1. Directed Technological Change

Consider the following model. There is a representative household with a utility function:

$$U = \int_0^\infty e^{-\rho t} \log C(t)\, dt$$

(I will drop time t in all the variables when no ambiguity occurs). The household simultaneously and inelastically supplies two types of labor, H units of high skilled labor and L units of low skilled labor. There is no population growth.

There are three types of firms in the economy. First, there is a perfectly competitive final good producer with technology:

$$Y = Y_H^{\varepsilon} Y_L^{1-\varepsilon}$$

where Y_H is a high-skill-intensive intermediate good with price p_H and Y_L is a low-skill-intensive intermediate good with price p_L. We will take the final good as our numeraire and, hence, all the other prices in the economy are expressed in terms of the final good. We will call the relative price of these two intermediate goods

$$p = \frac{p_H}{p_L}.$$

Second, there is a perfectly competitive producer of Y_H with technology

$$Y_H = \frac{1}{\alpha}\left(\int_0^{N_H} x_H(v)^{\alpha}\, dv\right) H^{1-\alpha}$$

that produces the good with inputs $x_H(v)$ with range $[0, N_H]$ and high skilled labor H paid a wage w_H and a perfectly competitive producer of Y_L with technology

$$Y_L = \frac{1}{\alpha}\left(\int_{N_H}^{N_H+N_L} x_L(v)^{\alpha}\, dv\right) L^{1-\alpha}$$

that produces the good with inputs $x_L(v)$ a range $[N_H, N_H + N_L]$ and low skilled labor L paid a wage w_L. All inputs fully used up in the production process. Also, we will define a relative wage

$$\omega = \frac{w_H}{w_L}.$$

Third, inputs are supplied by monopolists who innovated in the past and were awarded a patent. They sell their inputs at prices $p_H(v)$ and $p_L(v)$ respectively. The production of inputs is done at a (constant) marginal cost α

Innovation in inputs is given by a specification:

$$\begin{aligned} \dot{N}_H &= \eta_H Z_H \\ \dot{N}_L &= \eta_L Z_L \end{aligned}$$

with some initial $N_H(0)$ and $N_L(0)$ where Z_H and Z_L are the amount of research in each input sector and η_H and η_L are parameters. There is free-entry into innovation.

For this exercise, you can take as given the result that, since we are picking the final good as our numeraire, the price index satisfies:

$$1 = \left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon} \left(\frac{1}{\varepsilon}\right)^{\varepsilon} p_H^{\varepsilon} p_L^{1-\varepsilon}$$

(hint: use it only at step 6. below).

1. Write down the resource constraint of the economy in terms of the final good.
2. Write down the problem of the final good producer and find its optimality conditions.
3. Write down the problem of each intermediate good producer and find its optimality conditions.
4. Write down the problem of each input producer and find its optimality conditions.
5. Define an equilibrium for this economy.
6. Characterize the Balanced Growth Path (BGP) of this economy. Among other things, you need to:
    1. Find the price of each variety.
    2. Write the Hamiltonian equation of each monopolist.
    3. Argue that, indeed, there is a BGP and that is unique.
    4. Find an expression that relates H and L with the growth rate of the economy along the BGP.
    5. Find an expression for $\omega = \frac{w_H}{w_L}$.
7. Are there transitional dynamics in this economy? Justify your answer.

Finally C also grows at rate g and we find:

$$\begin{aligned}
\frac{\dot{C}}{C} &= r - \rho = g \\
g &= r - \rho \\
&= \eta_L \left(1-\alpha\right) \left(\left(1-\varepsilon\right)^{1-\alpha\varepsilon} \varepsilon^{\alpha\varepsilon}\right)^{\frac{1}{1-\alpha}} \left(\frac{N_H H}{N_L L}\right)^{\varepsilon} L - \rho \\
&= \left(1-\alpha\right) \left(1-\varepsilon\right)^{\left(1-\varepsilon\right)\frac{\alpha}{1-\alpha}} \varepsilon^{\frac{\varepsilon}{1-\alpha}} \left(\eta_H H\right)^{\varepsilon} \left(\eta_L L\right)^{1-\varepsilon} - \rho
\end{aligned}$$