and the nonparametric residual, as we saw through the sketch in Section 3.

Restriction 3 (Rank Conditions)**.** The following conditions hold for every $y \in \mathcal{Y}$:
(i) Heterogeneous Dynamics: the integral operator $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ defined by $P_y\xi(y') = \int f_{Y_3|Y_2U}(y' \mid y, u) \cdot \xi(u)du$ is bounded and invertible.
(ii) Nondegenerate Proxy Model: there exists $\delta > 0$ such that $0 < f_{Z|U}(1 \mid u) \leqslant 1 - \delta$ for all u.
Relevant Proxy: $f_{Z|U}(1 \mid u) \neq f_{Z|U}(1 \mid u')$ whenever $u \neq u'$.
(iii) No Extinction: $f_{D_2|Y_2U}(1 \mid y, u) > 0$ for all $u \in \mathcal{U}$.
(iv) Initial Heterogeneity: the integral operator $S_y : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$ defined by $S_y\xi(u) = \int f_{Y_2Y_1UD_1U}(y, y', u, 1) \cdot \xi(y')dy'$ is bounded and invertible.

Under the special case discussed in Section 3.1, Restriction 3 is equivalent to (3.8)–(3.12), by which the dynamic function g and the proxy model ζ were identified in that section. The notion of 'invertibility' depends on the normed linear spaces on which the operators are defined. We use $\mathcal{L}^2$ in order to exploit convenient properties of the Hilbert spaces.[19] A bounded linear operator between Hilbert spaces guarantees existence and uniqueness of its adjoint operator, which of course presumes a pre-Hilbert space structure in particular. Moreover, the invertibility guarantees that the adjoint operator is also invertible, which is an important property used to derive identification of the selection rule h and initial condition $F_{Y_1 U}$. Andrews (2011) shows that a wide variety of injective operators between $\mathcal{L}^2$ spaces can be constructed from an orthonormal basis, and that the completeness assumption 'generically' holds.

The first part of the rank condition (ii) requires that the proxy model (ζ, F_W) exhibits nondegeneracy. The second part of the rank condition (ii) requires that Z is a relevant proxy for unobserved heterogeneity U, as characterized by distinct proxy scores $f_{Z|U}(1 \mid u)$ across u. The rank condition (iii) requires that there continue to exist some survivors in each heterogeneous type, hence no type U goes extinct. This restriction is natural because one cannot learn about a dynamic structure of the group of individuals that goes extinct after the first time period.

Restriction 4 (Labeling of U in Nonseparable Models). $u \equiv f_{Z|U}(1 \mid u)$ for all $u \in \mathcal{U}$.

Due to its unobservability, U has neither intrinsic values nor units of measurement. This is a reason for potential non-uniqueness of fully nonseparable functions. The purpose of Restriction 4 is to attach concrete values to unobserved heterogeneity U; see also Hu and Schennach (2008). Restriction 4 is innocuous in nonseparable models in the sense that identification is considered up to observational equivalence $g(y, u, \varepsilon) \equiv g_\pi(y, \pi(u), \varepsilon)$ for any permutation π of $\mathcal{U}$. On the other hand, this restriction is redundant and too stringent for additively separable models, in which U has the same unit of measurement as Y by construction. In the latter case, we can replace Restriction 4 by the following alternative labeling assumption.

Restriction 4′ (Labeling of U in Separable Models). $u \equiv \mathrm{E}[g(y, u, \mathcal{E})] - \tilde{g}(y)$ for all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ for some function $\tilde{g}$.

This alternative labeling restriction is innocuous for separable models in the sense that it is automatically satisfied by additive models of the form $g(y, u, \varepsilon) = \tilde{g}(y) + u + \varepsilon$ with $\mathrm{E}[\mathcal{E}] = 0$.

4.2 Representation

Nonparametric identification of nonseparable functions is generally feasible only up to some equivalence classes (e.g., Matzkin, 2003, 2007). Representations of these equivalence classes are discussed as a preliminary step toward identification. Restrictions 1 and 2 allow representations of functions g, h, and ζ by normalizing the distributions of the independent errors.

Lemma 1 (Quantile Representations of Non-Decreasing Càglàd Functions).
(i) Suppose that Restrictions 1 and 2 (i) hold. Then $F_{Y_3|Y_2 U}$ uniquely determines g up to the observational equivalence classes represented by the normalization $\mathcal{E}_t \sim \mathit{Uniform}(0, 1)$.

[19] Carrasco, Florens, and Renault (2007) review some important properties of operators on Hilbert spaces.

(ii) Suppose that Restrictions 1 and 2 (ii) hold. Then $F_{D_2|Y_2U}$ uniquely determines h up to the observational equivalence classes represented by the normalization $V_t \sim$ Uniform$(0,1)$.
(iii) Suppose that Restrictions 1 and 2 (iii) hold. Then $F_{Z|U}$ uniquely determines ζ up to the observational equivalence classes represented by the normalization $W \sim$ Uniform$(0,1)$.

A proof is given in Appendix A.1. The representations under these assumptions and normalizations are established by the respective quantile regressions:

$$\begin{array}{rclcll} g(y,u,\varepsilon) &=& F^{-1}_{Y_3|Y_2U}(\varepsilon \mid y,u) &:=& \inf\{y' \mid \varepsilon \leq F_{Y_3|Y_2U}(y' \mid y,u)\} & \forall(y,u,\varepsilon) \\ h(y,u,v) &=& F^{-1}_{D_2|Y_2U}(v \mid y,u) &:=& \inf\{d \mid v \leq F_{D_2|Y_2U}(d \mid y,u)\} & \forall(y,u,v) \\ \zeta(u,w) &=& F^{-1}_{Z|U}(w \mid u) &:=& \inf\{z \mid w \leq F_{Z|U}(z \mid u)\} & \forall(u,w) \end{array}$$

The non-decreasing condition in Restriction 1 is sufficient for almost-everywhere equivalence of the quantile representations. Furthermore, we also require the càglàd condition of Restriction 1 for point-wise equivalence of the quantile representations. Given Lemma 1, it remains to show that the observed distributions $F_{Y_2Y_1ZD_1}(\,\cdot,\,\cdot,\,\cdot,1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,1,1))$ uniquely determine $(F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Z|U})$ as well as F_{Y_1U}.

4.3 The Main Identification Result

Section 4.2 shows that $F_{Y_3|Y_2U}$, $F_{D_1|Y_1U}$, and $F_{Z|U}$ uniquely determine g, h, and ζ, respectively, up to the aforementioned equivalence classes. Therefore, the model (g,h,F_{Y_1U},ζ) can be identified by showing the well-definition of the inverse DGP correspondence

$$(F_{Y_2Y_1ZD_1}(\,\cdot,\,\cdot,\,\cdot,1),\; F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,1,1)) \;\mapsto\; (F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Y_1U}, F_{Z|U}).$$

Lemma 2 (Identification)**.** *Under Restrictions 1, 2, 3, and 4, $(F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Y_1U}, F_{Z|U})$ is uniquely determined by $F_{Y_2Y_1ZD_1}(\,\cdot,\,\cdot,\,\cdot,1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,1,1)$.*

Combining Lemmas 1 and 2 yields the following main identification result of this paper.

Theorem 1 (Identification)**.** *Under Restrictions 1, 2, 3, and 4, the model (g,h,F_{Y_1U},ζ) is identified by $F_{Y_2Y_1ZD_1}(\,\cdot,\,\cdot,\,\cdot,1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,1,1)$ up to the equivalence classes represented by the normalizations $\mathcal{E}_t, V_t, W \sim$ Uniform$(0,1)$.*

A proof of Lemma 2 is given in Appendix A.2, and consists of six steps of spectral decompositions, operator inversions (solving Fredholm equations of the first kind), and algebra. Figure 1 illustrates how observed data uniquely determines the model (g,h,F_{Y_1U},ζ) through the six steps. Section 3.1 provided an informal sketch of the proof of the first step among others.

Remark 2. While the baseline model only induces a permanent dropout through $D_t = 0$, we can also allow for an entry through $D_t = 1$. See Appendix C.1. This is useful to model entry of firms as well as reentry of female workers into the labor market after child birth. This extension also accommodates general unbalanced panel data with various causes of selection.

Remark 3. Three additional time periods can be used as a substitute for a nonclassical proxy variable. In other words, $T = 6$ periods of panel data alone identifies the model, and a proxy variable is optional. See Section 3.2 and Appendix C.2.

Remark 4. The baseline model consists of the first-order Markov process. Appendix C.3 generalizes the baseline result to allow for higher-order lags in the functions g and h. Generally, $\tau + 2$ periods of unbalanced panel data identifies the model with τ-th order Markov process g and $(\tau - 1)$-st order Markov decision rule h. The baseline model is a special case with $\tau = 1$.

Remark 5. The baseline model only allows individual fixed effects U. Suppose that the dynamic model g_t involves time effects, for example, to reflect common macroeconomic shocks to income dynamics. Then the model $(\{g_t\}_{t=2}^{T}, h, F_{Y_1 U}, \zeta)$ can be identified by $T + 1$ periods of unbalanced panel data from $t = 0$ to $t = T \geqslant 3$. See Appendix C.4.

Remark 6. The rule for missing observations was defined in terms of a lagged selection indicator D_{t-1} which depends on Y_{t-1}. Data may instead be selected based on contemporaneous D_t which depends on Y_t. See Appendix C.5. Note that, in the latter case, we may not observe Y_t based on which the data is selected. These two selection criteria reflect the ex ante versus ex post Roy selection processes by rational agents.

Remark 7. Restriction 3 (i) implies the cardinality relation $|\text{supp}(U)| \leqslant |\text{supp}(Y_t)|$. This cardinality restriction in particular rules out binary Y_t with continuously distributed U. Furthermore, the relevant proxy in Restriction 3 (ii) implies $\dim(U) \leqslant 1$.

Remark 8. For the result using a proxy variable, the notation appears to suggest that a proxy is time-invariant. However, a time-variant proxy $Z_t = \zeta(U, W_t)$ may also be used as far as W_t satisfies the same independence restriction as W.

5 Estimation

The identification result is derived through six steps of spectral decompositions, operator inversions (solutions to Fredholm equations of the first kind), and algebra, as illustrated in Figure 1. A sample-analog or plug-in estimation following all these steps is practically infeasible. The present section therefore discusses how to turn this six-step procedure into a one-step procedure.

5.1 Constrained Maximum Likelihood

After showing nonparametric identification as in Section 4, one can generally proceed with the maximum likelihood estimation of parametric or semi-parametric sub-models. In our context, however, the presence of missing observations biases the standard maximum likelihood estimator. In this section, we apply the Kullback-Leibler information inequality to translate our main identification result (Lemma 2) into an *identification-preserving* criterion function, which is robust against selection or missing data.

Because the model $(g, h, F_{Y_1 U}, \zeta)$ is represented by a quadruple $(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U})$ (see Lemmas 1 and 4), we use $\mathcal{F}$ to denote the set of all the admissible model representations:

$$\mathcal{F} = \{(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}) \mid (g, h, F_{Y_1U}, \zeta) \text{ satisfies Restrictions 1, 2, 3, and 4}\}.$$

As a consequence of the main identification result, the true model $(F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$ can be characterized by the following criterion, allowing for a one-step plug-in estimator.

Corollary 1 (Constrained Maximum Likelihood)**.** *If the true model $(F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$ is an element of $\mathcal{F}$, then it is the unique solution to*

$$\max_{\left(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}\right) \in \mathcal{F}} \quad c_1 E\left[\log \int f_{Y_t|Y_{t-1}U}(Y_2 \mid Y_1, u) f_{D_t|Y_tU}(1 \mid Y_1, u) f_{Y_1U}(Y_1, u) f_{Z|U}(Z \mid u) d\mu(u) \middle| D_1 = 1\right] +$$
$$c_2 E\left[\log \int f_{Y_t|Y_{t-1}U}(Y_3 \mid Y_2, u) f_{Y_t|Y_{t-1}U}(Y_2 \mid Y_1, u) f_{D_t|Y_tU}(1 \mid Y_2, u) f_{D_t|Y_tU}(1 \mid Y_1, u) f_{Y_1U}(Y_1, u) f_{Z|U}(Z \mid u) d\mu(u) \middle| D_2 = D_1 = 1\right]$$

for any $c_1, c_2 > 0$ subject to

$$\int f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_1, u) = f_{D_1}(1) \qquad \textit{and}$$
$$\int f_{Y_t|Y_{t-1}U}(y_2 \mid y_1, u) f_{D_t|Y_tU}(1 \mid y_2, u) f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_2, y_1, u) = f_{D_2D_1}(1, 1).$$

A proof is found in Appendix B.1. The sense of uniqueness stated in the corollary is up to the equivalence classes identified by the underlying probability measures. Once a representing model $(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U})$ is parametrically or semi-/non-parametrically specified, the sample analog of the objective and constraints can be formed from observed data. The first term in the objective can be estimated since (Y_2, Y_1, Z) is observed conditionally on $D_1 = 1$. Similarly, the second term can be estimated since (Y_3, Y_2, Y_1, Z) is observed conditionally on $D_2 = D_1 = 1$. All the components in the two constraints are also computable from observed data since $f_{D_1}(1)$ and $f_{D_2D_1}(1, 1)$ are observable.

This criterion is related to the maximum likelihood. The objective consists of a convex combination of expected log likelihoods conditional on survivors. Using this objective alone therefore would incur a survivorship bias. To adjust for the selection bias, the constraints bind the model to correctly predict the observed selection probabilities. Any pair of positive values may be chosen for c_1 and c_2. However, there is a certain choice of these coefficients that makes the constrained optimization problem easier, as discussed in the following remark.

Remark 9. Solutions to constrained optimization problems like Corollary 1 are characterized by saddle points of the Lagrangean functional. Although it appears easier than the original six-step procedure, this saddle-point problem over a function space is still practically challenging. By an appropriate choice of c_1 and c_2, we can, however, turn this saddle point problem into an unconstrained maximization problem. Let λ_1 and λ_2 denote the Lagrange multipliers for the two constraints in the corollary. Under some regularity conditions (Fréchet differentiability of the objective and constraint functionals, differentiability of the solution to the selection probability, and the regularity of the solution for the constraint functionals), the choice of $c_1 = \Pr(D_1 = 1)$

and $c_2 = \Pr(D_2 = D_1 = 1)$ guarantees $\lambda_1^* = \lambda_2^* = 1$ at the optimum (see Appendix B.2). With this knowledge of the values of λ_1^* and λ_2^*, the solution to the problem in the corollary can now be characterized by a maximum rather than a saddle point. This fact is useful both for implementation of numerical solution methods and for availability of the existing large sample theories of parametric, semiparametric, and nonparametric M-estimators.

Remark 10. In case of using $T = 6$ periods of unbalanced panel data instead of a proxy variable, a similar one-step criterion to Corollary 1 can be derived. See Appendix C.2.

5.2 An Estimator

The six steps of the identification strategy do not admit a practically feasible plug-in estimator. On the other hand, Corollary 1 and Remark 9 together yield the standard M-estimator by the sample analog. We decompose the model set as $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{F}_4$, where $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ are sets of parametric or semi-/non-parametric models for $f_{Y_t|Y_{t-1}U}$, $f_{D_t|Y_tU}$, f_{Y_1U}, and $f_{Z|U}$, respectively. Accordingly, we denote an element of $\mathcal{F}$ by $f = (f_1, f_2, f_3, f_4)$ for brevity. With this notation, Corollary 1 and Remark 9 imply that the estimator of the true model f_0 can be characterized by a solution $\hat{f}$ to the maximization problem:

$$\max_{f \in \mathcal{F}_{k(n)}} \frac{1}{n} \sum_{i=1}^{n} l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; f)$$

for some sieve space $\mathcal{F}_{k(n)} = \mathcal{F}_{1,k_1(n)} \times \mathcal{F}_{2,k_2(n)} \times \mathcal{F}_{3,k_3(n)} \times \mathcal{F}_{4,k_4(n)} \subset \mathcal{F}$, where

$$\begin{aligned}
l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; f) \;&:=\; \mathbb{1}\{D_{i1} = 1\} \cdot l_1(Y_{i2}, Y_{i1}, Z_i; f) \\
&\quad + \mathbb{1}\{D_{i2} = D_{i1} = 1\} \cdot l_2(Y_{i3}, Y_{i2}, Y_{i1}, Z_i; f) - l_3(f) - l_4(f), \\
l_1(Y_{i2}, Y_{i1}, Z_i; f) \;&:=\; \log \int f_1(Y_{i2} \mid Y_{i1}, u) f_2(1 \mid Y_{i1}, u) f_3(Y_{i1}, u) f_4(Z_i \mid u) d\mu(u), \\
l_2(Y_{i3}, Y_{i2}, Y_{i1}, Z_i; f) \;&:=\; \log \int f_1(Y_{i3} \mid Y_{i2}, u) f_1(Y_{i2} \mid Y_{i1}, u) f_2(1 \mid Y_{i2}, u) f_2(1 \mid Y_{i1}, u) \\
&\quad \times\; f_3(Y_{i1}, u) f_4(Z_i \mid u) d\mu(u), \\
l_3(f) \;&:=\; \int f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_1, u), \qquad \text{and} \\
l_4(f) \;&:=\; \int f_1(y_2 \mid y_1, u) f_2(1 \mid y_2, u) f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_2, y_1, u).
\end{aligned}$$

Note that Y_{i3} and Y_{i2} may be missing in data, but the interactions with the indicators $\mathbb{1}\{D_{i1} = 1\}$ and $\mathbb{1}\{D_{i2} = D_{i1} = 1\}$ allow the expression $l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; f)$ to make sense even if they are missing.

Besides the identifying Restrictions 1, 2, 3, and 4 for the model set $\mathcal{F}$, we require additional technical assumptions, stated in the appendix for brevity of exposition, to guarantee a well-behaved estimator in large samples.

Proposition 1 (Consistency)**.** *Suppose that $\mathcal{F}$ satisfies Restrictions 1, 2, 3, 4, and the assumptions under Appendix B.2. If, in addition, Assumptions 1, 2, and 3 in Appendix B.3 restrict the model set $\mathcal{F}$, choice of the sieves $\{\mathcal{F}_{k(n)}\}_{n=1}^{\infty}$, and the data $F_{Y_3Y_2Y_1ZD_2D_1}$, then $\left\|\hat{f}-f_0\right\| = o_p(1)$ holds, where this norm $\|\cdot\|$ is defined in Appendix B.3.*

The estimator can also be adapted to semi-parametric and parametric sub-models which can be more relevant for empirical analysis. Appendix B.4 introduces a semi-parametric estimator and its asymptotic distribution. Parametric models may be estimated with the standard M-estimation theory.

5.3 Monte Carlo Evidence

This section shows Monte Carlo evidence to evaluate the estimation method proposed in this paper. The endogenously unbalanced panel data of $N = 1,000$ and $T = 3$ are generated using the following DGP:

$$\begin{cases} Y_t = \alpha_1 Y_{t-1} + U + \mathcal{E}_t & \mathcal{E}_t \sim \text{Normal}(0, \alpha_2) \\ D_t = \mathbb{1}\{\beta_0 + \beta_1 Y_t + \beta_2 U + V_t \geq 0\} & V_t \sim \text{Logistic}(0,1) \\ F_{Y_1 U} & (Y_1, U) \sim \text{Normal}\left(\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \begin{pmatrix} \gamma_3^2 & \gamma_3\gamma_4\gamma_5 \\ \gamma_3\gamma_4\gamma_5 & \gamma_4^2 \end{pmatrix}\right) \\ Z = \mathbb{1}\{\delta_0 + \delta_1 U + W \geq 0\} & W \sim \text{Normal}(0,1) \end{cases}$$

Monte Carlo simulation results of the constrained maximum likelihood estimation are displayed in the first four rows of Table 1. The first row shows simulated distributions of parameter estimates by the fully-parametric estimation using the true model. The inter-quartile ranges capture the true parameter value of 0.5 without suffering from attrition bias. The second row shows simulated distributions of parameter estimates by semiparametric estimation, where the distribution of F_{Y_1U} is assumed to be semiparametric with normality of the conditional distribution $F_{U|Y_1}$. The third row shows simulated distributions of parameter estimates by semiparametric estimation, where the distributions of $\mathcal{E}_t$ and V_t are assumed to be unknown. The fourth row shows simulated distributions of parameter estimates by semiparametric estimation combining the above two semiparametric assumptions. While the medians are slightly off the true values, the inter-quartile ranges again capture the true parameter value of 0.5.

If one is interested in only the dynamic model g, then the sample-analog estimation of the first step in the six-step identification strategy can be used instead of the constrained maximum likelihood. With the notations from Section 3.1, minimizing $\rho(\hat{L}_{y,1}\hat{L}_{y,0}^{-1},\ P_y(\alpha)Q_1Q_0^{-1}P_y^{-1}(\alpha))$ for some divergence measure or a metric ρ yields the first-step estimation. Using the square-integrated difference for ρ, the fifth row of Table 1 shows a semiparametric first-step estimation for the dynamic model g. The interquartile range of the MC-simulated distribution indeed captures the true value of 0.5, and it is reasonably tight for a semiparametric estimator. But why is the interquartile range of this first-step estimator tighter than those of the CMLE in the first four rows, despite the greater degree of nonparametric specification? Recall that the first-step estimation uses only two semi-/non-parametric functions (g, ζ) because the other elements have been nonparametrically differenced out. This is to be contrasted with the CMLE, which uses all the four semi-/non-parametric functions $(g, h, F_{Y_1 U}, \zeta)$. The first-step estimator therefore uses less sieve spaces than the CMLE, and incurs smaller mean square errors in finite sample.

If there were no missing observations from attrition, existing methods such as Arellano and Bond (1991) would consistently estimate α_1. Similarly, because V_t follows the logistic distribution, the fixed-effect logit method (which is the only $\sqrt{N}$-consistent binary response estimator in particular; Chamberlain, 2010) would consistently estimate β_1 if the counterfactual binary choice of dynamic selection were observable after attrition. However, missing data from attrition causes these estimators to be biased as shown in the bottom three rows of Table 1. Observe that the fixed effect logit estimator is not only biased, but the sign is even opposite to the truth. This fact evidences that ignorance of selection could lead to a misleading result, even if the true parametric and distributional model is known.

6 Empirical Illustration: SES and Mortality

6.1 Background

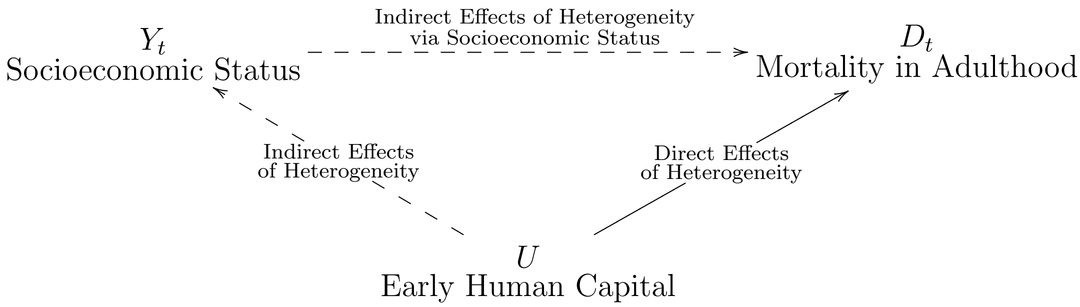
A large number of biological and socio-economic elements help to explain mortality (Cutler, Deaton, and Lleras-Muney, 2006). Among others, measures of socioeconomic status (SES) including earnings, employment, and income are important, yet puzzling as presumable economic determinants of mortality. The literature has reached no consensus on the sign of the effects of these measures of SES on mortality. On one hand, higher SES seems to play a malignant role. For example, at the macroeconomic unit of observations, recessions reduce mortality (Ruhm, 2000). For another example, higher social security income induces higher mortality (Snyder and Evans, 2006). On the other hand, higher SES has been reported to play a protective role. Deaton and Paxson (2001) and Sullivan and von Wachter (2009a) show that higher income reduces mortality. Job displacement, which results in a substantial drop in income, induces higher and long-lasting mortality (Sullivan and von Wachter, 2009b). The apparent discrepancy of the signs may be reconciled by the fact that these studies consider different sources of income and different units of observations.

A major concern in empirical analysis is the issue of endogeneity. Design-based empirical analysis often provides a solution. However, while non-labor income may allow exogenous variations to facilitate natural and quasi-experimental studies (e.g., Snyder and Evans, 2006), labor outcome is often harder to control exogenously. An alternative approach is to explicitly control the common factors that affect both SES and mortality. Education, in particular, is an important observable common factor, e.g., Lleras-Muney (2005) reports causal effects of education on adult mortality. Controlling for this common factor may completely remove the effect of income on mortality. For example, Adams, Hurd, McFadden, Merrill, and Ribeiro

(2003) show that income conditional on education is not correlated with health.

Education may play a substantial role, but it may not be the only common factor that needs to be controlled for. A wide variety of early human capital (HC) besides those reflected on education is considered to affect SES and/or adult health in the long run. Case, Fertig, and Paxson (2005) report long-lasting direct and indirect effects of childhood health on health and well-being in adulthood. Maccini and Yang (2009) find that the natural environment at birth affects adult health. Almond and Mazumder (2005) and Almond (2006) show evidence that HC acquired in utero affects long-run health. Early HC could contain a wide variety of categories of HC, such as genetic expression, acquired health, knowledge, and skills, all of which develop in an interactive manner with inter-temporal feedbacks during childhood (e.g., Heckman, 2007; Cunha and Heckman, 2008; Cunha, Heckman, and Schennach, 2010).

Failure to control for these components of early HC would result in identification of "spurious dependence" (e.g., Heckman, 1981ab, 1991). Early HC may directly affect adult mortality via the development of chronic conditions in childhood. Early HC may also affect earnings, which may in turn affect adult mortality, as illustrated below.



Identification of these two competing causal effects of Y_t and U on D_t, or distinction between the two channels in the above diagram, requires to control for the unobserved heterogeneity of early HC. Unlike education, however, most components of early HC are unobservable from the viewpoint of econometricians. Suppose that early HC develops into fixed characteristics by adulthood. How can we control for these heterogeneous characteristics? Because of the strong cross-section correlation between SES and these heterogeneous characteristics, a variation over time is useful to disentangle their competing effects on mortality, e.g., Deaton and Paxson (2001) and Sullivan and von Wachter (2009a). I extend these ideas by treating early HC as a fixed unobserved heterogeneity to be distinguished from time-varying observed measures of SES. To account for both nonseparable heterogeneity and the survivorship bias, I use the econometric method developed in this paper.

6.2 Empirical Model

Sullivan and von Wachter (2009b) show elaborate evidence on the malignant effects of job displacement on mortality, carefully ruling out the competing hypothesis of selective displacement. Sullivan and von Wachter (2009a), focusing on income as a measure of SES, find that there are protective effects of higher SES on mortality whereas there is no or little evidence of causal effects of unobserved attributes such as patience on mortality. Using the econometric methods developed in this paper, I attempt to complement these analyses by explicitly modeling unobserved heterogeneity and survival selection.

The following econometric model represents the causal relationship described in the above diagram.

$$\begin{cases} (i) \quad Y_{it} = g(Y_{i,t-1}, U_i, \mathcal{E}_{it}) & \text{SES Dynamics} \\ (ii) \quad D_{it} = h(Y_{it}, U_i, V_{it}) & \text{Survival Selection} \\ (iii) \; F_{Y_1 U} & \text{Initial Condition} \\ (iv) \quad Z_i = \zeta(U_i, W_i) & \text{Nonclassical Proxy} \end{cases}$$

where Y_{it}, D_{it}, and U_i denote SES, survival, and unobserved heterogeneity, respectively. As noted earlier, the heterogeneity U reflects early human capital (HC) acquired prior to the start of the panel data, which play the role of sustaining employment dynamics in model (i). This early HC may include acquired and innate abilities, knowledge, skills, patience, diligence, and chronic health conditions, which may affect the survival selection (ii) as well as the income dynamics. The initial condition (iii) models a statistical summary of the initial observation of SES that has developed cumulatively and dependently on the early HC prior to the first observation by econometrician.

For this empirical application, we consider the model in which all the random variables are binary as in Section 3. Specifically, Y_{it} indicates that individual i is (0) unemployed or (1) employed, D_{it} indicates that individual i is (0) dead or (1) alive, and U_i indicates that individual i belongs to (0) type I or (1) type II. Several proxy variables are used for Z_i as means of showing robustness of empirical results. The heterogeneous type U does not yet have any intrinsic meaning at this point, but it turns out empirically to have a consistent meaning in terms of a pattern of employment dynamics as we will see in Section 6.4.

Besides unobserved heterogeneity, other main sources of endogeneity in analysis of SES and mortality are cohort effects and age effects. In parametric regression analysis, one can usually control for these effects by inserting additive dummies or polynomials of age. Since additive controls are infeasible for our setup of nonseprable models, we implement the econometric analysis for each bin of age categories in order to mitigate the age and cohort effects.

6.3 Data

The NLS Original Cohorts: Older Men consist of 5,020 individuals aged 46–60 as of April 1, 1966. The subjects were surveyed annually or biennially starting in 1966. Attrition is frequent in this panel data. In order for the selection model to exactly represent the survival selection, we remove those individuals with attrition due to reasons other than death.

It is important to rule out competing hypotheses that obscure the credibility of our empirical results. For example, health as well as wealth is an important factor of retirement deicision (Bound, Stinebrickner, and Waidmann, 2010). It is not unlikely that individuals who have chosen to retire from jobs for health problems subsequently die. If so, we would erroneously impute death to voluntary retirements. To eliminate this confounding factor, we consider the subsample of individuals who reported that health problems do not limit work in 1971. Furthermore, we also consider the subsample of individuals who died from acute diseases such as heart attacks and strokes, because workers dying unexpectedly from acute diseases are less likely to change labor status before a sudden death than those who die from cancer or diabetes. Death certificates are used to classify causes of deaths to this end.

Recall that the econometric methods presented in this paper offer two paths of identification. One is to use a panel of $T = 3$ with a nonclassical proxy variable, and the other is to use a panel of $T = 6$ without a proxy. While the the survey was conducted at more than six time points, the list of survey years do not exhibit equal time intervals (1966, 67, 68, 69, 71, 73, 75, 76, 78, 80, 81, 83, and 90). None of annual or biennial sequences consist consecutive six periods from this anomalistic list of years. Therefore we choose the method of proxy variables. Because one of the proxy variables is collected only once in 1973, we need to set T=1 or T=2 to year 1973 in order to satisfy the identifying restriction. We thus set $T = 2$ to year 1973 to exploit a larger size of data, hence using the three-period data from years 71, 73, and 75 in our analysis. The subjects are aged 51–65 in 1971, but we focus on the younger cohorts not facing the retirement age.

We use height, mother's occupation, and father's occupation, as potential candidates for proxy variables. Height reflects health investments in childhood (Schultz, 2002). Mother's education and father's occupation reflect early endowments and investments in human capital in the form of intergenerational inheritance; e.g., Currie and Moretti (2003) show evidence of intergenerational transmission of human capital. We use these three proxies to aim to show robustness of our empirical results.

6.4 Empirical Results

Table 2 summarizes estimates of the first-order Markov process of employment dynamics and the conditional two-year survival probabilities using height as a proxy variable. The top and bottom panels correspond to younger cohorts (aged 51–54 in 1971) and older cohorts (aged 55–58 in 1971), respectively. The left and right columns correspond to Type I ($U_i = 0$) and Type II ($U_i = 1$), respectively. These unobserved types exhibit a consistent pattern: off-diagonal elements of the employment Markov matrices for Type I dominate those of Type II. In other words, Type I and Type II can be characterized as movers and stayers, respectively. In view of the survival probabilities in the top panel (young cohorts), we find that individuals almost surely stay alive as far as they are employed. On the other hand, the two-year survival probabilities drop by about 10% if individuals are unemploed. While the data indicates statistical significance of their difference only for Type I, the magnitudes of differences in the point estimates are almost identical between the two types. The same qualitative pattern persists in the older cohorts.

To show a robustness of this baseline result, we repeat this estimation using the other two proxy variables, mother's education and father's occupation. Figure 2 graphs estimates of Markov probabilities of employment. The shades in the bars indicate different proxy variables used for estimation. We see that these point estimates are robust across the three proxy variables, implying that a choice of a particular proxy does not lead to an irregular result in favor of certain claims. Table 3 graphs estimates of conditional two-year survival probabilities. Again, the point estimates are robust across the three proxy variables.

As mentioned earlier, selective or voluntary retirement is a potential source of bias. To rule out this possibility, we consider two subpopulations: 1. those individuals who reported that health problems do not limit their work in 1971; and 2. those individuals who eventually died from acute diseases. Figures 4 and 5 show estimates for the first subpopulation. Figures 6 and 7 show estimates for the second subpopulation. Again, robustness across the three proxies persists, and the qualitative pattern remains the same as the baseline result. The relatively large variations in the estimates for the second subpopulation is imputed to small sample sizes due to the limited availability of death certificates from which we identify causes of deaths.

In summary, we obtain the following two robust results. First, accounting for unobserved heterogeneity and survivorship bias as well as voluntary retirements, employment status has protective effects on survival selection. This reinforces the results of Sullivan and von Wachter (2009b). Second, there is no evidence of the effects of unobserved attributes on survival selection, since the conditional survival probabilities are almost the same between type I and type II. This is in accord with the claim of Sullivan and von Wachter (2009a), who deduce that lagged SES has little effect on mortality conditionally on the SES of immediate past.

Using the estimated Markov model g and the estimated initial condition $F_{Y_1 U}$, we can simulate the counterfactual employment rates assuming that all the individuals were to remain alive throughout the entire period. Figure 8 shows actual employment rates (black lines) and counterfactual employment rates (grey lines) for each cohort category for each proxy variable. I again remark that the qualitative patterns are the same across three proxy variables for each cohort category. Not shockingly, if it were not for deaths, the counterfactual employment rates would have been even lower than what we observed from actual data. In other words, deaths of working age population are saving the actual figures of employment rates to look higher.

7 Summary

This paper proposes a set of nonparametric restrictions to point-identify dynamic panel data models by nonparametrically differencing out both nonseparable heterogeneity and selection. Identification requires either $T = 3$ periods of panel data and a proxy variable or $T = 6$ periods of panel data without an outside proxy variable. As a consequence of the identification result, the constrained maximum likelihood criterion follows, which corrects for selection and allows for one-step estimation. Monte Carlo simulations are used to evidence the effectiveness of the estimators. In the empirical application, I find protective effects of employment on survival selection, and the result is robust.

Appendix

A Proofs for Identification

A.1 Lemma 1 (Representation)

Proof. (i) First, we show that there exists a function $\bar{g}$ such that $(\bar{g}, \text{Uniform}(0,1))$ is observationally equivalent to $(g, F_{\mathcal{E}})$ for any $(g, F_{\mathcal{E}})$ satisfying Restrictions 1 and 2. By the absolute continuity and the convex support in Restriction 1, $F_{\mathcal{E}}$ is invertible. Hence, we can define $h := F_{\mathcal{E}}^{-1}$. Now, define $\bar{g}$ by $\bar{g}(y, u, \ \cdot\ ) := g(y, u, \ \cdot\ ) \circ h^{-1}$ for each (y, u). Note that, under Restriction 2, $(\bar{g}, F_{h(\mathcal{E})})$ is observationally equivalent to $(g, F_{\mathcal{E}})$ by construction. However, we have $h(\mathcal{E}) \sim \text{Uniform}(0,1)$ by the definition of h. It follows that $(\bar{g}, \text{Uniform}(0,1))$ is observationally equivalent to $(g, F_{\mathcal{E}})$.

In light of the previous paragraph, we can impose the normalization $\mathcal{E}_t \sim \text{Uniform}(0,1)$. Let $\Lambda(y, u, \varepsilon)$ denote the set $\Lambda(y, u, \varepsilon) = \{y' \in g(y, u, (0,1)) \mid \varepsilon \leq F_{Y_3|Y_2U}(y' \mid y, u)\}$, where $g(y, u, (0,1))$ denotes the set $\{g(y, u, \varepsilon) \mid \varepsilon \in (0,1)\}$. I claim that $g(y, u, \varepsilon) = \inf \Lambda(y, u, \varepsilon)$.

First, we note that $g(y, u, \varepsilon) \in \Lambda(y, u, \varepsilon)$. To see this, calculate

$$
\begin{aligned}
F_{Y_3|Y_2U}(g(y,u,\varepsilon) \mid y, u) &= \Pr(g(y,u,\mathcal{E}_3) \leq g(y,u,\varepsilon) \mid Y_2 = y, U = u) \\
&= \Pr(g(y,u,\mathcal{E}_3) \leq g(y,u,\varepsilon)) \ \geq \ \Pr(\mathcal{E}_3 \leq \varepsilon) \ = \ \varepsilon,
\end{aligned}
$$

where the first equality follows from $Y_3 = g(y, u, \mathcal{E}_3)$ given $(Y_2, U) = (y, u)$, the second equality follows from Restriction 2 (i), the next inequality follows from the non-decrease of $g(y, u, \cdot)$ by Restriction 1 together with monotonicity of the probability measure, and the last equality is due to $\mathcal{E}_t \sim U(0,1)$. This shows that $\varepsilon \leq F_{Y_3|Y_2U}(g(y,u,\varepsilon) \mid y, u)$, hence $g(y, u, \varepsilon) \in \Lambda(y, u, \varepsilon)$.

Second, I show that $g(y, u, \varepsilon)$ is a lower bound of $\Lambda(y, u, \varepsilon)$. Let $y' \in \Lambda(y, u, \varepsilon)$. Since g is non-decreasing and càglàd (left-continuous) in the third argument by Restriction 1, we can define $\varepsilon' := \max\{\varepsilon \in (0,1) \mid g(y, u, \varepsilon) = y'\}$. But then,

$$
\begin{aligned}
F_{Y_3|Y_2U}(y' \mid y, u) &= \Pr(g(y,u,\mathcal{E}_3) \leq y' \mid Y_2 = y, U = u) \\
&= \Pr(g(y,u,\mathcal{E}_3) \leq y') \ = \ \varepsilon',
\end{aligned}
$$

where the first equality follows from $Y_3 = g(y, u, \mathcal{E}_3)$ given $(Y_2, U) = (y, u)$, the second equality follows from Restriction 2 (i), and the last equality follows from the definition of ε' together with the non-decrease of $g(y, u, \cdot)$ by Restriction 1 and $\mathcal{E}_t \sim U(0,1)$. Using this result, in turn, yields

$$
g(y,u,\varepsilon) \ \leq \ g(y, u, F_{Y_3|Y_2U}(y' \mid y, u)) \ = \ g(y, u, \varepsilon') \ = \ y',
$$

where the first inequality follows from $\varepsilon \leq F_{Y_3|Y_2U}(y' \mid y, u)$ by definition of y' as well as the non-decrease of $g(y, u, \cdot)$ by Restriction 1, the next equality follows from the previous result $F_{Y_3|Y_2U}(y' \mid y, u) = \varepsilon'$, and the last equality follows from the definition of ε'. Since y' was chosen as an arbitrary element of $\Lambda(y, u, \varepsilon)$, this shows that $g(y, u, \varepsilon)$ is indeed a lower bound of it. Therefore, $g(y, u, \varepsilon) = \inf\{y' \in g(y, u, (0,1)) \mid \varepsilon \leq F_{Y_3|Y_2U}(y' \mid y, u)\}$, and g is uniquely determined by $F_{Y_3|Y_2U}$. (Moreover, note that $\inf\{y' \in g(y, u, (0,1)) \mid \varepsilon \leq F_{Y_3|Y_2U}(y' \mid y, u)\}$ coincides with the definition of the quantile regression $F^{-1}_{Y_3|Y_2U}(\ \cdot\ \mid y, u)$, hence g is identified by this quantile regression, i.e., $g(y, u, \varepsilon) = F^{-1}_{Y_3|Y_2U}(\varepsilon \mid y, u)$.)

Part (ii) of the lemma can be proved in exactly the same way as in the proof of part (i). In particular, h is identified by the quantile regression: $h(y, u, v) = F^{-1}_{D_2|Y_2U}(v \mid y, u)$. Similarly, part (iii) of the the lemma can be proved in the same way, and ζ is identified by the quantile regression: $\zeta(u, w) = F^{-1}_{Z|U}(w \mid u)$. $\square$

A.2 Lemma 2 (Identification)

Proof. We will construct six steps for a proof of this lemma. The first step shows that the observed joint distributions uniquely determine $F_{Y_3|Y_2U}$ and $F_{Z|U}$ by a spectral decomposition of a composite linear operator. The second step is auxiliary, and shows that $F_{Y_2Y_1UD_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$ is uniquely determined from the observed joint distributions together with inversion of the operators identified in the first step. The third step again uses spectral decomposition to identify an auxiliary operator with the kernel represented by $F_{Y_1|Y_2UD_2D_1}(\,\cdot\,\mid\,\cdot\,,\,\cdot\,,1,1)$. In the fourth step, solving an integral equation with the adjoint of this auxiliary operator in turn yields another auxiliary operator with the multiplier represented by $F_{Y_2UD_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$. The fifth step uses the three operators identified in Steps 2, 3, and 4 to identify an operator with the kernel represented by $F_{D_2|Y_2U}$ by solving a linear inverse problem. The last step uses results from Steps 1, 2, and 5 to show that the initial joint distribution F_{Y_1U} is uniquely determined from the observed joint distributions. These six steps together prove that the observed joint distributions $F_{Y_2Y_1ZD_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot, \,\cdot, \,\cdot, \,\cdot, 1, 1)$ uniquely determine the model $(F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Y_1U}, F_{Z|U})$ as claimed in the lemma.

Given fixed y and z, define the operators $L_{y,z} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$, $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$, $Q_z : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $R_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $S_y : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, $T_y : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, and $T'_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$ by

$$
\begin{aligned}
(L_{y,z}\xi)(y_3) &= \int f_{Y_3Y_2Y_1ZD_2D_1}(y_3, y, y_1, z, 1, 1) \cdot \xi(y_1) dy_1, \\
(P_y\xi)(y_3) &= \int f_{Y_3|Y_2U}(y_3 \mid y, u) \cdot \xi(u) du, \\
(Q_z\xi)(u) &= f_{Z|U}(z \mid u) \cdot \xi(u), \\
(R_y\xi)(u) &= f_{D_2|Y_2U}(1 \mid y, u) \cdot \xi(u), \\
(S_y\xi)(u) &= \int f_{Y_2Y_1UD_1}(y, y_1, u, 1) \cdot \xi(y_1) dy_1, \\
(T_y\xi)(u) &= \int f_{Y_1|Y_2UD_2D_1}(y_1 \mid y, u, 1, 1) \cdot \xi(y_1) dy_1, \\
(T'_y\xi)(u) &= f_{Y_2UD_2D_1}(y, u, 1, 1) \cdot \xi(u)
\end{aligned}
$$

respectively. We consider $\mathcal{L}^2$ spaces as the normed linear spaces on which these operators are defined, particularly in order to guarantee the existence of its adjoint operator T^*_y to be introduced in Step 4. (Recall that a bounded linear operator between Hilbert spaces admits existence of its adjoint operator.) Identification of the operator leads to that of the associated conditional density (up to null sets), and vice versa. Here, the operators $L_{y,z}$, P_y, S_y, and T_y are integral operators whereas Q_z, R_y, and T'_y are multiplication operators. Note that $L_{y,z}$ is identified from observed joint distribution $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$.

Figure 1 illustrates six steps toward identification of $(F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Y_1U}, F_{Z|U})$ from the observed joint distributions $F_{Y_2Y_1ZD_1}(\,\cdot, \,\cdot, \,\cdot, 1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot, \,\cdot, \,\cdot, \,\cdot, 1, 1)$. The four objects (g, h, F_{Y_1U}, ζ) of interest are enclosed by double lines. The objects that can be observed from data are enclosed by dashed-lines All the other objects are intermediary, and are enclosed by solid lines. Starting out with the observed objects, we show in each step that the intermediary objects are uniquely determined. These uniquely determined intermediary objects in turn show the uniqueness of the four objects (g, h, F_{Y_1U}, ζ) of interest.

Step 1: Uniqueness of $F_{Y_3|Y_2U}$ and $F_{Z|U}$

The kernel $f_{Y_3Y_2Y_1ZD_2D_1}(\ \cdot\ , y,\ \cdot\ , z, 1, 1)$ of the integral operator $L_{y,z}$ can be rewritten as

$$\begin{aligned} f_{Y_3Y_2Y_1ZD_2D_1}(y_3, y, y_1, z, 1, 1) \ &= \ \int f_{Y_3|Y_2Y_1ZUD_2D_1}(y_3 \mid y, y_1, z, u, 1, 1) \\ &\quad \times f_{Z|Y_2Y_1UD_2D_1}(z \mid y, y_1, u, 1, 1) \\ &\quad \times f_{D_2|Y_2Y_1UD_1}(1 \mid y, y_1, u, 1) \cdot f_{Y_2Y_1UD_1}(y, y_1, u, 1)\ du \end{aligned} \tag{A.1}$$

But by Lemma 3 (i), (iv), and (iii), respectively, Restriction 2 implies that

$$\begin{aligned} f_{Y_3|Y_2Y_1ZUD_2D_1}(y_3 \mid y, y_1, z, u, 1, 1) \ &= \ f_{Y_3|Y_2U}(y_3 \mid y, u), \\ f_{Z|Y_2Y_1UD_2D_1}(z \mid y, y_1, u, 1, 1) \ &= \ f_{Z|U}(z \mid u), \\ f_{D_2|Y_2Y_1UD_1}(1 \mid y, y_1, u, 1) \ &= \ f_{D_2|Y_2U}(1 \mid y, u). \end{aligned}$$

Equation (A.1) thus can be rewritten as

$$\begin{aligned} f_{Y_3Y_2Y_1ZD_2D_1}(y_3, y, y_1, z, 1, 1) \ &= \ \int f_{Y_3|Y_2U}(y_3 \mid y, u) \cdot f_{Z|U}(z \mid u) \\ &\quad \times f_{D_2|Y_2U}(1 \mid y, u) \cdot f_{Y_2Y_1UD_1}(y, y_1, u, 1)\ du \end{aligned}$$

But this implies that the integral operator $L_{y,z}$ is written as the operator composition

$$L_{y,z} = P_y Q_z R_y S_y.$$

Restriction 3 (i), (ii), (iii), and (iv) imply that the operators P_y, Q_z, R_y, and S_y are invertible, respectively. Hence so is $L_{y,z}$. Using the two values $\{0, 1\}$ of Z, form the product

$$L_{y,1} L_{y,0}^{-1} = P_y Q_{1/0} P_y^{-1}$$

where $Q_{z/z'} := Q_z Q_{z'}^{-1}$ is the multiplication operator with proxy odds defined by

$$(Q_{1/0}\xi)(u) = \frac{f_{Z|U}(1 \mid u)}{f_{Z|U}(0 \mid u)} \xi(u).$$

By Restriction 3 (ii), the operator $L_{y,1} L_{y,0}^{-1}$ is bounded. The expression $L_{y,1} L_{y,0}^{-1} = P_y Q_{1/0} P_y^{-1}$ thus allows unique eigenvalue-eigenfunction decomposition similarly to that of Hu and Schennach (2008).

The distinct proxy odds as in Restriction 3 (ii) guarantee distinct eigenvalues and single dimensionality of the eigenspace associated with each eigenvalue. Within each of the single-dimensional eigenspace is a unique eigenfunction pinned down by $\mathcal{L}^1$-normalization because of the unity of integrated densities. The eigenvalues $\lambda(u)$ yield the multiplier of the operator $Q_{1/0}$, hence $\lambda(u) = f_{Z|U}(1 \mid u)/f_{Z|U}(0 \mid u)$. This proxy odds in turn identifies $f_{Z|U}(\ \cdot\ \mid u)$ since Z is binary. The corresponding normalized eigenfunctions are the kernels of the integral operator P_y, hence $f_{Y_3|Y_2U}(\ \cdot\ \mid y, u)$. Lastly, Restriction 4 facilitates unique ordering of the eigenfunctions $f_{Y_3|Y_2U}(\ \cdot\ \mid y, u)$ by the distinct concrete values of $u = \lambda(u)$. This is feasible because the eigenvalues $\lambda(u) = f_{Z|U}(1 \mid u)/f_{Z|U}(0 \mid u)$ are invariant from y. That is, eigenfunctions $f_{Y_3|Y_2U}(\ \cdot\ \mid y, u)$ of the operator $L_{y,1} L_{y,0}^{-1}$ across different y can be uniquely ordered in u invariantly from y by the common set of ordered distinct eigenvalues $u = \lambda(u)$.

Therefore, $F_{Y_3|Y_2U}$ and $F_{Z|U}$ are uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_2D_1}(\ \cdot\ ,\ \cdot\ ,\ \cdot\ ,\ \cdot\ , 1, 1)$. Equivalently, the operators P_y and Q_z are uniquely determined for each y and z, respectively.

Step 2: Uniqueness of $F_{Y_2 Y_1 U D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$

By Lemma 3 (ii), Restriction 2 implies $f_{Y_2|Y_1 U D_1}(y' \mid y,u,1) = f_{Y_2|Y_1 U}(y' \mid y,u)$. Using this equality, write the density of the observed joint distribution $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$ as

$$\begin{aligned} f_{Y_2 Y_1 D_1}(y',y,1) &= \int f_{Y_2|Y_1 U D_1}(y' \mid y,u,1) f_{Y_1 U D_1}(y,u,1) du \\ &= \int f_{Y_2|Y_1 U}(y' \mid y,u) f_{Y_1 U D_1}(y,u,1) du \end{aligned} \tag{A.2}$$

By Lemma 4 (i), $F_{Y_3|Y_2 U}(y' \mid y,u) = F_{Y_2|Y_1 U}(y' \mid y,u)$ for all y', y, u. Therefore, we can write the operator P_y as

$$(P_y \xi)(y') = \int f_{Y_3|Y_2 U}(y' \mid y,u) \cdot \xi(u) du = \int f_{Y_2|Y_1 U}(y' \mid y,u) \cdot \xi(u) du.$$

With this operator notation, it follows from (A.2) that

$$f_{Y_2 Y_1 D_1}(\,\cdot\,,y,1) = P_y f_{Y_1 U D_1}(y,\,\cdot\,,1).$$

By Restriction 3 (i), this operator equation can be solved for $f_{Y_1 U D_1}(y,\,\cdot\,,1)$ as

$$f_{Y_1 U D_1}(y,\,\cdot\,,1) = P_y^{-1} f_{Y_2 Y_1 D_1}(\,\cdot\,,y,1) \tag{A.3}$$

Recall that P_y was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$. The function $f_{Y_2 Y_1 D_1}(\,\cdot\,,y,1)$ is also uniquely determined by the observed joint distribution $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$. Therefore, (A.2) shows that $f_{Y_1 U D_1}(\,\cdot\,,\,\cdot\,,1)$ is uniquely determined by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$.

Using the solution to the above inverse problem, we can write the kernel of the operator S_y as

$$\begin{aligned} f_{Y_2 Y_1 U D_1}(y',y,u,1) &= f_{Y_2|Y_1 U D_1}(y' \mid y,u,1) \cdot f_{Y_1 U D_1}(y,u,1) \\ &= f_{Y_2|Y_1 U}(y' \mid y,u) \cdot f_{Y_1 U D_1}(y,u,1) \\ &= f_{Y_3|Y_2 U}(y' \mid y,u) \cdot f_{Y_1 U D_1}(y,u,1) \\ &= f_{Y_3|Y_2 U}(y' \mid y,u) \cdot [P_y^{-1} f_{Y_2 Y_1 D_1}(\,\cdot\,,y,1)](u) \end{aligned}$$

where the second equality follows from Lemma 3 (ii), the third equality follows from Lemma 4 (i), and the forth equality follows from (A.3). Since $f_{Y_3|Y_2 U}$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $[P_y^{-1} f_{Y_2 Y_1 D_1}(\,\cdot\,,y,1)]$ was shown in the previous paragraph to be uniquely determined for each y by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$, it follows that $f_{Y_2 Y_1 U D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$ too is uniquely determined by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$. Equivalently, the operator S_y is uniquely determined for each y.

Step 3: Uniqueness of $F_{Y_1|Y_2 U D_2 D_1}(\,\cdot \mid \cdot\,,\,\cdot\,,1,1)$

First, note that the kernel of the composite operator $T_y' T_y$ can be written as

$$\begin{aligned} f_{Y_2 U D_2 D_1}(y,u,1,1) \cdot f_{Y_1|Y_2 U D_2 D_1}(y_1 \mid y,u,1,1) &= f_{Y_2 Y_1 U D_2 D_1}(y,y_1,u,1,1) \qquad \text{(A.4)} \\ &= f_{D_2|Y_2 Y_1 U D_1}(1 \mid y,y_1,u,1) \cdot f_{Y_2 Y_1 U D_1}(y,y_1,u,1) \\ &= f_{D_2|Y_2 U}(1 \mid y,u) \cdot f_{Y_2 Y_1 U D_1}(y,y_1,u,1) \end{aligned}$$

where the last equality is due to Lemma 3 (iii). But the last expression corresponds to the kernel of the composite operator $R_y S_y$, thus showing that $T'_y T_y = R_y S_y$. But then, $L_{y,z} = P_y Q_z R_y S_y = P_y Q_z T'_y T_y$. Note that the invertibility of R_y and S_y as required by Assumption 3 implies invertibility of T'_y and T_y as well, for otherwise the equivalent composite operator $T'_y T_y = R_y S_y$ would have a nontrivial nullspace.

Using Restriction 3, form the product of operators as in Step 1, but in the opposite order as

$$L_{y,0}^{-1} L_{y,1} = T_y^{-1} Q_{1/0} T_y$$

The disappearance of T'_y is due to commutativity of multiplication operators. By the same logic as in Step 1, this expression together with Restriction 3 (ii) admits unique left eigenvalue-eigenfunction decomposition. Moreover, the point spectrum is exactly the same as the one in Step 1, as is the middle multiplication operator $Q_{1/0}$. This equivalence of the spectrum allows consistent ordering of U with that of Step 1. Left eigenfunctions yield the kernel of T_y pinned down by the normalization of unit integral. This shows that the operator T_y is uniquely determined by the observed joint distribution $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$.

Step 4: Uniqueness of $F_{Y_2 U D_2 D_1}(\,\cdot\,,\,\cdot\,,1,1)$
Equation (A.4) implies that

$$\int f_{Y_1|Y_2 U D_2 D_1}(y_1 \mid y, u, 1, 1) \cdot f_{Y_2 U D_2 D_1}(y, u, 1, 1) du = f_{Y_2 Y_1 D_2 D_1}(y, y_1, 1, 1)$$

hence yielding the linear operator equation

$$T_y^* f_{Y_2 U D_2 D_1}(y,\,\cdot, 1, 1) = f_{Y_2 Y_1 D_2 D_1}(y,\,\cdot, 1, 1)$$

where T_y^* denotes the adjoint operator of T_y. Since T_y is invertible, so is its adjoint T_y^*. But then, the multiplier of the multiplication operator T'_y can be given by the unique solution to the above linear operator equation, i.e.,

$$f_{Y_2 U D_2 D_1}(y,\,\cdot, 1, 1) = (T_y^*)^{-1} f_{Y_2 Y_1 D_2 D_1}(y,\,\cdot, 1, 1)$$

Note that T_y hence T_y^* was shown to be uniquely determined by $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ in Step 3, and $f_{Y_2 Y_1 D_2 D_1}(\,\cdot,\,\cdot, 1, 1)$ is also available from observed data. Therefore, the operator T'_y is uniquely determined by $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$.

Step 5: Uniqueness of $F_{D_2|Y_2 U}(1 \mid \,\cdot\,,\,\cdot\,)$
First, the definition of the operators R_y, S_y, T_y, and T'_y and Lemma 3 (iii) yield the operator equality $R_y S_y = T'_y T_y$, where T_y and T'_y have been shown to be uniquely determined by the observed joint distribution $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ in Steps 3 and 4, respectively. Recall that S_y was also shown in Step 2 to be uniquely determined by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$. Restriction 3 (iv) guarantees invertibility of S_y. It follows that the operator inversion $R_y = (R_y S_y) S_y^{-1} = (T'_y T_y) S_y^{-1}$ yields the operator R_y, in turn showing that its multiplier $f_{D_2|Y_2 U}(1 \mid y,\,\cdot\,)$ is uniquely determined for each y by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$.

Step 6: Uniqueness of $F_{Y_1 U}$
Recall from Step 2 that $f_{Y_2 Y_1 U D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$ is uniquely determined by the observed joint distributions $F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ and $F_{Y_2 Y_1 D_1}(\,\cdot\,,\,\cdot\,,1)$. We can write

$$\begin{aligned}
f_{Y_2 Y_1 U D_1}(y', y, u, 1) &= f_{Y_2|Y_1 U D_1}(y' \mid y, u, 1) f_{D_1|Y_1 U}(1 \mid y, u) f_{Y_1 U}(y, u) \\
&= f_{Y_2|Y_1 U}(y' \mid y, u) f_{D_1|Y_1 U}(1 \mid y, u) f_{Y_1 U}(y, u) \\
&= f_{Y_3|Y_2 U}(y' \mid y, u) f_{D_2|Y_2 U}(1 \mid y, u) f_{Y_1 U}(y, u),
\end{aligned}$$

where the second equality follows from Lemma 3 (ii), and the third equality follows from Lemma 4 (i) and (ii). For a given (y, u), there must exist some y' such that $f_{Y_3|Y_2U}(y' \mid y, u) > 0$ by a property of conditional density functions. Moreover, Restriction 3 (iii) requires that $f_{D_2|Y_2U}(1 \mid y, u) > 0$ for a given y for all u. Therefore, for such a choice of y', we can write

$$f_{Y_1U}(y, u) = \frac{f_{Y_2Y_1UD_1}(y', y, u, 1)}{f_{Y_3|Y_2U}(y' \mid y, u) f_{D_2|Y_2U}(1 \mid y, u)}$$

Recall that $f_{Y_3|Y_2U}(\,\cdot \mid \cdot\,,\,\cdot\,)$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,, 1, 1)$, $f_{Y_2Y_1UD_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,, 1)$ was shown in Step 2 to be uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,, 1, 1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,, 1)$, and $f_{D_2|Y_2U}(1 \mid \cdot\,,\,\cdot\,)$ was shown in Step 5 to be uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,, 1, 1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,, 1)$. Therefore, it follows that the initial joint density f_{Y_1U} is uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,, 1, 1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,, 1)$. $\square$

A.3 Lemma 3 (Independence)

Lemma 3 (Independence)**.** *The following implications hold:*
(i) Restriction 2 (i) $\Rightarrow \mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, W) \Rightarrow Y_3 \perp\!\!\!\perp (Y_1, D_1, D_2, Z) \mid (Y_2, U)$.
(ii) Restriction 2 (i) $\Rightarrow \mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1, W) \Rightarrow Y_2 \perp\!\!\!\perp (D_1, Z) \mid (Y_1, U)$.
(iii) Restriction 2 (ii) $\Rightarrow V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1) \Rightarrow D_2 \perp\!\!\!\perp (Y_1, D_1) \mid (Y_2, U)$.
(iv) Restriction 2 (iii) $\Rightarrow W \perp\!\!\!\perp (Y_1, \mathcal{E}_2, V_1, V_2) \Rightarrow Z \perp\!\!\!\perp (Y_2, Y_1, D_2, D_1) \mid U$.

Proof. In order to prove the lemma, we use the following two properties of conditional independence:
CI.1. $A \perp\!\!\!\perp B$ implies $A \perp\!\!\!\perp B \mid \phi(B)$ for any Borel function ϕ.
CI.2. $A \perp\!\!\!\perp B \mid C$ implies $A \perp\!\!\!\perp \phi(B, C) \mid C$ for any Borel function ϕ.

(i) First, note that Restriction 2 (i) $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, W)$ together with the structural definition $Z = \zeta(U, W)$ implies $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, Z)$. Applying CI.1 to this independence relation $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, Z)$ yields

$$\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, Z) \mid (g(Y_1, U, \mathcal{E}_2), U).$$

Since $Y_2 = g(Y_1, U, \mathcal{E}_2)$, it can be rewritten as $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, Z) \mid (Y_2, U)$. Next, applying CI.2 to this conditional independence yields

$$\mathcal{E}_3 \perp\!\!\!\perp (Y_1, h(Y_1, U, V_1), h(Y_2, U, V_2), Z) \mid (Y_2, U).$$

Since $D_t = h(Y_t, U, V_t)$ for each $t \in \{1, 2\}$, it can be rewritten as $\mathcal{E}_3 \perp\!\!\!\perp (Y_1, D_1, D_2, Z) \mid (Y_2, U)$. Lastly, applying CI.2 again to this conditional independence yields

$$g(Y_2, U, \mathcal{E}_3) \perp\!\!\!\perp (Y_1, D_1, D_2, Z) \mid (Y_2, U).$$

Since $Y_3 = g(Y_2, U, \mathcal{E}_3)$, it can be rewritten as $Y_3 \perp\!\!\!\perp (Y_1, D_1, D_2, Z) \mid (Y_2, U)$.

(ii) Note that Restriction 2 (i) $\mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1, W)$ together with the structural definition $Z = \zeta(U, W)$ implies $\mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1, Z)$. Applying CI.1 to this independence relation $\mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1, Z)$ yields

$$\mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1, Z) \mid (Y_1, U).$$

Next, applying CI.2 to this conditional independence yields

$$g(Y_1, U, \mathcal{E}_2) \perp\!\!\!\perp (U, Y_1, V_1, Z) \mid (Y_1, U).$$

Since $Y_2 = g(Y_1, U, \mathcal{E}_2)$, it can be rewritten as $Y_2 \perp\!\!\!\perp (U, Y_1, V_1, Z) \mid (Y_1, U)$. Lastly, applying CI.2 again to this conditional independence yields

$$Y_2 \perp\!\!\!\perp (h(Y_1, U, V_1), Z) \mid (Y_1, U).$$

Since $D_1 = h(Y_1, U, V_1)$, it can be rewritten as $Y_2 \perp\!\!\!\perp (D_1, Z) \mid (Y_1, U)$.

(iii) Applying CI.1 to Restriction 2 (ii) $V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1)$ yields

$$V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1) \mid (g(Y_1, U, \mathcal{E}_2), U).$$

Since $Y_2 = g(Y_1, U, \mathcal{E}_2)$, it can be rewritten as $V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1) \mid (Y_2, U)$. Next, applying CI.2 to this conditional independence yields

$$V_2 \perp\!\!\!\perp (Y_1, h(Y_1, U, V_1)) \mid (Y_2, U).$$

Since $D_1 = h(Y_1, U, V_1)$, it can be rewritten as $V_2 \perp\!\!\!\perp (Y_1, D_1) \mid (Y_2, U)$. Lastly, applying CI.2 to this conditional independence yields

$$h(Y_2, U, V_2) \perp\!\!\!\perp (Y_1, D_1) \mid (Y_2, U).$$

Since $D_2 = h(Y_2, U, V_2)$, it can be rewritten as $D_2 \perp\!\!\!\perp (Y_1, D_1) \mid (Y_2, U)$.

(iv) Note that Restriction 2 (iii) $W \perp\!\!\!\perp (Y_1, \mathcal{E}_2, V_1, V_2)$ together with the structural definition $Z = \zeta(U, W)$ yields $Z \perp\!\!\!\perp (Y_1, \mathcal{E}_2, V_1, V_2) \mid U$. Applying CI.2 to this conditional independence relation $Z \perp\!\!\!\perp (Y_1, \mathcal{E}_2, V_1, V_2) \mid U$ yields

$$Z \perp\!\!\!\perp (Y_1, g(Y_1, U, \mathcal{E}_2), h(Y_1, U, V_1), h(g(Y_1, U, \mathcal{E}_2), U, V_2)) \mid U.$$

Since $D_t = h(Y_t, U, V_t)$ for each $t \in \{1, 2\}$ and $Y_2 = g(Y_1, U, \mathcal{E}_2)$, this conditional independence can be rewritten as $Z \perp\!\!\!\perp (Y_1, Y_2, D_1, D_2) \mid U$. □

A.4 Lemma 4 (Invariant Transition)

Lemma 4 (Invariant Transition).
(i) Under Restrictions 1 and 2 (i), $F_{Y_3|Y_2U}(y' \mid y, u) = F_{Y_2|Y_1U}(y' \mid y, u)$ *for all* y', y, u.
(ii) Under Restrictions 1 and 2 (ii), $F_{D_2|Y_2U}(d \mid y, u) = F_{D_1|Y_1U}(d \mid y, u)$ *for all* d, y, u.

Proof. (i) First, note that Restriction 2 (i) $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2, W)$ implies $\mathcal{E}_3 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2)$, which in turn implies that $\mathcal{E}_3 \perp\!\!\!\perp (g(Y_1, U, \mathcal{E}_2), U)$, hence $\mathcal{E}_3 \perp\!\!\!\perp (Y_2, U)$. Second, Restriction 2 (i) in particular yields $\mathcal{E}_2 \perp\!\!\!\perp (Y_1, U)$. Using these two independence results, we obtain

$$\begin{aligned} F_{Y_3|Y_2U}(y' \mid y, u) &= \Pr[g(y, u, \mathcal{E}_3) \leq y' \mid Y_2 = y, U = u] \\ &= \Pr[g(y, u, \mathcal{E}_3) \leq y'] \\ &= \Pr[g(y, u, \mathcal{E}_2) \leq y'] \\ &= \Pr[g(y, u, \mathcal{E}_2) \leq y' \mid Y_1 = y, U = u] \;=\; F_{Y_2|Y_1U}(y' \mid y, u) \end{aligned}$$

for all y', y, u, where the second equality follows from $\mathcal{E}_3 \perp\!\!\!\perp (Y_2, U)$, the third equality follows from identical distribution of $\mathcal{E}_t$ by Restriction 1, and the forth equality follows from $\mathcal{E}_2 \perp\!\!\!\perp (Y_1, U)$.

(ii) Restriction 2 (ii) $V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_1, \mathcal{E}_2, V_1)$ implies that $V_2 \perp\!\!\!\perp (g(Y_1, U, \mathcal{E}_2), U)$, hence $V_2 \perp\!\!\!\perp (Y_2, U)$. Restriction 2 (ii) also implies $V_1 \perp\!\!\!\perp (Y_1, U)$. Using these two independence results, we obtain

$$\begin{aligned} F_{D_2|Y_2U}(d \mid y, u) &= \Pr[h(y, u, V_2) \le d \mid Y_2 = y, U = u] \\ &= \Pr[h(y, u, V_2) \le d] \\ &= \Pr[h(y, u, V_1) \le d] \\ &= \Pr[h(y, u, V_1) \le d \mid Y_1 = y, U = u] \;=\; F_{D_1|Y_1U}(d \mid y, u) \end{aligned}$$

for all d, y, u, where the second equality follows from $V_2 \perp\!\!\!\perp (Y_2, U)$, the third equality follows from identical distribution of V_t from Restriction 1, and the forth equality follows from $V_1 \perp\!\!\!\perp (Y_1, U)$. $\square$

B Proofs for Estimation

B.1 Corollary 1 (Constrained Maximum Likelihood)

Proof. Denote the supports of conditional densities by $I_1 = \{(y_2, y_1, z) \mid f_{Y_2Y_1Z|D_2D_1}(y_2, y_1, z \mid 1) > 0\}$ and $I_2 = \{(y_3, y_2, y_1, z) \mid f_{Y_3Y_2Y_1Z|D_2D_1}(y_3, y_2, y_1, z \mid 1, 1) > 0\}$. The Kullback-Leibler information inequality requires that

$$\int_{I_1} \log \left[\frac{f_{Y_2Y_1Z|D_1}(y_2, y_1, z \mid 1)}{\varphi(y_2, y_1, z)} \right] f_{Y_2Y_1Z|D_1}(y_2, y_1, z \mid 1) d\mu(y_2, y_1, z) \ge 0 \qquad \text{and}$$
$$\int_{I_2} \log \left[\frac{f_{Y_3Y_2Y_1Z|D_2D_1}(y_3, y_2, y_1, z \mid 1, 1)}{\psi(y_3, y_2, y_1, z)} \right] f_{Y_3Y_2Y_1Z|D_2D_1}(y_3, y_2, y_1, z \mid 1, 1) d\mu(y_3, y_2, y_1, z) \ge 0$$

for all non-negative measurable functions φ and ψ such that $\int \varphi = \int \psi = 1$. These two inequalities hold with equalities if and only if $f_{Y_2Y_1Z|D_1}(\,\cdot\,, \,\cdot\,, \,\cdot \mid 1) = \varphi$ and $f_{Y_3Y_2Y_1Z|D_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot \mid 1, 1) = \psi$, respectively. Let the set of such pairs of functions (φ, ψ) satisfying the above two Kullback-Leibler inequalities be denoted by

$$\Lambda = \left\{ (\varphi, \psi) \,\middle|\, \varphi \text{ and } \psi \text{ are non-negative measurable functions with } \int \varphi = \int \psi = 1 \right\}.$$

With this notation, the maximization problem

$$\max_{(\varphi,\psi)\in\Lambda} c_1 \mathrm{E}\left[\log \varphi(Y_2, Y_1, Z) \middle| D_1 = 1\right] + c_2 \mathrm{E}\left[\log \psi(Y_3, Y_2, Y_1, Z) \middle| D_2 = D_1 = 1\right] \tag{B.1}$$

has the unique solution $(\varphi, \psi) = (f_{Y_2Y_1Z|D_1}(\,\cdot\,, \,\cdot\,, \,\cdot \mid 1), f_{Y_3Y_2Y_1Z|D_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot \mid 1, 1))$.

Now, let $F(\,\cdot\,; M)$ denote a distribution function generated by model $M \in \mathcal{F}$. For the true model $M^* := (F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$, we have

$$\begin{aligned} F_{Y_2Y_1ZD_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1) &= F_{Y_2Y_1ZD_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1; M^*) \\ F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1) &= F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1; M^*) \end{aligned}$$

Moreover, the identification result of Lemma 2 showed that this true model M^* is the unique element in $\mathcal{F}$ that generates the observed parts of the joint distributions $F_{Y_2Y_1ZD_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1)$ and $F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1)$, i.e.,

$$\begin{aligned} F_{Y_2Y_1ZD_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1) &= F_{Y_2Y_1ZD_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1; M) \text{ if and only if } M = M^* \\ F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1) &= F_{Y_3Y_2Y_1ZD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1; M) \text{ if and only if } M = M^* \end{aligned}$$

But this implies that F^* is the unique model that generates the observable conditional densities $f_{Y_2Y_1Z|D_1}(\ \cdot,\ \cdot,\ \cdot \mid 1)$ and $f_{Y_3Y_2Y_1Z|D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot \mid 1,1)$ among those models $M \in \mathcal{F}$ that are compatible with the observed selection frequencies $f_{D_1}(1)$ and $f_{D_2D_1}(1,1)$, i.e.,

$$\begin{aligned} f_{Y_2Y_1Z|D_1}(\ \cdot,\ \cdot,\ \cdot \mid 1) &= f_{Y_2Y_1Z|D_1}(\ \cdot,\ \cdot,\ \cdot \mid 1; M) \text{ if and only if } M = M^* \\ &\quad \text{given } f_{D_1}(1;M) = f_{D_1}(1), \text{ and} \qquad \text{(B.2)} \\ f_{Y_3Y_2Y_1Z|D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot \mid 1,1) &= f_{Y_3Y_2Y_1Z|D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot \mid 1,1; M) \text{ if and only if } M = M^* \\ &\quad \text{given } f_{D_2D_1}(1,1;M) = f_{D_2D_1}(1,1) \qquad \text{(B.3)} \end{aligned}$$

Since $(\varphi, \psi) = (f_{Y_2Y_1Z|D_1}(\ \cdot,\ \cdot,\ \cdot \mid 1), f_{Y_3Y_2Y_1Z|D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot \mid 1,1))$ is the unique solution to (B.1), the statements (B.2) and (B.3) imply that the true model M^* is the unique solution to

$$\begin{aligned} \max_{M \in \mathcal{F}} \quad & c_1 \mathrm{E}\left[\log f_{Y_2Y_1Z|D_1}(Y_2, Y_1, Z \mid 1; M) \middle| D_1 = 1\right] \\ & + c_2 \mathrm{E}\left[\log f_{Y_3Y_2Y_1Z|D_2D_1}(Y_3, Y_2, Y_1, Z \mid 1,1; M) \middle| D_2 = D_1 = 1\right] \\ \text{s.t.} \quad & f_{D_1}(1;M) = f_{D_1}(1) \quad \text{and} \quad f_{D_2D_1}(1,1;M) = f_{D_2D_1}(1,1) \end{aligned}$$

or equivalently

$$\begin{aligned} \max_{M \in \mathcal{F}} \quad & c_1 \mathrm{E}\left[\log f_{Y_2Y_1ZD_1}(Y_2, Y_1, Z, 1; M) \middle| D_1 = 1\right] \\ & + c_2 \mathrm{E}\left[\log f_{Y_3Y_2Y_1ZD_2D_1}(Y_3, Y_2, Y_1, Z, 1,1; M) \middle| D_2 = D_1 = 1\right] \\ \text{s.t.} \quad & f_{D_1}(1;M) = f_{D_1}(1) \quad \text{and} \quad f_{D_2D_1}(1,1;M) = f_{D_2D_1}(1,1) \qquad \text{(B.4)} \end{aligned}$$

since what have been omitted are constants due to the constraints.

By using Lemmas 3 and 4, we can write the equalities

$$\begin{aligned} f_{Y_2Y_1ZD_1}(y_2, y_1, z, 1; M) &= \int f_{Y_t|Y_{t-1}U}(y_2 \mid y_1, u) f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) f_{Z|U}(z \mid u) d\mu(u) \\ f_{D_1}(1;M) &= \int f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_1, u) \end{aligned}$$

$$\begin{aligned} f_{Y_3Y_2Y_1ZD_2D_1}(y_3, y_2, y_1, z, 1, 1; M) &= \int f_{Y_t|Y_{t-1}U}(y_3 \mid y_2, u) f_{Y_t|Y_{t-1}U}(y_2 \mid y_1, u) f_{D_t|Y_tU}(1 \mid y_2, u) \\ &\quad f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) F_{Z|U}(z \mid u) d\mu(u) \\ f_{D_2D_1}(1,1;M) &= \int f_{Y_t|Y_{t-1}U}(y_2 \mid y_1, u) f_{D_t|Y_tU}(1 \mid y_2, u) \\ &\quad f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_2, y_1, u) \end{aligned}$$

for any model $M := (F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}) \in \mathcal{F}$. Substituting these equalities in (B.4), we conclude that the true model $(F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$ is the unique solution to

$$\max_{\left(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}\right) \in \mathcal{F}} \quad c_1 \mathrm{E}\left[\log \int f_{Y_t|Y_{t-1}U}(Y_2 \mid Y_1, u) f_{D_t|Y_tU}(1 \mid Y_1, u) f_{Y_1U}(Y_1, u) f_{Z|U}(Z \mid u) d\mu(u) \middle| D_1 = 1\right] +$$

$$c_2 \mathrm{E}\left[\log \int f_{Y_t|Y_{t-1}U}(Y_3 \mid Y_2, u) f_{Y_t|Y_{t-1}U}(Y_2 \mid Y_1, u) f_{D_t|Y_tU}(1 \mid Y_2, u) f_{D_t|Y_tU}(1 \mid Y_1, u) f_{Y_1U}(Y_1, u) F_{Z|U}(Z \mid u) d\mu(u) \middle| D_2 = D_1 = 1\right]$$

subject to

$$\int f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_1, u) = f_{D_1}(1) \qquad \text{and}$$

$$\int f_{Y_t|Y_{t-1}U}(y_2 \mid y_1, u) f_{D_t|Y_tU}(1 \mid y_2, u) f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_2, y_1, u) = f_{D_2D_1}(1,1)$$

as claimed. $\square$

B.2 Remark 9 (Unit Lagrange Multipliers)

For short-hand notation, we write $f = (f_1, f_2, f_3, f_4) \in \mathcal{F}$ for an element of $\mathcal{F}$, $p_1 := \Pr(D_1 = 1)$, $p_2 := \Pr(D_2 = D_1 = 1)$, and $p := (p_1, p_2)'$. The solution $f^*(\cdot\ ; p) \in \mathcal{F}$ has Lagrange multipliers $\lambda^*(p) = (\lambda_1^*(p), \lambda_2^*(p))' \in \Lambda$ such that $(f^*(\cdot\ ; p), \lambda^*(p))$ is a saddle point of the Lagrangean functional

$$L(f, \lambda; p) = p_1 L_1(f; p_1) + p_2 L_2(f; p_2) - \lambda_1(L_3(f) - p_1) - \lambda_2(L_4(f) - p_2),$$

where the functionals $L_1, \cdots, L_4$ are defined as

$$\begin{aligned}
L_1(f; p_1) &= \int \left[\log \int f_1(y_2 \mid y_1, u) f_2(1 \mid y_1, u) f_3(y_1, u) f_4(z \mid u) d\mu(u)\right] \\
&\quad \times f_{Y_2 Y_1 Z \mid D_1}(y_2, y_1, z \mid 1) d\mu(y_2, y_1, z) \\
L_2(f; p_2) &= \int \left[\log \int f_1(y_3 \mid y_2, u) f_1(y_2 \mid y_1, u) f_2(1 \mid y_2, u) f_2(1 \mid y_1, u) f_3(y_1, u) f_4(z \mid u) d\mu(u)\right] \\
&\quad \times f_{Y_3 Y_2 Y_1 Z \mid D_2 D_1}(y_3, y_2, y_1, z \mid 1, 1) d\mu(y_3, y_2, y_1, z) \\
L_3(f) &= \int f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_1, u) \qquad \text{and} \\
L_4(f) &= \int f_1(y_2 \mid y_1, u) f_2(1 \mid y_2, u) f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_2, y_1, u).
\end{aligned}$$

Moreover, $f^*(\cdot\ ; p)$ maximizes $L(f, \lambda^*(p); p)$ given λ is restricted to $\lambda^*(p)$. We want to claim that $\lambda^*(p) = (1, 1)'$. The following assumptions are imposed to this end.

Assumption (Regularity for Unit Lagrange Multipliers).
(i) Selection probabilities are positive: $p_1, p_2 > 0$.
(ii) The functionals L_1, L_2, L_3, and L_4 are Fréchet differentiable with respect to f at the solution $f^*(\cdot\ ; p)$ for some norm $\|\cdot\|$ on a linear space containing $\mathcal{F}$.
(iii) The solution $(f^*(\cdot\ ; p), \lambda^*(p))$ is differentiable with respect to p.
(iv) The solution $f^*(\cdot\ ; p)$ is a regular point of the constraint functionals L_3 and L_4.

A sufficient condition for part (ii) of this assumption will be provided later in terms of a concrete normed linear space.

Proof. Since the Chain Rule holds for a composition of Fréchet differentiable transformations (cf. Luenberger, 1969; pp.176), we have

$$\begin{aligned}
\frac{d}{dp_1} L(f^*(\cdot\ ; p), \lambda^*(p); p) &= D_{f,\lambda} L(f^*(\cdot\ ; p), \lambda^*(p); p) \cdot D_{p_1}(f^*(\cdot\ ; p), \lambda^*(p)) \\
&\quad + \frac{\partial}{\partial p_1} L(f^*(\cdot\ ; p), \lambda^*(p); p) \;=\; \frac{\partial}{\partial p_1} L(f^*(\cdot\ ; p), \lambda^*(p); p)
\end{aligned}$$

where the second equality follows from the equality constraints and the stationarity of $L(\cdot, \lambda^*(p); p)$ at $f^*(\cdot\ ; p)$, which is a regular point of the constraint functionals L_3 and L_4 by assumption.

On one hand, the partial derivative is

$$\frac{\partial}{\partial p_1} L(f^*(\cdot\ ; p), \lambda^*(p); p) = \lambda_1^*(p).$$

On the other hand, the complementary slackness yields

$$\frac{d}{dp_1} L(f^*(\cdot\ ; p), \lambda^*(p); p) \;=\; \frac{d}{dp_1}[p_1 L_1(f^*(\cdot\ ; p); p_1)].$$

In order to evaluate the last term, we first note that

$$\begin{aligned} p_1 L_1(f; p_1) \;=\; & \int \left[\log \int f_1(y_2 \mid y_1, u) f_2(1 \mid y_1, u) f_3(y_1, u) f_4(z \mid u) d\mu(u)\right] \\ & \times f_{Y_2 Y_1 Z D_1}(y_2, y_1, z, 1) d\mu(y_2, y_1, z). \end{aligned}$$

In view of the proof of Corollary 1, we recall that $f^*(\cdot\ ; p)$ maximizes $p_1 L_1(\cdot\ ; p_1)$, and the solution $f^*(\cdot\ ; p)$ satisfies

$$\int f_1^*(y_2 \mid y_1, u; p) f_2^*(1 \mid y_1, u; p) f_3^*(y_1, u; p) f_4^*(z \mid u; p) d\mu(u) = f_{Y_2 Y_1 Z | D_1}(y_2, y_1, z \mid 1) \cdot p_1,$$

where the conditional density $f_{Y_2 Y_1 Z | D_1}(\cdot, \cdot, \cdot \mid 1)$ is invariant from variations in p, and the scale of the integral varies by p_1 which defines the $\mathcal{L}^1$-equivalence class of non-negative functions over which the Kullback-Leibler information inequality is satisfied. Therefore, we have

$$\frac{d}{dp_1}\left[\log \int f_1^*(y_2 \mid y_1, u; p) f_2^*(1 \mid y_1, u; p) f_3^*(y_1, u; p) f_4^*(z \mid u; p) d\mu(u)\right] = \frac{1}{p_1}.$$

It then follows that

$$\frac{d}{dp_1} L(f^*(\cdot\ ; p), \lambda^*(p); p) \;=\; \frac{d}{dp_1}[p_1 L_1(f^*(\cdot\ ; p); p_1)] = \frac{1}{p_1} \int f_{Y_2 Y_1 Z D_1}(y_2, y_1, z, 1) d\mu(y_2, y_1, z) = 1,$$

showing that $\lambda_1^*(p) = 1$. Similar lines of argument prove $\lambda_2^*(p) = 1$. □

Part (ii) of the above assumption is ambiguous about the definition of underlying topological spaces, as we did not explicitly define the norm. In order to complement for it, here we consider a sufficient condition. Write $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{F}_4$. Define a norm on $\mathcal{F}$ by $\|f\|_s := \|f_1\|_2 + \|f_2\|_2 + \|f_3\|_2 + \|f_4\|_2$, where $\|\cdot\|_2$ denotes the $\mathcal{L}^2$-norm. Also, define the set $B_j(M) = \{f_j \in \mathcal{F}_j \mid \|f_j\|_\infty \leqslant M\}$ for $M \in (0, \infty)$ for each $j = 1, 2, 3, 4$. The following uniform boundedness and integrability together imply part (ii).

Assumption (A Sufficient Condition for Part (ii))**.** There exists $M < \infty$ such that $\mathcal{F}_1 \subset \mathcal{L}^1 \cap B_1(M)$, $\mathcal{F}_2 \subset \mathcal{L}^1 \cap B_2(M)$, $\mathcal{F}_3 \subset \mathcal{L}^1 \cap B_3(M)$, and $\mathcal{F}_4 \subset \mathcal{L}^1 \cap B_4(M)$ hold with the respective Lebesgue measurable spaces.

Note that $\mathcal{F}_1 \subset \mathcal{L}^1 \cap \mathcal{L}^\infty$, $\mathcal{F}_2 \subset \mathcal{L}^1 \cap \mathcal{L}^\infty$, $\mathcal{F}_3 \subset \mathcal{L}^1 \cap \mathcal{L}^\infty$, and $\mathcal{F}_4 \subset \mathcal{L}^1 \cap \mathcal{L}^\infty$ follow from this assumption, since $B(M) \subset \mathcal{L}^\infty$ for each $j = 1, 2, 3, 4$. But then, each of these sets is also square integrable as $\mathcal{L}^1 \cap \mathcal{L}^\infty \subset \mathcal{L}^2$ (cf. Folland, 1999; pp. 185).

To see the Fréchet differentiability of L_1, observe that for any $\eta \in \mathcal{F}$

$$
\begin{aligned}
& \left\| \int (f_1+\eta_1)(f_2+\eta_2)(f_3+\eta_3)(f_4+\eta_4)d\mu(u) - \int f_1 f_2 f_3 f_4 d\mu(u) - DL_1(f;\eta) \right\|_1 \\
\leqslant\ & \|f_1 f_2 \eta_3 \eta_4\|_1 + \|f_1 \eta_2 \eta_3 f_4\|_1 + \|\eta_1 \eta_2 f_3 f_4\|_1 + \|f_1 \eta_2 f_3 \eta_4\|_1 + \|\eta_1 f_2 \eta_3 f_4\|_1 + \|\eta_1 f_2 f_3 \eta_4\|_1 \\
& + \|f_1 \eta_2 \eta_3 \eta_4\|_1 + \|\eta_1 f_2 \eta_3 \eta_4\|_1 + \|\eta_1 \eta_2 f_3 \eta_4\|_1 + \|\eta_1 \eta_2 \eta_3 f_4\|_1 + \|\eta_1 \eta_2 \eta_3 \eta_4\|_1 \\
\leqslant\ & \|f_1\|_\infty \|f_2\|_\infty \|\eta_3\|_2 \|\eta_4\|_2 + \|f_1\|_\infty \|f_4\|_\infty \|\eta_2\|_2 \|\eta_3\|_2 + \|f_3\|_\infty \|f_4\|_\infty \|\eta_1\|_2 \|\eta_2\|_2 \\
& + \|f_1\|_\infty \|f_3\|_\infty \|\eta_2\|_2 \|\eta_4\|_2 + \|f_2\|_\infty \|f_4\|_\infty \|\eta_1\|_2 \|\eta_3\|_2 + \|f_2\|_\infty \|f_3\|_\infty \|\eta_1\|_2 \|\eta_4\|_2 \\
& + \|f_1\|_\infty \|\eta_2\|_\infty \|\eta_3\|_2 \|\eta_4\|_2 + \|\eta_1\|_\infty \|f_2\|_\infty \|\eta_3\|_2 \|\eta_4\|_2 + \|\eta_1\|_\infty \|f_3\|_\infty \|\eta_2\|_2 \|\eta_4\|_2 \\
& + \|\eta_1\|_\infty \|f_4\|_\infty \|\eta_2\|_2 \|\eta_3\|_2 + \|\eta_1\|_\infty \|\eta_2\|_\infty \|\eta_3\|_2 \|\eta_4\|_2 \\
\leqslant\ & (\|f_1\|_\infty \|f_2\|_\infty + \|f_1\|_\infty \|f_4\|_\infty + \|f_3\|_\infty \|f_4\|_\infty + \|f_1\|_\infty \|f_3\|_\infty + \|f_2\|_\infty \|f_4\|_\infty \\
& + \|f_2\|_\infty \|f_3\|_\infty + \|f_1\|_\infty \|\eta_2\|_\infty + \|\eta_1\|_\infty \|f_2\|_\infty + \|\eta_1\|_\infty \|f_3\|_\infty + \|\eta_1\|_\infty \|f_4\|_\infty \\
& + \|\eta_1\|_\infty \|\eta_2\|_\infty) \|\eta\|_s^2 \ \leqslant\ 11M^2 \|\eta\|_s^2 ,
\end{aligned}
$$

where the $\mathcal{L}^1$-norm in the first line is by integration with respect to (y_2, y_1, z), all the remaining $\mathcal{L}^p$-norms are by integration with respect to (y_2, y_1, z, u), $DL_1(f;\eta) := \int (f_1 f_2 f_3 \eta_4 + f_1 f_2 \eta_3 f_4 + f_1 \eta_2 f_3 f_4 + \eta_1 f_2 f_3 f_4) d\mu(u)$, the first inequality follows from the triangle inequality, the second inequality follows from the Hölder's inequality, the third inequality follows from our definition of the norm on $\mathcal{F}$, and the last inequality follows from our assumption. But then,

$$
\lim_{\|\eta\|_s \to 0} \frac{\left\| \int (f_1+\eta_1)(f_2+\eta_2)(f_3+\eta_3)(f_4+\eta_4)d\mu(u) - \int f_1 f_2 f_3 f_4 d\mu(u) - DL_1(f;\eta) \right\|_1}{\|\eta\|_s} = 0,
$$

showing that $DL_1(f;\eta)$ is the Fréchet derivative of the operator $f \mapsto \int f_1 f_2 f_3 f_4 d\mu(u)$. This in turn implies Fréchet differentiability of the functional L_1 at the solution $f^*(\cdot\ ;p)$, since the functional $\mathcal{L}^1 \ni \eta \mapsto \int \log \eta dF_{Y_2 Y_1 Z|D_1=1}$ is Fréchet differentiable at $\eta = f_{Y_2 Y_1 Z D_1}(\cdot,\cdot,\cdot,1)$. Similar lines of arguments will show Fréchet differentiability of the other functionals L_2, L_3, and L_4 at $f^*(\cdot\ ;p)$.

B.3 Proposition 1 (Consistency of the Nonparametric Estimator)

As a setup, we define a normed linear space $(\mathcal{L}, \|\cdot\|)$ containing the model set $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{F}_4$ as follows. We define the uniform norm of f as the essential supremum

$$
\|f\|_\infty = \text{ess sup}_x |f(x)| .
$$

Following Newey and Powell (2003) and others, we also define the following version of the uniform norm when characterizing compactness:

$$
\|f\|_{R,\infty} = \text{ess sup}_x |f(x)(1 + x'x)| .
$$

Noted that $\|\cdot\|_\infty \leqslant \|\cdot\|_{R,\infty}$ holds. Similarly define the version of the $\mathcal{L}^1$ norm

$$
\|f\|_{R,1} = \int |f(x)| (1 + x'x) dx.
$$

Define a norm $\|\cdot\|$ on a linear space containing $\mathcal{F}$ by

$$
\|f\| := \|f_1\|_{R,\infty} + \|f_2\|_{R,\infty} + \|f_3\|_{R,\infty} + \|f_4\|_{R,\infty} .
$$

We consider $\mathcal{F}$ with the subspace topology of this normed linear space, where Assumption 3 below imposes restrictions on how to appropriately choose such a subset $\mathcal{F}$.

We assume that the data is i.i.d.

Assumption 1 (Data)**.** The data $\{(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{2i}, D_{1i})\}_{i=1}^n$ is i.i.d.

In order to model the rate at which the complexity of sieve spaces evolve with sample size n, we introduce the notation $N(\cdot\ ,\cdot\ , \|\cdot\|)$ for the covering numbers without bracketing. Let $B(f, \varepsilon) = \{f' \in \mathcal{F} \mid \|f - f'\| < \varepsilon\}$ denote the ε-ball around $f \in \mathcal{F}$ with respect to the norm $\|\cdot\|$ defined above. For each $\varepsilon > 0$ and n, let $N(\varepsilon, \mathcal{F}_{k(n)}, \|\cdot\|)$ denote the minimum number of such ε-balls covering $\mathcal{F}_{k(n)}$, i.e., $\min\{|C| \mid \cup_{f\in C} B(f, \varepsilon) \supset \mathcal{F}_{k(n)}\}$. With this notation, we assume the following restriction.

Assumption 2 (Sieve Spaces)**.**
(i) $\{\mathcal{F}_{k(n)}\}_{n=1}^\infty$ is an increasing sequence, $\mathcal{F}_{k(n)} \subset \mathcal{F}$ for each n, and there exists a sequence $\{\pi_{k(n)} f_0\}_{n=1}^\infty$ such that $\pi_{k(n)} f_0 \in \mathcal{F}_{k(n)}$ for each n.
(ii) $\log N(\varepsilon, \mathcal{F}_{k(n)}, \|\cdot\|) = o(n)$ for all $\varepsilon > 0$.

The next assumption facilitates compactness of the model set and Hölder continuity of the objective functional, both of which are important for nice large sample behavior of the estimator. We assume that the true model f_0 belongs to $\mathcal{F}$ satisfying the following.

Assumption 3 (Model Set)**.**
(i) $\mathcal{L}^1$ Compactness: Each of $\mathcal{F}_2$ and $\mathcal{F}_3$ is compact with respect to $\|\cdot\|_{R,1}$. Thus, let $M < \infty$ be a number such that $\sup_{f_i \in \mathcal{F}_i} \|f_i\|_{R,1} \leqslant M$ for each $i = 2, 3$.
(ii) $\mathcal{L}^\infty$ Compactness: Each of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ is compact with respect to $\|\cdot\|_{R,\infty}$. Thus, let $M_\infty < \infty$ be a number such that $\sup_{f_i \in \mathcal{F}_i} \|f_i\|_{R,\infty} \leqslant M_\infty$ for each $i = 1, 2, 3, 4$.
(iii) Uniformly Bounded Density of $\mathcal{E}$: There exists $M_1 < \infty$ such that

$$\sup_{f_1 \in \mathcal{F}_1} \sup_{y_2, y_1} \int |f_1(y_2 \mid y_1, u)|\, du \leqslant M_1.$$

(iv) Uniformly Bounded Density of Y_1: There exists $M_3 < \infty$ such that

$$\sup_{f_3 \in \mathcal{F}_3} \sup_{y_1} \int |f_3(y_1, u)|\, du \leqslant M_3.$$

(v) Bounded Objective:

$$\mathrm{E}\left[\left(\inf_{f\in\mathcal{F}} \int f_1(Y_2 \mid Y_1, u) f_2(1 \mid Y_1, u) f_3(Y_1, u) f_4(Z \mid u) d\mu(u)\right)^{-1}\right] < \infty \quad \text{and}$$
$$\mathrm{E}\left[\left(\inf_{f\in\mathcal{F}} \int f_1(Y_3 \mid Y_2, u) f_1(Y_2 \mid Y_1, u) f_2(1 \mid Y_2, u) f_2(1 \mid Y_1, u) f_3(Y_1, u) f_4(Z \mid u) d\mu(u)\right)^{-1}\right] < \infty$$

Part (i) of this assumption is not redundant, since f_i are not densities, but conditional densities. Despite their appearance, parts (iii) and (iv) of this assumption are not so stringent. Suppose, for example, that the true model f_0 consists of the traditional additively separable dynamic model $Y_t = \alpha Y_{t-1} + U + \mathcal{E}_t$ with a uniformly bounded density of $\mathcal{E}_t$. In this case, the true model f_0 can indeed reside in an $\mathcal{F}$ satisfying the restriction of part (iii) for a suitable choice of M_1. Similarly, the true model f_0 can reside in an $\mathcal{F}$ satisfying the restriction of part (iv) for a suitable choice of M_3, whenever the density of Y_1 is uniformly bounded.

Proof. We show the consistency claim of Proposition 1 by showing that Conditions 3.1, 3.2, 3.4, and 3.5M of Chen (2007) are satisfied by our assumptions (Restrictions 1, 2, 3, and 4 and Assumptions 1, 2, and 3). Restrictions 1, 2, 3, and 4 imply her Condition 3.1 by our identification result yielding Corollary 1 together with Remark 9. Her Condition 3.2 is directly assumed by our Assumption 2 (i). Her Condition 3.4 is implied by our Assumption 3 (ii) applied to the Tychonoff's Theorem. Her Conditions 3.5M (i) and (iii) are directly assumed by our Assumptions 1 and 2 (ii), respectively, provided that we will prove her Condition 3.5M (ii) with $s = 1$. It remains to prove Hölder continuity of $l(y_3, y_2, y_1, z, d_2, d_1;\ \cdot\ ) : (\mathcal{F}, \|\cdot\|) \to \mathbb{R}$ for each $(y_3, y_2, y_1, z, d_2, d_1)$, which in turn implies her Condition 3.5M (ii).

In order to show Hölder continuity of the functional $l(y_3, y_2, y_1, z, d_2, d_1;\ \cdot\ )$, it suffices to prove that of $l_1(y_2, y_1, z;\ \cdot\ )$, $l_2(y_3, y_2, y_1, z;\ \cdot\ )$, l_3, and l_4. First, consider $l_1(y_2, y_1, z;\ \cdot\ )$. For a fixed (y_2, y_1, z), observe

$$\begin{aligned}
& \left|\exp(l_1(y_2, y_1, z; f)) - \exp(l_1(y_2, y_1, z; \bar{f}))\right| \\
\leqslant\ & \left|\int (f_1 - \bar{f}_1) f_2 f_3 f_4 du\right| + \left|\int \bar{f}_1 (f_2 - \bar{f}_2) f_3 f_4 du\right| \\
& + \left|\int \bar{f}_1 \bar{f}_2 (f_3 - \bar{f}_3) f_4 du\right| + \left|\int \bar{f}_1 \bar{f}_2 \bar{f}_3 (f_4 - \bar{f}_4) du\right| \\
\leqslant\ & \left\|f_1 - \bar{f}_1\right\|_\infty \|f_2\|_\infty \int |f_3|\, du\, \|f_4\|_\infty + \left\|\bar{f}_1\right\|_\infty \left\|f_2 - \bar{f}_2\right\|_\infty \int |f_3|\, du\, \|f_4\|_\infty \\
& + \int \left|\bar{f}_1\right| du \left\|\bar{f}_2\right\|_\infty \left\|f_3 - \bar{f}_3\right\|_\infty \|f_4\|_\infty + \int \left|\bar{f}_1\right| du \left\|\bar{f}_2\right\|_\infty \left\|\bar{f}_3\right\|_\infty \left\|f_4 - \bar{f}_4\right\|_\infty \\
\leqslant\ & 2M_\infty^2 (M_1 + M_3) \left\|f - \bar{f}\right\|,
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from the Hölder's inequality, and the third inequality uses Assumption 3 (ii), (iii), and (iv), together with the fact that $\|\cdot\|_\infty \leqslant \|\cdot\|_{R,\infty}$. By Assumption 3 (v), there exists a function κ_1 such that $\mathrm{E}[\kappa_1(Y_2, Y_1, Z)] < \infty$ and

$$\left|l_1(y_2, y_1, z; f) - l_1(y_2, y_1, z; \bar{f})\right| \leqslant 2M_\infty^2 (M_1 + M_3) \left\|f - \bar{f}\right\| \kappa_1(y_2, y_1, z).$$

This shows Hölder (in particular Lipschitz) continuity of the functional $l_1(y_2, y_1, z;\ \cdot\ )$.

By similar calculations using Assumption 3 (ii) and (iii), we obtain

$$\begin{aligned}
& \left|\exp(l_2(y_3, y_2, y_1, z; f)) - \exp(l_2(y_3, y_2, y_1, z; \bar{f}))\right| \\
\leqslant\ & \left\|f_1 - \bar{f}_1\right\|_\infty \int |f_1|\, du\, \|f_2\|_\infty^2 \|f_3\|_\infty \|f_4\|_\infty + \int \left|\bar{f}_1\right| du \left\|f_1 - \bar{f}_1\right\|_\infty \|f_2\|_\infty^2 \|f_3\|_\infty \|f_4\|_\infty \\
& + \int \left|\bar{f}_1\right| du \left\|\bar{f}_1\right\|_\infty \left\|f_2 - \bar{f}_2\right\|_\infty \|f_2\|_\infty \|f_3\|_\infty \|f_4\|_\infty \\
& + \int \left|\bar{f}_1\right| du \left\|\bar{f}_1\right\|_\infty \left\|\bar{f}_2\right\|_\infty \left\|f_2 - \bar{f}_2\right\|_\infty \|f_3\|_\infty \|f_4\|_\infty \\
& + \int \left|\bar{f}_1\right| du \left\|\bar{f}_1\right\|_\infty \left\|\bar{f}_2\right\|_\infty^2 \left\|f_3 - \bar{f}_3\right\|_\infty \|f_4\|_\infty + \int \left|\bar{f}_1\right| du \left\|\bar{f}_1\right\|_\infty \left\|\bar{f}_2\right\|_\infty^2 \left\|\bar{f}_3\right\|_\infty \left\|f_4 - \bar{f}_4\right\|_\infty \\
\leqslant\ & 6M_\infty^4 M_1 \left\|f - \bar{f}\right\|.
\end{aligned}$$

By Assumption 3 (v), there exists a function κ_2 such that $\mathrm{E}[\kappa_2(Y_3, Y_2, Y_1, Z)] < \infty$ and

$$\left|l_2(y_3, y_2, y_1, z; f) - l_2(y_3, y_2, y_1, z; \bar{f})\right| \leqslant 6M_\infty^4 M_1 \left\|f - \bar{f}\right\| \kappa_2(y_3, y_2, y_1, z).$$

This shows Lipschitz continuity of the functional $l_2(y_3, y_2, y_1, z;\ \cdot\ )$.

Next, using Assumption 3 (i) yields

$$\begin{aligned}\left|l_3(f) - l_3(\bar{f})\right| &\leqslant \left|\int (f_2 - \bar{f}_2) f_3\right| + \left|\int \bar{f}_2 (f_3 - \bar{f}_3)\right| \\ &\leqslant \left\|f_2 - \bar{f}_2\right\|_\infty \|f_3\|_1 + \left\|\bar{f}_2\right\|_1 \left\|f_3 - \bar{f}_3\right\|_\infty \leqslant 2M \left\|f - \bar{f}\right\|,\end{aligned}$$

Similarly, using Assumption (i) and (ii) yields

$$\begin{aligned}\left|l_4(f) - l_4(\bar{f})\right| &\leqslant \left\|f_1 - \bar{f}_1\right\|_\infty \|f_2\|_\infty^2 \|f_3\|_1 + \left\|\bar{f}_1\right\|_\infty \left\|f_2 - \bar{f}_2\right\|_\infty \|f_2\|_\infty \|f_3\|_1 \\ &\quad + \left|\bar{f}_1\right|_\infty \left\|\bar{f}_2\right\|_\infty \left\|f_2 - \bar{f}_2\right\|_\infty \|f_3\|_1 + \left|\bar{f}_1\right|_\infty \left\|\bar{f}_2\right\|_\infty \|f_2\|_1 \left\|f_3 - \bar{f}_3\right\|_\infty \\ &\leqslant 4MM_\infty^2 \left\|f - \bar{f}\right\|.\end{aligned}$$

It follows that l_3 and l_4 are also Lipschitz continuous. These in particular implies Hölder continuity of the functionals $l_1(y_2, y_1, z;\ \cdot\ )$, $l_2(y_3, y_2, y_1, z;\ \cdot\ )$, l_3, and l_4, hence $l(y_3, y_2, y_1, z, d_2, d_1;\ \cdot\ )$. Therefore, Chen's Condition 3.5M (ii) is satisfied with $s = 1$ by our assumptions. □

B.4 Semiparametric Estimation

Section 5.2 proposed an estimator which treats the quadruple $(f_{Y_t|Y_{t-1}U}, f_{D_t|Y_tU}, f_{Y_1U}, f_{Z|U})$ of the density functions nonparametrically. In practice, it may be more useful to specify one or more of these densities semi-parametrically. For example, the dynamic model g is conventionally specified by

$$g(y, u, \varepsilon) = \alpha y + u + \varepsilon.$$

By denoting the nonparametric density functions of $\mathcal{E}_t$ by $f_{\mathcal{E}}$, we can represent the density $f_{Y_t|Y_{t-1}U}$ by $f_{Y_t|Y_{t-1}U}(y' \mid y, u) = f_{\mathcal{E}}(y' - \alpha y - u)$. Consequently, a model is represented by $(\alpha, f_{\mathcal{E}}, f_{D_t|Y_tU}, f_{Y_1U}, f_{Z|U})$. For ease of writing, let this model be denoted by $\theta = (\alpha, \tilde{f}_1, f_2, f_3, f_4)$. Accordingly, write a set of such models by $\Theta = \mathcal{A} \times \tilde{F}_1 \times F_2 \times F_3 \times F_4$

Under these notations, Corollary 1 and Remark 9 characterize a sieve semiparametric estimator $\hat{\theta}$ of θ_0 as the solution to

$$\max_{\theta \in \Theta_{k(n)}} \frac{1}{n} \sum_{i=1}^{n} l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; \theta)$$

for some sieve space $\Theta_{k(n)} = \mathcal{A}_{k(n)} \times \tilde{\mathcal{F}}_{1,k_1(n)} \times \mathcal{F}_{2,k_2(n)} \times \mathcal{F}_{3,k_3(n)} \times \mathcal{F}_{4,k_4(n)} \subset \Theta$, where

$$\begin{aligned}l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; \theta) &:= \mathbb{1}\{D_{i1} = 1\} \cdot l_1(Y_{i2}, Y_{i1}, Z_i; \theta) \\ &\quad + \mathbb{1}\{D_{i2} = D_{i1} = 1\} \cdot l_2(Y_{i3}, Y_{i2}, Y_{i1}, Z_i; \theta) - l_3(\theta) - l_4(\theta), \\ l_1(Y_{i2}, Y_{i1}, Z_i; \theta) &:= \log \int \tilde{f}_1(Y_{i2} - \alpha Y_{i1} - u) f_2(1 \mid Y_{i1}, u) f_3(Y_{i1}, u) f_4(Z_i \mid u) d\mu(u), \\ l_2(Y_{i3}, Y_{i2}, Y_{i1}, Z_i; \theta) &:= \log \int \tilde{f}_1(Y_{i3} - \alpha Y_{i2} - u) f_1(Y_{i2} - \alpha Y_{i1} - u) \\ &\quad \times f_2(1 \mid Y_{i2}, u) f_2(1 \mid Y_{i1}, u) f_3(Y_{i1}, u) f_4(Z_i \mid u) d\mu(u), \\ l_3(\theta) &:= \int f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_1, u), \qquad \text{and} \\ l_4(\theta) &:= \int \tilde{f}_1(y_2 - \alpha y_1 - u) f_2(1 \mid y_2, u) f_2(1 \mid y_1, u) f_3(y_1, u) d\mu(y_2, y_1, u).\end{aligned}$$

The asymptotic distribution of $\hat{\alpha}$ can be derived by following the method of Ai and Chen (2003), which was also used in Blundell, Chen, Kristensen (2007) and Hu and Schennach (2008). First, I introduce auxiliary notations. Define the path-wise derivative

$$l'_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; \theta - \theta_0) = \lim_{r \to 0} \frac{l(y_3, y_2, y_1, z, d_2, d_1; \theta(\theta_0, r)) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0)}{r}$$

where $\theta(\theta_0, \cdot) : \mathbb{R} \to \Theta$ denotes a path such that $\theta(\theta_0, 0) = \theta_0$ and $\theta(\theta_0, 1) = \theta$. Similarly define the path-wise derivative with respect to each component of $(\tilde{f}_1, f_2, f_3, f_4)$ by

$$\begin{aligned}
\frac{d}{d\tilde{f}_1} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; \tilde{f}_1 - \tilde{f}_{10}) &= \lim_{r \to 0} \frac{l(y_3, y_2, y_1, z, d_2, d_1; \tilde{f}_1(\theta_0, r)) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0)}{r} \\
\frac{d}{df_2} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_2 - f_{20}) &= \lim_{r \to 0} \frac{l(y_3, y_2, y_1, z, d_2, d_1; f_2(\theta_0, r)) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0)}{r} \\
\frac{d}{df_3} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_3 - f_{30}) &= \lim_{r \to 0} \frac{l(y_3, y_2, y_1, z, d_2, d_1; f_3(\theta_0, r)) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0)}{r} \\
\frac{d}{df_4} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_4 - f_{40}) &= \lim_{r \to 0} \frac{l(y_3, y_2, y_1, z, d_2, d_1; f_4(\theta_0, r)) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0)}{r}
\end{aligned}$$

where $\tilde{f}_1(\theta_0, \cdot) : \mathbb{R} \to \tilde{F}_1$, $f_2(\theta_0, \cdot) : \mathbb{R} \to F_2$, $f_3(\theta_0, \cdot) : \mathbb{R} \to F_3$, and $f_4(\theta_0, \cdot) : \mathbb{R} \to F_4$ denote paths as before.

Recenter the set of parameters by $\Omega = \Theta - \theta_0$ so that

$$\langle v_1, v_2 \rangle = \mathrm{E}\left[l'_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; v_1) l'_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; v_2) \right]$$

defines an inner product on Ω. Furthermore, by taking the closure $\overline{\Omega}$, we obtain a complete space $\overline{\Omega}$ with respect to the topology induced by $\langle \cdot, \cdot \rangle$, hence a Hilbert space $(\overline{\Omega}, \langle \cdot, \cdot \rangle)$. It can be written as $\overline{\Omega} = \mathbb{R} \times \overline{W}$ where $W = \tilde{F}_1 \times F_2 \times F_3 \times F_4 - (\tilde{f}_{10}, f_{20}, f_{30}, f_{40})$. Given these notations, define

$$\begin{aligned}
w^* := (\tilde{f}_1^*, f_2^*, f_3^*, f_4^*) = \arg\min_{w \in \overline{W}} \mathrm{E}\Bigg[\Bigg(& \frac{d}{d\alpha} l(Y_3, Y_2, Y_1, Z, D_2, D_1; \theta_0) - \frac{d}{d\tilde{f}_1} l_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; \tilde{f}_1) \\
& - \frac{d}{df_2} l_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; f_2) - \frac{d}{df_3} l_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; f_3) \\
& - \frac{d}{df_4} l_{\theta_0}(Y_3, Y_2, Y_1, Z, D_2, D_1; f_4) \Bigg)^2 \Bigg].
\end{aligned}$$

Given this w^*, next define

$$\begin{aligned}
\Phi_{w^*}(y_3, y_2, y_1, z, d_2, d_1) := & \frac{d}{d\alpha} l(y_3, y_2, y_1, z, d_2, d_1; \theta_0) - \frac{d}{d\tilde{f}_1} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; \tilde{f}_1) \\
& - \frac{d}{df_2} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_2) - \frac{d}{df_3} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_3) \\
& - \frac{d}{df_4} l_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; f_4).
\end{aligned}$$

The next assumption sets a moment condition.

Assumption 4 (Bounded Second Moment)**.** $\sigma := \mathrm{E}\left[\Phi_{w^*}(Y_3, Y_2, Y_1, Z, D_2, D_1)^2\right] < \infty$

The mapping $\theta - \theta_0 \overset{s}{\mapsto} \alpha - \alpha_0$ is a linear functional on $\overline{\Omega}$. Since $(\overline{\Omega}, \langle \cdot, \cdot \rangle)$ is a Hilbert space, the Riesz Representation Theorem guarantees the existence of $v^* \in \overline{\Omega}$ such that $s(\theta - \theta_0) = \langle v^*, \theta - \theta_0 \rangle$ for all $\theta \in \overline{\Theta}$ under Assumption 4. Moreover, this representing vector has the explicit formula $v^* = (\sigma^{-1}, -\sigma^{-1} w^*)$. Using Corollary 1 of Shen (1997) yields asymptotic distribution of $\sqrt{N}(\alpha - \alpha_0) = \sqrt{N}\left\langle v^*, \hat{\theta} - \theta_0 \right\rangle$, which is $N(0, \sigma^{-1})$.

In order to invoke Shen's corollary, a couple of additional notations need to be introduced. The remainder of the linear approximation is

$$\begin{aligned} r(y_3, y_2, y_1, z, d_2, d_1; \theta - \theta_0) \quad := \quad & l(y_3, y_2, y_1, z, d_2, d_1; \theta) - l(y_3, y_2, y_1, z, d_2, d_1; \theta_0) \\ & - l'_{\theta_0}(y_3, y_2, y_1, z, d_2, d_1; \theta - \theta_0) \end{aligned}$$

A divergence measure is defined by

$$K(\theta_0, \theta) := \frac{1}{N} \sum_{i=1}^{N} \mathrm{E}\left[l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; \theta_0) - l(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}; \theta) \right].$$

Denote the empirical process induced by g by

$$\nu_n(g) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(g(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}) - \mathrm{E}g(Y_{i3}, Y_{i2}, Y_{i1}, Z_i, D_{i2}, D_{i1}) \right)$$

For a perturbation ϵ_n such that $\epsilon_n = o(n^{-1/2})$, let $\theta^*(\theta, \epsilon_n) = (1 - \epsilon_n)\theta + \epsilon_n(u^* + \theta_0)$ where $u^* = \pm v^*$. Lastly, P_n denote the projection $\Theta \to \Theta_n$. The following high-level assumptions of Shen (1997) guarantees asymptotic normality of $\sqrt{N}\left\langle v^*, \hat{\theta} - \theta_0 \right\rangle$, or equivalently of $\sqrt{N}(\alpha - \alpha_0)$.

Assumption 5 (Regularity)**.** (i) $\sup_{\{\theta \in \Theta_n \mid \|\theta - \theta_0\| \leqslant \delta_0\}} n^{-1/2} \nu_n(r(Y_3, Y_2, Y_1, Z, D_2, D_1; \theta - \theta_0) - r(Y_3, Y_2, Y_1, Z, D_2, D_1; P_n(\theta^*(\theta, \epsilon_n)) - \theta_0)) = O_p(\epsilon_n^2)$. (ii) $\sup_{\{\theta \in \Theta_n \mid 0 < \|\theta - \theta_0\| \leqslant \delta_n\}} \left[K(\theta_0, P_n(\theta^*(\theta, \epsilon_n)) - K(\theta_0, \theta) \right] - \frac{1}{2}\left[\|\theta^*(\theta, \epsilon_n) - \theta_0\|^2 - \|\theta - \theta_0\|^2 \right] = O(\epsilon_n^2)$. (iii) $\sup_{\{\theta \in \Theta_n \mid 0 < \|\theta - \theta_0\| \leqslant \delta_n\}} \|\theta^*(\theta, \epsilon_n) - P_n(\theta^*(\theta, \epsilon_n))\| = O(\delta_n^{-1} \epsilon_n^2)$. (iv) $\sup_{\{\theta \in \Theta_n \mid \|\theta - \theta_0\| \leqslant \delta_n\}} n^{-1/2} \nu_n(l'_{\theta_0}(\cdots; \theta^*(\theta, \epsilon_n) - P_n(\theta^*(\theta, \epsilon_n)))) = O_p(\epsilon_n^2$. (v) $\sup_{\{\theta \in \Theta_n \mid \|\theta - \theta_0\| \leqslant \delta_n\}} n^{-1/2} \nu_n(l'_{\theta_0}(\cdots; \theta - \theta_0)) = O_p(\epsilon_n)$.

Proposition 2 (Asymptotic Distribution of a Semiparametric Estimator)**.** *Suppose that Restrictions 1, 2, 3, and 4 and Assumptions 4 and 5 hold. Then,* $\sqrt{N}(\alpha - \alpha_0) \xrightarrow{d} N(0, \sigma^{-1})$.

C Special Cases and Generalizations of the Baseline Model

C.1 A Variety of Missing Observations

While the baseline model considered in the paper induces a permanent dropout from data by a hazard selection, variants of the model can be encompassed as special cases under which the main identification remains to hold. Specifically, we consider the following Classes 1 and 2 as special models of Class 3.

Class 1 (Nonseparable Dynamic Panel Data Model).

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T \qquad \text{(State Dynamics)} \\ F_{Y_1 U} & \text{(Initial joint distribution of } (Y_1, U)) \\ Z = \zeta(U, W) & \text{(Optional: nonclassical proxy of } U) \end{cases}$$

□

Class 2 (Nonseparable Dynamic Panel Data Model with Missing Observations).

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T \qquad \text{(State Dynamics)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T-1 \qquad \text{(Selection)} \\ F_{Y_1 U} & \text{(Initial joint distribution of } (Y_1, U)) \\ Z = \zeta(U, W) & \text{(Optional: nonclassical proxy of } U) \end{cases}$$

where Y_t is censored by the binary indicator D_t of sample selection as follows:

$$\begin{cases} Y_t \text{ is observed} & \text{if } D_{t-1} = 1 \text{ or } t = 1. \\ Y_t \text{ is unobserved} & \text{if } D_{t-1} = 0 \text{ and } t > 1. \end{cases}$$

□

A representative example of this instantaneous selection is the Roy model such as $h(y, u, v) = \mathbb{1}\{\mathrm{E}[\pi(g(y, u, \mathcal{E}_{t+1}), u)] \geqslant c(u, v)\}$ where π measures payoffs and c measures costs. The following is the baseline model considered in the paper.

Class 3 (Nonseparable Dynamic Panel Data Model with Hazards).

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T \qquad \text{(State Dynamics)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T-1 \qquad \text{(Hazard Model)} \\ F_{Y_1 U} & \text{(Initial joint distribution of } (Y_1, U)) \\ Z = \zeta(U, W) & \text{(Optional: nonclassical proxy of } U) \end{cases}$$

where $D_t = 0$ induces a hazard of permanent dropout in the following manner:

$$\begin{cases} Y_1 \text{ is observed}, & \\ Y_2 \text{ is observed} & \text{if } D_1 = 1, \\ Y_3 \text{ is observed} & \text{if } D_1 = D_2 = 1. \end{cases}$$

□

The present appendix section proves that identification of Class 3 implies identification of Classes 1 and 2. The observable parts of the joint distributions in each of the three classes include (but are not limited to) the following:

Class 1:	Observe	$F_{Y_3 Y_2 Y_1 Z D_2 D_1}$, $F_{Y_2 Y_1 Z D_1}$, $F_{Y_3 Y_1 Z D_2}$, and $F_{Y_3 Y_2 Z D_2}$
Class 2:	Observe	$F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1)$, $F_{Y_2 Y_1 Z D_1}(\cdot, \cdot, \cdot, 1)$, and $F_{Y_3 Y_1 Z D_2}(\cdot, \cdot, \cdot, 1)$
Class 3:	Observe	$F_{Y_3 Y_2 Y_1 Z D_2 D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1)$ and $F_{Y_2 Y_1 Z D_1}(\cdot, \cdot, \cdot, 1)$

Since the selection variable D_t in Class 1 is not defined, we assume without loss of generality that it is degenerate at $D_t = 1$ in Class 1.

The problem of identification under each class can be characterized by the well-definition of the following maps:

$$\begin{aligned}
&\text{Class 1:} \quad (F_{Y_3Y_2Y_1ZD_2D_1}, F_{Y_2Y_1ZD_1}, F_{Y_3Y_1ZD_2}, F_{Y_3Y_2ZD_2}) \overset{\iota_1}{\mapsto} (g, F_{Y_1U}, \zeta) \\
&\text{Class 2:} \quad (F_{Y_3Y_2Y_1ZD_2D_1}(\cdot,\cdot,\cdot,\cdot,1,1), F_{Y_2Y_1ZD_1}(\cdot,\cdot,\cdot,1), F_{Y_3Y_1ZD_2}(\cdot,\cdot,\cdot,1)) \overset{\iota_2}{\mapsto} (g, h, F_{Y_1U}, \zeta) \\
&\text{Class 3:} \quad (F_{Y_3Y_2Y_1ZD_2D_1}(\cdot,\cdot,\cdot,\cdot,1,1), F_{Y_2Y_1ZD_1}(\cdot,\cdot,\cdot,1) \overset{\iota_3}{\mapsto} (g, h, F_{Y_1U}, \zeta)
\end{aligned}$$

The main identification result of this paper was to show well-definition of the map ι_3. Therefore, in order to argue that identification of Class 3 implies identification of Classes 1 and 2, it suffices to claim that the well-definition of ι_3 implies well-definition of the maps ι_1 and ι_2.

First, note that the trivial projections

$$\begin{aligned}
&(F_{Y_3Y_2Y_1ZD_2D_1}, F_{Y_2Y_1ZD_1}, F_{Y_3Y_1ZD_2}, F_{Y_3Y_2ZD_2}) \overset{\pi_1}{\mapsto} (F_{Y_3Y_2Y_1ZD_2D_1}, F_{Y_2Y_1ZD_1}, F_{Y_3Y_1ZD_2}) \quad \text{and} \\
&(F_{Y_3Y_2Y_1ZD_2D_1}(\cdot,\cdot,\cdot,\cdot,1,1), F_{Y_2Y_1ZD_1}(\cdot,\cdot,\cdot,1), F_{Y_3Y_1ZD_2}(\cdot,\cdot,\cdot,1)) \\
&\qquad \overset{\pi_2}{\mapsto} (F_{Y_3Y_2Y_1ZD_2D_1}(\cdot,\cdot,\cdot,\cdot,1,1), F_{Y_2Y_1ZD_1}(\cdot,\cdot,\cdot,1))
\end{aligned}$$

are well-defined. Second, by the construction of degenerate random variable $D_t = 1$ in Class 1, the map

$$\begin{aligned}
&(F_{Y_3Y_2Y_1ZD_2D_1}, F_{Y_2Y_1ZD_1}, F_{Y_3Y_1ZD_2}, F_{Y_3Y_2ZD_2}) \\
&\qquad \overset{\kappa_1}{\mapsto} (F_{Y_3Y_2Y_1ZD_2D_1}(\cdot,\cdot,\cdot,\cdot,1,1), F_{Y_2Y_1ZD_1}(\cdot,\cdot,\cdot,1), F_{Y_3Y_1ZD_2}(\cdot,\cdot,\cdot,1))
\end{aligned}$$

is well-defined in Class 1. Third, the trivial projection

$$(g, h, F_{Y_1U}, \zeta) \overset{\rho}{\mapsto} (g, F_{Y_1U}, \zeta)$$

is well-defined.

Now, notice that

$$\begin{aligned}
\iota_1 &= \rho \circ \iota_3 \circ \kappa_1 \circ \pi_1 && \text{in Class 1, and} \\
\iota_2 &= \iota_3 \circ \pi_2 && \text{in Class 2.}
\end{aligned}$$

Therefore, the well-definition of ι_3 implies well-definition of ι_1 and ι_2 in particular. Therefore, identification of Class 3 implies identification of Classes 1 and 2.

C.2 Identification without a Nonclassical Proxy Variable

The main result of this paper assumed use of a nonclassical proxy variable Z. However, this use was mentioned to be optional, and one can substitute a slightly longer panel $T = 6$ for use of a proxy variable. In this section we show how the model (g, h, F_{Y_1U}) can be identified from the endogenously censored joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,1,1,1,1,1)$ that follows from $T = 6$ time periods of unbalanced panel data without additional information Z.

Restriction 5 (Independence)**.**
(i) Exogeneity of $\mathcal{E}_t$: $\quad \mathcal{E}_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s<t}, \{V_s\}_{s<t}, W)$ for all $t \geq 2$.
(ii) Exogeneity of V_t: $\quad V_t \perp\!\!\!\perp (U, Y_1, \{\mathcal{E}_s\}_{s\leqslant t}, \{V_s\}_{s<t})$ for all $t \geq 1$.

For simplicity of notation, we compress the nondegenerate random variable Y_3 into a binary random variable $Z := \eta(Y_3)$ with a known transformation η such that part (iii) of the following rank condition is satisfied. As the notation suggests, this Z serves as a substitute for a nonclassical proxy variable.

Restriction 6 (Rank Conditions)**.** The following conditions hold for every $y \in \mathcal{Y}$:
(i) Heterogeneous Dynamics: the integral operator $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ defined by $P_y\xi(y') = \int f_{Y_3|Y_2U}(y' \mid y, u) \cdot \xi(u)du$ is bounded and invertible.
(ii) There exist y_4 and y_2 satisfying the following conditions:
Nondegeneracy: $f_{Y_4ZD_5D_4D_3|Y_2U}(y_4, 1, 1, 1, 1 \mid y_2, u)$ is bounded away from 0 and 1 for all u.
Relevance: $\frac{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1|y_2,u)}{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1|y_2,u)} \neq \frac{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1|y_2,u')}{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1|y_2,u')}$ whenever $u \neq u'$.
(iii) No Extinction: $f_{D_2|Y_2U}(1 \mid y, u) > 0$ for all $u \in \mathcal{U}$.
(iv) Initial Heterogeneity: the integral operator $S_y : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$ defined by $S_y\xi(u) = \int f_{Y_2Y_1UD_1U}(y, y', u, 1) \cdot \xi(y')dy'$ is bounded and invertible.

Lemma 5 (Independence)**.** *The following implications hold:*
(i) Restriction 5 (i) $\Rightarrow \mathcal{E}_6 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_1, V_2, V_3, V_4, V_5)$
$\Rightarrow Y_6 \perp\!\!\!\perp (Y_1, Y_2, Y_3, Y_4, D_1, D_2, D_3, D_4, D_5) \mid (Y_5, U)$.
(ii) Restriction 5 (i) & (ii) $\Rightarrow (\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_3, V_4, V_5) \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2)$
$\Rightarrow (Y_3, Y_4, Y_5, D_3, D_4, D_5) \perp\!\!\!\perp (Y_1, D_1, D_2) \mid (Y_2, U)$.
(iii) Restriction 5 (i) $\Rightarrow \mathcal{E}_2 \perp\!\!\!\perp (U, Y_1, V_1) \Rightarrow Y_2 \perp\!\!\!\perp D_1 \mid (Y_1, U)$.
(iv) Restriction 5 (ii) $\Rightarrow V_2 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1) \Rightarrow D_2 \perp\!\!\!\perp (Y_1, D_1) \mid (Y_2, U)$.

Proof. In order to prove the lemma, we use the following two properties of conditional independence:
CI.1. $A \perp\!\!\!\perp B$ implies $A \perp\!\!\!\perp B \mid \phi(B)$ for any Borel function ϕ.
CI.2. $A \perp\!\!\!\perp B \mid C$ implies $A \perp\!\!\!\perp \phi(B, C) \mid C$ for any Borel function ϕ.

(i) First, applying CI.1 to the independence $\mathcal{E}_6 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_1, V_2, V_3, V_4, V_5)$ and using the definition of g yield

$$\mathcal{E}_6 \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_1, V_2, V_3, V_4, V_5) \mid (Y_5, U).$$

Next, applying CI.2 to this conditional independence and using the definitions of g and h yield

$$\mathcal{E}_6 \perp\!\!\!\perp (Y_1, Y_2, Y_3, Y_4, D_1, D_2, D_3, D_4, D_5, Z) \mid (Y_5, U).$$

Applying CI.2 again to this conditional independence and using the definition of g yield

$$Y_6 \perp\!\!\!\perp (Y_1, Y_2, Y_3, Y_4, D_1, D_2, D_3, D_4, D_5, Z) \mid (Y_5, U).$$

(ii) First, applying CI.1 to the independence $(\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_3, V_4, V_5) \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2)$ and using the definition of g yield

$$(\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_3, V_4, V_5) \perp\!\!\!\perp (U, Y_1, \mathcal{E}_2, V_1, V_2) \mid (Y_2, U)$$

Next, applying CI.2 to this conditional independence and using the definitions of g and h yield

$$(\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, V_3, V_4, V_5) \perp\!\!\!\perp (Y_1, D_1, D_2) \mid (Y_2, U)$$

Applying CI.2 again to this conditional independence and using the definition of g yield

$$(Y_3, Y_4, Y_5, D_3, D_4, D_5) \perp\!\!\!\perp (Y_1, D_1, D_2) \mid (Y_2, U)$$

(iii) The proof is the same as that of Lemma 3 (ii).
(iv) The proof is the same as that of Lemma 3 (iii). □

Lemma 6 (Invariant Transition)**.**
(i) Under Restrictions 1 and 5 (i), $F_{Y_t|Y_{t-1}U}(y' \mid y, u) = F_{Y_{t'}|Y_{t'-1}U}(y' \mid y, u)$ *for all* y', y, u, t, t'.
(ii) Under Restrictions 1 and 5 (ii), $F_{D_2|Y_2U}(d \mid y, u) = F_{D_1|Y_1U}(d \mid y, u)$ *for all* d, y, u.

This lemma can be proved similarly to Lemma 4.

Lemma 7 (Identification)**.** *Under Restrictions 1, 4, 5, and 6,* $(F_{Y_3|Y_2U}, F_{D_2|Y_2U}, F_{Y_1U})$ *is uniquely determined by* $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1, 1, 1, 1, 1)$ *and* $F_{Y_2Y_1D_1}(\,\cdot\,,\ \cdot\,, 1)$.

Proof. Given fixed (y_5, y_4, z, y_2), define the operators $L_{y_5,y_4,z,y_2} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$, $P_{y_5} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$, $Q_{y_5,y_4,z,y_2} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $R_{y_2} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $S_{y_2} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, $T_{y_2} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, and $T'_{y_2} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$ by

$$
\begin{aligned}
(L_{y_5,y_4,z,y_2}\xi)(y_6) &= \int f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(y_6, y_5, y_4, z, y_2, y_1, 1, 1, 1, 1, 1) \cdot \xi(y_1) dy_1, \\
(P_{y_5}\xi)(y_3) &= \int f_{Y_6|Y_5U}(y_6 \mid y_5, u) \cdot \xi(u) du, \\
(Q_{y_5,y_4,z,y_2}\xi)(u) &= f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5, y_4, z, 1, 1, 1 \mid y_2, u) \cdot \xi(u), \\
(R_{y_2}\xi)(u) &= f_{D_2|Y_2U}(1 \mid y_2, u) \cdot \xi(u), \\
(S_{y_2}\xi)(u) &= \int f_{Y_2Y_1UD_1}(y_2, y_1, u, 1) \cdot \xi(y_1) dy_1, \\
(T_{y_2}\xi)(u) &= \int f_{Y_1|Y_2UD_2D_1}(y_1 \mid y_2, u, 1, 1) \cdot \xi(y_1) dy_1, \\
(T'_{y_2}\xi)(u) &= f_{Y_2UD_2D_1}(y_2, u, 1, 1) \cdot \xi(u)
\end{aligned}
$$

respectively.
Step 1: Uniqueness of $F_{Y_6|Y_5U}$ and $F_{Y_5Y_4ZD_5D_4D_3|Y_2U}(\,\cdot,\ \cdot,\ \cdot, 1, 1, 1 \mid\ \cdot,\ \cdot)$
The kernel $f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(\,\cdot, y_5, y_4, z, y_2,\ \cdot, 1, 1, 1, 1, 1)$ of the integral operator L_{y_5,y_4,z,y_2} can be rewritten as

$$
\begin{aligned}
& f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(y_6, y_5, y_4, z, y_2, y_1, 1, 1, 1, 1, 1) \\
= & \int f_{Y_6|Y_5Y_4ZY_2Y_1UD_5D_4D_3D_2D_1}(y_6 \mid y_5 y_4, z, y_2, y_1, u, 1, 1, 1, 1, 1) \\
& \quad \times f_{Y_5Y_4ZD_5D_4D_3|Y_2Y_1UD_2D_1}(y_5, y_4, z, 1, 1, 1 \mid y_2, y_1, u, 1, 1) \\
& \quad \times f_{D_2|Y_2Y_1UD_1}(1 \mid y_2, y_1, u, 1) \cdot f_{Y_2Y_1UD_1}(y_2, y_1, u, 1)\ du
\end{aligned}
\tag{C.1}
$$

But by Lemma 5 (i), (ii), and (iv), respectively, Restriction 5 implies that

$$
\begin{aligned}
f_{Y_6|Y_5Y_4ZY_2Y_1UD_5D_4D_3D_2D_1}(y_6 \mid y_5, y_4, z, y_2, y_1, u, 1, 1, 1, 1, 1) &= f_{Y_6|Y_5U}(y_5 \mid y_5, u), \\
f_{Y_5Y_4ZD_5D_4D_3|Y_2Y_1UD_2D_1}(y_5, y_4, z, 1, 1, 1 \mid y_2, y_1, u, 1, 1) &= f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5, y_4, z, 1, 1, 1 \mid y_2, u), \\
f_{D_2|Y_2Y_1UD_1}(1 \mid y_2, y_1, u, 1) &= f_{D_2|Y_2U}(1 \mid y_2, u).
\end{aligned}
$$

Equation (C.1) thus can be rewritten as

$$
\begin{aligned}
& f_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(y_6, y_5, y_4, z, y_2, y_1, 1, 1, 1, 1, 1) \\
= & \int f_{Y_6|Y_5U}(y_6 \mid y_5, u) \cdot f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5, y_4, z, 1, 1, 1 \mid y_2, u) \\
& \quad \times f_{D_2|Y_2U}(1 \mid y_2, u) \cdot f_{Y_2Y_1UD_1}(y_2, y_1, u, 1)\ du
\end{aligned}
$$

But this implies that the integral operator L_{y_4,y_3,y_2} is written as the operator composition

$$
L_{y_5,y_4,z,y_2} = P_{y_5} Q_{y_5,y_4,z,y_2} R_{y_2} S_{y_2,y_1}.
$$

Restriction 6 (i), (ii), (iii), and (iv) imply that the operators P_{y_5}, Q_{y_5,y_4,z,y_2}, R_{y_2}, and S_{y_2,y_1} are invertible, respectively. Hence so is L_{y_5,y_4,z,y_2}. Using the two values $\{0,1\}$ of Z, form the product

$$L_{y_5,y_4,1,y_2}L^{-1}_{y_5,y_4,0,y_2} = P_{y_5}Q_{y_4,1/0,y_2}P^{-1}_{y_4}$$

where $Q_{y_4,1/0,y_2} = Q_{y_5,y_4,1,y_2}Q^{-1}_{y_5,y_4,0,y_2}$ is the multiplication operator with proxy odds defined by

$$\begin{aligned}
(Q_{y_4,1/0,y_2}\xi)(u) &= \frac{f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5,y_4,1,1,1,1 \mid y_2,u)}{f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5,y_4,0,1,1,1 \mid y_2,u)}\xi(u)\\
&= \frac{f_{Y_5|Y_4U}(y_5 \mid y_4,u)\cdot f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1 \mid y_2,u)}{f_{Y_5|Y_4U}(y_5 \mid y_4,u)\cdot f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1 \mid y_2,u)}\xi(u)\\
&= \frac{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1 \mid y_2,u)}{f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1 \mid y_2,u)}\xi(u).
\end{aligned}$$

Note the invariance of this operator in y_5, hence the notation. By Restriction 6 (ii), the operator $L_{y_5,y_4,1,y_2}L^{-1}_{y_5,y_4,0,y_2}$ is bounded. The expression $L_{y_5,y_4,1,y_2}L^{-1}_{y_5,y_4,0,y_2} = P_{y_4}Q_{y_4,1/0,y_2}P^{-1}_{y_4}$ thus allows unique eigenvalue-eigenfunction decomposition as in the proof of Lemma 2.

The distinct proxy odds as in Restriction 6 (ii) guarantee distinct eigenvalues and single dimensionality of the eigenspace associated with each eigenvalue. Within each of the single-dimensional eigenspace is a unique eigenfunction pinned down by $\mathcal{L}^1$-normalization because of the unity of integrated densities. The eigenvalues $\lambda(u)$ yield the multiplier of the operator $Q_{y_4,1/0,y_2}$, hence $\lambda_{y_4,y_2}(u) = f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1 \mid y_2,u)/f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1 \mid y_2,u)$. This proxy odds in turn identifies $f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,\ \cdot,1,1,1 \mid y_2,u)$ since Z is binary. The corresponding normalized eigenfunctions are the kernels of the integral operator P_{y_5}, hence $f_{Y_6|Y_5U}(\ \cdot\ \mid y_5,u)$. Lastly, Restriction 4 facilitates unique ordering of the eigenfunctions $f_{Y_6|Y_5U}(\ \cdot\ \mid y_5,u)$ by the distinct concrete values of $u = \lambda_{y_4,y_2}(u)$. This is feasible because the eigenvalues $\lambda_{y_4,y_2}(u) = f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,1,1,1,1 \mid y_2,u)/f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,0,1,1,1 \mid y_2,u)$ are invariant from y_5. That is, eigenfunctions $f_{Y_6|Y_5U}(\ \cdot\ \mid y_5,u)$ of the operator $L_{y_5,y_4,1,y_2}L^{-1}_{y_5,y_4,0,y_2}$ across different y_5 can be uniquely ordered in u invariantly from y_5 by the common set of ordered distinct eigenvalues $u = \lambda_{y_4,y_2}(u)$.

Therefore, $F_{Y_6|Y_5U}$ and $F_{Y_4ZD_5D_4D_3|Y_2U}(y_4,\ \cdot,1,1,1 \mid y_2,u)$ are uniquely determined by the joint distribution $F_{Y_6Y_5Y_4ZY_2Y_1D_5D_4D_3D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,1,1,1,1,1)$, which in turn is uniquely determined by the observed joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,1,1,1,1,1)$. The multiplier of the operator Q_{y_5,y_4,z,y_2} is of the form

$$\begin{aligned}
f_{Y_5Y_4ZD_5D_4D_3|Y_2U}(y_5,y_4,z,1,1,1 \mid y_2,u) &= f_{Y_5|Y_4U}(y_5 \mid y_4,u)\cdot f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,z,1,1,1 \mid y_2,u)\\
&= f_{Y_6|Y_5U}(y_5 \mid y_4,u)\cdot f_{Y_4ZD_5D_4D_3|Y_2U}(y_4,z,1,1,1 \mid y_2,u)
\end{aligned}$$

by Lemma 6 (i), where the right-hand side object has been identified. Consequently, the operators P_{y_5} and Q_{y_5,y_4,z,y_2} are uniquely determined for each combination of y_5, y_4, z, y_2.

Step 2: Uniqueness of $F_{Y_2Y_1UD_1}(\ \cdot\ ,\ \cdot\ ,\ \cdot\ ,1)$
By Lemma 5 (iii), Restriction 5 implies $f_{Y_2|Y_1ZUD_1}(y' \mid y,z,u,1) = f_{Y_2|Y_1U}(y' \mid y,u)$. Using this equality, write the density of the observed joint distribution $F_{Y_2Y_1ZD_1}(\ \cdot\ ,\ \cdot\ ,\ \cdot\ ,1)$ as

$$\begin{aligned}
f_{Y_2Y_1D_1}(y_2,y_1,1) &= \int f_{Y_2|Y_1UD_1}(y_2 \mid y_1,u,1)f_{Y_1UD_1}(y_1,u,1)du\\
&= \int f_{Y_2|Y_1U}(y_2 \mid y_1,u)f_{Y_1UD_1}(y_1,u,1)du \qquad \text{(C.2)}
\end{aligned}$$

By Lemma 4 (i), $F_{Y_6|Y_5U}(y' \mid y,u) = F_{Y_2|Y_1U}(y' \mid y,u)$ for all y', y, u. Therefore, we can write the operator P_y as

$$(P_{y_1}\xi)(y_2) \;=\; \int f_{Y_6|Y_5U}(y_2 \mid y_1, u)\cdot \xi(u)du \;=\; \int f_{Y_2|Y_1U}(y_2 \mid y_1, u)\cdot \xi(u)du.$$

With this operator notation, it follows from (C.2) that

$$f_{Y_2Y_1D_1}(\,\cdot\,, y_1, 1) = P_{y_1} f_{Y_1UD_1}(y_1,\,\cdot\,, 1).$$

By Restriction 6 (i) and (ii), this operator equation can be solved for $f_{Y_1UD_1}(y,\,\cdot\,,1)$ as

$$f_{Y_1UD_1}(y_1,\,\cdot\,,1) = P_{y_1}^{-1} f_{Y_2Y_1D_1}(\,\cdot\,, y_1, 1) \tag{C.3}$$

Recall that P_y was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$. The function $f_{Y_2Y_1D_1}(\,\cdot\,, y, 1)$ is also uniquely determined by the observed joint distribution $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,,1)$. Therefore, (C.2) shows that $f_{Y_1UD_1}(\,\cdot\,,\,\cdot\,,1)$ is uniquely determined by the pair of the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,,1)$.

Using the solution to the above inverse problem, we can write the kernel of the operator S_{y_2} as

$$\begin{aligned}
f_{Y_2Y_1UD_1}(y_2, y_1, u, 1) &= f_{Y_2|Y_1UD_1}(y_2 \mid y_1, u, 1)\cdot f_{Y_1UD_1}(y_1, u, 1)\\
&= f_{Y_2|Y_1U}(y_2 \mid y_1, u)\cdot f_{Y_1UD_1}(y_1, u, 1)\\
&= f_{Y_6|Y_5U}(y_2 \mid y_1, u)\cdot f_{Y_1UD_1}(y_1, u, 1)\\
&= f_{Y_6|Y_5U}(y_2 \mid y_1, u)\cdot [P_{y_1}^{-1} f_{Y_2Y_1D_1}(\,\cdot\,, y_1, 1)](u)
\end{aligned}$$

where the second equality follows from Lemma 5 (iii), the third equality follows from Lemma 4 (i), and the forth equality follows from (C.3). Since $f_{Y_6|Y_5U}$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$ and $[P_{y_1}^{-1} f_{Y_2Y_1D_1}(\,\cdot\,, y_1, 1)]$ was shown in the previous paragraph to be uniquely determined for each y_1 by the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,,1)$, it follows that $f_{Y_2Y_1UD_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1)$ too is uniquely determined by the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\,\cdot\,,1)$. Equivalently, the operator S_{y_2} is uniquely determined for each y_2.

Step 3: Uniqueness of $F_{Y_1|Y_2UD_2D_1}(\,\cdot \mid \cdot\,,\,\cdot\,,1,1)$
This step is the same as Step 3 in the proof of Lemma 2, except that $L_{y,z}$ and $Q_{1/0}$ are replaced by L_{y_5,y_4,z,y_2} and $Q_{y_4,1/0,y_2}$, respectively, which were defined in Step 1 of this proof. $F_{Y_1|Y_2UD_2D_1}(\,\cdot \mid \cdot\,,\,\cdot\,,1,1)$ or the operator T_y is uniquely determined by the observed joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$.

Step 4: Uniqueness of $F_{Y_2UD_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$
This step is the same as Step 4 in the proof of Lemma 2. $F_{Y_2UD_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$ or the auxiliary operator T'_y is uniquely determined by the pair of the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot,\,\cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$.

Step 5: Uniqueness of $F_{D_2|Y_2U}(1 \mid \cdot\,,\,\cdot\,)$
This step is the same as Step 5 in the proof of Lemma 2. $F_{D_2|Y_2U}(1 \mid \cdot\,,\,\cdot\,)$ or the auxiliary operator T'_y is uniquely determined by the pair of the observed joint distributions

$F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\ \cdot\,, 1,1)$.

Step 6: Uniqueness of F_{Y_1U}

Recall from Step 2 that $f_{Y_2Y_1UD_1}(\,\cdot\,,\ \cdot\,,\ \cdot\,, 1)$ is uniquely determined by the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\ \cdot\,, 1)$. We can write

$$\begin{aligned} f_{Y_2Y_1UD_1}(y_2, y_1, u, 1) &= f_{Y_2|Y_1UD_1}(y_2 \mid y_1, u, 1) f_{D_1|Y_1U}(1 \mid y_1, u) f_{Y_1U}(y_1, u) \\ &= f_{Y_2|Y_1U}(y_2 \mid y_1, u) f_{D_1|Y_1U}(1 \mid y_1, u) f_{Y_1U}(y_1, u) \\ &= f_{Y_6|Y_5U}(y_2 \mid y_1, u) f_{D_2|Y_2U}(1 \mid y_1, u) f_{Y_1U}(y_1, u), \end{aligned}$$

where the second equality follows from Lemma 5 (iii), and the third equality follows from Lemma 4 (i) and (ii). For a given (y_1, u), there must exist some y_2 such that $f_{Y_6|Y_5U}(y_2 \mid y_1, u) > 0$ by a property of conditional density functions. Moreover, Restriction 6 (iii) requires that $f_{D_2|Y_2U}(1 \mid y_1, u) > 0$ for a given y_1 for all u. Therefore, for such a choice of y_2, we can write

$$f_{Y_1U}(y_1, u) = \frac{f_{Y_2Y_1UD_1}(y_2, y_1, u, 1)}{f_{Y_6|Y_5U}(y_2 \mid y_1, u) f_{D_2|Y_2U}(1 \mid y_1, u)}$$

Recall that $f_{Y_6|Y_5U}(\,\cdot \mid \cdot\,,\ \cdot\,)$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$, $f_{Y_2Y_1UD_1}(\,\cdot\,,\ \cdot\,,\ \cdot\,, 1)$ was shown in Step 2 to be uniquely determined by the pair of the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\ \cdot\,, 1)$, and $f_{D_2|Y_2U}(1 \mid \cdot\,,\ \cdot\,)$ was shown in Step 5 to be uniquely determined by the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\ \cdot\,, 1)$. Therefore, it follows that the initial joint density f_{Y_1U} is uniquely determined by the observed joint distributions $F_{Y_6Y_5Y_4Y_3Y_2Y_1D_5D_4D_3D_2D_1}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot, 1,1,1,1,1)$ and $F_{Y_2Y_1D_1}(\,\cdot\,,\ \cdot\,, 1)$. □

We next discuss an identification-preserving criterion analogously to Corollary 1. Let $\mathcal{F}$ denote the set of all the admissible model representations

$$\mathcal{F} = \{(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}) \mid (g, h, F_{Y_1U}, \zeta) \text{ satisfies Restrictions 1, 4, 5, and 6}\}.$$

A natural consequence of the main identification result of Lemma 7 is that the true model $(F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$ is the unique maximizer of the following criterion.

Corollary 2 (Constrained Maximum Likelihood)**.** *If the true model $(F^*_{Y_t|Y_{t-1}U}, F^*_{D_t|Y_tU}, F^*_{Y_1U}, F^*_{Z|U})$ is an element of $\mathcal{F}$, then it is the unique solution to*

$$\max_{\left(F_{Y_t|Y_{t-1}U}, F_{D_t|Y_tU}, F_{Y_1U}, F_{Z|U}\right) \in \mathcal{F}} \quad c_1 E\left[\log \int f_{Y_t|Y_{t-1}U}(Y_2 \mid Y_1, u) f_{D_t|Y_tU}(1 \mid Y_1, u) f_{Y_1U}(Y_1, u) d\mu(u) \middle| D_1 = 1\right] +$$
$$c_2 E\left[\log \int \prod_{s=1}^{5} f_{Y_t|Y_{t-1}U}(Y_{s+1} \mid Y_s, u) f_{D_t|Y_tU}(1 \mid Y_s, u) f_{Y_1U}(Y_1, u) d\mu(u) \middle| D_5 = \cdots = D_1 = 1\right]$$

for any $c_1, c_2 > 0$ subject to

$$\int f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_1, u) = f_{D_1}(1) \qquad \textit{and}$$

$$\int \prod_{s=2}^{5} f_{Y_t|Y_{t-1}U}(y_s \mid y_{s-1}, u) f_{D_t|Y_tU}(1 \mid y_s, u) f_{D_t|Y_tU}(1 \mid y_1, u) f_{Y_1U}(y_1, u) d\mu(y_2, y_1, u)$$
$$= f_{D_5D_4D_3D_2D_1}(1,1,1,1,1).$$

C.3 Models with Higher-Order Lags

The model discussed in this paper can be extended to the following model

$$\begin{cases} Y_t = g(Y_{t-1}, \cdots, Y_{t-\tau}, U, \mathcal{E}_t) & \text{for } t = \tau+1, \cdots, T \\ D_t = h(Y_t, \cdots, Y_{t-\tau+1}, U, V_t) & \text{for } t = \tau, \cdots, T-1 \\ F_{Y_\tau \cdots Y_1 U D_{\tau-1} \cdots D_1}(\cdots, \ \cdot\ , (1)) & \\ Z = \zeta(U, W) & \end{cases}$$

where g is a τ-th order Markov process with heterogeneity U, and the attrition model depends on the past as well as the current state. In this set up, we can observe the parts, $F_{Y_{\tau+2} \cdots Y_1 Z D_{\tau+1} \cdots D_1}(\cdots, \ \cdot\ , (1))$ and $f_{Y_{\tau+1} \cdots Y_1 Z D_\tau \cdots D_1}(\cdots, \ \cdot\ , (1))$, of the joint distributions if $T = \tau+2$. I claim that $T = \tau+2$ suffices for identification. In other words, it can be shown that $(g, h, F_{Y_\tau \cdots Y_1 U D_{\tau-1} \cdots D_1}(\cdots, \ \cdot\ , (1)), \zeta)$ is uniquely determined by $F_{Y_{\tau+2} \cdots Y_1 Z D_{\tau+1} \cdots D_1}(\cdots, \ \cdot\ , (1))$ and $f_{Y_{\tau+1} \cdots Y_1 Z D_\tau \cdots D_1}(\cdots, \ \cdot\ , (1))$ up to equivalence classes. To this end, we replace Restrictions 2 and 3 by the following restrictions.

Restriction 7 (Independence)**.**
(i) Exogeneity of $\mathcal{E}_t$: $\quad \mathcal{E}_t \perp\!\!\!\perp (U, \{Y_s\}_{s=1}^{\tau}, \{D_s\}_{s=1}^{\tau}, \{\mathcal{E}_s\}_{s<t}, \{V_s\}_{s<t}, W)$ for all $t \geq \tau+1$.
(ii) Exogeneity of V_t: $\quad V_t \perp\!\!\!\perp (U, \{Y_s\}_{s=1}^{\tau-1}, \{D_s\}_{s=1}^{\tau-1}, \{\mathcal{E}_s\}_{s\leqslant t}, \{V_s\}_{s<t})$ for all $t \geq \tau$.
(iii) Exogeneity of W: $\quad W \perp\!\!\!\perp (\{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \{\mathcal{E}_t\}_t, \{V_t\}_t)$.

Restriction 8 (Rank Conditions)**.** The following conditions hold for every $(y) \in \mathcal{Y}^\tau$:
(i) Heterogeneous Dynamics: the integral operator $P_{(y)} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ defined by $P_{(y)}\xi(y') = \int f_{Y_{\tau+2}|Y_{\tau+1} \cdots Y_2 U}(y' \mid (y), u) \cdot \xi(u) du$ is bounded and invertible.
(ii) Nondegenerate Proxy Model: $f_{Z|U}(1 \mid u)$ is bounded away from 0 and 1 for all u.
Relevant Proxy: $f_{Z|U}(1 \mid u) \neq f_{Z|U}(1 \mid u')$ whenever $u \neq u'$.
(iii) No Extinction: $f_{D_{\tau+1}|Y_{\tau+1} \cdots Y_2 U}(1 \mid (y), u) > 0$ for all $u \in \mathcal{U}$.
(iv) Initial Heterogeneity: the integral operator $S_{(y)} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$ defined by $S_{(y)}\xi(u) = \int f_{Y_{\tau+1} \cdots Y_2 Y_1 U D_\tau \cdots D_1}((y), y', u, (1)) \cdot \xi(y') dy'$ is bounded and invertible.

Lemma 8 (Independence)**.** *The following implications hold:*
(i) Restriction 7 (i) $\Rightarrow Y_{\tau+2} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau+1}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$.
(ii) Restriction 7 (i) $\Rightarrow Y_{\tau+1} \perp\!\!\!\perp (\{D_t\}_{t=1}^{\tau}, Z) \mid (\{Y_t\}_{t=1}^{\tau}, U)$.
(iii) Restriction 7 (ii) $\Rightarrow D_{\tau+1} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau}) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$.
(iv) Restriction 7 (iii) $\Rightarrow Z \perp\!\!\!\perp (\{Y_t\}_{t=1}^{\tau+1}, \{D_t\}_{t=1}^{\tau+1}) \mid U$.

Proof. As in the proof of Lemma 3, we use the following two properties of conditional independence:
CI.1. $A \perp\!\!\!\perp B$ implies $A \perp\!\!\!\perp B \mid \phi(B)$ for any Borel function ϕ.
CI.2. $A \perp\!\!\!\perp B \mid C$ implies $A \perp\!\!\!\perp \phi(B, C) \mid C$ for any Borel function ϕ.

(i) First, note that Restriction 2 (i) $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}, W)$ together with the structural definition $Z = \zeta(U, W)$ implies $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}, V_\tau, Z)$. Applying CI.1 to this independence relation $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}, Z)$ yields

$$\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}, Z) \mid (g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), \{Y_t\}_{t=2}^{\tau}, U).$$

Since $Y_{\tau+1} = g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1})$, this conditional independence relation can be rewritten as $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$. Next, applying CI.2 to this conditional independence yields

$$\mathcal{E}_{\tau+2} \perp\!\!\!\perp (Y_1, h(Y_{\tau+1}, \cdots, Y_2, U, V_{\tau+1}), \{D_t\}_{t=1}^{\tau}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U).$$

Since $D_{\tau+1} = h(Y_{\tau+1}, \cdots, Y_2, U, V_{\tau+1}))$, it can be rewritten as $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau+1}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$. Lastly, applying CI.2 again to this conditional independence yields

$$g(Y_{\tau+1}, \cdots, Y_2, U, \mathcal{E}_{\tau+2}) \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau+1}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U).$$

Since $Y_{\tau+2} = g(Y_{\tau+1}, \cdots, Y_2, U, \mathcal{E}_{\tau+2})$, this conditional independence relation can be rewritten as $Y_{\tau+2} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau+1}, Z) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$.

(ii) Note that Restriction 2 (i) $\mathcal{E}_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, W)$ together with the structural definition $Z = \zeta(U, W)$ implies $\mathcal{E}_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, Z)$. Applying CI.1 to this independence relation yields

$$\mathcal{E}_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, Z) \mid (\{Y_t\}_{t=1}^{\tau}, U).$$

Next, applying CI.2 to this conditional independence yields

$$g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}) \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, Z) \mid (\{Y_t\}_{t=1}^{\tau}, U).$$

Since $Y_{\tau+1} = g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1})$, this conditional independence relation can be rewritten as $Y_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, Z) \mid (D_1, U)$. Lastly, applying CI.2 again to this conditional independence yields $Y_{\tau+1} \perp\!\!\!\perp (\{D_t\}_{t=1}^{\tau}, Z) \mid (\{Y_t\}_{t=1}^{\tau}, U)$.

(iii) Applying CI.1 to Restriction 2 (ii) $V_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau-1}, \mathcal{E}_{\tau+1}, V_\tau)$ yields

$$V_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau-1}, \mathcal{E}_{\tau+1}, V_\tau) \mid (g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), \{Y_t\}_{t=2}^{\tau}, U).$$

Since $Y_{\tau+1} = g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1})$, it can be rewritten as $V_{\tau+1} \perp\!\!\!\perp (U, \{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau-1}, \mathcal{E}_{\tau+1}, V_\tau) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$. Next, applying CI.2 to this conditional independence yields

$$V_{\tau+1} \perp\!\!\!\perp (Y_1, h(Y_\tau, \cdots, Y_1, U, V_\tau), \{D_t\}_{t=1}^{\tau-1}) \mid (\{Y_t\}_{t=2}^{\tau+1}, U).$$

Since $D_\tau = h(Y_\tau, \cdots, Y_1, U, V_\tau)$, it can be rewritten as $V_{\tau+1} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau}) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$. Lastly, applying CI.2 to this conditional independence yields

$$h(Y_{\tau+1}, \cdots, Y_2, U, V_2) \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau}) \mid (\{Y_t\}_{t=2}^{\tau+1}, U).$$

Since $D_{\tau+1} = h(Y_{\tau+1}, \cdots, Y_2, U, V_{\tau+1})$, it can be rewritten as $D_{\tau+1} \perp\!\!\!\perp (Y_1, \{D_t\}_{t=1}^{\tau}) \mid (\{Y_t\}_{t=2}^{\tau+1}, U)$.

(iv) Note that Restriction 2 (iii) $W \perp\!\!\!\perp (\{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1})$ together with the structural definition $Z = \zeta(U, W)$ yields $Z \perp\!\!\!\perp (\{Y_t\}_{t=1}^{\tau}, \{D_t\}_{t=1}^{\tau}, \mathcal{E}_{\tau+1}, V_{\tau+1}) \mid U$. Applying CI.2 to this conditional independence relation yields

$$Z \perp\!\!\!\perp (g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), \{Y_t\}_{t=1}^{\tau}, h(g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), U, V_{\tau+1}), \{D_t\}_{t=1}^{\tau}) \mid U.$$

Since $Y_{\tau+1} = g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1})$ and $D_{\tau+1} = h(Y_{\tau+1}, U, V_{\tau+1})$, this conditional independence can be rewritten as $Z \perp\!\!\!\perp (\{Y_t\}_{t=1}^{\tau+1}, \{D_t\}_{t=1}^{\tau+1}) \mid U$. □

Lemma 9 (Invariant Transition).
(i) Under Restrictions 1 and 7 (i), $F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) = F_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u)$ *for all* $y', (y), u$.
(ii) Under Restrictions 1 and 7 (ii), $F_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(d \mid (y), u) = F_{D_1|Y_\tau \cdots Y_1 U}(d \mid (y), u)$ *for all* $d, (y), u$.

Proof. (i) First, note that Restriction 7 (i) implies $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (U, Y_\tau, \cdots, Y_1, \mathcal{E}_{\tau+1})$, which in turn implies that $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (g(Y_\tau, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), Y_\tau, \cdots, Y_2, U)$, hence $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (Y_{\tau+1}, \cdots, Y_2, U)$. Second, Restriction 7 (i) in particular yields $\mathcal{E}_{\tau+1} \perp\!\!\!\perp (Y_\tau, \cdots, Y_1, U)$. Using these two independence results, we obtain

$$
\begin{aligned}
F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) &= \Pr[g((y), u, \mathcal{E}_{\tau+2}) \leq y' \mid (Y_{\tau+1}, \cdots, Y_2) = (y), U = u] \\
&= \Pr[g((y), u, \mathcal{E}_{\tau+2}) \leq y'] \\
&= \Pr[g((y), u, \mathcal{E}_{\tau+1}) \leq y'] \\
&= \Pr[g((y), u, \mathcal{E}_{\tau+1}) \leq y' \mid (Y_\tau, \cdots, Y_1) = (y), U = u] \\
&= F_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u)
\end{aligned}
$$

for all $y', (y), u$, where the second equality follows from $\mathcal{E}_{\tau+2} \perp\!\!\!\perp (Y_{\tau+1}, \cdots, Y_2, U)$, the third equality follows from identical distribution of $\mathcal{E}_t$ by Restriction 1, and the forth equality follows from $\mathcal{E}_{\tau+1} \perp\!\!\!\perp (Y_\tau, \cdots, Y_1, U)$.

(ii) Restriction 7 (ii) implies that $V_{\tau+1} \perp\!\!\!\perp (g(Y_{\tau+1}, \cdots, Y_1, U, \mathcal{E}_{\tau+1}), Y_\tau, \cdots, Y_1, U)$, hence $V_{\tau+1} \perp\!\!\!\perp (Y_{\tau+1}, \cdots, Y_2, U)$. Restriction 7 (ii) also implies $V_\tau \perp\!\!\!\perp (Y_\tau, \cdots, Y_1, U)$. Using these two independence results, we obtain

$$
\begin{aligned}
F_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(d \mid (y), u) &= \Pr[h((y), u, V_{\tau+1}) \leq d \mid (Y_{\tau+1}, \cdots, Y_2) = (y), U = u] \\
&= \Pr[h((y), u, V_{\tau+1}) \leq d] \\
&= \Pr[h((y), u, V_\tau) \leq d] \\
&= \Pr[h((y), u, V_\tau) \leq d \mid (Y_\tau, \cdots, Y_1) = (y), U = u] \\
&= F_{D_1|Y_\tau \cdots Y_1 U}(d \mid (y), u)
\end{aligned}
$$

for all $d, (y), u$, where the second equality follows from $V_{\tau+1} \perp\!\!\!\perp (Y_{\tau+1}, \cdots, Y_2, U)$, the third equality follows from identical distribution of V_t from Restriction 1, and the forth equality follows from $V_\tau \perp\!\!\!\perp (Y_\tau, \cdots, Y_1, U)$. □

Lemma 10 (Identification). *Under Restrictions 1, 4, 7, and 8,* $(F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}, F_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}, F_{Y_\tau \cdots Y_1 U D_{\tau-1}\cdots D_1}(\cdots, \cdot, (1)), F_{Z|U})$ *is uniquely determined by* $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ *and* $F_{Y_{\tau+1}\cdots Y_1 Z D_\tau \cdots D_1}(\cdots, \cdot, (1))$.

Proof. Given fixed (y) and z, define the operators $L_{(y),z} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$, $P_{(y)} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$, $Q_z : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $R_{(y)} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $S_{(y)} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, $T_{(y)} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_U)$, and $T'_{(y)} : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$ by

$$
\begin{aligned}
(L_{(y),z}\xi)(y_{\tau+2}) &= \int f_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(y_{\tau+2}, (y), y_1, z, (1)) \cdot \xi(y_1) dy_1, \\
(P_{(y)}\xi)(y_{\tau+2}) &= \int f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y_{\tau+2} \mid (y), u) \cdot \xi(u) du, \\
(Q_z\xi)(u) &= f_{Z|U}(z \mid u) \cdot \xi(u), \\
(R_{(y)}\xi)(u) &= f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u) \cdot \xi(u), \\
(S_{(y)}\xi)(u) &= \int f_{Y_{\tau+1}\cdots Y_2 Y_1 U D_\tau \cdots D_1}((y), y_1, u, (1)) \cdot \xi(y_1) dy_1, \\
(T_{(y)}\xi)(u) &= \int f_{Y_1|Y_{\tau+1}\cdots Y_2 U D_{\tau+1} D_1}(y_1 \mid (y), u, (1)) \cdot \xi(y_1) dy_1, \\
(T'_{(y)}\xi)(u) &= f_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}((y), u, (1)) \cdot \xi(u)
\end{aligned}
$$

respectively. The operators $L_{(y),z}$, $P_{(y)}$, $S_{(y)}$, and $T_{(y)}$ are integral operators whereas Q_z, $R_{(y)}$, and $T'_{(y)}$ are multiplication operators. Note that $L_{(y),z}$ is identified from observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots,\ \cdot\ ,(1))$.

Step 1: Uniqueness of $F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}$ and $F_{Z|U}$

The kernel $f_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\ \cdot\ ,(y),\ \cdot\ ,z,(1))$ of the integral operator $L_{(y),z}$ can be rewritten as

$$\begin{aligned} f_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(y_{\tau+2},(y),y_1,z,(1)) \;=\; & \int f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_1 Z U D_{\tau+1}\cdots D_1}(y_{\tau+2}\mid (y),y_1,z,u,(1)) \\ & \times f_{Z|Y_{\tau+1}\cdots Y_1 U D_{\tau+1}\cdots D_1}(z\mid (y),y_1,u,(1)) \\ & \times f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_1 U D_{\tau}\cdots D_1}(1\mid (y),y_1,u,(1)) \\ & \times f_{Y_{\tau+1}\cdots Y_1 U D_{\tau}\cdots D_1}((y),y_1,u,(1))\, du \end{aligned} \tag{C.4}$$

But by Lemma 8 (i), (iv), and (iii), respectively, Restriction 7 implies that

$$\begin{aligned} f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_1 Z U D_{\tau+1}\cdots D_1}(y_{\tau+2}\mid (y),y_1,z,u,(1)) &= f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y_{\tau+2}\mid (y),u), \\ f_{Z|Y_{\tau+1}\cdots Y_1 U D_{\tau+1}\cdots D_1}(z\mid (y),y_1,u,(1)) &= f_{Z|U}(z\mid u), \\ f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_1 U D_{\tau}\cdots D_1}(1\mid (y),y_1,u,(1)) &= f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1\mid (y),u). \end{aligned}$$

Equation (C.4) thus can be rewritten as

$$\begin{aligned} f_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(y_{\tau+2},(y),y_1,z,(1)) \;=\; & \int f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y_{\tau+2}\mid (y),u)\cdot f_{Z|U}(z\mid u) \\ & \times f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1\mid (y),u) \\ & \times f_{Y_{\tau+1}\cdots Y_1 U D_{\tau}\cdots D_1}((y),y_1,u,(1))\, du \end{aligned}$$

But this implies that the integral operator $L_{y,z}$ is written as the operator composition

$$L_{(y),z} = P_{(y)} Q_z R_{(y)} S_{(y)}.$$

Restriction 8 (i), (ii), (iii), and (iv) imply that the operators $P_{(y)}$, Q_z, $R_{(y)}$, and $S_{(y)}$ are invertible, respectively. Hence so is $L_{(y),z}$. Using the two values $\{0,1\}$ of Z, form the product

$$L_{(y),1} L_{y,0}^{-1} = P_{(y)} Q_{1/0} P_{(y)}^{-1}$$

where $Q_{z/z'} := Q_z Q_{z'}^{-1}$. By Restriction 8 (ii), the operator $L_{(y),1} L_{(y),0}^{-1}$ is bounded. The expression $L_{(y),1} L_{(y),0}^{-1} = P_{(y)} Q_{1/0} P_{(y)}^{-1}$ thus allows unique eigenvalue-eigenfunction decomposition.

The distinct proxy odds as in Restriction 8 (ii) guarantee distinct eigenvalues and single dimensionality of the eigenspace associated with each eigenvalue. Within each of the single-dimensional eigenspace is a unique eigenfunction pinned down by $\mathcal{L}^1$-normalization because of the unity of integrated densities. The eigenvalues $\lambda(u)$ yield the multiplier of the operator $Q_{1/0}$, hence $\lambda(u) = f_{Z|U}(1\mid u)/f_{Z|U}(0\mid u)$. This proxy odds in turn identifies $f_{Z|U}(\ \cdot\ \mid u)$ since Z is binary. The corresponding normalized eigenfunctions are the kernels of the integral operator $P_{(y)}$, hence $f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(\ \cdot\ \mid (y),u)$. Lastly, Restriction 4 facilitates unique ordering of the eigenfunctions $f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(\ \cdot\ \mid (y),u)$ by the distinct concrete values of $u = \lambda(u)$. This is feasible because the eigenvalues $\lambda(u) = f_{Z|U}(1\mid u)/f_{Z|U}(0\mid u)$ are invariant from (y). That is, eigenfunctions $f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(\ \cdot\ \mid (y),u)$ of the operator $L_{(y),1} L_{(y),0}^{-1}$ across different (y) can be uniquely ordered in u invariantly from (y) by the common set of ordered distinct eigenvalues $u = \lambda(u)$.

Therefore, $F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}$ and $F_{Z|U}$ are uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \ \cdot\ , (1))$. Equivalently, the operators $P_{(y)}$ and Q_z are uniquely determined for each (y) and z, respectively.

Step 2: Uniqueness of $F_{Y_{\tau+1}\cdots Y_1 U D_\tau \cdots D_1}(\cdots, \ \cdot\ , (1))$
By Lemma 8 (ii), Restriction 7 implies $f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U D_\tau \cdots D_1}(y' \mid (y), u, (1)) = f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u)$. Using this equality, write the density of the observed joint distribution $F_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\cdots, (1))$ as

$$\begin{aligned} f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(y', (y), (1)) &= \int f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U D_\tau \cdots D_1}(y' \mid (y), u, (1)) \\ &\quad \times f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), u, (1)) du \\ &= \int f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u) \\ &\quad \times f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), u, (1)) du \end{aligned} \tag{C.5}$$

By Lemma 9 (i), $F_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) = F_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u)$ for all $y', (y), u$. Therefore, we can write the operator $P_{(y)}$ as

$$(P_{(y)}\xi)(y') = \int f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) \cdot \xi(u) du = \int f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u) \cdot \xi(u) du.$$

With this operator notation, it follows from (C.5) that

$$f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\ \cdot\ , (y), (1)) = P_{(y)} f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), \ \cdot\ , (1)).$$

By Restriction 8 (i) and (ii), this operator equation can be solved for $f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), \ \cdot\ , (1))$ as

$$f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), \ \cdot\ , (1)) = P_{(y)}^{-1} f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\ \cdot\ , (y), (1)) \tag{C.6}$$

Recall that $P_{(y)}$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \ \cdot\ , (1))$. The function $f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\ \cdot\ , (y), (1))$ is also uniquely determined by the observed joint distribution $f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\cdots, (1))$. Therefore, (C.5) shows that $f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}(\cdots, \ \cdot\ , (1))$ is uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \ \cdot\ , (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\cdots, (1))$.

Using the solution to the above inverse problem, we can write the kernel of the operator $S_{(y)}$ as

$$\begin{aligned} f_{Y_{\tau+1}\cdots Y_1 U D_\tau \cdots D_1}(y', (y), u, (1)) &= f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U D_\tau \cdots D_1}(y' \mid (y), u, (1)) \cdot f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), u, (1)) \\ &= f_{Y_{\tau+1}|Y_\tau \cdots Y_1 U}(y' \mid (y), u) \cdot f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), u, (1)) \\ &= f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) \cdot f_{Y_\tau \cdots Y_1 U D_\tau \cdots D_1}((y), u, (1)) \\ &= f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) \\ &\qquad \times [P_{(y)}^{-1} f_{Y_{\tau+1}\cdots Y_1 Z D_\tau \cdots D_1}(\ \cdot\ , (y), z, (1))](u) \end{aligned}$$

where the second equality follows from Lemma 8 (ii), the third equality follows from Lemma 9 (i), and the forth equality follows from (C.6). Since $f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \ \cdot\ , (1))$ and $[P_{(y)}^{-1} f_{Y_{\tau+1}\cdots Y_1 Z D_\tau \cdots D_1}(\ \cdot\ , (y), z, (1))]$ was shown in the previous paragraph to be uniquely determined for each y by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \ \cdot\ , (1))$ and

$f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\cdots,(1))$, it follows that $f_{Y_{\tau+1}\cdots Y_1 U D_\tau \cdots D_1}(\cdots, \cdot, (1))$ too is uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1} \cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau \cdots D_1}(\cdots,(1))$. Equivalently, the operator $S_{(y)}$ is uniquely determined for each (y).

Step 3: Uniqueness of $F_{Y_1 | Y_{\tau+1}\cdots Y_2 U D_{\tau+1} \cdots D_1}(\cdot \mid \cdots, \cdot, (1))$
First, note that the kernel of the composite operator $T'_{(y)} T_{(y)}$ can be written as

$$\begin{aligned}
& f_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}((y), u, (1)) \cdot f_{Y_1|Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}(y_1 \mid (y), u, (1)) \\
= \; & f_{Y_{\tau+1}\cdots Y_1 U D_{\tau+1}\cdots D_1}((y), y_1, u, (1)) \\
= \; & f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_1 U D_\tau \cdots D_1}(1 \mid (y), y_1, u, (1)) \cdot f_{Y_{\tau+1}\cdots Y_1 U D_\tau\cdots D_1}((y), y_1, u, (1)) \\
= \; & f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u) \cdot f_{Y_{\tau+1}\cdots Y_1 U D_\tau \cdots D_1}((y), y_1, u, (1))
\end{aligned} \tag{C.7}$$

where the last equality is due to Lemma 8 (iii). But the last expression corresponds to the kernel of the composite operator $R_{(y)} S_{(y)}$, thus showing that $T'_{(y)} T_{(y)} = R_{(y)} S_{(y)}$. But then, $L_{(y),z} = P_{(y)} Q_z R_{(y)} S_{(y)} = P_{(y)} Q_z T'_{(y)} T_{(y)}$. Note that the invertibility of $R_{(y)}$ and $S_{(y)}$ as required by Assumption 8 implies invertibility of $T'_{(y)}$ and $T_{(y)}$ as well, for otherwise the equivalent composite operator $T'_{(y)} T_{(y)} = R_{(y)} S_{(y)}$ would have a nontrivial nullspace.

Using Restriction 8, form the product of operators as

$$L^{-1}_{(y),0} L_{(y),1} = T^{-1}_{(y)} Q_{1/0} T_{(y)}$$

The disappearance of $T'_{(y)}$ is due to commutativity of multiplication operators. By the same logic as in Step 1, this expression together with Restriction 8 (ii) admits unique left eigenvalue-eigenfunction decomposition. Moreover, the point spectrum is exactly the same as the one in Step 1, as is the middle multiplication operator $Q_{1/0}$. This equivalence of the spectrum allows consistent ordering of U with that of Step 1. Left eigenfunctions yield the kernel of $T_{(y)}$ pinned down by the normalization of unit integral. This shows that the operator $T_{(y)}$ is uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$.

Step 4: Uniqueness of $F_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$
Equation (C.7) implies that

$$\begin{aligned}
\int f_{Y_1|Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}(y_1 \mid (y), u, (1)) \cdot f_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}((y), u, (1)) du \\
= f_{Y_{\tau+1}\cdots Y_1 D_{\tau+1}\cdots D_1}((y), y_1, (1))
\end{aligned}$$

hence yielding the linear operator equation

$$T^*_{(y)} f_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}((y), \cdot, (1)) = f_{Y_{\tau+1}\cdots Y_1 D_{\tau+1}\cdots D_1}((y), \cdot, (1))$$

where $T^*_{(y)}$ denotes the adjoint operator of $T_{(y)}$. Since $T_{(y)}$ is invertible, so is its adjoint operator $T^*_{(y)}$. But then, the multiplier of the multiplication operator $T'_{(y)}$ can be given by the unique solution to the above linear operator equation, i.e.,

$$f_{Y_{\tau+1}\cdots Y_2 U D_{\tau+1}\cdots D_1}((y), \cdot, (1)) = (T^*_{(y)})^{-1} f_{Y_{\tau+1}\cdots Y_1 D_{\tau+1}\cdots D_1}((y), \cdot, (1))$$

Note that $T_{(y)}$ hence $T^*_{(y)}$ was shown to be uniquely determined by $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ in Step 3, and $f_{Y_{\tau+1}\cdots Y_1 D_{\tau+1}\cdots D_1}(\cdots,(1))$ is also available from observed data. Therefore, the operator $T'_{(y)}$ is uniquely determined by $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$.

Step 5: Uniqueness of $F_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid \cdots, \cdot)$
First, the definition of the operators $R_{(y)}$, $S_{(y)}$, $T_{(y)}$, and $T'_{(y)}$ and Lemma 8 (iii) yield the operator equality $R_{(y)}S_{(y)} = T'_{(y)}T_{(y)}$, where $T_{(y)}$ and $T'_{(y)}$ have been shown to be uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ in Steps 3 and 4, respectively. Recall that $S_{(y)}$ was also shown in Step 2 to be uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$. Restriction 8 (iv) guarantees invertibility of $S_{(y)}$. It follows that the operator inversion $R_{(y)} = (R_{(y)}S_{(y)})S^{-1}_{(y)} = (T'_{(y)}T_{(y)})S^{-1}_{(y)}$ yields the operator $R_{(y)}$, in turn showing that its multiplier $f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), \cdot)$ is uniquely determined for each (y) by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$.

Step 6: Uniqueness of $F_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}(\cdots, \cdot, (1))$
Recall from Step 2 that $f_{Y_{\tau+1}\cdots Y_1 U D_\tau\cdots D_1}(\cdots, \cdot, (1))$ is uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$. We can write

$$\begin{aligned}
f_{Y_{\tau+1}\cdots Y_1 U D_\tau\cdots D_1}(y', (y), u, (1)) &= f_{Y_{\tau+1}|Y_\tau\cdots Y_1 U D_\tau\cdots D_1}(y' \mid (y), u, (1)) \\
&\quad \times f_{D_\tau|Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}(1 \mid (y), u, (1)) \\
&\quad \times f_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}((y), u, (1)) \\
&= f_{Y_{\tau+1}|Y_\tau\cdots Y_1 U}(y' \mid (y), u) \cdot f_{D_\tau|Y_\tau\cdots Y_1 U}(1 \mid (y), u) \\
&\quad \times f_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}((y), u, (1)) \\
&= f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) \cdot f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u) \\
&\quad \times f_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}((y), u, (1)),
\end{aligned}$$

where the second equality follows from Lemma 8 (ii), and the third equality follows from Lemma 9 (i) and (ii). For a given $((y), u)$, there must exist some y' such that $f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) > 0$ by a property of conditional density functions. Moreover, Restriction 8 (iii) requires that $f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u) > 0$ for a given (y) for all u. Therefore, for such a choice of y', we can write

$$f_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}((y), u, (1)) = \frac{f_{Y_{\tau+1}\cdots Y_1 U D_\tau\cdots D_1}(y', (y), u, (1))}{f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u) \cdot f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u)}$$

$f_{Y_{\tau+2}|Y_{\tau+1}\cdots Y_2 U}(y' \mid (y), u)$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$, $f_{Y_{\tau+1}\cdots Y_1 U D_\tau\cdots D_1}(y', (y), u, (1))$ was shown in Step 2 to be uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$, and $f_{D_{\tau+1}|Y_{\tau+1}\cdots Y_2 U}(1 \mid (y), u)$ was shown in Step 5 to be uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$. Therefore, it follows that the joint density $f_{Y_\tau\cdots Y_1 U D_{\tau-1}\cdots D_1}(\cdots, \cdot, (1))$ is uniquely determined by the observed joint distributions $F_{Y_{\tau+2}\cdots Y_1 Z D_{\tau+1}\cdots D_1}(\cdots, \cdot, (1))$ and $f_{Y_{\tau+1}\cdots Y_1 D_\tau\cdots D_1}(\cdots, (1))$. $\square$

C.4 Models with Time-Specific Effects

The baseline model (3.1) that we considered in this paper assumes that the dynamic model g is time-invariant. It is often more realistic to allow this model to have time-specific effects.

Consider the following variant of the model (3.1).

$$\begin{cases} Y_t = g_t(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T \qquad \text{(State Dynamics)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T-1 \qquad \text{(Hazard Model)} \\ F_{Y_1 U} & \text{(Initial joint distribution of } (Y_1, U)) \\ Z = \zeta(U, W) & \text{(Optional: nonclassical proxy of } U) \end{cases}$$

The differences from (3.1) are the t subscripts under g.

The objective is to identify the model $(\{g_t\}_{t=2}^T, h, F_{Y_1U}, \zeta)$. The main obstacle is that the invariant transition of Lemma 4 (i) is no longer useful. As a result, Steps 2 and 6 in the proof of Lemma 2 break down. In order to remedy this hole, we need to observe data of an additional time period prior to the start of the data, i.e., $t = 0$. For brevity, we show this result for the case of $T = 3$.

Lemma 11 (Identification)**.** *Suppose that Restrictions 1, 2, 3, and 4 hold conditionally on* $Pr(D_0 = 1)$*. Then the model* $(\{F_{Y_t|Y_{t-1}U}\}_{t=2}^3, F_{D_t|Y_tU}, F_{Y_1U|D_0=1}, F_{Z|U})$ *is uniquely determined by the observed joint distributions* $F_{Y_1Y_0ZD_0}(\,\cdot,\ \cdot\ ,\ \cdot\ ,1)$, $F_{Y_2Y_1Y_0ZD_1D_0}(\,\cdot,\ \cdot,\ \cdot\ ,\ \cdot\ ,1,1)$*, and* $F_{Y_3Y_2Y_1Y_0ZD_2D_1D_0}(\,\cdot,\ \cdot,\ \cdot,\ \cdot,\ \cdot,1,1,1)$*.*

Proof. Many parts of the proof Lemma 2 remains available. However, under the current model with time-specific transition, the operator P_y is time-specific. Therefore, we use two operators $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ and $P'_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ for each y defined as

$$\begin{aligned} (P_y\xi)(y_2) &= \int f_{Y_2|Y_1U}(y_2 \mid y, u) \cdot \xi(u)du, \\ (P'_y\xi)(y_3) &= \int f_{Y_3|Y_2U}(y_3 \mid y, u) \cdot \xi(u)du, \end{aligned}$$

Accordingly, we employ the two observable operators $L_{y,z} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$ and $L'_{y,z} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$ for each (y, z) defined as

$$\begin{aligned} (L_{y,z}\xi)(y_2) &= \int f_{Y_2Y_1Y_0ZD_1D_0}(y_2, y, y_0, z, 1, 1) \cdot \xi(y_0)dy_0, \\ (L'_{y,z}\xi)(y_3) &= \int f_{Y_3Y_2Y_1ZD_2D_1D_0}(y_3, y, y_1, z, 1, 1, 1) \cdot \xi(y_1)dy_1. \end{aligned}$$

All the other operators directly carry over from the proof of Lemma 2 as:

$$\begin{aligned} (Q_z\xi)(u) &= f_{Z|U}(z \mid u) \cdot \xi(u), \\ (R_y\xi)(u) &= f_{D_2|Y_2U}(1 \mid y, u) \cdot \xi(u), \\ (S_y\xi)(u) &= \int f_{Y_2Y_1UD_1D_0}(y, y_1, u, 1, 1) \cdot \xi(y_1)dy_1, \\ (T_y\xi)(u) &= \int f_{Y_1|Y_2UD_2D_1D_0}(y_1 \mid y, u, 1, 1, 1) \cdot \xi(y_1)dy_1, \\ (T'_y\xi)(u) &= f_{Y_2UD_2D_1D_0}(y, u, 1, 1, 1) \cdot \xi(u) \end{aligned}$$

except that the additional argument $D_0 = 1$ is attached to the kernels of S_y and T_y and the multiplier of T'_y.

The first task is to identify the kernels of these two integral operators. Following Step 1 of the proof of Lemma 2 by using the observed operator $L'_{y,z}$ shows that P'_y and Q_z are identified. Equivalently, $F_{Y_3|Y_2U}$ and $F_{Z|U}$ are identified. Similarly, following Step 1 by using the observed operator $L_{y,z}$ shows that P_y and Q_z are identified. Equivalently, $F_{Y_2|Y_1U}$ is identified as well.

Next, follow Step 2 of the proof of Lemma 2, except that we use our current definition of P_y instead of P'_y. It follows that

$$f_{Y_2Y_1UD_1D_0}(y', y, u, 1, 1) \;=\; f_{Y_2|Y_1U}(y' \mid y, u) \cdot [P_y^{-1} f_{Y_2Y_1D_1D_0}(\;\cdot\;, y, 1, 1)](u)$$

where $f_{Y_2|Y_1U}$ was identified as the kernel of P_y in the previous step, P_y was identified in the previous step, and $f_{Y_2Y_1D_1D_0}(\;\cdot,\;\cdot, 1, 1)$ is observable from data. This shows that the operator S_y is identified for each y.

Steps 3–5 analogously follow from the proof of Lemma 2 except that the current definitions of $L_{y,z}$, R_y, S_y, T_y, and T'_y are used. These steps show that R_y in particular are identified for each y.

Lastly, extending the argument of Step 6 in the proof of Lemma 2 yields

$$f_{Y_1U|D_0}(y, u \mid 1) = \frac{f_{Y_2Y_1UD_1D_0}(y', y, u, 1, 1)}{f_{Y_2|Y_1U}(y' \mid y, u) f_{D_2|Y_2U}(1 \mid y, u) f_{D_0}(1)}$$

where $f_{Y_2Y_1UD_1D_0}(\;\cdot,\;\cdot,\;\cdot, 1, 1)$ was identified in the second step, $f_{Y_2|Y_1U}$ was identified in the first step, $f_{D_2|Y_2U}$ was identified in the previous step, and $f_{D_0}(1)$ is observable from data. It follows that $F_{Y_1U|D_0=1}$ is identified. □

C.5 Censoring by Contemporaneous D_t instead of Lagged D_t

For the main identification result discussed, we assumed that lagged selection indicator D_t induces censored observation of Y_t as follows:

$$\begin{aligned} &\text{observe } Y_1, \\ &\text{observe } Y_2 \quad \text{if } D_1 = 1, \\ &\text{observe } Y_3 \quad \text{if } D_1 = D_2 = 1. \end{aligned}$$

In many application, contemporaneous D_t instead of lagged D_t may induce censored observation of Y_t as follows:

$$\begin{aligned} &\text{observe } Y_1, \quad \text{if } D_1 = 1 \\ &\text{observe } Y_2 \quad \text{if } D_1 = D_2 = 1, \\ &\text{observe } Y_3 \quad \text{if } D_1 = D_2 = D_3 = 1. \end{aligned}$$

where the model follows a slight modification of (3.1):

$$\begin{cases} Y_t = g(Y_{t-1}, U, \mathcal{E}_t) & t = 2, \cdots, T \qquad \text{(State Dynamics)} \\ D_t = h(Y_t, U, V_t) & t = 1, \cdots, T \qquad \text{(Hazard Model)} \\ F_{Y_1U} & \text{(Initial joint distribution of } (Y_1, U)) \\ Z = \zeta(U, W) & \text{(Optional: nonclassical proxy of } U) \end{cases}$$

(The difference from the baseline model (3.1) is that the hazard model is defined for all $t = 1, \cdots, T$.) In this model, the problem of identification is to show the well-definition of

$$(F_{Y_2Y_1ZD_2D_1}(\;\cdot,\;\cdot,\;\cdot, 1, 1),\; F_{Y_3Y_2Y_1ZD_3D_2D_1}(\;\cdot,\;\cdot,\;\cdot,\;\cdot, 1, 1, 1)) \;\mapsto\; (g, h, F_{Y_1U|D_1=1}, \zeta).$$

First, consider the following auxiliary lemma, which can be proved similarly to Lemma 3.

Lemma 12 (Independence)**.** *The following implications hold:*
(i) Restriction 2 (i) $\Rightarrow Y_3 \perp\!\!\!\perp (Y_1, D_1, D_2, D_3, Z) \mid (Y_2, U)$.
(ii) Restriction 2 (i) $\Rightarrow Y_2 \perp\!\!\!\perp (D_1, D_2, Z) \mid (Y_1, U)$.
(iii) Restriction 2 (ii) $\Rightarrow D_3 \perp\!\!\!\perp Y_2 \mid (Y_3, U)$.
(iv) Restriction 2 (iii) $\Rightarrow Z \perp\!\!\!\perp (Y_2, Y_1, D_3, D_2, D_1) \mid U$.

Some of the rank conditions of Restriction 3 are replaced as follows.

Restriction 9 (Rank Conditions)**.** The following conditions hold for every $y \in \mathcal{Y}$:
(i) Heterogeneous Dynamics: the integral operator $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$ defined by $P_y\xi(y') = \int f_{Y_3|Y_2U}(y' \mid y, u) \cdot \xi(u)du$ is bounded and invertible.
(ii) Nondegenerate Proxy Model: there exists $\delta > 0$ such that $\delta \leqslant f_{Z|U}(1 \mid u) \leqslant 1 - \delta$ for all u.
Relevant Proxy: $f_{Z|U}(1 \mid u) \neq f_{Z|U}(1 \mid u')$ whenever $u \neq u'$.
(iii) No Extinction: $f_{D_2|Y_2U}(1 \mid y, u) > 0$ for all $u \in \mathcal{U}$.
(iv) Initial Heterogeneity: the two integral operators $\tilde{L}_y : \mathcal{L}^2(Y_t) \to \mathcal{L}^2(U)$, and $S_y : \mathcal{L}^2(U) \to \mathcal{L}^2(Y_t)$ respectively defined by $\tilde{L}_y\xi(u) = \int f_{Y_2Y_1UD_3D_2D_1}(y, y_1, u, 1, 1, 1) \cdot \xi(y_1)dy_1$ and $S_y\xi(y_1) = \int f_{Y_2Y_1UD_2D_1}(y, y_1, u, 1, 1) \cdot \xi(u)du$ are bounded and invertible.

Lemma 13 (Identification)**.** *Under Restrictions 1, 2, 4, and 9,* $(F_{Y_3|Y_2U}, F_{D_3|Y_3U}, F_{Y_1U|D_1=1}, F_{Z|U})$ *is uniquely determined by* $F_{Y_2Y_1ZD_2D_1}(\,\cdot, \,\cdot, \,\cdot, 1, 1)$ *and* $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot, \,\cdot, \,\cdot, \,\cdot, 1, 1, 1)$.

Proof. Given fixed y and z, define the operators $L_{y,z} : \mathcal{L}^2(F_{Y_t}) \to \mathcal{L}^2(F_{Y_t})$, $P_y : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_{Y_t})$, $Q_z : \mathcal{L}^2(F_U) \to \mathcal{L}^2(F_U)$, $\tilde{L}_y : \mathcal{L}^2(Y_t) \to \mathcal{L}^2(U)$, and $S_y : \mathcal{L}^2(U) \to \mathcal{L}^2(Y_t)$ by

$$\begin{aligned}
(L_{y,z}\xi)(y_3) &= \int f_{Y_3Y_2Y_1ZD_3D_2D_1}(y_3, y, y_1, z, 1, 1, 1) \cdot \xi(y_1)dy_1, \\
(P_y\xi)(y_3) &= \int f_{Y_3|Y_2U}(y_3 \mid y, u) \cdot \xi(u)du, \\
(Q_z\xi)(u) &= f_{Z|U}(z \mid u) \cdot \xi(u), \\
(\tilde{L}_y\xi)(u) &= \int f_{Y_2Y_1UD_3D_2D_1}(y, y_1, u, 1, 1, 1) \cdot \xi(y_1)dy_1, \\
(S_y\xi)(y_1) &= \int f_{Y_2Y_1UD_2D_1}(y, y_1, u, 1, 1) \cdot \xi(u)du
\end{aligned}$$

respectively. Similarly to the proof of Lemma 2, the operator $L_{y,z}$ is identified from observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot, \,\cdot, \,\cdot, \,\cdot, 1, 1, 1)$.
Step 1: Uniqueness of $F_{Y_3|Y_2U}$ and $F_{Z|U}$
The kernel $f_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,, y, \,\cdot\,, z, 1, 1, 1)$ of the integral operator $L_{y,z}$ can be rewritten as

$$\begin{aligned}
f_{Y_3Y_2Y_1ZD_3D_2D_1}(y_3, y, y_1, z, 1, 1, 1) = \int & f_{Y_3|Y_2Y_1ZUD_3D_2D_1}(y_3 \mid y, y_1, z, u, 1, 1, 1) \\
& \times f_{Z|Y_2Y_1UD_3D_2D_1}(z \mid y, y_1, u, 1, 1, 1) \\
& \times f_{Y_2Y_1UD_3D_2D_1}(y, y_1, u, 1, 1, 1)\, du
\end{aligned} \tag{C.8}$$

But by Lemma 12 (i) and (iv) respectively, Restriction 2 implies that

$$\begin{aligned}
f_{Y_3|Y_2Y_1ZUD_3D_2D_1}(y_3 \mid y, y_1, z, u, 1, 1, 1) &= f_{Y_3|Y_2U}(y_3 \mid y, u), \\
f_{Z|Y_2Y_1UD_3D_2D_1}(z \mid y, y_1, u, 1, 1, 1) &= f_{Z|U}(z \mid u).
\end{aligned}$$

Equation (C.8) thus can be rewritten as

$$\begin{aligned} f_{Y_3Y_2Y_1ZD_3D_2D_1}(y_3, y, y_1, z, 1, 1, 1) &= \int f_{Y_3|Y_2U}(y_3 \mid y, u) \cdot f_{Z|U}(z \mid u) \\ &\qquad \times f_{Y_2Y_1UD_3D_2D_1}(y, y_1, u, 1, 1, 1)\, du \end{aligned}$$

But this implies that the integral operator $L_{y,z}$ is written as the operator composition

$$L_{y,z} = P_y Q_z \tilde{L}_y$$

Restriction 9 (i), (ii), and (iv) imply that the operators P_y, Q_z, and $\tilde{L}_y$ are invertible, respectively. Hence so is $L_{y,z}$. Using the two values $\{0,1\}$ of Z, form the product

$$L_{y,1}L_{y,0}^{-1} = P_y Q_{1/0} P_y^{-1}$$

where $Q_{z/z'} := Q_z Q_{z'}^{-1}$ is the multiplication operator with proxy odds defined by

$$(Q_{1/0}\xi)(u) = \frac{f_{Z|U}(1 \mid u)}{f_{Z|U}(0 \mid u)}\xi(u).$$

The rest of Step 1 is analogous to that of the proof of Lemma 2. Therefore, $F_{Y_3|Y_2U}$ and $F_{Z|U}$ are uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$. Equivalently, the operators P_y and Q_z are uniquely determined for each y and z, respectively.

Step 2: Uniqueness of $F_{Y_2Y_1UD_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1)$
By Lemma 12 (ii), Restriction 2 implies $f_{Y_2|Y_1UD_2D_1}(y' \mid y, u, 1, 1) = f_{Y_2|Y_1U}(y' \mid y, u)$. Using this equality, write the density of the observed joint distribution $F_{Y_2Y_1D_2D_1}(\,\cdot\,, \,\cdot\,, 1, 1)$ as

$$\begin{aligned} f_{Y_2Y_1D_2D_1}(y', y, 1, 1) &= \int f_{Y_2|Y_1UD_2D_1}(y' \mid y, u, 1, 1) f_{Y_1UD_2D_1}(y, u, 1, 1) du \\ &= \int f_{Y_2|Y_1U}(y' \mid y, u) f_{Y_1UD_2D_1}(y, u, 1, 1) du \end{aligned} \tag{C.9}$$

By Lemma 4 (i), $F_{Y_3|Y_2U}(y' \mid y, u) = F_{Y_2|Y_1U}(y' \mid y, u)$ for all y', y, u. Therefore, we can write the operator P_y as

$$(P_y\xi)(y') = \int f_{Y_3|Y_2U}(y' \mid y, u) \cdot \xi(u) du = \int f_{Y_2|Y_1U}(y' \mid y, u) \cdot \xi(u) du.$$

With this operator notation, it follows from (C.9) that

$$f_{Y_2Y_1D_2D_1}(\,\cdot\,, y, 1, 1) = P_y f_{Y_1UD_2D_1}(y, \,\cdot\,, 1, 1).$$

By Restriction 9 (i), this operator equation can be solved for $f_{Y_1UD_2D_1}(y, \,\cdot\,, 1, 1)$ as

$$f_{Y_1UD_2D_1}(y, \,\cdot\,, 1, 1) = P_y^{-1} f_{Y_2Y_1D_2D_1}(\,\cdot\,, y, 1, 1) \tag{C.10}$$

Recall that P_y was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, 1, 1, 1)$. The function $f_{Y_2Y_1D_2D_1}(\,\cdot\,, y, 1, 1)$ is also uniquely determined by the observed joint distribution $F_{Y_2Y_1D_2D_1}(\,\cdot\,, \,\cdot\,, 1, 1)$. Therefore, (C.9) shows that $f_{Y_1UD_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$ is uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$.

Using the solution to the above inverse problem, we can write the kernel of the operator S_y as

$$\begin{aligned} f_{Y_2Y_1UD_2D_1}(y',y,u,1,1) &= f_{Y_2|Y_1UD_2D_1}(y' \mid y,u,1,1)\cdot f_{Y_1UD_2D_1}(y,u,1,1) \\ &= f_{Y_2|Y_1U}(y' \mid y,u)\cdot f_{Y_1UD_2D_1}(y,u,1,1) \\ &= f_{Y_3|Y_2U}(y' \mid y,u)\cdot f_{Y_1UD_2D_1}(y,u,1,1) \\ &= f_{Y_3|Y_2U}(y' \mid y,u)\cdot [P_y^{-1} f_{Y_2Y_1D_2D_1}(\,\cdot\,,y,1,1)](u) \end{aligned}$$

where the second equality follows from Lemma 12 (ii), the third equality follows from Lemma 4 (i), and the forth equality follows from (C.10). Since $f_{Y_3|Y_2U}$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $[P_y^{-1} f_{Y_2Y_1D_2D_1}(\,\cdot\,,y,1,1)]$ was shown in the previous paragraph to be uniquely determined for each y by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$, it follows that $f_{Y_2Y_1UD_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1)$ too is uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$. Equivalently, the operator S_y is identified for each y.

Step 3: Uniqueness of $F_{Y_3D_3|Y_3U}(\,\cdot\,,1 \mid \,\cdot\,,\,\cdot\,)$

The density of the observed joint distribution $F_{Y_3Y_2Y_1D_3D_2D_1}(y_3,y_2,y_1,1,1,1)$ can be decomposed as

$$\begin{aligned} f_{Y_3Y_2Y_1D_3D_2D_1}(y_3,y_2,\,\cdot\,,1,1,1) &= \int f_{Y_3D_3|Y_2Y_1UD_2D_1}(y_3,1 \mid y_2,y_1,u,1,1) \\ &\qquad \times f_{Y_2Y_1UD_2D_1}(y_2,y_1,u,1,1)du \\ &= \int f_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,u)\cdot f_{Y_2Y_1UD_2D_1}(y_2,y_1,u,1,1)du \\ &= S_{y_2}\cdot f_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,\,\cdot\,) \end{aligned}$$

for each y_3 and y_2, where the second equality follows from Lemma 12 (i) and (iii). By Restriction 9 (iv), S_{y_2} is invertible, and we can rewrite the above equality as

$$f_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,\,\cdot\,) = S_{y_2}^{-1} f_{Y_3Y_2Y_1D_3D_2D_1}(y_3,y_2,\,\cdot\,,1,1,1).$$

Recall that S_{y_2} was shown to be uniquely determined in Step 2 by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$. Therefore, $F_{Y_3D_3|Y_2U}(\,\cdot\,,1 \mid \,\cdot\,,\,\cdot\,)$ is identified by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,1,1,1)$ and $F_{Y_2Y_1D_2D_1}(\,\cdot\,,\,\cdot\,,1,1)$.

Step 4: Uniqueness of $F_{D_3|Y_3U}(1 \mid \,\cdot\,,\,\cdot\,)$

The density of the observed joint distribution $F_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,u)$ can be decomposed as

$$\begin{aligned} f_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,u) &= f_{D_3|Y_3Y_2U}(1 \mid y_3,y_2,u)\cdot f_{Y_3|Y_2U}(y_3 \mid y_2,u) \\ &= f_{D_3|Y_3U}(1 \mid y_3,u)\cdot f_{Y_3|Y_2U}(y_3 \mid y_2,u) \end{aligned}$$

where the second equality follows from Lemma 12 (iii). For each pair (y_3,u) in the support, there exists y_2 such that $f_{Y_3|Y_2U}(y_3 \mid y_2,u) > 0$. For such y_2, rewrite the above equation as

$$f_{D_3|Y_3U}(1 \mid y_3,u) = \frac{f_{Y_3D_3|Y_2U}(y_3,1 \mid y_2,u)}{f_{Y_3|Y_2U}(y_3 \mid y_2,u)}.$$

Recall that Step 1 showed that $F_{Y_3|Y_2U}$ is uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$, and Step 3 showed that $F_{Y_3D_3|Y_2U}(\cdot, 1 \mid \cdot, \cdot)$ is identified by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$. Therefore, $F_{D_3|Y_3U}(1 \mid \cdot, \cdot)$ is identified by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$.

Step 5: Uniqueness of $F_{Y_1U|D_1=1}$
Recall from Step 2 that $f_{Y_2Y_1UD_2D_1}(\cdot, \cdot, \cdot, 1, 1)$ is uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$. We can write

$$\begin{aligned}
f_{Y_2Y_1UD_2D_1}(y', y, u, 1, 1) &= f_{D_2|Y_2Y_1UD_1}(1 \mid y', y, u, 1) f_{Y_2|Y_1UD_1}(y' \mid y, u, 1) f_{Y_1UD_1}(y, u, 1) \\
&= f_{D_2|Y_2U}(1 \mid y', u) f_{Y_2|Y_1U}(y' \mid y, u) f_{Y_1UD_1}(y, u, 1) \\
&= f_{D_3|Y_3U}(1 \mid y', u) f_{Y_3|Y_2U}(y' \mid y, u) f_{Y_1UD_1}(y, u, 1)
\end{aligned}$$

where the second equality follows from Lemma 12 (ii), and the third equality follows from Lemma 4 (i) and (ii). For a given (y, u), there must exist some y' such that $f_{Y_3|Y_2U}(y' \mid y, u) > 0$ by a property of conditional density functions. Moreover, Restriction 9 (iii) requires that $f_{D_3|Y_3U}(1 \mid y', u) > 0$ for a given y' for all u. Therefore, for such a choice of y', we can write

$$f_{Y_1UD_1}(y, u, 1) = \frac{f_{Y_2Y_1UD_2D_1}(y', y, u, 1, 1)}{f_{Y_3|Y_2U}(y' \mid y, u) f_{D_3|Y_3U}(1 \mid y', u)}$$

Recall that $f_{Y_3|Y_2U}(\cdot \mid \cdot, \cdot)$ was shown in Step 1 to be uniquely determined by the observed joint distribution $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$, $f_{Y_2Y_1UD_2D_1}(\cdot, \cdot, \cdot, 1, 1)$ was shown in Step 2 to be uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$, and $f_{D_3|Y_3U}(1 \mid \cdot, \cdot)$ was shown in Step 4 to be uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$. Therefore, it follows that the initial joint density $f_{Y_1U|D_1=1}$ is uniquely determined by the observed joint distributions $F_{Y_3Y_2Y_1ZD_3D_2D_1}(\cdot, \cdot, \cdot, \cdot, 1, 1, 1)$ and $F_{Y_2Y_1D_2D_1}(\cdot, \cdot, 1, 1)$. $\square$

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True Parameter Values: $\alpha_1 = \alpha_2 = \beta_0 = \beta_1 = \beta_2 = 0.5$

		Dynamic Model		Hazard Model		
	Percentile	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
Parametric CMLE	75%	0.556	0.519	0.673	0.855	1.293
	50%	0.502	0.502	0.517	0.523	0.569
	25%	0.454	0.483	0.403	0.169	−0.182
Semi-parametric* CMLE	75%	0.566	0.524	0.758	0.882	1.212
	50%	0.513	0.508	0.555	0.523	0.475
	25%	0.465	0.489	0.428	0.153	−0.288
Semi-parametric** CMLE	75%	0.558	—	—	0.589	
	50%	0.459	—	—	0.418	0.500
	25%	0.368	—	—	0.204	(Fixed)
Semi-parametric*** CMLE	75%	0.686	—	—	1.049	
	50%	0.436	—	—	0.719	0.500
	25%	0.271	—	—	0.403	(Fixed)
Semi-parametric† 1st Step	75%	0.585	—	—	—	—
	50%	0.495	—	—	—	—
	25%	0.385	—	—	—	—
Arellano-Bond	75%	0.464	—	—	—	—
	50%	0.412	—	—	—	—
	25%	0.352	—	—	—	—
Fixed-Effect Logit	75%	—	—	—	−0.134	—
	50%	—	—	—	−0.287	—
	25%	—	—	—	−0.441	—
Random-Effect Logit	75%	—	—	—	0.793	—
	50%	—	—	—	0.729	—
	25%	—	—	—	0.672	—

$*$ The distribution of $F_{Y_1 U}$ is semi-parametric.

$**$ The distributions of $\mathcal{E}_t$ and V_t are nonparametric.

$***$ The distribution of $F_{Y_1 U}$ is semi-parametric, and the distributions of $\mathcal{E}_t$ and V_t are nonparametric.

† The distribution $(\mathcal{E}_t, V_t, Y_1, U)$ and the functions g and h are nonparametric.

Table 1: MC-simulated distributions of parameter estimates.

Birth year cohorts 1917–1920 (aged 51–54 in 1971)

$N = 822$	Type I ($U = 0$)	Type II ($U = 1$)
	54.2%	45.8%

Type I ($U = 0$)

Markov Matrix		$Y_{t-1} = 0$	$Y_{t-1} = 1$
Y_t	0	0.930	0.128
	1	0.070	0.872

2-Year Survival Probability

Y_t	0	0.899 (0.044)
	1	1.000 (0.000)
H_0: equal probability		p-value = 0.021**

Type II ($U = 1$)

Markov Matrix		$Y_{t-1} = 0$	$Y_{t-1} = 1$
Y_t	0	1.000	0.025
	1	0.000	0.975

2-Year Survival Probability

Y_t	0	0.878 (0.149)
	1	0.999 (0.038)
H_0: equal probability		p-value = 0.445

Birth year cohorts 1913–1916 (aged 55–58 in 1971)

$N = 727$	Type I ($U = 0$)	Type II ($U = 1$)
	53.9%	46.1%

Type I ($U = 0$)

Markov Matrix		$Y_{t-1} = 0$	$Y_{t-1} = 1$
Y_t	0	0.954	0.162
	1	0.046	0.838

2-Year Survival Probability

Y_t	0	0.912 (0.036)
	1	1.000 (0.000)
H_0: equal probability		p-value = 0.013**

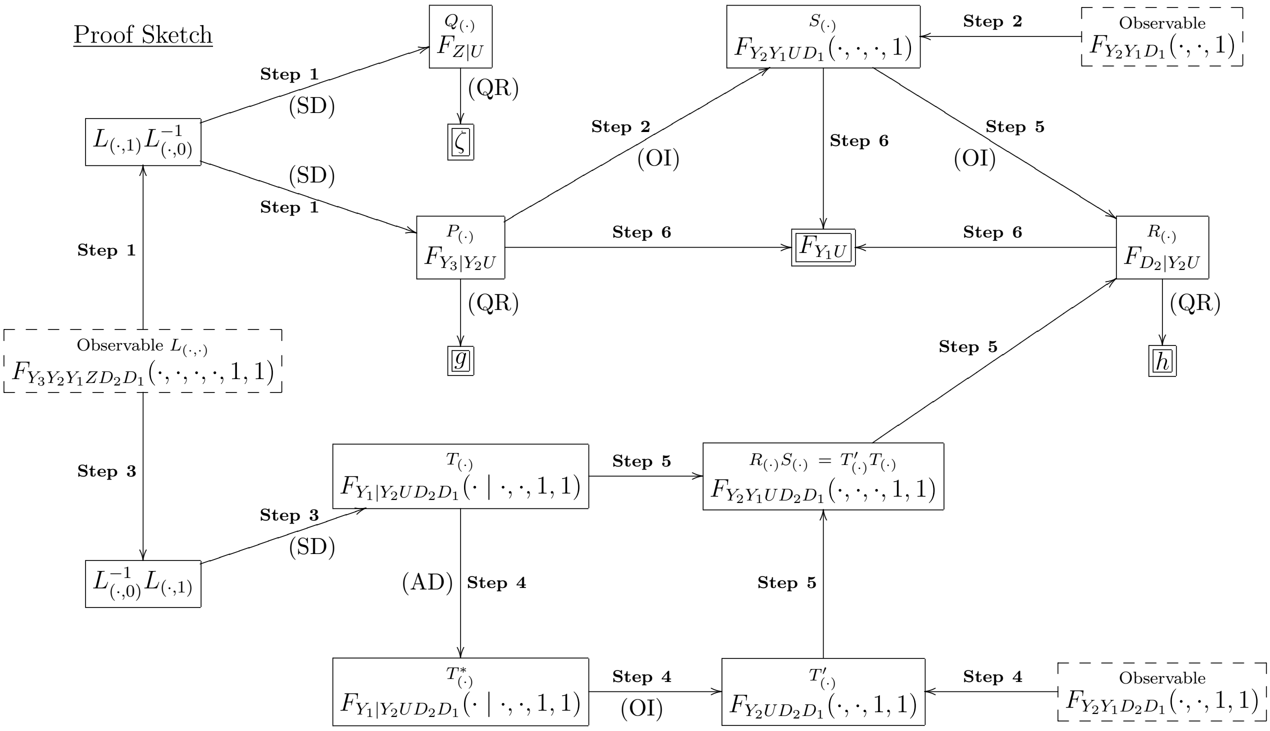
Type II ($U = 1$)

Markov Matrix		$Y_{t-1} = 0$	$Y_{t-1} = 1$
Y_t	0	1.000	0.066
	1	0.000	0.934

2-Year Survival Probability

Y_t	0	0.890 (0.107)
	1	0.983 (0.042)
H_0: equal probability		p-value = 0.416

Table 2: Model estimates with height as a proxy.



(AD) – Adjoint Operator (OI) – Operator Inversion
(QR) – Quantile Representation (SD) – Spectral Decomposition

Figure 1: A sketch of the proof of the identification strategy. The four objects enclosed by double lines are to be identified. The three objects enclosed by dashed lines are observable from data. All the other objects enclosed by solid lines are intermediaries.

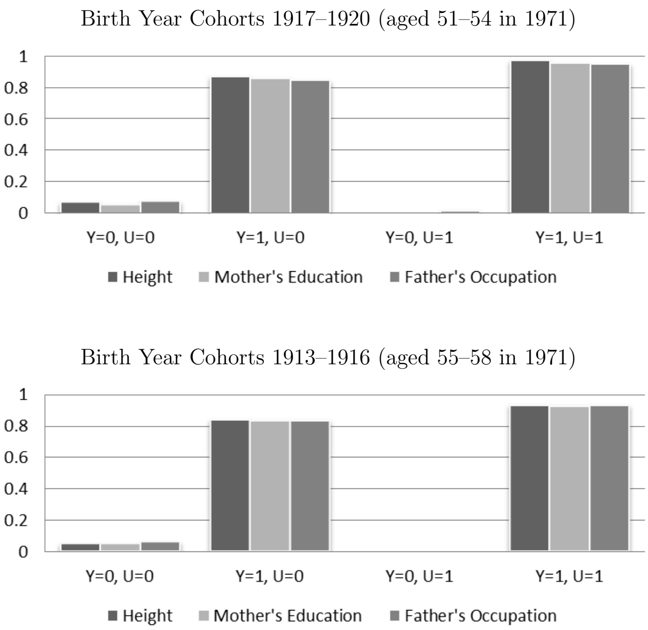


Figure 2: Markov probabilities of employment in the next two years. Colors vary by the type of proxy used.

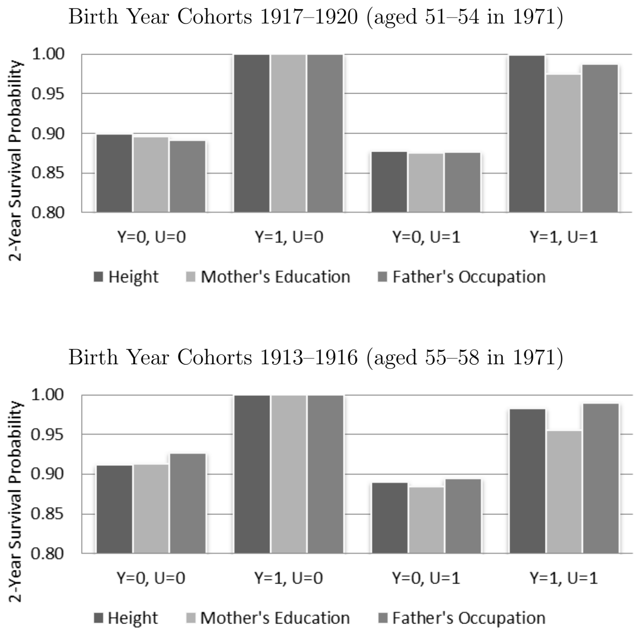


Figure 3: Conditional survival probabilities in the next two years. Colors vary by the type of proxy used.

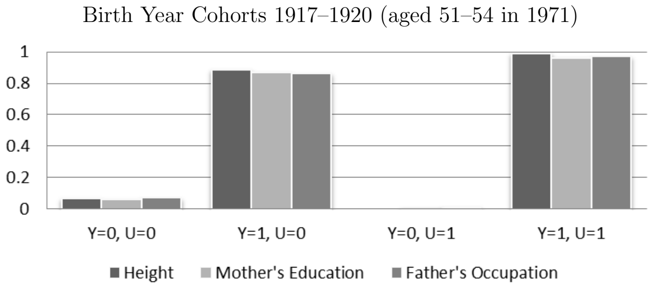


Figure 4: Markov probabilities of employment in the next two years among the subpopulation of individuals who reported health problems that limit work in 1971. Colors vary by the type of proxy used.

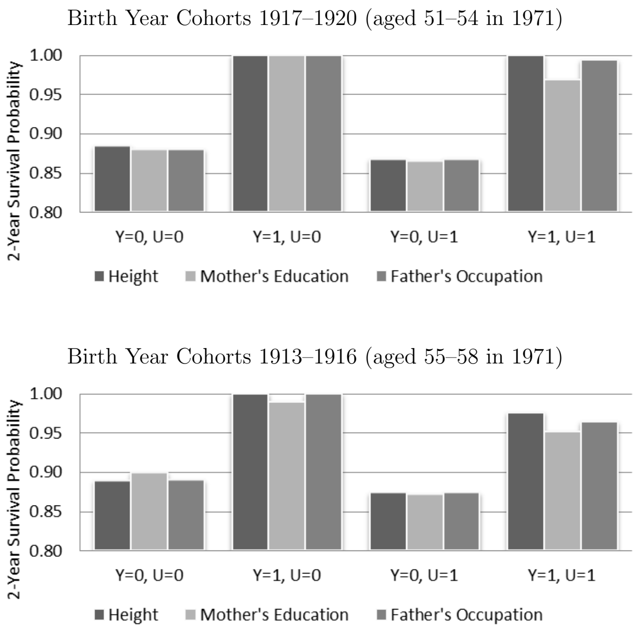


Figure 5: Conditional survival probabilities in the next two years among the subpopulation of individuals who reported health problems that limit work in 1971. Colors vary by the type of proxy used.

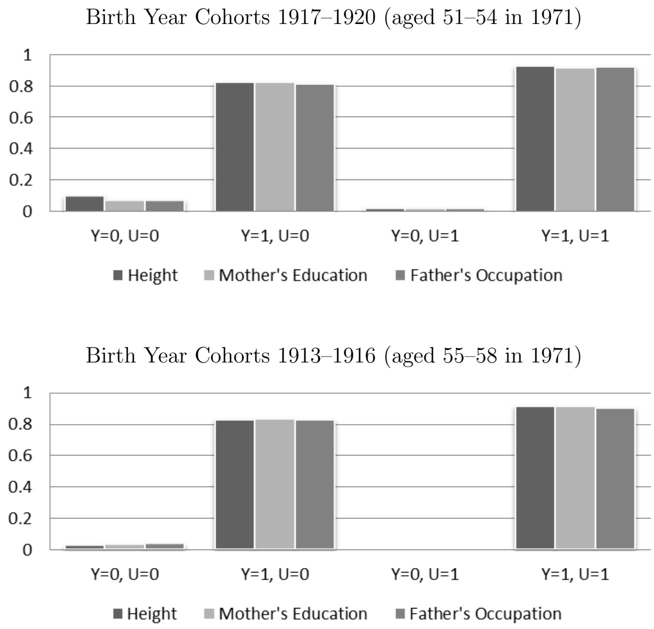


Figure 6: Markov probabilities of employment in the next two years among the subpopulation of individuals who eventually died from acute diseases according to death certificates. Colors vary by the type of proxy used.

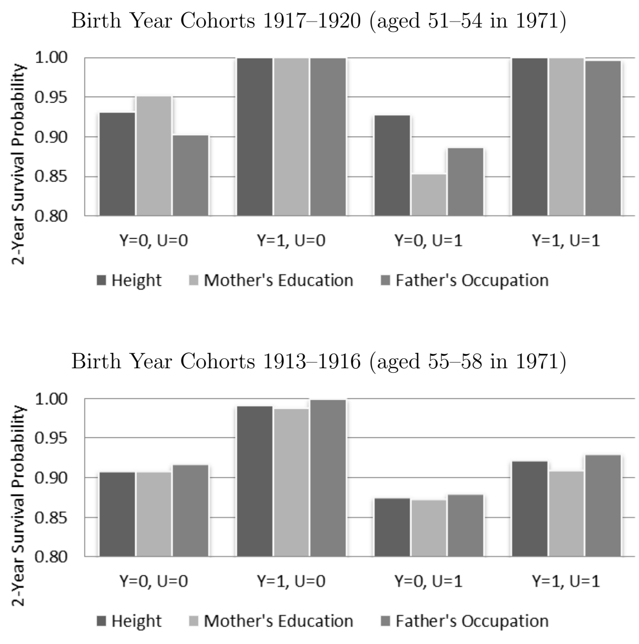


Figure 7: Conditional survival probabilities in the next two years among the subpopulation of individuals who eventually died from acute diseases according to death certificates. Colors vary by the type of proxy used.

Figure 8: Black lines indicate the actual employment rates among survivors in the data. Grey lines indicate the counterfactual employment rates if all the individuals alive in the first period in the data were to remain alive throughout the entire period.