SUPPLEMENTAL MATERIAL 1:

THE STRUCTURE OF THE SUPPLEMENTAL MATERIALS AND EXACT CONDITIONS FOR THEOREM 1

In the following Supplemental Materials, we prove Theorem 1 (folk theorem) for a general game without cheap talk or public randomization in steps. Remember that the arguments in the main text before Section 8 are valid for all the steps.

We offer an overview of the structure and summarize exactly what generic conditions we need to prove theorem 1 in each step.

16 Structure

First, we show Theorem 1 for a general two-player game with the perfect cheap talk, noisy cheap talk and public randomization.

Second, we show Theorem 1 for a general game with more than two players with the perfect cheap talk, noisy cheap talk and public randomization.

Third, we dispense with the perfect cheap talk, noisy cheap talk and public randomization in the two-player game. We proceed in steps. In the coordination block, we replace the perfect cheap talk with the noisy cheap talk. Then, we dispense with the noisy cheap talk in the coordination and main blocks. On the other hand, in the report block, we dispense with public randomization, after which we replace the perfect cheap talk with conditionally independent noisy cheap talk. Then, we dispense with the conditionally independent cheap talk.

Fourth, we dispense with the perfect cheap talk, noisy cheap talk and public randomization in the more-than-two-player game. The main difference from the two-player case is how to construct the coordination block without the perfect cheap talk but with the noisy cheap talk.

17 Assumptions

Given the above structure, we mention what generic assumptions we need to prove Theorem 1 in each step. Again, all the assumptions are generic under Assumption 2.

17.1 Two-Player General Games with Cheap Talk and Public Randomization

In the two-player general game with the perfect cheap talk, noisy cheap talk and public randomization, no additional assumption is necessary, that is, Assumptions 1, 3, 4 and 5 are sufficient.

17.2 More-Than-Two-Player General Games with Cheap Talk and Public Randomization

With more than two players, we modify Assumptions 4 and 5 to deal with the fact that in addition to player i and her monitor player $i-1$, there are players $-(i-1,i)$.

Assumptions 1 and 3 are maintained as it is. We replace Assumption 4 with its counterparts for more than two players, Assumptions 6, 7 and 8. In addition, Assumption 5 is replaced with Assumption 9.

17.3 Two-Player General Games withOUT Cheap Talk

Now we consider how to dispense with the perfect cheap talk, noisy cheap talk and public randomization device in a general two-player game.

17.3.1 Coordination Block

In the main paper, each player communicates x_i via perfect cheap talk.

Noisy Cheap Talk First, we replace the perfect cheap talk in the coordination block with the noisy cheap talk. To do so, we do not need any new assumptions.

Messages via Actions Second, we replace the noisy cheap talk with messages via actions. Since we replace the perfect cheap talk in the coordination block with the noisy cheap talk, this step enables us to dispense with the perfect cheap talk in the coordination block and the noisy cheap talk in the main blocks. In this step, we need an assumption to make sure that we can create a message protocol to preserve the important features of the inferences which were guaranteed with the noisy cheap talk (see Lemma 2). A sufficient condition is Assumption 10.

17.3.2 Report Block

In the main paper, the players coordinate on who will report the history by the public randomization device. In addition, the picked player reports the history via perfect cheap talk.

Dispensing with Public Randomization We first dispense with the public randomization device. So that the players can coordination through their actions and private signals, we need Assumption 11.

Conditionally Independent Noisy Cheap Talk We second replace the perfect cheap talk with conditionally independent noisy cheap talk. For this step, no new assumption is necessary (except for the availability of the conditionally independent noisy cheap talk).

Messages via Actions We third replace the conditionally independent noisy cheap talk with messages via actions. To do so, we need to create a statistics of a receiver to infer the messages from a sender so that the sender cannot get any information about the realization of the statistics through her private signals. See Assumption 12. Note that we do *not* assume that $2|Y_i| \leq |A_j||Y_j|$ for all i, j with $i \neq j$. Hence, we cannot use the method that Fong, Gossner, Hörner and Sannikov (2010) create $\lambda^j(y^j)$ in their Lemma 1, which preserves the conditional independence property.

17.4 More-Than-Two-Player General Games withOUT Cheap Talk

Finally, we consider how to dispense with the perfect cheap talk, noisy cheap talk and public randomization device in a general more-than-two-player game.

17.4.1 Coordination Block

In the Supplemental Material 3, each player communicates x_i via perfect cheap talk in the coordination block.

Noisy Cheap Talk We first replace the perfect cheap talk with the noisy cheap talk. As we have explained in Section 4, with more than two players, it is important to create a message protocol so that, while the players exchange messages and infer the other players' messages in order to coordinate on x_i, there is no player who can induce a situation where some players infer x_i is G while the others infer x_i is B. Since the signals from the noisy cheap talk is private as we will see in Section 29, we need a more sophisticated communication protocol than the case with two players. For that purpose, we add Assumption 14.

Messages via Actions Then, we replace the noisy cheap talk with messages via actions. So that we can create a message protocol to preserve the important features that were satisfied by the noisy cheap talk, we need Assumption 15.

17.4.2 Report Block

We need the more-than-two-player-case counterparts of Assumptions 11 and 12 to dispense with the public randomization and perfect cheap talk in the report block. So that the players can coordinate through their actions and private signals, we add Assumption 16.

In addition, to construct a statistics to preserve the conditional independence property, we need Assumption 17.

SUPPLEMENTAL MATERIAL 2:

PROOF OF THEOREM 1 for a General Two-Player Game With CHEAP TALK

In this Supplemental Material, we prove Theorem 1 (folk theorem) for a general two-player game with the perfect cheap talk, noisy cheap talk and public randomization devices.

Since there are only two players, when we say players i and j, unless otherwise specified, player i is different from player j.

18 Valid Lemmas

Since we maintain Assumptions 3, 4 and 5, Lemmas 3, 4, 5 and 6 are still valid. Also, since the noisy cheap talk is available, Lemma 2 holds.

19 Intuitive Explanation

The basic structure is the same as in the prisoners' dilemma: In each finitely repeated game, there are L review rounds and several supplemental rounds. In each review round, player j monitors player i by making a reward function linearly increase in player j's score about player i: $X_j(l) = \sum_t \Psi_{j,t}^{a(x)}$. If the realization of $\sum_t \Psi_{j,t}^{a(x)}$ is far from the ex ante mean, then player j will switch to a constant reward.

Remember that player i with $x_i = B$ and $\lambda_i(l) = B$ needs to give a *non-negative constant* reward. On the other hand, player i with harsh strategy $\sigma_i(B)$ needs to ensure that player j's payoff is below $\underline{v}_j$ *regardless of player* j*'s strategy*.

In the prisoners' dilemma, $a_i(x)$ with $x_i = B$ defined to satisfy (6) and (7) happens to be the minimiaxing action. Hence, player j's payoff is below $\underline{v}_j$ regardless of player j's strategy with non-negative reward. However, in a general game, $a_i(x)$ with $x_i = B$ is not always a minimaxing action. Therefore, player i with $x_i = B$ and $\lambda_i(l) = B$ needs to switch to the minimaxing action if player i thinks that player j has deviated.

For this purpose, in each lth review round, player j constructs a statistics such that if its realization is low, then player j *allows player i to minimax player j* from the next review round, that is, from the $(l+1)$th review round. See "player j's score about player j's own deviation" in Section 4 for the intuitive explanation.

The key lemma to construct such a statistics is as follows:

Lemma 10 *If Assumption 4 is satisfied, then there exist $q_2 > q_1$ such that, for all $j \in I$ and $a \in A$, there exists a function $\gamma_j^a : Y_j \to (0,1)$ such that,*

1. *The ex ante value of player i's conditional expectation of $\gamma_j^a(y_j)$ distinguishes whether player j takes a_j or not:*

$$\begin{aligned}
\mathbb{E}_{y_i}\left[\mathbb{E}_{y_j}\left[\gamma_j^a(y_j) \mid a, y_i\right] \mid \tilde{a}_j, a_i\right] &\equiv \sum_{y_j}\left\{\sum_{y_i} q(y_j \mid a, y_i) q(y_i \mid \tilde{a}_j, a_i)\right\}\gamma_j^a(y_j) \\
&= \begin{cases} q_2 & \text{if } \tilde{a}_j = a_j, \\ q_1 & \text{otherwise.} \end{cases}
\end{aligned}$$

2. *Player i cannot change the expected value of γ_j^a: For all $\tilde{a}_i \in A_i$,*

$$\mathbb{E}\left[\gamma_j^a(y_j) \mid \tilde{a}_i, a_j\right] \equiv \sum_{y_j} q(y_j \mid a_j, \tilde{a}_i)\gamma_j^a(y_j) = q_2.$$

Proof. It suffices that $Q_2(\tilde{a}_j, a_i)$ with $\tilde{a}_j \in A_i$ and $Q_1(\tilde{a}_i, a_j)$ with $\tilde{a}_i \neq a_i$ are linearly independent (see the proof of Lemma 3 for the definition of Q_1 and Q_2). Assumption 4 guarantees this.

Further, we can assume that q_1 and q_2 in Lemmas 3 and 10 are the same after applying an appropriate affine transformation of ψ's and γ's. The same caution is applicable whenever we say q_1 and q_2. ■

For each lth review round, player j allows player i to minimax player j from the $(l+1)$th round if "player j's score about player j's own deviation," $\sum_{t \in T(l)} \gamma_j^{a(x)}(y_{j,t})$, is very low. Then, player i with lower conditional expectation of $\sum_{t \in T(l)} \gamma_j^{a(x)}(y_{j,t})$ is willing to minimax player j as seen in Section 4. Since the conditional expectation decreases if player j deviates from Condition 1, the punishment is triggered properly. In addition, Condition 2 guarantees that player i cannot manipulate this statistics.

Intuitively, $d_j(l+1) \in \{G, B\}$ indicates whether or not player j allows player i to minimax: $d_j(l+1) = G$ implies that player j does not allow player i to minimax player j while $d_j(l+1) = B$ implies that player j allows after observing low $\sum_{t \in T(l)} \gamma_j^{a(x)}(y_{j,t})$.

On the other hand, as player i constructs $\hat{\lambda}_j(l+1)$ to infer $\lambda_j(l+1)$, player i constructs $\hat{d}_j(l+1) \in \{G, B\}$ to infer $d_j(l+1)$. $\hat{d}_j(l+1) = G$ implies that player i is not willing to minimax while $\hat{d}_j(l+1) = B$ implies that player i believes that player j allows player i to minimax player j and so player i is willing to minimax.

20 Structure of the Phase

We introduce supplemental rounds for $d_1(l+1)$ and $d_2(l+1)$ so that, in the supplemental rounds for $d_i(l+1)$, player i can send $d_i(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.

In addition, player j with $\lambda_j(l+1) = B$ and $\hat{d}_i(l+1) = B$ minimaxes player i in the $(l+1)$th review round. Since player j's reward function is constant after $\lambda_j(l+1) = B$, player i wants to take the best response to player j's action. To best respond to player j, player i wants to know $\hat{d}_i(l+1)$ which indicates whether player j minimaxes player i or not. Therefore, we also introduce supplemental rounds for $\hat{d}_2(l+1)$ and $\hat{d}_1(l+1)$ so that player j can send $\hat{d}_i(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$. The truthtelling incentive will be verified later.

Therefore, the whole structure of the phase is as follows:

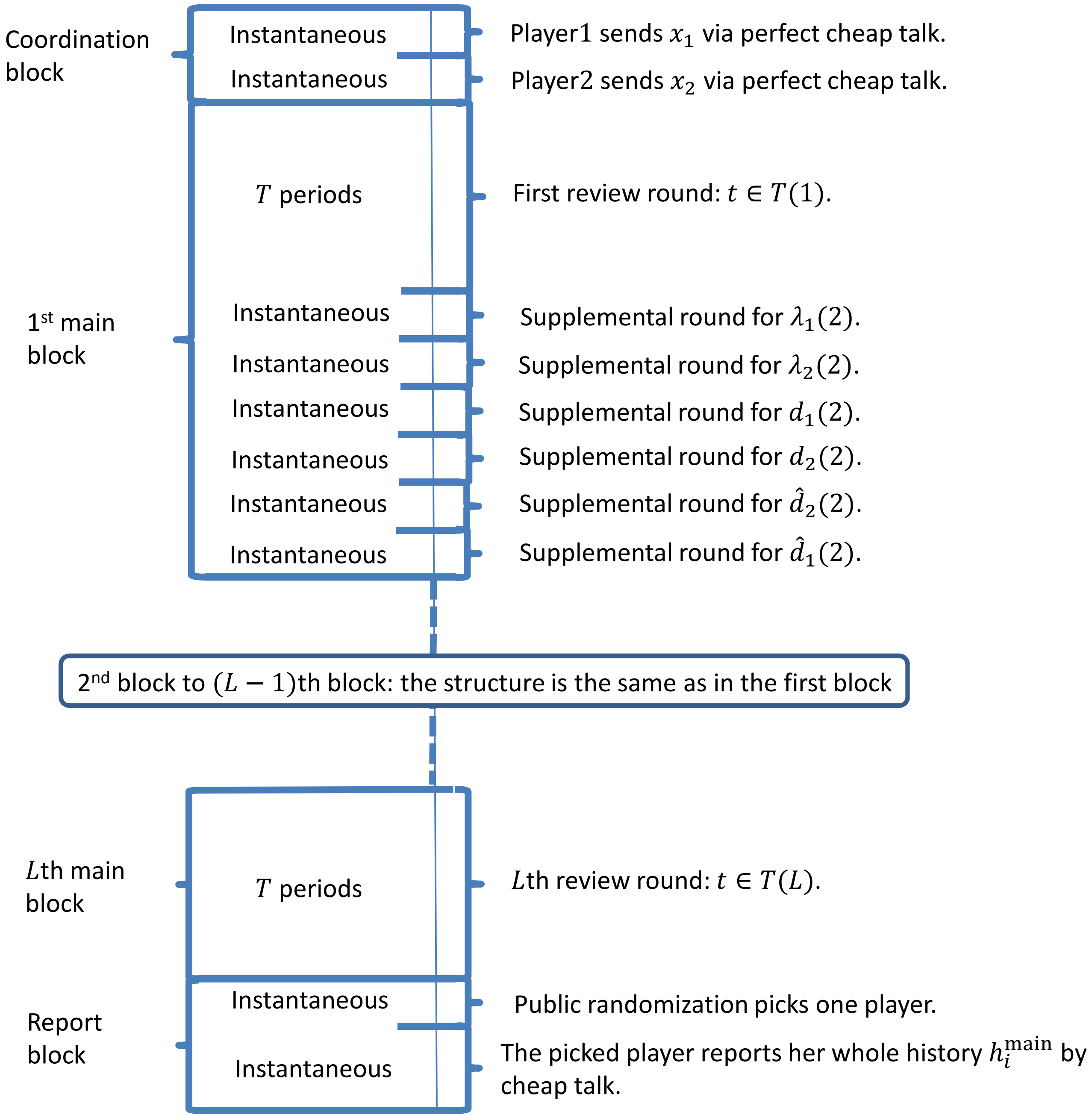

Figure 1 of the Supplemental Material 2:
Structure of the Phase

21 Strictness

For the rest of the proof, we assume that the static best response is unique and that the minimaxing action plan is unique. That is, we assume that there exists $\underline{u} > 0$ such that

1. For any i and $a_j \in A_j$, there exists a unique $BR_i(a_j)$ such that, for any $\tilde{a}_i \neq BR_i(a_j)$,

$$u_i(BR_i(a_j), a_j) - u_i(\tilde{a}_i, a_j) > \underline{u}. \tag{68}$$

   This is without loss: Otherwise, with $\{BR_i(a_j)\}_{a_j}$,[62] we can add a small reward $-\sum_t \frac{\underline{u}}{q_2 - q_1}\left(1 - \Psi_{j,t}^{BR_j(a_{j,t}), a_{j,t}}\right) < 0$ if $x_j = G$ or $\sum_t \frac{\underline{u}}{q_2 - q_1}\Psi_{j,t}^{BR_j(a_{j,t}), a_{j,t}} > 0$ if $x_j = B$ to restore this property. If we take $\underline{u}$ small, then (3), (4) and (5) are still satisfied.

2. For any i, there exists $\alpha_j^{\min\max} \in \Delta(A_j)$ such that

   (a) For any a_i, $u_i(a_i, \alpha_j^{\min\max}) \leq v_i^*$. That is, $\alpha_j^{\min\max}$ is a minimaxing strategy.

   (b) For any $a_j \in A_j$ (pure action plan) with $a_j \neq \alpha_j^{\min\max}$,

$$u_i(BR_i(a_j), a_j) - u_i(BR_i(\alpha_j^{\min\max}), \alpha_j^{\min\max}) > \underline{u}. \tag{69}$$

   Again, this is without loss: We can always add a small reward $-\sum_t \mathbf{1}\left\{\alpha_{j,t} = \alpha_j^{\min\max}\right\}\underline{u} < 0$ if $x_j = G$ or $\sum_t \mathbf{1}\left\{\alpha_{j,t} \in A_j \setminus \{\alpha_j^{\min\max}\}\right\}\underline{u}$ if $x_j = B$ to have this property, keeping (3), (4) and (5).

22 Perfect Monitoring

In this section, we consider a *one-shot* game with *perfect monitoring* parameterized with $l \in \mathbb{N}$ with the same sets of players and their possible actions. The result of this section is used when we consider how player j punishes player i in the T_P*-period* finitely repeated game with *private monitoring*.

[62] If there are multiple best responses to a_j, pick one arbitrarily for $BR_i(a_j)$.

In the game with parameter $l \in \{1, ..., L-1\}$, player j takes $a_j(x)$. Depending on player i's action, $\hat{d}_i(l+1) \in \{G, B\}$ is determined. If player i takes $a_i(x)$, then $\hat{d}_i(l+1) = G$ with probability one. If player i takes $a_i \neq a_i(x)$, then $\hat{d}_i(l+1) = B$ with probability $p_j^{l+1}(x)$ and $\hat{d}_i(l+1) = G$ with the remaining probability $1 - p_j^{l+1}(x)$. The payoff of player i is determined as

$$V_i^l = \max_{a_i} \frac{1}{L-l+1} u_i(a_i, a_j(x)) + \frac{L-l}{L-l+1} \mathbb{E}\left[W_i^{l+1}(\hat{d}_i(l+1)) \mid a_i, a_j(x)\right]$$

with

$$\begin{aligned} u_i^*(x) &= u_i(BR_i(a_j(x)), a_j(x)), && (70) \\ W_i^{l+1}(G) &= \frac{(L-l-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l} + \eta, && (71) \\ W_i^{l+1}(B) &= v_i^* + \eta, && (72) \end{aligned}$$

where $\eta > 0$ is a small number defined in Section 24.

In this game, we can show the following lemma:

Lemma 11 *For any $L \geq 2$, $q_2 > q_1$ and $\eta > 0$, there exist $\varepsilon^{\min\max} > 0$ and $\{\bar{p}_j^{l+1}(x)\}_{l=1}^{L-1} \in [0,1]$ such that, for any $\varepsilon < \varepsilon^{\min\max}$ and $\{p_j^{l+1}(x)\}_{l=1}^{L-1}$ with*

$$p_j^{l+1}(x) \in \left[\bar{p}_j^{l+1}(x) \frac{q_2 - q_1 - 2\varepsilon}{q_2 - q_1}, \bar{p}_j^{l+1}(x)\right]$$

for all $l = 1, ..., L-1$, it is uniquely optimal for player i to take $BR_i(a_j(x))$ and

$$V_i^l \leq W_i^l(G) = \frac{(L-l)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l+1} + \eta.$$

Proof. If $a_j(x)$ is $\alpha_j^{\min\max}$, then $w_i(x) \leq v_i^* = u_i^*(x)$ and

$$W_i^{l+1}(G) = W_i^{l+1}(B) = v_i^* + \eta.$$

Hence, for any $p_j^{l+1}(x) \in [0,1]$, it is uniquely optimal for player i to take $BR_i(a_j(x))$ and

$$V_i^l = v_i^* + \eta \leq W_i^l(G)$$

for all $l \in \{1, ..., L-1\}$ as desired. Uniqueness follows from (68): Since $\alpha_j^{\min\max} = a_j(x) \in A_j$ is pure, the static best response is unique.

Hence, we concentrate on the case with $a_j(x) \neq \alpha_j^{\min\max}$. Then, from (69), we have $u_i^*(x) > v_i^*$ and so

$$W_i^{l+1}(G) = \frac{(L-l-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l} + \eta > v_i^* + \eta = W_i^{l+1}(B) \tag{73}$$

for all $l \in \{1, ..., L-1\}$.

Further, if $BR_i(a_j(x)) = a_i(x)$, then with $\bar{p}_j^{l+1}(x) = p_j^{l+1}(x) = 0$, it is uniquely optimal for player i to take $BR_i(a_j(x))$ and

$$\begin{aligned} V_i^l &= \frac{1}{L-l+1}u_i^*(x) + \frac{L-l}{L-l+1}\left(\frac{(L-l-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l} + \eta\right) \\ &< \frac{1}{L-l+1}w_i(x) + \frac{(L-l-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l+1} + \eta \\ &\leq W_i^l(G) \end{aligned}$$

for all $l \in \{1, ..., L-1\}$ as desired. The strict inequality follows from the following two: (i) Since $BR_i(a_j(x)) = a_i(x)$, $u_i^*(x)$ is equal to w_i. (ii) $\eta > 0$ and $\frac{L-l}{L-l+1} < 1$.

Hence, we concentrate on the case with $u_i^*(x) > v_i^*$ and $BR_i(a_j(x)) \neq a_i(x)$. The latter means that $w_i(x) < u_i^*(x)$ from (68), which implies that $W_i^{l+1}(G) > W_i^{l+1}(B)$ for all $l \in \{1, ..., L-1\}$.

Note that the value after taking $a_i(x)$ is

$$\frac{1}{L-l+1}w_i(x) + \frac{L-l}{L-l+1}W_i^{l+1}(G)$$

for all $l \in \{1, ..., L-1\}$. On the other hand, if player i takes $BR_i(a_j(x))$, then the value is

$$\frac{1}{L-l+1} u_i^*(x) + \frac{L-l}{L-l+1}\left(1 - p_j^{l+1}(x)\right) W_i^{l+1}(G) + \frac{L-l}{L-l+1} p_j^{l+1}(x) W_i^{l+1}(B)$$

for all $l \in \{1, ..., L-1\}$. If

$$p_j^{l+1}(x) < \frac{u_i^*(x) - \max\{w_i(x), v_i^*\}}{(L-l-1)(\max\{w_i(x), v_i^*\} - v_i^*) + u_i^*(x) - v_i^*}$$

for all $l \in \{1, ..., L-1\}$, then it is strictly optimal to take $BR_i(a_j(x))$.

On the other hand, if

$$p_j^{l+1}(x) = \frac{u_i^*(x) - \max\{w_i(x), v_i^*\}}{(L-l-1)(\max\{w_i(x), v_i^*\} - v_i^*) + u_i^*(x) - v_i^*} \in (0,1), \tag{74}$$

then we have

$$V_i^l = \frac{(L-l)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l+1} + \frac{L-1}{L-l+1}\eta < W_i^l(G). \tag{75}$$

(74) follows from $w_i(x) < u_i^*(x)$. Since the last inequality in (75) is strict, if we take $\bar{p}_j^{l+1}(x)$ sufficiently close but smaller than

$$\frac{u_i^*(x) - \max\{w_i(x), v_i^*\}}{(L-l-1)(w_i(x) - \max\{w_i(x), v_i^*\}) + u_i^*(x) - v_i^*},$$

then the statement of this lemma holds for $\left\{\bar{p}_j^{l+1}(x)\right\}_{l=1}^{L-1}$. Taking ε sufficiently small, we are done. ■

23 Equilibrium Strategy

As in Section 11, we define $\sigma_i(x_i)$ and $\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$. In Section 23.1, we define the state variables that will be used to define the action plans and rewards. Given the states, Section 23.2 defines the action plan $\sigma_i(x_i)$ and Section 23.3 defines the reward function

$\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$. Finally, Section 23.4 determines the transition of the states defined in Section 23.1.

23.1 States x_i, $\lambda_i(l+1)$, $\hat{\lambda}_j(l+1)$, $d_i(l+1)$, $\hat{d}_j(l+1)$, $\hat{d}_i(l+1)(i)$, $\theta_i(l)$, $\theta_i(\lambda_i(l+1))$, $\theta_i(d_i(l+1))$ and $\theta_i(\hat{d}_j(l+1))$

The state $x_i \in \{G, B\}$ is determined at the beginning of the phase and fixed. By the perfect cheap talk, x becomes common knowledge.

As in the prisoners' dilemma, $\lambda_i(l+1) \in \{G, B\}$ is player i's state, indicating whether player i's score about player j has been erroneous. On the other hand, $\hat{\lambda}_j(l+1) \in \{G, B\}$ indicates whether player i believes that $\lambda_j(l+1) = G$ or $\lambda_j(l+1) = B$ is likely.

As seen in Section 19 (with the roles of players i and j reversed), $d_i(l+1) \in \{G, B\}$ is player i's state, indicating whether or not player i allows player j to minimax player i.

On the other hand, when player i decides whether or not to minimax player j in the $(l+1)$th review round, it is natural to calculate the belief about $d_j(l+1) = G$. The space for player i's possible beliefs in each period t in the $(l+1)$th review round is $[0, 1]$ and it depends on the details of a history h_i^t. However, we classify the set of player i's histories into two partitions: The set of histories labeled as $\hat{d}_j(l+1) = G$ and that labeled as $\hat{d}_j(l+1) = B$. If and only if $\hat{d}_j(l+1) = B$, player i believes that $d_j(l+1) = B$ (player j allows player i to minimax) and is willing to minimax player j.

To make the equilibrium tractable, $\hat{d}_j(l+1)$ depends only on player i's history at the beginning of the $(l+1)$th review round and is fixed during the $(l+1)$th review block, as $\hat{\lambda}_j(l+1)$.

As we have just mentioned, player j with $\hat{d}_i(l+1) = B$ minimaxes player i in the $(l+1)$th review round (the indices i and j are reversed). With $\hat{\lambda}_j(l+1) = B$, that is, when player i believes that the reward function is constant, player i wants to take the best response to player j's action. To best respond to player j, player i wants to know whether $\hat{d}_i(l+1)$ is G or B (that is, whether player j takes $a_j(x)$ or minimaxes player i). Therefore, we classify the set of player i's histories into two partitions: One with $\hat{d}_i(l+1)(i) = G$ and the other with $\hat{d}_i(l+1)(i) = B$. Intuitively, player i believes $\hat{d}_i(l+1)(i) = \hat{d}_i(l+1)$. Player i best responses to $a_j(x)$ if player i believes that $\hat{d}_i(l+1) = G$ (that is, if $\hat{d}_i(l+1)(i)$ is G) and best responses to $\alpha_j^{\min\max}$ if player i believes that $\hat{d}_i(l+1) = B$ (that is, if $\hat{d}_i(l+1)(i)$ is B).

Further, as in the prisoners' dilemma, player i makes player j indifferent between any action profile sequence after some history. If she does in the lth review round, then π_j^{main} will be $\sum_\tau \pi_j^{x_i}(a_{i,\tau}, y_{i,\tau})$ for period τ in the lth review round and after. $\theta_i(l) \in \{G, B\}$, $\theta_i(\lambda_i(l+1)) \in \{G, B\}$, $\theta_i(d_i(l+1)) \in \{G, B\}$ and $\theta_i(\hat{d}_j(l+1)) \in \{G, B\}$ are indices of whether player i uses such a reward because of the history in the lth main block. See Section 11.3 for how the reward function depends on these four states.

23.2 Player i's Action

In the coordination block, each player sends x_i truthfully via perfect cheap talk. Then, the state profile x becomes common knowledge.

In the lth review round, player i's strategy depends on $\hat{\lambda}_j(l)$, $\hat{d}_j(l)$ and $\hat{d}_i(l)(i)$. If $\hat{\lambda}_j(l) = G$, then player i takes $a_i(x)$ if $\hat{d}_j(l) = G$ and $\alpha_i^{\min\max}$ if $\hat{d}_j(l) = B$. If $\hat{\lambda}_j(l) = B$, then player i takes $BR_i(a_j(x))$ if $\hat{d}_i(l)(i) = G$ and $BR_i(\alpha_j^{\min\max})$ if $\hat{d}_i(l)(i) = B$.

In the supplemental rounds for $\lambda_i(l+1)$, $d_i(l+1)$ and $\hat{d}_j(l+1)$, respectively, player i sends the message $\lambda_i(l+1)$, $d_i(l+1)$ and $\hat{d}_j(l+1)$, respectively, truthfully via noisy cheap talk with precision $p = \frac{1}{2}$.

23.3 Reward Function

In this subsection, we explain player j's reward function on player i, $\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$.

Score First, since Lemma 3 is valid, we can define $X_j(l)$ as in (27).

Slope Second, take $\bar{L}$ such that (28) holds.

Reward Function As in the prisoners' dilemma, the reward $\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$ is written as

$$\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta) = \sum_{l=1}^{L} \sum_{t \in T(l)} \pi_i^{\delta}(t, a_{j,t}, y_{j,t}) + \begin{cases} -\bar{L}T + \sum_{l=1}^{L} \pi_i^{\text{main}}(x, h_j^{\text{main}}, l) & \text{if } x_j = G, \\ \bar{L}T + \sum_{l=1}^{L} \pi_i^{\text{main}}(x, h_j^{\text{main}}, l) & \text{if } x_j = B. \end{cases} \tag{76}$$

Remember that $T(l)$ is the set of periods in the lth review round.

Reward Function for Each Round If $\theta_j(\tilde{l}) = B$, $\theta_j(\lambda_j(\tilde{l}+1)) = B$, $\theta_j(d_j(\tilde{l}+1)) = B$ or $\theta_j(\hat{d}_i(\tilde{l}+1)) = B$ happens for some $\tilde{l} \leq l-1$, then player j makes player i indifferent between any action profile by

$$\pi_i^{\text{main}}(x, h_j^{\text{main}}, l) = \sum_{t \in T(l)} \pi_i^{x_j}(a_{j,t}, y_{j,t}). \tag{77}$$

Remember that $\pi_i^{x_j}$ is defined in Lemma 5.

Otherwise, that is, if $\theta_j(\tilde{l}) = \theta_j(\lambda_j(\tilde{l}+1)) = \theta_j(d_j(\tilde{l}+1)) = \theta_j(\hat{d}_i(\tilde{l}+1)) = G$ for all $\tilde{l} \leq l-1$, then player j's reward on player i is based on the state profile x, the index of the past erroneous history $\lambda_j(l)$, index of minimaxing $\hat{d}_i(l)$ and player j's score about player i, $X_j(l)$:

$$\pi_i^{\text{main}}(x, h_j^{\text{main}}, l) = \begin{cases} \bar{\pi}_i(x, G, \hat{d}_i(l), l) + \bar{L}\left(X_j(l) - (q_2 T + 2\varepsilon T)\right) & \text{if } x_j = G \text{ and } \lambda_j(l) = G, \\ \bar{\pi}_i(x, B, \hat{d}_i(l), l) & \text{if } x_i = G \text{ and } \lambda_j(l) = B, \\ \bar{\pi}_i(x, G, \hat{d}_i(l), l) + \bar{L}\left(X_j(l) - (q_2 T - 2\varepsilon T)\right) & \text{if } x_j = B \text{ and } \lambda_j(l) = G, \\ \bar{\pi}_i(x, B, \hat{d}_i(l), l) & \text{if } x_j = B \text{ and } \lambda_j(l) = B. \end{cases} \tag{78}$$

$\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ will be determined later so that (3), (4) and (5) are satisfied.

23.4 Transition of the States

In this subsection, we explain the transition of player i's states. Since x_i is fixed in the phase, we consider the following nine states:

23.4.1 Transition of $\lambda_j(l+1) \in \{G, B\}$

The transition of $\lambda_j(l+1) \in \{G, B\}$ is exactly the same as in Section 11.4.1.

23.4.2 Transition of $\hat{\lambda}_j(l+1) \in \{G, B\}$

The transition of $\hat{\lambda}_j(l+1) \in \{G, B\}$ is the same as in Section 11.4.2 except that, if player i has $\hat{\lambda}_j(l) = G$ and $\hat{d}_j(l) = B$, then $\hat{\lambda}_j(l+1) = G$ with high probability. As explained in Section 19, if $\hat{d}_j(l) = B$, then player i believes that player j allows player i to minimax player j, that is, any action is optimal for player i. Hence, the belief about $\lambda_j(l)$ is irrelevant.

That is, $\hat{\lambda}_j(1) = G$. If $\hat{\lambda}_j(l) = B$, then $\hat{\lambda}_j(l+1) = B$. If $\hat{\lambda}_j(l) = G$, then $\hat{\lambda}_j(l+1) \in \{G, B\}$ is defined as follows:

1. If (35) and (36) are satisfied or $\hat{d}_j(l) = B$, then player i randomly picks the following two procedures:
   (a) With large probability $1 - \eta$, player i has $\hat{\lambda}_j(l+1) = G$ regardless of the signals of the noisy cheap talk about $\lambda_j(l+1)$.
   (b) With small probability $\eta > 0$, player i will use the signal from the noisy cheap talk message: $\hat{\lambda}_j(l+1)$ is determined by (39). This is almost optimal by the same reason as in Section 11.4.2.
2. If "(35) is not satisfied or (36) is not satisfied" and $\hat{d}_j(l) = G$, then $\hat{\lambda}_j(l+1)$ is determined by (39). As 1-(b), this is almost optimal.

23.4.3 Transition of $d_j(l+1) \in \{G, B\}$

As Ψ_j^a and ψ_j^a, we firstly define $\Gamma_j^a \in \{0,1\}$ from $\gamma_j^a(y_j)$. Player j, after believing a being taken and observing y_j, constructs a random variable $\Gamma_j^a \in \{0,1\}$ from $\gamma_j^a(y_j)$ as player j constructs Ψ_j^a from $\psi_j^a(y_j)$.

Given $\left\{\Gamma_{j,t}^a\right\}_{t \in T(l)}$, we define the transition as follows: The initial condition is $d_j(1) = G$. For $l = 1, ..., L-1$, if $d_j(l) = B$, then $d_j(l+1) = B$ as for $\lambda_j(l+1)$. If $d_j(l) = G$, then player j calculates

$$G_j(l) = \sum_{t \in T_j(l)} \Gamma_{j,t}^{a(x)} + \mathbf{1}_{t_j(l)}.$$

Again, a random period $t_j(l)$ is excluded from monitoring. This is "player j's score about player j's own deviation" in Section 4. Given $G_j(l)$, $d_j(l+1)$ is determined as

$$d_j(l+1) = \begin{cases} G & \text{if } G_j(l) \in [q_2 T - \varepsilon T, q_2 T + \varepsilon T] \text{ or } \hat{d}_i(l) = B, \\ B & \text{if } G_j(l) \notin [q_2 T - \varepsilon T, q_2 T + \varepsilon T] \text{ and } \hat{d}_i(l) = G. \end{cases}$$

Note that, compared to $\lambda_j(l+1)$, $2\varepsilon T$ is replaced with εT. In addition, if $\hat{d}_i(l) = B$, then player j takes $\alpha_j^{\min\max} \neq a_j(x)$ to minimax player i. Although player j does not take $a_j(x)$, this is not a deviation from the equilibrium strategy. Hence, player j does not allow player i to minimax player j if $\hat{d}_i(l) = B$.

23.4.4 Transition of $\hat{d}_j(l+1) \in \{G, B\}$

As we have mentioned in Section 23.1, $\hat{d}_j(l+1) \in \{G, B\}$ is the partition of the set of player i's histories. Intuitively, player i believes that $d_j(l+1) = \hat{d}_j(l+1)$ with high probability.

Since $d_j(1) = G$ is common knowledge, define $\hat{d}_j(1) = G$.

In addition, $\hat{d}_j(l+1) = B$ once $\hat{d}_j(\tilde{l}) = B$ has happened for some $\tilde{l} \leq l$. Hence, we are left to specify, for each l, conditional on $\hat{d}_j(\tilde{l}) = G$ with all $\tilde{l} \leq l$, how $\hat{d}_j(l+1) \in \{G, B\}$ is determined.

Intuitively, as for $\hat{\lambda}_j(l+1)$, at the end of the lth review round, player i calculates $\mathbb{E}\left[G_j(l) \mid a(x), \{y_{i,t}\}_{t \in T(l)}\right]$. Then, in the supplemental round for $d_j(l+1)$, player j sends

$d_j(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$ and player i receives a signal $f[i](d_j(l+1))$. Based on $\mathbb{E}\left[G_j(l) \mid a(x), \{y_{i,t}\}_{t\in T(l)}\right]$ and $f[i](d_j(l+1))$, player i constructs $\hat{d}_j(l+1)$.

Instead of calculating the conditional expectation of $G_j(l)$ directly, player i calculates

$$\sum_{t\in T_i(l)} \mathbb{E}\left[\Gamma_{j,t}^{a(x)} \mid a(x), y_{i,t}\right] + q_2.$$

As we have mentioned for $\hat{\lambda}_j(l+1)$ in the prisoners' dilemma, player i uses $T_i(l)$, not $T_j(l)$. Since $T_i(l)$ and $T_j(l)$ differ at most for two periods, we can neglect the fact that $T_i(l)$ and $T_j(l)$ can be different for almost optimality.

Further, rather than using $\sum_{t\in T_i(l)} \mathbb{E}\left[\Gamma_{j,t}^{a(x)} \mid a(x), y_{i,t}\right]$, player i constructs $(E_i\Gamma_j^{a(x)})_t \in \{0,1\}$ from $\mathbb{E}\left[\Gamma_{j,t}^{a(x)} \mid a(x), y_{i,t}\right]$ as player i constructs $(E_i\Psi_j^{a(x)})_t$ from $\mathbb{E}\left[\Psi_{j,t}^{a(x)} \mid a(x), y_{i,t}\right]$. Let

$$E_iG_j(l) = \sum_{t\in T_i(l)} (E_i\Gamma_j^{a(x)})_t + q_2.$$

As for $E_iX_j(l)$, for all a_t and y_t, the *ex post* probability *given* $\{a_t, y_t\}_{t\in T(l)}$ that

$$\left|\sum_{t\in T_i(l)} \mathbb{E}\left[\Gamma_{j,t}^{a(x)} \mid a(x), y_{i,t}\right] + q_2 - E_iG_j(l)\right| \leq \frac{1}{4}\varepsilon T. \tag{79}$$

is $1 - \exp(-O(T))$ by the central limit theorem.

Consider player i's belief about $d_j(l+1)$ in the case with (79) and

$$q_2T - 2\varepsilon T > E_iG_j(l). \tag{80}$$

These two implies that

$$\mathbb{E}\left[G_j(l) \mid a(x), \{y_{i,t}\}_{t\in T(l)}, T_i(l), T_j(l)\right] < q_2T - \frac{3}{2}\varepsilon T$$

since $T_i(l)$ and $T_j(l)$ differ at most for two periods.

Since $d_j(l+1) = B$ if $G_j(l) \notin [q_2T - \varepsilon T, q_2T + \varepsilon T]$ and $\hat{d}_i(l) = G$, player i believes that

$d_j(l+1) = B$ if $\hat{d}_i(l) = G$ with probability $1 - \exp(-O(T))$.

On the other hand, forget about the belief about $d_j(l+1)$ and suppose that player i could know $d_j(l+1)$ (she cannot in private monitoring). Consider the two possible realizations of the signals in the supplemental round for $d_j(l+1)$. If $f[i](d_j(l+1)) = d_j(l+1)$, then player i receives a correct message. If $f[i](d_j(l+1)) \neq d_j(l+1)$, then with probability $1 - \exp(-T^{\frac{1}{2}})$, player j should receive the signal telling that player i does not received the correct signal, that is, $g[j](d_j(l+1)) = E$. If $g[j](d_j(l+1)) = E$, then player j will make player i indifferent between any action profile sequence (see Section 23.4.6). Therefore, if player i uses

$$\hat{d}_j(l+1) = f[i](d_j(l+1)), \tag{81}$$

then it can be shown that $\sigma_i(x_i)$ defined in Section 23.2 is almost optimal.

Given the observations above, we consider the following transition of $\hat{d}_j(l+1)$:

1. If (79) is satisfied, then
   (a) If $\lambda_i(l) = G$, then player i randomly picks the following two procedures:
      i. With probability $1 - \eta$, $\hat{d}_j(l+1) = G$.
      ii. With probability η, player i will use the signal from the noisy cheap talk message and $\hat{d}_j(l+1)$ is determined by (81).
   (b) If $\lambda_i(l) = B$, then there are following two cases:
      i. If $x_i = G$, then player i randomly picks the following two procedures:
         A. With probability $1 - \eta$, $\hat{d}_j(l+1) = G$.
         B. With probability η, $\hat{d}_j(l+1)$ is determined by (81).
      ii. If $x_i = B$, then player i randomly picks the following two procedures:
         A. With probability $1-\eta$, player i disregards the signal from the noisy cheap talk and determines $\hat{d}_j(l+1)$ from $E_iG_j(l)$. With probability

$$\bar{p}_i^{l+1}(x) \min\left\{1, \frac{\{q_2T - 2\varepsilon T - E_iG_j(l)\}_+}{q_2T - q_1T}\right\}, \tag{82}$$

$\hat{d}_j(l+1) = B$ and with the remaining probability, $\hat{d}_j(l+1) = G$. Remember that $\bar{p}_i^{l+1}(x)$ is determined in Lemma 11.

B. With probability η, $\hat{d}_j(l+1)$ is determined by (81).

2. If (79) is not satisfied, then $\hat{d}_j(l+1)$ is determined by (81).

There are several remarks: First, 1-(b)-ii-A is the only case where player i switches to $\hat{d}_j(l+1) = B$ based on $E_i G_j(l)$. (82) implies that, if $\hat{d}_j(l+1) = B$, then (79) and (80) are the case and player j believes that $d_j(l+1) = B$ with high probability.

Second, if 1-(a) is the case, then the reward function on player j is increasing in $X_i(l)$ and player j will take $a_j(x)$. Hence, player i needs not to punish player j. Hence, with high probability $1-\eta$, $\hat{d}_j(l+1) = G$.

Third, if 1-(b)-i is the case, then since player i is generous to player j, player i needs not to punish player j. However, if 1-(b)-ii is the case, then since player i is harsh on player j and player j's reward on player i is a non-negative constant, with high probability, player i punishes player j based on $E_i G_j(l)$.

Fourth, after any history, player i uses the signal from the noisy cheap talk with probability at least η. As seen in Section 4, this guarantees that player j believes that an error happens in communication if player j realizes that player i's continuation play is different from what player j expected (with the roles of players i and j reversed).

23.4.5 Transition of $\hat{d}_i(l+1)(i)$

Player j sends $\hat{d}_i(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$ in the supplemental round for $\hat{d}_i(l+1)$. Player i obeys the signal:

$$\hat{d}_i(l+1)(i) = f[i](\hat{d}_i(l+1)). \tag{83}$$

23.4.6 Transition of $\theta_i(l) \in \{G, B\}$, $\theta_i(\lambda_i(l+1)) \in \{G, B\}$, $\theta_i(d_i(l+1)) \in \{G, B\}$ and $\theta_i(\hat{d}_j(l+1)) \in \{G, B\}$

As we have seen in Section 11.3, $\theta_i(\tilde{l}) = B$, $\theta_i(\lambda_i(\tilde{l}+1)) = B$, $\theta_i(d_i(\tilde{l}+1)) = B$ or $\theta_i(\hat{d}_j(\tilde{l}+1)) = B$ with some $\tilde{l} \leq l-1$ implies that player j is indifferent between any action profile (except for the incentives from π_j^{report}).

$\theta_i(l) = B$ if at least one of the following four conditions is satisfied:

1. $G_i(l) \notin [q_2T - \varepsilon T, q_2T + \varepsilon T]$ and $\hat{d}_j(l) = G$.

2. 1-(b) or 2 is the case when player i creates $\hat{\lambda}_j(l+1)$ in Section 23.4.2.

3. (79) is satisfied and player i picks a case that happens with probability η when player i creates $\hat{d}_j(l+1)$ in Section 23.4.4.

4. (79) is not satisfied.

Otherwise, $\theta_i(l) = G$.

On the other hand, $\theta_i(\lambda_i(l+1)) = B$, $\theta_i(d_i(l+1)) = B$ and $\theta_i(\hat{d}_j(l+1)) = B$, respectively, if and only if player i receives $g[i](\lambda_i(l+1)) = E$, $g[i](d_i(l+1)) = E$ and $g[i](\hat{d}_j(l+1)) = E$, respectively. That is, if player i realizes that player j may receive a wrong signal in the supplemental rounds, then player i makes player j indifferent between any action profile.

23.4.7 Summary of the Transitions of θ_j

We summarize the implications of the transitions of θ_j. Since we want to consider player i's incentive, we consider θ_j, not θ_i. Hence, reverse the roles of players i and j if necessary when we refer to the previous sections.

First, suppose that $a(x)$ is taken in the lth review round and that $\hat{d}_i(l) = G$. In this case, if $G_j(l) \notin [q_2T - \varepsilon T, q_2T + \varepsilon T]$ happens, then player j makes player i indifferent from any action profile sequence from 1 of Section 23.4.6.

Further, from 1-(a) and 2 in Section 23.4.4 and 3 and 4 of Section 23.4.6, $\hat{d}_i(l) = B$ and $\lambda_j(l) = G$ implies that player j has made player i indifferent between any action profile sequence.

Therefore, suppose that player i knew that $a(x)$ is taken in the lth review round and $\lambda_j(l) = G$. If 1-(b)-ii-A is the case in Section 23.4.4, then player i at the end of the lth review round puts a belief no less than $1 - \exp(-O(T))$ on the event that any action is optimal at the end of the lth review round.

Second, if player j receives a signal indicating player i's mistake when player j sends a message m in a supplemental round, then $\theta_j(m) = B$ happens.

In total, we have shown the following lemma for $\hat{d}_j(l+1)$:

Lemma 12 *For sufficiently large T, for all $x \in \{G, B\}^2$ and $l \in \{1, ..., L\}$,*

1. *Suppose that player i knew that $a(x)$ is taken in the lth review round and $\lambda_j(l) = G$. If 1-(b)-ii-A is the case in Section 23.4.4, then player i at the end of the lth review round puts a belief no less than $1 - \exp(-O(T))$ on the event that any action is optimal.*

2. *If $\hat{d}_j(l+1) = G$ is determined by (81), then conditional on $d_j(l+1) \in \{G, B\}$, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that $d_j(l+1) = G$ or any action is optimal.*

3. *If $\hat{d}_j(l+1) = B$ is determined by (81), then conditional on $d_j(l+1) \in \{G, B\}$, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that any action is optimal.*

In addition, from (83), player i always uses the signals from the noisy cheap talk to construct $\hat{d}_i(l+1)(i)$. Therefore, similarly to the above lemma, we have the following:

Lemma 13 *Conditional on $\hat{d}_i(l+1) \in \{G, B\}$, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that $\hat{d}_i(l+1)(i) = \hat{d}_i(l+1)$ or any action is optimal.*

Fourth, in the supplemental rounds between the lth review round and $(l+1)$th review round, whenever player i's message affects player j's future actions, then $\theta_i(l) = B$ has happened and player i is indifferent between any action profile in the future review rounds.

Specifically, when player j creates $\hat{\lambda}_i(l+1)$ in Section 23.4.2, if player j uses the signal from the noisy cheap talk message, then 1-(b) or 2 is the case, which implies that $\theta_i(l) = B$ has happened.

In addition, when player j creates $\hat{d}_i(l+1)$ in Section 23.4.4, player j uses the signal from the noisy cheap talk message only if player j picks a case that happens with probability η or (79) is not satisfied. Both implies $\theta_j(l) = B$, as desired.

Further, from Section 23.2, the action by player j depends on $\hat{d}_j(l+1)(j)$ only if $\hat{\lambda}_i(l+1) = B$. $\hat{\lambda}_i(l+1) = B$ implies that 1-(b) or 2 is the case in Section 23.4.2, which implies that $\theta_i(l) = B$ happened.

Fifth, suppose that $\theta_j(\tilde{l}) = \theta_j(\lambda_j(\tilde{l}+1)) = \theta_j(d_j(\tilde{l}+1)) = \theta_j(\hat{d}_i(\tilde{l}+1)) = G$ for all $\tilde{l} \leq l-1$ (otherwise, player i is indifferent between any action profile except for π_i^{report}). We will show that the distribution of $\theta_j(l)$, $\theta_j(\lambda_j(l+1))$, $\theta_j(d_j(l+1))$ and $\theta_j(\hat{d}_i(l+1))$ is independent of player i's action in the lth review round with probability $1 - \exp(-O(T^{\frac{1}{2}}))$. To see why, consider the following three reasons:

1. (35) and (79) are not the case. This happens with the *ex post* probability $\exp(-O(T))$ *given* $\{a_t, y_t\}_{t \in T(l)}$.

2. Suppose that (35) and (79) are the case.

   In Section 11.4.2,

   (a) If $\hat{\lambda}_i(l) = B$, then nothing happens.

   (b) If $\hat{\lambda}_i(l) = G$ and $\hat{d}_i(l) = B$, then $\theta_j(l) = B$ happens by player j's own mixture and it is out of player i's control.

   (c) If $\hat{\lambda}_i(l) = \hat{d}_i(l) = G$, then from Section 23.2, player j takes $a_j(x)$. From Lemma 3, the distribution of $\left(E_j \Psi_i^{a(x)}\right)_t$ is independent of player i's action. Hence, whether (36) is satisfied or not is independent of player i's action. Conditional on that (36) is satisfied, whether 1-(a) or 1-(b) is the case depends on player j's mixture and is independent of player i's action.

Therefore, in Section 11.4.2, the distribution of $\theta_j(l)$ is independent of player i's strategy.

In Section 23.4.3, if $\hat{d}_i(l) = B$, then $d_j(l+1) = G$ does not newly happen. If $\hat{d}_i(l) = G$, then player j takes $a_j(x)$ since otherwise $\hat{\lambda}_i(l) = B$ and "$\hat{\lambda}_i(l) = B$ and $\hat{d}_i(l) = G$" contradict to $\theta_j(\tilde{l}) = \theta_j(\lambda_j(\tilde{l}+1)) = \theta_j(d_j(\tilde{l}+1)) = \theta_j(\hat{d}_i(\tilde{l}+1)) = G$ for all $\tilde{l} \leq l-1$. Therefore, by Lemma 10, the distribution of $G_j(l)$ is independent of player i's strategy in the lth review round.

In Section 23.4.4, $\theta_j(l) = B$ happens by player j's own mixture and it is out of player i's control.

3. $\theta_j(m) = B$ in a supplemental round if and only if $g[j](m) = E$ for some message m sent in the supplemental round, which happens with probability $\exp(-O(T^{\frac{1}{2}}))$ regardless of the message.

24 Variables

In this section, we show that we can take all the variables necessary for the equilibrium construction appropriately: q_2, q_1, $\bar{u}$, $\bar{L}$, L, η and ε.

Lemmas 3 and 10 determine q_1 and q_2 and Lemma 5 determines $\bar{u}$.

We take $\bar{L}$ sufficiently large so that

$$\bar{L}\left(q_2 - q_1\right) > \max_{a,i} 2\left|u_i\left(a\right)\right|. \tag{84}$$

We are left to pin down $L \in \mathbb{N}$, $\varepsilon > 0$ and $\eta > 0$. Take L sufficiently large sufficiently small such that

$$\max_{x:x_j=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \frac{\bar{L}}{L} < \underline{v}_i < \overline{v}_i < \min_{x:x_j=G} w_i(x) - \frac{\bar{L}}{L}. \tag{85}$$

Then, given L, take $\eta > 0$ sufficiently small so that

$$\begin{aligned} & \max_{x:x_j=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \eta + \frac{\bar{L}}{L} + 2L\eta\left(\bar{u} - \min_{i,x} w_i(x)\right) \\ < \ & \underline{v}_i < \overline{v}_i < \min_{x:x_j=G} w_i(x) - \frac{\bar{L}}{L} - 2L\eta\left(\bar{u} + \max_{i,x} w_i(x)\right). \end{aligned} \tag{86}$$

Finally, take $\varepsilon > 0$ sufficiently small so that

$$\begin{aligned} & \max_{x:x_j=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \frac{\bar{L}}{L} + 2\varepsilon\bar{L} + 2L\eta\left(\bar{u} - \min_{i,x} w_i(x)\right) \\ < \ & \underline{v}_i < \overline{v}_i < \min_{x:x_j=G} w_i(x) - \frac{\bar{L}}{L} - 2\varepsilon\bar{L} - 2L\eta\left(\bar{u} + \max_{i,x} w_i(x)\right). \end{aligned} \tag{87}$$

and

$$\varepsilon < \varepsilon^{\min\max}, \tag{88}$$

where $\varepsilon^{\min\max}$ is defined in Lemma 11 given q_2, q_1, L and η fixed above.

25 Almost Optimality

We now show that if we properly define $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$, then $\sigma_i(x_i)$ and π_i^{main} satisfy (8), (4) and (5).

25.1 Opponent's Action

First, we summarize what player i can believe about the opponent's action in each lth review round.

If $\hat{\lambda}_i(l) = B$, then from Sections 23.4.2 and 23.4.6, $\theta_j(\tilde{l}) = B$ with some $\tilde{l} \leq l-1$.

If $\hat{\lambda}_i(l) = G$, then from Sections 23.4.4 and 23.4.6, $\hat{d}_i(l) = B$ with "$x_j = G$ or $\lambda_j(l) = G$" implies $\theta_j(\tilde{l}) = B$ with some $\tilde{l} \leq l-1$. Hence, "$\hat{d}_i(l) = B$ and $\theta_j(\tilde{l}) = G$ for all $\tilde{l} \leq l-1$" imply $x_j = \lambda_j(l) = B$.

Then, from Section 23.2,

- player j takes $a_j(x)$ if $x_j = G$ or $\lambda_j(l) = G$ and takes "$a_j(x)$ if $\hat{d}_i(l) = G$ and $\alpha_j^{\min\max}$ if $\hat{d}_i(l) = B$" if $x_j = \lambda_j(l) = B$, or

- $\theta_j(\tilde{l}) = B$ with $\tilde{l} \leq l - 1$.

25.2 Almost Optimality of the Inferences

Second, we show the almost optimality of $\hat{\lambda}_j(l+1)$, $\hat{d}_j(l+1)$ and $\hat{d}_i(l+1)(i)$. Specifically, we want to show that, for any lth review round, for any h_i^t with period t in the lth review round, player i puts high posteriors on the following events:

1. If $\hat{\lambda}_j(l) = G$, then

   (a) "player j takes $a_j(x)$ and $\lambda_j(l) = G$" or any action is optimal.

   (b) If $\hat{d}_j(l) = B$, then any action is optimal.

2. If $\hat{\lambda}_j(l) = B$, then

   (a) If $\hat{d}_i(l)(i) = G$, then "player j takes $a_j(x)$ and $\lambda_j(l) = B$" or any action is optimal.

   (b) If $\hat{d}_i(l)(i) = B$, then "player j takes $\alpha_j^{\min\max}$ and $\lambda_j(l) = B$" or any action is optimal.

The basic logic is the same as Lemma 8 although the argument is more complicated since player j switches to the minimaxing action after some history.

For notational convenience, Let $\alpha_j(l)$ be player j's action plan in the lth review round[63] and $\boldsymbol{\alpha}_j(l) = (\alpha_j(1), ..., \alpha_j(l))$ be the sequence of player j's action plans from the first review round to the lth review round (excluding what messages player j sent by the noisy cheap talk in the supplemental rounds).

[63] Note that player j takes an *i.i.d.* action plan within a review round.

Lemma 14 *For any lth review round, for any h_i^t with period t in the lth review round, conditional on $\boldsymbol{\alpha}_j(l)$, player i puts a belief no less than*

$$1 - \exp(-O(T^{\frac{1}{2}})) \tag{89}$$

on the events that

1. *If $\hat{\lambda}_j(l) = G$, then*
   (a) *"player j takes $a_j(x)$ and $\lambda_j(l) = G$" or any action is optimal.*
   (b) *If $\hat{d}_j(l) = B$, then any action is optimal.*
2. *If $\hat{\lambda}_j(l) = B$, then*
   (a) *If $\hat{d}_i(l)(i) = G$, then "player j takes $a_j(x)$ and $\lambda_j(l) = B$" or any action is optimal.*
   (b) *If $\hat{d}_i(l)(i) = B$, then "player j takes $\alpha_j^{\min\max}$ and $\lambda_j(l) = B$" or any action is optimal.*

Proof. First, we show that player i can believe that $\hat{\lambda}_j(l) = \lambda_j(l)$ or any action is optimal. As in Lemma 8, there exists a unique l^* such that $\lambda_j(l)$ switches from G to B at the end of the l^*th review round. In addition, there exists $\hat{l}^*$ such that $\hat{\lambda}_j(l)$ switches from G to B at the end of the $\hat{l}^*$th main block. If $\lambda_j(L) = G$ ($\hat{\lambda}_j(L) = G$, respectively), then define $l^* = L$ ($\hat{l}^* = L$, respectively).

Then, there are following cases:

- If $l^* \geq \hat{l}^*$, then the proof is the same as Lemma 8.
- If $l^* < \hat{l}^*$, then there are following two cases:
  - If 1-(b) or 2 is the case when player i creates $\hat{\lambda}_j(l^*+1)$ in Section 23.4.2, then, again, the proof is the same as Lemma 8.

- If 1-(a) is the case, then there are following two cases: $\hat{d}_j(l^*) = G$ or $\hat{d}_j(l^*) = B$. If $\hat{d}_j(l^*) = G$, then player i takes $a_i(x)$. Further, from Section 25.1, player i can believe that player j with $\lambda_j(l^*) = G$ takes $a_j(x)$. Hence, player i can believe that $a(x)$ is taken in the l^*th review round. The rest of the proof is the same as Lemma 8. Note that player j's continuation play is determined by $\hat{\lambda}_i(\tilde{l})$, $\hat{d}_i(\tilde{l})$ and $\hat{d}_j(\tilde{l})(j)$ and that errors happen with probability $\exp(-O(T^{\frac{1}{2}}))$ in the supplemental rounds since player j uses the signals from noisy cheap talk with probability at least η to construct these states. Hence, learning does not change the posterior.

  If $\hat{d}_j(l^*) = B$, then there was $\tilde{l} < l^*$ such that player i switches from $\hat{d}_j(\tilde{l}-1) = G$ to $d_j(\tilde{l}) = B$. In the $(\tilde{l}-1)$th review round, since $\hat{d}_j(\tilde{l}-1) = G$ and $\lambda_j(\tilde{l}-1) = G$ by assumption, player i can believe that $a(x)$ was taken. Therefore, Lemma 12 implies that player i believes that any action is optimal with high probability from the $\tilde{l}$th review round. Note that learning from the continuation play does not change the posterior by the same reason as above.

Second, given the above observation, we can prove the statements in the lemma.

1. If $\hat{\lambda}_j(l) = G$, then player i believes that $\lambda_j(l) = G$ or any action is optimal with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ from the discussion above. If the former is the case, then player j takes $a_j(x)$ or any action is optimal from Section 25.1. Hence, 1-(a) is satisfied.

   Further, the proof for the case with $l^* < \hat{l}^*$ above implies that 1-(b) is satisfied.

2. If $\hat{\lambda}_j(l) = B$, then player i believes that $\lambda_j(l) = B$ or any action is optimal with probability $1 - \exp(-O(T^{\frac{1}{2}}))$. If the former is the case, then "player j takes $a_j(x)$ if $\hat{d}_i(l) = G$ and $\alpha_j^{\min\max}$ if $\hat{d}_i(l) = B$" if $\hat{\lambda}_i(l) = G$. If $\hat{\lambda}_i(l) = B$, then any action is optimal for player i. Therefore, in total, player i believes that "player j takes $a_j(x)$ if $\hat{d}_i(l) = G$ and $\alpha_j^{\min\max}$ if $\hat{d}_i(l) = B$" or any action is optimal with probability $1 - \exp(-O(T^{\frac{1}{2}}))$.

   Since Lemma 13 guarantees that player i can believe that $\hat{d}_i(l)(i) = \hat{d}_i(l)$ or any action is optimal with probability $1 - \exp(-O(T^{\frac{1}{2}}))$, 2-(a) and 2-(b) are satisfied.

■

25.3 Determination of $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$

Based on Lemma 14, we determine $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ such that $\sigma_i(x_i)$ and π_i^{main} satisfy (8), (4) and (5).

We show that $\sigma_i(x_i)$ satisfies the following proposition by backward induction:

Proposition 2 *For all $i \in I$, there exists $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ such that*

1. *$\sigma_i(x_i)$ is almost optimal: For each $l \in \{1, ..., L\}$,*
   (a) *For any period t in the lth review round, (8) holds.*
   (b) *When player i sends the noisy cheap talk messages in the supplemental rounds, (8) holds.*
2. *(4) is satisfied with π_i replaced with π_i^{main}. Since each $x_i \in \{G, B\}$ gives the same value conditional on x_j, the strategy in the coordination block is optimal.*[64]
3. *π_i^{main} satisfies (5).*

1-(b) follows from the following two facts: First, as seen in Section 23.4.7, whenever player i's message changes player j's action, $\theta_j(\tilde{l}) = B$ has happened. Second, Lemma 14 implies that player i can infer player j's private state with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ (or any action is optimal) by taking the equilibrium strategy. Therefore, the equilibrium strategy is almost optimal.

To show 3, as in the prisoners' dilemma, it suffices to have

$$\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l) \begin{cases} \leq 0 & \text{if } x_j = G, \\ \geq 0 & \text{if } x_j = B, \end{cases} \tag{90}$$

$$\left| \bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l) \right| \leq \max_{i,a} 2 \left| u_i(a) \right| T \tag{91}$$

[64] As in the prisoners' dilemma, even after the adjustment of the report block, any $x_i \in \{G, B\}$ still gives exactly the same value.

for all $x \in \{G, B\}^2$, $\lambda_j(l) \in \{G, B\}$, $\hat{d}_i(l) \in \{G, B\}$ and $l \in \{1, ..., L\}$.

We are left to construct $\bar{\pi}_i$ so that 1-(a) and 2 are satisfied together with (90) and (91). Remember that, from Section 25.1, player i can believe that, if (78) is being used in the lth review round, then with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$, player j takes

1. $a_j(x)$ if $x_j = G$ or $\lambda_j(l) = G$, and

2. "$a_j(x)$ if $\hat{d}_i(l) = G$ and $\alpha_j^{\min\max}$ if $\hat{d}_i(l) = B$" if $x_j = \lambda_j(l) = B$.

Together with Lemmas 13 and 14, player i can believe that, if (78) is being used, then with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$, player j takes

1. $a_j(x)$ if $x_j = G$ or $\hat{\lambda}_j(l) = G$, and

2. "$a_j(x)$ if $\hat{d}_i(l)(i) = G$ and $\alpha_j^{\min\max}$ if $\hat{d}_i(l)(i) = B$" if $x_j = \hat{\lambda}_j(l) = B$.

Below, we consider the cases with $x_j = G$ and $x_j = B$ separately.

25.3.1 Case 1: $x_j = G$

We start backward induction from the Lth review round.

Suppose that player j uses (78) in the Lth round. This together with $x_j = G$ implies that $\hat{d}_i(l) = G$. If $\lambda_j(L) = \hat{\lambda}_j(L)$ and $\hat{d}_i(L)(i) = \hat{d}_i(L)$, then from (78) and Section 23.2, if player i obeys $\sigma_i(x_i)$, then player i's average continuation payoff except for $\bar{\pi}_i$ is equal to $w_i(x) - 2\varepsilon \bar{L}$ if $\lambda_j(L) = G$ and $u_i(BR_i(a_j(x)), a_j(x)) \geq w_i(x)$ if $\hat{\lambda}_j(L) = B$.

Therefore, for $l = L$, there exists $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ with (90) and (91) such that player i's average continuation payoff is equal to $w_i(x) - 2\varepsilon \bar{L}$ if (78) is used, $\lambda_j(L) = \hat{\lambda}_j(L)$, and $\hat{d}_i(L)(i) = \hat{d}_i(L)$.

Consider the almost optimality of $\sigma_i(x_i)$. For almost optimality, Lemma 14 guarantees that player i can always believe that (78) is used, that $\lambda_j(L) = \hat{\lambda}_j(L)$, and that $\hat{d}_i(L)(i) = \hat{d}_i(L)$ (these imply that "$\lambda_j(L) = \hat{\lambda}_j(L)$ implies $\hat{d}_i(L)(i) = \hat{d}_i(L)$").

Therefore, it is almost optimal for player i to take $a_i(x)$ if $\hat{\lambda}_j(L) = G$ and "$BR_i(a_j(x))$ if $\hat{d}_i(L)(i) = G$ and take $BR_i\left(\alpha_j^{\min\max}\right)$ if $\hat{d}_i(L)(i) = B$" if $\hat{\lambda}_j(L) = B$, as desired.

In the $(L-1)$th review round, as for $l=L$, there exists $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ with (90) and (91) such that player i's average payoff from the $(L-1)$th review round is equal to $w_i(x) - 2\varepsilon \bar{L}$ if (78) is used, $\lambda_j(L) = \hat{\lambda}_j(L)$, $\hat{d}_i(L)(i) = \hat{d}_i(L)$, and player i obeys $\sigma_i(x_i)$.

If (78) is used in the $(L-1)$th review round, then (77) will be used in the Lth review round with probability no more than 2η (player j uses (39) or (81)) plus $\exp(-O(T^{\frac{1}{2}}))$. When (77) is used, per period payoff is bounded by $[-\bar{u}, \bar{u}]$ by Lemma 5.

Therefore, for $l = L-1$, retaking $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ if necessary, player i's average continuation payoff for the next two review rounds is equal to $w_i(x) - 2\varepsilon \bar{L} - 2L\eta\left(\bar{u} + \max_{i,a} w_i\left(a\right)\right)$ in the limit as $\delta \to 1$if player i obeys $\sigma_i(x_i)$.

For the almost optimality of $\sigma_i(x_i)$, only difference from the Lth review round is that, if $\lambda_j(L-1) = G$, then player i's action in the $(L-1)$th review round can affect the distribution of $\lambda_j(L)$ and $\hat{d}_i(L)$.

However, this effect is negligible by the following reasons:

First, we define $\bar{\pi}_i(x, \lambda_j(L), \hat{d}_i(L), L)$ such that player i's value from the Lth review round is independent of $\lambda_j(L)$ as long as $\lambda_j(L) = \hat{\lambda}_j(L)$ and $\hat{d}_i(L) = \hat{d}_i(L)(i) = G$.[65] Second, Lemma 14 implies that player i in the main blocks does not put a belief more than $\exp(-O(T^{\frac{1}{2}}))$ on the events that "$\lambda_j(L) \neq \hat{\lambda}_j(L)$ or $\hat{d}_i(L) \neq \hat{d}_i(L)(i)$" and player i's payoff is not independent of the action profile in the Lth review round. Third, Section 23.4.7 guarantees that the probability that player j will newly make player i indifferent between any action profile from the Lth review round is almost independent of player i's strategy in the $(L-1)$th main block. Therefore, for almost optimality, we can assume that player i in the $(L-1)$th review round maximizes

$$\sum_{t \in T(L-1)} u_i(a_t) + \pi_i^{\text{main}}(x, h_j^{\text{main}}, L-1), \tag{92}$$

assuming $\lambda_j(L-1) = \hat{\lambda}_j(L-1)$ and $\hat{d}_i(L-1) = \hat{d}_i(L-1)(i)$.

[65] In the above discussion, we have verified that this claim is correct for the case with $\theta_j = G$ unil the Lth review round.

For the other cases, player i is indifferent between any action profile sequence, which implies that player i's value is constant for any action profile, as desired.

Therefore, the same proof as $l = L$ works for $l = L - 1$.

Recursively, for $l = 1$, 1-(a) is satisfied and the average ex ante payoff of player i from the first review round is $w_i(x) - 2\varepsilon\bar{L} - 2L\eta\left(\bar{u} + \max_{i,a} w_i\left(a\right)\right)$ if $x_j = G$. Note that, in the first review round, $\lambda_j(1) = \hat{\lambda}_j(1) = G$, $d_j(1) = \hat{d}_j(1) = G$ and $\hat{d}_i(1) = \hat{d}_i(i)(1) = G$.

Taking the first term $-\bar{L}T$ in (76) into account, the average ex ante payoff is $w_i(x) - \frac{\bar{L}}{L} - 2\varepsilon\bar{L} - 2L\eta\left(\bar{u} + \max_{i,a} w_i\left(a\right)\right)$ if $x_j = G$.

From (87), we can further modify $\bar{\pi}_i\left(x, G, G, 1\right)$ with (90) and (91) such that $\sigma_i(x_i)$ gives $\bar{v}_i$ if $x_j = G$. Therefore, 2 is satisfied.

25.3.2 Case 2: $x_j = B$

The main difference from the case with $x_j = G$ is as follows: When $x_j = G$, then from Section 23.4.4, player j has $\hat{d}_i(l) = G$ with probability at least $1 - \eta$ and does not minimax player i with probability more than η. Hence, player i in each lth review round can neglect the effect on the probability of being minimaxed in the next review round.

On the other hand, when $x_j = B$, player i needs to take into account that player i's action in the lth review round will affect the probability of being minimaxed in the next review round.

As in the case with $x_j = G$, the distribution of θ_j is almost independent of player i's strategy. Since $\theta_j = B$ does not happen with probability more than $2\eta + \exp(-O(T^{\frac{1}{2}}))$ in each main block, we can deal with the effect of $\theta_j = B$ on the continuation payoff as we do in the case with $x_j = G$. Therefore, we assume that $\theta_j = G$ for each round.

Further, by Lemma 14, player i can neglect the possibility of the mis-coordination. Therefore, for almost optimality, we assume that, for each l, $\hat{\lambda}_j(l) = \lambda_j(l)$, $\hat{d}_i(l)(i) = \hat{d}_i(l)$, $\hat{\lambda}_i(l) = G$, and "$\lambda_j(l) = G$ implies $\hat{d}_i(l) = \hat{d}_i(l+1) = G$." The last one comes from the fact that $\lambda_j(l) = G$ and $\hat{d}_i(l+1) = B$ imply $\theta_j(l) = B$ from Sections 23.4.4 and 23.4.6.

In the Lth review round, consider the following cases:

1. If player j uses (78) and $\lambda_j(L) = \hat{\lambda}_j(L) = G$, then $a_i(x)$ is strictly optimal as in the case with $x_j = G$. The average payoff of player i in the Lth review round except for

$\bar{\pi}_i$ is $w_i(x) + 2\varepsilon\bar{L}$.

2. If player j uses (78) and $\lambda_j(L) = \hat{\lambda}_j(L) = B$, then there are following two cases:

   (a) If $\hat{d}_i(L) = \hat{d}_i(L)(i) = G$, then $BR_i(a_j(x))$ is optimal and gives the average payoff $u_i^*(x)$.

   (b) If $\hat{d}_i(L) = \hat{d}_i(L)(i) = B$, then $BR_i(\alpha_j^{\min\max})$ is optimal and gives the average payoff at most v_i^*.

Therefore, for $l = L$, there exists $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ with (90) and (91) such that player i's average continuation payoff is equal to $u_i^*(x) + 2\varepsilon\bar{L}$ if 1 or 2-(a) is the case and $\max\{w_i(x), v_i^*\} + \eta + 2\varepsilon\bar{L}$ if 2-(b) is the case. Note that the former is higher than the latter by

$$u_i^*(x) - \max\{w_i(x), v_i^*\} - \eta. \tag{93}$$

In the $(L-1)$th review round,

1. If player j uses (78) and $\lambda_j(L-1) = \hat{\lambda}_j(L-1) = G$, then $a_i(x)$ is optimal since (i) the payoffs from $\lambda_j(L) = \hat{\lambda}_j(L) = G$ and $\lambda_j(L) = \hat{\lambda}_j(L) = B$ with $\hat{d}_i(L) = \hat{d}_i(L)(i) = G$ are the same and (ii) player i can neglect the effect of player i's action in the $(L-1)$th review round on $\hat{d}_i(L)$. For (ii), we use the assumption that $\lambda_j(l) = G$ implies $\hat{d}_i(l) = \hat{d}_i(l+1) = G$.

   The average payoff of player i from the $(L-1)$ and Lth review rounds except for $\bar{\pi}_i(x, L-1, \lambda_j(L-1), \hat{d}_i(L-1))$ is

$$\frac{w_i(x) + u_i^*(x)}{2} + 2\varepsilon\bar{L} \leq \frac{\max\{w_i(x), v_i^*\} + u_i^*(x)}{2} + 2\varepsilon\bar{L}.$$

2. Suppose that player j uses (78) and $\lambda_j(L-1) = \hat{\lambda}_j(L-1) = B$. Now, $\lambda_j(L)$ is fixed at B. Hence, the relevant cases are the following two:

   (a) If $\hat{d}_i(L-1) = \hat{d}_i(L-1)(i) = G$, then $BR_i(a_j(x))$ is optimal.

To see why, remember that player j will have $\hat{d}_i(L) = B$ with probability (82). The marginal decrease of this probability by not taking the static best response is bounded by

$$\bar{p}_j^L(x) \frac{\text{marginal decrease of } E_j G_i(l)}{q_2 T - q_1 T} \leq \frac{\bar{p}_j^L(x)}{T} \frac{T-1}{T}.$$

Here, $\frac{T-1}{T}$ represents the probability that an arbitrary period in $T(l)$ is excluded from $T_j(l)$ and is not used to monitor player i. On the other hand, the maximum gain of preventing $\hat{d}_i(L) = B$ is equal to T times (93). Therefore, the expected gain is bounded by $\bar{p}_j^L(x)\frac{T-1}{T}$ times (93).

Since (93) corresponds to the gain of preventing $\hat{d}_i(L) = B$ for $l = L-1$ in Lemma 11, player j should take $BR_i(a_j(x))$ for sufficiently large δ.[66]

Given player i's strategy, if $\lambda_j(L-1) = \hat{\lambda}_j(L-1) = B$ and $\hat{d}_i(L-1) = \hat{d}_i(L-1)(i) = G$, then, conditional on $\theta_j(L) = G$, $\hat{d}_i(L)$ happens with probability

$$\bar{p}_j^{l+1}(x) \min \left\{ 1, \frac{q_2 T - 2\varepsilon T - q_1 T}{q_2 T - q_1 T} \right\}$$

from (82) and Lemma 10. Therefore, Lemma 11 guarantees that the average payoff from the $(L-1)$th and Lth main block is no more than

$$\frac{\max\{w_i(x), v_i^*\} + u_i^*(x)}{2} + \eta + 2\varepsilon \bar{L}.$$

(b) If $\hat{d}_i(L-1) = \hat{d}_i(L-1)(i) = B$, then $\hat{d}_i(L)$ is fixed at B. Therefore, $BR_i(\alpha_j^{\min\max})$ is optimal and this gives the average payoff at most v_i^*.

Therefore, for $l = L-1$, there exists $\bar{\pi}_i(x, \lambda_j(l), \hat{d}_i(l), l)$ with (90) and (91) such that player i's average continuation payoff is equal to $\frac{\max\{w_i(x), v_i^*\} + u_i^*(x)}{2} + \eta + 2\varepsilon \bar{L}$ if 1 or 2-(a) is the case and $v_i^* + 2\varepsilon \bar{L}$ if 2-(b) is the case.

[66] And so large T from (1).

Recursively, for $l = 1$, Proposition 1 is satisfied and the average ex ante payoff of player i at the first review round is $\frac{(L-1)\max\left\{w_i(x), v_i^*\right\} + u_i^*(x)}{L} + \eta + 2\varepsilon \bar{L}$. Note that, in the first review round, $\lambda_j(1) = \hat{\lambda}_j(1) = G$, $d_j(1) = \hat{d}_j(1) = G$ and $\hat{d}_i(1) = \hat{d}_i(i)(1) = G$.

Taking the probability of having $\theta_j = B$ and the first term $\bar{L}T$ in (76) into account, the average ex ante payoff is $\frac{(L-1)\max\left\{w_i(x), v_i^*\right\} + u_i^*(x)}{L} + \eta + \frac{\bar{L}}{L} + 2\varepsilon \bar{L} + 2L\eta\left(\bar{u} - \min_{i,a} w_i\left(a\right)\right)$.

From (87), we can further modify $\bar{\pi}_i\left(x, G, G, 1\right)$ with (90) and (91) such that $\sigma_i(x_i)$ gives $\underline{v}_i$ if $x_j = B$. Therefore, 2 is satisfied.

26 Exact Optimality

The report block is the same as in the prisoners' dilemma except that we change the definition of f_i to deal with the fact that the players take a mixed strategy to punish the opponent. We maintain the restriction (64) on f_i. Therefore, the incentive to tell the truth is satisfied and Π_i cancels out the ex ante punishment from (54), (55), (56), (57), (58), (59), (60) and (61).

For round r not corresponding to a review round, f_i is the same as in the prisoners' dilemma.

For round r corresponding to a review round, if $\hat{\mathfrak{h}}_i^r$ is an off-path history or a history where player i should not take a mixed minimax strategy in round r, then the reward is the same as in Section 15.7.[67]

If $\hat{\mathfrak{h}}_i^r$ is an on-path history where player i should take a mixed minimax strategy in round r, then we change f_i as follows.

If round r is the last review round, then player j gives

$$f_i(\hat{\mathfrak{h}}_i^r, \hat{\#}_i^r, \alpha_j(r)) = 2\left(v(\hat{\mathfrak{h}}_i^r, \left(T^*(r, a_i)\right)_{a_i}, \alpha_j(r)) - v(\hat{\mathfrak{h}}_i^r, \left(T(r, a_i)\right)_{a_i}, \alpha_j(r))\right) \tag{94}$$

to player i so that player i is indifferent between any action plan ex ante. Again, the first coefficient 2 represents (67). Since we condition on $\alpha_j(r)$, learning from $\left\{y_{i,t}\right\}_{t\in T(r)}$ is

[67] The definition of $\hat{\mathfrak{h}}_i^r$ is still valid.

irrelevant if round r is the last review round. Therefore, ex ante optimality is equivalent to sequential optimality for player i to take any action sequence and so to take $\alpha_i^{\min\max}$.

If round r is not the last review round, then f_i is the summation of the following two. First, remember that the history in round r, $\#_i^r$, changes the ex ante value at the beginning of the next round by affecting the belief about the best responses at the beginning of the next round. Let $V_i(\hat{\mathfrak{h}}_i^r, \#_i^r, x_j)$ be the value at the beginning of the next round given x_j.[68] To cancel out the effect of learning, player j first gives

$$2\left(\min_{\#_i^r} V_i(\hat{\mathfrak{h}}_i^r, \#_i^r, x_j) - V_i(\hat{\mathfrak{h}}_i^r, \hat{\#}_i^r, x_j)\right). \tag{95}$$

(95) is bounded by $[-O\left(T^{-r-6}\right), O\left(T^{-r-6}\right)]$ since (i) $\hat{\mathfrak{h}}_i^r, \hat{\#}_i^r$ which tells player i to minimax puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that π_i^{main} makes any action profile sequence indifferent, (ii) f_i for round $\tilde{r} \geq r+1$ is bounded by $[-T^{-r-6}, T^{-r-6}]$ from the inductive hypothesis and (64), (iii) we have established the incentive to tell the truth, and (iv) from (iii) and Π_i, the ex ante punishments from (54), (55), (56), (57), (58), (59), (60) and (61) are zero.

Then, player j calculates the ex ante payoff taking $\left(T(r, a_i)\right)_{a_i}$ in round r. Let $U(\hat{\mathfrak{h}}_i^r, \left(T(r, a_i)\right)_{a_i}, \alpha(r))$ be this payoff from round r. Since (95) cancels out the effect on the continuation values from the next round, the effects on (95) and the continuation payoff from the next round are neglected.

In addition to (95), player j gives

$$2\left(\min_{\left(T(r,a_i)\right)_{a_i}} U(\hat{\mathfrak{h}}_i^r, \left(T(r, a_i)\right)_{a_i}, \alpha(r)) - U(\hat{\mathfrak{h}}_i^r, (\hat{T}(r, a_i))_{a_i}, \alpha(r))\right) \tag{96}$$

to player i so that player i is indifferent between any action plan ex ante.

That is, f_i is the summation of (95) and (96).

[68] By backward induction, given $\left(\hat{\mathfrak{h}}_i^r, \#_i^r\right)$, the ex ante value at the beginning of the next round can be calculated, assuming that player i will take a best response after $\left(\hat{\mathfrak{h}}_i^r, \#_i^r\right)$. This is well defined even after player i's deviation since player j treats each period within a round identically. See Section 15.7.

Given this, it is optimal for player i to take $\alpha_i^{\min\max}$ after any history since (i) learning from $\{y_{i,t}\}_{t \in T(r)}$ for the reward function for the current round r is irrelevant after conditioning $\alpha_j(r)$ and (ii) learning for the future reward is canceled out with (95).

SUPPLEMENTAL MATERIAL 3:

PROOF OF THEOREM 1 for a General N-Player Game With CHEAP TALK

In this Supplemental Material, we prove Theorem 1 (folk theorem) for a general N-player game with cheap talk and public randomization. We concentrate on $N \geq 3$. See the Supplemental Material 2 for the case with $N = 2$ and the Supplemental Material 5 for the dispensability of cheap talk and public randomization.

In this Supplemental Material, when we say player i with $i \notin \{1, ..., N\}$, it means player $i \pmod N$. In particular, player 0 is player N and player $N+1$ is player 1.

Fix $v \in \text{int}(F^*)$ arbitrarily. We will find $\{\sigma_i(x_i)\}_{i,x_i}$ and $\left\{\pi_i(x_{i-1}, h_{i-1}^{T_P+1} : \delta)\right\}_{i,x_{i-1}}$ in the finitely repeated game with (3), (4) and (5).

As in the main text, let

$$v_i^* \equiv \min_{\alpha_{-i} \in \Pi_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i(a_i, \alpha_{-i})$$

be the minimax payoff (by independently mixed strategies). In addition, let $\alpha_{-i}^{\min\max} \equiv (\alpha_{j,i}^{\min\max})_{j \neq i}$ be the solution for the above problem, that is, $\alpha_{j,i}^{\min\max}$ is player j's stage game strategy when players $-i$ minimax player i.

27 Intuitive Explanation

Before proceeding to the proof, we offer an intuitive explanation. As in the two-player case, we have L main blocks, where L will be defined in Section 34. Player $i-1$ incentivizes player i by the reward function. Similarly to $\lambda_j(l)$ and $\hat{\lambda}_j(l)$ in the two-player case, player $i-1$ has $\lambda_{i-1}(l) \in \{G, B\}$ indicating whether player $i-1$ has observed an "erroneous history" and player i has $\hat{\lambda}_{i-1}(l) \in \{G, B\}$ indicating what is the optimal action for player i.

As we have mentioned in Section 4.6.3, the more-than-two-player case has two differences from the two-player case. In the Supplemental Material 3, we focus on the first difference (see the Supplemental Material 5 for the second difference): Since we use perfect and public cheap talk to coordinate on x_i, each player infers the same x_i. Therefore, we concentrate on how players $-i$ coordinate on minimaxing player i after the histories where player i is likely to have deviated. To deal with this problem, we consider a mechanism to coordinate on the punishment different from the two-player case.

For each player i, there are two monitors, players $i-1$ and $i+1$. In other words, each player n monitors players $n-1$ and $n+1$.

After the lth review round, each player j constructs a variable $d_j(l+1) \in \{0, j-1, j+1\}$. $d_j(l+1) = 0$ implies that player j thinks that there was no deviator in players $j-1$ and $j+1$ in the lth review round. $d_j(l+1) = j-1$ implies that player j thinks that player $j-1$ deviated in the lth review round. $d_j(l+1) = j+1$ implies that player j thinks that player $j+1$ deviated. Player j sends the message $d_j(l+1)$ to each player $n \in -j$ by noisy cheap talk with precision $p = \frac{1}{2}$.[69] Each player $n \neq j$ constructs the inference of $d_j(l+1)$, $d_j(l+1)(n) \in \{0, j-1, j+1\}$, from the private signal of the noisy cheap talk.

Each player n minimaxes player i by $\alpha_{n,i}^{\min\max}$ if and only if player n infers that the two monitors $i-1$ and $i+1$ think that player i has deviated: $d_{i-1}(l+1)(n) = d_{i+1}(l+1)(n) = i$.

To incentivize the players to tell the truth about $d_j(l+1)$, whenever player j's message has an impact on the decision of minimax, we make player j indifferent between any action profile. This happens only if there is "player $j' \in -j$ with $d_{j'}(l+1) \neq 0, j$" or "players $j' \in -j$ and $n \in -j$ with $d_{j'}(l+1)(n) \neq 0, j$." With the noisy cheap talk with precision $p = \frac{1}{2}$, the latter does not happen with probability more than $\exp(-O(T^{\frac{1}{2}}))$. Therefore, if we construct $d_{j'}(l+1)$ such that player j cannot manipulate $d_{j'}(l+1)$, then player j follows the equilibrium strategy.

28 Almost Optimality

As seen in Section 7, we first show that player i's strategy is "almost optimal."

[69] Precisely, since $d_n(l+1)$ is ternary while the noisy cheap talk can send a binary message, player n sends a sequence of binary messages. See Section 33.2.

We divide the reward function into two parts:

$$\pi_i(x_{i-1}, h_{i-1}^{T_P+1} : \delta) = \pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta) + \pi_i^{\text{report}}(x_{i-1}, h_{i-1}^{T_P+1}, h_{i-1}^{\text{rereport}} : \delta).$$

Contrary to the two-player case, we have $h_{i-1}^{\text{rereport}}$ in π_i^{main} and π_i^{report}. We will define h_{i-1}^{main}, $h_{i-1}^{T_P+1}$ and $h_{i-1}^{\text{rereport}}$ below.

With more than two players, player $i-1$ wants to use the information owned by players $-(i-1,i)$ to construct player $(i-1)$'s reward function on player i. Hence, as we will see in Section 31, after the report block where player i reports h_i^{main}, we have the "re-report block" where players $-(i-1,i)$ send their history to player $i-1$. This information is used only for π_i and does not affect the value of players $-(i-1,i)$. Therefore, we can assume that players $-(i-1,i)$ tell the truth. Further, since the information in the re-report block is used only for the reward (not for the action plan $\sigma_{i-1}(x_{i-1})$), it is sufficient for player $i-1$ to know the information by the end of the review phase.

Let h_{i-1}^{main}, $h_{i-1}^{T_P+1}$, and $h_{i-1}^{\text{rereport}}$, respectively, be the history of player $i-1$ in the main blocks, "in the coordination, main and report blocks," and in the re-report block, respectively.

We first construct $\sigma_i(x_i)$ and $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$ satisfying (8), (4) and (5) if we neglect the report block. After constructing such π_i^{main}, we construct the strategy in the report block and π_i^{report} such that $\pi_i = \pi_i^{\text{main}} + \pi_i^{\text{report}}$ satisfies (3), (4) and (5).

29 Special Case

We still focus on the spacial case where the perfect cheap talk, noisy cheap talk and public randomization are available:

Perfect Cheap Talk We assume that the perfect cheap talk is available. In addition, we assume that perfect cheap talk is public, that is, when player i sends a message to another player, all the players can know exactly what is the message.

Noisy Cheap Talk Between j and n with Precision $p = \frac{1}{2}$ We assume that, for each pair of players j and n with $j \neq n$, each player j has an access to a noisy cheap talk device to send a binary message $m \in \{G, B\}$ to player n.

With more than two players, we use only the noisy cheap talk with precision $p = \frac{1}{2}$. Hence, from now on, when we consider the game with more than two players, we assume that $p = \frac{1}{2}$.

When player j sends m to player n via noisy cheap talk, it generates player n's private signals $f[n](m) \in \{G, B\}$ with the following probability:

$$\Pr(\{f[n](m) = f\} \mid m) = \begin{cases} 1 - \exp(-T^{\frac{1}{2}}) & \text{for } f = m, \\ \exp(-T^{\frac{1}{2}}) & \text{for } f = \{G, B\} \setminus \{m\}. \end{cases}$$

That is, $f[n](m)$ is the correct signal with high probability.

Given player n's signal $f[n](m)$, it generates player $(n-1)$'s private signal $g[n-1](m) \in \{m, E\}$ with following probability:

$$\Pr(\{g[n-1](m) = E\} \mid m, f[n](m)) = 1 - \exp(-T^{\frac{1}{2}})$$

for all $(m, f[n](m))$ with $f[n](m) \neq m$. Note that, contrary to the two-player case, not player j (the sender) but player $n-1$ (the controller of player n's payoff) receives this message. We do not specify the probability for the other cases except that

- anything happens with probability at least $\exp(-O(T^{\frac{1}{2}}))$:

$$\Pr(\{(f[n](m), g[n-1](m)) = (f, g)\} \mid m) \geq \exp(-O(T^{\frac{1}{2}}))$$

  for all m and (f, g), and

- unconditionally on $f[n](m)$, $g[n-1](m) = m$ with high probability:

$$\Pr(\{g[n-1](m) = m\} \mid m) \geq 1 - \exp(-O(T^{\frac{1}{2}}))$$

for all m.

Finally, player $j-1$ (the controller of the sender) observes a private signal $f_2\left[j-1\right](m) \in \{G,B\}$ and player $n-1$ (the controller of the receiver) observes a private signal $g_2[n-1](m) \in \{G,B\}$. The role of f_2, g_2 is the same as in the two-player case: To incentivize the sender to tell the truth about m and the receiver to tell the truth about $f[n](m)$ in the report block.

We assume that $f_2\left[j-1\right](m)$ and $g_2\left[n-1\right](m)$ are very imprecise signals compared to $f\left[n\right](m)$ and $g\left[n-1\right](m)$ but $f_2\left[j-1\right](m)$ and $g_2\left[n-1\right](m)$ have some information about the other players' information. That is, there exists $\eta > 0$ such that:

- For all $f_2 \in \{G,B\}$ and $g_2 \in \{G,B\}$,

$$\Pr\left(\left\{f_2\left[j-1\right](m) = f_2, g_2\left[n-1\right](m) = g_2\right\} \mid m, f[n](m), g\left[n-1\right](m)\right) \geq \eta. \tag{97}$$

  By (97), even after observing any $f_2[j-1](m)$, player n still believes that if $f\left[n\right](m) \neq m$, then $g[n-1](m) = E$ with probability $1 - \exp(-O(T^{\frac{1}{2}}))$.[70]

  Therefore, for almost optimality, player n can neglect $f_2[j-1](m)$ even if she can observe $f_2[j-1](m)$, that is, even if n is equal to $j-1$.

- For any $m \in \{G,B\}$, $g[n-1](m) \in \{G,B\}$, $f[n](m), f[n](m)' \in \{G,B\}$ and $f_2[j-1](m), f_2[j-1](m)' \in \{G,B\}$, if $(f[n](m), f_2[j-1](m)) \neq (f[n](m)', f_2[j-1](m)')$, then

$$\left\| \begin{array}{c} \mathbb{E}\left[\mathbf{1}_{g_2[n-1](m)} \mid m, g[n-1](m), f[n](m), f_2[j-1](m)\right] \\ -\mathbb{E}\left[\mathbf{1}_{g_2[n-1](m)} \mid m, g[n-1](m), f[n](m)', f_2[j-1](m)'\right] \end{array} \right\| > \eta. \tag{98}$$

  This implies that, in the report block,

  - If $j-1 \neq n$, then player n reports $f[n](m)$ in the report block and player $n-1$ punishes player n by (98) with $f[n](m)$ replaced with player n's report of $f[n](m)$, $\widehat{f[n](m)}$. So that player $n-1$ can calculate (98), in the re-report block, player

[70] If $j-1 = n$, then player n does observe $f_2[j-1](m)$.

$j-1$ informs player $n-1$ of $f_2[j-1](m)$. Even after knowing m, player n has the incentive to tell the truth about $f[n](m)$.

– If $j-1=n$, then player n reports $f[n](m)$ and $f_2[j-1](m)$ in the report block and player $n-1$ punishes player n by (98) with $f[n](m)$ replaced with $\widehat{f[n](m)}$ and $f_2[j-1](m)$ replaced with $f_2[\widehat{j-1]}(m)$. Even after knowing m, player n has the incentive to tell the truth about $f[n](m)$ and $f_2[j-1](m)$.

As we will see below, $g[n-1](m)$ and $g_2[n-1](m)$ are not revealed to player n in the main blocks.

- For any $m, m' \in \{G, B\}$, $f[n](m) \in \{G, B\}$, $g[n-1](m), g[n-1](m)' \in \{G, B\}$ and $g_2[n-1](m), g_2[n-1](m)' \in \{G, B\}$, if $(m, g[n-1](m), g_2[n-1](m)) \neq (m', g[n-1](m)', g_2[n-1](m)')$, then

$$\left\| \begin{array}{c} \mathbb{E}\left[\mathbf{1}_{f_2[j-1](m)} \mid m, g[n-1](m), g_2[n-1](m), f[n](m)\right] \\ -\mathbb{E}\left[\mathbf{1}_{f_2[j-1](m)} \mid m', g[n-1](m)', g_2[n-1](m)', f[n](m)\right] \end{array} \right\| > \eta. \tag{99}$$

This implies that, in the report block,

– If $n-1 \neq j$, then player j reports m in the report block and player $j-1$ punishes player j by (99) with m replaced with player j's report of m, $\hat{m}$. So that player $j-1$ can calculate (99), in the re-report block, player $n-1$ informs player $j-1$ of $g[n-1](m)$ and $g_2[n-1](m)$, and player n informs player $j-1$ of $f[n](m)$. Even after knowing $f[n](m)$, player j has the incentive to tell the truth about m.

– If $n-1=j$, then player j reports m, $g[n-1](m)$ and $g_2[n-1](m)$ in the report block and player $j-1$ punishes player j by (99) with m, $g[n-1](m)$ and $g_2[n-1](m)$ replaced with $\hat{m}$, $g[\widehat{n-1]}(m)$ and $g_2[\widehat{n-1]}(m)$, respectively. So that player $j-1$ can calculate (99), in the re-report block, player n informs player $j-1$ of $f[n](m)$. Even after knowing $f[n](m)$, player j has the incentive to tell the truth about m, $g[n-1](m)$ and $g_2[n-1](m)$.

As we will see below, $f_2[j-1](m)$ is not revealed to player j in the main blocks.

We assume that all the signals are private and so

- player j knows only m,
- player n knows only $f[n](m)$,
- player $n-1$ knows only $g[n-1](m)$ and $g_2[n-1](m)$, and
- player $j-1$ knows only $f_2[j-1](m)$.[71]

As for Lemma 2, we can summarize the important features of the noisy cheap talk in the following lemma:

Lemma 15 *The signals by the noisy cheap talk satisfies the following conditions:*

1. *For any $m \in \{G, B\}$, player n's signal $f[n](m)$ is correct with high probability:*

$$\Pr(\{f[n](m) = m\} \mid m) \geq 1 - \exp(-T^{\frac{1}{2}}).$$

2. *For any $m \in \{G, B\}$, $f[n](m) \in \{G, B\}$ and $f_2[j-1](m) \in \{G, B\}$, after knowing m, $f[n](m)$ and $f_2[j-1](m)$, player n puts a high belief on the events that either $f[n](m)$ is correct or $g[n-1](m) = E$. That is,*

$$\begin{aligned} & \Pr(\{f[n](m) = m \text{ or } g[n-1](m) = E\} \mid m, f[n](m), f_2[j-1](m)) \\ \geq \ & 1 - \exp(-T^{\frac{1}{2}}). \end{aligned}$$

3. *For any $m \in \{G, B\}$, any signal profile can happen with positive probability:*

$$\begin{aligned} & \Pr(\{(f[n](m), g[n-1](m), f_2[j-1](m), g_2[n-1](m)) = (f, g, f_2, g_2)\} \mid m) \\ \geq \ & \exp(-O(T^{\frac{1}{2}})) \end{aligned}$$

[71] If there is a player whose index appears multiple times, then we assume that the player knows all the signals of the players with that index.
For example, if player j and player $n-1$ are the same player, she knows m, $g[n-1](m)$ and $g[n-1](m)$.

for all (f, g, f_2, g_2).

We do not have a condition corresponding to Condition 3 of Lemma 2.

Public Randomization As in the two-player case, the players use public randomization in the report block to determine who will report the history h_i^{main}.

30 Assumptions

In addition to Assumptions 1 and 3, we need multi-player counterparts of Assumptions 4 and 5.

30.1 Identifiability for the Reward

As Assumption 4, for each player $i \in I$ and action profile $a \in A$, to incentivize player i, it is important that the controller of her payoff (player $i-1$) statistically identifies player i's deviation. That is, we want to create a statistics $\psi_{i-1}^a(y_{i-1})$ whose expectation is higher when player i follows the prescribed action a_i than $\tilde{a}_i \neq a_i$: With some $q_2 > q_1$,

$$\mathbb{E}\left[\psi_{i-1}^a(y_{i-1}) \mid \tilde{a}_i, a_{-i}\right] \equiv \sum_{y_{i-1}} q(y_{i-1} \mid \tilde{a}_i, a_{-i})\psi_{i-1}^a(y_{i-1}) = \begin{cases} q_2 & \text{if } \tilde{a}_i = a_i, \\ q_1 & \text{if } \tilde{a}_i \neq a_i. \end{cases} \tag{100}$$

In addition, with more than two players, it is important that player $j \neq i-1, i$ cannot change the ex ante value of $\psi_{i-1}^a(y_{i-1})$ by unilateral deviation: For any $j \neq i-1, i$ and $\tilde{a}_j \in A_j$,

$$\mathbb{E}\left[\psi_{i-1}^a(y_{i-1}) \mid \tilde{a}_j, a_{-j}\right] \equiv \sum_{y_{i-1}} q(y_{i-1} \mid \tilde{a}_j, a_{-j})\psi_{i-1}^a(y_{i-1}) = q_2. \tag{101}$$

Further, as in the two-player case, player i calculates the conditional expectation of $\psi_{i-1}^a(y_{i-1})$ after observing y_i, believing that a is taken:

$$\sum_{y_{i-1}} \psi_{i-1}^a(y_{i-1}) q(y_{i-1} \mid a, y_i).$$

With more than two players, we want to make sure that not only player $i-1$ but also all the other players than player i cannot change the ex ante value of this conditional expectation by unilateral deviation: For each $j \neq i$ and $\tilde{a}_j \in A_j$,

$$\sum_{y_i} \left(\sum_{y_{i-1}} \psi_{i-1}^a(y_{i-1}) q(y_{i-1} \mid a, y_i) \right) q\left(y_i \mid \tilde{a}_j, a_{-j}\right) = q_2.$$

Note that this expression is equivalent to

$$\sum_{y_{i-1}} \left(\sum_{y_i} q(y_{i-1} \mid a, y_i) q\left(y_i \mid \tilde{a}_j, a_{-j}\right) \right) \psi_{i-1}^a(y_{i-1}) = q_2. \tag{102}$$

A sufficient condition for the existence of such ψ_{i-1}^a is as follows: Let $Q_1(j, \tilde{a}_j, a_{-j}) \equiv \left(q(y_{i-1} \mid \tilde{a}_j, a_{-j})\right)_{y_{i-1}}$ be the vector expression of the distribution of player $(i-1)$'s signals conditional on $\tilde{a}_j, a_{-j}$. In addition, let $Q_2(j, \tilde{a}_j, a_{-j}) \equiv \left(\sum_{y_i} q(y_{i-1} \mid a, y_i) q(y_i \mid \tilde{a}_j, a_{-j})\right)_{y_{i-1}}$ be the vector expression of ex ante distribution of player $(i-1)$'s signals when y_i is first generated according to $q(y_i \mid \tilde{a}_j, a_{-j})$ and then y_{i-1} is generated according to $q(y_{i-1} \mid a, y_i)$. We assume that all the vectors $Q_1(i, \tilde{a}_i, a_{-i})$ with $\tilde{a}_i \in A_i$, $Q_1(j, \tilde{a}_j, a_{-j})$ with $j \neq i, i-1$ and $\tilde{a}_j \neq a_j$ and $Q_2(j, \tilde{a}_j, a_{-j})$ with $j \neq i$ and $\tilde{a}_j \neq a_j$ are linearly independent:

Assumption 6 *For any $i \in I$ and $a \in A$, $Q_1(i, \tilde{a}_i, a_{-i})$ with $\tilde{a}_i \in A_i$, $Q_1(j, \tilde{a}_j, a_{-j})$ with $j \neq i, i-1$ and $\tilde{a}_j \neq a_j$ and $Q_2(j, \tilde{a}_j, a_{-j})$ with $j \neq i$ and $\tilde{a}_j \neq a_j$ are linearly independent.*

This assumption is generic if $|Y_{i-1}| \geq |A_i| + |A_{i-1}| - 1 + 2\sum_{j \neq i-1, i} (|A_j| - 1)$, which is guaranteed by Assumption 2. We can show that Assumption 6 is sufficient for (100), (101) and (102).

Lemma 16 *If Assumption 6 is satisfied, then there exist $q_2 > q_1$ such that, for each $i \in I$ and $a \in A$, there exists a function $\psi_{i-1}^a : Y_{i-1} \to (0, 1)$ such that (100), (101) and (102) are satisfied.*

Proof. The same as Lemma 3. ■

30.2 Identifiability for Minimaxing

In addition, to coordinate on punishing (minimaxing) player i, not only player $i-1$ but also player $i+1$ monitors player i.

It is important to have $\gamma_{i+1}^{a}(y_{i+1})$ by which player $i+1$ can distinguish whether player i takes a_i or not:

$$\mathbb{E}[\gamma_{i+1}^{a}(y_{i+1}) \mid \tilde{a}_i, a_{-i}] \equiv \sum_{y_{i+1}} q(y_{i+1} \mid \tilde{a}_i, a_{-i})\gamma_{i+1}^{a}(y_{i+1}) = \begin{cases} q_2 & \text{if } \tilde{a}_i = a_i, \\ q_1 & \text{otherwise.} \end{cases} \tag{103}$$

Also, we want to make sure that the other players $-(i, i+1)$ cannot change the expectation of γ_{i+1}^{a}: For all $j \neq i, i+1$ and $\tilde{a}_j \in A_j$,

$$\mathbb{E}[\gamma_{i+1}^{a}(y_{i+1}) \mid \tilde{a}_j, a_{-j}] \equiv \sum_{y_{i+1}} q(y_{i+1} \mid \tilde{a}_j, a_{-j})\gamma_{i+1}^{a}(y_{i+1}) = q_2. \tag{104}$$

A sufficient condition for the existence of such γ_{i+1}^{a} is as follows: Let $Q^{\min\max}(j, \tilde{a}_j, a_{-j}) \equiv (q(y_{i+1} \mid \tilde{a}_j, a_{-j}))_{y_{i+1}}$ be the vector expression of the distribution of player $(i+1)$'s signals conditional on $\tilde{a}_j, a_{-j}$. We assume that all the vectors $Q^{\min\max}(i, \tilde{a}_i, a_{-i})$ with $\tilde{a}_i \in A_i$ and $Q^{\min\max}(j, \tilde{a}_j, a_{-j})$ with $j \neq i, i+1$ and $\tilde{a}_j \neq a_j$ are linearly independent:

Assumption 7 *For any $i \in I$ and $a \in A$, $Q^{\min\max}(i, \tilde{a}_i, a_{-i})$ with $\tilde{a}_i \in A_i$ and $Q^{\min\max}(j, \tilde{a}_j, a_{-j})$ with $j \neq i, i+1$ and $\tilde{a}_j \neq a_j$ are linearly independent.*

This assumption is generic if $|Y_{i+1}| \geq |A_i| + \sum_{j \neq i, i+1} (|A_j| - 1)$, which is guaranteed by Assumption 2. We can show that Assumption 7 is sufficient for (103) and (104).

Lemma 17 *If Assumption 7 is satisfied, then there exist $q_2 > q_1$ such that, for all $i \in I$ and $a \in A$, there exists a function $\gamma_{i+1}^{a} : Y_{i+1} \to (0, 1)$ such that (103) and (104) are satisfied.*

Proof. The same as Lemma 3. ■

30.3 Individual Identifiability

In the two-player game, for each player i, the controller of her payoff (player $i-1=j$) knows which action player $-i=j$ takes for each t. This is not the case with more than two players since players $-i$ now contains players $-(i-1,i)$.

Suppose that the controller of player i's payoff (player $i-1$) knows which *strategy* players $-i$ play. If the coordination does not go well, then it is possible that each player $j \in -i$ takes some $a_j \in A_j$ or $\alpha_{j,n_j}^{\min\max}$ (player j's strategy when players $-n_j$ minimax player $n_j \in -n$) with different n_j's for different j's.[72]

For each i, for each α_{-i} such that each player $j \in -i$ takes a pure strategy $a_j \in A_j$ or $\alpha_{j,n_j}^{\min\max}$ for some player $n_j \in -j$, we want to construct $\pi_i^{x_{i-1}}(\alpha_{-i}, y_{i-1})$ such that player i's payoff is constant regardless of α_{-i} and player i's strategy.

A sufficient condition is that all the vectors of player $(i-1)$'s signal distributions are linearly independent with respect to $a_i \in A_i$ if player $i-1$ knew α_{-i}. That is, we assume the following:

Assumption 8 *For any $i \in I$ and α_{-i} such that each player $j \in -i$ takes a pure strategy $a_j \in A_j$ or $\alpha_{j,n_j}^{\min\max}$ for some player $n_j \in -j$, $\left(q_{i-1}\left(y_{i-1} \mid a_i, \alpha_i\right)\right)_{y_{i-1}}$ is linearly independent with respect to $a_i \in A_i$.*

This is generic if $|Y_{i-1}| \geq |A_i|$. Note that we assume that player $i-1$ knew α_{-i}. As we have mentioned, players $-(i-1,i)$ in the re-report block tells player $i-1$ what strategy α_{-i} they take in the main blocks.

Then, we can construct the two reward: One is to cancel out discounting and the other is to make player i indifferent between any action profile sequence:

Lemma 18 *If Assumption 8 is satisfied, then for each $i \in I$, there exists $\pi_i^{\delta} : \mathbb{N} \times \Delta\left(A_{-i}\right) \times Y_{i-1} \to \mathbb{R}$ such that, for all $a_{i,t} \in A_i$ and $\alpha_{-i,t}$ such that each player $j \in -i$ takes a pure*

[72] As will be seen in Section 33.2, $\alpha_{j,j}^{\min\max}$ is defined to be a pure strategy.

strategy $a_j \in A_j$ or $\alpha_{j,n_j}^{\min\max}$ for some player $n_j \in -j$, we have

$$\delta^{t-1} u_i\left(a_{i,t}, \alpha_{-i,t}\right) + \mathbb{E}\left[\pi_i^{\delta}\left(t, \alpha_{-i,t}, y_{i-1,t}\right) \mid a_{i,t}, \alpha_{-i,t}\right] = u_i\left(a_{i,t}, \alpha_{-i,t}\right) \text{ for all } t \in \{1, ..., T_P\}$$

and

$$\lim_{\delta \to 1} \frac{1-\delta}{1-\delta^{T_P}} \sum_{t=1}^{T_P} \sup_{\alpha_{-i,t}, y_{i-1,t}} \left|\pi_i^{\delta}\left(t, \alpha_{-i,t}, y_{i-1,t}\right)\right| = 0 \tag{105}$$

for all $T_P = O(T)$ with $T = (1-\delta)^{-\frac{1}{2}}$. Here, the supremum is taken for $\alpha_{-i,t}$ satisfying the condition above.

Proof. The same as Lemma 4. ■

As we will see in Section 33.3, we add

$$\sum_{t=1}^{T_P} \pi_i^{\delta}\left(t, \alpha_{-i,t}, y_{i-1,t}\right) \tag{106}$$

to π_i^{main} so that we can ignore discounting.

Lemma 19 *If Assumption 8 is satisfied, then, there exists $\bar{u} > 0$ such that, for each $i \in I$, there exist $\pi_i^G(\alpha_{-i}, \cdot) : Y_{i-1} \to [-\bar{u}, 0]$ and $\pi_i^B(\alpha_{-i}, \cdot) : Y_{i-1} \to [0, \bar{u}]$ such that*

$$\begin{aligned} u_i\left(a_i, \alpha_{-i}\right) + \mathbb{E}\left[\pi_i^G\left(\alpha_{-i}, y_{i-1}\right) \mid a_i, \alpha_{-i}\right] &= \text{constant} \in [-\bar{u}, \bar{u}], \\ u_i\left(a_i, \alpha_{-i}\right) + \mathbb{E}\left[\pi_i^B\left(\alpha_{-i}, y_{i-1}\right) \mid a_i, \alpha_{-i}\right] &= \text{constant} \in [-\bar{u}, \bar{u}] \end{aligned}$$

for all $a_i \in A_i$ and α_{-i} such that each player $j \in -i$ takes a pure strategy $a_j \in A_j$ or $\alpha_{j,n_j}^{\min\max}$ for some player $n_j \in -j$.

Proof. The same as Lemma 5. ■

30.4 Slight Correlation

As in the two-player case, we need to establish the truthtelling incentive in the report block. When player i reports her history $(a_{i,t}, y_{i,t})$ in some period t in the main blocks, intuitively, player $i-1$ punishes player i proportionally tow

$$\left\| \mathbf{1}_{y_{j,t}} - \mathbb{E}\left[\mathbf{1}_{y_{j,t}} \mid \hat{a}_{i,t}, \hat{y}_{i,t}, a_{-i,t}, \{y_{n,t}\}_{n \in -(j,i)} \right] \right\|^2$$

with

$$j = \begin{cases} i-1 & \text{if } i \neq 1, \\ 2 & \text{if } i = 1. \end{cases} \tag{107}$$

Hence, player i wants to minimize

$$\mathbb{E}\left[\begin{array}{c} \left\| \mathbf{1}_{y_{j,t}} - \mathbb{E}\left[\mathbf{1}_{y_{j,t}} \mid \hat{a}_{i,t}, \hat{y}_{i,t}, a_{-i,t}, \{y_{n,t}\}_{n \in -(j,i)} \right] \right\|^2 \\ \mid a_{i,t}, y_{i,t}, a_{-i,t}, \{y_{n,t}\}_{n \in -(j,i)} \end{array} \right]. \tag{108}$$

Compared to the two-player case, we assume that player i knows the action profile by players $-i$ and signal observations by players $-(j,i)$.

We assume that a different $(a_{i,t}, y_{i,t})$ has different information about $y_{i-1,t}$ conditional on $a_{-i,t}, \{y_{n,t}\}_{n \in -(j,i)}$:

Assumption 9 *For any $i \in I$, j with (107), $a_{-i} \in A_{-i}$, $\{y_{n,t}\}_{n \in -(j,i)} \in \prod_{n \in -(j,i)} Y_n$, $a_i, a_i' \in A_i$ and $y_i, y_i' \in Y_i$, if $(a_i, y_i) \neq (a_i', y_i')$, then*

$$\mathbb{E}\left[\mathbf{1}_{y_j} \mid a_i, y_i, a_{-i}, \{y_{n,t}\}_{n \in -(j,i)} \right] \neq \mathbb{E}\left[\mathbf{1}_{y_j} \mid a_i', y_i', a_{-i}, \{y_{n,t}\}_{n \in -(j,i)} \right].$$

Given Assumption 9, the truthtelling is uniquely optimal.

Lemma 20 *If Assumption 9 is satisfied, then for any $a_t \in A$, $\{y_{n,t}\}_{n \in -(j,i)} \in \prod_{n \in -(j,i)} Y_n$ and $y_{i,t} \in Y_i$, $(\hat{a}_{i,t}, \hat{y}_{i,t}) = (a_{i,t}, y_{i,t})$ is a unique minimizer of (108).*

31 Structure of the Phase

In this section, we explain the structure of the finitely repeated game. As in the two-player game, we have the coordination block at the beginning, where each player takes turns to send the cheap talk message $x_i \in \{G, B\}$: First, player 1 sends x_1, second, player 2 sends x_2, and so on until player N sends x_N. Note that x will become common knowledge for the rest of the game.

After the coordination block, we have L main blocks. The first $(L-1)$ blocks is further divided into $1 + N + N(N-1)$ rounds. That is, for $l \in \{1, ..., L-1\}$, the lth main block consists of the following rounds: First, the players play a T-period review round.

After that, as indicated in Section 27, each player $i-1$ sends $\lambda_{i-1}(l+1)$ to player i by the noisy cheap talk between $i-1$ and i. The players take turns: Player 1 sends $\lambda_1(l+1)$ to player 2, player 2 sends $\lambda_2(l+1)$ to player 3, and so on until player N sends $\lambda_N(l+1)$ to player 1. We call the instance where player $i-1$ sends $\lambda_{i-1}(l+1)$ to player i "supplemental round for $\lambda_{i-1}(l+1)$."

After that, each player j sends $d_j(l+1)$ to each player $n \in -j$ by the noisy cheap talk between j and i.[73] The players take turns: Player 1 sends $d_1(l+1)$ to player 2, player 1 sends $d_1(l+1)$ to player 3, and so on until player 1 sends $d_1(l+1)$ to player N. Then, player 2 sends $d_2(l+1)$ to player 1, player 2 sends $d_2(l+1)$ to player 3, and so on until player 2 sends $d_2(l+1)$ to player N. This step continues until player N sends $d_N(l+1)$ to player $N-1$. We call the instance where player j sends $d_j(l+1)$ to player n "supplemental round for $d_j(l+1)$ between j and n."

The last Lth main block has only the T-period review round.

Let $T(l)$ be the set of T periods in the lth review round.

After the last main block, there is the report block, where each player i reports the whole history h_i^{main}.

Finally, after the report block, there is the re-report block, where each player i reports the whole history h_i^{main} again. This time, player i's message is used only for the reward $\pi_j^{\text{main}}(x_{j-1}, h_{j-1}^{\text{main}}, h_{j-1}^{\text{rereport}} : \delta)$ with $j \neq i$. That is, player i's message does not affect player i.

Therefore, the whole structure of the phase is as follows:

[73] While the noisy cheap talk is for a binary message, $d_j(l+1) \in \{0, j-1, j+1\}$ is ternary. The players use a sequence of binary messages to send $d_j(l+1)$ as we will see in Section 33.2.

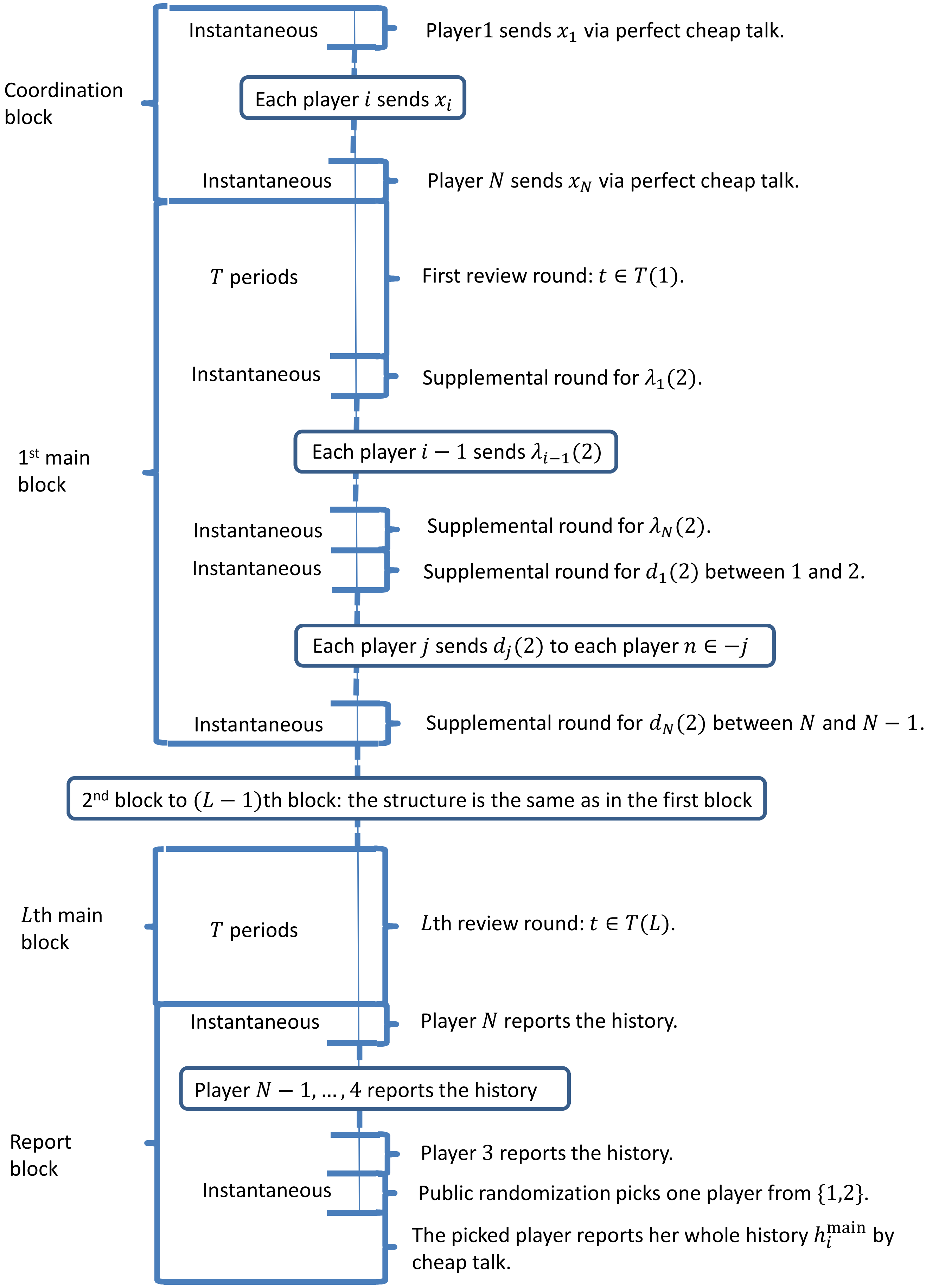

Figure 1 of the Supplemental Material 3: Structure of the Phase

32 Perfect Monitoring

As in the two-player game, we consider a *one-shot* game with *perfect monitoring* parameterized with $l \in \mathbb{N}$. In the game with parameter $l \in \{1, ..., L-1\}$, players $-i$ takes $a_{-i}(x)$. Depending on player i's action, $c(l+1) \in \{0, i\}$ is determined. If player i takes $a_i(x)$, then $c(l+1) = 0$ with probability one. If player i takes $a_i \neq a_i(x)$, then $c(l+1) = i$ with probability $p_{i+1}^{l+1}(x)$ and $c(l+1) = 0$ with the remaining probability $1 - p_{i+1}^{l+1}(x)$. The payoff of player i is determined as

$$V_i^l = \max_{a_i} \frac{1}{L-l+1} u_i(a_i, a_{-i}(x)) + \frac{L-l}{L-l+1} \mathbb{E}\left[W_i^{l+1}(c(l+1)) \mid a_i, a_{-i}(x)\right]$$

with

$$\begin{aligned} u_i^*(x) &= u_i(BR_i(a_{-i}(x)), a_{-i}(x)), \\ W_i^{l+1}(G) &= \frac{(L-l-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l} + \eta, \\ W_i^{l+1}(B) &= v_i^*. \end{aligned}$$

As in the two-player case, we can show the following lemma:

Lemma 21 *For any $L \in \mathbb{N}$ and $\eta > 0$, there exist $\{p_{i+1}^{l+1}(x)\}_{l=1}^{L-1} \in [0,1]$ such that it is uniquely optimal for player i to take $BR_i(a_{-i}(x))$ and*

$$V_i^l \leq \frac{(L-l)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L-l+1} + \eta = W_i^l(G),$$

Proof. The same as Lemma 11. With Assumption 6, we can assume that the gain in the instantaneous utility from taking a best response is strict as in Section 21. ■

Note that, contrary to the two-player case, we do not have the term $\frac{q_2 - q_1 - 2\varepsilon}{q_2 - q_1}$. This comes from the fact that we will use the different coordination device on minimaxing player i.

33 Equilibrium Strategy

In this section, we define $\sigma_i(x_i)$ and $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$. In Section 33.1, we define the state variables that will be used to define the action plans and rewards. Given the states, Section 33.2 defines player i's action plan $\sigma_i(x_i)$ and Section 33.3 defines player $(i-1)$'s reward function $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$ on player i. Finally, Section 33.4 determines the transition of the states defined in Section 33.1.

33.1 States x_i, $\lambda_i(l+1)$, $\hat{\lambda}_{i-1}(l+1)$, $d_i(l+1)$, $d_j(l+1)(i)$, $c_i(l+1)$, $\theta_i(l)$, $\theta_i(\lambda_j(l+1))$ and $\theta_i(d_j(l+1))$

The intuitive meaning of $x_i \in \{G, B\}$, $\lambda_i(l+1) \in \{G, B\}$ and $\hat{\lambda}_{i-1}(l+1) \in \{G, B\}$ is the same as in the two-player case with j replaced with $i-1$.

As seen in Section 27, $d_i(l+1) \in \{0, i-1, i+1\}$ indicates what player i thinks about a deviation by players $i-1$ and $i+1$.

Player $j \neq i$ sends $d_j(l+1)$ via noisy cheap talk to player i in the supplemental round for $d_j(l+1)$ between j and i. Let $d_j(l+1)(i)$ be player i's inference of the message, which will be determined in Section 33.4.4.

Player i minimaxes player n by $\alpha_{i,n}^{\min\max}$ if and only if player i infers that the two monitors $n-1$ and $n+1$ think that player n has deviated: $d_{n-1}(l+1)(i) = d_{n+1}(l+1)(i) = n$. If there is a unique player with such an "agreement," then player i thinks that player n's deviation is "confirmed." $c_i(l+1) = n$ implies such a situation. Otherwise, we have $c_i(l+1) = 0$. Hence, $c_i(l+1) \in \{0\} \cup I$.

The intuitive meaning of $\theta_i(l) \in \{G, B\}$, $\theta_i(\lambda_j(l+1)) \in \{G, B\}$ and $\theta_i(d_j(l+1)) \in \{G, B\}$ is the same as in the two-player case except for the following: Player i controls player $(i+1)$'s payoff. Hence, player i makes player $i+1$ indifferent between any action profile sequence after $\theta_i(l) = B$, $\theta_i(\lambda_j(l+1)) = B$ and $\theta_i(d_j(l+1)) = B$. The precise definitions are given in Sections 33.4.6, 33.4.7 and 33.4.8.

33.2 Player i's Action

In the coordination block, each player sends x_i truthfully. Then, the state profile x becomes common knowledge.

In the lth review round, player i's strategy depends on $\hat{\lambda}_{i-1}(l)$ and $c_i(l)$. If $\hat{\lambda}_{i-1}(l) = G$, then player i takes $a_i(x)$ if $c_i(l) = 0$ and $\alpha_{i,n}^{\min\max}$ if $c_i(l) = j \in I$ (define $\alpha_{i,i}^{\min\max}$ as an arbitrary pure action). If $\hat{\lambda}_{i-1}(l) = B$, then player i takes $BR_i(a_{-i}(x))$ if $c_i(l) = 0$ and $BR_i(\alpha_{-i}^{\min\max})$ if $c_i(l) \in I$.

In the supplemental round for $\lambda_i(l+1)$, player i sends $\lambda_i(l+1)$ truthfully via noisy cheap talk to player $i+1$.

In the supplemental round for $d_i(l+1)$ between i and n, player i sends $d_i(l+1)$ truthfully via noisy cheap talk to player n.

Since $d_i(l+1)$ is ternary while the noisy cheap talk can send a binary message, we attach a sequence of binary messages to each $d_i(l+1)$. Specifically, given $d_i(l+1) \in \{0, i-1, i+1\}$, player i define a sequence $d_i(l+1)\{1\}, d_i(l+1)\{2\} \in \{G, B\}^2$: If $d_i(l+1) = 0$, then $d_i(l+1)\{1\} = G$ and $d_i(l+1)\{2\} = B$ with probability $\frac{1}{2}$ and $d_i(l+1)\{1\} = B$ and $d_i(l+1)\{2\} = G$ with probability $\frac{1}{2}$. If $d_i(l+1) = i-1$, then $d_i(l+1)\{1\} = G$ and $d_i(l+1)\{2\} = G$. If $d_i(l+1) = i+1$, then $d_i(l+1)\{1\} = B$ and $d_i(l+1)\{2\} = B$.

Player i with $d_i(l+1)$ sends $d_i(l+1)\{1\}$ and $d_i(l+1)\{2\}$ truthfully via noisy cheap talk. We define

$$f[n](d_i(l+1)) = \begin{cases} 0 & \text{if } f[n](d_i(l+1)\{1\}) \neq f[n](d_i(l+1)\{2\}), \\ i-1 & \text{if } f[n](d_i(l+1)\{1\}) = f[n](d_i(l+1)\{2\}) = G, \\ i+1 & \text{if } f[n](d_i(l+1)\{1\}) = f[n](d_i(l+1)\{2\}) = B, \end{cases}$$

$$\begin{aligned} & g[n-1](d_i(l+1)) \\ = & \begin{cases} E & \text{if } g[n-1](d_i(l+1)\{1\}) = E \text{ or } g[n-1](d_i(l+1)\{2\}) = E, \\ d_i(l+1) & \text{otherwise,} \end{cases} \end{aligned}$$

$$g_2[n-1](d_i(l+1)) = (g_2[n-1](d_i(l+1)\{1\}), g_2[n-1](d_i(l+1)\{2\}))\,,$$

and

$$f_2[i-1](d_i(l+1)) = (f_2[i-1](d_i(l+1)\{1\}), f_2[i-1](d_i(l+1)\{2\}))\,.$$

Then, the message transmits correctly with probability no less than $1-\exp(-O(T^{\frac{1}{2}}))$ and given $d_i(l+1)\{1\}, d_i(l+1)\{2\}$, player n puts a conditional belief on the events that $f[n](d_i(l+1)) = d_i(l+1)$ or $g[n-1](d_i(l+1)) = E$ with probability no less than $1-\exp(-O(T^{\frac{1}{2}}))$. In addition, given $d_i(l+1)\{1\}, d_i(l+1)\{2\}$, any signal sequences can happen with probability no less than $\exp(-O(T^{\frac{1}{2}}))$. Therefore, Lemma 15 holds as if player i sent $d_i(l+1)$ via noisy cheap talk.

33.3 Reward Function

In this subsection, we explain player $(i-1)$'s reward function on player i, $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$.

Score As in the two-player case, each player $i-1$ constructs $\Psi_{i-1,t}^{a(x)} \in \{0,1\}$ from $\psi_{i-1}^{a(x)}(y_{i-1,t})$. Player $i-1$ picks $t_{i-1}(l)$ randomly from $T(l)$. With $T_{i-1}(l) \equiv T(l) \setminus \{t_{i-1}(l)\}$, player $i-1$ calculates the score about player i

$$X_{i-1}(l) = \sum_{t \in T_{i-1}(l)} \Psi_{i-1,t}^{a(x)} + \mathbf{1}_{t_{i-1}(l)}.$$

Slope Second, take $\bar{L}$ sufficiently large so that

$$\bar{L}\,(q_2 - q_1) > \max_{a,i} 2\,|u_i\,(a)|\,.$$

Reward Function As in the two-player case, the reward $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$ is written as

$$\begin{aligned} & \pi_i^{\text{main}}\left(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta\right) \\ = & \sum_{l=1}^{L} \sum_{t \in T(l)} \pi_i^{\delta}\left(t, \alpha_{-i,t}, y_{i-1,t}\right) + \begin{cases} -\bar{L}T + \sum_{l=1}^{L} \pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) & \text{if } x_{i-1} = G, \\ \bar{L}T + \sum_{l=1}^{L} \pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) & \text{if } x_{i-1} = B. \end{cases} \end{aligned} \tag{109}$$

Note that we add (106) to ignore discounting.

Reward Function for Each Round If $\theta_{i-1}(\tilde{l}) = B$, $\theta_{i-1}(\lambda_j(\tilde{l}+1)) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1)) = B$ happens for some $\tilde{l} \leq l-1$ and $j \in I$, then player $i-1$ makes player i indifferent between any action profile sequence by

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) = \sum_{t \in T(l)} \pi_i^{x_{i-1}}(\alpha_{i-1}, y_{i-1,t}). \tag{110}$$

Otherwise, that is, if $\theta_{i-1}(\tilde{l}) = \theta_{i-1}(\lambda_j(\tilde{l}+1)) = \theta_{i-1}(d_j(\tilde{l}+1)) = G$ for all $\tilde{l} \leq l-1$ and $j \in I$, then player $(i-1)$'s reward on player i is based on x, $\lambda_{i-1}(l)$ and $c_{i-1}(l)$. The formal description is given by

$$\begin{aligned} & \pi_i^{\text{main}}\left(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l\right) \\ = & \begin{cases} \bar{\pi}_i(x, G, c_{i-1}(l), l) + \bar{L}\{X_{i-1}(l) - (q_2 T + 2\varepsilon T)\} & \text{if } x_{i-1} = G \text{ and } \lambda_{i-1}(l) = G, \\ \bar{\pi}_i(x, B, c_{i-1}(l), l) & \text{if } x_i = G \text{ and } \lambda_{i-1}(l) = B, \\ \bar{\pi}_i(x, G, c_{i-1}(l), l) + \bar{L}\{X_{i-1}(l) - (q_2 T - 2\varepsilon T)\} & \text{if } x_{i-1} = B \text{ and } \lambda_{i-1}(l) = G, \\ \bar{\pi}_i(x, B, c_{i-1}(l), l) & \text{if } x_{i-1} = B \text{ and } \lambda_{i-1}(l) = B. \end{cases} \end{aligned} \tag{111}$$

Here, $\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)$ will be determined later so that (8), (4) and (5) are satisfied.

33.4 Transition of the States

In this subsection, we explain the transition of player i's states. Since x_i is fixed in the phase, we consider the following eight:

33.4.1 Transition of $\lambda_{i-1}(l+1) \in \{G, B\}$

The transition of $\lambda_{i-1}(l+1) \in \{G, B\}$ is the same as in Section 23.4.1 with j replaces with $i-1$ except that $c_{i-1}(l) = 0$ is necessary to transit from $\lambda_{i-1}(l) = G$ to $\lambda_{i-1}(l+1) = B$. Player $i-1$ with $c_{i-1}(l) = n \in I$ will minimax player n. In that case, the action profile taken in the lth review round may not be $a(x)$ and so $\Psi_{i-1}^{a(x)}$ is not a correct statistics. Therefore, we have $c_{i-1}(l) = 0$ as a condition for $\lambda_{i-1}(l)$ to transit from G to B. As will be seen in Section 33.5, player $i-1$ with $\lambda_{i-1}(l) = G$ does not have $c_{i-1}(l) = i$ unless player i is indifferent between any action profile. In addition, if player $i-1$ has $c_{i-1}(l) = n \in -i$, then player i is indifferent between any action profile. Therefore, conditioning on $c_{i-1}(l) = 0$ does not cause a problem in player i's incentive to follow $\sigma_i(x_i)$.

Now, we define the transition of $\lambda_{i-1}(l)$: The initial condition is $\lambda_{i-1}(1) = G$. If $\lambda_{i-1}(l) = B$, then $\lambda_{i-1}(l+1) = B$. If $\lambda_{i-1}(l) = G$, then

1. If $X_{i-1}(l) \notin [q_2T - 2\varepsilon T, q_2T + 2\varepsilon T]$ and $c_{i-1}(l) = 0$, then $\lambda_{i-1}(l+1) = B$.

2. If $X_{i-1}(l) \in [q_2T - 2\varepsilon T, q_2T + 2\varepsilon T]$ or $c_{i-1}(l) \neq 0$, then $\lambda_{i-1}(l+1) = G$.

33.4.2 Transition of $\hat{\lambda}_{i-1}(l+1) \in \{G, B\}$

The transition of $\hat{\lambda}_{i-1}(l+1) \in \{G, B\}$ is the same as in the two-player case with j replaced with $i-1$, except that if $c_i(l) = j \in -i$, then $\hat{\lambda}_{i-1}(l+1) = G$ with high probability. As will be seen in Section 33.5, player i with $c_i(l) = j \in -i$ believes that player i is indifferent between any action profile. This implies that player $i-1$ does not monitor player i by $\Psi_{i-1}^{a(x)}$ and that player i does not need to infer $\lambda_{i-1}(l+1)$ seriously.

That is, $\hat{\lambda}_{i-1}(1) = G$. If $\hat{\lambda}_{i-1}(l) = B$, then $\hat{\lambda}_{i-1}(l+1) = B$. If $\hat{\lambda}_{i-1}(l) = G$, then $\hat{\lambda}_{i-1}(l+1) \in \{G, B\}$ is defined as follows.

Intuitively, player i calculates the conditional expectation of $X_{i-1}(l)$, believing that $a(x)$ was taken: $\mathbb{E}\left[X_{i-1}(l) \mid a(x), \{y_{i,t}\}_{t\in T(l)}\right]$. Instead, as in the two-player case, player i calculates

$$\sum_{t\in T_i(l)} \mathbb{E}\left[\Psi^{a(x)}_{i-1,t} \mid a(x), y_{i,t}\right] + q_2.$$

Further, instead of using $\mathbb{E}\left[\Psi^{a(x)}_{i-1,t} \mid a(x), y_{i,t}\right]$ directly, player i constructs $(E_i\Psi^{a(x)}_{i-1})_t \in \{0,1\}$ and $E_iX_{i-1}(l) = \sum_{t\in T_i(l)}(E_i\Psi^{a(x)}_{i-1})_t$ as in the two-player case.

With ex post (given $\{a_t, y_t\}_{t\in T(l)}$) probability $1-\exp(-O(T))$, we have

$$\left|\sum_{t\in T_i(l)} \mathbb{E}\left[\Psi^{a(x)}_{i-1,t} \mid a(x), y_{i,t}\right] + q_2 - E_iX_{i-1}(l)\right| \leq \frac{1}{4}\varepsilon T. \tag{112}$$

In addition, if (112) is the case and

$$E_iX_{i-1}(l) \in [q_2T - \frac{1}{2}\varepsilon T, q_2T + \frac{1}{2}\varepsilon T], \tag{113}$$

then since $T_i(l)$ and $T_{i-1}(l)$ are different at most for two periods, player i has

$$\mathbb{E}\left[X_{i-1}(l) \mid a(x), \{y_{i,t}\}_{t\in T(l)}, T_i(l), T_{i-1}(l)\right] \in [q_2T - \varepsilon T, q_2T + \varepsilon T]. \tag{114}$$

Given above statistics, there are following cases:

1. If "(112) and (113) are satisfied" or $c_i(l) = j \in -i$, then player i randomly picks the following two procedures:

   (a) With large probability $1-\eta$, player i has $\hat{\lambda}_{i-1}(l+1) = G$ regardless of the signals from the noisy cheap talk about $\lambda_{i-1}(l+1)$.

   (b) With small probability $\eta > 0$, player i will use the signal from the noisy cheap talk: $\hat{\lambda}_{i-1}(l+1)$ is determined by

$$\hat{\lambda}_{i-1}(l+1) = f[i](\lambda_{i-1}(l+1)). \tag{115}$$

2. If "(112) or (113) is not satisfied" and $c_i(l) \in \{0, i\}$, then $\hat{\lambda}_{i-1}(l+1)$ is determined by (115).

33.4.3 Transition of $d_i(l+1)$

We define the transition of $d_i(l+1) \in \{0, i-1, i+1\}$: $d_i(l+1) = 0$ implies that player i believes that neither player $i-1$ nor $i+1$ has deviated in the lth review round; $d_i(l+1) = i-1$ implies that player i believes that player $i-1$ has unilaterally deviated; $d_i(l+1) = i+1$ implies that player i believes that player $i+1$ has unilaterally deviated.

As player $i-1$ constructs $\Psi_{i-1,t}^{a(x)} \in \{0,1\}$ from $\psi_{i-1}^{a(x)}(y_{i-1,t})$, player i constructs $\Gamma_i^{a(x)}$ from $\gamma_i^{a(x)}(y_i)$. Remember that player i picks $t_i(l)$ randomly from $T(l)$ when she constructs $X_i(l)$. With the same $t_i(l)$, player i calculates another score about player $i-1$:

$$G_i(l) = \sum_{t \in T_i(l)} \Gamma_{i,t}^{a(x)} + \mathbf{1}_{t_i(l)}.$$

From Lemma 17, player i can monitor player $i-1$ by $G_i(l)$. That is, a low realization of $G_i(l)$ implies player $(i-1)$'s deviation.

Define $d_i(1) = 0$. For $l = 1, ..., L-1$, $d_i(l+1)$ is determined as follows:

1. If $c_i(l) \neq 0$, then $d_i(l+1) = d_i(l)$.

2. If $c_i(l) = 0$ and $\lambda_i(l) = G$, that is, if player i has not observed an erroneous history, then player i picks one of the following transitions randomly:

   (a) With small probability 2η, $d_i(l+1) = 0$.

   (b) With small probability 2η, $d_i(l+1) = i-1$.

   (c) With small probability 2η, $d_i(l+1) = i+1$.

   (d) With large probability $1-6\eta$, $d_i(l+1)$ is determined as follows. Conditional on 1-(d), $d_i(l+1) = i-1$ with probability

$$p_i^{l+1}(x) \min\left\{1, \frac{\{q_2 T - G_i(l)\}_+}{q_2 T - q_1 T}\right\} \tag{116}$$

and $d_i(l+1)=0$ with the remaining probability.

3. If $c_i(l)=0$ and $\lambda_i(l)=B$, that is, if player i has observed an erroneous history, then

   (a) If $x_i=G$, then player i picks one of the following transitions randomly:

      i. With small probability 3η, $d_i(l+1)=i-1$.

      ii. With small probability 3η, $d_i(l+1)=i+1$.

      iii. With large probability $1-6\eta$, $d_i(l+1)=0$.

   (b) If $x_i=B$, then player i picks one of the following transition randomly:

      i. With small probability 3η, $d_i(l+1)=0$.

      ii. With small probability 3η, $d_i(l+1)=i-1$.

      iii. With large probability $1-6\eta$, $d_i(l+1)=i+1$.

We postpone the intuitive explanation of this transition until Section 33.4.5.

33.4.4 Transition of $d_j(l+1)(i)$

If $j=i$, then player i knows $d_j(l+1)$. Hence, $d_j(l+1)(i)=d_j(l+1)$.

For each $j\neq i$, player i constructs $d_j(l+1)(i)$ by

$$d_j(l+1)(i)=f[i](d_j(l+1)) \tag{117}$$

using the signals that arrives when player j sends the message about $d_j(l+1)$ via noisy cheap talk between players j and i.

33.4.5 Transition of $c_i(l+1)\in\{0\}\cup I$

As seen in Section 33.2, $c_i(l+1)\in\{0\}\cup I$ is player i's index about whom to minimax. $c_i(l+1)=0$ implies that player i does not minimax any player in the $(l+1)$th review round while $c_i(l+1)=n\in -i$ implies that player i minimaxes player n. The action after $c_i(l+1)=i$ depends on $\hat{\lambda}_{i-1}(l+1)$.

Player i constructs a variable $c_i(l+1) \in \{0\} \cup I$ as follows: $c_i(1) = 0$ (no player minimaxes any player in the initial review round). For $l \geq 1$, if $c_i(l) \in I$, then $c_i(l+1) = c_i(l)$ (intuitively, once player i decides to minimax player j, she will keep minimaxing player j). If $c_i(l) = 0$, then if there exists a unique $j \in I$ such that $d_{j-1}(l+1)(i) = d_{j+1}(l+1)(i) = j$, then $c_i(l+1) = j$. In this case, we say player j's unilateral deviation is "confirmed." Otherwise, $c_i(l+1) = 0$. As explained in Section 27, the consensus between players $j-1$ and $j+1$ is necessary to confirm player j's deviation (if the communication about $\{d_j(l+1)\}_j$ does not have a noise).

Let us explain the basic structure of the coordination on the punishment. For a simple explanation, for a while, assume that there is no noise in the communication: $d_j(l+1) = d_j(l+1)(n)$ for all j and n.

See Section 33.4.3. Since we want to consider player i's incentive, we consider the transition of $d_j(l+1)$ for the two monitors of player i, players $i-1$ and $i+1$. Neglect the events that happen with probability no more than 3η. Then, the transition is as follows: For $j \in \{i-1, i+1\}$,

1. If $c_j(l) \neq 0$, then $d_j(l+1) = d_j(l)$.

2. If $c_j(l) = 0$ and $\lambda_j(l) = G$, then $d_j(l+1) = j-1$ with probability

$$p_j^{l+1}(x) \min\left\{1, \frac{\{q_2 T - G_j(l)\}_+}{q_2 T - q_1 T}\right\}$$

   and $d_j(l+1) = 0$ with the remaining probability.

3. If $c_j(l) = 0$ and $\lambda_j(l) = B$, then

   (a) If $x_j = G$, then $d_j(l+1) = 0$.

   (b) If $x_j = B$, then $d_j(l+1) = j+1$.

From 2, while $\lambda_{i-1}(l) = G$, player $i-1$ monitors player $i-2$, not player i. That is, $d_{i-1}(l+1) \neq i$. Remember that the consensus between players $i-1$ and $i+1$ is necessary for the confirmation of player i's deviation. Therefore, while $\lambda_{i-1}(l) = G$, player i is not minimaxed.

Consider the case with $\lambda_{i-1}(l) = B$. If $x_{i-1} = G$, then from 3-(a), $d_{i-1}(l+1) \neq i$. By the same reason, player i is not minimaxed. If $x_{i-1} = B$, then from 3-(b), player $i-1$ always infers player i's deviation. Hence, it is all up to player $i+1$ to confirm player i's deviation. Suppose that player $i+1$ has $\lambda_{i+1}(l) = G$. Then, from 1, player $i+1$ monitors player i as player j with $x_j = \lambda_j(l) = B$ monitors player i in the two-player case: See (82) and (116). Therefore, together with Lemma 21, if player i believes that $\lambda_{i-1}(l) = B$, then player i should take the static best response to players $(-i)$'s action: Player i with $\hat{\lambda}_{i-1}(l) = B$ and $c_i(l) = 0$ takes $BR_i(a_{-i}(x))$.

33.4.6 Transition of $\theta_{i-1}(l) \in \{G, B\}$

As in the two-player case, $\theta_{i-1}(\tilde{l}) = B$ with for some $\tilde{l} \leq l-1$ implies that player i is indifferent between any action profile (except for the incentives from π_i^{report}). Here, we consider player $(i-1)$'s state since we want to consider player i's incentive and player i's value is affected by player $(i-1)$'s state.

$\theta_{i-1}(l) = B$ if one of the following four conditions is satisfied:

1. There exists player $j \neq i-1, i$ such that 1 is the case when player j creates $\lambda_j(l+1)$ in Section 33.4.1 (replace $i-1$ with j).

2. There exists player $j \neq i$ such that 1-(b) or 2 is the case when player j creates $\hat{\lambda}_{j-1}(l+1)$ in Section 33.4.2 (replace i with j).

3. There exists player $j \neq i$ who picks a case that happens with probability at most 3η when player j creates $d_j(l+1)$ in Section 33.4.3 (replace i with j).

4. There exists player $j \neq i, i+1$ who has $d_j(l+1) = j-1$ in 2-(d) of Section 33.4.3 (replace i with j).

Except for the above four cases, $\theta_{i-1}(l) = G$.

33.4.7 Transition of $\theta_{i-1}(\lambda_j(l+1)) \in \{G, B\}$

$\theta_{i-1}(\lambda_j(\tilde{l}+1)) = B$ with for some $j \in I$ and $\tilde{l} \leq l-1$ implies that player i is indifferent between any action profile (except for the incentives from π_i^{report}). Compared to the two-player case, player $i-1$ takes care of the miscommunication between players j and $j+1$ with $j \neq i$.

$\theta_{i-1}(\lambda_j(l+1)) = B$ if one of the following conditions is satisfied:

1. For $j = i-1$, when player $i-1$ sends $\lambda_{i-1}(l+1)$ to player i via noisy cheap talk, player $i-1$ receives the signal indicating that player i's signal may be wrong: $g[i-1](\lambda_{i-1}(l+1)) = E$.

2. For $j \neq i-1, i$, when player j sends $\lambda_j(l+1)$ to player $j+1$ via noisy cheap talk, player $j+1$ receives a wrong signal: $f[j+1](\lambda_j(l+1)) \neq \lambda_j(l+1)$.

Note that in order to know 2 is the case, player $i-1$ needs to know $\lambda_j(l+1)$ and $f[j+1](\lambda_j(l+1))$. These variables are sent in the re-report block and so included in $h_{i-1}^{\text{rereport}}$. Since θ_{i-1} only affects the reward function (does not affect $\sigma_{i-1}(x_{i-1})$), it suffices that player $i-1$ knows the information by the end of the review phase.

Except for the above two cases, $\theta_{i-1}(\lambda_j(l+1)) = G$.

33.4.8 Transition of $\theta_{i-1}(d_j(l+1)) \in \{G, B\}$

The intuitive explanation is the same as $\theta_{i-1}(\lambda_j(l+1))$.

$\theta_{i-1}(d_j(l+1)) = B$ if one of the following two conditions is satisfied:

1. For $j \in -i$ and i, when player j sends $d_j(l+1)$ to player i via noisy cheap talk, player $i-1$ receives the signal indicating that player i's signal may be wrong: $g[i-1](d_j(l+1)) = E$.

2. For $j \in -i$ and $n \in -i$, when player j sends $d_j(l+1)$ to player j via noisy cheap talk, player n receives a wrong signal: $f[n](d_j(l+1)) \neq d_j(l+1)$.

Again, the information about $d_j(l+1)$ with $j \neq i-1, i$ and $f[n](d_j(l+1))$ with $n \neq i-1, i$ is sent in the re-report block and so included in $h_{i-1}^{\text{rereport}}$.

Except for the above two cases, $\theta_{i-1}(d_j(l+1)) = G$.

33.5 Summary of the Transitions of θ_{i-1}

First, we consider the implication of the transitions of θ_{i-1} on $\left(\hat{\lambda}_{j-1}\right)_{j \neq i}$ and $(\lambda_{j-1})_{j \neq i, i+1}$.

If $\hat{\lambda}_{j-1}(l) = B$ for some $j \neq i$, then 1-(b) or 2 is the case when player j creates $\hat{\lambda}_{j-1}(\tilde{l}+1)$ in Section 23.4.2 (replace i with j). From Section 33.4.6, we have $\theta_{i-1}(\tilde{l}) = B$ for some $\tilde{l} \leq l-1$. This also implies that, whenever player i's message $\lambda_i(\tilde{l}+1)$ changes player $(i+1)$'s continuation play, player i has been indifferent between any action profile sequence.

If $\lambda_{j-1}(l) = B$ for some $j \neq i, i+1$, then from Case 1 of Section 33.4.6, we have $\theta_{i-1}(\tilde{l}) = B$ for some $\tilde{l} \leq l-1$.

In summary, we can concentrate on the case with $\hat{\lambda}_{j-1}(l) = G$ for all $j \neq i$ and $\lambda_{j-1}(l) = G$ for all $j \neq i, i+1$.

Second, we consider the implication of the transitions of θ_{i-1} on player i's incentive to tell the truth about $d_i(l+1)$ in the supplemental rounds.

Suppose that there exists player $n \in I$ for whom $d_i(l+1)(n)$ has an impact on $c_n(l+1)$. We will show that this implies $\theta_{i-1}(\tilde{l}) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1))$ for some $j \in I$ and $\tilde{l} \leq l$.

1. If there exists player $j \neq i$ whose message player $n \in I$ misinterprets, that is, if $d_j(l+1)(n) \neq d_j(l+1)$, then

   (a) If $n = i$, then from Case 1 of Section 33.4.8, player i believes that $\theta_{i-1}(d_j(l+1)) = B$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Since the continuation play by players $-i$ is independent of $g[i-1](d_j(l+1))$, this is true for all the main blocks.

   (b) If $n \in -i$, then from Case 2 of Section 33.4.8, we have $\theta_{i-1}(d_j(l+1)) = B$.

   Hence, for almost optimality, player i can believe that $d_j(l+1)(n) = d_j(l+1)$ for all $j \neq i$ and $n \in I$.

2. If player $j \neq i$ picks a case which occurs with probability at most 3η, then from Case 3 of Section 33.4.6, we have $\theta_{i-1}(\tilde{l}) = B$ for some $\tilde{l} \leq l$.

   Hence, we can assume that $d_j(\tilde{l}+1)$ with $j \neq i$ and $\tilde{l} \leq l$ would transit as if described in Section 33.4.5.

   Given above, there are two possible cases where $d_i(l+1)(n)$ has an impact on whether $c_n(l+1)$ for some $n \in I$:

3. $d_{i-2}(l+1) = i-1$. This implies that player $i-2 \neq i-1, i$ has monitored player $i-1$. From Section 33.4.5, this implies that $\lambda_{i-2}(l) = B$. Since $i-2 \neq i-1, i$ (or $\lambda_{j-1}(l) = B$ with $j \neq i, i+1$), from Case 1 of Section 33.4.6, we have $\theta_{i-1}(\tilde{l}) = B$ for some $\tilde{l} \leq l$.

4. $d_{i+2}(l+1) = i+1$. This implies that $d_{j+1}(l+1) = j$ with $j \neq i, i+1$. From Case 4 of Section 33.4.6, we have $\theta_{i-1}(\tilde{l}) = B$ for some $\tilde{l} \leq l$.

Therefore, if there exists player $n \in I$ for whom $d_i(l+1)(n)$ has an impact on $c_n(l+1)$, then $\theta_{i-1}(\tilde{l}) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1))$ for some $j \in I$ and $\tilde{l} \leq l$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.

Third, we consider the implication of θ_{i-1} on the mis-coordination on $(c_n)_{n \in I}$. Suppose that there exist $n, n' \in I$ with $n \neq n'$ such that $c_n(l) \neq c_{n'}(l)$. There are two possible cases:

1. If $c_n(l)$ and $c_{n'}(l)$ have been determined independently of $\left\{d_i(\tilde{l}+1)(n)\right\}_{\tilde{l} \leq l-1}$ and $\left\{d_i(\tilde{l}+1)(n')\right\}_{\tilde{l} \leq l-1}$, then it means that there exists player $j \neq i$ such that there was a miscommunication between j and "n or n'." By the same reason as above, player i believes that $\theta_{i-1}(d_j(\tilde{l}+1)) = B$ for some $\tilde{l} \leq l-1$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.

2. If $\left\{d_i(\tilde{l}+1)(n)\right\}_{\tilde{l} \leq l}$ or $\left\{d_i(\tilde{l}+1)(n')\right\}_{\tilde{l} \leq l}$ has an impact on $c_n(l)$ or $c_{n'}(l)$, then from the discussion above, this implies $\theta_{i-1}(\tilde{l}) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1))$ for some j and $\tilde{l} \leq l-1$.

Therefore, if there exist players $n \in I$ and $n' \in I$ with $c_n(l) \neq c_{n'}(l)$, then $\theta_{i-1}(\tilde{l}) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1))$ for some j and $\tilde{l} \leq l-1$.

Fourth, we will show that, if player i is not indifferent between any action profile sequence, then $c_n(l)$ must be i or 0 for all $n \in I$. Suppose that there exist $n \in I$ and $j \in -i$ with $c_n(l) = j$. We will show that this implies $\theta_{i-1}(\hat{l}) = B$ or $\theta_{i-1}(d_{j'}(\hat{l}+1))$ for some j' and $\hat{l} \leq l-1$.

As above, for almost optimality, we can concentrate on the case with $d_{j'}(\tilde{l}+1)(n) = d_{j'}(\tilde{l}+1)$ for all $n \in I$, $j' \in -i$ and $\tilde{l} \leq l-1$. In addition, we can assume that $d_{j'}(\tilde{l}+1)$ with $j' \in -i$ would transit as if described in Section 33.4.5.

1. If $j \neq i+1$, then $d_{j-1}(\tilde{l}+1) = j$. From Section 33.4.5, this implies that $\lambda_{j-1}(l) = B$. Since $j-1 \neq i-1, i$ (or $j \neq i, i+1$), this implies $\theta_{i-1}(\hat{l}) = B$ for some $\hat{l} \leq l-1$.

2. If $j = i+1$, then $d_{j+1}(\tilde{l}+1) = j$. Since $j+1 = i+2 \neq i, i+1$, from Case 4 of Section 33.4.6, we have $\theta_{i-1}(\hat{l}) = B$ for some $\hat{l} \leq l-1$.

Therefore, $\theta_{i-1}(\tilde{l}) = \theta_{i-1}(\lambda_j(\tilde{l}+1)) = \theta_{i-1}(d_j(\tilde{l}+1)) = G$ for all j and $\tilde{l} \leq l-1$ implies that $c_n(l) = i$ for all $n \in I$ or $c_n(l) = 0$ for all $n \in I$.

Finally, we consider when player i can be minimaxed. Suppose that if $c_n(l) = i$ for all $n \in I$, then except for the case with miscommunications between $i-1$ and n,[74] we have $d_{i-1}(\tilde{l}+1) = i$. Then, player $i-1$ picks a case that happens with probability at least 3η or $x_{i-1} = \lambda_{i-1}(\tilde{l}) = B$ from Section 33.4.3. If the former is the case, then we have $\theta_{i-1}(\tilde{l}) = B$. Hence, player i can believe that player i is minimaxed with $\theta_{i-1}(\tilde{l}) = \theta_{i-1}(\lambda_j(\tilde{l}+1)) = \theta_{i-1}(d_j(\tilde{l}+1)) = G$ for all j and $\tilde{l} \leq l-1$ only if $x_{i-1} = \lambda_{i-1}(\tilde{l}) = B$.

In total, we have shown the following statements:

1. We can concentrate on the following two cases:

   (a) $c_n(l) = 0$ for all $n \in -i$. In this case, $a_{-i}(x)$ is played in the lth review round.

   In (a), from Lemmas 16 and 17, the distribution of $X_j(l)$ with $j \neq i-1, i$, $E_j X_{j-1}(l)$ with $j \neq i$ and $G_j(l)$ with $j \neq i-1, i$ is independent of player i's strategy. Therefore, the distribution of $\theta_{i-1}(l)$ is independent of player i's strategy.

[74] If so, then $\theta_{i-1}(d_{i-1}(\tilde{l}+1)) = B$ with probability at least $1 - \exp(-O(T^{\frac{1}{2}}))$ from the discussion above.

(b) $x_{i-1} = \lambda_{i-1}(l) = B$ and $c_n(l) = i$ for all $n \in -i$. In this case, $\alpha_{-i}^{\min\max}$ is played in the lth review round.

In (b), the distribution of $\theta_{i-1}(l)$ is independent of player i's strategy since

i. For all $j \in -(i-1, i)$, $\lambda_j(l+1)$ is fixed.

ii. For all $j \in -i$, $\hat{\lambda}_{j-1}(l+1)$ is determined by player j's mixture.

iii. For all $j \in -i$, $d_j(l+1)$ is fixed.

2. If player i constructs $c_i(l)$ as prescribed, then $c_n(\tilde{l}) = c_{n'}(\tilde{l})$ for all $\tilde{l} \leq l$.

3. Whenever player i's message $\lambda_i(l+1)$ changes player $(i+1)$'s continuation play, player i is indifferent between any action profile.

4. Whenever there exists player $n \in I$ for whom $d_i(l+1)(n)$ matters for $c_n(l+1)$, then player i is indifferent between any action profile.

Therefore, we have shown the following lemma:

Lemma 22 *For all $i \in I$ and $l = 1, ..., L$, for any history of player i in the lth review round, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the following events:*

1. *Player $(-i)$'s action profile in the lth review round satisfies one of the following four:*

   (a) $\lambda_{i-1}(l) = G$, $c_i(l) = 0$ and $a_{-i}(x)$ is played,

   (b) $\lambda_{i-1}(l) = B$, $c_i(l) = 0$ and $a_{-i}(x)$ is played,

   (c) $x_{i-1} = \lambda_{i-1}(l) = B$, $c_i(l) = i$, and $\alpha_{-i}^{\min\max}$ is played, or

   (d) $\theta_{i-1}(\tilde{l}) = B$, $\theta_{i-1}(\lambda_j(\tilde{l}+1)) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1)) = B$ for some j and $\tilde{l} \leq l-1$.

2. *If $\theta_{i-1}(\tilde{l}) = \theta_{i-1}(\lambda_j(\tilde{l}+1)) = \theta_{i-1}(d_j(\tilde{l}+1)) = G$ for all $j \in I$ and $\tilde{l} \leq l-1$, then the distribution of $\theta_{i-1}(l)$, $\theta_{i-1}(\lambda_j(l+1))$ and $\theta_{i-1}(d_j(l+1))$ for all $j \in I$ is independent of player i's strategy.*

3. *If player i's message $\lambda_i(l+1)$ changes player $(i+1)$'s continuation play, then $\theta_{i-1}(l) = B$.*

4. *If there exists player $n \in I$ for whom $d_i(l+1)(n)$ matters for $c_n(l+1)$, then $\theta_{i-1}(\tilde{l}) = B$, $\theta_{i-1}(\lambda_j(\tilde{l}+1)) = B$ or $\theta_{i-1}(d_j(\tilde{l}+1)) = B$ for some $j \in I$ and $\tilde{l} \leq l-1$.*

34 Variables

In this section, we finish defining the variables necessary for the equilibrium construction: q_2, q_1, $\bar{u}$, $\bar{L}$, L, η and ε. $\bar{u}$, q_1 and q_2 are determined in Lemmas 16, 17 and 19. We take $\bar{L}$ sufficiently large so that

$$\bar{L}\left(q_2 - q_1\right) > \max_{a,i} 2\left|u_i\left(a\right)\right|.$$

We are left to pin down L, $\varepsilon > 0$ and $\eta > 0$. Take L sufficiently large and $\varepsilon > 0$ sufficiently small such that

$$\begin{aligned} & \max_{x:x_{i-1}=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \frac{\bar{L}}{L} + 2\varepsilon\bar{L} \\ < \;& \underline{v}_i < \overline{v}_i < \min_{x:x_{i-1}=G} w_i(x) - \frac{\bar{L}}{L} - 2\varepsilon\bar{L} \end{aligned}$$

Then, take $\eta > 0$ sufficiently small so that

$$\begin{aligned} & \max_{x:x_{i-1}=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \eta + \frac{\bar{L}}{L} + 2\varepsilon\bar{L} + 7\left(N-1\right)L\eta\left(\bar{u} - \min_{i,a} w_i\left(a\right)\right) \\ < \;& \underline{v}_i < \overline{v}_i < \min_{x:x_{i-1}=G} w_i(x) - \frac{\bar{L}}{L} - 2\varepsilon\bar{L} - 7\left(N-1\right)L\eta\left(\bar{u} + \max_{i,a} w_i\left(a\right)\right). \end{aligned} \tag{118}$$

35 Almost Optimality

Again, we want to show the almost optimality first: We construct $\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)$ such that (8), (4) and (5) are satisfied.

35.1 Almost Optimality of the Inferences

We show the almost optimality of $\hat{\lambda}_{i-1}(l+1)$ and $c_i(l)$. The basic logic is the same as in Lemma 8. Let $\alpha_{-i}(l)$ be player $(-i)$'s action plan in the lth review round[75] and $\boldsymbol{\alpha}_{-i}(l) = (\alpha_{-i}(1), ..., \alpha_{-i}(l))$ be the sequence of player $(-i)$'s action plans from the first review round to the lth review round (excluding what messages players $-i$ sent by the noisy cheap talk).

Lemma 23 *For any lth review round, for any h_i^t with period t in the lth review round, conditional on $\boldsymbol{\alpha}_{-i}(l)$, player i puts a belief no less than*

$$1 - \exp(-O(T^{\frac{1}{2}})) \tag{119}$$

on the events that

1. *If $\hat{\lambda}_{i-1}(l) = G$, then*
   (a) *"players $-i$ take $a_{-i}(x)$ and $\lambda_{i-1}(l) = G$" or any action is optimal.*
   (b) *If $c_i(l) = j \neq 0$, then any action is optimal.*
2. *If $\hat{\lambda}_{i-1}(l) = B$, then*
   (a) *If $c_i(l) \neq i$, then "players $-i$ take $a_{-i}(x)$ and $\lambda_{i-1}(l) = B$" or any action is optimal.*
   (b) *If $c_i(l) = i$, then "players $-i$ take $\alpha_{-i}^{\min\max}$ and $\lambda_{i-1}(l) = B$" or any action is optimal.*

Proof. From Lemma 22, it suffices to show that player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that $\lambda_{i-1}(l) = \hat{\lambda}_{i-1}(l)$ or any action is optimal.

As in the two-player case, there exists a unique l^* such that $\lambda_{i-1}(l)$ switches from G to B at the end of the l^*th review round. In addition, there exists $\hat{l}^*$ such that $\hat{\lambda}_{i-1}(l)$ switches

[75] Note that players $-i$ take an *i.i.d.* action plan within a review round.

from G to B at the end of the $\hat{l}^*$th main block. If $\lambda_{i-1}(L) = G$ ($\hat{\lambda}_{i-1}(L) = G$, respectively), then define $l^* = L$ ($\hat{l}^* = L$, respectively).

Then, there are following three cases:

- If $l^* \geq \hat{l}^*$, then the proof is the same as Lemma 8.
- If $l^* < \hat{l}^*$, then there are following two cases:
  - If 1-(b) or 2 is the case when player i creates $\hat{\lambda}_{i-1}(l^* + 1)$ in Section 33.4.2, then, again, the proof is the same as Lemma 8. We assume that, whenever player i has deviated from $\sigma_i(x_i)$ in the l^*th review round or before, player i creates $\hat{\lambda}_{i-1}(l^*+1)$ by (115).
  - If 1-(a) is the case in Section 33.4.2, then there are following three cases:
    * If $c_i(l^*) \neq 0$, then since $\lambda_{i-1}(l^*) = G$, from 4 of Lemma 22, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that any action is optimal, as desired.
    * If players $-i$ played $a_{-i} \neq a_{-i}(x)$ in the l^*th review round, then from 1 of Lemma 22, player i puts the belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that any action is optimal, as desired.
    * $c_n(l^*) = 0$ for all $n \in I$. Then, the players play $a(x)$. The rest of the proof is the same as Lemma 8. Note that players $(-i)$'s continuation play is determined by $\left(\hat{\lambda}_{j-1}(\tilde{l})\right)_{j\in -i}$ and $\left(c_n(\tilde{l})\right)_{n\in -i}$. For $\hat{\lambda}_{j-1}(\tilde{l})$, errors happen with probability $\exp(-O(T^{\frac{1}{2}}))$ since each player j uses the signals from the noisy cheap talk with probability at least η. For $c_n(\tilde{l})$, each player n always uses $d_j(\tilde{l}+1)(n)$ for all $j \in I$ and any $d_j(\tilde{l}+1)(n)$ happens with probability at least 2η as long as $c_n(\tilde{l}) = 0$. Therefore, learning from the continuation play does not update the belief more than $\exp(O(T^{\frac{1}{2}}))$.

■

35.2 Determination of $\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)$

Based on Lemmas 21, 22 and 23, we determine $\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)$ such that $\sigma_i(x_i)$ and π_i^{main} satisfy (8), (4) and (5).

Proposition 3 *For all $i \in I$, there exists $\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)$ such that*

1. *$\sigma_i(x_i)$ is almost optimal: For each $l \in \{1, ..., L\}$,*
   (a) *For any period t in the lth review round, (8) holds.*
   (b) *When player i sends the noisy cheap talk messages in the supplemental rounds, (8) holds.*[76]
2. *(4) is satisfied with π_i replaced with π_i^{main}. Since each $x_i \in \{G, B\}$ gives the same value conditional on x_{-i}, the strategy in the coordination block is optimal.*[77]
3. *π_i^{main} satisfies (5).*

1-(b) follows from the following two facts: First, 3 and 4 of Lemma 22 imply that, whenever player i's message changes some player's action, player i has been indifferent between any action profile. Second, Lemma 23 implies that player i can infer $(c_n(l))_{n \in -i}$ and $\lambda_{i-1}(l)$ with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ (or any action is optimal) by taking the equilibrium strategy. Therefore, the equilibrium strategy is almost optimal.

As in the two-player case, for 3, it suffices to have

$$\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l) \begin{cases} \leq 0 & \text{if } x_{i-1} = G, \\ \geq 0 & \text{if } x_{i-1} = B, \end{cases} \tag{120}$$

$$|\bar{\pi}_i(x, \lambda_{i-1}(l), c_{i-1}(l), l)| \leq \max_{i,a} 2 |u_i(a)| T \tag{121}$$

for all $x \in \{G, B\}^N$, $\lambda_{i-1}(l) \in \{G, B\}$, $c_{i-1}(l) \in \{0\} \cup I$ and $l \in \{1, ..., L\}$.

[76] If $l = L$, then this is redundant.

[77] As in the two-player case, even after the adjustment of the report block, any $x_i \in \{G, B\}$ still gives exactly the same value.

We are left to construct $\bar{\pi}_i$ so that 1-(a) and 2 are satisfied together with (120) and (121).

Remember that, from Lemma 23, it is almost optimal to take any action after $c_i(l) = j \in -i$. Hence, we verified the incentive to minimax player j if $\hat{\lambda}_{i-1}(l) = G$ and $c_i(l) = j$ as desired.

Further, from Lemma 23, we can concentrate on the following five cases:

- $x_{i-1} = G$, $\hat{\lambda}_{i-1}(l) = \lambda_{i-1}(l) = G$, $c_i(l) = 0$ and players $-i$ take $a_{-i}(x)$. Remember that player i takes $a_i(x)$ from Section 33.2.

- $x_{i-1} = G$, $\hat{\lambda}_{i-1}(l) = \lambda_{i-1}(l) = B$, $c_i(l) = 0$ and players $-i$ take $a_{-i}(x)$. Remember that player i takes $BR_i(a_{-i}(x))$ from Section 33.2.

- $x_{i-1} = B$, $\hat{\lambda}_{i-1}(l) = \lambda_{i-1}(l) = G$, $c_i(l) = 0$ and players $-i$ take $a_{-i}(x)$. Remember that player i takes $a_i(x)$ from Section 33.2.

- $x_{i-1} = B$, $\hat{\lambda}_{i-1}(l) = \lambda_{i-1}(l) = B$, $c_i(l) = 0$ and players $-i$ take $a_{-i}(x)$. Remember that player i takes $BR_i(a_{-i}(x))$ from Section 33.2.

- $x_{i-1} = B$, $\hat{\lambda}_{i-1}(l) = \lambda_{i-1}(l) = B$, $c_i(l) = i$ and players $-i$ take $\alpha_{-i}^{\min\max}$. Remember that player i takes $BR_i(\alpha_{-i}^{\min\max})$ from Section 33.2.

Therefore, almost optimality of $\sigma_i(x_i)$ and the existence of $\bar{\pi}_i$ with (120) and (121) can be shown as in the two-player case. Remember that if the fourth bullet point is the case, then player $i-1$ with $\lambda_{i-1}(l) = B$ has $d_{i-1}(l+1) = i$ with probability at least $1 - 6\eta$ and it is up to player $i+1$ to monitor player i (see Section 33.4.3). (116) implies that player $i+1$ monitors player i as player j monitors player i in the two-player case. Therefore, from Lemma 21, it is optimal for player i to take $BR_i(a_{-i}(x))$, as desired.

Note that the slack in (118) is enough since, for each lth main block, except for the cases that happen with probability no more than $\exp(-O(T^{\frac{1}{2}}))$, $\theta_{i-1}(l) = B$, $\theta_{i-1}(\lambda_j(l+1)) = B$ or $\theta_{i-1}(d_j(l+1)) = B$ happens only if

- There is player $j \neq i$ who uses the signal from the noisy cheap talk to create $\hat{\lambda}_{j-1}(l+1)$ (Case 1-(b) or 2 in Section 33.4.2 with i replaced with j). This happens with probability no more than $(N-1)\,\eta$.
- There is player $j \neq i$ who has $d_j(l+1)$ that occurs with probability at most 3η. This happens with probability no more than $(N-1)\,6\eta$.

36 Report Block

We are left to construct the report and re-report blocks to attain the exact optimality of the equilibrium strategies. In this section, we explain the report block.

36.1 Structure of the Report Block

The report block proceeds as follows:

1. Player N sends the message about h_N^{main}.
2. Player $N-1$ sends the message about h_{N-1}^{main}.

   $\vdots$

3. Player 3 sends the message about h_3^{main}.
4. Then, public randomization y^p is drawn.
5. Player 1 reports h_1^{main} if $y^p \leq \frac{1}{2}$ and player 2 reports h_2^{main} if $y^p > \frac{1}{2}$.

We explain each step in the sequel.

36.2 Player i sends h_i^{main}

Since there is a chronological order for the rounds and r is a generic serial number of rounds, the notations $\#_i^r$, $\#_i^r(k)$, $T(r,k)$ and $\{a_{i,t}, y_{i,t}\}_{t \in T(r,k)}$ defined in the Appendix of the main paper is still valid except that

- If player i sends m to player n via noisy cheap talk in round r, then $\#_i^r$ contains m for sure. In addition, if and only if player $n-1$ is player i, $g[n-1](m)$ and $g_2[n-1](m)$ are also included in $\#_i^r$.

- If player i receives m from player n via noisy cheap talk in round r, then $\#_i^r$ contains $f[i](m)$ for sure. In addition, if and only if player $n-1$ is player i, $f_2[n-1](m)$ is also included.

Player i sends the message about h_i^{main} in the same way as player i sends the message in Section 15.7. That is, for each round r,

- If round r corresponds to a review round, then
  - First, player i reports the summary $\#_i^r$.
  - Second, for each subround k, player i reports the summary $\#_i^r(k)$.
  - Third, public randomization is drawn such that each subround k is randomly picked with probability $T^{-\frac{3}{4}}$. Let $k(r)$ be the subround picked by the public randomization.
  - Fourth, for $k(r)$, player i reports the whole history $\{a_{i,t}, y_{i,t}\}_{t \in T(r,k(r))}$ in the $k(r)$th subround.

- If player i sends or receives a noisy cheap talk message in round r, then player i reports $\#_i^r$.

Again, the necessary number of binary messages to send the information is

$$O(T^{\frac{1}{4}}). \tag{122}$$

36.3 Reward Function π_i^{report}

We are left to define the reward function π_i^{report}. As a preparation, we prove the following lemma:

Lemma 24 *Let h_i be player i's history right before player i sends the message about h_i^{main} in the report block. If Assumption 9 is satisfied, then there exists $\bar{\varepsilon} > 0$ such that*

1. *For each $l \in \{1, ..., L\}$, in the lth review round, there exists $g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, a_i, y_i)$ such that, for period $t \in T(l)$, it is better for player i to report $(a_{i,t}, y_{i,t})$ truthfully: For all h_i,*

$$\begin{aligned} & \mathbb{E}\left[g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{a}_{i,t}, \hat{y}_{i,t}) \mid h_i, (\hat{a}_{i,t}, \hat{y}_{i,t}) = (a_{i,t}, y_{i,t})\right] \\ > \ & \mathbb{E}\left[g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{a}_{i,t}, \hat{y}_{i,t}) \mid h_i, (\hat{a}_{i,t}, \hat{y}_{i,t}) \neq (a_{i,t}, y_{i,t})\right] + \bar{\varepsilon} T^{-1}, \end{aligned} \tag{123}$$

where $(\hat{a}_{i,t}, \hat{y}_{i,t})$ is player i's message in the report block.

2. *For round r where player i sends or receives the noisy cheap talk message, it is better for player i to report player i's history $\#_i^r$ truthfully:*

$$\begin{aligned} & \mathbb{E}\left[g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) \mid h_i, \hat{\#}_i^r = \#_i^r\right] \\ > \ & \mathbb{E}\left[g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) \mid h_i, \hat{\#}_i^r \neq \#_i^r\right] + \bar{\varepsilon} T^{-1}, \end{aligned} \tag{124}$$

where $\hat{\#}_i^r$ is player i's message about $\#_i^r$ in the report block.

Proof.

1. By the same proof as Lemma 9, we can show that

$$g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{a}_{i,t}, \hat{y}_{i,t}) = -\mathbf{1}\{t_j(r) = t\} \left\| \mathbf{1}_{y_{j,t}} - \mathbb{E}[\mathbf{1}_{y_{j,t}} \mid \hat{a}_{i,t}, \hat{y}_{i,t}, a_{-i,t}, \left(y_{n,t}, \varphi_{n,t}\right)_{n \neq j,i}] \right\|^2$$

with

$$j = \begin{cases} i-1 & \text{if } i \neq 1, \\ 2 & \text{if } i = 1 \end{cases} \tag{125}$$

works. Note that, compared to Lemma 9,

   (a) $\varphi_n \equiv (\Psi_n^{a(x)}, (E_{n-1}\Psi_n), \Gamma_n)$ is a statistics that player n constructs in the lth review round.

(b) We condition the history $(y_{n,t}, \varphi_{n,t})_{n \neq j,i}$ of all the players except for player i herself and player j. (125) and the structure of the report block explained in Section 36.1 imply that player i cannot know player j's history from the report block.

As we will see, players $-(i-1,i)$ sends the information $t_j(r), a_{-(i-1,i)}, (y_n, \varphi_n)_{n \neq j,i}$ to player $i-1$ in the re-report block and so $t_j(r)$, $a_{-(i-1,i)}$ and $(y_n, \varphi_n)_{n \neq j,i}$ are in $h_{i-1}^{\text{rereport}}$.

2. If player i sends m to player n via noisy cheap talk in round r, then

(a) If $\#_i^r$ contains $g[n-1](m)$ and $g_2[n-1](m)$, then

$$g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) = -\left\| \mathbf{1}_{f_2[i-1](m)} - \mathbb{E}[\mathbf{1}_{f_2[i-1](m)} \mid f[n](m), \hat{\#}_i^r] \right\|^2 .$$

(b) If $\#_i^r$ does not contain $g[n-1](m)$ and $g_2[n-1](m)$, then

$$g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) = -\left\| \mathbf{1}_{f_2[i-1](m)} - \mathbb{E}[\mathbf{1}_{f_2[i-1](m)} \mid f[n](m), \hat{\#}_i^r, g[n-1](m), g_2[n-1](m)] \right\|^2 .$$

If player i receives m from player n via noisy cheap talk in round r, then

(a) If $\#_i^r$ contains $f_2[n-1](m)$, then

$$g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) = -\left\| \mathbf{1}_{g_2[i-1](m)} - \mathbb{E}[\mathbf{1}_{g_2[i-1](m)} \mid g[i-1](m), \hat{\#}_i^r] \right\|^2 .$$

(b) If $\#_i^r$ does not contain $f_2[n-1](m)$, then

$$g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r) = -\left\| \mathbf{1}_{g_2[i-1](m)} - \mathbb{E}[\mathbf{1}_{g_2[i-1](m)} \mid g[i-1](m), \hat{\#}_i^r, f_2[n-1](m)] \right\|^2 .$$

Then, (98) and (99) imply that truthtelling is optimal.

As we will see, player $n-1 \in -(i-1,i)$ sends $g[n-1](m)$, $g_2[n-1](m)$, $f[n-1](m)$ and $f_2[n-1](m)$ to player $i-1$ in the re-report block.

■

Given these preparations, by backward induction, we construct $\pi_i^{\text{report}}\left(h_{i-1}^{T_P+1}, h_{i-1}^{\text{rereport}}, \hat{h}_i^{r+1}, r\right)$ for each r such that

$$\pi_i^{\text{report}}(x_{i-1}, h_{i-1}^{T_P+1}, h_{i-1}^{\text{rereport}}) = \sum_r \pi_i^{\text{report}}\left(h_{i-1}^{r+1}, h_{i-1}^{\text{rereport}}, \hat{h}_i^{r+1}, r\right)$$

makes it optimal to tell the truth in the report block and $\sigma_i(x_i)$ is exactly optimal.

Formally, $\pi_i^{\text{report}}\left(h_{i-1}^{r+1}, h_{i-1}^{\text{rereport}}, \hat{h}_i^{r+1}, r\right)$ is the summation of the following rewards and punishments.

Punishment for a Lie As in the two-player case, we punish a lie. For round r corresponding to a review round, the punishment is the summation of the following three:

- The number indicating player i's lie about $\{a_{i,t}, y_{i,t}\}_{t\in T(r,k(r))}$:

$$\sum_{t\in T(r,k(r))} T^{-3} g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{a}_{i,t}, \hat{y}_{i,t}). \tag{126}$$

- The number indicating player i's lie about $\#_i^r(k)$:

$$T^{-3} \times T^{\frac{3}{4}} \times \mathbf{1}\left\{\hat{\#}_i^r(k(r)) \neq \sum_{t\in T(r,k(r))} \mathbf{1}_{\hat{a}_{i,t},\hat{y}_{i,t}}\right\}, \tag{127}$$

  where $\mathbf{1}_{\hat{a}_{i,t},\hat{y}_{i,t}}$ is a vector whose element corresponding to (a_i, y_i) is equal to 1 if $(\hat{a}_{i,t}, \hat{y}_{i,t}) = (a_i, y_i)$ and 0 otherwise.

- The number indicating player i's lie about $\#_i^r$:

$$T^{-3} \times \mathbf{1}\left\{\hat{\#}_i^r \neq \sum_k \hat{\#}_i^r(k)\right\}. \tag{128}$$

For round r where player i sends or receives a message m, player $i-1$ punishes player i if it is likely for player i to tell a lie by

$$T^{-3} g_i(h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, \hat{\#}_i^r). \tag{129}$$

Cancel Out the Expected Punishment by Telling the Truth As in the two-player case, we cancel out the differences in ex ante value of the punishment between difference actions and messages: If player i reports the history (player $i \in \{1,2\}$ needs to be picked by the public randomization to report the history), then we add the following variable to π_i^{main}:

- if round r is a review round, then

$$\sum_{t \in T(r)} \mathbf{1}\{t \in T(r, k(r))\} \mathbf{1}\{t_j(l) = t\} T^{-3} \Pi_i(a_{-i,t}, y_{i-1,t}),$$

- if player i sends the message in round r, then

$$T^{-3} \Pi_i(f[n](m)),$$

  where player n is the receiver of the message, and

- if player i receives the message in round r, then

$$T^{-3} \Pi_i(n, m)$$

  where player n is the sender of the message m.

Here, $\Pi_i(a_{-i}, y_{i-1})$ is defined so that the differences in (126) among action a_i's are canceled out ex ante before taking a_i. Since we assume that player $i-1$ knew a_{-i}, Assumption 8 is sufficient to construct such $\Pi_i(a_{-i}, y_{i-1})$. As we will see, player $i-1$ gets the information about a_{-i} from players $-(i-1, i)$ in the re-report block.

Similarly, $\Pi_i(f[n](m))$ ($\Pi_i(n,m)$, respectively) is defined so that the differences in (129) among messages are canceled out ex ante before sending (receiving, respectively) the message. Again, player $n \in -(i-1,i)$ sends $f[n](m)$ (m, respectively) to player $i-1$ in the re-report block. The identifiability to construct such Π_i is guaranteed by Lemma 15.

Reward for Optimal Action and Incentive to Tell the Truth This is the same as in the general two-player case. We construct the reward f_i so that, for each round r, for any period t in round r, for any history h_i^t, conditional on $\mathcal{A}_{-i}(r)$, $\sigma_i(x_i)$ is optimal. Here, $\mathcal{A}_{-i}(r)$ represents

- which state $x_{-i} \in \{G,B\}$ players $-i$ is in, and
- for each review round l that is before or equal to round r, which action plan α_j each player $j \in -i$ takes in the lth review rounds.

See Section 26 for the construction of f_i.

37 Re-Report Block

This is the block for each player $i-1$ to collect the information owned by players $-(i-1,i)$ which is necessary to construct player$(i-1)$'s reward on player i, π_i.

In the re-report block, we have the following rounds in this chronological order:

- Players $-(N-1,N)$ send the information to player $N-1$ to construct π_N.
- Players $-(N-2,N-1)$ send the information to player $N-2$ to construct π_{N-1}.

  $\vdots$
- Players $-(1,2)$ send the information to player 1 to construct π_2.
- Players $-(N,1)$ send the information to player N to construct π_1.

We explain what information is sent for each step:

37.1 Information Sent by Players $-(i-1, i)$ to Player $i-1$

First, each player $n \in -(i-1, i)$ sends the information about their histories in the coordination and main blocks:

- Which state x_n player n has.
- For each lth review round, what strategy $\alpha_n(l)$ player n took. Remember that player n's strategy within a round is *i.i.d.*
- For each lth review round, for each (a_n, y_n, φ_n), how many times player n observed (a_n, y_n, φ_n).
- For each lth review round, which period $t_n(l)$ is excluded from $T_n(l)$.
- At the end of each lth review round, all the realizations of player n's randomization for the construction of each state. For example, when player n constructs $\hat{\lambda}_{n-1}(l+1)$ in Section 33.4.2, which of 1-(a), 1-(b) or 2 is the case.
- For each supplemental round when player n sends a message, which message m player n sent.
- For each supplemental round when player n receives a message, which signal $f[n](m)$ player n had.

Second, the players communicate about the histories related to the report block.

- For each round r corresponding to a review round,
  - First, player $i-1$ sends which $k(r)$ player $i-1$ and i coordinate about sending $(a_{i,t}, y_{i,t})$ for each r corresponding to the review round. With public randomization, $k(r)$ is public. Expecting that we replace public randomization with coordination through private signals, we let player $i-1$ speak $k(r)$ here.

- Then, for each r, based on player $(i-1)$'s report $k(r)$, each player $n \in -(j,i)$ with j defined in (125) sends $\left(y_{n,t}, \varphi_{n,t}\right)_{t \in T(r,k(r))}$ and each player $n \in -(i-1,i)$ sends $(a_{n,t})_{t \in T(r,k(r))}$.

- For a supplemental round, each player $n \in -(i-1,i)$ sends $g[n](m)$ and $g_2[n](m)$ if player i receives a message and $f[n](m)$ and $f_2[n](m)$ if player i sends a message.

Then, player $i-1$ collects all the information necessary to construct player $(i-1)$'s reward on player i, π_i. Further, the cardinality of the messages sent in the re-report block is

$$O(T^{\frac{1}{4}}) \tag{130}$$

by the same calculation as for (50).

SUPPLEMENTAL MATERIAL 4:

PROOF OF THEOREM 1 for a General Two-Player Game withOUT CHEAP TALK

In this Supplemental Material, we prove the dispensability of the perfect cheap talk, noisy cheap talk and public randomization in the proof of Theorem 1.

After we summarize new notations and assumptions in Section 38, we show that the players can communicate and coordinate via actions and private signals. We take the following steps to dispense with the perfect and noisy cheap talk and public randomization device.

Remember that the coordination block uses the perfect cheap talk to communicate x, that the supplemental rounds for $\lambda_i(l+1)$, $d_i(l+1)$ and $\hat{d}_j(l+1)$ use the noisy cheap talk, and that the report block uses the public randomization and perfect cheap talk.

First, in Section 39, we replace the perfect cheap talk in the coordination block with the noisy cheap talk. Although x_i is no longer common knowledge, by exchanging the messages via noisy cheap talk several times, each player can construct an inference of x_i such that, given the opponent's inference, each player puts a belief no less than $1-\exp(-O(T^{\frac{1}{2}}))$ on the event that if their inferences are different, then the opponent has made her indifferent between any action profile sequence in the main blocks. No additional assumption is necessary for this step.

Second, in Section 41, we dispense with the noisy cheap talk in the coordination block (given the first step above) and supplemental rounds. See Section 4 for the intuition and Section 38.1 for a new assumption necessary for this step.

Third, in Section 44.2, we dispense with the public randomization in the report block, keeping the perfect cheap talk. Now, the players coordinate with their actions and private signals. Section 38.2 offers a sufficient condition for this step.

Fourth, in Section 44.3, we replace the perfect cheap talk in the report block with "conditionally independent noisy cheap talk." In the report block, the receiver does not have a strict incentive to infer the messages correctly since the messages are used only for the reward on the sender. Hence, we can disregard the incentives for the receiver. For the sender, since the cheap talk is conditionally independent, she always believes that each message transmits correctly with high probability. We can show that the cardinality of the messages sent in the report block is sufficiently small compared to the precision of the conditionally independent noisy cheap talk. Therefore, all the messages transmit correctly with high probability, which is enough to construct π_i^{report} to make $\sigma_i(x_i)$ exactly optimal. This step does not require any assumption in addition to the availability of the conditionally independent cheap talk.

Fifth, in Section 44.4, we replace the conditionally independent noisy cheap talk with messages via actions. This step is novel since we do not assume anything about the differences in each player's number of signals. See Section 38.3 for what generic assumption is necessary.

When we say players i and j in this Supplemental Material, unless otherwise specified, it implies that $i \neq j$. In addition, without loss of generality, we assume that

$$|A_1| \, |Y_1| \geq |A_2| \, |Y_2| \, . \tag{131}$$

38 Notations and Assumptions

38.1 Assumption for Dispensing with the Noisy Cheap Talk

When we dispense with the noisy cheap talk with precision $p \in (0,1)$ about a binary message $m \in \{G, B\}$, with η being a small number to be defined, the sender (say player j) determines

$$z_j(m) = \begin{cases} m & \text{with probability } 1 - 2\eta, \\ \{G, B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

and player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2} a_j^G + \frac{1}{2} a_j^B & \text{if } z_j(m) = M \end{cases}$$

for T^p periods. That is, player j sends the "true" message $\alpha_j^{z_j(m)} = a_j^m$ with high probability $1-2\eta$ while player j "tells a lie" with probability 2η. With probability η, player j sends the opposite message $z_j(m) = \{G, B\} \setminus \{m\}$. With probability η, player j "mixes" two messages: $z_j(m) = M$ and $\alpha_j^M = \frac{1}{2}a_j^G + \frac{1}{2}a_j^B$.

Player i (receiver) takes a_i^G.

Let $\mathbf{y}_j$ be the vector whose element corresponding to y_j represents the frequency that player j observes y_j while taking $\alpha_j^{z_j(m)}$. Define $\mathbf{y}_i$ symmetrically for the receiver. In addition, let $\mathbf{q}_j(a) = (q_j(y_j \mid a))_{y_j}$ ($\mathbf{q}_i(a) = (q_i(y_i \mid a))_{y_i}$, respectively) be the distribution of player j's (player i's, respectively) signals.

Our task is to create a mapping from $\mathbf{y}_j$ to $g[j](m) \in \{m, E\}$ and that from $\mathbf{y}_i$ to $f[i](m) \in \{G, B\}$ such that important features of Lemma 2 are satisfied. The mapping from $\mathbf{y}_i$ to $f[i](m)$ cannot depend on m since the receiver does not know the true message.

First, when player j tells a lie, player j makes player i indifferent between any action. That is, $g[j](m) = E$ if $z_j(m) \neq m$.

Second, regardless of player i's deviation, as long as $z_j(m) = m$, $\mathbf{y}_j$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^m, a_i)\}_{a_i})$ (affine hull of player j's signal distributions with respect to player i's deviations) with high probability. As we will see, if not, then player j makes player i indifferent between any action profile in the continuation game. That is, $g[j](m) = E$ if $\mathbf{y}_j$ is not close to $\text{aff}(\{\mathbf{q}_j(a_j^m, a_i)\}_{a_i})$.

Using 1 and 2 of Notation 1 below, $g[j](m) = E$ if $\mathbf{y}_j \notin \mathcal{H}_j[\varepsilon](m)$ for small ε to be determined.

Therefore, in total, we define

1. $g[j](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$.

2. $g[j](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_j \notin \mathcal{H}_j[\varepsilon](m)$.

By the law of large numbers (remember that the message is repeated for T^p periods), $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$ with probability $1 - \exp(-O(T^p))$. Therefore, $g[j](m) = m$ with probability $1 - 2\eta - \exp(-O(T^p))$.

On the other hand, player j wants to infer the message. For the moment, since $z_j(m) \neq m$ implies that $g[j](m) = E$, let player i assume that $z_j(m) = m$. Player i wants to infer that the message is $\hat{m}$ if player i's conditional expectation of $\mathbf{y}_j$ given $\hat{m}$ and $\mathbf{y}_i$ is close to $\mathcal{H}_j[\varepsilon](\hat{m})$. Consider the following mapping:

- $f[i](m) = G$ if the conditional expectation of $\mathbf{y}_j$ given $m = G$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^G, a_i)\}_{a_i})$, and
- $f[i](m) = B$ if the conditional expectation of $\mathbf{y}_j$ given $m = B$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^B, a_i)\}_{a_i})$.

Suppose that this is well defined. That is, there is no $\mathbf{y}_i$ such that the conditional expectation of $\mathbf{y}_j$ given $m = G$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^G, a_i)\}_{a_i})$, and, at the same time, the conditional expectation of $\mathbf{y}_j$ given $m = B$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^B, a_i)\}_{a_i})$.

Then, 2 of Lemma 2 is satisfied. To see why, suppose that $m = G$ and $f[i](m) = B$. This means that, the first bullet is *not* the case. That is, given $m = G$ (the true message) and $\mathbf{y}_i$, the conditional expectation of $\mathbf{y}_j$ is *not* close to $\text{aff}(\{\mathbf{q}_j(a_j^G, a_i)\}_{a_i})$. Hence, given m, $g[j](m) = E$ with probability $1 - \exp(-O(T^p))$. The symmetric argument holds for $m = B$.

However, the above mapping from $\mathbf{y}_i$ to $f[i](m)$ is not always well defined. If $|Y_i| > 2(|Y_j| - |A_i|) + 1$, then it is possible to have $\mathbf{y}_i$ such that the conditional expectation of $\mathbf{y}_j$ given $m = G$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^G, a_i)\}_{a_i})$ *and* that the conditional expectation of $\mathbf{y}_j$ given $m = B$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^B, a_i)\}_{a_i})$.

Therefore, we restrict our attention to $\mathbf{y}_i$ that is close to $\text{aff}(\{\mathbf{q}_i(a_i^G, a_j)\}_{a_j})$ (affine hull of player i's signal distributions with respect to player j's actions). Regardless of player j's message and deviation, $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_i(a_i^G, a_j)\}_{a_j})$ with probability $1 - \exp(-O(T^p))$. Later, we will care about $\mathbf{y}_i$ that is *not* close to $\text{aff}(\{\mathbf{q}_i(a_i^G, a_j)\}_{a_j})$.

That is,

- If $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_i(a_i^G, a_j)\}_{a_j})$, then
  - $f[i](m) = G$ if the conditional expectation of $\mathbf{y}_j$ given $m = G$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^G, a_i)\}_{a_i})$, and

– $f[i](m) = B$ if the conditional expectation of $\mathbf{y}_j$ given $m = B$ and $\mathbf{y}_i$ is close to $\text{aff}(\{\mathbf{q}_j(a_j^B, a_i)\}_{a_i})$.

- If $\mathbf{y}_i$ is not close to $\text{aff}(\{\mathbf{q}_i(a_i^G, a_j)\}_{a_j})$, then we will consider the extra care later.

Using 4 of Notation 1 below,

- If $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$,[78] then

  – $f[i](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](G)$, and

  – $f[i](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](B)$.

- If $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$, then we will consider the extra care later.

Then, for the first bullet, 2 of Lemma 2 is satisfied. In addition, by the law of large numbers (remember the message is repeated for T^p periods), $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$ and $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](m)$ with probability $1-\exp(-O(T^p))$. Therefore, $f[i](m) = m$ with probability $1-\exp(-O(T^p))$.

To sustain 3 of Lemma 2, we further modify player i's inference so that player j with $g[j](m) = m$ can believe that $f[j](m) = m$ or $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$. As we will see, if $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$, then player i makes player j indifferent between any action profile sequence. This does not affect player j's incentive to send the message truthfully since $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$ with probability $1 - \exp(-O(T^p))$ regardless of player j's strategy.

Suppose that player i infers

- $f[i](m) = G$ if there exists $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](G)$ such that $\mathbf{y}_i$ is close to player j's conditional expectation of the empirical distribution of player i's signals given $\mathbf{y}_j$ and $m = G$.

- $f[i](m) = B$ if there exists $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](B)$ such that $\mathbf{y}_i$ is close to player j's conditional expectation of the empirical distribution of player i's signals given $\mathbf{y}_j$ and $m = B$.

[78] Here, we use 1 of Notation 1 with j replaced with i. Since the receiver takes a_i^G, this is comparable to the situation where player i is the sender and sends $m = G$ after taking the affine hull with respect to the opponent's actions.

Using 5 of Notation 1 below, player i infers

- $f[i](m) = G$ if $\mathbf{y}_i \in \mathcal{I}_i[\varepsilon](G)$, and
- $f[i](m) = B$ if $\mathbf{y}_i \in \mathcal{I}_i[\varepsilon](B)$.

Player j with $g[j](m) = m$ should have $z_j(m) = m$ and $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$. Since player j knows that she takes a_j^m, player j calculates the conditional expectation of the empirical distribution of player i's signals given $\mathbf{y}_j$ and m. The above inference implies that if player i's signal observation is close to this conditional expectation (player j believes that this is the case), then player i infers $f[i](m) = m$ as desired.

In total, we define a mapping from $\mathbf{y}_i$ to $f[i](m)$ such that

1. If $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$, then
   (a) $f[i](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$,
   (b) $f[i](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](B) \cup \mathcal{I}_i[\varepsilon](B)$ or $\mathbf{y}_i \notin \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$, and
2. If $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$, then we will consider the extra care later.

See 1-(b). For completeness, we require $f[i](m) = B$ if $\mathbf{y}_i$ is not included in either $\mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$ or $\mathcal{H}_{j,i}[\varepsilon](B) \cup \mathcal{I}_i[\varepsilon](B)$.

Let us consider Case 2. Here, player i infers $z_j(m)$ from the likelihood, taking the possibility that $z_j(m) \neq m$ into account. Given $m \in \{G, B\}$ and $\mathbf{y}_i$, the conditional likelihood ratio between $z_j(m) = z_j \in \{G, B, M\}$ and $z_j(m) = z_j' \in \{G, B, M\}$ is

$$\frac{\Pr\left(z_j(m) = z_j \mid m, \mathbf{y}_i\right)}{\Pr\left(z_j(m) = z_j' \mid m, \mathbf{y}_i\right)} = \frac{\Pr\left(\mathbf{y}_i \mid z_j(m) = z_j\right)}{\Pr\left(\mathbf{y}_i \mid z_j(m) = z_j'\right)} \frac{\Pr\left(z_j(m) = z_j \mid m\right)}{\Pr\left(z_j(m) = z_j' \mid m\right)}.$$

$\log \frac{\Pr(\mathbf{y}_i \mid z_j(m) = z_j)}{\Pr\left(\mathbf{y}_i \mid z_j(m) = z_j'\right)}$ is expressed as $T^p\left(\mathcal{L}(\mathbf{y}_i, z_j) - \mathcal{L}(\mathbf{y}_i, z_j')\right)$ with

$$\mathcal{L}(\mathbf{y}_i, z_j) = y_{i,1} \log q(y_{i,1} | a_i^G, \alpha_j^{z_j}) + \cdots + y_{i,|Y_i|} \log q(y_{i,|Y_i|} | a_i^G, \alpha_j^{z_j}).$$

If $\mathcal{L}(\mathbf{y}_i, z_j)$ is strictly concave with respect to the mixture of a_j^G and a_j^B for all possible $\mathbf{y}_i$'s, then there exists $\kappa > 0$ such that one of the following is true:

1. $z_j(m) = G$ is sufficiently more likely than $z_j(m) = B$: $\mathcal{L}(\mathbf{y}_i, G) - \kappa \geq \mathcal{L}(\mathbf{y}_i, B)$.

2. $z_j(m) = B$ is sufficiently more likely than $z_j(m) = G$: $\mathcal{L}(\mathbf{y}_i, B) - \kappa \geq \mathcal{L}(\mathbf{y}_i, G)$.

3. If $z_j(m) = G$ and $z_j(m) = B$ are equally likely, then since $\mathcal{L}(\mathbf{y}_i, z_j)$ is strictly concave, $z_j(m) = M$ is most likely: $\mathcal{L}(\mathbf{y}_i, M) - \kappa \geq \mathcal{L}(\mathbf{y}_i, G), \mathcal{L}(\mathbf{y}_i, B)$.

Suppose that 1 is the case. This means that $\frac{\Pr(z_j(m)=G|m,\mathbf{y}_i)}{\Pr(z_j(m)=B|m,\mathbf{y}_i)} \geq \exp(\kappa T^p)\frac{\eta}{1-2\eta}$ for all $m \in \{G, B\}$. Remember that $z_j(m) = M$ implies that player j told a lie and that $g[j](m) = E$. Hence, given any $m \in \{G, B\}$, player i puts a conditional belief no less than $1 - \exp(O(-T^p))$ on the event that $m = G$ or $g[j](m) = E$. Similarly, if 2 is the case, then given any $m \in \{G, B\}$, player i puts a conditional belief no less than $1 - \exp(O(-T^p))$ on the event that $m = B$ or $g[j](m) = E$. Finally, if 3 is the case, then given any $m \in \{G, B\}$, player i puts a conditional belief no less than $1 - \exp(O(-T^p))$ on the event that $g[j](m) = E$. In this case, player i can infer m arbitrarily for almost optimality.

Hence, using the likelihood, there exists a mapping from $\mathbf{y}_i$ to $f[i](m) \in \{G, B\}$ such that, given any $m \in \{G, B\}$, player i puts a conditional belief more than $1 - \exp(O(-T^p))$ on the event that $m = f[i](m)$ or $g[j](m) = E$.

In total,

1. If $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$, then

   (a) $f[i](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$,

   (b) $f[i](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](B) \cup \mathcal{I}_i[\varepsilon](B)$ or $\mathbf{y}_i \notin \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$, and

2. If $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$, then player i infers $f[i](m)$ from the likelihood.

After we introduce the notations, Assumption 10 gives a sufficient condition for the above mapping to be well defined, that is, there is no $\mathbf{y}_i$ such that the above mapping maps $\mathbf{y}_i$ to both G and B.

Notation 1 *For $m, \hat{m} \in \{G, B\}$, we define*

1. *A $(|Y_j| - |A_i| + 1) \times |Y_j|$ matrix $H_j(m)$ and a $(|Y_j| - |A_i| + 1) \times 1$ vector $\mathbf{p}_j(m)$ such that the affine hull of player j's signal distributions with respect to player i's action when player j takes a_j^m is represented by*

$$\operatorname{aff}(\{\mathbf{q}_j(a_j^m, a_i)\}_{a_i}) \cap \mathbb{R}_+^{|Y_j|} = \left\{\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|} : H_j(m)\mathbf{y}_j = \mathbf{p}_j(m)\right\}.$$

2. *The set of hyperplanes generated by perturbing RHS of 1: for $\varepsilon \geq 0$,*

$$\mathcal{H}_j[\varepsilon](m) \equiv \left\{\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_j|-|A_i|+1} \text{ such that } \left\{\begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ H_j(m)\mathbf{y}_j = \mathbf{p}_j(m)+\boldsymbol{\varepsilon} \end{array}\right\}\right\}.$$

3. *The matrix projecting player i's signal frequency $\mathbf{y}_i$ on the conditional expectation of player j's signal frequency $\mathbf{y}_j$ given an action profile a:*

$$Q_{j,i}(a) = \begin{bmatrix} q(y_{j,1} \mid a, y_{i,1}) & \cdots & q(y_{j,1} \mid a, y_{i,|Y_i|}) \\ \vdots & & \vdots \\ q(y_{j,|Y_j|} \mid a, y_{i,1}) & \cdots & q(y_{j,|Y_j|} \mid a, y_{i,|Y_i|}) \end{bmatrix}.$$

4. *The set of player i's signal frequencies $\mathbf{y}_i$'s such that the conditional expectation of player j's signal frequency $\mathbf{y}_j$ given $\hat{m}$ and $\mathbf{y}_i$ is close to $\mathcal{H}_j[\varepsilon](\hat{m})$:*

$$\mathcal{H}_{j,i}[\varepsilon](\hat{m}) = \left\{\begin{array}{c} \mathbf{y}_i \in \mathbb{R}_+^{|Y_i|} \text{ such that} \\ \text{there exist } \boldsymbol{\varepsilon}_1 \in \mathbb{R}^{|Y_j|},\ \boldsymbol{\varepsilon}_2 \in \mathbb{R}^{|Y_j|-|A_i|+1} \text{ and } \mathbf{y}_j \in \mathbb{R}_+^{|Y_j|} \text{ with} \\ \left\{\begin{array}{c} \mathbf{y}_j = Q_{j,i}(a_j^{\hat{m}}, a_i^G)\mathbf{y}_i + \boldsymbol{\varepsilon}_1, \\ H_j(\hat{m})\mathbf{y}_j = \mathbf{p}_j(\hat{m}) + \boldsymbol{\varepsilon}_2, \\ \|\boldsymbol{\varepsilon}_1\|, \|\boldsymbol{\varepsilon}_2\| \leq \varepsilon \end{array}\right. \end{array}\right\}.$$

5. *The set of player i's frequencies $\mathbf{y}_i$'s such that, if player j believes that $(a_j^{\hat{m}}, a_i^G)$ is taken and observes $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](\hat{m})$, then player j's conditional expectation of $\mathbf{y}_i$ given*

$\hat{m}$ *and* $\mathbf{y}_j$ *is close to the set:*

$$\mathcal{I}_i[\varepsilon](\hat{m}) \equiv \left\{ \begin{array}{c} \mathbf{y}_i \in \mathbb{R}_+^{|Y_i|} : \exists \mathbf{y}_j \in \mathbb{R}_+^{|Y_j|},\ \boldsymbol{\varepsilon}_j \in \mathbb{R}^{|Y_j|-|A_j|+1} \text{ and } \boldsymbol{\varepsilon}_i \in \mathbb{R}^{|Y_i|} \text{ such that} \\ \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}_j\| \le \varepsilon, \|\boldsymbol{\varepsilon}_i\| \le \varepsilon, \\ H_j(\hat{m})\mathbf{y}_j = \mathbf{p}_j(\hat{m}) + \boldsymbol{\varepsilon}_j, \\ \mathbf{y}_i = Q_{i,j}(a_j^{\hat{m}}, a_i^G)\mathbf{y}_j + \boldsymbol{\varepsilon}_i \end{array} \right. \end{array} \right\}.$$

Before stating a sufficient condition, we prove one lemma that will turn out to be useful:

Lemma 25 *For each* $j \in I$*, we can take* $H_j(G)$ *and* $H_j(B)$ *such that all the elements are in* $(0,1)$*.*

Proof. Let m_H be the minimum element of $H_j(m)$ and M_H be the maximum element of $H_j(m)$. Let $\tilde{H}_j(m)$ be the matrix whose (l,n) element is $\frac{(H_j(m))_{l,n}+|m_H|+1}{|M_H|+2|m_H|+2} \in (0,1)$ and $\tilde{\mathbf{p}}_j(m)$ be the vector whose lth element is $\frac{(\mathbf{p}_j(m))_l+|m_H|+1}{|M_H|+2|m_H|+2}$.

We will show

$$\left\{\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|} : H_j(m)\mathbf{y}_j = \mathbf{p}_j(m)\right\} \equiv \mathcal{H}_j(m) = \tilde{\mathcal{H}}_j(m) \equiv \left\{\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|} : \tilde{H}_j(m)\mathbf{y}_j = \tilde{\mathbf{p}}_j(m)\right\}.$$

1. $\mathcal{H}_j(m) \subset \tilde{\mathcal{H}}_j(m)$

   Suppose that $\mathbf{y}_j \in \mathcal{H}_j(m)$. Since $\mathbf{y}_j \in \text{aff}(\{\mathbf{q}_j(a_j^m, a_i)\}_{a_i}) \subset \text{aff}\left(\{\mathbf{1}_{y_j}\}_{y_j \in Y_j}\right)$,[79] $\tilde{H}_j(m)\mathbf{y}_j = \tilde{\mathbf{p}}_j(m)$ as desired.

2. $\mathcal{H}_j(m) \supset \tilde{\mathcal{H}}_j(m)$

   Suppose that $\mathbf{y}_j \notin \mathcal{H}_j(m)$. Since $\text{aff}(\{\mathbf{q}_j(a_j^m, a_i)\}_{a_i}) \subset \text{aff}\left(\{\mathbf{1}_{y_j}\}_{y_j \in Y_j}\right)$, without loss of generality, we can assume that one row of $H_j(m)$ is parallel to $(1,...,1)$ and that the element of $\mathbf{p}_j(m)$ corresponding to that row is 1. If $(1,...,1)\mathbf{y}_j \neq 1$, then $\left(\frac{1+|m_H|+1}{|M_H|+2|m_H|+2}, ..., \frac{1+|m_H|+1}{|M_H|+2|m_H|+2}\right)\mathbf{y}_j = \frac{1+|m_H|+1}{|M_H|+2|m_H|+2}(1,...,1)\mathbf{y}_j \neq \frac{1+|m_H|+1}{|M_H|+2|m_H|+2}$ and $\mathbf{y}_j \notin$

[79] Remember that $\mathbf{1}_{y_j}$ is a $|Y_j| \times 1$ vector such that the element corresponding to y_i is 1 and the others are 0.

$\tilde{\mathcal{H}}_j(m)$ as desired. If $(1,...,1)\mathbf{y}_j = 1$, then there is another row $\mathbf{h}_j(m)$ and the corresponding element $p_j(m)$ of $\mathbf{p}_j(m)$ such that

$$\mathbf{h}_j(m)\mathbf{y}_j \neq p_j(m).$$

Let $\tilde{\mathbf{h}}_j(m)$ be the corresponding row of $\tilde{H}_j(m)$ and $\tilde{p}_j(m)$ is the corresponding element of $\tilde{\mathbf{p}}_j(m)$. Then,

$$\begin{aligned}
\tilde{\mathbf{h}}_j(m)\mathbf{y}_j &= \frac{1}{|M_H| + 2|m_H| + 2}\left(\mathbf{h}_j(m) + (|m_H| + 1)(1,...,1)\right)\mathbf{y}_j \\
&= \frac{1}{|M_H| + 2|m_H| + 2}\left(\mathbf{h}_j(m)\mathbf{y}_j + |m_H| + 1\right) \\
&\neq \frac{1}{|M_H| + 2|m_H| + 2}\left(p_j(m) + |m_H| + 1\right) = \tilde{p}_j(m)
\end{aligned}$$

and so $\mathbf{y}_j \notin \tilde{\mathcal{H}}_j(m)$.

■

Now, we state our sufficient condition:

Assumption 10 *For each $j \in I$, there exists $a_j^G, a_j^B \in A_j$ such that the following five conditions are satisfied:*

1. *There exists $\mathbf{x} \in \mathbb{R}^{3|Y_i|+2|Y_j|-|A_j|-2|A_i|+5}$ such that*

$$\begin{bmatrix} H_i(G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{0} \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0,$$

2. *There exists* $\mathbf{x} \in \mathbb{R}^{|Y_i|+2|Y_j|-|A_j|-2|A_i|+3}$ *such that*

$$\begin{bmatrix} H_i(G) \\ H_j(G)Q_{j,i}(a_j^G, a_i^G) \\ H_j(B)Q_{j,i}(a_j^B, a_i^G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0,$$

3. *There exists* $\mathbf{x} \in \mathbb{R}^{2|Y_i|+2|Y_j|-|A_j|-2|A_i|+4}$ *such that*

$$\begin{bmatrix} H_i(G) & O \\ H_j(B)Q_{j,i}(a_j^B, a_i^G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(B) \\ \mathbf{0} \\ \mathbf{p}_j(G) \end{bmatrix} \cdot \mathbf{x} > 0,$$

4. *There exists* $\mathbf{x} \in \mathbb{R}^{2|Y_i|+2|Y_j|-|A_j|-2|A_i|+4}$ *and* $\mathbf{y}_j \in \mathbb{R}_+^{|Y_i|}$ *such that*

$$\begin{bmatrix} H_i(G) & O \\ H_j(G)Q_{j,i}(a_j^G, a_i^G) & O \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0$$

5. *For each* $k \in \{1, \ldots, |Y_i|\}$, *we have*

$$q(y_{i,k}|a_i^G, \alpha_j^G) \neq q(y_{i,k}|a_i^G, \alpha_j^B).$$

It will be apparent from the proof of Theorem 1 below that a_j^G for the case where player j sends the message $m = G$ and a_j^G for the case where player j is the receiver can be different. However, for notational simplicity, we assume that these two a_j^G's are the same.

Taking into account the fact that some of rows are parallel to $\mathbf{1}$, we have the following:

1. $\mathbf{x}$ has $3|Y_i| + 2|Y_j| - |A_j| - 2|A_i| + 1$ degrees of freedom while the condition has $|Y_i| + |Y_j| + 1$ constraints.

2. $\mathbf{x}$ has $|Y_i| + 2|Y_j| - |A_j| - 2|A_i| + 1$ degrees of freedom while the condition has $|Y_i| + 1$ constraints.

3. $\mathbf{x}$ has $2|Y_i| + 2|Y_j| - |A_j| - 2|A_i| + 1$ degrees of freedom while the condition has $|Y_i| + |Y_j| + 1$ constraints.

4. $\mathbf{x}$ has $2|Y_i| + 2|Y_j| - |A_j| - 2|A_i| + 1$ degrees of freedom while the condition has $|Y_i| + |Y_j| + 1$ constraints.

Therefore, Assumption 10 is generic if Assumption 2 is satisfied.

The next lemma shows that Assumption 10 is actually sufficient so that the above inference $f[i](m)$ is well defined.

Lemma 26 *If Assumption 10 is satisfied, then there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$, for any $i \in I$, there is at most one $\hat{m} \in \{G, B\}$ such that $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G) \cap (\mathcal{H}_{j,i}[\varepsilon](\hat{m}) \cup \mathcal{I}_i[\varepsilon](\hat{m}))$.*

Proof. It suffices to show that, for sufficiently small ε,

$$\begin{aligned}
\mathcal{H}_i[\varepsilon](G) \cap \mathcal{I}_i[\varepsilon](G) \cap \mathcal{I}_i[\varepsilon](B) &= \emptyset, \\
\mathcal{H}_i[\varepsilon](G) \cap \mathcal{H}_{j,i}[\varepsilon](G) \cap \mathcal{H}_{j,i}[\varepsilon](B) &= \emptyset, \\
\mathcal{H}_i[\varepsilon](G) \cap \mathcal{H}_{j,i}[\varepsilon](B) \cap \mathcal{I}_i[\varepsilon](G) &= \emptyset, \\
\mathcal{H}_i[\varepsilon](G) \cap \mathcal{H}_{j,i}[\varepsilon](G) \cap \mathcal{I}_i[\varepsilon](B) &= \emptyset.
\end{aligned}$$

That is, for sufficiently small ε, it is equivalent to have the following four:

- For any $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$, there are no $\mathbf{y}_i \in \mathbb{R}_+^{|Y_i|}$ and $\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|}$ such that[80]

$$\begin{bmatrix} H_i(G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix} \begin{bmatrix} \mathbf{y}_i \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{0} \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} + \boldsymbol{\varepsilon},$$

- For any $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$, there is no $\mathbf{y}_i \in \mathbb{R}_+^{|Y_i|}$ such that

$$\begin{bmatrix} H_i(G) \\ H_j(G) Q_{j,i}(a_j^G, a_i^G) \\ H_j(B) Q_{j,i}(a_j^B, a_i^G) \end{bmatrix} \mathbf{y}_i = \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{p}_j(B) \end{bmatrix} + \boldsymbol{\varepsilon},$$

- For any $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$, there are no $\mathbf{y}_i \in \mathbb{R}_+^{|Y_i|}$ and $\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|}$ such that

$$\begin{bmatrix} H_i(G) & O \\ H_j(B) Q_{j,i}(a_j^B, a_i^G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \end{bmatrix} \begin{bmatrix} \mathbf{y}_i \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(B) \\ \mathbf{0} \\ \mathbf{p}_j(G) \end{bmatrix} + \boldsymbol{\varepsilon},$$

- For any $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$, there are no $\mathbf{y}_i \in \mathbb{R}_+^{|Y_i|}$ and $\mathbf{y}_j \in \mathbb{R}_+^{|Y_j|}$ such that

$$\begin{bmatrix} H_i(G) & O \\ H_j(G) Q_{j,i}(a_j^G, a_i^G) & O \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix} \begin{bmatrix} \mathbf{y}_i \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} + \boldsymbol{\varepsilon}.$$

[80] One row of each of $H_i(G)$, $H_j(m)$, $H_j(G)Q_{j,i}(a_j^m, a_i^G)$ and $Q_{i,j}(a_j^m, a_i^G)$ is parallel to $\mathbf{1}$ for each $m \in \{G, B\}$. This reduces the number of constraints. We change the freedom of $\mathbf{x}$ after we take the dual accordingly. The same caution is applicable for the other three equations.

By Farkas Lemma,[81] if $\varepsilon = 0$, then it suffices to have the following:

- There exists $\mathbf{x} \in \mathbb{R}^{3|Y_i|+2|Y_j|-|A_j|-2|A_i|+5}$ such that

$$\begin{bmatrix} H_i(G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{0} \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0,$$

- There exists $\mathbf{x} \in \mathbb{R}^{|Y_i|+2|Y_j|-|A_j|-2|A_i|+3}$ such that

$$\begin{bmatrix} H_i(G) \\ H_j(G)Q_{j,i}(a_j^G, a_i^G) \\ H_j(B)Q_{j,i}(a_j^B, a_i^G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0,$$

- There exists $\mathbf{x} \in \mathbb{R}^{2|Y_i|+2|Y_j|-|A_j|-2|A_i|+4}$ such that

$$\begin{bmatrix} H_i(G) & O \\ H_j(B)Q_{j,i}(a_j^B, a_i^G) & O \\ -E & Q_{i,j}(a_j^G, a_i^G) \\ O & H_j(G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(B) \\ \mathbf{0} \\ \mathbf{p}_j(G) \end{bmatrix} \cdot \mathbf{x} > 0,$$

[81] Farkas Lemma has a constraint that each element of $\mathbf{x}$ should be non-negative. However, since we have equality constraints, not inequality constraints, for each l with $x_l < 0$, we can multiply -1 to x_l, each element of lth row of the matrix in the LHS, and lth element of the vector in RHS. Therefore, non-negativity constraint is redundant.

- There exists $\mathbf{x} \in \mathbb{R}^{2|Y_i|+2|Y_j|-|A_j|-2|A_i|+4}$ such that

$$\begin{bmatrix} H_i(G) & O \\ H_j(G)Q_{j,i}(a_j^G, a_i^G) & O \\ -E & Q_{i,j}(a_j^B, a_i^G) \\ O & H_j(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{p}_i(G) \\ \mathbf{p}_j(G) \\ \mathbf{0} \\ \mathbf{p}_j(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

Since the second inequality of each condition is strict, for sufficiently small ε, the same $\mathbf{x}$'s work for all $\boldsymbol{\varepsilon}$ with $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$. Therefore, we are done. ■

In addition, the following lemma shows that Assumption 10 is also sufficient to construct $f[i](m)$ based on the likelihood:

Lemma 27 *If Assumption 10 is satisfied, then for each $m \in \{G, B\}$ and $j \in I$, there exists a mapping from $\mathbf{y}_i$ to $f[i](m) \in \{G, B\}$ such that, for any $\mathbf{y}_i$, given m, player i puts a belief no less than $1 - \exp(-O(T^p))$ on the events that $f[i](m) = m$ or $g[j](m) = E$.*

Proof. From the above discussion, it suffices to show that there exists $\kappa > 0$ such that, for

$$\mathcal{L}(\mathbf{y}_i, z_j, z_j') \equiv \mathcal{L}(\mathbf{y}_i, z_j) - \mathcal{L}(\mathbf{y}_i, z_j'),$$

for any $\mathbf{y}_i \in \Delta\left(\{\mathbf{1}_{y_i}\}_{y_i \in Y_i}\right)$, one of the following is true:

1. $\mathcal{L}(\mathbf{y}_i, G, B) \geq \kappa$,

2. $\mathcal{L}(\mathbf{y}_i, B, G) \geq \kappa$,

3. $\mathcal{L}(\mathbf{y}_i, M, G) \geq \kappa$ and $\mathcal{L}(\mathbf{y}_i, M, B) \geq \kappa$.

Let $\alpha_j^\lambda = \lambda a_j^G + (1-\lambda)\, a_j^B$ for $\lambda \in [0,1]$ and consider

$$\mathcal{L}(\mathbf{y}_i, \lambda) = y_{i,1} \log q(y_{i,1} | a_i^G, \alpha_j^\lambda) + \cdots + y_{i,|Y_i|} \log q(y_{i,|Y_i|} | a_i^G, \alpha_j^\lambda).$$

Then,

$$\frac{d^2\mathcal{L}(\mathbf{y}_i,\lambda)}{d\lambda^2} = -\sum_{k=1}^{|Y_i|} y_{i,k}\left\{\frac{q(y_{i,k}|a_i^G,\alpha_j^G)-q(y_{i,k}|a_i^G,\alpha_j^B)}{q(y_{i,k}|a_i^G,\alpha_j^\lambda)}\right\}^2 < 0$$

for any $\mathbf{y}_i$ because of 5 of Assumption 10. Hence, $\mathcal{L}(\mathbf{y}_i,\lambda)$ is strictly concave. Therefore, since $\mathcal{L}(\mathbf{y}_i,z_j,\tilde{z}_j)$ is the difference in $\mathcal{L}(\mathbf{y}_i,\lambda)$, we have

$$\max\left\{\mathcal{L}(\mathbf{y}_i,G,B),\mathcal{L}(\mathbf{y}_i,B,G),\min\left\{\mathcal{L}(\mathbf{y}_i,M,G),\mathcal{L}(\mathbf{y}_i,M,B)\right\}\right\} > 0.$$

Since LHS is continuous in $\mathbf{y}_i$ and $\Delta\left(\{\mathbf{1}_{y_i}\}_{y_i\in Y_i}\right)$ is compact, there exists $\kappa > 0$ such that

$$\max\left\{\mathcal{L}(\mathbf{y}_i,G,B),\mathcal{L}(\mathbf{y}_i,B,G),\min\left\{\mathcal{L}(\mathbf{y}_i,M,G),\mathcal{L}(\mathbf{y}_i,M,B)\right\}\right\} > \kappa$$

for all $\mathbf{y}_i \in \Delta\left(\{\mathbf{1}_{y_i}\}_{y_i\in Y_i}\right)$ as desired. ■

Therefore, from Lemmas 26 and 27, if Assumption 10 is satisfied, then for $\varepsilon < \bar{\varepsilon}$, the following mapping preserves the important features of Lemma 2:[82] For the sender,

1. $g[j](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$.
2. $g[j](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_j \notin \mathcal{H}_j[\varepsilon](m)$.

And for the receiver,

1. If $\mathbf{y}_i \in \mathcal{H}_i[\varepsilon](G)$, then
   (a) $f[i](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$,
   (b) $f[i](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{j,i}[\varepsilon](B) \cup \mathcal{I}_i[\varepsilon](B)$ or $\mathbf{y}_i \notin \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$, and
2. If $\mathbf{y}_i \notin \mathcal{H}_i[\varepsilon](G)$, then player i infers $f[i](m)$ from the likelihood.

If ε defined in (88) does not satisfy $\varepsilon < \bar{\varepsilon}$, then re-take ε such that

$$\varepsilon < \bar{\varepsilon}.$$

[82] We replace $\exp(-T^p)$ in the probability of $g[j](m) = E$ in Lemma 2 with $2\eta + \exp(-T^p)$.

This does not affect the consistency among the variables defined in Section 24.

38.2 Assumption for Dispensing with the Public Randomization

When we dispense with the public randomization in the report block, the players use actions and private signals to coordinate. Fix $i \in I$ arbitrarily.

The players play some action profile a^G. Then, each player observes her own private signal.

Player j partitions the set of her signals into non-empty subsets $Y_{j,1}^i$ and $Y_{j,2}^i$ with $Y_j = Y_{j,1}^i \cup Y_{j,2}^i$.

Player i tries to infer which set player j's signal belongs to. With some $\bar{p}_i \in (0,1)$, player i classifies the set of her signals into two classes: The set of signals with which player i thinks that player j observes $y_j \in Y_{j,1}^i$ with probability more than $\bar{p}_i$ and the set of signals with which player i thinks that player j observes $y_j \in Y_{j,1}^i$ with probability less than $\bar{p}_i$. That is,

$$Y_{i,1}^i \equiv \left\{ y_i \in Y_i : \Pr(\{y_j \in Y_{j,1}^i\} \mid a^G, y_i) > \bar{p}_i \right\} \tag{132}$$

$$Y_{i,2}^i \equiv \left\{ y_i \in Y_i : \Pr(\{y_j \in Y_{j,1}^i\} \mid a^G, y_i) < \bar{p}_i \right\}. \tag{133}$$

We assume that there exists $\bar{p}_i \in (0,1)$ such that $Y_{i,1}^i$ and $Y_{i,2}^i$ are non-empty partitions of Y_i.

Assumption 11 *For each $i \in I$, there exists $a^G \in A$ such that there exist $Y_{j,1}^i$, $Y_{j,2}^i$, $\bar{p}_i$, $Y_{i,1}^i$ and $Y_{i,2}^i$ such that $Y_{i,1}^i$ and $Y_{i,2}^i$ satisfy (132), (133) and*

$$Y_{i,1}^i \neq \emptyset, Y_{i,2}^i \neq \emptyset, Y_i = Y_{i,1}^i \cup Y_{i,2}^i.$$

For notational convenience, we assume that a^G is the same for each player and the same as in Assumption 10.

38.3 Assumption for Dispensing with Conditionally Independent Cheap Talk

As mentioned, after we replace the perfect cheap talk in the report block with the conditionally independent noisy cheap talk, we dispense with the conditionally independent cheap talk.

To do so, we want to construct a statistics that preserves the conditional independence property for player 2. Player 2 sends a binary message $m \in \{G, B\}$ by taking $a_2^m \in \{a_2^G, a_2^B\}$. Player 1 takes some mixed action $\bar{\alpha}_1 \in \Delta(A_1)$. Based on the realization of the mixture a_1 and signal observation y_1, player 1 calculates $\phi_1(a_1, y_1)$. We want to make sure that, regardless of player 2's signal observation, player 2 believes that player 1 statistically infers player 2's signal properly: There exist $q_2 > q_1$ such that

$$\mathbb{E}\left[\phi_1(a_1, y_1) \mid \bar{\alpha}_1, a_2, y_2\right] = \begin{cases} q_2 & \text{if } a_2 = a_2^G, \\ q_1 & \text{if } a_2 \neq a_2^G \end{cases} \tag{134}$$

for all $y_2 \in Y_2$.

A sufficient condition for the existence of such ϕ is as follows: Let $\bar{Q}_1(\bar{\alpha}_1, a_2, y_2) \equiv (q_1(a_1, y_1 \mid \bar{\alpha}_1, a_2, y_2))_{a_1, y_1}$ be the vector expression of the conditional probability of (a_1, y_1) after player 2 plays a_2 and observes y_2. It is sufficient that $\bar{Q}_1(\bar{\alpha}_1, a_2, y_2)$ is linearly independent with respect to $(a_2, y_2) \in A_2 \times Y_2$.

Assumption 12 *There exists $\bar{\alpha}_1 \in \Delta(A_1)$ such that $\bar{Q}_1(\bar{\alpha}_1, a_2, y_2)$ is linearly independent with respect to a_2, y_2.*

Note that this is generic since we assume (131).

The following lemma shows that Assumption 12 is sufficient to have ϕ with (134).

Lemma 28 *If Assumption 12 is satisfied for $\bar{\alpha}_1 \in \Delta(A_1)$, then there exist $q_2 > q_1$ and $\phi_1 : A_1 \times Y_1 \to (0, 1)$ such that (134) holds for all $y_2 \in Y_2$.*

Note that we do not assume the existence of such ϕ when player 1 sends the message.

39 Replacing the Perfect Cheap Talk in the Coordination Block with the Noisy Cheap Talk

Remember that, with the perfect cheap talk, the players communicate about x in the coordination block in the following way: First, player 1 tells x_1 and second, player 2 tells x_2.

We divide the step where player i sends the message about x_i into the following steps (remember that this step is called the "round for x_i" with the perfect cheap talk in Section 10):

1. First, player i sends x_i to player j via noisy cheap talk with precision $p = \frac{1}{2}$. Among other things,[83] $f[j](x_i, 1) \in \{G, B\}$ and $g[i](x_i, 1) \in \{x_i, E\}$ are generated. With abuse of notation, instead of x_i, we use $(x_i, 1)$ since, as we will see, player i will re-send the message x_i and we want to distinguish the result of the first message and the second message.

2. Second, player i sends x_i to player j via noisy cheap talk with precision $p = \frac{2}{3}$. Among other things,[84] $f[j](x_i, 2) \in \{G, B\}$ and $g[i](x_i, 2) \in \{x_i, E\}$ are generated.

It is important to realize that the precision is higher for the second step. Given these two steps, player j constructs an inference of x_i, $x_i(j) \in \{G, B\}$.

3. Third, player j sends $x_i(j)$ to player i via noisy cheap talk with precision $p = \frac{1}{2}$. Among other things, $f[i](x_i(j)) \in \{G, B\}$ and $g[j](x_i(j)) \in \{x_i, E\}$ are generated.

Given these three steps, player i constructs an inference of x_i, $x_i(i) \in \{G, B\}$. Each player $n \in \{1, 2\}$ plays the continuation game as if x_i were $x_i(n)$.

In addition, after some events, player i (player j, respectively) makes player j (player i, respectively) indifferent between any action profile sequence in the main blocks by using $\pi_j^{x_i}(a_{i,t}, y_{i,t})$ ($\pi_i^{x_j}(a_{j,t}, y_{j,t})$, respectively) for π_j^{main} (π_i^{main}, respectively).

[83] Precisely, in addition to $f_2[j](x_i, 1)$ and $g_2[i](x_i, 1)$.

[84] Precisely, in addition to $f_2[j](x_i, 2)$ and $g_2[i](x_i, 2)$.

Intuitively, the coordination goes as follows: If player i observes $g[i](x_i, 2) = x_i$, then with high probability, player i infers $x_i(i) = x_i$. With small probability, player i uses the signal from player j's message: $x_i(i) = f[i](x_i(j))$. In addition, if the latter is the case, then player i makes player j indifferent between any action profile in the main blocks. If $g[i](x_i, 2) \neq x_i$, then player i uses the signal from player j's message: $x_i(i) = f[i](x_i(j))$.

On the other hand, player j uses the signals from the second message from player i and $x_i(j) = f[j](x_i, 2)$ with high probability. With small probability, player j uses the signal from the first message: $x_i(j) = f[j](x_i, 1)$. In addition, if the latter is the case, then player j makes player i indifferent between any action profile in the main blocks.

Consider player i's inference. If player i uses $f[i](x_i(j))$, then since player j's continuation play is independent of $g[j](x_i(j))$, 2 of Lemma 2 implies that player i can always believe that player i's inference is correct or player j knows the mistake with high probability.

If player i adheres to x_i after $g[i](x_i, 2) = x_i$, then player i before observing player j's continuation play believes that $f[j](x_i, 2) = g[j](x_i, 2)$ by 3 of Lemma 2 with high probability. Hence, player i believes that $x_i(j) = f[j](x_i, 2) = g[j](x_i, 2)$ or any action profile is optimal. When player i realizes that $x_i(j) \neq g[j](x_i, 2)$, player i believes that player j uses $f[j](x_i, 1)$ rather than $f[j](x_i, 2)$. Here is where we use the assumption that the precision of the second message is higher than the first message. Since the precision of the second message is higher than the first message, player i after observing $x_i(j)$ contradictory to player i's expectation from the second message believes that player j uses the first message (this happens with positive probability) and that there was an error in the first message. Remember that player j makes player i indifferent between any action profile in the main blocks if player j uses the first message. Therefore, after observing $x_i(j)$ contradictory to x_i, player i believes that any action is optimal with high probability.

Therefore, player i is willing to obey the same strategy as in the case with the perfect cheap talk with x_i replaced with $x_i(i)$.

Consider player j's inference. If player j uses the first message: $x_i(j) = f[j](x_i, 1)$, then since player i's continuation play is independent of $g[i](x_i, 1)$, 2 of Lemma 2 implies that this is almost optimal.

Suppose that player j uses $f[j](x_i, 2)$ and realizes $x_i(i) \neq f[j](x_i, 2)$. Then, since the precision of player i's second message is higher than player j's message, player j believes that player i uses player j's message (this happens with positive probability) and that there was an error in player j's message. Remember that player i makes player j indifferent between any action profile in the main blocks if player i uses player j's message. Therefore, after observing $x_i(i)$ contradictory to $f[j](x_i, 2)$, player j believes that any action is optimal with high probability.

Verify that $g[i](x_i, 2) = x_i$ and that player i adheres to x_i with high probability regardless of player j's message and so player j cannot manipulate $x_i(i)$.

The following lemma formalizes the argument:

Lemma 29 *We can define $(x_i(1), x_i(2))_{i \in I}$ and the events that a player makes her opponent indifferent between any action profile sequence such that, conditional on $x \in \{G, B\}^2$, for each $i \in I$, the inferences in the coordination block satisfy the following:*

1. *Given the true state x_i and player j's inference $x(j)$, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that $x_i(j) = x_i(i)$ or player i is indifferent between any action profile.*

2. *Given the true state x_i and player i's inference $x(i)$, player j puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the events that $x_i(i) = x_i(j)$ or player j is indifferent between any action profile.*

3. *It is almost optimal for the players to send the messages truthfully.*

We first define $x_i(i)$ and $x_i(j)$. Player i constructs $x_i(i)$ as follows:

1. If $g[i](x_i, 2) = x_i$ in the second step, then player i mixes the following two:

   (a) With probability $1 - \eta$, $x_i(i) = x_i$. That is, with high probability, player i adheres to her own state.

(b) With probability η, $x_i(i) = f[i](x_i(j))$. That is, with low probability, player i uses the signal from player j's message.

2. If $g[i](x_i) = E$ in the second step, then player i always uses the signal from player j's message: $x_i(i) = f[i](x_i(j))$.

For completeness, if player i deviates in the step 1 or 2 of the communication, then player i always uses the signal from player j's message: $x_i(i) = f[i](x_i(j))$.

Player j mixes the following two:

1. With probability $1-\eta$, $x_i(j) = f[j](x_i, 2)$. That is, with high probability, player j uses the signal from player i's second message.

2. With probability η, $x_i(j) = f[j](x_i, 1)$. That is, with low probability, player j uses the signal from player i's first message.

Second, we identify after what history player i (player j, respectively) makes player j (player i, respectively) indifferent between any action profile sequence.

Player i makes player j indifferent between any action profile sequence if (and only if based on the round for x_i)[85] $g[i](x_i, 1) = E$, $g[i](x_i, 2) = E$, or "1-(b) or 2 is the case for the construction of $x_i(i)$."

Player j makes player i indifferent between any action profile sequence if (and only if based on the round for x_i) $g[j](x_i(j)) = E$ or 2 is the case for the construction of $x_i(j)$.

Given the above preparation, we prove the theorem:

Proof of 1 of Lemma 29: If 1-(b) or 2 of the construction of $x_i(i)$ is the case, then 2 of Lemma 2 guarantees the result. Note that player j's continuation play in the main blocks does not reveal $g[j](x_i(j))$.

[85] With abuse of notation, for the multimple steps to coordinate on x_i, we use the same notation "the round for x_i" as in the case with the perfect cheap talk. In Section 40, we introduce a different notation from the case with the perfect cheap talk.

If 1-(a) is the case, then without conditioning on $x_i(j)$, by 3 of Lemma 2, player i puts a belief no less than $1 - \exp(-O(T^{\frac{2}{3}}))$ on the event that $f[j](x_i, 2) = x_i = x_i(i)$. Whenever player j uses $f[j](x_i, 1)$ for $x_i(j)$, player j makes player i indifferent between any action profile sequence. Hence, without conditioning on $x_i(j)$, player i puts the belief no less than $1 - \exp(-O(T^{\frac{2}{3}}))$ on the event that $x_i(j) = x_i(i)$ or player i is indifferent between any action profile.

Suppose that player i learns $x_i(j) \neq x_i(i)$. Remember that with probability η, player j uses the signal of the first message $f[j](x_i, 1)$. Since the precision of the first message is $p = \frac{1}{2}$, 4 of Lemma 2 implies that player i believes that any $f[j](x_i, 1)$ could happen with probability $\exp(-O(T^{\frac{1}{2}}))$ regardless of $g[i](x_i, 1)$ and $g_2[i](x_i, 1)$. Since player i's prior on $f[j](x_i, 2) = x_i(i)$ is $1 - \exp(-O(T^{\frac{2}{3}}))$, after learning $x_i(j) \neq x_i(i)$, player i puts a posterior no less than $1 - \exp(-O(T^{\frac{2}{3}}))/\exp(-O(T^{\frac{1}{2}})) = 1 - \exp(-O(T^{\frac{2}{3}}))$ on the event that player j uses the signal of the first message $f[j](x_i, 1)$ and that $f[j](x_i, 1)$ was wrong. In that event, player j makes player i indifferent between any action profile sequence. Therefore, we are done.

Proof of 2 of Lemma 29: If 2 of the construction of $x_i(j)$ is the case, then 2 of Lemma 2 guarantees the result. Note that $x_i(i)$ never reveals $g[i](x_i, 1)$.

If 1 of the construction of $x_i(j)$ is the case, then 2 of Lemma 2 implies that, without conditioning on $x_i(i)$, player j puts the belief no less than $1 - \exp(-O(T^{\frac{2}{3}}))$ on the events that $x_i(i) = x_i(j)$ or player j is indifferent between any action profile since (i) if $g[i](x_i, 2) = E$, then player j is indifferent between any action profile and (ii) if $g[i](x_i, 2) = x_i$ and player i uses $f[i](x_i(j))$, then player j is indifferent between any action profile.

Suppose that player j learns that $x_i(i) \neq x_i(j)$ and $x_i(i) = x_i$.[86] If player j puts a high belief on $x_i(i) = g[i](x_i, 2) = x_i$, then this lemma does not hold. However, with probability η, player i uses the signal from player j's message, $f[i](x_i(j))$. Since the precision of this message is $p = \frac{1}{2}$, 4 of Lemma 2 implies that player j believes that any $f[i](x_i(j))$ could happen with

[86] In the other cases, either $x_i(i) = x_i(j)$ or "$x_i(i) \neq x_i$ and so player j is indifferent between any action profile in the main blocks."

probability $\exp(-O(T^{\frac{1}{2}}))$ regardless of $g[j](x_i(j))$ and $g_2[j](x_i(j))$. Since player j's prior on the event "$x_i = f[i](x_i, 2)$ or $g[i](x_i, 2) = E$" is $1 - \exp(-O(T^{\frac{2}{3}}))$, after learning $x_i(i) \neq x_i(j)$ and $x_i(i) = x_i$, player i puts a posterior no less than $1 - \exp(-O(T^{\frac{2}{3}}))/\exp(-O(T^{\frac{1}{2}})) = 1 - \exp(-O(T^{\frac{2}{3}}))$ on the event that player i uses the result of player j's message $f[i](x_i(j))$ and that $f[i](x_i(j))$ happened to be x_i. In that event, player i makes player j indifferent between any action profile sequence. Therefore, we are done.

Proof of 3 of Lemma 29: Let us consider player i's incentive. First, the probability that player j makes player i indifferent is almost independent of player i's strategy: $g[j](x_i(j)) = E$ happens with probability no more than $\exp(-O(T^{\frac{1}{2}}))$ regardless of $x_i(j)$. In addition, whether 1 or 2 is the case for the construction of $x_i(j)$ is determined by player j's own randomization.

Since x_i controls player j's value, not player i's value, player i does not have an incentive to deviate to coordinate on a different x_i. Since 1 of Lemma 29 guarantees that player i can infer player j's inference $x_i(j)$ correctly or player i is indifferent between any action profile, we are done.

Next, we consider player j's incentive. First, the probability that player i makes player j indifferent is independent of player j's strategy: The distribution of $g[i](x_i, 1)$ and $g[i](x_i, 2)$ is independent of player j's strategy. In addition, whether 1-(a) or 1-(b) is the case for the construction of $x_i(i)$ is determined by player i's own randomization.

Second, by 2 of Lemma 29, the equilibrium strategy enables player j to infer player i's inference $x_i(i)$ correctly or player j is indifferent between any action profile with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.

Third, whenever player i uses the signal from player j's message, $f[i](x_i(j))$, 1-(b) or 2 is the case for the construction of $x_i(i)$ and player i makes player j indifferent.

Therefore, the truthtelling incentive for $x_i(j)$ is satisfied.

40 Structure of the Review Phase

Replacing the perfect cheap talk in the coordination block with the noisy cheap talk, we have the following structure of the review phase:

Now, the coordination block has six rounds with the following chronological order:

- The round for $(x_1, 1)$ where player 1 sends x_1 via noisy cheap talk with precision $p = \frac{1}{2}$.
- The round for $(x_1, 2)$ where player 1 sends x_1 via noisy cheap talk with precision $p = \frac{2}{3}$.
- The round for $(x_1, 3)$ where player 2 sends $x_1(2)$ via noisy cheap talk with precision $p = \frac{1}{2}$.
- The round for $(x_2, 1)$ where player 2 sends x_2 via noisy cheap talk with precision $p = \frac{1}{2}$.
- The round for $(x_2, 2)$ where player 2 sends x_2 via noisy cheap talk with precision $p = \frac{2}{3}$.
- The round for $(x_2, 3)$ where player 1 sends $x_2(1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.

After that, we have L review blocks. For each lth main block with $l = 1, ..., L - 1$, there are following seven rounds in the following chronological order:

- The lth review round where the players play the stage game for T periods.
- The supplemental round for $\lambda_1(l + 1)$ where player 1 sends $\lambda_1(l + 1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.
- The supplemental round for $\lambda_2(l + 1)$ where player 2 sends $\lambda_2(l + 1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.
- The supplemental round for $d_1(l + 1)$ where player 1 sends $d_1(l + 1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.
- The supplemental round for $d_2(l + 1)$ where player 2 sends $d_2(l + 1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.

- The supplemental round for $\hat{d}_2(l+1)$ where player 1 sends $\hat{d}_2(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.

- The supplemental round for $\hat{d}_1(l+1)$ where player 2 sends $\hat{d}_1(l+1)$ via noisy cheap talk with precision $p = \frac{1}{2}$.

The last main block only has the Lth review round where the players play the stage game for T periods.

After that, we have the report block, which will be explained fully in Section 44.

As we can see, there is a chronological order for the rounds. Hence, we can number all the rounds serially. For example, the round for $(x_1, 1)$ is round 1, the round for $(x_1, 2)$ is round 2, and so on.

In addition, if we replace the noisy cheap talk with precision p with messages via actions, then the round r where player i sends the message via noisy cheap talk with precision p consists of T^p periods. For example, in the round for $(x_1, 1)$, the players play the stage game for $T^{\frac{1}{2}}$ periods.

Finally, let $T(r)$ be the set of periods in round r.

41 Replacing the Noisy Cheap Talk with Messages via Actions

Consider round r where player j sends m via noisy cheap talk message with precision p. We replace the noisy cheap talk with precision p with messages via actions as follows. Now, round r consists of T^p periods.

As we have mentioned in Section 38.1, to send message m with precision p, player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2}a_j^G + \frac{1}{2}a_j^B & \text{if } z_j(m) = M \end{cases}$$

with

$$z_j(m) = \begin{cases} m & \text{with probability } 1-2\eta, \\ \{G,B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

for T^p periods. Player i takes a_i^G for T^p periods.

Instead of using $g[j](m)$ and $f[i](m)$ in Section 38.1, we formally re-define $g[j](m)$ and $f[i](m)$ below. Although the intuitive meaning is the same as in Section 38.1, we slightly change the definition to deal with the incentives to send the message and to establish the truthtelling incentive in the report block.

41.1 Formal: $g[j](m)$

If $z_j(m) \neq m$, then $g[j](m) = E$ as in Section 38.1. In addition, player j randomly picks $t_j(r)$ from the set of periods in round r. Define $T_j(r) \equiv T(r) \setminus \{t_j(r)\}$.

Let us concentrate on the case with $z_j(m) = m$. Let $\mathbf{y}_j(r)$ be the frequency of player j's signal observations in round r. Instead of using $\mathbf{y}_j(r)$ directly as in Section 38.1, we consider the following procedure to construct $g[j](m)$.

Player j constructs random variables $\left\{\mathbf{\Omega}_{j,t}^H\right\}_{t \in T_j(r)}$ as follows. After taking a_j^m (remember that we concentrate on the case with $z_j(m) = m$) and observing $y_{j,t}$, player j calculates $H_j(m)\mathbf{1}_{y_{j,t}}$. Then, player j draws $(|Y_j| - |A_i| + 1)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $H_j(m)\mathbf{1}_{y_{j,t}}$, then the lth element of $\mathbf{\Omega}_{j,t}^H$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_{j,t}^H$ is equal to 0. From Lemma 25,

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{j,t}^H\right)_l = 1\right\} \mid a, y\right) = \left(H_j(m)\mathbf{1}_{y_{j,t}}\right)_l \in (0,1) \tag{135}$$

for all a and y.

We define $g[j](m) = m$ if and only if

$$\left\| \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_j(r)} \mathbf{\Omega}_{j,t}^H - \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_j(r)} H_j(m) \mathbf{1}_{y_{j,t}} \right\| \leq \frac{\varepsilon}{4} \tag{136}$$

and

$$\left\| \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_j(r)} \mathbf{\Omega}_{j,t}^H - \mathbf{p}_j(m) \right\| \leq \frac{\varepsilon}{2}. \tag{137}$$

In summary, there are following cases:

1. $g[j](m) = E$ if $z_j(m) \neq m$, (136) is not satisfied, or (137) is not satisfied.

2. $g[j](m) = m$ if $z_j(m) = m$, (136) is satisfied, and (137) is satisfied.

41.2 Formal: $f[i](m)$

Let $\mathbf{y}_i(r)$ be the frequency of player i's signal observations in round r. Instead of using $\mathbf{y}_i(r)$ directly as in Section 38.1, we consider the following procedure to construct $f[i](m)$.

First, player i randomly picks $t_i(r)$ from the set of periods in round r. With $T_i(r) \equiv T(r) \setminus \{t_i(r)\}$, player i constructs $f[i](m)$ based only on $\{y_{i,t}\}_{t \in T_i(r)}$. For notational convenience, let $\mathbf{y}_i(r, T_i(r))$ be the frequency of player i's signal observations in $T_i(r)$.

$f[i](m)$ is determined as in Section 38.1 with $\mathbf{y}_i(r)$ replaced with $\mathbf{y}_i(r, T_i(r))$.

1. If $\mathbf{y}_i(r, T_i(r)) \in \mathcal{H}_i[\varepsilon](G)$, then

   (a) $f[i](m) = G$ if $\mathbf{y}_i(r, T_i(r)) \in \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$,

   (b) $f[i](m) = B$ if $\mathbf{y}_i(r, T_i(r)) \in \mathcal{H}_{j,i}[\varepsilon](B) \cup \mathcal{I}_i[\varepsilon](B)$ or $\mathbf{y}_i(r, T_i(r)) \notin \mathcal{H}_{j,i}[\varepsilon](G) \cup \mathcal{I}_i[\varepsilon](G)$, and

2. If $\mathbf{y}_i(r, T_i(r)) \notin \mathcal{H}_i[\varepsilon](G)$, then player i infers $f[i](m)$ from the likelihood using $\mathbf{y}_i(r, T_i(r))$ (neglecting $y_{i,t_i(r)}$).

By Assumption 3 (full support), neglecting $\left(a_{i,t_i(r)}, y_{i,t_i(r)}\right)$ does not affect the posterior so much.

41.3 Formal: $\theta_i(j \to_m i)$

In addition, player i constructs $\theta_i(j \to_m i) \in \{G, B\}$ for a round where player i receives a message m from player j. Intuitively, $\theta_i(j \to_m i) = B$ implies that player i makes player j indifferent between any action profile sequence in the subsequent rounds.

First, player i creates $\{\mathbf{\Omega}_{i,t}^H\}_{t \in T_i(r)}$ as player j constructs $\{\mathbf{\Omega}_{j,t}^H\}_{t \in T_j(r)}$ with $m = G$. Since the receiver (player i) takes a_i^G and we take the affine hull with respect to player j's actions for the definition of $H_i(G)$, the situation is as player i were a sender and sent a message G. Hence, the distribution of $\mathbf{\Omega}_{i,t}^H$ is independent of player j's strategy.

We define $\theta_i(j \to_m i) = G$ if and only if

$$\left\| \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_i(r)} \mathbf{\Omega}_{i,t}^H - \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_i(r)} H_i(G) \mathbf{1}_{y_{i,t}} \right\| \leq \frac{\varepsilon}{4} \tag{138}$$

and

$$\left\| \frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_i(r)} \mathbf{\Omega}_{i,t}^H - \mathbf{p}_i(G) \right\| \leq \frac{\varepsilon}{2}. \tag{139}$$

Otherwise, $\theta_i(j \to_m i) = B$. Note that $\theta_i(j \to_m i) = B$ if $\mathbf{y}_i(r, T_i(r)) \notin \mathcal{H}_i[\varepsilon](G)$ by the triangle inequality.

In summary, we can show the following lemma:

Lemma 30 *There exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, for sufficiently large T, for any $i, j \in I$, the above mappings satisfy the following:*

1. *For any $m \in \{G, B\}$, $f[i](m) = m$ with probability $1 - \exp(-O(T^p))$ and $g[j](m) = m$ with probability $1 - 2\eta - \exp(-O(T^p))$.*

2. *For any $m \in \{G, B\}$, given m and any $\mathbf{y}_i(r)$, player i puts a belief no less than $1 - \exp(-O(T^p))$ on the event that $f[i](m) = m$ or $g[j](m) = E$.*

3. *For any $m \in \{G, B\}$, given m and any $\mathbf{y}_j(r)$, player j with $g[j](m) = m$ puts a belief no less than $1 - \exp(-O(T^p))$ on the event that $f[i](m) = m$ or $\theta_i(j \to_m i) = B$.*

4. *For any $m \in \{G, B\}$, any $(f[i](m), g[j](m))$ happens with probability at least $\exp(-O(T^p))$.*

5. *The probability of $g[j](m)$ being E does not react to player i's strategy by more than $\exp(-O(T^p))$.*

6. *The distribution of $\theta_i(j \to_m i)$ is independent of player j's strategy with probability no less than $1 - \exp(-O(T^p))$.*

Note that, compared to the noisy cheap talk, Condition 3 implies that player j with $g[j](m) = m$ believes that $f[i](m) = m$ or $\theta_i(j \to_m i) = B$ with probability no less than $1 - \exp(-O(T^p))$, instead of believing $f[i](m) = m$. However, since $\theta_i(j \to_m i) = B$ implies that player j is indifferent between any action sequence, the inference defined in Section 39 is still almost optimal. In addition, Condition 6 guarantees that player j does not have an incentive to deviate to manipulate $\theta_i(j \to_m i)$. Further, as $f_2[i](m)$ is not revealed in the main blocks, $y_{i,t_i(r)}$ is not revealed by player i's continuation strategy in the main block. Similarly, as $g_2[j](m)$ is not revealed in the main blocks, $y_{j,t_j(r)}$ is not revealed by player j's continuation strategy in the main block. This fact will be important to incentivize the players to tell the truth in the report block.

Proof.

1. This follows from the law of large numbers.

2. If $f[i](m) = m$, then we are done. Suppose not.

   Note that the definition of $g[j](m)$ implies that $g[j](m) = m$ only if $z_j(m) = m$ and (136) and (137) are satisfied. Therefore, $g[j](m) = m$ only if $z_j(m) = m$ and $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$.

   $f[i](m) \neq m$ implies that either

   (a) $\mathbf{y}_i(r, T_i(r)) \in \mathcal{H}_{j,i}[\varepsilon](m)$ is not the case, or

   (b) player i infers $f[i](m)$ from the likelihood using $\mathbf{y}_i(r, T_i(r))$ (neglecting $y_{i,t_i(r)}$).

If (a) is the case, then by Hoeffding's inequality, given m, player i puts a belief no less than $1-\exp(-O(T^p))$ on the event that $\mathbf{y}_j \notin \mathcal{H}_j[\varepsilon](m)$. If (b) is the case, then by Lemma 27, given m, player i puts a belief no less than $1-\exp(-O(T^p))$ on the event that $z_j(m) \neq m$. Note that, by Assumption 3 (full support), neglecting $\left(a_{j,t_j(r)}, y_{j,t_j(r)}\right)$ and $\left(a_{i,t_i(r)}, y_{i,t_i(r)}\right)$ does not affect the posteriors so much.

3. As we have mentioned, $g[j](m) = m$ implies that $z_j(m) = m$ and $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$. Hence, player j's conditional expectation of $\mathbf{y}_i$ is $Q_{i,j}(a_j^m, a_i^G)\mathbf{y}_j$. Since $\mathbf{y}_j \in \mathcal{H}_j[\varepsilon](m)$ implies $H_j(m)\mathbf{y}_j = \mathbf{p}_j(m) + \boldsymbol{\varepsilon}_j$ with some $\|\boldsymbol{\varepsilon}_j\| \leq \varepsilon$, by Hoeffding's inequality and the definition of $\mathcal{I}_i[\varepsilon](m)$, player j puts a belief no less than $1-\exp(-O(T^p))$ on the event that $\mathbf{y}_i(r, T_i(r)) \in \mathcal{I}_i[\varepsilon](m)$. Again, by Assumption 3 (full support), neglecting $\left(a_{j,t_j(r)}, y_{j,t_j(r)}\right)$ and $\left(a_{i,t_i(r)}, y_{i,t_i(r)}\right)$ does not affect the order of the posteriors. If $\mathbf{y}_i(r, T_i(r)) \in \mathcal{I}_i[\varepsilon](m)$, then either $f[i](m) = m$ or $\mathbf{y}_i(r, T_i(r)) \notin \mathcal{H}_i[\varepsilon](G)$. If the latter is the case, then $\theta_i(j \to_m i) = B$, as desired.

4. Given m, any $(y_t)_{t\in T(r)}$ can occur with probability at least

$$\left\{\min_{y,a} q(y \mid a)\right\}^{T^p}.$$

Assumption 3 (full support) implies that this probability is $\exp(-O(T^p))$.

5. $z_j(m) \neq m$ happens with probability 2η regardless of m and player i's strategy. The distribution of $\boldsymbol{\Omega}_{j,t}^H$ is independent of player i's strategy in period t and (136) is satisfied ex post (conditional on $\{a_t, y_t\}_{t\in T(r)}$) with probability $1-\exp(-O(T^p))$ by the law of large numbers. Finally, even if the message m can depend on player i's past strategy, the probability of $g[j](m)$ being E does not react to m by more than $\exp(-O(T^p))$. Therefore, the probability of $g[j](m)$ being E does not react to player i's strategy by more than $\exp(-O(T^p))$.

6. The distribution of $\boldsymbol{\Omega}_{i,t}^H$ is independent of player j's strategy (note that player i takes a_i^G after any history) and (138) is satisfied ex post with probability $1-\exp(-O(T^p))$

by the law of large numbers.

■

42 Equilibrium Strategies

In this section, we define $\sigma_i(x_i)$ and π_i^{main}.

42.1 States

The states $\lambda_i(l+1)$, $\hat{\lambda}_j(l+1)$, $d_i(l+1)$, $\hat{d}_j(l+1)$, $\hat{d}_i(l+1)(i)$, $\theta_i(l)$, $\theta_i(\lambda_i(l+1))$, $\theta_i(d_i(l+1))$ and $\theta_i(\hat{d}_j(l+1))$ are defined as in the Supplemental Material 2 except that x is replaced with $x(i) = (x_1(i), x_2(i))$ defined in Section 39.

If we replace the noisy cheap talk with messages via actions, then we use $f[i](m)$ (when player i is a receiver) and $g[i](m)$ (when player i is a sender) defined in Section 41. $\theta_i(\lambda_i(l+1))$, $\theta_i(d_i(l+1))$ and $\theta_i(\hat{d}_j(l+1))$ are still valid.

In addition, remember that each player makes the opponent indifferent between any action profile sequence if the following events happen in the coordination block: Each player i makes player j indifferent between any action profile sequence if $g[i](x_i, 1) = E$, $g[i](x_i, 2) = E$, $g[i](x_j(i)) = E$, "1-(b) or 2 is the case for the construction of $x_i(i)$," or 2 is the case for the construction of $x_j(i)$.

We create a new state $\theta_i(\text{c}) \in \{G, B\}$ to summarize these events. For player i, if the events listed above happen, then we say $\theta_i(\text{c}) = B$. Otherwise, $\theta_i(\text{c}) = G$. Note that $\theta_i(\text{c})$ is well defined for the coordination block with and without the noisy cheap talk.

42.2 Player i's Action

42.2.1 With the Noisy Cheap Talk

In the coordination block, the players play the game as explained in Section 39. For the other blocks, $\sigma_i(x_i)$ prescribes the same action with x replaced with $x(i)$ except for the report block. See Section 44 for the strategy in the report block.

42.2.2 Without the Noisy Cheap Talk

In a round where player i would send a message m via noisy cheap talk with precision p if it were available, the players' strategies are explained in Section 41. Here, since player i is sender, reverse i and j: Player i (sender) takes $\alpha_i^{z_i(m)}$ and player j (receiver) takes a_j^G. $f[j](m) \in \{G, B\}$ and $g[i](m) \in \{m, E\}$ are determined as in Section 41.

42.3 Reward Function

In this subsection, we explain player j's reward function on player i, $\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$.

42.3.1 With the Noisy Cheap Talk

The reward function is the same as in the Supplemental Material 2 except that x replaced with $x(j)$ and that if $\theta_j(\mathrm{c}) = B$ happens, then player j uses

$$\pi_i^{\text{main}}(x_j, h_j^{\text{main}}, l) = \sum_{t \in T(l)} \pi_i^{x_j}(a_{j,t}, y_{j,t}).$$

for all the review rounds.

42.3.2 Without the Noisy Cheap Talk

Without cheap talk, $\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta)$ is defined by

$$\pi_i^{\text{main}}(x_j, h_j^{\text{main}} : \delta) = \sum_{l=1}^{L} \sum_{t \in T(l)} \pi_i^{\delta}(t, a_{j,t}, y_{j,t}) + \sum_{r} \pi_i^{\text{main}}(x_j, h_j^{\text{main}}, r : \delta),$$

where $\pi_i^{\text{main}}(x_j, h_j^{\text{main}}, r : \delta)$ is the reward for each round r. Note that we add (23) only for the review rounds. For the other rounds where the players communicate, the reward for round r, $\pi_i^{\text{main}}(x_j, h_j^{\text{main}}, r : \delta)$, directly takes discounting into accounting.

For round r corresponding to a review round, the reward function is the same as in the case with the noisy cheap talk except that, if $\theta_j(i \to_m j) = B$ happens in some previous round where player i sends a message m to player j, then player j uses

$$\pi_i^{\text{main}}(x_j, h_j^{\text{main}}, r) = \sum_{t \in T(r)} \pi_i^{x_j}(a_{j,t}, y_{j,t})$$

to make player i indifferent between any action profile sequence.

For round r where the players communicate, player j makes player i indifferent between any action profile sequence by

$$\pi_i^{\text{main}}(x_j, h_j^{\text{main}}, r) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_j}(a_{j,t}, y_{j,t}). \tag{140}$$

Note that we take discounting into account.[87]

43 Almost Optimality of $\sigma_i(x_i)$

We first consider player i's incentive to receive a message m. When player i receives the message, Lemma 30 implies that a_i^G gives player i the inference $f[i](m)$ satisfying $f[i](m) = m$ or $g[j](m) = E$ with $1 - \exp(-O(T^{\frac{1}{2}}))$. In addition, the probability of $g[j](m)$ being E is almost independent of player i's strategy. Since (140) cancels out the difference in the instantaneous utilities, it is almost optimal to take a_i^G.

Second, we verify player i's incentive to receive a message. Consider the rounds for $(x_i, 1)$ or $(x_i, 2)$. With the noisy cheap talk, we are done with Lemma 29. Suppose that we replace the noisy cheap talk with messages via actions. Remember that if player i deviates in these rounds, then player i uses the signal from player j's message in the round for $(x_i, 3)$.

[87] In the review rounds, there is a positive probability that the opponent uses the reward that is linear in $X_j(l)$, which does not take discounting into account. Therefore, to make each round symmetric, we add π_i^δ to cancel out discounting. See footnote 60 for the explanation of why this is important.

On the other hand, in the rounds where the players communicate, it is common knowledge that the opponent uses the reward that takes discounting into account. Hence, we do not need to add π_i^δ.

Therefore, from Lemmas 29 and 30, "$x(i) = x(j)$," "$\theta_j(i \rightarrow_{x_i} j) = B$ for the round $(x_i, 1)$ or $(x_i, 2)$" or "$\theta_j(\mathrm{c}) = B$" with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ and the coordination on the same inference of x_i is achieved with high probability, regardless of player i's strategy in the rounds for $(x_i, 1)$ or $(x_i, 2)$. In addition, the distribution of $\theta_j(i \rightarrow_{x_i} j) = B$ and $\theta_j(\mathrm{c}) = B$ is almost independent of player i's strategy from Lemmas 29 and 30. Since x_i controls only player j's payoff, this implies that player i is almost indifferent between any strategies in the rounds for $(x_i, 1)$ or $(x_i, 2)$.

For the other rounds where player i sends the message, whenever player i's message affects player j's strategy (action or reward), $\theta_j(i \rightarrow_{x_i} j) = B$ or $\theta_j(\mathrm{c}) = B$ has happened before and player i is indifferent between any actions.

Therefore, when player i sends a message, player i is almost indifferent between any strategy.

For review rounds, from Lemmas 29 and 30, given $x(j)$, for any t in the main blocks and any h_i^t, player i puts a conditional belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$, $\theta_j(i \rightarrow_{x_i} j) = B$ or $\theta_j(\mathrm{c}) = B$. Therefore, the same proof for Proposition 2 works except that now player j makes player i indifferent between any action profile sequence with higher probability:

1. $\theta_j(\mathrm{c}) = B$ with probability

$$\begin{aligned} & \underbrace{2\eta}_{\substack{z_j(x_j)\neq x_j \\ \text{in the round} \\ \text{for } (x_j,1)}} + \underbrace{2\eta}_{\substack{z_j(x_j)\neq x_j \\ \text{in the round} \\ \text{for } (x_j,2)}} + \underbrace{\eta}_{\substack{\text{1-(b)} \\ \text{is the case} \\ \text{for } x_j(j)}} + \underbrace{2\eta}_{\substack{z_j(x_i(j))\neq x_i(j) \\ \text{in the round} \\ \text{for } (x_i,3)}} + \underbrace{\eta}_{\substack{2 \\ \text{is the case} \\ \text{for } x_i(j)}} \\ \leq\ & 8\eta. \end{aligned}$$

plus negligible probability $\exp(-O(T^{\frac{1}{2}}))$.

2. For each supplemental rounds where player j sends a message m,

$$\underbrace{2\eta}_{z_j(m)\neq m}$$

plus negligible probability $\exp(-O(T^{\frac{1}{2}}))$.

Therefore, instead of (87), we re-take η sufficiently small such that[88]

$$\begin{aligned} & \max_{x:x_j=B} \frac{(L-1)\max\{w_i(x), v_i^*\} + u_i^*(x)}{L} + \eta + \frac{\bar{L}}{L} + 2\varepsilon\bar{L} + (8+6L)\eta\left(\bar{u} - \min_{i,a} w_i(a)\right) \\ < \ & \underline{v}_i < \overline{v}_i < \min_{x:x_j=G} w_i(x) - \frac{\bar{L}}{L} - 2\varepsilon\bar{L} - (8+6L)\eta\left(\bar{u} + \max_{i,a} w_i(a)\right). \end{aligned} \tag{141}$$

Finally, since the review round has T period while the other round has at most $T^{\frac{2}{3}}$ periods, the payoffs from the rounds other than the review rounds are negligible. Therefore, Proposition 2 holds.

44 Report Block

We are left to construct the report block. First, we explain the report block with the perfect cheap talk and public randomization. Although this is the same setup as in the main text, since we replace the cheap talk in the coordination block and supplemental rounds with messages via actions, we need to change the structure accordingly.

Second, we construct the report block with the perfect cheap talk but without public randomization device.

Third, we replace the perfect cheap talk with conditionally independent noisy cheap talk.

Finally, we replace the conditionally independent noisy cheap talk with messages via actions.

Whenever the players play the stage game, we cancel out the difference in the instantaneous utilities and discounting by adding

$$\delta^{t-1}\pi_i^{x_j}(a_{j,t}, y_{j,t}).$$

Since the report block lasts for $O(T^{\frac{1}{3}})$ periods, this does not affect the equilibrium payoff.

[88] Precisely speaking, ε is re-taken after η is re-taken so that Lemma 11 is satisfied.

44.1 Report Block with the Perfect Cheap Talk and Public Randomization

We formally construct π_i^{report} assuming that the players send messages via actions in the coordination block and supplemental rounds, keeping the perfect cheap talk and public randomization in the report block.

Remember that r is a serial number of the rounds. Let $\mathcal{A}_j(r)$ be the set of information up to and including round r consisting of

- What state x_j player j is in, and
- For each lth review round, what action plan $\alpha_j(l)$ player j took in the lth review round if round r is the lth review round or after.

Remember that, for each round r, for any period t in round r and any history h_i^t, conditional on $\mathcal{A}_j(r)$, $\sigma_i(x_i)$ is almost optimal.

The reward π_i^{report} is the same as in the case with the cheap talk in the coordination block and supplemental rounds except for the following differences:

Subrounds As we divide a review round into review subrounds whose length is $T^{\frac{1}{4}}$ periods, we divide each round into subrounds whose length is $T^{\frac{1}{4}}$ periods.

Since the rounds for $(x_1, 2)$ and $(x_2, 2)$ have $T^{\frac{2}{3}}$ periods, there are $T^{\frac{2}{3}-\frac{1}{4}}$ subrounds. Since the other rounds for communication have $T^{\frac{1}{2}}$ periods, there are $T^{\frac{1}{4}}$ subrounds.

The coordination on $k(r)$ is analogously modified.

Truthtelling Incentive Since the players communicate via actions, we use (51) to give the incentive for player i to tell the truth instead of (52) and (53). Note that player j's action plan is independent of the signal observation in period $t_j(r)$ and that player i cannot learn it from the history in the coordination and main blocks.

The Rounds where Player i Sends the Message Note that player i takes a mixed strategy in the round where player i sends the message. Moreover, the history in this round affects the belief about the best responses at the beginning of the next round.

Therefore, as we incentivize player i to take a mixed strategy for minimaxing player j, we cancel out the difference of the values coming from learning at the beginning of the next round by (95). Then, we cancel out the difference in the payoffs in the current round by (96).

Since player i is almost indifferent between any strategies (see Section 43), the effect of this adjustment is sufficiently small.

44.2 Report Block withOUT the Public Randomization

In this subsection, we keep the availability of the perfect cheap talk and dispense with the public randomization device. We use the public randomization device to coordinate on the following two: First, who reports the history in the report block. Second, for each round r, the picked player sends the message about $(a_{i,t}, y_{i,t})$ for t included in $T(r, k(r))$ for some $k(r)$ determined by the public randomization. We explain how to dispense with the public randomization for each of them.

44.2.1 Coordination on Who will Report the History

Remember that the problem is that we want to require that (i) for each i, there is a positive probability that π_i^{report} adjusts the reward function and that (ii) each player should not learn about the opponent's history from the opponent's messages in the report block.

Instead of using the public randomization device, we use actions and private signals to coordinate as follows:

1. First, the players take the action profile a^G. Then, each player i observes $y_i \in Y_i$.

2. Player 2 sends the message whether player 2 observed $y_2 \in Y_{2,1}^2$ or $y_2 \in Y_{2,2}^2$ in Step 1. See Assumption 11 to review the notation. Player 1 adjusts player 1's reward function

on player 2 if and only if player 2 says that player 2 observed $y_2 \in Y_{2,1}^2$.

3. Player 2 sends the messages first. Player 2 has the following two cases:

   (a) If player 2 observed $y_2 \in Y_{2,1}^2$, then player 2 sends the messages about her history h_2^{main} truthfully.

   (b) If player 2 observed $y_2 \in Y_{2,2}^2$, then player 2 sends a meaningless message $\{\emptyset\}$.

4. Player 1 sends the messages about her history h_1^{main} truthfully.

In Step 4, player 2 adjusts player 2's reward function on player 1 if and only if player 2 observed $y_2 \in Y_{2,2}^2$. Therefore, the probability that player j's reward on player i is adjusted is now $\Pr(y_2 \in Y_{2,2}^2 \mid a^G)$ for $i = 1$ and $\Pr(y_2 \in Y_{2,1}^2 \mid a^G)$ for $i = 2$. The term representing

$$\frac{1}{\Pr(\text{player } i \text{ is picked by the public randomization device})}$$

in π_i^{report} of Section 15.7 is analogously modified to

$$\begin{aligned} \Pr(y_2 &\in Y_{2,2}^2 \mid a^G) \text{ for } i = 1, \\ \Pr(y_2 &\in Y_{2,1}^2 \mid a^G) \text{ for } i = 2. \end{aligned}$$

Consider player 1's incentives. By Assumptions 3 and 11, there is a positive probability that $y_2 \in Y_{2,2}^2$ and player 2's reward on player 1 is adjusted. In addition, when player 1 sends the message in Step 4, player 1 conditions that player 2's message in Step 3 does not reveal h_2^{main}, as desired.

Therefore, we need to verify the incentives that the players take a^G in Step 1 and player 2 tells the truth in Step 2 and 3. To establish the incentives, we add the following rewards:

- In Step 1, to incentivize the players to take a^G, each player j gives a reward on a_i^G:

$$T^{-1}\Psi_j^{a^G}. \tag{142}$$

- In Step 2, to incentivize player 2 to tell the truth, player 1 punishes player 2 by $-T^{-2}$ if player 2 sends the message $y_2 \in Y_{2,1}^2$ while player 1 observed $y_1 \in Y_{1,2}^2$ and by $-\frac{1}{\bar{p}_2}T^{-2}$ if player 2 sends the message $y_2 \in Y_{2,2}^2$ while player 1 observed $y_1 \in Y_{1,1}^2$.

- In Step 3,

  – If player 2 sent the message $y_2 \in Y_{2,1}^2$ in Step 2, then player 1's reward on player 2 is the same as π_2^{report} so that player 2 sends the messages about her history h_2^{main} truthfully.

  – If player 2 sent the message $y_2 \in Y_{2,2}^2$ in Step 2, then player 1 changes π_2^{report} so that player 1 gives a small reward on $\{\emptyset\}$:

$$\pi_2^{\text{report}}(x_1, h_1^{T_P+1} : \delta) = T^{-3}\mathbf{1}\{\text{player 2 sends } \{\emptyset\}\} - T^{-3}.$$

Then, we can show the incentive compatibility of the above strategy.

- In Step 1, since all the rewards affected by player i's action except for (142) are bounded by $O(T^{-2})$, it is strictly optimal to take a^G.

- In Step 2, since all the rewards affected by player i's current and future actions are bounded by $O(T^{-3})$,[89] if

$$\Pr(\{y_1 \in Y_{1,1}^2\} \mid a^G, y_2) > \frac{\bar{L}T^{-2} + O(T^{-3})}{\frac{1}{\bar{p}_2}\bar{L}T^{-2} - O(T^{-3})} \to \bar{p}_2,$$

  then it is optimal to send $y_2 \in Y_{2,1}^2$ and if

$$\Pr(\{y_1 \in Y_{1,1}^2\} \mid a^G, y_2) < \frac{\bar{L}T^{-2} - O(T^{-3})}{\frac{1}{\bar{p}_2}\bar{L}T^{-2} + O(T^{-3})} \to \bar{p}_2,$$

  then it is optimal to send $y_2 \in Y_{2,2}^2$, as desired.

[89] For the punishment in Step 5 in the coordination on $k(r)$ below, there is a punishment of order T^{-2}. However, as we will see, by backward induction, this punishment is not affected by Step 2 here.

- In Step 3, since player 2 believes that the message in Step 2 was transmitted correctly, it is optimal to tell the truth by the same reason as in Section 15.7.

Remember that player 1 rewards player 2 in Step 3 based on player 2's message about y_2, not depending on player 1's signal y_1. Therefore, after sending the message about y_2 in Step 2, it is optimal for player 2 to follow the equilibrium strategy.

44.2.2 Coordination on $k(r)$

For each player i, while player i sends the messages about h_i^{main}, for each round r, the players coordinate on $k(r) \in \{1, \dots, K\}$ with $K \leq T^{\frac{3}{4}}$.

By abuse of language, in our equilibrium,

- in Step 3 in Section 44.2.1, even if player 2 sends $\{\emptyset\}$, the players play the following game for each round and player 1 punishes player 2.

- in Step 4 in Section 44.2.1, even if player 2 sent h_2^{main}, the players play the following game for each round and player 2 punishes player 1.

The players take a^G for $\log_2 K$ periods. We create a mapping between a sequence of $\{1, 2\}$, $\boldsymbol{i} \in \{1, 2\}^{\log_2 K}$, and $k(r)$ such that each $\boldsymbol{i}$ uniquely identifies $k(r)$ and that, for each $k(r)$, there is at least one $\boldsymbol{i}$ that is mapped into $k(r)$.

For each $n \in \{1, ..., \log_2 K\}$, the players coordinate on one element of $\{1, 2\}$ as in Steps 1, 2 and 3 in Section 44.2.1. That is,

1. The players take a^G for $\log_2 K$ times.

2. For each $n \in \{1, ..., \log_2 K\}$, if player j observes $y_j \in Y_{j,1}^i$, then player j infers that the nth element of $\boldsymbol{i}$ is 1. Otherwise, that is, if player j observes $y_j \in Y_{j,2}^i$, then player j infers that the nth element of $\boldsymbol{i}$ is 2. By doing so, player j infers $\boldsymbol{i}$. Let $\boldsymbol{i}(j)$ be player j's inference. Let $k(r, j)$ be player j's inference of $k(r)$ that corresponds to $\boldsymbol{i}(j)$.

3. On the other hand, for each $n \in \{1, ..., \log_2 K\}$, player i infers $\boldsymbol{i}(i)$ and $k(r,i)$ using the partitions $Y_{i,1}^i$ and $Y_{i,2}^i$ instead of $Y_{j,1}^i$ and $Y_{j,2}^i$.

4. Player i sends the sequence of binary messages $\boldsymbol{i}(i) \in \{1,2\}^{\log_2 K}$.

5. For each $n \in \{1, ..., \log_2 K\}$, player j punishes player i if player i's message $\boldsymbol{i}_n(i)$ is different from $\boldsymbol{i}_n(j)$. Here, $\boldsymbol{i}_n(1)$ and $\boldsymbol{i}_n(2)$ are the nth element of $\boldsymbol{i}(1)$ and $\boldsymbol{i}(2)$, respectively.

   Specifically, player j punishes player i by $-T^{-2}$ if player i sends the message $\boldsymbol{i}_n(i) = 1$ while player j observed $\boldsymbol{i}_n(j) = 2$ and by $-\frac{1}{\bar{p}_2}T^{-2}$ if player i sends the message $\boldsymbol{i}_n(i) = 2$ while player j observed $\boldsymbol{i}_n(j) = 1$.

   Note that this is the same as in Step 2 in Section 44.2.1.

6. From the message $\boldsymbol{i}(i)$, player j knows $k(r,i)$. Player j calculates the punishment (55) based on $k(r,i)$.

By backward induction, we can show that it is always optimal to follow the equilibrium strategy: Consider the message for the last element of the sequence, $\boldsymbol{i}_K(i)$, for the last round. Since the punishment from the previous messages about $\boldsymbol{i}(i)$ is sunk and the reward or punishment affected by player i's continuation strategy except for the punishment coming from $\boldsymbol{i}_K(i) \neq \boldsymbol{i}_K(j)$ is $O(T^{-3})$, if

$$\Pr(\{y_j \in Y_{j,1}^i\} \mid a^G, y_i) > \frac{\bar{L}T^{-2} + O(T^{-3})}{\frac{1}{\bar{p}_2}\bar{L}T^{-2} - O(T^{-3})} \to \bar{p}_i,$$

then it is optimal to send $y_i \in Y_{i,1}^i$ and if

$$\Pr(\{y_j \in Y_{j,1}^i\} \mid a^G, y_i) < \frac{\bar{L}T^{-2} - O(T^{-3})}{\frac{1}{\bar{p}_2}\bar{L}T^{-2} + O(T^{-3})} \to \bar{p}_i,$$

then it is optimal to send $y_i \in Y_{i,2}^i$, as desired.

For the second $\boldsymbol{i}_{K-1}(i)$, since the expected punishment from the last message $\boldsymbol{i}_K(i)$ is fixed by the equilibrium strategy, the same argument holds. Recursively, we can show the optimality of the equilibrium strategy.

As mentioned in footnote 89, when the players coordinate on who will report the history in Step 2 of Section 44.2.1, we can assume that the expected punishment from $\boldsymbol{i}_n(i) \neq \boldsymbol{i}_n(j)$ is fixed.

Although player j punishes player i for mis-coordination in Step 5, when player j calculates (55), player j uses player i's inference of $k(r)$, $k(r,i)$. Hence, once player i sends the messages about $\boldsymbol{i}(i)$, player i has the incentive to tell the truth about h_i^{main} based on her own inference $k(r,i)$.

Expected Punishment As we have mentioned, when the players coordinate on whether player 2 should send the message about h_2^{main}, player 1 rewards player 2 in Step 3 based on player 2's message about y_2. In addition, when the players coordinate on $k(r)$, player j uses $k(r,i)$ to calculate (55) and the term $T^{\frac{3}{4}}$ in (55) is replaced by

$$\frac{1}{\Pr(k(r,i) \text{ is realized in the coordination explained in Section 44.2.2})}.$$

Therefore, given the truthtelling incentive for y_2 and $k(r,i)$, the expected punishment from the coordination is independent of the players' history in $\left(h_1^{\text{main}}, h_2^{\text{main}}\right)$. Hence, this coordination in the report block does not affect any incentive in the coordination and main blocks.

44.3 Report Block with Conditionally Independent Cheap Talk

In this subsection, we replace the perfect cheap talk with conditionally independent noisy cheap talk. That is, each player has the conditionally independent noisy cheap talk communication device to send a binary message $m \in \{G, B\}$. When player i sends the message m, the receiver (player j) observes the correct message m with high probability $1 - \exp(-T^{\frac{1}{3}})$ while player j observes the erroneous message $\{G, B\} \setminus \{m\}$ with low probability $\exp(-T^{\frac{1}{3}})$.

Player i (sender) does not obtain any information about what message player j receives (conditional on m). Hence, the communication is conditionally independent.

When player j (receiver) constructs π_i^{report}, player j needs to take care of the possibility that player j receives an erroneous message.

Note that the number of binary messages sent in the report block is $O(T^{\frac{1}{4}})$. Therefore, all the messages transmit correctly with probability at least

$$1 - O(T^{\frac{1}{4}}) \exp(-T^{\frac{1}{3}}).$$

In addition, the cardinality of the information sent by all the messages is $\exp(O(T^{\frac{1}{4}}))$. Let $\mathcal{M}_i$ be the set of information possibly sent by player i in the report block with $|\mathcal{M}_i| = \exp(O(T^{\frac{1}{4}}))$. Let P_i be the $|\mathcal{M}_i| \times |\mathcal{M}_i|$ matrix whose (k, k') element represents

$$\Pr \begin{pmatrix} \text{player } j \text{ receives the message corresponding to the element } k' \text{ of } \mathcal{M}_i \\ \mid \text{player } i \text{ sends the message corresponding to the element } k \text{ of } \mathcal{M}_i \end{pmatrix}.$$

Since $|\mathcal{M}_i| = \exp(O(T^{\frac{1}{4}}))$ and all the messages transmit correctly with probability no less than $1 - O(T^{\frac{1}{4}}) \exp(-T^{\frac{1}{3}})$,

$$\left(1 - O(T^{\frac{1}{4}}) \exp(-T^{\frac{1}{3}})\right)^{\exp(O(T^{\frac{1}{4}}))} \geq 1 - \exp(O(T^{\frac{1}{4}}))O(T^{\frac{1}{4}}) \exp(-T^{\frac{1}{3}}) \to 1$$

as T goes to infinity and so

$$\lim_{\delta \to 1} P_i^{-1} = \lim_{T \to \infty} P_i^{-1} = E \text{ (identity matrix)}.$$

Let

$$\pi_i^{\text{report}}(x_j, h_j^{T_P+1}, k : \delta)$$

be the reward function that player j with $h_j^{T_P+1}$ would construct after the history corresponding to the element k via perfect cheap talk, that is, if P_i were E.[90] In addition, let $\boldsymbol{\pi}_i^{\text{report}}(x_j, h_j^{T_P+1} : \delta)$ be the vector stacking all $\pi_i^{\text{report}}(x_j, h_j^{T_P+1}, k : \delta)$'s with respect to k.

If we player j uses

$$P_i^{-1}\boldsymbol{\pi}_i^{\text{report}}(x_j, h_j^{T_P+1} : \delta),$$

then player i's incentive is the same as in the situation that the messages would always transmit correctly (as if P_i were E). Since the truthtelling incentive is strict, multiplying P_i^{-1} to $\boldsymbol{\pi}_i^{\text{report}}(x_j, h_j^{T_P+1} : \delta)$ does not affect the incentives in the report block if P_i^{-1} is sufficiently close to E.

44.4 Report Block withOUT the Conditionally Independent Cheap Talk

Finally, we dispense with the conditionally independent cheap talk.

Consider Step 2 of Section 44.2.1. If player 2 wants to send $y_2 \in Y_{2,1}^2$, then player 2 takes a_2^G for $T^{\frac{1}{3}}$ periods. On the other hand, if player 2 wants to send $y_2 \in Y_{2,2}^2$, then player 2 takes a_2^B for $T^{\frac{1}{3}}$ periods.

Player 1 takes $\bar{\alpha}_1$ for $T^{\frac{1}{3}}$ periods. After calculating $\phi_1(a_{1,t}, y_{1,t})$, player 1 constructs $\Phi_{1,t}$ from $\phi_1(a_{1,t}, y_{1,t})$ as she constructs $\Psi_{1,t}$ from $\psi_1^a(y_{1,t})$. Lemma 39 implies that

$$\Pr\left(\{\Phi_{1,t} = 1\} \mid \bar{\alpha}_{1,t}, a_{2,t}, y_{2,t}\right) = \begin{cases} q_2 & \text{if } a_{2,t} = a_2^G, \\ q_1 & \text{if } a_{2,t} \neq a_2^G \end{cases} \tag{143}$$

for all t and $y_{2,t}$.

Player 1 infers that player 2's message is G if

$$\frac{\sum_t \Phi_{1,t}}{T^{\frac{1}{3}}} > \frac{q_2 + q_1}{2} \tag{144}$$

and B otherwise. Here, the summation is taken over $T^{\frac{1}{3}}$ periods where player 2 sends the

[90] Here, we use $h_j^{T_P+1}$ instead of h_j^{main} since player j needs to use the signal observations while the players coordinate on who will report the history and $k(r)$.

message.

Next, consider Step 3 of Section 44.2.1. If 3-(a) is the case, then player 2 would send binary messages about h_2^{main} with the conditionally independent noisy cheap talk. Since all the messages are binary, we can see player 2 sending a binary message $m \in \{G, B\}$. Without the conditionally independent cheap talk, for each message m, the players spend $T^{\frac{1}{3}}$ periods. Player 2 takes a_2^m and player 1 takes $\bar{\alpha}_1$. Player 1 infers player 2's message by (144).

On the other hand, if 3-(b) of Section 44.2.1 is the case, then the players spend the same number of periods as in 3-(a). For periods where player 2 would send the message about h_2^{main} in 3-(a), player 2 takes a_2^G. On the other hand, for periods where player 2 sends the message about $k(r)$, player 2 sends the same message as in 3-(a). Player 1 always takes $\bar{\alpha}_1$.[91]

Player 1's reward on player 2 is determined as follows: While player 2 sends the message m corresponding to Step 2 of Section 44.2.1, player 1 gives the following reward: Let $t(m)$ be the first period when player 2 sends m.

1. At $t(m)$, both a_2^G and a_2^B are indifferent and are better than the other actions (if any). The reward is given by

$$\Psi_{1,t(m)}^{a_2^G, a_{1,t(m)}} + \Psi_{1,t(m)}^{a_2^B, a_{1,t(m)}} - (q_2 + q_1). \tag{145}$$

   By algebra, we can verify that

   (a) The expected payoff of taking a_2^G or a_2^B in period $t(m)$ is 0.

   (b) The expected payoff of taking another action is $-O(1)$.

2. After that, the constant action is optimal:

$$\sum_{t=t(m)+1}^{t(m)+T^{\frac{1}{3}}-1} \left(c + T^{-1}\bar{L}\frac{1}{q_2(1-q_1)(q_1-q_2)^2} \left(\begin{array}{c} (1-q_1)\mathbf{1}\{\Phi_{1,t(m)}=1\}\Psi_{1,t}^{a_2^G, a_{1,t}} \\ +q_2\mathbf{1}\{\Phi_{1,t(m)}=0\}\Psi_{1,t}^{a_2^B, a_{1,t}} \end{array} \right) \right). \tag{146}$$

[91] Note that player 1 takes the same action between in 3-(a) and 3-(b). Therefore, player 1 does not need to know which is the case.

Here, c is a constant such that the expected payoff of taking a_2^G after $a_{2,t(m)} = a_2^G$ is equal to 0.

From (143) and (146), we can verify that

(a) The expected payoff of taking a_2^m in period t after taking a_2^m in period $t(m)$ is 0.

(b) The expected payoff of taking another action in period t after taking a_2^m in period $t(m)$ is $-O(T^{-1})$.

Next, let us consider the reward for the messages corresponding to Step 3 of Section 44.2.1.

If player 1 infers that player 2's message corresponding to Step 2 says that player 2 observed $y_2 \in Y_{2,1}^2$, then for each message m in Step 3 of Section 44.2.1, player 1 gives the same reward as (145) and (146).

If player 1 infers that player 2's message in Step 2 says that player 2 observed $y_2 \in Y_{2,2}^2$, then for periods where player 2 is supposed to take a_2^G, player 1 gives

$$\sum_t \left(\Psi_{1,t}^{a_2^G, a_{1,t}} - q_2 \right), \tag{147}$$

so that player 2 takes a_2^G. Note that the expected payoff from (147) by taking a_2^G is zero.

For periods where player 2 sends the message about $k(r)$, player 1 gives the same reward as (145) and (146).

By backward induction, we can show the following: Suppose that player 2 constantly took a_2^G in Step 2 of Section 44.2.1. Then, with probability no less than $1 - \exp(-O(T^{\frac{1}{3}}))$, player 1 uses the reward (145) and (146).[92] Consider the last message m. All the rewards from (145) and (146) determined by the previous messages are sunk. In addition, the punishment and reward in the report block g_2 and f_i defined in Section 15.7 are bounded by $O(T^{-3})$. Hence, the difference between 1-(a) and 1-(b) is sufficiently large that, in period $t(m)$, player

[92] More precisely, given the truthtelling incentive, since player j takes into the account that P_i is not an identity matrix as seen in Section 44.3, player i believes that the messages transmit correctly with probability one.

2 takes either a_2^G or a_2^B. In addition, the difference between 2-(a) and 2-(b) is sufficiently large that after taking a_2^G or a_2^B in period $t(m)$, player 2 should take the constant action.

Since this equilibrium strategy makes the expected payoff from the last message, (145) and (146) equal to 0, the same argument holds until the first message.

Symmetrically, if player 2 constantly took a_2^B in Step 2 of Section 44.2.1, then it is optimal for player 2 in Step 3 to take a_2^G when player 2 is supposed to take a_2^G and to take a_2^G or a_2^B constantly for periods where player 2 should send a message for $k(r)$.

Finally, when player 2 sends the message about y_2 in Step 2 of Section 44.2.1, it is strictly optimal to take a constant action since (i) there is a strict incentive for the constant action and (ii) it gives player 2 the better idea about whether player 2 should tell the truth about the history or take a_2^G constantly to maximize (147).

Therefore, (i) this replacement of the conditionally independent cheap talk with messages via actions does not affect the payoff and since player 2 repeats the message for $T^{\frac{1}{3}}$ periods, (ii) player 1 infers the correct message m with high probability $1 - \exp(-O(T^{\frac{1}{3}}))$, (iii) player 2's private signal cannot update player 2's belief about player 1's inference of player 2's message (conditional independence) by (143), and (iv) since the number of necessary messages is $O(T^{\frac{1}{4}})$, the number of necessary periods for player 2 to send all the messages is

$$O(T^{\frac{1}{3}+\frac{1}{4}}) < O(T),$$

as desired.

For player 1, since we cannot assume that $|A_1|\,|Y_1| \leq |A_2|\,|Y_2|$, we cannot generically find a function $\phi_2(a_2, y_2)$ with the conditional independence property symmetric to $\phi_1(a_1, y_1)$. Therefore, after Step 4 in Section 44.2.1, we add an additional round where player 1 sends the messages about player 1's histories in Step 4. Based on the information that player 2 obtains in this additional round, player 2 creates a statistics to infer player 1's messages in Step 4, so that while player 1 sends the messages about h_1^{main} in Step 4 (before observing the history in the additional round), player 1 cannot update player 2's inference of the messages from player 1's signal observations.

44.4.1 Recovery of Conditional Independence

In Step 4 of Section 44.2.1, without cheap talk, player 1 takes $a_1 \in \{a_1^G, a_1^B\}$ to send the message. To send each message m, player 1 repeats a_1^m for $T^{\frac{1}{3}}$ periods. Player 2 takes a_2^G. As for player 2, this takes $O(T^{\frac{1}{3}+\frac{1}{4}})$ periods.

After this step is over, we have the following round named the "round for conditional independence." The intuitive structure is as follows. For each period t in Step 4, player 1 reports the history in period t to player 2 in the round for conditional independence. Player 2 infers player 1's messages in Step 4 combining player 2's signals in Step 4 and player 1's reports about player 1's history.

Player 2 gives the following rewards to player 1: (i) The adjustment of player 1's reward so that $\sigma_1(x_1)$ is exactly optimal; (ii) A reward that makes an optimal strategy in the round for conditional independence given player 1's history in Step 4 unique; (iii) We make sure that (i) is much smaller than (ii), so that player 1's history in Step 4 and the strictness of player i's incentive in the round for conditional independence completely determines player 1's strategy in the round for conditional independence, independently of h_1^{main}. (iv) Given player 1's continuation strategy in the round for conditional independence, player 2 in Step 4 changes player 1's continuation payoff so that ex ante (before player 1 takes an action in each period of Step 4), the difference in the expected payoffs from different actions in Step 4 is canceled out, taking (ii) into account. (iv) implies that the round for conditional independence does not affect player 1's payoff in Step 4.

Finally, since player 2 obtains rich information about player 1's history in Step 4 from the round for conditional independence, player 2 infers player 1's messages in Step 4 so that player 1 cannot update the belief about player 2's inference of player 1's message during Step 4. (ii) implies that the incentives in the round for conditional independence is not affected by this.

Now, we define the round fo conditional independence formally. For each period t in Step 4, we attach $S \log_2 |A_1| |Y_1|$ periods in the round for conditional independence. S is a fixed number to be determined. Hence, this new round also takes $O(T^{\frac{1}{3}+\frac{1}{4}})$ periods.

In these $S \log_2 |A_1| |Y_1|$ periods, player 1 sends the message about the history in each period t in Step 4, $(a_{1,t}, y_{1,t})$, as follows:

- We create a mapping between a sequence of $\{a_1^G, a_1^B\}$, $\mathbf{a}_1(a_1, y_1) \in \{a_1^G, a_1^B\}^{\log_2 |A_1||Y_1|}$, and $(a_1, y_1) \in |A_1| \times |Y_1|$, such that each $\mathbf{a}_1(a_1, y_1)$ uniquely identifies (a_1, y_1) and that, for each (a_1, y_1), there is at least one $\mathbf{a}_1(a_1, y_1)$ that is mapped into (a_1, y_1).
- Player 1's strategy is to be determined.
- Player 2 always takes a_2^G.
- $S \log_2 |A_1| |Y_1|$ periods are separated into $\log_2 |A_1| |Y_1|$ sets of S periods. In each S periods, player 2 infers that player 1 sends the message a_1^G if

$$\frac{\sum_s \Psi_{2,s}^{a^G}}{S} > \frac{q_2 + q_1}{2} \tag{148}$$

and a_1^B otherwise. From these inferences and the correspondence $\mathbf{a}_1(a_1, y_1)$, player 2 infers player 1's message $(\hat{a}_{1,t}, \hat{y}_{1,t})$.

Let

$$-\left\| \mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_2^G, \hat{a}_{1,t}, \hat{y}_{1,t} \right] \right\|^2 - f(h_2^{S \log_2 |A_1||Y_1|}), \tag{149}$$

be player 2's reward on player 1. Here, $h_2^{S \log_2 |A_1||Y_1|}$ is player 2's history in the $S \log_2 |A_1| |Y_1|$ periods where player 1 sends $(\hat{a}_{1,t}, \hat{y}_{1,t})$ and f will be determined in the following lemma:

Lemma 31 *There exists e_1 such that, for each $S \in \mathbb{N}$ and $\varepsilon > 0$, there generically exist f and $e_2 > 0$ such that, suppose that the players play the following game:*

1. *Nature chooses $a_{1,t}$ (t is introduced to make the notations consistent) and $(y_{1,t}, y_{2,t})$ is distributed according to $q(y_t \mid a_{1,t}, a_{2,t}^G)$. Player 1 can observe only $(a_{1,t}, y_{1,t})$.*
2. *The players play an $(S \log_2 |A_1| |Y_1|)$-period finitely repeated game where, in each period $s \in \{1, \ldots, S \log_2 |A_1| |Y_1|\}$, player 1 chooses $a_{1,s} \in A_1$, the signal profile $(y_{1,s}, y_{2,s})$ is*

generated by $q(y_s \mid a_{1,s}, a_{2,s}^G)$, player 1 can observe only $(a_{1,s}, y_{1,s})$, and there is no instantaneous utility.

3. *Player 2 infers $(\hat{a}_{1,t}, \hat{y}_{1,t})$ as explained above.*

4. *Player 1's utility is given by (149).*

Then,

(a) *The message transmits correctly with probability at least $1 - \varepsilon - e_1 \exp(-O(S^{\frac{1}{2}}))$, and*

(b) *For any two pure strategies σ_1 and $\tilde{\sigma}_1$, if there exists h_1 where $\sigma_1 \mid h_1 \neq \tilde{\sigma}_1 \mid h_1$ on the path after h_1, then the continuation payoff from h_1 is different by at least e_2. Here, with abuse of notation, σ_1 and h_1 are player 1's strategy and history in the game just defined. Let σ_1^* be the (unique) optimal strategy.*

Note that $e_2 > 0$ here corresponds to (ii) in the intuitive explanation above.

Proof. There exists $\bar{E}$ such that

$$\left\| \mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, \hat{a}_{1,t}, \hat{y}_{1,t} \right] \right\|^2 < \bar{E}$$

for all $y_{2,t}$, $a_{2,t}^G$, $\hat{a}_{1,t}$ and $\hat{y}_{1,t}$. In addition, define

$$e = \min_{(a_{1,t}, y_{1,t})} \left\{ \begin{array}{c} \min_{(\hat{a}_{1,t}, \hat{y}_{1,t}) \neq (a_{1,t}, y_{1,t})} \mathbb{E}\left[\left\| \mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, \hat{a}_{1,t}, \hat{y}_{1,t} \right] \right\|^2 \mid a_{1,t}, y_{1,t} \right] \\ -\mathbb{E}\left[\left\| \mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, a_{1,t}, y_{1,t} \right] \right\|^2 \mid a_{1,t}, y_{1,t} \right] \end{array} \right\}.$$

By Assumption 5, $e > 0$. Note that $\bar{E}$ and e are independent of S and f.

Fix S and $\varepsilon > 0$ arbitrarily. Lemma 3 implies that we can find and fix f and $e_2 > 0$ such that (b) is satisfied and that $f(h_2^{S \log_2 |A_1||Y_1|}) \in [-\frac{\varepsilon e}{2}, \frac{\varepsilon e}{2}]$ for all $h_2^{S \log_2 |A_1||Y_1|}$. Specifically, first, without loss of generality, we can make sure that player 1 has only two strategies since otherwise, Lemma 3 enables player 2 to give a very high punishment if player 1 takes an action other than a_1^G or a_1^B. Second, consider the last period of the game. If there is a history where a_1^G and a_1^B are indifferent, Lemma 3 guarantees that player 2 can break the ties by adding a small reward for a_1^G uniformly for all the histories. Making this adjustment sufficiently small in order not to affect the strict incentives in the other histories, we establish the strictness in the last period. Third, we can proceed by backward induction. Whenever player 2 breaks a tie for some period, it does not affect the strict incentives in the later periods since the reward in a certain period will be sunk in the later periods.

Let $\bar{\sigma}_1$ be such that player 1 constantly takes a_1^G or a_1^B for each S periods that correspond to the proper counterpart of $\mathbf{a}_1(a_{1,t}, y_{1,t})$. In addition, define

$$\begin{aligned}
R^*(a_{1,t}, y_{1,t}) &= -\mathbb{E}\left[\left\|\mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, a_{1,t}, y_{1,t}\right]\right\|^2 \mid a_{1,t}, y_{1,t}\right], \\
\bar{R}(a_{1,t}, y_{1,t}) &= -\min_{(\hat{a}_{1,t}, \hat{y}_{1,t}) \neq (a_{1,t}, y_{1,t})} \mathbb{E}\left[\left\|\mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, \hat{a}_{1,t}, \hat{y}_{1,t}\right]\right\|^2 \mid a_{1,t}, y_{1,t}\right], \\
\underline{R} &= -\max_{(a_{1,t}, y_{1,t})} \max_{(\hat{a}_{1,t}, \hat{y}_{1,t}) \neq (a_{1,t}, y_{1,t})} \mathbb{E}\left[\left\|\mathbf{1}_{y_{2,t}} - \mathbb{E}\left[\mathbf{1}_{y_{2,t}} \mid a_{2,t}^G, \hat{a}_{1,t}, \hat{y}_{1,t}\right]\right\|^2 \mid a_{1,t}, y_{1,t}\right].
\end{aligned}$$

Note that for all $(a_{1,t}, y_{1,t})$,

$$\begin{aligned}
R^*(a_{1,t}, y_{1,t}) - \underline{R} &\leq 2E, \\
R^*(a_{1,t}, y_{1,t}) - \bar{R}(a_{1,t}, y_{1,t}) &\geq e.
\end{aligned}$$

For (a), since the message transmits with ex ante probability $1 - \exp(-O(S^{\frac{1}{2}}))$ with $\bar{\sigma}_1$, the optimal strategy σ_1^* should guarantee

$$\begin{aligned}
& \left(1 - \Pr\left((\hat{a}_{1,t}, \hat{y}_{1,t}) \neq (a_{1,t}, y_{1,t})\right)\right) R^*(a_{1,t}, y_{1,t}) + \Pr\left((\hat{a}_{1,t}, \hat{y}_{1,t}) \neq (a_{1,t}, y_{1,t})\right) \bar{R}(a_{1,t}, y_{1,t}) \\
& -\mathbb{E}\left[f\left(h_2^{S \log_2 |A_1||Y_1|}\right) \mid \sigma_1^* \mid (a_{1,t}, y_{1,t})\right] \\
\geq \; & \left(1 - \exp(-O(S^{\frac{1}{2}}))\right) R^*(a_{1,t}, y_{1,t}) + \exp(-O(S^{\frac{1}{2}}))\underline{R} - \mathbb{E}\left[f\left(h_2^{S \log_2 |A_1||Y_1|}\right) \mid \bar{\sigma}_1 \mid (a_{1,t}, y_{1,t})\right]
\end{aligned}$$

or

$$
\begin{aligned}
& \Pr\left((\hat{a}_{1,t},\hat{y}_{1,t}) \neq (a_{1,t},y_{1,t})\right) \\
\leq \; & \frac{R^*(a_{1,t},y_{1,t}) - \underline{R}}{R^*(a_{1,t},y_{1,t}) - \bar{R}(a_{1,t},y_{1,t})} \exp(-O(S^{\frac{1}{2}})) \\
& + \frac{\mathbb{E}\left[f\left(h_2^{S\log_2|A_1||Y_1|}\right) \mid \bar{\sigma}_1 \mid (a_{1,t},y_{1,t})\right] - \mathbb{E}\left[f\left(h_2^{S\log_2|A_1||Y_1|}\right) \mid \sigma_1^* \mid (a_{1,t},y_{1,t})\right]}{R^*(a_{1,t},y_{1,t}) - \bar{R}(a_{1,t},y_{1,t})} \\
\leq \; & \frac{2E}{e}\exp(-O(S^{\frac{1}{2}})) + \varepsilon.
\end{aligned}
$$

Take

$$
e_1 = \frac{2E}{e} > 0,
$$

which is independent of S and f, then we are done. ■

Given that, for each period t in Step 4 of Section 44.2.1, player 1 will take $\sigma_1^* \mid (a_{1,t},y_{1,t})$ in the round for conditional independence and that player 2's reward on player 1 in the round for conditional independence is (149), the expected payoff from Nature's choice $a_{1,t}$ is determined. By Lemma 3, there exists $\bar{g}(a_1)$ such that

$$
\sum_{a_1} \bar{g}\left(a_1\right) \Psi_{2,t}^{a_2^G,a_1} \tag{150}
$$

cancels out the difference from (149). Player 2 adds (151) to player 2's reward on player 1 in the report block. Then, seeing $a_{1,t}$ as player 1's action in Step 4 of Section 44.2.1, any action gives ex ante payoff 0. Note that this corresponds to (iv) in the intuitive explanation above.

Then, based on the report σ_1^* in the round for conditional independence, player 2 can construct the statistics that indicates player 1's action with the conditional independence property from the perspective of Step 4 of Section 44.2.1.

Lemma 32 *There exist $S \in \mathbb{N}$, $\varepsilon > 0$ and $\phi_2 : \hat{a}_1, \hat{y}_1, y_2 \to (0,1)$ such that, for all $(a_{1,t},y_{1,t})$,* *if player* 1 *reports* $(a_{1,t}, y_{1,t})$ *by* σ_1^**, then for all* $y_{1,t}$*,*

$$\mathbb{E}\left[\phi_2(\hat{a}_{1,t}, \hat{y}_{1,t}, y_{2,t}) \mid a_{2,t}^G, a_{1,t}, y_{1,t}, \sigma_1^* \mid a_{1,t}, y_{1,t}\right] = \begin{cases} q_2 & \text{if } a_{1,t} = a_{1,t}^G, \\ q_1 & \text{if } a_{1,t} \neq a_{1,t}^G. \end{cases}$$

Proof. Since, from Condition (a) of Lemma 31, $(\hat{a}_{1,t}, \hat{y}_{1,t})$ transmits with probability no less than $1 - \varepsilon - e_1 \exp(-O(S^{\frac{1}{2}}))$, for sufficiently large S and small ε, player 2 has enough information. Note that e_1 does not depend on S. ■

Now, we are ready to construct Step 4 of Section 44.2.1. Fix $S \in \mathbb{N}$ and $\varepsilon > 0$ such that Lemma 32 holds. Then, fix f and e_2 such that Lemma 31 holds for those S and ε. Finally, take δ such that $e_2 > T^{-1} = (1-\delta)^{\frac{1}{2}}$. This implies that (iii) in the intuitive explanation is satisfied.

Step 4 of Section 44.2.1 For each $T^{\frac{1}{3}}$ periods when player 1 is supposed to take a constant action, player 2 infers that player 1's action is a_1^G if

$$\frac{\sum_t \Phi_{2,t}}{T^{\frac{1}{3}}} > \frac{q_2 + q_1}{2}$$

and a_2^B otherwise. Then, from Lemma 32, if player 1 uses $\sigma_1^* \mid a_{1,t}, y_{1,t}$ in the round for conditional independence, then $\Phi_{2,t}$ has the same property as $\Phi_{1,t}$ from Lemma 32.

If player 2 sent the message $y_2 \in Y_{2,2}^2$ ($y_2 \in Y_{2,1}^2$, respectively) in Step 2 of Section 44.2.1, then player 2's reward on player 1 in the Step 4 is symmetrically defined as player 1's reward on player 2 in Step 3 after inferring that player 2's message in Step 2 says that player 2 observed $y_2 \in Y_{2,1}^2$ ($y_2 \in Y_{2,2}^2$, respectively). On the top of that, player 2 gives the reward

$$\sum_{a_1} \bar{g}(a_1) \Psi_{2,t}^{a_2^G, a_1} \tag{151}$$

to cancel out the effect of the round for conditional independence. Note that player 2 does not need to know the true action $a_{1,t}$ to calculate (151).

Round for Conditional Independence For $S \log_2 |A_1| |Y_1|$ periods that correspond to period t in Step 4 of Section 44.2.1, player 1 plays $\sigma_1^* \mid a_{1,t}, y_{1,t}$ and player 2 plays a_2^G. The reward is given by (149).

Optimality of Player 1's Strategy Note that all the rewards in Step 4 of Section 44.2.1 affected by the messages in the round for conditional independence is bounded by T^{-1}. Since we take T such that $e_2 > T^{-1}$, from Condition 2 of Lemma 31, regardless of the history in Step 4, the optimal strategy in the round for conditional independence is σ_1^* (note that (151) is sunk in the round for conditional independence).

Then, (151) together with the expected reward in the round for conditional independence makes player 1 indifferent between all the actions in terms of the expected reward in the round for conditional independence and yield 0 regardless of the history.

Therefore, the same argument as in Step 3 of Section 44.2.1 for player 2 establishes the result since, given that player 1 takes σ_1^*, the conditional independence property holds.

SUPPLEMENTAL MATERIAL 5:

PROOF OF THEOREM 1 for a General N-Player Game withOUT CHEAP TALK

In this Supplemental Material, we prove the dispensability of cheap talk and public randomization in the proof of Theorem 1 for a general N-player game with $N \geq 3$ (see the Supplemental Material 3 for the proof with cheap talk and public randomization). Remember that in the Supplemental Material 3, the coordination block uses the perfect cheap talk, the supplemental rounds use the noisy cheap talk, the report block uses the public randomization and perfect cheap talk, and the re-report block uses the perfect cheap talk.

First, in Section 46, we replace the perfect cheap talk in the coordination block with the noisy cheap talk. As seen in Section 4.7.2, with more than two players, we need to make sure that while the players exchange messages and infer the other players' messages from private signals in order to coordinate on x_i, there is no player who can induce a situation where some players infer x_i is G while the others infer x_i is B in order to increase her own equilibrium payoff. For this purpose, we need to use the communication through actions and to make new assumptions. In Section 45.1, we introduce these new assumptions and explain why they are necessary.

Second, in Section 48, we dispense with the noisy cheap talk in the coordination block (given the first step above) and supplemental rounds. See Section 45.2 for what assumption is necessary for this step.

Third, in Section 51, we dispense with the public randomization and the perfect cheap talk in the report and re-report blocks. See 45.3 for new assumptions for this step.

In this Supplemental Material, when we say player $i \notin \{1, ..., N\}$, without otherwise specified, it means player $i \pmod N$. In addition, without loss of generality, assume that

$$|A_1| |Y_1| \geq \cdots \geq |A_N| |Y_N| . \tag{152}$$

45 Notations and Assumptions

45.1 Assumptions for Dispensing with the Perfect Cheap Talk in Coordination Block

We explain how to replace the perfect cheap talk with the noisy cheap talk in the coordination block. As explained in Section 29, the noisy cheap talk is "private" in that when player j sends the message to player n via noisy cheap talk, the main signal $f[n](m)$ is only observed by player n.

This creates the second problem in Section 4.6.3: If player i sent the message x_i to each of the other players $-i$ via noisy cheap talk separately, then player i could create a situation where some players infer x_i is G while the others infer x_i is B by telling a lie. Since the action that will be taken in the main blocks may not be included in $\{a(x)\}_x$ and we do not have any bound on player i's payoff in such a situation, it might be of player i's interest to tell a lie.

To prevent this situation, we consider the following message protocol: Let $N(i) = \{i, i+1, i+2\}$ be the set of players whose index is in $\{i, i+1, i+2\}$. In addition, let

$$n^*(i) \in \arg\min_{j \in \{i, i+2\}} |A_j| \, |Y_j| \tag{153}$$

be the player whose $|A_j| \, |Y_j|$ is smaller among $\{i, i+2\}$. Let

$$n^{**}(i) = \{i, i+2\} \setminus \{n^*(i)\} \tag{154}$$

be the other player. Note that $N(i) = \{n^*(i), i+1, n^{**}(i)\}$. The players communicate as follows:

1. First, player i sends the message about x_i to player $n^*(i)$.

2. Then, player $n^*(i)$ sends the message about x_i to players $N(i)$ via actions. This corresponds to "Phase 1" of Hörner and Olszewski (2006).

3. After that, each player j in $N(i)$ sends the message about x_i to each player $n \neq j$ via noisy cheap talk.

4. Finally, each player n infers x_i based on the messages from $N(i)$. This corresponds to "Phase 2" of Hörner and Olszewski (2006).

As Hörner and Olszewski (2006), to incentive each player $j \in N(i)$ to tell the truth in Step 3, for each $j \in N(i)$, if there exists player $n \in -j$ such that player n's inference of player j's message changes player n's inference of x_i in Step 4 (that is, if player j is "pivotal"), then player $j-1$ makes player j indifferent between any action profile sequence.

Given above, we will show that player $n^*(i)$ does not want to deviate in Step 2 in order to create a situation where player $n^*(i)$ herself will be pivotal with high probability in Step 3. Remember that we take $n^*(i)$ such that the set of player $n^*(i)$'s action-signal pairs is smaller than that of player $n^{**}(i)$ in (153). Heuristically, this guarantees that player $n^*(i)$ cannot infer player $n^{**}(i)$'s inference precisely, which prevents player $n^*(i)$ from creating the situation where player $n^*(i)$ is pivotal.

Given player $n^*(i)$'s truthtelling strategy in Step 2, the probability that player i is pivotal in Step 3 is almost independent of player i's strategy in Step 1. Since x_i controls player $(i+1)$'s payoff, players i and $n^*(i) \neq i+1$ do not have an incentive to manipulate the communication in Step 1.

Below, we explain which step requires exactly what assumption.

Let us consider Step 1 first. Suppose that player i wants to send the message $x_i \in \{G, B\}$ to player $n^*(i)$. If $n^*(i) = i$, then this is redundant. Otherwise, player i sends x_i by taking $a_i^{x_i}$ for $T^{\frac{1}{2}}$ periods. The other players are supposed to take a_{-i}^G. We want to make sure that player $n^*(i)$ can statistically infer player i's message regardless of deviations by the other players $-(i, n^*(i))$.

More generally, for each $i \in I$ and $n \in -i$, we want to construct a statistics $\psi_n^i(y_n)$ with which player n can infer player i's binary message regardless of the other players' deviation.

That is,

$$\mathbb{E}\left[\psi_n^i(y_n) \mid a_i, a_j, a_{-(i,j)}^G\right] = \begin{cases} q_2 & \text{if } a_i = a_i^G, \\ q_1 & \text{if } a_i = a_i^B \end{cases} \tag{155}$$

for all $j \in -(i,n)$ and $a_j \in A_j$.

A sufficient condition is as follows: Let $Q_n^i(a_i, a_j, a_{-(i,j)}^G) \equiv \left(q(y_n \mid a_i, a_j, a_{-(i,j)}^G)\right)_{y_n}$ be the vector expression of player n's signal distribution conditional on $a_i, a_j, a_{-(i,j)}^G$. It suffices to assume that all the vectors $Q_n^i(a_i, a_j, a_{-(i,j)}^G)$ with $j \in -(i,n)$, $a_i \in \{a_i^G, a_i^B\}$ and $a_j \in A_j$ are linearly independent.

Assumption 13 *For any $i \in I$ and $n \in -i$, there exist $\left\{a_i^G, a_i^B\right\} \subset A_i$ and $a_{-i}^G \in A_{-i}$ such that $Q_n^i(a_i, a_j, a_{-(i,j)}^G)$ with $j \in -(i,n)$, $a_i \in \left\{a_i^G, a_i^B\right\}$ and $a_j \in A_j$ are linearly independent.*

For notational convenience, we assume that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

This assumption is generic since Assumption 2 implies that $|Y_n| \geq 2\sum_{j\neq i,n}|A_j|$. The following lemma shows that this assumption is sufficient for the existence of ψ_n^i.

Lemma 33 *If Assumption 13 holds, then for each $i \in I$ and $n \in -i$, there exist $q_2 > q_1$ and $\psi_n^i : Y_n \to (0,1)$ satisfying (155).*

Proof. The same as Lemma 3. ■

See (168) for how player $n^*(i)$ infers x_i using $\psi_n^i(y_n)$.

After player $n^*(i)$ infers x_i, player $n^*(i)$ sends the message about her inference of x_i to players $N(i) = \{n^*(i), i+1, n^{**}(i)\}$. To distinguish player $n^*(i)$'s inference of x_i from the true state x_i, let $w_i \in \{G, B\}$ denote player $n^*(i)$'s inference of x_i.

While player $n^*(i)$ sends w_i, player $n^*(i)$ takes $a_{n^*(i)}^{w_i}$, player $i+1$ takes $\alpha_{i+1}^* \in \Delta(A_{i+1})$, player $n^{**}(i)$ takes $\alpha_{n^{**}(i)}^* \in \Delta\left(A_{n^{**}(i)}\right)$, and each player $j \notin N(i)$ takes a_j^G for $T^{\frac{1}{2}}$ periods. That is, in equilibrium, the players take

$$\alpha(i, w_i) \equiv \left(a_{n^*(i)}^{w_i}, \alpha_{i+1}^*, \alpha_{n^{**}(i)}^*, \left\{a_j^G\right\}_{j\notin N(i)}\right)$$

for $T^{\frac{1}{2}}$ periods.

Take $n \in N(i) \setminus n^*(i)$. Suppose that player $j = N(i) \setminus \{n^*(i), n\}$ unilaterally deviates and takes $a_j \in A_j$. Then, the distribution of player n's action-signal pairs is

$$\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i)) \equiv \left(q\left(a_n, y_n \mid a_j, \alpha_{-j}(i, w_i)\right)\right)_{a_n \in A_n, y_n \in Y_n}.$$

Consider the following linear equations: For any $a_j \in A_j$,

$$\boldsymbol{i}_n(i)\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i)) = \begin{cases} q_2 & \text{if } w_i = G, \\ q_1 & \text{if } w_i = B. \end{cases} \tag{156}$$

Here, $\boldsymbol{i}_n(i)$ is a $1 \times |A_n| |Y_n|$ vector. Intuitively, if player n uses $\boldsymbol{i}_n(i)\mathbf{1}_{a_{n,t}, y_{n,t}}$ after the history $(a_{n,t}, y_{n,t})$ to infer w_i, then player j cannot manipulate player n's inference.

Solve (156) for $\boldsymbol{i}_n(i)$. Suppose that there are $L_n(i)$ linearly independent solutions. Then, let

$$I_n(i) = \left(\boldsymbol{i}_n^l(i)\right)_{l=1}^{L_n(i)} \tag{157}$$

be the $L_n(i) \times |A_n| |Y_n|$ matrix collecting all the linearly independent $\boldsymbol{i}_n(i)$'s. Suppose that player n infers w_i is equal to $\hat{w}_i \in \{G, B\}$ if the realized frequency $\mathbf{x}$ of action-signal pairs satisfies

$$I_n(i)\mathbf{x} + \boldsymbol{\varepsilon} = q(\hat{w}_i)\mathbf{1}$$

for some $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$ (imagine that ε is a small number). Here,

$$q(\hat{w}_i) = \begin{cases} q_2 & \text{if } \hat{w}_i = G, \\ q_1 & \text{if } \hat{w}_i = B. \end{cases}$$

We will take care of the case where there is no such $\hat{w}_i \in \{G, B\}$ later. Note that (156) implies that player $j = N(i) \setminus \{n^*(i), n\}$ cannot manipulate this inference.

In addition, consider the matrix projecting player $n^*(i)$'s history on the conditional expectation of player n's history given an action profile by players $-n^*(i)$ being equal to

$\alpha_{-n^*(i)}(i, w_i)$:

$$\underset{\left(|A_n||Y_n|\times\left|A_{n^*(i)}\right|\left|Y_{n^*(i)}\right|\right)}{Q_{n,n^*(i)}(i)},$$

where the element corresponding to (a_n, y_n), $\left(a_{n^*(i)}, y_{n^*(i)}\right)$ is the conditional probability that player n observes (a_n, y_n) given $\left(a_{n^*(i)}, y_{n^*(i)}\right)$ and $\alpha_{-n^*(i)}(i, w_i)$:

$$q(a_n, y_n|\alpha_{-n^*(i)}(i, w_i), a_{n^*(i)}, y_{n^*(i)}).$$

Since $\alpha_{-n^*(i)}(i, w_i) = \left(\alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, \left\{a^G_j\right\}_{j\notin N(i)}\right)$ is independent of w_i, $Q_{n,n^*(i)}$ is independent of w_i.

Given $Q_{n,n^*(i)}(i)$, the set of player $n^*(i)$'s histories such that player $n^*(i)$ believes that player n infers $\hat{w}_i \in \{G, B\}$ with a non-negligible probability is expressed by

$$\mathcal{I}_{n,n^*(i)}[\varepsilon](i, \hat{w}_i) \equiv \left\{\begin{array}{c} \mathbf{x} \in \mathbb{R}_+^{\left|A_{n^*(i)}\right|\left|Y_{n^*(i)}\right|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{L_n(i)} \text{ such that} \\ \left\{\begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ I_n(i)Q_{n,n^*(i)}(i)\mathbf{x} = q(\hat{w}_i)\mathbf{1} + \boldsymbol{\varepsilon}. \end{array}\right. \end{array}\right\}$$

So that player $n^*(i)$ cannot induce the situation that players $n^{**}(i)$ and $i+1$ infer the different states, we want to make sure that, for sufficiently small ε,

$$\mathcal{I}_{n^{**}(i),n^*(i)}[\varepsilon](i, G) \cap \mathcal{I}_{i+1,n^*(i)}[\varepsilon](i, B) = \emptyset \tag{158}$$

and

$$\mathcal{I}_{n^{**}(i),n^*(i)}[\varepsilon](i, B) \cap \mathcal{I}_{i+1,n^*(i)}[\varepsilon](i, G) = \emptyset. \tag{159}$$

Therefore, we give a sufficient condition for (158) and (159).

In addition, we want to incentives each player $i' \in I$ to take a prescribed action by the reward function $\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1})$ such that

- If player i' is player $n^*(i)$, then the ex ante payoff of player i' is constant for all $a_{i'} \in A_{i'}$:

$$\begin{aligned} & u_{i'}\left(a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)}\right) + \mathbb{E}\left[\begin{array}{c} \pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \\ \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)} \end{array}\right] \\ = & \text{ constant.} \end{aligned} \tag{160}$$

  A sufficient condition for this is that all the vectors of player $(i'-1)$'s signal distribution given $a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)}$ are linearly independent with respect to $a_{i'}$. That is,

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)})\right)_{y_{i'-1}}$$

  is linearly independent with respect to $a_{i'} \in A_{i'}$.

- If player i' is not player $n^*(i)$, then the ex ante payoff of player i' is constant for all $a_{i'} \in A_{i'}$ and player $n^*(i)$'s possible messages:

$$\begin{aligned} & u_{i'}\left(a_{i'}, \alpha_{-i'}(i,G)\right) + \mathbb{E}\left[\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \mid a_{i'}, \alpha_{-i'}(i,G)\right] \\ = & u_i\left(a_{i'}, \alpha_{-i'}(i,B)\right) + \mathbb{E}\left[\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \mid a_{i'}, \alpha_{-i'}(i,B)\right] \\ = & \text{ constant.} \end{aligned} \tag{161}$$

  A sufficient condition for this is that all the vectors of player $(i'-1)$'s signal distribution given $a_{i'}, \alpha_{-i'}(i, w_i)$ are linearly independent with respect to $a_{i'}$ and w_i. That is,

$$(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha_{-i'}(i, w_i)))_{y_{i'-1}}$$

  is linearly independent with respect to $a_{i'} \in A_{i'}$ and $w_i \in \{G, B\}$.

In total, the following assumption is sufficient.

Assumption 14 *For any $i \in I$, there exist $\left\{a^G_{n^*(i)}, a^B_{n^*(i)}\right\}$, $\alpha^*_{i+1} \in \Delta\left(A_{i+1}\right)$, $\alpha^*_{n^{**}(i)} \in \Delta\left(A_{n^{**}(i)}\right)$, $a^G_{-N(i)}$, q_2 and q_1 such that*

1. $q_2, q_1 \in (0,1)$ *and* $q_2 > q_1$.

2. *There exists* $\mathbf{x} \in \mathbb{R}^{L_{i+1}(i)+L_{n^{**}(i)}(i)}$ *such that*

$$\begin{bmatrix} I_{i+1}(i) Q_{i+1,n^*(i)}(i) \\ I_{n^{**}(i)}(i) Q_{n^{**}(i),n^*(i)}(i) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} q_2 \mathbf{1} \\ q_1 \mathbf{1} \end{bmatrix} \cdot \mathbf{x} > 0.$$

3. *There exists* $\mathbf{x} \in \mathbb{R}^{L_{i+1}(i)+L_{n^{**}(i)}(i)}$ *such that*

$$\begin{bmatrix} I_{i+1}(i) Q_{i+1,n^*(i)}(i) \\ I_{n^{**}(i)}(i) Q_{n^{**}(i),n^*(i)}(i) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} q_1 \mathbf{1} \\ q_2 \mathbf{1} \end{bmatrix} \cdot \mathbf{x} > 0.$$

4. *For* $i' = n^*(i)$,

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)})\right)_{y_{i'-1}}$$

is linearly independent with respect to $a_{i'} \in A_{i'}$.

5. *For* $i' \in I \setminus \{n^*(i)\}$,

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha_{-i'}(i, w_i))\right)_{y_{i'-1}}$$

is linearly independent with respect to $a_{i'} \in A_{i'}$ *and* $w_i \in \{G, B\}$.

Since all the expressions are linear and $\mathbf{q}_n$ is a probability distribution, we can make sure that each element in $I_n(i)$ is in $(0,1)$. Further, for notational simplicity, we assume that $\left(a_j^G, a_j^B\right)_{j \in I}$ in Assumption 13 satisfies Assumption 14 for each i.[93]

This assumption is generic by the following reason: (156) puts $2\left(|A_j| - 1\right)$ constraints while we have $|A_n|\,|Y_n| - 1$ degrees of freedom for $\boldsymbol{i}_n(i)$ if $\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i))$ is linearly independent for each w_i and a_j except for the constraint that "if we add all the elements up, then it should be one." Hence, generically $L_n(i)$ is equal to $|A_n|\,|Y_n| - 2\,|A_j| + 1$. Therefore, for each one of Conditions 2 and 3, we have $|A_{i+1}|\,|Y_{i+1}| + \left|A_{n^{**}(i)}\right| \left|Y_{n^{**}(i)}\right| - 2\,|A_{i+1}| - 2\left|A_{n^{**}(i)}\right| + 1$

[93] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

degrees of freedom for $\mathbf{x}$,[94] while we have $\left|A_{n^*(i)}\right| \left|Y_{n^*(i)}\right| + 1$ constraints. Hence, Assumption 2 together with (153) implies that we can generically find $\mathbf{x}$ for Conditions 2 and 3.

In addition, Condition 4 is generic if $|Y_{i'-1}| \geq |A_{i'}|$ and Condition 5 is generic if $|Y_{i'-1}| \geq 2|A_{i'}|$. Note that Assumption 2 implies that these inequalities are satisfied.

The following lemma shows that Assumption 14 is sufficient for (158) and (159).

Lemma 34 *If Assumption 14 is satisfied, then there exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon < \bar{\varepsilon}$, for any $i \in I$, (158) and (159) are satisfied.*

Proof. The same as Lemma 26. ■

In addition, the following lemma shows that Assumption 14 is sufficient for the construction of the reward stated above:

Lemma 35 *There exists $\bar{u}$ such that, for each i and i', there exist $\pi_{i'}^G(n^*(i) \to N(i), \cdot, \cdot) : A_{i'-1} \times Y_{i'-1} \to [-\bar{u}, 0]$ and $\pi_{i'}^B(n^*(i) \to N(i), \cdot, \cdot) : A_{i'-1} \times Y_{i'-1} \to [0, \bar{u}]$ such that*

1. *(160) is satisfied for $i' = n^*(i)$ and*

2. *(161) is satisfied for $i' \in -n^*(i)$.*

Proof. The same as Lemma 3. ■

45.2 Assumption for Dispensing with the Noisy Cheap Talk

We explain how player j sends a binary message $m \in \{G, B\}$ to player n via actions instead of the noisy cheap talk. Since we only use the noisy cheap talk with precision $p = \frac{1}{2}$, we concentrate on the case with $p = \frac{1}{2}$.

As in the two-player case, with η being a small number to be defined, the sender (player j) determines

$$z_j(m) = \begin{cases} m & \text{with probability } 1 - 2\eta, \\ \{G, B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

[94] Note that two rows are parallel to $\mathbf{1}$.

and player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2}a_j^G + \frac{1}{2}a_j^B & \text{if } z_j(m) = M \end{cases}$$

for $T^{\frac{1}{2}}$ periods. The other players $-j$ take a_{-j}^G.

For each $i \in I$, let $\mathbf{y}_i$ be the realized frequency of player i's signal observation while player j sends m. In addition, let $\mathbf{q}_i(a) = (q_i(y_i \mid a))_{y_i}$ be player i's signal distribution with action profile a.

We want to construct $f[n](m) \in \{G, B\}$ from $\mathbf{y}_n$ and $g[n-1](m) \in \{m, E\}$ from $\mathbf{y}_{n-1}$ such that

- Player n infers the message correctly with high probability,
- Player $n-1$ has $g[n-1](m) = m$ with high probability,
- Given m, player n believes that $f[n](m) = m$ or $g[n-1](m) = E$ with high probability,
- Player n cannot manipulate $g[n-1](m)$, and
- The players other then the sender and receiver cannot manipulate $f[n](m)$ to increase their payoff.

As in the two player case, $g[n-1](m) = E$ if and only if $z_j(m) \neq m$ or $\mathbf{y}_{n-1}$ is not close to the affine hull of player $(n-1)$'s signal distribution with respect to player n's deviation, $\text{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n})$. Using 2 of Notation 2 below,

1. $g[n-1](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon](m)$.
2. $g[n-1](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon](m)$.

Here, we assume that player $n-1$ knew the true message m. As will be seen in Section 52, player j informs player $n-1$ of m in the re-report block. Since $g[n-1](m)$ only affects the reward function (does not affect $\sigma_{n-1}(x_{n-1})$), it suffices that player $n-1$ knows the information by the end of the review phase.

On the other hand, regardless of any player's deviation, with high probability, player n (receiver) receives $\mathbf{y}_n$ close to the affine hull of player n's signal distributions with respect to player i's action with $i \in -i$, that is,

$$\text{aff}(\{\mathbf{q}_n(a_n^G, a_j, a_{-(j,n)}^G)\}_{a_j \in A_j}) \cup \text{aff}(\{\mathbf{q}_n(a_n^m, a_j^G, a_i, a_{-(i,j,n)}^G)\}_{m \in \{G,B\}, i \neq j,n, a_i \in A_i}). \tag{162}$$

Using 4 of Notation 2, $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$ with high probability.

If $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$, then as in the two-player case, player n constructs $f[n](m)$ such that

- $f[n](m) = G$ if the conditional expectation of $\mathbf{y}_{n-1}$ given $m = G$ and $\mathbf{y}_n$ is close to $\mathcal{H}_{n-1}[\varepsilon](G)$, and
- $f[n](m) = B$ if the conditional expectation of $\mathbf{y}_{n-1}$ given $m = B$ and $\mathbf{y}_n$ is close to $\mathcal{H}_{n-1}[\varepsilon](B)$.

Using 6 of Notation 2,

- $f[n](m) = G$ if $\mathbf{y}_n \in \mathcal{H}_{n-1,n}[\varepsilon](G)$, and
- $f[n](m) = B$ if $\mathbf{y}_n \in \mathcal{H}_{n-1,n}[\varepsilon](B)$.

Further, so that players $-(j,n)$ cannot manipulate player n's inference (if $z_j(m) = m$), player n infers that m is $\hat{m} \in \{G, B\}$ if $\mathbf{y}_n$ is close to the affine full of player n's signal distributions under the message $\hat{m}$ with respect to a unilateral deviation of each player $i \in -(j,n)$, that is, if $\mathbf{y}_n$ is close to $\text{aff}(\{\mathbf{q}_n(a_j^{\hat{m}}, a_i, a_{-(i,j)}^G)\}_{i \neq j,n, a_i \in A_i})$.

Using 8 of Notation 2,

- $f[n](m) = G$ if $\mathbf{y}_n \in \mathcal{J}_n[\varepsilon](G)$, and
- $f[n](m) = B$ if $\mathbf{y}_n \in \mathcal{J}_n[\varepsilon](B)$.

In total,

1. If $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$, then

   (a) $f[n](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{n-1,n}[\varepsilon]\,(G) \cup \mathcal{J}_n[\varepsilon]\,(G)$,

   (b) $f[n](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{n-1,n}[\varepsilon]\,(B) \cup \mathcal{J}_n[\varepsilon]\,(B)$ or $\mathbf{y}_i \notin \mathcal{H}_{n-1,n}[\varepsilon]\,(G) \cup \mathcal{J}_n[\varepsilon]\,(G)$, and

2. If $\mathbf{y}_n \notin \mathcal{G}_n[\varepsilon]$, then player n infers $f[n](m)$ from the likelihood as in the two-player case.

Here, compared to the two-player case, $\mathcal{I}_i[\varepsilon](\hat{m})$ is not introduced since Lemma 15 does not have a counterpart of 3 of Lemma 2.

In addition, we want to incentives each player $i \in I$ to take a prescribed action by the reward function $\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1})$ such that

- If player i is player j (sender), then the ex ante payoff of player i is constant for all $a_i \in A_i$:
$$u_i\left(a_i, a_{-i}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_{-i}^G\right] = \text{constant.} \tag{163}$$
  A sufficient condition for this is that all the vectors of player $(i-1)$'s signal distribution given a_i, a_{-i}^G are linearly independent with respect to a_i. That is,
$$\left(q_{i-1}(y_{i-1} \mid a_i, a_{-i}^G)\right)_{y_{i-1}}$$
  is linearly independent with respect to $a_i \in A_i$.

- If player i is not player j, then the ex ante payoff of player i is constant for all $a_i \in A_i$ regardless of player j's message:
$$\begin{aligned} & u_i\left(a_i, a_j^G, a_{-(i,j)}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_j^G, a_{-(i,j)}^G\right] \\ = \; & u_i\left(a_i, a_j^B, a_{-(i,j)}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_j^B, a_{-(i,j)}^G\right]. \end{aligned} \tag{164}$$

A sufficient condition for this is that all the vectors of player $(i-1)$'s signal distribution given $a_i, a_j^m, a_{-(i,j)}^G$ are linearly independent with respect to a_i and m. That is,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_j^m, a_{-(i,j)}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$ and $m \in \{G, B\}$.

We first give notations and then give a sufficient condition so that the above inference is well defined and that the reward function exists.

Notation 2 *For $a_j^G, a_j^B \in A_j$ and $a_{-j}^G \in A_{-j}$, for $m \in \{G, B\}$, we define the following:*

1. *A $(|Y_{n-1}| - |A_n| + 1) \times |Y_{n-1}|$ matrix $H_{n-1}(m)$ and a $(|Y_{n-1}| - |A_n| + 1) \times 1$ vector $\mathbf{p}_{n-1}(m)$ such that the affine hull of player $(n-1)$'s signal distributions with respect to player n's action when the other players take $a_j^m, a_{-(j,n)}^G$ is represented by*

$$\begin{aligned} & \operatorname{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n \in A_n}) \cap \mathbb{R}_+^{|Y_{n-1}|} \\ = & \left\{\mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} : H_{n-1}(m)\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(m)\right\}. \end{aligned}$$

2. *The set of hyperplanes that are generated by perturbing RHS of the characterization of $\operatorname{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n \in A_n}) \cap \mathbb{R}_+^{|Y_{n-1}|}$: For $\varepsilon \geq 0$,*

$$\mathcal{H}_{n-1}[\varepsilon](m) = \left\{ \begin{array}{c} \mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_{n-1}| - |A_n| + 1} \text{ such that} \\ \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ H_{n-1}(m)\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(m) + \boldsymbol{\varepsilon} \end{array} \right. \end{array} \right\}.$$

3. *Let G_i be a $(|Y_n| - |A_j| - 2\sum_{i \neq j,n} |A_i| + 1) \times |Y_n|$ matrix and $\mathbf{g}_n$ be a $(|Y_n| - |A_j| - 2\sum_{i \neq j,n} |A_i| + 1) \times 1$ vector such that (162) is represented by*

$$\left\{\mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : G_n\mathbf{y}_n = \mathbf{g}_n\right\}.$$

4. *The set of hyperplanes that are generated by perturbing RHS of the above characterization: For* $\varepsilon \geq 0$*,*

$$\mathcal{G}_i[\varepsilon] \equiv \left\{ \begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1} \\ \textit{such that} \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon \\ G_n \mathbf{y}_n = \mathbf{g}_n + \boldsymbol{\varepsilon} \end{array} \right. \end{array} \right\}.$$

5. *The matrix projecting the distributions of player* n*'s signals on the conditional distribution of player* $(n-1)$*'s signals given an action profile* a*:*

$$Q_{n-1,n}(a) = \begin{bmatrix} q(y_{n-1,1} \mid a, y_{n,1}) & \cdots & q(y_{n-1,1} \mid a, y_{n,|Y_n|}) \\ \vdots & & \vdots \\ q(y_{n-1,|Y_{n-1}|} \mid a, y_{n,1}) & \cdots & q(y_{n-1,|Y_{n-1}|} \mid a, y_{n,|Y_n|}) \end{bmatrix}.$$

6. *For* $\hat{m} \in \{G, B\}$*, the set of player* n*'s signal frequencies such that player* n*'s conditional expectation of player* $(n-1)$*'s signal frequency is in* $\mathcal{H}_{n-1}[\varepsilon](\hat{m})$ *when the players take* $a_j^{\hat{m}}, a_{-j}^G$*:*

$$\mathcal{H}_{n-1,n}[\varepsilon](\hat{m}) = \left\{ \begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_{n-1}|} \textit{ such that} \\ \textit{there exist } \boldsymbol{\varepsilon}_1 \in \mathbb{R}^{|Y_{n-1}|},\ \boldsymbol{\varepsilon}_2 \in \mathbb{R}^{|Y_{n-1}|-|A_n|+1} \textit{ and } \mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} \textit{ satisfying} \\ \left\{ \begin{array}{c} \mathbf{y}_{n-1} = Q_{n-1,n}(a_j^{\hat{m}}, a_{-j}^G)\mathbf{y}_n + \boldsymbol{\varepsilon}_1, \\ H_{n-1}(\hat{m})\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(\hat{m}) + \boldsymbol{\varepsilon}_2, \\ \|\boldsymbol{\varepsilon}_1\|, \|\boldsymbol{\varepsilon}_2\| \leq \varepsilon \end{array} \right. \end{array} \right\}.$$

7. *For* $\hat{m} \in \{G, B\}$*, a* $(|Y_n| - \sum_{i\neq j,n} |A_i| + 1) \times |Y_n|)$ *matrix* $J_n(\hat{m})$ *and a* $(|Y_n| - \sum_{i\neq j,n} |A_i| + 1) \times 1$ *vector* $\mathbf{r}_n(\hat{m})$ *such that the affine hull of player* n*'s signal distributions with respect to player* i*'s deviation with* $i \in -(j, n)$ *when the other players take* $a_j^m, a_{-(i,j)}^G$

is represented by

$$\begin{aligned}&\operatorname{aff}(\{\mathbf{q}_n(a_j^{\hat{m}}, a_i, a_{-(i,j)}^G)\}_{i\neq j,n,a_i\in A_i}) \cap \mathbb{R}_+^{|Y_n|} \\ = \ &\left\{\mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : J_n(\hat{m})\mathbf{y}_n = \mathbf{r}_n(\hat{m})\right\}.\end{aligned}$$

8. *The set of hyperplanes that are generated by perturbing RHS of the above characterization: For $\varepsilon \geq 0$,*

$$\mathcal{J}_n[\varepsilon](m) = \left\{\begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_n| - \sum_{i\neq j,n}|A_i|+1} \text{ such that} \\ \left\{\begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ J_n(\hat{m})\mathbf{y}_n = \mathbf{r}_n(\hat{m}) + \boldsymbol{\varepsilon} \end{array}\right. \end{array}\right\}.$$

Similar to Lemma 25, we can take $H_{n-1}(G)$, $H_{n-1}(B)$, $J_n(G)$ and $J_n(B)$ so that all the elements of all the matrices are in $(0,1)$.

Assumption 15 *For each $j \in I$ and $n \in -j$, there exist $a_j^G, a_j^B \in A_j$ and a_{-j}^G such that the following seven conditions are satisfied:*

1. *There exists $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+2(|Y_{n-1}|-|A_n|+1)}$ such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(G)Q_{n-1,n}(a_j^G, a_{-j}^G) \\ H_{n-1}(B)Q_{n-1,n}(a_j^B, a_{-j}^G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(G) \\ \mathbf{p}_{n-1}(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

2. *There exists $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+2\left(|Y_n|-\sum_{i\neq j,n}|A_i|+1\right)}$ such that*

$$\begin{bmatrix} G_n \\ J_n(G) \\ J_n(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{r}_n(G) \\ \mathbf{r}_n(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

3. *There exists* $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+(|Y_{n-1}|-|A_n|+1)+|Y_n|-\sum_{i\neq j,n}|A_i|+1}$ *such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(G)Q_{n-1,n}(a_j^G, a_{-j}^G) \\ J_n(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(G) \\ \mathbf{r}_n(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

4. *There exists* $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+(|Y_{n-1}|-|A_n|+1)+|Y_n|-\sum_{i\neq j,n}|A_i|+1}$ *such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(B)Q_{n-1,n}(a_j^B, a_{-j}^G) \\ J_n(G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(B) \\ \mathbf{r}_n(G) \end{bmatrix} \cdot \mathbf{x} > 0.$$

5. *For each* $k \in \{1, \ldots, |Y_n|\}$, *we have*

$$q(y_{n,k}|a_j^G, \alpha_{-j}^G) \neq q(y_{n,k}|a_j^G, \alpha_{-j}^B).$$

6. *For* $i = j$,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_{-i}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$.

7. *For* $i \in -j$,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_j^m, a_{-(i,j)}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$ *and* $m \in \{G, B\}$.

For notational simplicity, we assume that $\left(a_j^G, a_j^B\right)_{j\in I}$ in Assumption 13 satisfies Assumption 15 for each j.[95]

As Assumption 13, we can show that Assumption 2 implies that we can generically find $\mathbf{x}$'s for each condition of Assumption 15 and that Conditions 6 and 7 are satisfied.

[95] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

The next two lemmas show that Assumption 15 is actually sufficient so that the above inference $f[n](m)$ is well defined.

Lemma 36 *If Assumption 15 is satisfied, then there exists* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon < \bar{\varepsilon}$*, for each* $j \in I$ *and* $n \in -j$*, for any* $\mathbf{y}_n \in \Delta\left(\{\mathbf{1}_{y_n}\}_{y_n \in Y_n}\right)$*, at most one* $\hat{m} \in \{G, B\}$ *satisfies* $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon] \cap (\mathcal{H}_{n-1,n}[\varepsilon](\hat{m}) \cup \mathcal{J}_n[\varepsilon](\hat{m}))$.

Proof. The same as in Lemma 26. ■

Lemma 37 *For each* $m \in \{G, B\}$*,* $j \in I$ *and* $n \in -j$*, if Assumption 15 is satisfied, then there exists a mapping from* $\mathbf{y}_n \in \Delta\left(\{\mathbf{1}_{y_n}\}_{y_n \in Y_n}\right)$ *to* $f[n](m) \in \{G, B\}$ *such that, for any* m *and* $\mathbf{y}_n$*, given* m*, player* n *puts a belief no less than* $1 - \exp(-O(T^{\frac{1}{2}}))$ *on the events that* $f[n](m) = m$ *or* $g[n-1](m) = E$.

Proof. The same as in Lemma 27. ■

We also provide the lemma to show that Assumption 15 is sufficient to construct the reward:

Lemma 38 *There exists* $\bar{u}$ *such that, for each* $j \in I$ *and* $i \in I$*, there exist* $\pi_i^G(j,\cdot,\cdot) : A_{i-1} \times Y_{i-1} \to [-\bar{u}, 0]$ *and* $\pi_i^B(j,\cdot,\cdot) : A_{i-1} \times Y_{i-1} \to [0, \bar{u}]$ *such that*

1. *(163) is satisfied for* $i = j$ *and*

2. *(164) is satisfied for* $i \in -j$.

If ε defined in (118) does not satisfy $\varepsilon < \bar{\varepsilon}$ in Lemmas 34 and 36, then re-take ε such that ε is smaller than $\bar{\varepsilon}$. This does not affect the consistency among the variables defined in Section 34.

45.3 Assumptions for Dispensing with the Public Randomization and Perfect Cheap Talk

First, to dispense with the public randomization, we need an assumption comparable to Assumption 11 in the two-player case. For each $i \in I$, with player j replaced with player

$i-1$ (the controller of player i's payoff), all the definitions about a^G, $Y^i_{i-1,1}$, $Y^i_{i-1,2}$, (132), (133), $Y^i_{i,1}$ and $Y^i_{i,2}$ in Section 38.2 are valid with more than two players.

Now, we formally state the more-than-two-player analogue of Assumption 11:

Assumption 16 *For each $i \in I$, there exists $a^G \in A$ such that there exist $Y^i_{i-1,1}$, $Y^i_{i-1,2}$, $\bar{p}_i$, $Y^i_{i,1}$ and $Y^i_{i,2}$ such that $Y^i_{i,1}$ and $Y^i_{i,2}$ satisfy*

1. *(132) and (133) with j replaced with $i-1$, and*

2. $$Y^i_{i,1} \neq \emptyset, Y^i_{i,2} \neq \emptyset, Y_i = Y^i_{i,1} \cup Y^i_{i,2}, Y_{i-1} = Y^i_{i-1,1} \cup Y^i_{i-1,2}.$$

For notational convenience, we assume that a^G is the same for each player and the same as in Assumption 13.[96]

Second, when player i with $i \geq 2$ sends the message, player $i-1$ wants to construct a statistics $\phi_{i-1}(a_{i-1}, y_{i-1})$ such that player $i-1$ can infer player i's message statistically and that the conditional independence property holds for player i, as $\phi_j(a_j, y_j)$ in Lemma 28: For some $a^G_i \in A_i$, $\bar{\alpha}_{i-1} \in \Delta(A_{i-1})$, $a^G_{-(i-1,i)} \in A_{-(i-1,i)}$, for all $y_i \in Y_i$,

$$\mathbb{E}\left[\phi_{i-1}(a_{i-1}, y_{i-1}) \mid \bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i\right] = \begin{cases} q_2 & \text{if } a_i = a^G_i, \\ q_1 & \text{if } a_i \neq a^G_i. \end{cases} \tag{165}$$

A sufficient condition for the existence of such $\phi_{i-1}(a_{i-1}, y_{i-1})$ is as follows: Let $\bar{Q}_{i-1}(\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i) \equiv (q_{i-1}(a_{i-1}, y_{i-1} \mid \bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i))_{a_{i-1}, y_{i-1}}$ be the vector expression of the conditional probability of (a_{i-1}, y_{i-1}) after player i plays a_i and observes y_i, assuming that players $-i$ take $\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}$. It is sufficient that $\bar{Q}_i(\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i)$ is linearly independent with respect to a_i, y_i.

At the same time, while player i sends a message by taking different a_i's, each player $n-1$ needs to incentivize player n to take the equilibrium strategy. To do so, we want to

[96] Remember that in Assumption 13, we assumed that a^G_i that is used for player i to send the message is the same as a^G_i that is player i's action in a^G_{-j} when player $j \in -i$ sends the message.

construct the reward to cancel out the differences in the instantaneous utilities: If we pick a_i^B from $A_i \diagdown \{a_i^G\}$ properly, then for each $n \in I$, there exists a reward $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$ such that the ex ante payoff of player n is constant for all $a_n \in A_n$ and $a_i \in \{a_i^G, a_i^B\}$:

$$\begin{aligned} & u_i(a_n, \alpha_{-n}) + \mathbb{E}\left[\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1}) \mid a_n, \alpha_{-n}\right] \\ = \; & \text{constant} \end{aligned} \tag{166}$$

for all $a_n \in A_n$ and

$$\begin{cases} \alpha_{-n} \in \left\{\left(\bar{\alpha}_{i-1}, a^G_{-(i-1,i,n)}, a_i^G\right), \left(\bar{\alpha}_{i-1}, a^G_{-(i-1,i,n)}, a_i^B\right)\right\} & \text{if player } n \text{ is not player } i \text{ (sender)}, \\ \alpha_{-n} \in \left\{\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}\right\} & \text{if player } n \text{ is player } i. \end{cases} \tag{167}$$

A sufficient condition for the existence of such $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$ is as follows: Let
$\bar{Q}_{n-1}(i, a_n, \alpha_{-n}) = (q_{n-1}(y_{n-1} \mid a_n, \alpha_{-n}))_{y_{n-1}}$ be the vector expression of the conditional probability of y_{n-1} after the players play a_n, α_{-n}. It is sufficient that $\bar{Q}_n(i, a_n, \alpha_{-n})$ is linearly independent with respect to $a_n \in A_n$ and α_{-n} with (167).

Assumption 17 *For each $i \geq 2$, there exist $\bar{\alpha}_{i-1} \in A_{i-1}$ and $a^G_{-(i-1,i)}$ such that $\bar{Q}_i(\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i)$ is linearly independent with respect to a_i, y_i. Further, there exist a_i^G, a_i^B such that for each $n \in I$, $\bar{Q}_n(i, a_n, \alpha_{-n})$ is linearly independent with respect to $a_n \in A_n$ and α_{-n} with (167).*

The former requirement is generic since we assume (152). In addition, the latter requirement is generic since $|Y_{n-1}| \geq 2|A_n|$.

Again, for notational convenience, for each i, a_i^G that player i uses to send a message and a_i^G that player i takes in $a^G_{-(j-1,j)}$ when another player j is a sender are the same. Moreover, assume that (a_i^G, a_i^B) is the same as in Assumption 13.[97]

We can show that Assumption 17 is sufficient to have ϕ_{i-1} with conditionally independent property:

[97] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a^G_{-j} when player $j \in -i$ sends the message.

Lemma 39 *If Assumption 17 is satisfied, then there exist $q_2 > q_1$ such that for all $i \in \{2, ..., N\}$, there exist $\phi_{i-1} : A_{i-1} \times Y_{i-1} \to (0,1)$ such that (165) is satisfied.*

Proof. The same as Lemma 28. ■

In addition, Assumption 17 is sufficient to have $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$:

Lemma 40 *There exists $\bar{u} > 0$ such that, for each $i \in I$ and $n \in I$, there exist $\pi_n^G(\text{report}, i, \cdot, \cdot) : A_{n-1} \times Y_{n-1} \to [-\bar{u}, 0]$ and $\pi_n^B(\text{report}, i, \cdot, \cdot) : A_{n-1} \times Y_{n-1} \to [0, \bar{u}]$ such that (166) is satisfied.*

46 Coordination Block with the Noisy Cheap Talk

We consider the coordination block without the perfect cheap talk but with the noisy cheap talk with precision $p = \frac{1}{2}$.

As mentioned in Section 45.1, we define

$$\begin{aligned} N(i) &= \{i, i+1, i+2\}, \\ n^*(i) &\in \arg\min_{j \in \{i, i+2\}} |A_j| \, |Y_j| \, , \\ n^{**}(i) &= \{i, i+2\} \setminus \{n^*(i)\}. \end{aligned}$$

First, player i sends the message about $x_i \in \{G, B\}$ to player $n^*(i)$ via actions. Let $w_i \in \{G, B\}$ be player $n^*(i)$'s inference of this message. Second, player $n^*(i)$ sends the message about w_i to players $N(i)$ via actions. Each player $n \in N(i)$ constructs player n's inference of w_i, denoted $w_i(n) \in \{G, M, B\}$. Here, the inference M ("middle") is introduced so that it prevents player $n^*(i)$ from creating a situation where player $n^*(i)$ is pivotal. See 45.1 for the definition of "pivotal."

46.1 Structure of the Coordination Block

Formally, the coordination block proceeds as follows:

- The periods where the players coordinate on x_1:

- The coordination round 1 for x_1. Player 1 sends the message about x_1 to player $n^*(1)$ via actions. If $n^*(1) = 1$, then this round does not exist.
- The coordination round 2 for x_1. Player $n^*(1)$ sends the message about w_1 to players $N(1)$ via actions. Player $n \in N(1)$ creates the inference $w_1(n)$.
- For each $j \in N(1) = \{1, 2, 3\}$ and $n \in -j$, we have the coordination rounds 3 for x_1 between j and n, where player j sends the message $w_1(j)$ via noisy cheap talk. The players take turns: First, player 1 sends $w_1(1)$ to player 2, second, player 1 sends $w_1(1)$ to player 3, and so on until player 1 sends $w_1(1)$ to player N. Then, player 2 sends $w_1(2)$ to player 1, and so on until player 2 sends $w_1(2)$ to player N. After player 2, player 3 sends $w_1(3)$ for each of the opponents -3 sequentially.

$\vdots$

- The periods where the players coordinate on x_i:
  - The coordination round 1 for x_i. Player i sends the message about x_i to player $n^*(i)$ via actions. If $n^*(i) = i$, then this round does not exist.
  - The coordination round 2 for x_i. Player $n^*(i)$ sends the message about w_i to players $N(i)$ via actions. Player $n \in N(i)$ creates the inference $w_i(n)$.
  - For each $j \in N(i)$ and $n \in -j$, we have the coordination rounds 3 for x_i between j and n, where player j sends the message $w_i(j)$ via noisy cheap talk. Again, the players take turns.

$\vdots$

- The periods where the players coordinate on x_N:
  - The coordination round 1 for x_N. Player N sends the message about x_N to player $n^*(N)$ via actions. If $n^*(N) = N$, then this round does not exist.
  - The coordination round 2 for x_N. Player $n^*(N)$ sends the message about w_N to players $N(N)$ via actions. Player $n \in N(N)$ creates the inference $w_N(n)$.

– For each $j \in N(N)$ and $n \in -j$, we have the coordination rounds 3 for x_N between j and n, where player j sends the message $w_N(j)$ via noisy cheap talk. Again, the players take turns.

For notational convenience, let $T(i \to_{x_i} n^*(i))$ be the set of periods in the coordination round 1 for x_i, where player i sends the message x_i to player $n^*(i)$ via actions. Similarly, let $T(n^*(i) \to_{w_i} N(i))$ be the set of periods in the coordination round 2 for x_i, where player $n^*(i)$ sends the message w_i to players $N(i)$ via actions

We explain each round in the sequel.

46.2 Coordination Round 1 for x_i

If player i is the same person as player $n^*(i)$, then this round does not exist. Let $w_i = x_i$ be player $n^*(i)$'s inference (player i's inference in other words).

Otherwise, player i takes $a_i^{x_i}$ and the other players $-i$ take a_{-i}^G for $T^{\frac{1}{2}}$ periods. Remember that $T(i \to_{x_i} n^*(i))$ be the set of periods in this round.

Player $n^*(i)$ creates her inference of x_i denoted by w_i as follows: First, player $n^*(i)$ creates $\Psi^i_{n^*(i),t} \in \{0,1\}$ from $\psi^i_{n^*(i)}(y_{n^*(i),t})$ as player i creates $\Psi^{a(x)}_{i,t}$ from $\psi^{a(x)}_i(y_{i,t})$. See Lemma 33 for the definition of $\psi^i_{n^*(i)}(y_{n^*(i),t})$.

Second, player $n^*(i)$ randomly picks $t_{n^*(i)}(i \to_{x_i} n^*(i))$ from $T(i \to_{x_i} n^*(i))$.

Finally, player $n^*(i)$ infers x_i from $\left\{\Psi^i_{n^*(i),t}\right\}_{T(i \to_{x_i} n^*(i))}$ but excludes period $t_{n^*(i)}(i \to_{x_i} n^*(i))$. That is, with $T_{n^*(i)}(i \to_{x_i} n^*(i)) \equiv T(i \to_{x_i} n^*(i)) \setminus \left\{t_{n^*(i)}(i \to_{x_i} n^*(i))\right\}$, player $n^*(i)$ infers $w_i = G$ if

$$\frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_{n^*(i)}(i \to_{x_i} n^*(i))} \Psi^i_{n^*(i),t} \geq \frac{q_1 + q_2}{2} \tag{168}$$

and $w_i = B$ otherwise.

Lemma 33 directly implies the following:

Lemma 41 *For any $i \in I$ and $x_i \in \{G, B\}$, if players i and $n^*(i)$ follow the equilibrium* *strategy, then*

$$\Pr\left(\{w_i = x_i\} \mid x_i\right) \geq 1 - \exp(-O(T^{\frac{1}{2}}))$$

and the conditional distribution of w_i given x_i is independent of another player $j \in -(i, n^*(i))$'s unilateral deviation.

46.3 Coordination Round 2 for x_i

This is the round where player $n^*(i)$ sends w_i to players $N(i)$. Player $n^*(i)$ takes $a_{n^*(i)}^{w_i}$, player $i+1$ takes α_{i+1}^*, player $n^{**}(i)$ takes $\alpha_{n^{**}(i)}^*$, and each player $j \notin N(i)$ takes a_j^G for $T^{\frac{1}{2}}$ periods. See Assumption 14 for the definition of α_{i+1}^* and $\alpha_{n^{**}(i)}^*$. Remember that $T(n^*(i) \to_{w_i} N(i))$ be the set of periods in this round.

See (156) and (157) for the definition of the $L_n(i) \times |A_n| |Y_n|$ matrix $I_n(i)$. Based on $I_n(i)$, each player $n \in N(i) \setminus \{n^*(i)\}$ constructs a random variable $\boldsymbol{I}_{n,t}(i)$ as follows: After taking a_n and observing y_n, player n calculates $I_n(i)\,\mathbf{1}_{a_n,y_n}$. Here, $\mathbf{1}_{a_n,y_n}$ is a $|A_n| |Y_n| \times 1$ vector such that the element corresponding to a_n, y_n is equal to 1 and the other elements are 0. Hence, $I_n(i)\,\mathbf{1}_{a_n,y_n}$ is a $L_n(i) \times 1$ vector. Then, player n draws $L_n(i)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $I_n(i)\,\mathbf{1}_{a_n,y_n}$, then the lth element of $\boldsymbol{I}_n(i)$ is equal to 1. Otherwise, the lth element of $\boldsymbol{I}_n(i)$ is equal to 0. We have

$$\Pr(\{(\boldsymbol{I}_n(i))_l = 1\} \mid a, y) = \boldsymbol{i}_n^l(i)\mathbf{1}_{a_n,y_n}. \tag{169}$$

Given $\{\boldsymbol{I}_{n,t}(i)\}_{t \in T(n^*(i) \to_{w_i} N(i))}$, player $n \in N(i)$ infers w_i as follows: Player n randomly picks $t_n(n^*(i) \to_{w_i} N(i))$ from $T(n^*(i) \to_{w_i} N(i))$. Player n infers w_i from $\{\boldsymbol{I}_{n,t}(i)\}_{T(n^*(i) \to_{w_i} N(i))}$ but excludes period $t_n(n^*(i) \to_{w_i} N(i))$. Specifically, with $T_n(n^*(i) \to_{w_i} N(i)) \equiv T(n^*(i) \to_{w_i} N(i)) \setminus \{t_n(n^*(i) \to_{w_i} N(i))\}$,

1. Player $n^*(i)$ infers her own message straightforwardly: $w_i(n^*(i)) = w_i$.

2. Player $n \in N(i) \setminus \{n^*(i)\}$ infers as follows:

(a) If

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_n(n^*(i) \to_{w_i} N(i))} \boldsymbol{I}_{n,t}(i) - q_2 \mathbf{1} \right\| \leq \varepsilon,$$

then $w_i(n) = G$.

(b) If

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_n(n^*(i) \to_{w_i} N(i))} \boldsymbol{I}_{n,t}(i) - q_1 \mathbf{1} \right\| \leq \varepsilon,$$

then $w_i(n) = B$.

(c) Otherwise, $w_i(n) = M$ (the posterior is not skewed enough for $w_i = G$ or $w_i = B$ and so player $n^*(i)$ infers that the message is "middle").

Assumption 14 implies the following Lemma:

Lemma 42 *For any $\varepsilon < \bar{\varepsilon}$, for any $i \in I$ and $w_i \in \{G, B\}$,*

1. *For any $n \in N(i^*)$,*

   (a) *If players $n^*(i)$ and n follow the equilibrium strategy, then*

   $$\Pr(\{w_i(n) = w_i\} \mid w_i) \geq 1 - \exp(-O(T^{\frac{1}{2}})).$$

   (b) *The distribution of $w_i(n)$ given w_i is independent of player $j = N(i) \setminus \{n^*(i), n\}$'s unilateral deviation.*

2. *For any history of player $n^*(i)$ at the end of the coordination round 2 for x_i, player $n^*(i)$ puts a belief no more than $\exp(-O(T^{\frac{1}{2}}))$ on the event*

$$\{G, B\} \ni w_i(n^{**}(i)) \neq w_i(i+1) \in \{G, B\}.$$

Proof.

1. Follows from (156) and (169).

2. Follows from Lemma 34 and Hoeffding's inequality. By Assumption 3, excluding period $t_n(n^*(i) \to_{w_i} N(i))$ does not affect the probability so much.

■

As we will see, as long as the noisy cheap talk by the other players transmits correctly in the coordination round 3 for x_i (this is true ex ante at the end of the coordination round 2 for x_i), player $n^*(i)$ is pivotal for some player's inference of x_i if and only if $\{G, B\} \ni w_i(n^{**}(i)) \neq w_i(i+1) \in \{G, B\}$. 2 of Lemma 42 guarantees that, after any history (including those after player $n^*(i)$'s deviation), the probability that player $n^*(i)$ is pivotal is negligible for the almost optimality.

For each $n \in N(i) \setminus \{n^*(i)\}$, consider player $j = N(i) \setminus \{n^*(i), n\}$. As we will see, player j is not pivotal if players $n^*(i)$ and n infer the same state w_i. Therefore, 1 of Lemma 42 guarantees that player j cannot manipulate player n's inference to create a situation where player j is pivotal.

46.4 Coordination Round 3 for x_i Between Players j and n

This is the round where player $j \in N(i)$ sends $w_i(j)$ to player $n \in I$. Let $w_i(j)(n) \in \{G, B, M\}$ be player n's inference of player j's message. Here, we assume that the noisy cheap talk is available. See Section 48 for how to dispense with the noisy cheap talk.

If player j is the same player as player n, then $w_i(j)(n) = w_i(j)$, that is, player j infers her own message straightforwardly.

Otherwise, player j sends messages as follows. From $w_i(j) \in \{G, M, B\}$, player j constructs a sequence of two binary messages $w_i(j)\{1\}, w_i(j)\{2\} \in \{G, B\}^2$: If $w_i(j) = G$, then $w_i(j)\{1\} = w_i(j)\{2\} = G$; If $w_i(j) = B$, then $w_i(j)\{1\} = w_i(j)\{2\} = B$; If $w_i(j) = M$, then $w_i(j)\{1\} = G$ and $w_i(j)\{2\} = B$ with probability $\frac{1}{2}$ and $w_i(j)\{1\} = B$ and $w_i(j)\{2\} = G$ with probability $\frac{1}{2}$.

Player j sends the two messages $w_i(j)\{1\}$ and $w_i(j)\{2\}$ sequentially via noisy cheap talk.

With abuse of notation, we define $g[n-1](w_i(j)) \in \{w_i(j), E\}$ and $f[n](w_i(j)) \in \{G, M, B\}$ as follows: For $g[n-1](w_i(j))$,

1. $g[n-1](w_i(j)) = w_i(j)$ if and only if player $n-1$ thinks that there is no error for $f[n](w_i(j)\{1\})$ and $f[n](w_i(j)\{2\})$, that is, $g[n-1](w_i(j)\{1\}) = w_i(j)\{1\}$ and $g[n-1](w_i(j)\{2\}) = w_i(j)\{2\}$.

2. $g[n-1](w_i(j)) = E$ otherwise.

For $f[n](w_i(j))$, player i infers $f[n](w_i(j))$ from $f[n](w_i(j)\{1\})$ and $f[n](w_i(j)\{2\})$, using the mapping between $w_i(j)$ and $w_i(j)\{1\}, w_i(j)\{2\}$.

1. $f[n](w_i(j)) = G$ if and only if $f[n](w_i(j)\{1\}) = f[n](w_i(j)\{2\}) = G$.

2. $f[n](w_i(j)) = B$ if and only if $f[n](w_i(j)\{1\}) = f[n](w_i(j)\{2\}) = B$.

3. $f[n](w_i(j)) = M$ if and only if "$f[n](w_i(j)\{1\}) = G$ and $f[n](w_i(j)\{2\}) = B$" or "$f[n](w_i(j)\{1\}) = B$ and $f[n](w_i(j)\{2\}) = G$."

$g_2[n-1](w_i(j))$ and $f_2[j-1](w_i(j))$ are analogously defined.

Finally, player n infers $w_i(j)$ as $w_i(j)(n) = f[n](w_i(j))$.

46.5 Player n's Inference of x_i

Based on these rounds, player n infers x_i as follows. Let $x_i(n) \in \{G, B\}$ be player n's inference of x_i. From $\{w_i(j)(n)\}_{j \in N(i)}$, player n constructs $x_i(n)$ such that

$$x_i(n) = \begin{cases} G & \text{if} \begin{cases} w_i(n^{**}(i))(n) = w_i(i+1)(n) = G, \\ w_i(n^{**}(i))(n) = M, w_i(i+1)(n) = G, \\ w_i(n^{**}(i))(n) = G, w_i(i+1)(n) = M, \\ w_i(n^{**}(i))(n) = B, w_i(i+1)(n) = G, w_i(n^{*}(i))(n) = G, \\ w_i(n^{**}(i))(n) = G, w_i(i+1)(n) = B, w_i(n^{*}(i))(n) = G, \end{cases} \\ B & \text{otherwise.} \end{cases} \tag{170}$$

Finally, let

$$x(n) = \{x_i(n)\}_{i \in I}$$

be the profile of the inferences.

46.6 Definition of $\theta_{i-1}(\mathrm{c}) \in \{G, B\}$

Based on the realization of the coordination block, if some events happen, then player $i-1$ makes player i indifferent between any action profile sequence in the main blocks. $\theta_{i-1}(\mathrm{c}) = B$ implies that such an event happens while $\theta_{i-1}(\mathrm{c}) = G$ implies that such an event does not happen.

We will define the events to induce $\theta_{i-1}(\mathrm{c}) = B$: For each $j \in I$, while the players coordinate on x_j,

1. There exists player $j' \in -i$ with $j' \in N(j)$ such that when player j' sends the message $w_j(j')$ to player i, player $i-1$ has $g[i-1](w_j(j')) = E$.

2. There exist players $j' \in -i$ and $n \in -i \cap N(j)$ such that when player j' sends the message $w_j(j')$ to player n, player n has a wrong signal $f[n](w_j(j')) \neq w_j(j')$.

3. Player i is in $N(j)$ and consider the following inference:

$$x_j(n) = \begin{cases} G & \text{if } \begin{cases} w_j(n^{**}(j)) = w_j(j+1) = G, \\ w_j(n^{**}(j)) = M, w_j(j+1) = G, \\ w_j(n^{**}(j)) = G, w_j(j+1) = M, \\ w_j(n^{**}(j)) = B, w_j(j+1) = G, w_j(n^{*}(j)) = G, \\ w_j(n^{**}(j)) = G, w_j(j+1) = B, w_j(n^{*}(j)) = G, \end{cases} \\ B & \text{otherwise.} \end{cases} \tag{171}$$

Note that this is what we replace player n's inference of the messages in the coordination round 3 in (170) with the true messages. We have $\theta_{i-1}(\mathrm{c}) = B$ if there exist $n \in I$ and $j \in I$ such that player i's message $w_j(i)$ matters for $x_j(n)$ in (171). That is,

(a) If player i is $n^*(j)$, then

$$\{G, B\} \ni w_j(n^{**}(j)) \neq w_j(j+1) \in \{G, B\}. \tag{172}$$

(b) If player i is in $N(j) \setminus \{n^*(j)\}$, then

$$w_j \equiv w_j(n^*(j)) \neq w_j(i'). \tag{173}$$

with $i' = N(j) \setminus \{i, n^*(j)\}$.

Note that, although player n can be player i herself, whether or not $w_j(i)$ matters in (171) is determined by the other players' messages $\{w_j(i')\}_{i' \neq i}$.

In the definition of $\theta_{i-1}(\mathrm{c})$, player $i-1$ uses the information owned by players $-(i-1,i)$. Section 52 explains how players $-(i-1,i)$ inform player $i-1$ of their history necessary to create $\theta_{i-1}(\mathrm{c})$ in the re-report block. Since $\theta_{i-1}(\mathrm{c})$ only affects the reward function (that is, does not affect $\sigma_{i-1}(x_{i-1})$), it suffices that player $i-1$ knows the information by the end of the review phase.

We verify that the distribution of $\theta_{i-1}(\mathrm{c})$ is almost independent of player i's strategy: For Cases 1 and 2, we need to verify that player i cannot manipulate $\theta_{i-1}(\mathrm{c})$ by affecting some player's message m. The definition of the noisy cheap talk implies that the probability of $g[i-1](m) = E$ when player i is a receiver and that of $f[n](m) \neq m$ when player $j \in -i$ is a sender and player $n \in -i$ is a receiver are almost independent of m.[98]

For Case 3-(a), 2 of Lemma 42 implies that player i puts a belief no more than $\exp(-O(T^{\frac{1}{2}}))$ on (172) after *any history* (including those after player i's deviation) at the end of the coordination round 2 for x_j. Since $w_j(n^{**}(j))$ and $w_j(j+1)$ are fixed at the end of the coordination round 2 for x_j, whether (172) happens or not is almost independent of player i's strategy.

For Case 3-(b), note that if $w_j(n^{**}(j)) = w_j$, then (173) is not the case. In addition, regardless of w_j, this event happens with probability no more than $\exp(-O(T^{\frac{1}{2}}))$ from the

[98] Note that m can be affected by player i's strategy before the round where player j sends m to player n.

perspective at the end of the coordination round 1 for x_j by 1 of Lemma 42.[99] Therefore, no player can change the distribution of $\theta_{i-1}(\mathrm{c})$ by more than $\exp(-O(T^{\frac{1}{2}}))$.

In summary, we have shown the following lemma:

Lemma 43 *If*

1. *the probability of $g[i-1](m) = E$ when player i is a receiver of a message m is almost independent of m and*

2. *the probability of $f[n](m) \neq m$ when player $j \in -i$ is a sender of a message m and player $n \in -i$ is a receiver is almost independent of m,*

then, the distribution of $\theta_{i-1}(\mathrm{c}) \in \{G, B\}$ is almost independent of player i's strategy.

The premise of lemma is stated to clarify what assumption about the noisy cheap talk is used, expecting that we will dispense with it later.

46.7 Incentives in the Coordination Block

First, Lemma 43 implies that player i does not have an incentive to manipulate $\theta_{i-1}(\mathrm{c})$.

Second, we consider player i's incentive to tell the truth about $w_n(i)$ with $i \in N(n)$ for the coordination round 3 for x_n between i and $i' \in -i$. If player i' with $i' \in -i$ received a wrong signal $f[i'](w_n(j))$ for some $n \in I$ and $j \in -i$, then Case 2 of $\theta_{i-1}(\mathrm{c})$ implies $\theta_{i-1}(\mathrm{c}) = B$. Hence, together with Case 3 of $\theta_{i-1}(\mathrm{c})$, whenever player i's message matters for $x_n(i')$ for some $i' \in -i$, then $\theta_{i-1}(\mathrm{c}) = B$ and player i is indifferent between any action profile sequence. Therefore, it is optimal for player i to tell the truth.

Third, we consider the incentive of player i in the coordination rounds 1 and 2 for x_n. If player i is player $n^*(n)$, then since x_n controls the value of player $n+1 \neq n^*(n)$, player $n^*(n)$ is indifferent between coordinating on $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. (170) and 2 of of Lemma 42 imply that, if the messages in the coordination round

[99] Note that w_j can be affected by some player's strategy before the end of the coordination round 1 for x_j.

3 transmit correctly if a sender is not player $n^*(n)$ (this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}})))$, then player $n^*(n)$ at the end of the coordination round 2 puts a conditional belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$ regardless of player $n^*(n)$'s history. Therefore, player $n^*(n)$ (player i) is almost indifferent between any strategy in the coordination rounds 1 and 2 for x_n.

If player i is player n (the initial holder of state x_n) but not player $n^*(n)$, then again, since x_n controls the value of player $n+1 \neq n$, player n is indifferent between coordinating on $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. 1 of of Lemma 42 implies that regardless of player n's strategy in the coordination rounds 1 and 2 for x_n, players $n^*(n)$ and $n+1$ have $w_n(n^*(n)) = w_n(n+1) \in \{G, B\}$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Then, (170) implies that, if the messages in the coordination round 3 transmit correctly if a sender is not player n (again, this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}})))$, then either $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. Therefore, player n (player i) is almost indifferent between any strategy in the coordination rounds 1 and 2.

If player i is not player n or player $n^*(n)$, then Lemma 41 and 1 of of Lemma 42 imply that, regardless of player i's strategy in the coordination rounds 1 and 2 for x_n, players $n^*(n)$ and at least one player $i' \in N(n) \setminus \{i\}$ have $w_n(n^*(n)) = w_n(i') = x_n$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Then, (170) implies that, if the messages in the coordination round 3 transmit correctly if a sender is not player i (this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}})))$, then either $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. Therefore, player i is almost indifferent between any strategy in the coordination rounds 1 and 2.

Finally, we show that the definition of $\theta_{i-1}(\mathrm{c}) = B$ implies that, for any i, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$ by the following reasons: (i) If player i's signal $f[i](w_n(j))$ was wrong for some $n \in I$ and $j \in -i$, then, given $w_n(j)$, $g[i-1](w_n(j)) = E$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Since $g[i-1](w_n(j))$ is not revealed by players $(-i)$'s continuation strategy in the main blocks, player i believes that $\theta_{i-1}(\mathrm{c}) = B$

because of Case 1. (ii) If player i' with $i' \in -i$ received a wrong signal $f[i'](w_n(j))$ for some $n \in I$ and $j \in -i$, then Case 2 of $\theta_{i-1}(\mathrm{c})$ implies $\theta_{i-1}(\mathrm{c}) = B$. From (i) and (ii), player i who considers almost optimality can condition that $f[i'](w_n(j)) = w_n(j)$ for all $i' \in I$, $n \in I$ and $j \in -i$. (iii) If player i is pivotal for player i''s inference of x_n with $i' \in I$ and $n \in I$, then $\theta_{i-1}(\mathrm{c}) = B$. Hence, $w_n(i)$ does not matter for player i's value. Therefore, in total, $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$.

The following lemma summarizes the above discussion:

Lemma 44 *The following two statements are true:*

1. *If, for each player $i \in I$,*

   (a) *the probability of $g[i-1](m) = E$ when player i is a receiver of a message m is almost independent of m,*

   (b) *the probability of $f[n](m) \neq m$ when player $j \in -i$ is a sender of a message m and player $n \in -i$ is a receiver is almost independent of m, and*

   (c) *for all n with $i \neq n+1$, player i's value is almost the same between $x_n(j) = G$ for all $j \in I$ and $x_n(j) = B$ for all $j \in I$ regardless of $\{x_{n'}(j)\}_{j \in I, n' \leq n-1}$ ($n' \leq n-1$ implies that the coordination rounds for $x_{n'}$ comes before those for x_n),*

   then it is almost optimal for player i to follow the equilibrium strategy in the coordination block.

2. *For any $i \in I$, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$.*

Note that, for the second statement, 1-(a), 1-(b) and 1-(c) are not necessary.

47 Structure of the Review Phase

Replacing the perfect cheap talk in the coordination block with the noisy cheap talk, the structure of the coordination block is as explained in Section 46.1. Now, the coordination block has at most $N(1+1+3(N-1))$ rounds.[100] After the coordination block, the structure is the same as in Section 31 of the Supplemental Material 3. As in the Supplemental Material 3, let r be a generic serial number for a round.

If we replace the noisy cheap talk with messages via actions, then as we will see in Section 48, we treat rounds where a player sends one message and rounds where a player send two messages separately. Each round where the sender sends one message has $T^{\frac{1}{2}}$ periods. Section 48 explains how the sender sends the message. On the other hand, each round where the sender would send two messages via noisy cheap talk is now divided into two rounds each of which has $T^{\frac{1}{2}}$ without the noisy cheap talk. Using the first $T^{\frac{1}{2}}$-period round, the sender sends the first message as we will explain in Section 48. After that, using the second $T^{\frac{1}{2}}$-period round, the sender sends the second message. With abuse of notation, let r again be a generic serial number for a round and $T(r)$ be the set of periods in round r.

48 Dispensing with the Noisy Cheap Talk

We consider how player j sends a binary message $m \in \{G, B\}$ to player i in some round. Let r be the serial number of this round and $T(r)$ be the set of periods in round r.

As mentioned in Section 45.2, with η being a small number to be defined, the sender (player j) determines

$$z_j(m) = \begin{cases} m & \text{with probability } 1-2\eta, \\ \{G, B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

and player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2}a_j^G + \frac{1}{2}a_j^B & \text{if } z_j(m) = M \end{cases}$$

[100] The precise number depends on whether $n^*(i) = i$ or not for each i.

for $T^{\frac{1}{2}}$ periods. The other players $-j$ take a_{-j}^{G}.

Then, intuitively, as in Section 45.2, $g[n-1](m)$ is determined as follows:

1. $g[n-1](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon](m)$.

2. $g[n-1](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon](m)$.

Instead of using $\mathbf{y}_{n-1}$ directly, as in the two-player case, we consider the following construction of $g[n-1](m)$.

48.1 Formal: $g[n-1](m) \in \{m, E\}$

In the definition of $g[n-1](m)$, player $n-1$ uses m, which is the information owned by player j. Section 52 explains how player j informs player $n-1$ of m. Since $g[n-1](m)$ only affects the reward function (does not affect $\sigma_{n-1}(x_{n-1})$), it suffices that player $n-1$ knows the information by the end of the review phase.

If $z_j(m) \neq m$, then $g[n-1](m) = E$ as in Section 45.2. Let us concentrate on the case with $z_j(m) = m$. Let $\mathbf{y}_{n-1}(r)$ be the frequency of player $(n-1)$'s signals in round r.

First, player $n-1$ randomly picks $t_{n-1}(r)$ from $T(r)$, the set of periods in round r. With $T_{n-1}(r) \equiv T(r) \setminus \{t_{n-1}(r)\}$, player $n-1$ constructs random variables $\left\{\mathbf{\Omega}_{n-1,t}^{H}\right\}_{t \in T_{n-1}(r)}$ as follows. After taking a_{n-1} ($a_{n-1} = a_{n-1}^{m}$ if player $n-1$ is the sender ($n-1=j$) since we concentrate on $z_j(m) = m$ and $a_{n-1} = a_{n-1}^{G}$ if player $n-1$ is not the sender) and observing $y_{n-1,t}$, player $n-1$ calculates $H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}$. Then, player $n-1$ draws $(|Y_{n-1}| - |A_n| + 1)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}$, then the lth element of $\mathbf{\Omega}_{n-1,t}^{H}$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_{n-1,t}^{H}$ is equal to 0. Since all the elements of $H_{n-1}(m)$ are in $(0,1)$, we have

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{n-1,t}^{H}\right)_l = 1\right\} \mid a, y\right) = \left(H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}\right)_l \in (0,1) \tag{174}$$

for all a and y.

We define $g[n-1](m) = m$ if and only if

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} \mathbf{\Omega}_{n-1,t}^{H} - \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} H_{n-1}(m) \mathbf{1}_{y_{n-1,t}} \right\| \leq \frac{\varepsilon}{4} \tag{175}$$

and

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} \mathbf{\Omega}_{n-1,t}^{H} - \mathbf{p}_{n-1}(m) \right\| \leq \frac{\varepsilon}{2}. \tag{176}$$

In summary, there are following cases:

1. $g[n-1](m) = E$ if $z_{n-1}(m) \neq m$, (175) is not satisfied, or (176) is not satisfied.

2. $g[n-1](m) = m$ if $z_{n-1}(m) = m$, (175) is satisfied, and (176) is satisfied.

48.2 Formal: $f[n](m) \in \{G, B\}$

On the other hand, let us consider how the receiver (player n) infers the message. Let $\mathbf{y}_n(r)$ be the frequency of player n's signal observations in round r. Instead of using $\mathbf{y}_n(r)$ directly as in Section 45.2, we consider the following procedure to construct $f[n](m)$.

First, player n randomly picks $t_n(r)$ from $T(r)$, the set of periods in round r. With $T_n(r) \equiv T(r) \setminus \{t_n(r)\}$, player n constructs $f[n](m)$ only depending on $\{y_{n,t}\}_{t \in T_n(r)}$. For notational convenience, let $\mathbf{y}_n(r, T_n(r))$ be the frequency of player n's signal observations in $T_n(r)$.

$f[n](m)$ is determined as in Section 45.2 with $\mathbf{y}_n(r)$ replaced with $\mathbf{y}_n(r, T_n(r))$:

1. If $\mathbf{y}_n(r, T_n(r)) \in \mathcal{G}_n[\varepsilon]$, then

   (a) $f[n](m) = G$ if $\mathbf{y}_i(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](G) \cup \mathcal{J}_n[\varepsilon](G)$.

   (b) $f[n](m) = B$ if $\mathbf{y}_i(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](B) \cup \mathcal{J}_n[\varepsilon](B)$ or $\mathbf{y}_i(r, T_n(r)) \notin \mathcal{H}_{n-1,n}[\varepsilon](G) \cup \mathcal{J}_n[\varepsilon](G)$, and

2. If $\mathbf{y}_n(r, T_n(r)) \notin \mathcal{G}_n[\varepsilon]$, then player n infers $f[n](m)$ from the likelihood as in the two-player case.

By Assumption 3 (full support), neglecting $\left(a_{n,t_n(r)}, y_{n,t_n(r)}\right)$ does not affect the posterior so much.

48.3 Definition of $\theta_{i-1}(j \to_m n) \in \{G, B\}$

While player $j \in I$ sends a message m to player $n \in -j$, for each $i \in I$, player $i-1$ creates $\theta_{i-1}(j \to_m n) \in \{G, B\}$. As for $\theta_{i-1}(\mathrm{c})$, $\theta_{i-1}(j \to_m n) = B$ implies that player $i-1$ makes player i indifferent between any action profile sequence in the subsequent rounds.

Again, player $i-1$ uses the information owned by players $-(i-1, i)$. Section 52 explains how players $-(i-1, i)$ inform player $i-1$ in the re-report block. Since $\theta_{i-1}(j \to_m n)$ only affects the reward function (does not affect $\sigma_{i-1}(x_{i-1})$), it suffices that player $i-1$ knows the information by the end of the review phase.

To create $\theta_{i-1}(j \to_m n)$, player $i-1$ calculates the following variables:

Construction of $\mathbf{\Omega}_{n,\tau}^{G}$ If $i-1 \neq n$, then player n informs player $i-1$ of how many times player n observes y_n for each $y_n \in Y_n$ in $T(r)$ (while receiving the message). Let $T(r, y_n)$ be this number.

For each $y_n \in Y_n$, player $i-1$ calculates $G_n \mathbf{1}_{y_n}$. Then, repeat the following process $T(r, y_n)$ times: Player $i-1$ draws $\left(|Y_n| - |A_j| - 2\sum_{i' \neq j,n} |A_{i'}| + 1\right)$ random variables independently from the uniform distribution on $[0, 1]$. If the lth realization of these random variables is less than the lth element of $G_n \mathbf{1}_{y_n}$, then the lth element of $\mathbf{\Omega}_n^G$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_n^G$ is equal to 0. Since player $i-1$ repeats this process $T(r, y_n)$ times, it generates $T(r, y_n)$ *i.i.d.* random variables $\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}$. Since all the elements of G_n are in $(0, 1)$,

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{n,\tau}^G\right)_l = 1\right\}\right) = \left(G_n \mathbf{1}_{y_n}\right)_l .$$

In total, $\left\{\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is constructed.

Construction of $\mathbf{\Omega}_{n,\tau}^{J}(m)$ In addition to player n informing player $i-1$ of $T(r, y_n)$, player j informs player $i-1$ of m in the re-report block.

For each $y_n \in Y_n$, player $i-1$ calculates $J_n(m)\mathbf{1}_{y_n}$. Then, repeat the following process $T(r, y_n)$ times: Player $i-1$ draws $\left(|Y_n| - \sum_{i' \neq j,n} |A_{i'}| + 1\right)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $J_n(m)\mathbf{1}_{y_n}$, then the lth element of $\mathbf{\Omega}_n^J(m)$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_n^J(m)$ is equal to 0. Since player $i-1$ repeats this process $T(r, y_n)$ times, it generates $T(r, y_n)$ *i.i.d.* random variables $\left\{\mathbf{\Omega}_{n,\tau}^J(m)\right\}_{\tau=1}^{T(r,y_n)}$. Since all the elements of $J_n(m)$ are in $(0,1)$,

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{n,\tau}^J(m)\right)_l = 1\right\}\right) = \left(J_n(m)\mathbf{1}_{y_n}\right)_l.$$

In total, $\left\{\left\{\mathbf{\Omega}_{n,\tau}^J(m)\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is constructed.

Definition of $\theta_{i-1}(j \to_m n) \in \{G, B\}$ For player $i \in \{j, n\}$ (sender or receiver), $\theta_{i-1}(j \to_m n) = G$ for any history.

If player i is in players $-(j,n)$, then player $i-1$ has $\theta_{i-1}(j \to_m n) = G$ if

1. The frequency of $\left\{\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is close to $\mathbf{g}_n$:

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^G - \mathbf{g}_n \right\| \leq \frac{\varepsilon}{2}.$$

   Regardless of player i's deviation, this is the case with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by Notation 2 and the law of large numbers.[101]

2. The frequency of $\left\{\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is close to $\left\{\frac{T(r,y_n)}{T} G_n \mathbf{1}_{y_n}\right\}_{y_n}$ (the frequency of

[101] While player $n^*(i)$ excludes one period $t_{n^*(i)}(i \to_{x_i} n^*(i))$ from (168), player $i-1$ does not exclude a period from $\{T(r, y_n)\}_{y_n}$.

The reason why player $n^*(i)$ excludes one period $t_{n^*(i)}(i \to_{x_i} n^*(i))$ from (168) is to prevent the continuation play of player $n^*(i)$ from revealing player $n^*(i)$'s signal observation too much. This is important to incentivize player $n^*(i)+1$ to tell the truth in the report block.

On the other hand, since $\theta_{i-1}(j \to_m n)$ is not revealed by player $(i-1)$'s continuation play in the main blocks, player $i-1$ does not need to exclude one period here.

The same causion is applicable for the other three inequalities to determine $\theta_{i-1}(j \to_m n)$.

$G_n \mathbf{1}_{y_n}$ using player n's true signal observation):

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{G} - \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} G_n \mathbf{1}_{y_n} \right\| \leq \frac{\varepsilon}{4}.$$

Ex post (after conditioning $\{a_t, y_t\}_{t \in T(r)}$), this is true with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers.

3. The frequency of $\left\{ \{\mathbf{\Omega}_{n,\tau}^{J}(m)\}_{\tau=1}^{T(r,y_n)} \right\}_{y_n \in Y_n}$ is close to $\mathbf{r}_n(m)$:

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{J}(m) - \mathbf{r}_n(m) \right\| \leq \frac{\varepsilon}{2}.$$

Regardless of player i's deviation, this is the case with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by Notation 2 and the law of large numbers.

4. The frequency of $\left\{ \{\mathbf{\Omega}_{n,\tau}^{J}(m)\}_{\tau=1}^{T(r,y_n)} \right\}_{y_n \in Y_n}$ is close to $\left\{ \frac{T(r,y_n)}{T} J_n(m) \mathbf{1}_{y_n} \right\}_{y_n}$ (the frequency of $J_n(m)\mathbf{1}_{y_n}$ using player n's true signal observation):

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{J}(m) - \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} J_n(m) \mathbf{1}_{y_n} \right\| \leq \frac{\varepsilon}{4}.$$

Ex post (after conditioning $\{a_t, y_t\}_{t \in T(r)}$), this is true with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers.

If player i is in players $-(j,n)$ and at least one of the above four conditions is not satisfied, then player $i-1$ has $\theta_{i-1}(j \to_m n) = B$.

48.4 Summary of the Properties of $g[n-1](m)$, $f[n](m)$ and $\theta_{i-1}(j \to_m n)$

In summary, we can show the following lemma:

Lemma 45 *For sufficiently large T, for any $j \in I$ and $n \in -j$, the above communication protocol satisfies the following:*

1. *$g[n-1](m) = E$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ for any $m \in \{G, B\}$.*

2. *Given any $m \in \{G, B\}$ and any $\mathbf{y}_n(r)$, player n puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $f[n](m) = m$ or $g[n-1](m) = E$.*

3. *Given $m \in \{G, B\}$, any $f[n](m)$ happens with probability at least $\exp(-O(T^{\frac{1}{2}}))$.*

4. *The probability of $g[n-1](m)$ being equal to E does not react to player n's strategy by more than $\exp(-O(T^{\frac{1}{2}}))$.*

5. *For $i \in -(j, i)$, whenever player n does not have $f[n](m) = m$, $\theta_{i-1}(j \to_m n) = B$.*

6. *For each $i \in I$, the distribution of $\theta_{i-1}(j \to_m n)$ is independent of player i's strategy with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.*

Proof.

1. This follows from the law of large numbers.

2. If $f[n](m) = m$, then we are done. Suppose not. Note that the definition of $g[n-1](m)$ implies that $g[n-1](m) = m$ only if $z_j(m) = m$ and (175) and (176) are satisfied. Therefore, $g[n-1](m) = m$ only if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon](m)$.

   $f[n](m) \neq m$ implies that either

   (a) $\mathbf{y}_n(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](m)$ is not the case, or

   (b) player i infers $f[n](m)$ from the likelihood using $\mathbf{y}_n(r, T_n(r))$ (neglecting $y_{n,t_n(r)}$)

   is the case. If (a) is the case, then by Hoeffding's inequality, player n believes that $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon](m)$ given m with probability $1 - \exp(-O(T^{\frac{1}{2}}))$. If (b) is the case, then by Lemma 37, player n believes that $z_j(m) \neq m$ given m with probability $1 - \exp(-O(T^{\frac{1}{2}}))$.

Note that, by Assumption 3 (full support), neglecting $\left(a_{n,t_n(r)}, y_{n,t_n(r)}\right)$ does not affect the posterior so much.

3. Given m, any $(y_t)_{t\in T(r)}$ can occur with probability at least

$$\left\{\min_{y,a} q(y \mid a)\right\}^{T^{\frac{1}{2}}}.$$

Assumption 3 (full support) implies that this probability is $\exp(-O(T^{\frac{1}{2}}))$.

4. Ex ante, $g[n-1](m) = E$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ regardless of m. Therefore, the probability of $g[n-1](m)$ being equal to E does not react to player n's strategy before round r by more than $\exp(-O(T^{\frac{1}{2}}))$.

   In addition, the distribution of $\mathbf{\Omega}^H_{n-1,t}$ is independent of player n's strategy in period t and (175) is satisfied ex post (conditional on $\{a_t, y_t\}_{t\in T(r)}$) with probability $1-\exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers. Therefore, the probability of $g[n-1](m)$ being equal to E does not react to player n's strategy in round r by more than $\exp(-O(T^{\frac{1}{2}}))$.

5. Follows from the triangle inequality.

6. For player $i \in \{j, n\}$, $\theta_{i-1}(j \to_m n) = G$ always. If $i \notin \{j, n\}$, then ex ante, $\theta_{i-1}(j \to_m n) = G$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ regardless of m. Therefore, the distribution of $\theta_{i-1}(j \to_m n)$ is not changed by more than $\exp(-O(T^{\frac{1}{2}}))$ by player i's strategy before round r.

   In addition, by Notation 2, the distribution of $\mathbf{\Omega}^G_{n,\tau}$ and $\mathbf{\Omega}^J_{n,\tau}(m)$ is independent of player i's strategy in round r and Cases 2 and 4 in the definition of $\theta_{i-1}(j \to_m n)$ is satisfied ex post (conditional on $\{a_t, y_t\}_{t\in T(r)}$) with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers. Therefore, the distribution of $\theta_{i-1}(j \to_m n)$ is independent of player i's strategy in round r with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.

■

49 Equilibrium Strategies

In this section, we define $\sigma_i(x_i)$ and π_i^{main}.

49.1 States

The states $\lambda_i(l+1)$, $\hat{\lambda}_{i-1}(l+1)$, $d_i(l+1)$, $d_j(l+1)(i)$, $c_i(l+1)$, $\theta_i(l)$, $\theta_i(\lambda_j(l+1))$ and $\theta_i(d_j(l+1))$ are defined as in the Supplemental Material 3 except that x is replaced with $x(i)$ defined in Section 46.5.

If we replace the noisy cheap talk with messages via actions, then we use $f[i](m)$ (when player i is a receiver) and $g[i](m)$ (when player $i+1$ is a receiver) defined in Section 48. In addition, each player i makes player $i+1$ indifferent between any action profile sequence if the following events happen:

- In the coordination block, $\theta_i(\text{c}) = B$ happens.
- In a round where player $j \in I$ sends a message m to player $n \in -j$, $\theta_i(j \to_m n) = B$ happens.

49.2 Player i's Action

49.2.1 With the Noisy Cheap Talk

In the coordination block, the players play the game as explained in Section 46. For the other blocks, $\sigma_i(x_i)$ prescribes the same action with x replaced with $x(i)$ except for the report and re-report blocks. See Sections 51 and 52 for the strategy in the report and re-report blocks.

49.2.2 Without the Noisy Cheap Talk

When player $j \in I$ sends a message m to player $n \in -j$, then the strategies are determined in Section 48.

49.3 Reward Function

In this subsection, we explain player $i-1$'s reward function on player i, $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$. In general, the total reward $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$ is the summation of rewards for each round r:

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta) = \sum_{l=1}^{L} \sum_{t \in T(l)} \pi_i^{\delta}(t, \alpha_{-i,t}, y_{i-1,t}) + \sum_r \pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta).$$

Note that we add (106) to ignore discounting only for the review rounds. As we will see, for the round where the players communicate, we use reward function that take discounting into account directly.

We define $\pi_i^{\text{main}}(x, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta)$ for each r.

49.3.1 With the Noisy Cheap Talk

In the coordination block, for round r where player j sends message x_{i-1} to player $n^*(i)$, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$$

to make player i indifferent between any action profile sequence.[102] Note that we take discounting into account.

In the coordination block, for round r where player $n^*(j)$ sends message w_j to player $N(j)$, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(n^*(j) \to N(j), a_{i-1,t}, y_{i-1,t}).$$

In the main blocks, the reward function is the same as in the Supplemental Material 3

[102] $\pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$ is defined in Lemma 38. Here, we use the assumption that the same a_j^G, a_j^B in Assumption 13 satisfy Assumption 15 for each j.
If not, assume that $\left(a_j^G, a_j^B\right)_j$ in Assumption 13 satisfy 6 and 7 in Assumption 15.

except that x replaced with $x(i-1)$ and that if $\theta_{i-1}(\mathrm{c}) = B$ happens, then player $i-1$ uses

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) = \sum_{t \in T(l)} \pi_i^{x_{i-1}}(\alpha_{-i,t}, y_{i-1,t})$$

for all the review rounds to make player i indifferent between any action profile.[103]

49.3.2 Without the Noisy Cheap Talk

For round r corresponding to a review round, the reward function is the same as in the case with the noisy cheap talk except that if there is round $\tilde{r} \leq r-1$ (before r) such that player j sends a message m to player n in round $\tilde{r}$ and $\theta_{i-1}(j \to_m n) = B$ happens, then player $i-1$ uses

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) = \sum_{t \in T(l)} \pi_i^{x_{i-1}}(\alpha_{-i,t}, y_{i-1,t})$$

to make player i indifferent between any action profile.

For round r where player j sends a message, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$$

defined in Lemma 38. Again, we take discounting into account.

50 Almost Optimality of the Strategy

We want to verity (8), (4) and (5) are satisfied. First, by definition in Section 49.3, (5) is satisfied.

Second, since the length of the rounds other than the review rounds is $T^{\frac{1}{2}}$, the payoff from the review rounds approximately determines the payoff from the review phase for sufficiently large δ (and so sufficiently large T). Therefore, we neglect the payoffs from the rounds other

[103] Since $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l)$ does not depend on δ, with abuse of notation, we omit δ from $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta)$ with r corresponding to the lth review round.

than the review rounds.

Third, we consider (8) and (4) in the case with the noisy cheap talk. Suppose that $x(j) = x(i)$ for all $i, j \in I$ at the end of the coordination block. Then, (8) and (4) are shown as in the case with the perfect cheap talk.

This implies that the premises of Lemma 44 are satisfied. Therefore, (i) the incentive in the coordination block is satisfied and (ii) we can concentrate on the case with $x(j) = x(i)$ for all $i, j \in I$.

(i) and (ii) implies (8). In addition, by the law of large numbers, $x(j) = x$ for all $j \in -i$ in the coordination block with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Therefore, (4) is satisfied at the beginning of the review phase.

Finally, we consider (8) and (4) in the case without the noisy cheap talk. Again, suppose that $x(j) = x(i)$ for all $i, j \in I$ at the end of the coordination block. Then, (8) and (5) are verified as in the case with the perfect cheap talk except for the following two points:

- Player $i-1$ makes player i indifferent between any action profile sequence because of $g[i-1](m) = E$ or $\theta_{i-1}(j \rightarrow_m n) = B$ with higher probability. However, the probability of $g[i-1](m) = E$ or $\theta_{i-1}(j \rightarrow_m n)$ is bounded by $O(\eta)$. Hence, re-taking η sufficiently small as we do in (141), we can deal with this problem as in the two-player case.

- When player $j \in -i$ sends a message m to player $n \in -(i, j)$, player i can manipulate the distribution of $f[n](m)$. However, Lemma 45 implies that player i cannot manipulate $\theta_{i-1}(j \rightarrow_m n)$. $f[n](m)$ matters for player i's continuation payoff if and only if $\theta_{i-1}(j \rightarrow_m n) = G$. Hence, the relevant events for player i are

  - $f[n](m) = m$ and $\theta_{i-1}(j \rightarrow_m n) = G$, or
  - $f[n](m) \neq m$ or $\theta_{i-1}(j \rightarrow_m n) = B$.

  Since $f[n](m) \neq m$ implies $\theta_{i-1}(j \rightarrow_m n) = B$, the relevant histories for player i are

  - $\theta_{i-1}(j \rightarrow_m n) = G$, or

- $\theta_{i-1}(j \to_m n) = B$.

  Since player i cannot manipulate $\theta_{i-1}(j \to_m n)$ by Lemma 45, player i does not have an incentive to manipulate $f[n](m)$.

To verify the incentives in the coordination block, we consider the premises of Lemmas 43 and 44 in the case without the noisy cheap talk.

The premise 1 of Lemmas 43 and premise 1-(a) of Lemma 44 are satisfied by Lemma 45.

As we have mentioned above, when player $j \in -i$ sends a message m to player $n \in -(i,j)$, player i does not have an incentive to manipulate $f[n](m)$. Therefore, the premise 2 of Lemmas 43 and premise 1-(b) of Lemma 44 are satisfied.

We are left to verify the premise 1-(c) of Lemma 44: Player i's value is almost the same between $x_n(j) = G$ for all $j \in I$ and $x_n(j) = B$ for all $j \in I$ regardless of $\{x_{n'}(j)\}_{j \in I, n' \neq n}$. To formally show this, we proceed backward from player N's state. There are following two cases:

- Suppose that $x_{n'}(j) \neq x_{n'}(j')$ happens for some $n' \in \{1, ..., N-1\}$, $j \in I$ and $j' \in -j$. Then, by definition of $\{\theta_{i-1}(j'' \to_m n'')\}_{j'',n''}$ and 2 of Lemma 44,[104] player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $\theta_{i-1}(\mathrm{c}) = B$ in the coordination rounds for $x_{n'}$ or that there exist $j'' \in I$ and $n'' \in -j''$ such that $\theta_{i-1}(j'' \to_m n'') = B$ happens when player $j'' \in N(n')$ sends a message m to player n'' in the coordination round 3 for $x_{n'}$. Therefore, if $x_{n'}(j) \neq x_{n'}(j')$ happens for some $n' \in \{1, ..., N-1\}$, $j \in I$ and $j' \in -j$, then player i is almost indifferent between any action profile sequence, which implies player i's value is almost constant.

- Suppose that $x_{n'}(j) = x_{n'}(j')$ for all $n \in \{1, ..., N-1\}$ and $j, j' \in I$. Then, if either $x_N(j) = G$ for all $j \in I$ or $x_N(j) = B$ for all $j \in I$ is the case, then we have verified that (4) holds with x replaced with $x(j)$. Since $i \neq N+1$, player i's value is almost the same between $x_N(j) = G$ for all $j \in I$ and $x_N(j) = B$ for all $j \in I$.

[104] 2 of Lemma 44 does not use the premises 1-(a), 1-(b) and 1-(c).

Therefore, 1-(c) of Lemma 44 holds for $n = N$. This implies that each player follows the equilibrium path in the coordination rounds for x_N. Hence, at the end of the coordination rounds for x_{N-1}, each player i believes that $x_N(j) = x_N$ for all $j \in I$ or $\theta_{i-1}(j \rightarrow_m n) = B$ in the coordination round 3 for x_N between some $j \in I$ and $n \in -j$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Hence, the same argument as for $n = N$ holds for $n = N - 1$. By induction, we are done.

Therefore, all the premises in Lemmas 43 and Lemma 44 are satisfied. This implies that

1. It is almost optimal for player i to follow the equilibrium strategy in the coordination block.

2. For any i, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or "$\theta_{i-1}(j \rightarrow_m n) = B$ or $\theta_{i-1}(\mathrm{c}) = B$ happens in the coordination block."

Note that 1 implies the almost optimality of $\sigma_i(x_i)$ in the coordination block and that 2 implies the almost optimality of $\sigma_i(x_i)$ in the main blocks. Hence, (8) is verified.

Since we have verified (4) for $x(j) = x(i)$ for all $i, j \in I$, we are left to show (4) at the beginning of the review phase. Compared to the case with the noisy cheap talk, we need to deal with the fact that $g[n-1](m) = E$ and $\theta_{i-1}(j \rightarrow_m n) = B$ can happen when player j sends a message m to player n in the coordination block with higher probability. However, since the ex ante probability of $\theta_{i-1}(j \rightarrow_m n) = B$ for some $j \in I$, $n \in -j$ and m is bounded by $O(\eta)$, re-taking η smaller if necessary, we are done.

51 Report Block

We are left to construct the report and re-report blocks to attain the exact optimality of the equilibrium strategies. In this section, we explain the report block.

Contrary to the two-player case, we directly construct the report block without public randomization or any cheap talk.

51.1 Structure of the Report Block

The report block proceeds as follows:

1. Player N sends the messages about h_N^{main}.

2. Player $N-1$ sends the messages about h_{N-1}^{main}.

   $\vdots$

3. Player 3 sends the messages about h_3^{main}.

4. As in the two-player case, players 1 and 2 coordinate on which of them will send messages:

   (a) Each player takes a^G and each player i observes her private signal y_i.

   (b) If player 2 observes $y_2 \in Y_{2,1}^2$, then player 2 sends the message that $y_2 \in Y_{2,1}^2$ to player 1. Otherwise, that is, if player 2 observes $y_2 \in Y_{2,2}^2$, then player 2 sends the message that $y_2 \in Y_{2,2}^2$ to player 1.

5. If player 2 has sent the message $y_2 \in Y_{2,1}^2$, then player 2 sends the meaningful messages about h_2^{main}. If player 2 has sent the message $y_2 \in Y_{2,2}^2$, then player 2 takes a_2^G for the periods where player 2 would send the messages about h_2^{main} otherwise.

6. Player 1 sends the message about h_1^{main}.

7. The players play the round for conditional independence.

We explain each step in the sequel.

51.2 Player $i \geq 3$ sends h_i^{main}

Since there is a chronological order for the rounds and r is a generic serial number of rounds, the notations $\#_i^r$, $\#_i^r(k)$, $T(r,k)$ and $\{a_{i,t}, y_{i,t}\}_{t \in T(r,k)}$ defined in the Supplemental Material 3 is still valid.

Player i sends the messages about h_i^{main} in the same way as player 2 sends the messages in the Supplemental Material 4 with two players.

That is, for each round r,

1. First, player i reports $\#_i^r$.

2. Second, player i reports $\{\#_i^r(k)\}_{k\in\{1,\dots,K\}}$. See Section 44.2.2 for the definition of K.

3. Third, players i and $i-1$ coordinate on $k(r)$ as players 2 and 1 coordinate on $k(r)$ in Section 44.2.2.

4. Fourth, player i sends $\{a_{i,t}, y_{i,t}\}_{t\in T(r,k(r,i))}$. $k(r,i)$ is the result of the coordination on $k(r)$.

In Steps 1, 2 and 4, player i sends a message as player 2 does in the Supplemental Material 4 and player $i-1$ interprets the message as player 1 does in the Supplemental Material 4: Player i takes $a_i \in \{a_i^G, a_i^B\}$, player $i-1$ takes $\bar{\alpha}_{i-1}$ and players $-(i-1,i)$ take $a_{-(i-1,i)}^G$. Player $i-1$ constructs $\Phi_{i-1} \in \{0,1\}$ from $\phi_{i-1}(a_{i-1}, y_{i-1})$ as player 1 constructs $\Phi_1 \in \{0,1\}$ from $\phi_1(a_1, y_1)$ in the Supplemental Material 4. Then, player $i-1$ infers player i's message from Φ_{i-1} as player 1 infers player 2's message from Φ_1. Then, from Lemma 39, player i cannot infer Φ_{i-1} from player i's signals.

In Step 3, the coordination between player i and $i-1$ is the same as in Section 44.2.2 with j replaced with $i-1$. Assumption 16 implies that this is a well defined procedure.

51.3 Player 2 sends h_2^{main}

Player 2 sends the messages about h_2^{main} as player $i \geq 3$ if and only if player 2 observed $y_2 \in Y_{2,1}^2$ in Step 4 of Section 51.1. If player 2 observes $y_2 \in Y_{2,2}^2$, then player 2 takes a_2^G for periods where player 2 would send $\#_2^r$, $\{\#_2^r(k)\}_{k\in\{1,\dots,K\}}$ and $\{a_{2,t}, y_{2,t}\}_{t\in T(r,k(r,2))}$ otherwise. In addition, the coordination on $k(r)$ between players 2 and 1 is the same as in the Supplemental Material 4 (with the other players $-(1,2)$ taking $a_{-(i,2)}^G$). Assumption 16 implies that this is a well defined procedure.

As for the case with $i \geq 3$, player 2 takes $a_2 \in \{a_2^G, a_2^B\}$, player 1 takes $\bar{\alpha}_1$ and players $-(1,2)$ take $a_{-(1,2)}^G$. Player 1 constructs $\Phi_1 \in \{0,1\}$ from $\phi_1(a_1, y_1)$ to infer player 2's message. Lemma 39 guarantees that player 2 cannot infer Φ_1 from player 2's signals.

51.4 Player 1 sends h_1^{main}

Player 1 sends the messages about h_1^{main} to player N as player $i \geq 3$. As in the two-player case, player 1 takes $a_1 \in \{a_1^G, a_1^B\}$ and players -1 take a_{-1}^G.

After that, player 1 sends the histories in the report block to player N as player 1 does to player 2 in the round for conditional independence in Section 44.4.1. Again, this set of periods is called "the round for conditional independence." In this round, player 1 takes some action $a_1 \in A_1$ and players -1 take a_{-1}^G. Player N infers this message from y_N. By 7 of Assumption 15, player N can statistically identify player 1's action.[105]

From the history in the round for conditional independence, player N constructs Φ_N. Compared to the two-player case, player 2 is replaced with player N.

51.5 Reward Function π_i^{report}

First, for each i, player $i-1$ gives the reward for player i that cancels out the instantaneous utility. When player $n \in -1$ sends the message about h_n^{main}, player $i-1$ gives

$$\delta^{t-1}\pi_i^{x_{i-1}}(\text{report}, n, a_{i-1}, y_{i-1})$$

to player i. (166) implies the payoff of each player i is constant for any action.

When player 1 reports h_1^{main}, player 1 takes $\left\{a_1^G, a_1^B\right\}$ and players -1 take a_{-i}^G.[106] Hence, by 7 of Assumption 15, for each player i, player $i-1$ can cancel out the differences in player i's instantaneous utilities by

$$\delta^{t-1}\pi_i^{x_{i-1}}(1, a_{i-1}, y_{i-1}).$$

[105] Since we use Assumption 15, $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15, not in Assumption 17. Note that, for notational simplicity, we use the same notations for different assumptions.

[106] Remember that $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15.

Next, we consider the reward in the round for conditional independence. As we will see, player 1 sends in the re-report block what action player 1 takes in each period in the round for conditional independence. Hence, for player $i \in -1$, player $i-1$ will know a_1 from the re-report block and player $i-1$ knows that players $-(1,i)$ take $a^G_{-(1,i)}$. For player $i=1$, since players -1 take a^G_{-1}, player $i-1=N$ knows a_{-i} without the messages in the re-report block. For each i, player $i-1$ gives

$$\delta^{t-1}\pi_i^{x_{i-1}}(a_{-i}, y_{i-1})$$

to cancel out the difference of player i's instantaneous utilities.

On the top of that, while the players should take a^G to coordinate on $k(r)$ or whether player 2 reports the history, player $i-1$ incentivizes player i to take a_i^G. Since player $i-1$ knows that players $-i$ take $a^G_{-i} \in A_{-i}$, player $i-1$ can construct a strict reward on a_i^G from Lemma 16.

In the report block, when player i sends the message, no player $j \in -i$ has an incentive to manipulate player $(i-1)$'s inference of player i's message since player i's message only affects player $(i-1)$'s reward on player i and we construct the structure of the report block in Section 51.1 and the punishment for telling a lie, $g_j(h^{\text{main}}_{j-1}, h^{\text{rereport}}_{j-1}, \hat{a}_{j,t}, \hat{y}_{j,t})$, so that player j does not have an incentive to learn player i's history from the report block.

Finally, we construct π_i^{report} that makes $\sigma_i(x_i)$ exactly optimal. This step is the same as in Section 36 except for the following:

- $\varphi_{n,t}$ for each round r is defined as follows:
  - If round r corresponds to the coordination round 1 for x_j with some $j \in I$ where player n infers player j's message x_j (that is, player n is $n^*(j)$), then $\varphi_{n,t}$ is $\Psi^j_{n,t}$ defined in (168).
  - If round r corresponds to the coordination round 1 for x_j with some $j \in I$ where player n is not $n^*(j)$, then $\varphi_{n,t}$ is $\{\emptyset\}$.

- If round r corresponds to the coordination round 2 for x_j with some $j \in I$ where player n receives a message from $n^*(j)$ (that is, player n is in $N(j) \setminus \{n^*(j)\}$), then $\varphi_{n,t}$ is $\boldsymbol{I}_{n,t}(j)$ defined in (169).
- If round r corresponds to the coordination round 2 for x_j with some $j \in I$ where player n is not in $N(j) \setminus \{n^*(j)\}$, then $\varphi_{n,t}$ is $\{\emptyset\}$.
- If round r corresponds to the coordination round 3 or the supplemental round, then $\varphi_{n,t}$ is $\{\emptyset\}$.
- If round r corresponds to the review round, then $\varphi_{n,t}$ is the same as in Section 36.

- $t_{i-1}(r)$ is not defined for a round in the coordination block or supplemental round if player $(i-1)$'s continuation strategy does not depend on player $(i-1)$'s history in that round. In that case, player $i-1$ randomly picks one.

- For a round in the coordination block where player i takes a mixed strategy to send a message, we (i) first cancel out the effect of the history in the round on learning about the best responses from the next rounds, and (ii) second make any action sequence is indifferent ex ante. Since player i believes that player i is almost indifferent between any strategies whenever player i sends a message, this treatment is the same as we incentivize player i to take a mixed minimaxing strategy in the review round.

We are left to deal with the probability that the message does not transmit correctly with probability 1. We deal with this problem in Section 53 after we explain the re-report block.

52 Re-Report Block

As in Section 37, we introduce the re-report block so that, for each player i, player $i-1$ can collect the information necessary to construct π_i from players $-(i-1,i)$.

The basic structure of the re-report block is the same as in Section 37:

- Players $-(N-1,N)$ sends the information to player $N-1$ to construct π_N.
- Players $-(N-2,N-1)$ sends the information to player $N-2$ to construct π_{N-1}.

  $\vdots$

- Players $-(1,2)$ sends the information to player 1 to construct π_2.
- Players $-(N,1)$ sends the information to player N to construct π_1.

When players $-(i-1,i)$ sends the information to player $i-1$, each player takes turns to send the information:

- Player 1 sends the information to player $i-1$ if $1 \in -(i-1,i)$. If $1 \notin -(i-1,i)$, then skip this step.

  $\vdots$

- Player N sends the information to player $i-1$ if $N \in -(i-1,i)$. If $N \notin -(i-1,i)$, then skip this step.

When player $n \in -(i-1,i)$ sends the information about her history, she sends the following information chronologically:

- For each round r, what strategy α_n player n took in round r. Note that this contains the information about what message player n sent if player n sends a message in that round. The cardinality of this message is no more than

$$|A_n| + \underbrace{N-1}_{\substack{\text{the mixed strategy is taken} \\ \text{only if player } n \text{ sends } z_j(m)=M \\ \text{or minimaxes another player}}}.$$

- For each round r, for each (a_n, y_n, φ_n), how many times player n observed (a_n, y_n, φ_n). Note that this contains the information about what was player n's inference of a message if player n receives a message in that round. The cardinality of this message is no more than $T^{O(1)}$.

- Note that the above two pieces of information is sufficient for player $i-1$ to construct $\theta_{i-1}(j \to_m n)$.

- For each round r, what was $t_n(r)$. The cardinality of this message is no more than T.

- At the end of each lth review round, what was the realization of player n's randomization for the construction of some states. The cardinality of this message is a finite fixed number.

- So that player $i-1$ knows $(a_{-i,t})_{t \in T(r,k(r,i))}$ and $\left(y_{n,t}, \varphi_{n,t}\right)_{t \in T(r,k(r,i))}$,

  - first, for each r, player $i-1$ sends the message about $k(r,i)$ to players $-(i-1,i)$.[107] Each player $n \in -(i-1,i)$ infers $k(r,i)$ from their private signals. Let $k_n(r,i)$ be player n's inference. The cardinality of this message is no more than $T^{\frac{3}{4}}$.

  - Second, player n sends the messages about $\left(a_{n,t}, y_{n,t}, \varphi_{n,t}\right)_{t \in T(r,k_n(r,i))}$ to player $i-1$. The cardinality of this message is $\exp(O(T^{\frac{1}{4}}))$.

- If player n is player 1, then player 1 sends the message about player 1's history in the round for conditional independence: $(a_{1,t}, y_{1,t})$ for all t in the round for conditional independence. Since the length of the round for conditional independence is $S \sum_r \left| T\left(r, \hat{k}_n(r,i)\right) \right| = O(T^{\frac{1}{4}})$, the cardinality of this message is $\exp(O(T^{\frac{1}{4}}))$.

Therefore, the cardinality of the whole message is $\exp(O(T^{\frac{1}{4}}))$ and the length of the sequence of binary messages $\{G, B\}$ necessary to encode the information is $O(T^{\frac{1}{4}})$. To send a binary message $m \in \{G, B\}$, player n repeats a_n^m for $T^{\frac{1}{3}}$ times to increase the precision. The other players $-n$ take a_{-n}^G.[108]

By 7 of Assumption 15, player $i-1$ can statistically identify player n's action. Also, for each player j, player $j-1$ can cancel out the differences in player j's instantaneous utilities by

[107] We assume that player $i-1$ knew player i's inference $k(r,i)$. See Section 53 for how to deal with the small probability that player $i-1$ misinterprets player i's message about $k(r,i)$.

[108] Since we use Assumption 15, $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15, not in Assumption 17. Note that, for notational simplicity, we use the same notations for different assumptions.

the reward. The incentive to tell the truth is automatically satisfied since player n's message is used only for the reward on player i with $i \neq n$ except for the round for conditional independence. The incentive in the round for conditional independence is established as in Lemma 31.

53 The Probability of Errors in the Report and Re-Report Blocks

Note that the cardinality of the whole messages in the report and re-report blocks is $\exp(O(T^{\frac{1}{4}}))$. Hence, the length of the sequence of binary messages $\{G, B\}$ that each player takes to send the messages in the report or re-report block is $O(T^{\frac{1}{4}})$.

Since all the messages transmit correctly with probability at least

$$1 - O(T^{\frac{1}{4}}) \exp(-O(T^{\frac{1}{3}})),$$

by the same treatment as in Section 44.3, we can assume as if all the messages would transmit correctly. We do not apply this procedure for the messages in the round for conditional independence. As seen in Lemma 31, the incentive in the round for conditional independence is established taking into account the probability of mis-transmission.