

# Robust Data-Driven Efficiency Guarantees in Auctions

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## Abstract

Analysis of welfare in auctions comes traditionally via one of two approaches: precise but fragile inference of the exact details of a setting from data or robust but coarse theoretical price of anarchy bounds that hold in any setting. As markets get more and more dynamic and bidders become more and more sophisticated, the weaknesses of each approach are magnified.

In this paper, we provide tools for analyzing and estimating the *empirical price of anarchy* of an auction. The empirical price of anarchy is the worst case efficiency loss of any auction that *could have produced the data*, relative to the optimal.

Our techniques are based on inferring simple properties of auctions: primarily the expected revenue and the expected payments and allocation probabilities from possible bids. These quantities alone allow us to empirically estimate the revenue covering parameter of an auction which allows us to re-purpose the theoretical machinery of Hartline et al. [2014] for empirical purposes. Moreover, we show that under general conditions the revenue covering parameter estimated from the data approaches the true parameter with the error decreasing at the rate proportional to the square root of the number of auctions and at most polynomially in the number of agents. While we focus on the setting of position auctions, and particularly the generalized second price auction, our techniques are applicable far more generally.

Finally, we apply our techniques to a selection of advertising auctions on Microsoft's Bing and find empirical results that are a significant improvement over the theoretical worst-case bounds.

## 1 Introduction

Evaluation of the revenue and welfare of market mechanisms has been one of the key questions in Economics. A typical question of interest is the comparison of a currently deployed mechanism with the best solution implemented by a central planner, taking into account the incentives of participating Economic agents. The *price of anarchy*, first introduced in Koutsoupias and Papadimitriou [1999] for network routing games, provides a bound on the ratio of the revenue or welfare from the implementation of the second best and the current mechanism over all possible uncertainties in the market.

The worst-case nature of price-of-anarchy bounds results in very robust results, but this robustness can come at the cost of bounds that are coarse for an analyst interested in understanding the performance of a currently deployed mechanism. In some types of games this is not a problem because performance can be empirically measured: processing time or memory usage can be measured; route choice and delay in a network can be tracked, and compared to another benchmark.

However, in auctions and other settings where agents have private information that impacts the objectives of a system, estimating the performance has traditionally required learning that private information. Oftentimes the results from this style of analysis are also very sensitive to the exact decision making of bidders, and for instance are not robust to bidders who choose only the approximately best action, or play learning strategies.

In this paper, we bridge the robust but coarse theoretical price of anarchy bounds and precise but fragile inference based bounds, by integrating data directly into the price of anarchy style analysis. Instead of quantifying over all settings and uncertainties, we take the worst case over all settings and uncertainties that could induce the observed data. The more we know about the data generated by a mechanism, the higher the potential for an accurate bound.

Many real-world mechanisms are implemented in highly dynamic settings, where the assumptions of perfect best-response are highly suspect. Cases in point include advertising auctions on Google, Facebook and Bing, markets for electricity and treasury bill auctions. In such dynamic settings it becomes hard to justify the common knowledge assumptions needed for the full evaluation of the Bayes-Nash equilibrium. For instance, in the sponsored search auctions the bidders use sophisticated algorithmic bidding tools that dynamically explore the structure of the empirical best response correspondence and optimize bids in continuous time. Moreover, given the dynamic market structure, the underlying primitives of the model, such as valuations of participating agents may significantly change over time. As a result, the welfare analysis in the context of traditional structural modeling approach, which requires first to estimate the primitives of the model and then to compute the equilibrium outcome of the current and the optimal mechanism, becomes very hard if not impossible.

Our approach benefits from the inherent robustness of worst-case analysis to realistic market features such as differences in details of mechanisms or agents who only approximately best-respond. At the same time, our approach uses the data and effectively informs the price of anarchy bound regarding the “worst case scenario” distributions of uncertainty that are clearly inconsistent with the observed data. That allows us to improve the welfare and revenue bounds given by the theoretical price of anarchy.

## 1.1 Methods

**Theoretical** Our theoretical techniques for proving empirical Price of Anarchy bounds are largely an empirical application of the revenue-covering framework of Hartline et al.

[2014], targetted to position auctions. First, we analyze the optimization problem of a bidder, comparing actions in the auction based on their expected price-per-click (or first-price equivalent bid in the terminology of Hartline et al. [2014]).

Second, we relate the revenue of an auction to a threshold quantity, which is based on how expensive allocation is. We call this empirical revenue covering, and differs from revenue covering of Hartline et al. [2014] only in that we measure it for a given instance of an auction instead of proving it for every possible strategy profile. As a result, our empirical revenue covering framework applies even more broadly than theoretical revenue covering: it can be measured for *any Bayes-Nash Equilibrium* of *any* single-parameter auction in the independent, private values model.

Finally, we consider and measure how agents would react to the optimization problem that they are faced with. In the terminology of Hartline et al. [2014], we measure the value-covering of the auction, which improves on the  $1 - \frac{1}{e}$  term. This can be done both with precise knowledge of price-per-click allocation rule, or with rough knowledge of concentration bounds on the price-per-click allocation rule.

Our general approach can also be seen as reducing the empirical analysis of an auction to the econometric question of estimating the revenue of an auction and estimating the allocations and prices-per-click of actions in the auction.

**Econometrics** Our econometric approach is based on recovering the price and allocation functions from the auction data: provided that we observe the realizations of uncertainty (regarding the bids of participating agents and the scores assigned to the agents by the mechanism), the empirical approximation to the price and allocation functions for each agent are based on computing the average price and average allocation for each possible bid across historical auctions. We demonstrate that if the auction logic is “relatively simple” (i.e. it is based on the finite list of ranking and comparison operations), then both price and allocation functions can be recovered accurately uniformly over the bid space. This property further allows us establish the convergence of the empirical price of anarchy bound to its population counterpart. As find that the statistical noise *only has a second order effect* on the recovered empirical price of anarchy. We also note that our results extend beyond the standard i.i.d. data settings allowing us to consider complex serially correlated time series data that satisfy the  $\beta$ -mixing conditions which is compatible with various learning dynamics.

## Robustness

Our results adopt a robustness to changes in the mechanism or the setting that is similar to the inherent robustness of results from the revenue covering or smoothness frameworks.

- **Beyond Position Auctions.** While all of our analysis is based on the position auction model with uncertain quality scores, the analysis is general and can be applied for Bayes-Nash Equilibria of other auctions and feasibility environments as well.

- **Changes in the Mechanism.** As the thresholds we calculate are based on the price-per-click allocation rule of a bidder, threshold quantities can be compared and computed no matter what the mechanism is as long as these quantities can be estimated. If for instance an auctioneer is A/B testing many different auctions, the same analysis can be used for their comparisons as long as the prices-per-click and allocations from actions can be learned.
- **Approximate Equilibrium.** If the agents in an auction only  $\epsilon$ -best respond to the optimization problem that they are faced with, then our efficiency results only degrade by that  $\epsilon$ .  
Moreover, if some agents are irrational and some are rational, then our results can be broken out to give efficiency results only for the rational bidders.
- **Learning Quality Scores.** We model the quality score of a bidder as coming from a known distribution. This distribution should be interpreted as the auctioneer's knowledge of the quality score of the bidder. Our efficiency results give a comparison to the optimal auction *subject to the same knowledge of quality scores of the bidders*. This distribution moreover can have arbitrary correlations, as it only really affects the space of feasible allocations.

## 1.2 Contributions

Our primary contributions are the following:

- **Empirical Price of Anarchy.** We introduce the *empirical Price of Anarchy*(EPoA) benchmark for welfare, representing the worst case efficiency loss of a game consistent with a distribution of data from the game.
- **Empirical Revenue Covering.** We refine the revenue-covering framework of Hartline et al. [2014] for proving robust EPoA bounds, and show that we can empirically estimate the empirical revenue covering of the Generalized Second Price auction with very fast convergence properties.
- **Data.** We apply and bound the empirical price of anarchy from GSP advertising auctions run in Microsoft's Bing, and show that we get EPoA bounds that are significantly stronger than the relevant theoretical bounds.

## 1.3 Related Work

Our approach for empirical revenue covering is primarily an empirical application and refinement of the revenue covering framework in Hartline et al. [2014], which uses the revenue-covering property to prove theoretical bounds for auctions that always satisfy the revenue-covering property. The revenue covering approach itself is a refinement of the

smooth-games and mechanisms frameworks of Roughgarden [2009, 2012]; Syrgkanis and Tardos [2013] for the Bayesian setting. Hartline et al. [2014] also give revenue approximation results for the first-price auction with the optimal reserve prices, which we do not. Our techniques could be used for improved bounds for revenue if the optimal reserve prices were known and implemented, or with the assumption of more symmetry in the setting.

The notion of thresholds and revenue covering is also strongly related to the threshold and the  $c$ -threshold approximate concept in Syrgkanis and Tardos [2013].

The efficiency of the Generalized Second-Price auction (GSP) was originally modeled and studied in full-information settings in Edelman et al. [2007] and Varian [2009]. Gomes and Sweeney [2014] characterize equilibrium in the Bayesian setting, and give conditions on the existence of efficient equilibria. Athey and Nekipelov [2010] give a structural model of GSP with varying quality scores, which are included in our model. Caragiannis et al. [2014] explores the efficiency of GSP in the Bayesian setting, and finds a theoretical price of anarchy for welfare of 2.927 when the value distributions are independent or correlated, and players do not overbid. Our results apply only for independent distributions of values, but do not need the no-overbidding assumption. However, our approach applies even to the modified GSP which is being used where the scoring rule used for ranking is not equal to the quality  $\gamma_i$  of a bidder. The latter theoretical results do not apply for this modified GSP. Hence, in principle we could observe higher inefficiency in the data than the theoretical bound above. Despite this fact we find in the data that only better inefficiency bounds are derived, with the exception of one search phrase where we almost exactly match the latter worst-case theoretical bound. The semi-smoothness based approach of Caragiannis et al. [2014] can be seen through our model as using a welfare covering property in place of revenue covering.

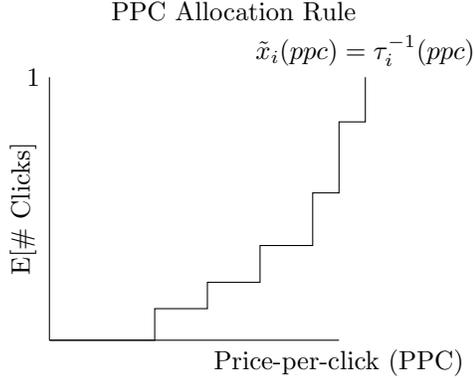
## 2 Preliminaries

We consider the position auction setting, with  $m$  positions and  $n$  bidders. Each bidder  $i$  has a private value  $v_i$  drawn independently from distribution  $F_i$  over the space of possible values  $V_i$ . We denote the joint value-space and distribution over values  $\mathbf{V} = \prod_i V_i$  and  $\mathbf{F} = \prod_i F_i$  respectively. Bidders have a linear utility, so if they pay  $P_i$  to receive a probability of service  $x_i$ , the utility of the bidder is  $u_i = v_i x_i - P_i$ .

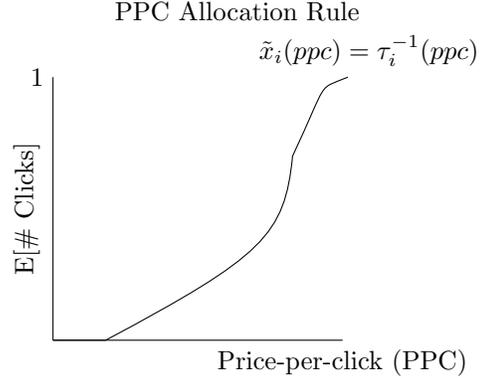
An outcome  $\pi$  in a position auction is an allocation of positions to bidders.  $\pi(j)$  denotes the bidder who is allocated position  $j$ ;  $\pi^{-1}(i)$  refers to the position assigned to bidder  $i$ . Henceforth we will adopt the terminology of ad auctions and refer to service as a ‘click’.

When bidder  $i$  is assigned to slot  $j$ , the probability of click  $c_{i,j}$  is the product of the click-through-rate of the slot  $\alpha_j$  and the quality score of the bidder,  $\gamma_i$ , so  $c_{i,j} = \alpha_j \gamma_i$ . We will generally assume that  $\gamma_i$  is drawn independently from distribution  $\Gamma_i$ , and is observable to the auctioneer, but *not* to the bidder themselves.

Since the auctioneer can use the quality scores in assigning bidders to slots and the



(a) In full-information GSP, the PPC allocation rule is piecewise constant with a step per slot, where the price-per-click corresponding with a slot is the bid of the agent in the next slot.



(b) In Bayesian GSP, the number of clicks from a bid is a smoother function. Note that since a bid might result in winning a number of different slots, the number of clicks comes from all possible slots.

quality scores impact the number of clicks that each agent sees, an allocation  $\mathbf{x}$  is feasible if and only if there is a quality-score dependent assignment of slots to bidders that gives rise to this allocation.

Denote by  $\rho(\gamma, \cdot)$  such an assignment, where  $\rho(\gamma, j)$  is the player who is assigned position  $j$  when the quality score profile is  $\gamma$  and  $\rho^{-1}(\gamma, i)$  is the position assigned to player  $i$ . Moreover, denote with  $\mathcal{M}$  the space of all such quality score dependent assignments. Then an allocation  $\mathbf{x}$  is feasible if there exists  $\rho \in \mathcal{M}$  such that for each bidder  $i$ :  $x_i = \mathbf{E}_\gamma[a_{\rho^{-1}(\gamma, i)} \gamma_i]$ . Call  $\mathbf{X}$  the set of all feasible allocations.

A position auction  $A$  consists of a bid space  $\mathcal{B}$ , allocation rule  $\mathbf{x} : \mathcal{B}^n \rightarrow \mathbf{X}$  mapping from bid profiles to feasible allocations and payment allocation rule  $\mathbf{P} : \mathcal{B}^n \rightarrow \mathbb{R}^n$  mapping from bid profiles to payments. A strategy profile  $\sigma : \mathbb{R}^n \rightarrow \mathcal{B}^n$  maps values of agents to bids. For a set of values  $\mathbf{v}$ , the utility generated for each bidder is  $U_i(\mathbf{b}; v_i) = v_i x_i(\sigma(\mathbf{b})) - P_i(\sigma(\mathbf{b}))$ .

Given a strategy profile  $\sigma$ , we will often use and consider the expected allocation and payment an agent expects to receive when playing an bid  $b_i$ , taking expectation over other agents values and the quality score  $\gamma_i$ . We call  $x_i(b_i) = \mathbf{E}_{\mathbf{v}_{-i}}[x_i(b_i, \sigma_{-i}(\mathbf{v}_{-i}))]$  the *interim bid allocation rule*. We define  $P_i(b_i)$  and  $u_i(b_i)$  analogously.

A strategy profile  $\sigma$  is in Bayes-Nash Equilibrium (BNE) if for all agents  $i$ ,  $\sigma_i(v_i)$  maximizes their interim expected utility: e.g., for all bids  $b'$ ,  $u_i(\sigma_i(v_i)) \geq u_i(b')$ .

The welfare from an allocation  $\mathbf{x}$  is the expected utility generated for both the bidders and the auctioneer,  $\sum_i x_i v_i$ . Thus the expected utility of a strategy profile  $\sigma$  is

$$\text{WELFARE}(A(\sigma)) = \mathbf{E}_{\mathbf{v}} \left[ \sum_i x_i(\sigma_i(v_i)) v_i \right] \quad (1)$$

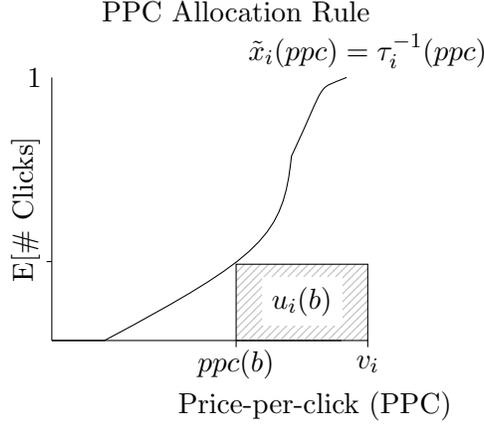


Figure 1: For any bid  $b$  with PPC  $ppc(b)$ , the area of a rectangle between  $(ppc(b), x_i(ppc(b)))$  and  $(v_i, 0)$  on the bid allocation rule is the expected utility  $u_i(b)$ . The BNE action  $b^*$  is chosen to maximize this area.

We will break down the welfare of the auction into the revenue paid to the auctioneer,  $\text{REV}(A(\sigma)) = \mathbf{E}_{\mathbf{v}}[\sum_i P_i(\sigma_i(\mathbf{v}))]$  payments made to the bidder and the utility derived from the agents,  $\text{UTIL}(A(\sigma)) = \mathbf{E}_{\mathbf{v}}[\sum_i u_i(\sigma(\mathbf{v}))]$ , with

$$\text{WELFARE}(A(\sigma)) = \text{REV}(A(\sigma)) + \text{UTIL}(A(\sigma))$$

Our benchmark for welfare will be the welfare of the auction that chooses a feasible allocation to maximize the welfare generated, thus  $\text{WELFARE}(\text{OPT}) = \mathbf{E}_{\mathbf{v}}[\max_{\mathbf{x}} \sum_i x_i v_i] = \mathbf{E}_{\mathbf{v}, \gamma}[\max_{\pi} \sum_i \gamma_i \alpha_{\pi^{-1}(i)} v_i]$ . We will denote the resulting optimal value-based allocation rule  $\mathbf{x}^*$ .

The (Bayesian) price-of-anarchy for welfare of an auction is defined as the worst-case ratio of welfare in the optimal auction to the welfare in an equilibrium, taken over all distributions and equilibriums

$$PoA(A) = \max_{\Gamma, \mathbf{F}, \sigma \in \text{BNE}(A, \mathbf{F})} \frac{\text{WELFARE}(\text{OPT})}{\text{WELFARE}(A)} \quad (2)$$

## 2.1 Sponsored Search Auction: model and data

We consider data generated by advertisers repeatedly participating in a sponsored search auction. The mechanism that is being repeated at each stage is an instance of a generalized second price auction triggered by a search query.

The rules of each auction are as follows<sup>1</sup>: Each advertiser  $i$  is associated with a click

<sup>1</sup>ignoring small details that we will ignore and are rarely in effect

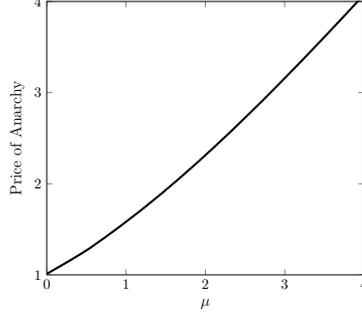
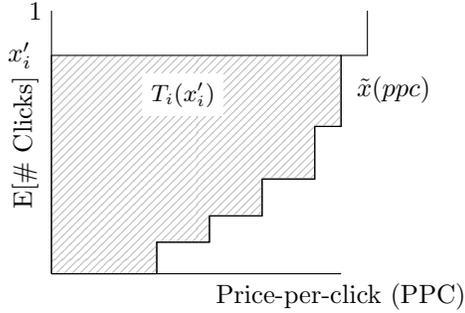
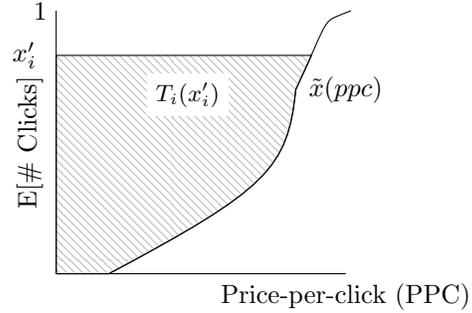


Figure 2: The price of anarchy of an auction which is  $\mu$ -revenue covered, either theoretically or empirically, is  $\frac{\mu}{1-e^{-\mu}}$ .



(a) In full-information GSP, the threshold for agent  $i$  corresponds to a summation over all worse slots than the one he receives.



(b) In Bayesian GSP, the threshold for bidder  $i$  in slot  $j$  is the area above the PPC allocation rule up to allocation of  $\alpha_j \gamma_i$  — note that it is *not* the expectation of the full-information threshold.

probability  $\gamma_i$  and a scoring coefficient  $s_i$  and is asked to submit a bid-per-click  $b_i$ . Advertisers are ranked by their rank-score  $q_i = s_i \cdot b_i$  and allocated positions in decreasing order of rank-score as long as they pass a rank-score reserve  $r$ . If advertisers also pass a higher mainline reserve  $m$ , then they may be allocated in the positions that appear in the mainline part of the page, but at most  $k$  advertisers are placed on the mainline.

If advertiser  $i$  is allocated position  $j$ , then he is clicked with some probability  $c_{i,j}$ , which we will assume to be separable into a part  $\alpha_j$  depending on the position and a part  $\gamma_i$  depending on the advertiser, and that the position related effect is the same in all the participating auctions:  $c_{i,j} = \alpha_j \cdot \gamma_i$ . We denote with  $\gamma = (\gamma_1, \dots, \gamma_m)$  the vector of position coefficients. All the mentioned sets of parameters  $\theta = (\mathbf{s}, \alpha, \gamma, r, m, k)$  and the bids  $\mathbf{b}$  are observable in the data. Moreover, the parameters and bids are known to the auctioneer at the allocation time. We will denote with  $\pi_{\mathbf{b},\theta}(j)$  the bidder allocated in slot  $j$  under a bid

profile  $\mathbf{b}$  and parameter profile  $\theta$ . We denote with  $\pi_{\mathbf{b},\theta}^{-1}(i)$  the slot allocated to bidder  $i$ .

If advertiser  $i$  is allocated position  $j$ , then he pays only when he is clicked and his payment, i.e. his cost-per-click (CPC) is the minimal bid he had to place to keep his position, which is:

$$\text{cpc}_{ij}(\mathbf{b}; \theta) = \max \left\{ \frac{s_{\pi_{\mathbf{b},\theta}(j+1)} \cdot b_{\pi_{\mathbf{b},\theta}(j+1)}}{s_i}, \frac{r}{s_i}, \frac{m}{s_i} \cdot \mathbf{1}\{j \in M\} \right\} \quad (3)$$

where with  $M$  we denote the set of mainline positions.

We also assume that each advertiser has a value-per-click (VPC)  $v_i$ , which is not observed in the data. If under a bid profile  $\mathbf{b}$ , advertiser  $i$  is allocated slot  $\pi_{\mathbf{b},\theta}^{-1}(i)$ , his expected utility is:

$$U_i(\mathbf{b}; v_i) = \mathbf{E}_\theta \left[ \alpha_{\pi_{\mathbf{b},\theta}^{-1}(i)} \cdot \gamma_i \cdot \left( v_i - \text{cpc}_{i\pi_{\mathbf{b},\theta}^{-1}(i)}(\mathbf{b}; \theta) \right) \right] \quad (4)$$

### 3 Price of Anarchy from Data

We begin by defining the empirical price-of-anarchy of an auction.

**Definition 1** (Empirical Price of Anarchy). *The Bayesian empirical price-of-anarchy for welfare of an auction and a distribution of data  $\mathcal{D}$  generated by the auction is the worst-case ratio of welfare in the optimal auction to the welfare in an equilibrium, taken over all distributions and equilibriums that could generate the distribution of data  $\mathcal{D}$ .*

$$EPoA(A, \mathcal{D}) = \max_{\Gamma, \mathbf{F}, \sigma \in BNE(A, \mathbf{F}, \Gamma) \cap D(A, \sigma) = \mathcal{D}} \frac{\text{WELFARE}(\text{OPT})}{\text{WELFARE}(A(\sigma))} \quad (5)$$

We use the notation  $D(A) = \mathcal{D}$  to denote that  $\mathcal{D}$  is the distribution of data produced by running the mechanism  $A$ , taken in expectation over all type distributions and the randomness of the auction. In our setting, these data will be quantities resulting from distributions of bids, including the expected revenue of the strategy profile.

#### 3.1 Empirical Revenue Covering Framework for Position Auctions

In this section, we refine the revenue covering framework of Hartline et al. [2014] for empirical bounds. Notably, we use a pointwise version of revenue-covering that applies for a given strategy profile and auction rather than taking the worst-case revenue covering over all strategy profiles.

The property of  $\mu$ -revenue covering is based only on the relationship between the expected revenue of an auction, and a property of the optimization problem that the bidders are solving (the expected threshold).

Both of these quantities are observable in the data, and hence by observing that an instance of an auction is  $\mu$ -revenue covered, we will get empirical price of anarchy bounds that apply for the auction we are observing.

The rest of this section will proceed in three parts:

1. Generate PPC Allocation rules: Analyze how to bid in the auction.
2. Measure  $\mu$ : Analyze the correspondence between thresholds and revenue in the auction.
3. Measure  $\lambda$ : Calculate the worst-case tradeoff between utility and thresholds in the auction.

**Generate PPC Allocation Rules** We first focus on the optimization problem each bidder faces. When bidding in an auction, each bidder must think about for each possible bid, how many clicks she will receive and how much she will have to pay on average for each click. In particular, the utility of an agent can be written to *only* include these terms:

$$\begin{aligned} u_i(b) &= v_i x_i(b) - P_i(b) \\ &= x_i(b) \left( v_i - \frac{P_i(b)}{x_i(b)} \right) \end{aligned} \tag{6}$$

The price-per-click term  $\frac{P_i(b)}{x_i(b)}$  term now plays exactly the same role in the utility function that the first-price bid does in the first price auction. We call this term  $ppc(b) = \frac{P_i(b)}{x_i(b)}$  the price-per-click of the bid in a position auction. Outside of position auctions, it is called the (first-price) equivalent bid in Hartline et al. [2014], because it plays the same role as a first-price bid does in a first-price style auction.

Our analysis will be based on the *price-per-click* allocation rule  $\tilde{x}(ppc)$ , which plots the expected number of clicks of bids against their prices-per-click. See Figure 1b for an illustration of the PPC allocation rule.

The utility of a bidder has a simple representation on the plot of the PPC allocation rule. The utility of a bidder is  $u_i(b) = x_i(b) (v_i - ppc(b))$ , which has a clean visualization on a plot of the PPC allocation rule: is the area of a rectangle between the points  $(ppc, \tilde{x}_i(ppc))$  and  $(v_i, 0)$ . See Figure 1 for an illustration.

**Thresholds & Revenue Covering** We will most often use the inverse of the PPC allocation rule for our analysis; let  $\tau_i(z) = \tilde{x}_i^{-1}(z)$  be the price-per-click of the cheapest bid that achieves allocation at least  $z$ . More formally,  $\tau_i(z) = \min_{b|x_i(b) \geq z} \{ppc(b)\}$ .

The *threshold* for agent  $i$  and expected probability of click  $x'_i$  is

$$T_i(x'_i) = \int_0^{x'_i} \tau_i(z) dz \tag{7}$$

See Figure 3b for an illustration of  $T_i(x'_i)$  on the plot of the price-per-click allocation rule.

The *total threshold* for the allocation  $\mathbf{x}'$  is then the sum of the thresholds across all agents,  $\sum_i T_i(x'_i)$ .<sup>2</sup> We now refine the notion of revenue-covering from Hartline et al. [2014] to apply for a specific strategy profile.

**Definition 2** (Revenue Covering). *Strategy profile  $\sigma$  of auction  $A$  is  $\mu$ -revenue covered if for any feasible allocation  $\mathbf{x}'$ ,*

$$\mu\text{REV}(A(\sigma)) \geq \sum_i T_i(x'_i). \quad (8)$$

If we can prove that for any strategy profile the auction and strategy profile are revenue covered, then we say the auction is  $\mu$ -revenue covered - this matches the definition of  $\mu$ -revenue covering in Hartline et al. [2014].

**Definition 3.** *Auction  $A$  is  $\mu$ -revenue covered if for any strategy profile  $\sigma$ ,  $\sigma$  and  $A$  are  $\mu$ -revenue covered.*

## Value Covering & PoA Results

**Lemma 4** (Value Covering). *For any bidder  $i$  with value  $v_i$  and allocation amount  $x'_i$ ,*

$$u_i(v_i) + \frac{1}{\mu} T_i(x'_i) \geq \frac{1 - e^{-\mu}}{\mu} x'_i v_i. \quad (9)$$

The proof is included in the appendix for completeness: it is a refinement of the proof of value covering in Hartline et al. [2014], matching the bound in Syrgkanis and Tardos [2013] for  $c$ -threshold approximate auctions.

Combining revenue covering of a strategy profile and value covering gives a welfare approximation result for that strategy profile:

**Theorem 5.** *The welfare in any  $\mu$ -revenue covered strategy profile  $\sigma$  of auction  $A$  is at least a  $\frac{\mu}{1 - e^{-\mu}}$ -approximation to the optimal welfare.*

*Proof of Theorem 5.* Let  $\mathbf{x}^*(\mathbf{v})$  be the welfare optimal allocation for valuation profile  $\mathbf{v}$ . Recall that the optimal allocation is also allowed to use the instantiation of the quality scores and is taken in expectation over the quality scores. Applying the value covering inequality of Equation (9) with respect to allocation quantity  $x_i^*(\mathbf{v})$  gives that for each bidder  $i$  with value  $v_i$ ,

$$u_i(v_i) + \frac{1}{\mu} T_i(x_i^*(\mathbf{v})) \geq \frac{1 - e^{-\mu}}{\mu} x_i^*(\mathbf{v}) v_i. \quad (10)$$

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<sup>2</sup>Note that while this is different than the general definition of expected thresholds in Hartline et al. [2014], it is the same as the definition of thresholds for the generalized-first-price position auction in Hartline et al. [2014]. It is also related to the threshold notion in Syrgkanis and Tardos [2013], which uses  $\tau(\mathbf{x}')$  as the threshold quantity rather than  $T(\mathbf{x}') = \int_0^z \tau(z) dz$ .

The quantity  $x_i^*(\mathbf{v})v_i$  is exactly agent  $i$ 's expected contribution to the welfare of the optimal auction. Applying the revenue covering inequality (8) for  $\mathbf{x}' = \mathbf{x}^*(\mathbf{v})$  and taking expectation over  $\mathbf{v}$  yields:

$$\mu \cdot \text{REV}(A(\sigma)) \geq \mathbf{E}_{\mathbf{v}} \left[ \sum_i T_i(x_i^*(\mathbf{v})) \right] \quad (11)$$

By Equations (10) and (11) we obtain:

$$\begin{aligned} \text{UTIL}(A(\sigma)) + \text{REV}(A(\sigma)) &\geq \mathbf{E}_{\mathbf{v}} \left[ \sum_i u_i(v_i) \right] + \mathbf{E}_{\mathbf{v}} \left[ \sum_i \frac{1}{\mu} T_i(x_i^*(\mathbf{v})) \right] \\ &= \sum_i \mathbf{E}_{\mathbf{v}} \left[ u_i(v_i) + \frac{1}{\mu} T_i(x_i^*(\mathbf{v})) \right] \\ &\geq \sum_i \mathbf{E}_{\mathbf{v}} \left[ \frac{1 - e^{-\mu}}{\mu} x_i^*(\mathbf{v})v_i \right] = \frac{1 - e^{-\mu}}{\mu} \text{WELFARE}(\text{OPT}) \end{aligned}$$

Since  $\text{WELFARE}(A(\sigma)) = \text{REV}(A(\sigma)) + \text{UTIL}(A(\sigma))$ , we have our desired result:

$$\text{WELFARE}(A(\sigma)) \geq \frac{1 - e^{-\mu}}{\mu} \text{WELFARE}(\text{OPT}).$$

□

See Figure 2 for a plot of the price of anarchy as a function of  $\mu$ .

### 3.2 Refining with Observational Data

We now discuss the calculation of  $\mu$  from the distribution of data  $\mathcal{D}$  generated by an auction. The revenue of an auction is observable in  $\mathcal{D}$ . If we can also upper bound  $\sum_i T_i(x'_i)$  for any feasible allocation  $\mathbf{x}'$ , then we have an upper bound on  $\mu$ . Define  $\mathbf{T}$  to be this upperbound, hence  $\mathbf{T} = \max_{\mathbf{x}} \sum_i T_i(x_i)$ .

Recall that as the auction gets to know the quality scores before deciding the allocation of positions, any feasible allocation corresponds to a quality score dependent assignment of slots to bidders.

If the quality scores  $\gamma$  were deterministic, then we could write

$$\mathbf{T}_{\text{fixed}} = \max_{\mathbf{x}} \sum_i T_i(x_i) = \max_{\rho} \sum_i T_i(\mathbf{E}_{\gamma} [\gamma_i \alpha_{\rho^{-1}(\gamma_i)}]) = \max_{\pi} \sum_i T_i(\gamma_i \alpha_{\pi^{-1}(i)}). \quad (12)$$

The latter optimization problem would simply be a bipartite weighted matching problem, where the weight of bidder  $i$  for position  $j$  would be  $T_i(\gamma_i \alpha_{\pi^{-1}(i)})$ . However, when the quality scores are random and their distribution has support of size  $K$ , then the space

of feasible assignments  $\mathcal{M}$  has size  $(n^m)^K$  and the problem does not have the structure of a matching problem anymore, since the functions  $T_i(\cdot)$  are arbitrary convex functions. Solving this complicated maximization problem seems hopeless. In fact it can be shown that the latter problem is NP-hard by a reduction from the maximum hypergraph matching problem, when the size of the support of the correlated distribution of  $\gamma$  is not constant. The hardness arises even if each  $\gamma_i$  is either 0 or 1. We defer the proof to the full version.

However, for the purpose of providing an upper bound on the empirical price of anarchy, it suffices to compute an upper bound on  $\mathbf{T}$  and then show that this upper bound is revenue covered. We will use the convexity of functions  $T_i(\cdot)$  to provide such an upper bound. Specifically, let

$$\bar{x}_i = \max_{\mathbf{x}} x_i = \max_{\rho \in \mathcal{M}} \mathbf{E}_{\gamma} [a_{\rho^{-1}(\gamma, i)} \gamma_i] = \alpha_1 \mathbf{E}_{\gamma} [\gamma_i]$$

denote the maximum possible allocation of bidder  $i$ . Then observe that by convexity for any feasible  $x_i$ :  $T_i(x_i) \leq x_i \frac{T_i(\bar{x}_i)}{\bar{x}_i}$ . Thus we will define:

$$\bar{\mathbf{T}}^1 = \max_{\mathbf{x}} \sum_i x_i \frac{T_i(\bar{x}_i)}{\bar{x}_i} \quad (13)$$

Then we immediately get the following observation:

**Observation 1.**  $\bar{\mathbf{T}}^1 \geq \mathbf{T}$ .

Computing  $\bar{\mathbf{T}}^1$  is a much easier computational problem than computing  $\mathbf{T}$ . Specifically, by linearity of expectation:

$$\begin{aligned} \bar{\mathbf{T}}^1 &= \max_{\mathbf{x}} \sum_i x_i \frac{T_i(\bar{x}_i)}{\bar{x}_i} = \max_{\rho \in \mathcal{M}} \sum_i \mathbf{E}_{\gamma} [\alpha_{\rho^{-1}(\gamma, i)} \gamma_i] \frac{T_i(\bar{x}_i)}{\bar{x}_i} \\ &= \max_{\rho \in \mathcal{M}} \mathbf{E}_{\gamma} \left[ \sum_i \alpha_{\rho^{-1}(\gamma, i)} \gamma_i \cdot \frac{T_i(\bar{x}_i)}{\bar{x}_i} \right] \\ &= \mathbf{E}_{\gamma} \left[ \max_{\pi \in \Pi} \sum_i \alpha_{\pi^{-1}(i)} \gamma_i \cdot \frac{T_i(\bar{x}_i)}{\bar{x}_i} \right] \end{aligned}$$

Now observe that the problem inside the expectation is equivalent to a welfare maximization problem where each player  $i$  has a value-per-click of  $v'_i = \frac{T_i(\bar{x}_i)}{\bar{x}_i}$  and we want to maximize the welfare:  $\sum_i \alpha_{\pi^{-1}(i)} \gamma_i \cdot v'_i$ . The optimal such allocation is simply the greedy allocation which assigns slots to bidders in decreasing order of  $\gamma_i \cdot v'_i$ . Thus computing  $\bar{\mathbf{T}}^1$  consists of running a greedy allocation algorithm for each quality score profile  $\gamma$  in the support of the distribution of quality scores, which would take time  $K \cdot (m + n \log(n))$ . When applying it to the data, we will simply compute the optimal greedy allocation for each instance of the quality scores that arrives in each auction (i.e. we compute the latter for the empirical distribution of quality score profiles).

**T Lower Bounds** If we can find a lower bound  $\underline{\mathbf{T}} \leq \mathbf{T}$ , then we can use  $\underline{\mathbf{T}}$  as a certificate for the optimality of our upper bound: we will know that at best the auction is  $\underline{\mathbf{T}}/\text{REV}(A)$ -revenue covered.

One such lower bound comes from considering the case that the auction does not know the quality scores when deciding the allocation. This case is equivalent to the case that the bidder has a deterministic quality  $\mathbf{E}_\gamma[\gamma_i]$ . The total threshold can be calculated just as in Equation (12): let  $\mathbf{T}_{\text{avg}} = \max_\pi \sum_i T_i(\mathbf{E}_\gamma[\gamma_i] \alpha_{\pi^{-1}(i)})$ .

If we show that  $\mathbf{T}_{\text{avg}}$  is revenue covered then we get an efficiency guarantee with respect to the optimal allocation problem that cannot condition on the quality scores

This can be seen as an interesting alternative welfare benchmark, even when the auction gets to see the quality scores at the allocation time.

As a fixed allocation independent of the quality scores is a feasible quality score dependent allocation, we immediately get that:  $\mathbf{T}_{\text{avg}} \leq \mathbf{T} \leq \overline{\mathbf{T}}^1$ . Thus we can use  $\mathbf{T}_{\text{avg}}$  as a certificate of approximate optimality of our upper bound  $\overline{\mathbf{T}}^1$  to check that it is not far from the true optimal threshold  $\mathbf{T}$ .

**Empirical Revenue Covering** If we can estimate the threshold upper-bound and the revenue, then this is now enough for a revenue-covering result for the strategy profile being played in the auction:

**Lemma 6.** *For any auction  $A$  with strategy profile  $\sigma$ , revenue  $\text{REV}(A)$  and threshold upper bound  $\overline{\mathbf{T}}$ ,  $A$  is  $\frac{\overline{\mathbf{T}}}{\text{REV}(A)}$ -revenue covered with (implicit) strategy profile  $\sigma$ .*

Combining this with Theorem 5 directly gives a welfare approximation result:

**Corollary 1.** *For any instance of an auction, with (unobservable) strategy profile  $\sigma$  and (observable) revenue  $\text{REV}(A)$  and threshold upper bound  $\overline{\mathbf{T}}$ , the empirical price of anarchy for auction  $A$  is at most*

$$\frac{\overline{\mathbf{T}}}{\text{REV}(A)} \frac{1}{1 - e^{-\overline{\mathbf{T}}/\text{REV}(A)}}. \quad (14)$$

**Empirical Value Covering** We can also use data to improve the  $\frac{1}{1-e^{-\mu}}$  factor in the approximation bound. This term comes from value covering (Lemma 4), which analyzes how bidders react to the price-per-click allocation rules they face. In the proof of value covering, it is shown that *no matter what the price-per-click allocation rule is*, it is always the case that  $u_i + \frac{1}{\mu} T_i(x'_i) \geq \frac{1-e^{-\mu}}{\mu} x'_i v_i$ . When we can observe the price-per-click allocation rules, we can simply take the worst case over the price-per-click allocation rules that we observe for each player, giving an improved price of anarchy result.

**Definition 7** (Empirical Value Covering). *Auction  $A$  and strategy profile  $\sigma$  are empirically  $\lambda$ -value covered if  $A$  is  $\mu$ -revenue covered, and for any bidder  $i$  with value  $v_i$  and allocation*

amount  $x'_i$ ,

$$u_i(v_i) + \frac{1}{\mu}T_i(x'_i) \geq \frac{\lambda}{\mu}x'_iv_i. \quad (15)$$

**Lemma 8.** *If auction  $A$  and strategy profile  $\sigma$  are empirically  $\mu$ -revenue covered and  $\lambda$ -value covered, then the empirical price of anarchy of  $A$  and  $\sigma$  is at most  $\frac{\mu}{\lambda}$ .*

*Proof.* The proof is analogous to the proof of Theorem 5, using the value covering parameter  $\lambda$  in place of the general value covering result, Lemma 4.  $\square$

As threshold quantities are observable in the data — and required to generate revenue covering results — one approach is to directly look, and find the worst case ratio of threshold and utility to value.

**Lemma 9.** *For a  $\mu$ -revenue covered strategy profile  $\sigma$  and auction  $A$  with maximum feasible probabilities of allocation  $\bar{x}_i$ , let  $\underline{\lambda}_i^\mu = \min_{v_i, x'_i} \frac{\mu u_i(v_i) + T_i(x'_i)}{x'_i v_i}$  and  $\underline{\lambda}^\mu = \min_i \underline{\lambda}_i^\mu$ .*

*Then  $A$  and  $\sigma$  are empirically  $\underline{\lambda}^\mu$ -value covered.*

In the case that an auction is shown to be  $\mu$ -revenue covered with respect to the upper bound  $\bar{\mathbf{T}}^1$ , the maximization can be simplified to only consider the allocation amount  $\bar{x}_i$ , hence  $\underline{\lambda}_i^\mu = \min_{v_i} \frac{\mu u_i + T_i(\bar{x}_i)}{v_i}$ .

**Concentration Bounds** We can also improve on the value covering term even if we only know some properties about the concentration of the price-per-click allocation rule. If the price-per-click allocation rule is highly concentrated, and the minimum feasible price per click is at least a  $(1 - \frac{1}{k})$  fraction of the maximum feasible price per click, we can get significantly improved bounds.

**Lemma 10.** *For any  $\mu$ -revenue covered auction  $A$  and strategy profile  $\sigma$  with  $\mu \geq 1$ , if  $\tau(\epsilon) \geq (1 - 1/k)\tau(\mathbf{x}')$  for any feasible allocation amount  $\mathbf{x}'$  and  $\epsilon > 0$ ,  $A$  and  $\sigma$  are empirically  $(1 - 1/k)$ -value covered.*

The proof is included in the appendix: see Table 1 for better numerical results.

## 4 Learning Agents and Empirical Price of Anarchy

We show that the exact same analysis as in the previous section extends even if the data we observe are not generated from a Bayes-Nash equilibrium of a stochastic i.i.d. valuation setting, but rather are generated from learning agents whose valuation is fixed and who are experimenting on how to play, using some no-regret learning algorithm.

In this setting, we assume we observe a sequence of data  $\mathcal{D}$  of  $T$  timesteps. The empirical price of anarchy takes the following definition

$\mu$	$k = 1$	$k = 2$	$k = 4$	$k = 10$	$k = 100$
0.5	1.271	1.	1.	1.	1.
0.75	1.421	1.116	1.	1.	1.
1	1.582	1.302	1.163	1.072	1.009
1.25	1.752	1.506	1.382	1.304	1.256
1.5	1.931	1.717	1.61	1.545	1.505
2	2.313	2.157	2.079	2.032	2.003
4	4.075	4.037	4.019	4.007	4.001
8	8.003	8.001	8.001	8.	8.

Table 1: Empirical Price of Anarchy when the price-per-click of getting any allocation is at least a  $(1 - \frac{1}{k})$  fraction of the price-per-click of getting the maximum allocation, with empirical revenue covering parameter  $\mu$ .

**Definition 11** (Empirical Price of Anarchy for Learning Agents). *The empirical price-of-anarchy for learning agents of an auction and a distribution of data  $\mathcal{D}$  generated by the auction is the worst-case ratio of welfare in the optimal auction to the welfare in an equilibrium, taken over all valuation profiles and learning outcomes that could generate the sequence of data  $\mathcal{D}$ .*

$$EPoA(A, \mathcal{D}) = \max_{\mathbf{v}, \mathcal{D} \text{ is no-regret for } \mathbf{v}} \frac{\sum_{t=1}^T \text{WELFARE}(\text{OPT}(\mathbf{v}, \gamma^t); \mathbf{v})}{\sum_{t=1}^T \text{WELFARE}(A(\mathbf{b}^t, \gamma^t); \mathbf{v})} \quad (16)$$

#### 4.1 Average Utility, Average Price-per-click

We first focus on the optimization problem each bidder faces. When bidding in a sequence of auctions, we assume that each bidder  $i$ , with some value  $v_i$ , is submitting a sequence of bids  $b_i^t$ , such that in the limit he achieves no-regret with respect to any fixed bid in hindsight:

$$\forall b'_i : \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T (U_i(b'_i, \mathbf{b}_{-i}^t; v_i) - U_i(\mathbf{b}^t; v_i)) \leq 0 \quad (17)$$

For simplicity, we will assume that for the sequence we observe, each bidder has zero regret for his value, with respect any fixed bid. Our results smoothly degrade if the regret is at most some small  $\epsilon$ . Hence, for now on, we will assume:

$$\forall b'_i : \frac{1}{T} \sum_{t=1}^T (U_i(b'_i, \mathbf{b}_{-i}^t; v_i) - u_i(\mathbf{b}^t; v_i)) \leq 0 \quad (18)$$

Consider the utility of an agent from a fixed bid  $b$ . We can re-write it as:

$$\frac{1}{T} \sum_{t=1}^T U_i(b, \mathbf{b}_{-i}^t; v_i) = v_i \frac{1}{T} \sum_{t=1}^T x_i(b, \mathbf{b}_{-i}^t) - \frac{1}{T} \sum_{t=1}^T P_i(b, \mathbf{b}_{-i}^t) \quad (19)$$

Given the sequence of bids and gamma profiles, we define for any fixed bid  $b$ :

$$u_i^T(b; v_i) = \frac{1}{T} \sum_{t=1}^T u_i(b, \mathbf{b}_{-i}^t; v_i) \quad (20)$$

$$x_i^T(b) = \frac{1}{T} \sum_{t=1}^T x_i(b, \mathbf{b}_{-i}^t) \quad (21)$$

$$P_i^T(b) = \frac{1}{T} \sum_{t=1}^T P_i(b, \mathbf{b}_{-i}^t) \quad (22)$$

the average allocation and the average payment as a function of the fixed bid. Then we can write:

$$u_i^T(b; v_i) = v_i x_i^T(b) - P_i^T(b) = x_i^T(b) \left( v_i - \frac{P_i^T(b)}{x_i^T(b)} \right) \quad (23)$$

The average price-per-click term  $\frac{P_i^T(b)}{x_i^T(b)}$  term now plays exactly the same role in the utility function that the first-price bid does in a one-shot first price auction. We call this term  $ppc^T(b) = \frac{P_i^T(b)}{x_i^T(b)}$  the average price-per-click of the bid in a position auction.

**Fixed bid thresholds.** We will use the inverse of the PPC allocation rule for our analysis; let  $\tau_i^T(z)$  be the price-per-click of the cheapest fixed bid that achieves average allocation at least  $z$ . More formally,

$$\tau_i^T(z) = \min_{b | x_i^T(b) \geq z} \{ ppc^T(b) \}. \quad (24)$$

The *threshold* for agent  $i$  and average probability of click  $x'_i$  is

$$T_i^T(x'_i) = \int_0^{x'_i} \tau_i^T(z) dz \quad (25)$$

## 4.2 Revenue and Value Covering for Learning Agents

First we show that the average utility of a bidder and the average threshold for any allocation satisfy a very useful inequality:

**Lemma 12** (Value Covering for Learning Agents). *For any bidder  $i$  with value  $v_i$  and average allocation amount  $x'_i$ ,*

$$\frac{1}{T} \sum_{t=1}^T U_i(\mathbf{b}^t; v_i) + \frac{1}{\mu} T_i^T(x'_i) \geq \frac{1 - e^{-\mu}}{\mu} x'_i v_i. \quad (26)$$

*Proof.* By the no-regret property we know that:

$$\forall b : \frac{1}{T} \sum_{t=1}^T U_i(\mathbf{b}^t; v_i) \geq u_i^T(b; v_i)$$

Thus it suffices to show that:

$$\max_b u_i^T(b; v_i) + \frac{1}{\mu} T_i^T(x'_i) \geq \frac{1 - e^{-\mu}}{\mu} x'_i v_i. \quad (27)$$

The latter follows exactly as in the proof of Lemma 12.  $\square$

**Definition 13** (Revenue Covering for Learning Agents). *A sequence of bid profiles  $\mathbf{b}^1, \dots, \mathbf{b}^T$  of auction  $A$  is  $\mu$ -revenue covered if for any feasible average allocation  $\mathbf{x}'$ ,*

$$\mu \frac{1}{T} \sum_{t=1}^T \text{REV}(A(\mathbf{b}^t)) \geq \sum_i T_i^T(x'_i). \quad (28)$$

Combining revenue covering of a strategy profile and value covering gives a welfare approximation result for that strategy profile:

**Theorem 14** (Empirical Price of Anarchy Bound for Learning Agents). *The average welfare in any  $\mu$ -revenue covered strategy profile  $\sigma$  of auction  $A$  produced by no-regret learning agents is at least a  $\frac{\mu}{1 - e^{-\mu}}$ -approximation to the average optimal welfare.*

*Proof of Theorem 5.* Let  $\mathbf{x}^*(\mathbf{v})$  be the welfare optimal average allocation for valuation profile  $\mathbf{v}$ , i.e. if  $\mathbf{x}^*(\mathbf{v}, \gamma)$  is the optimal allocation of clicks for valuation profile  $\mathbf{v}$  and quality scores  $\gamma$ , then  $\mathbf{x}^*(\mathbf{v}) = \frac{1}{T} \sum_{t=1}^T \mathbf{x}^*(\mathbf{v}, \gamma^t)$ .

Applying the value covering inequality of Equation (9) with respect to average allocation quantity  $x_i^*(\mathbf{v})$  gives that for each bidder  $i$  with value  $v_i$ ,

$$\frac{1}{T} \sum_{t=1}^T U_i(\mathbf{b}^t; v_i) + \frac{1}{\mu} T_i^T(x_i^*(\mathbf{v})) \geq \frac{1 - e^{-\mu}}{\mu} x_i^*(\mathbf{v}) v_i. \quad (29)$$

The quantity  $x_i^*(\mathbf{v}) v_i$  is exactly agent  $i$ 's expected contribution to the welfare of the optimal auction. Applying the revenue covering inequality (8) for  $\mathbf{x}' = \mathbf{x}^*(\mathbf{v})$  yields:

$$\mu \cdot \frac{1}{T} \sum_{t=1}^T \text{REV}(A(\mathbf{b}^t)) \geq \sum_i T_i^T(x_i^*(\mathbf{v})) \quad (30)$$

By Equations (29) and (30) we obtain:

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T (\text{UTIL}(A(\mathbf{b}^t); \mathbf{v}) + \text{REV}(A(\mathbf{b}^t))) &\geq \sum_i \left( \frac{1}{T} \sum_{t=1}^T U_i(\mathbf{b}^t; v_i) + \frac{1}{\mu} T_i^T(x_i^*(\mathbf{v})) \right) \\
&\geq \frac{1 - e^{-\mu}}{\mu} \sum_i x_i^*(\mathbf{v}) v_i \\
&= \frac{1 - e^{-\mu}}{\mu} \frac{1}{T} \sum_{t=1}^T \sum_i v_i x_i^*(\mathbf{v}, \gamma^t) \\
&= \frac{1 - e^{-\mu}}{\mu} \frac{1}{T} \sum_{t=1}^T \text{WELFARE}(\text{OPT}(\mathbf{v}, \gamma^t))
\end{aligned}$$

Since  $\text{WELFARE}(A(\mathbf{b})) = \text{REV}(A(\mathbf{b})) + \text{UTIL}(A(\mathbf{b}); \mathbf{v})$ , we have our desired result:

$$\frac{1}{T} \sum_{t=1}^T \text{WELFARE}(A(\mathbf{b}^t, \gamma^t)) \geq \frac{1 - e^{-\mu}}{\mu} \frac{1}{T} \sum_{t=1}^T \text{WELFARE}(\text{OPT}(\mathbf{v}, \gamma^t))$$

□

## 5 Statistical properties of the Empirical Price of Anarchy

Our empirical approach is based on the analysis of the *average cost per click* in a sponsored search auction defined as the ratio of the expected price to the expected click probability in a given auction instance. We are going to adhere to the settings where in each auction  $t$  we observe the bids of  $n$  eligible bidders  $i = 1, \dots, n$  denoted  $b_{i,t}$ .<sup>3</sup> We assume that we observe  $T$  instances of the auction for each bidder whose allocation we consider in the EPoA bound.

As we mentioned previously, in position auctions the additional randomness is introduced by the scoring rule that comes out of the scoring algorithm crafted by the search engine to estimate bidder-specific probabilities of a click. The scores  $s_{i,t}$  assigned to each bidder are derived from the click probabilities. Although, in many theoretical analyses it is assumed that  $\gamma_{i,t} \equiv s_{i,t}$ , in practice the scores are not always equal to the clickabilities, for instance, due to “squashing” or penalization of particular bidders (e.g. associated with fraudulent or harmful web content). These multipliers can play various roles from controlling the relevance of ads to the users to price discrimination of the participants of the auction. Our results extend to the case of arbitrary scoring algorithm (provided that the true bidder-specific click probabilities can also be recovered from the data).

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<sup>3</sup>The number of bidders is fixed without loss of generality. If the number of active bidders varies over time then  $n$  defines the upper bound on the number of participating bidders and the bids of the “inactive” bidders are set to zero.

We now characterize the structure of the allocation rule that defines “auction logic”, i.e. the function that takes the scores and bids as inputs and outputs the prices per click and allocations to the bidders. The auction logic is based on the customization of the order of bidders to each auction (corresponding to an individual user query) using the assignment rule  $\rho(b_t, s_t, j)$ . We recycle the notation  $\rho$  that we used before but add an argument  $b_t$  to note that the assignment of the bidder to slot  $j$  depends both on the vector of scores  $s_t$  and the bids  $b_t$  eligible for query  $t$ . We use the notation  $b_{i,t}$  to denote the bid of bidder  $i$  applied to auction  $t$  and  $s_{i,t}$  to denote the score of this bidder in that query. As in Athey and Nekipelov [2010] we also can incorporate the (random and fixed) query, main line and other reserve prices by adding “virtual bidders” to the set of actual bidders to incorporate.

**Assumption 1.** *In each query  $t$ , the allocation of bidders to slots is determined by the allocation rule  $\rho(b_t, s_t, j)$  for  $j = 1, \dots, J$  where each function  $\rho(b_t, s_t, j)$  can be represented as a finite superposition of:*

- (i) *Fixed linear functions of  $b_t$ ,  $s_t$  and the element-by-element product  $s_t * b_t$*
- (ii) *Indicator functions for  $\mathbf{1}\{\cdot > \cdot\}$*
- (iii) *Sums and differences  $\cdot \pm \cdot$*

Finally, for each bidder  $i$  the ratio  $\gamma_{i,t}/s_{i,t} < \Gamma$ .

Note that the allocation rule  $\rho(b_t, s_t, j)$  also determines the price once the pricing rule is known. For GSP the price of slot  $j$  can be computed as

$$p(b_t, s_t, j) = \frac{s_{\rho(b_t, s_t, j+1), t} b_{\rho(b_t, s_t, j+1), t}}{s_{\rho(b_t, s_t, j), t}}$$

An example of the implementation of the allocation and price rule for the generalized second price auction with a simple single reserve price is given in Edelman et al. [2007] and Varian [2009]. In that case the price and the allocation rule are determined solely by the score-weighted bid  $s_{it}b_{it}$  for each bidder. The allocation of bidders to slots is determined by the ranks of their score-weighted bids:

$$\sum_{k=1}^n \mathbf{1}\{s_{k,t}b_{k,t} > \gamma_{\rho(b_t, s_t, j), t} b_{\rho(b_t, s_t, j), t}\} = j - 1.$$

Our inference will be based on the idea that the customization of the order of bidders to users generates randomness, that in turn, allows us to apply the concentration inequalities to the prices and allocations averaged over  $T$  auction instances.

**Assumption 2.** *The combined score profile and the vector of bids  $(s_t, b_t)$  is independently drawn from its fixed joint distribution at each auction instance  $t$ .*

We note here that Assumption 2 is sufficient but not necessary to establish our bound inequalities below. Our general results only require the vector of bids and scores to follow a  $\beta$ -mixing over the course of arriving auctions. That allows for various adaptation and learning dynamics for bids and the scoring algorithm. We omit those more general results for the sake of brevity.

We allow general setting where the scores of different bidders can be correlated (i.e. there can be query-level features that affect the scores of all participating bidders). At the same time, we also notice that in our formulation we allow the values of the bidder to come from an arbitrary distribution and be correlated both across bidders and over time. In particular, the case of fixed values of the bidders (such as in Edelman et al. [2007], Varian [2009] and Athey and Nekipelov [2010]) is included as a special case.

Note that in the context of the random scores, the expected price and allocation rules are computed as expectations over the bid and the score distributions:

$$p_i(b) = E_{s, b_{-i}} \left[ \alpha_{\rho^{-1}(b_t, s_t, i)} \gamma_{i,t} \frac{s_{\rho(b_t, s_t, \rho^{-1}(b_t, s_t, i)+1), t} b_{\rho(b_t, s_t, \rho^{-1}(b_t, s_t, i)+1), t}}{s_{i,t}} \mid b_{i,t} = b \right]$$

and

$$x_i(b) = E_{s, b_{-i}} \left[ \alpha_{\rho^{-1}(b_t, s_t, i)} \gamma_{i,t} \mid b_{i,t} = b \right].$$

Now for position discounts  $\alpha_1, \dots, \alpha_J$  we estimate the expected price and allocation rule by replacing expectations with sample averages:

$$\hat{p}_i(b) = \frac{1}{T} \sum_{t=1}^T \alpha_{\rho^{-1}(\tilde{b}_t, s_t, i)} \gamma_{i,t} \frac{s_{\rho(\tilde{b}_t, s_t, \rho^{-1}(\tilde{b}_t, s_t, i)+1), t} b_{\rho(\tilde{b}_t, s_t, \rho^{-1}(\tilde{b}_t, s_t, i)+1), t}}{s_{i,t}} \quad (31)$$

and

$$\hat{x}_i(b) = \frac{1}{T} \sum_{t=1}^T \alpha_{\rho^{-1}(\tilde{b}_t, s_t, i)} \gamma_{i,t}, \quad (32)$$

where  $\tilde{b}_t = (b, b_{-i,t})$ , i.e. this is the bid profile where the bid of bidder  $i$  is set to  $a$  and the bids of remaining bidders are set to the empirically observed values.

In our subsequent analysis we assume that the bid space is bounded such that there is a universal constant  $B$  such that the support of bids is a compact subset of  $[0, B]$ . Next we characterize the properties of the presented estimators for the price and the allocation functions.

**Theorem 15.** *For estimators (31) and (32) there exist universal constants  $C_1, C_2$  and  $\phi$  such that*

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{p}_i(b) - p_i(b)| \right] \leq C_1 B \alpha_1 \Gamma n^\phi$$

and

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{x}_i(b) - x_i(b)| \right] \leq C_2 \alpha_1 n^\phi.$$

The result of the theorem suggests that under fixed (or bounded) number of bidders in the auctions, the estimated expected price and allocation rule have the error that contracts at the rate  $\sqrt{T}$ , i.e. behaves like the sample mean of the i.i.d. sequence of random variables. That also gives us a simple approach to computing the upper bound variance of the estimated price and the allocation rule by taking the maximum of

$$\widehat{\text{Var}}(\widehat{x}_i(b)) = \frac{1}{T} \sum_{t=1}^T \left( \alpha_{\rho^{-1}(\bar{b}_t, s_t, i)} \gamma_{i,t} - \widehat{x}_i(b) \right)^2$$

over the bid space, and obtain a similar evaluation for the variance of the price function.

Next we turn to the analysis of the quantities that form the EPoA, namely the integrals of functions  $\tau_i(z)$ . In that case the object of interest, can be written as

$$T_i(x) = \int_0^x \frac{p_i(x_i^{-1}(z))}{z} dz.$$

We need to ensure that this function is well-behaved if the true population allocation and pricing rule are available.

**Assumption 3.** *Functions  $p_i(\cdot)$  and  $x_i(\cdot)$  are continuous and strictly monotone. Moreover, there exist constants  $\kappa, \zeta > 0$  and  $\Delta > 0$  such that for  $b \in [0, \Delta]$*

$$\widehat{p}_i(b) < \kappa b + o(\Delta) \quad \text{and} \quad \widehat{x}_i(b) > \zeta b + o(\Delta).$$

Moreover, the population counterparts  $p_i(b)$  and  $x_i(b)$  satisfy

$$|p_i(b_1) - p_i(b_2)| \leq L_p |b_1 - b_2|, \quad \text{and} \quad |x_i(b_1) - x_i(b_2)| \geq L_x |b_1 - b_2|.$$

Now we define the empirical analog of  $T_i(x)$  obtained by the replacement of the true allocation and pricing functions with their empirical analogs:

$$\widehat{T}_i(x) = \int_0^x \frac{\widehat{p}_i(\widehat{x}_i^{-1}(z))}{z} dz.$$

Our next step will be to establish the uniform convergence of the estimated function  $\widehat{T}_i(x)$  to  $T_i(x)$ .

**Theorem 16.**

$$E \left[ \sup_{x \in [0,1]} \sqrt{T} \left( \widehat{T}_i(x) - T_i(x) \right) \right] \leq O \left( n^\phi \right)$$

Now we investigate how the replacement of the true thresholds with the empirical thresholds affect the outcome of maximization over possible allocations. Define  $\widehat{\mathbf{T}} = \max_{\mathbf{x}} \sum_i \widehat{T}_i(x_i)$ ,

**Corollary 2.**  $E \left[ \sqrt{T} \left| \widehat{\mathbf{T}} - \overline{\mathbf{T}} \right| \right] = O(n^{1+\phi})$

*Proof.* From Theorem 16 it follows that

$$\sup_{x_i} \sqrt{T} \left| \widehat{T}_i(x_i) - T_i(x_i) \right| = O_p(n^\phi).$$

Then

$$\sup_{\mathbf{x}} \sqrt{T} \left| \sum_i \widehat{T}_i(x_i) - \sum_i T_i(x_i) \right| = O_p(n^{\phi+1}).$$

Provided that for two non-negative functions  $\sup_{\mathbf{x}}(f(\mathbf{x}) - g(\mathbf{x})) \geq \sup_{\mathbf{x}} f(\mathbf{x}) - \sup_{\mathbf{x}} g(\mathbf{x})$ , we conclude that

$$\sup_{\mathbf{x}} \sum_i \widehat{T}_i(x_i) \leq \sup_{\mathbf{x}} \sum_i T_i(x_i) + O_p \left( \frac{n^{\phi+1}}{\sqrt{T}} \right).$$

□

This corollary states that the empirical analog of  $\overline{\mathbf{T}}$  approaches to its true value and the distance between the true and the empirical value is of order  $n^{\phi+1}/\sqrt{T}$ , i.e. this distance shrinks at the rate  $\sqrt{T}$ .

The last component is the estimation of the revenue. Note that the true revenue in our notation can be expressed as

$$\text{Rev}(M) = \frac{1}{T} \sum_{t=1}^T \sum_i p_i(b_{i,t}).$$

The corresponding empirical revenue is obtained by the replacement of the expected payment function with its estimated version, i.e.

$$\widehat{\text{Rev}}(M) = \sum_i \widehat{p}_i(b_i).$$

**Corollary 3.**

$$E \left[ \sqrt{T} \left| \widehat{\text{Rev}}(M) - \text{Rev}(M) \right| \right] = O(n^{\phi+1})$$

*Proof.* Note that

$$\begin{aligned} \sqrt{T}|\widehat{\text{Rev}}(M) - \text{Rev}(M)| &\leq \sum_i \sqrt{T}|\widehat{p}_i(b_{i,t}) - p_i(b_{i,t})| \\ &\leq n \max_i \sup_b \sqrt{T}|\widehat{p}_i(b) - p_i(b)| \\ &= O(n^{\phi+1}) \end{aligned}$$

□

Thus we established that both the empirical analog for  $\overline{\mathbf{T}}$  and the empirical analog for  $\text{Rev}(M)$  converge to their true counterparts at the rate  $\sqrt{T}$ .

## 6 Data Analysis

We run our analysis on the BingAds auctions. We analyzed eleven phrases from multiple thematic categories. For each phrase we retrieved data of auctions for the phrase for the period of a week. For each phrase and bidder that participated in the auctions for the phrase we computed the allocation curve and by simulating the auctions for the week and computing what would have happened at each auction for each possible bid an advertiser could submit. We discretized the bid space and assumed a hard upper bound on the bid amount.

For instance the left part of Figure 3 shows the allocation curves for a subset of the advertisers for a specific search phrase. Then we computed the threshold curves by numerically integrating the allocation curves. The threshold curves for the same subset of advertisers are depicted in the right part of Figure 3. Most of these keywords have a huge amount of heterogeneity across advertisers as can be seen by the very different bid levels of each advertiser and the very different quality score. For instance, in Figure 4 we depict the average bid, average quality score and average payment of each of the same subset of advertisers for which we depicted the allocation and threshold function in Figure 3.

Subsequently, we applied all the techniques we describe in Section 3 for each of the search phrases. We first computed the optimal upper bound on the thresholds  $\overline{\mathbf{T}}^1$  and by observing the revenue of the auctions from the data, we can compute an upper bound on the revenue covering of the auction for the phrase, i.e.  $\mu^1 = \overline{\mathbf{T}}^1/\text{REV}$ . Then for this  $\mu^1$  we optimized over  $\lambda$  by using the allocation curves and Lemma 9 and assuming some hard upper bound on the valuation of each advertiser and found the optimal such  $\lambda$ , denoted by  $\lambda^1$ . Then an upper bound on the empirical price of anarchy is  $\mu^1/\lambda^1$ .

Subsequently we tested the tightness of our analysis by computing the value of the true thresholds on the optimal allocation that was computed under the linear approximations of the thresholds. This is a feasible allocation and hence the true value of  $\mathbf{T}$  is at least the value of the thresholds for this allocation. Hence, by looking at the value of the thresholds

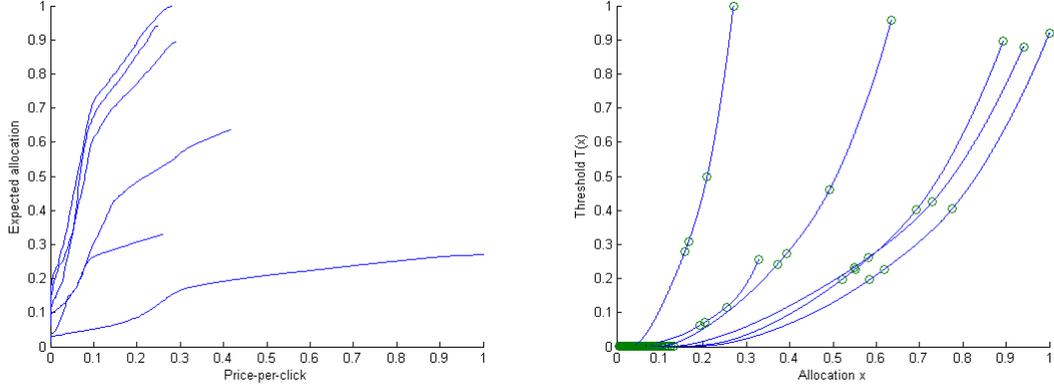


Figure 3: Examples of allocation curves (left) and threshold curves (right) for a subset of six advertisers for a specific keyword during the period of a week. All axes are normalized to 1 for privacy reasons. The circles in the left plot correspond to the expected allocation and expected threshold if bidder  $i$  was given the  $j$ -th slot in all the auctions, i.e. the circle corresponding to the highest allocation and threshold corresponds to the point  $(\alpha_1 \mathbf{E}[\gamma_i], T(\alpha_1 \mathbf{E}[\gamma_i]))$ , the next circle corresponds to  $(\alpha_2 \mathbf{E}[\gamma_i], T(\alpha_2 \mathbf{E}[\gamma_i]))$ , etc.

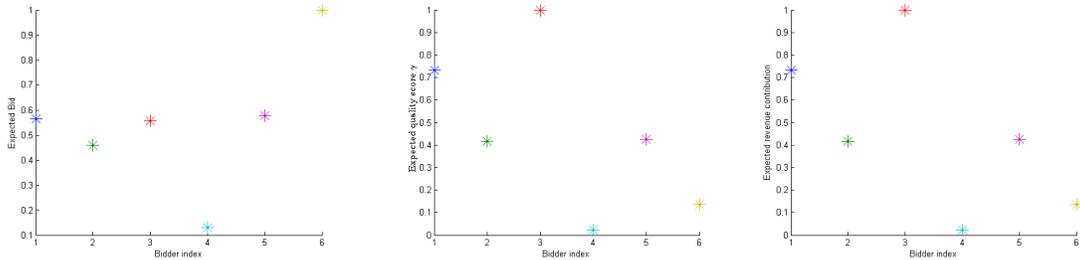


Figure 4: Average bid  $\mathbf{E}[b_i]$ , average quality factor  $\mathbf{E}[\gamma_i]$  and average revenue contribution  $\mathbf{E}[P_i]$ , correspondingly, for the same subset of six advertisers that participated in a specific keyword during the period of a week.  $y$  axes are normalized to 1 for privacy reasons.

at this allocation, denoted by  $LB - \mathbf{T}$  we can check how good our approximation of  $\mathbf{T}$  is  $\bar{\mathbf{T}}^1$ . Then we also computed the optimal thresholds for any quality score independent allocation rule. Apart from yielding yet another lower bound for  $\mathbf{T}$ , the latter analysis also yields an empirical price of anarchy with respect to such a handicapped optimal welfare, which can also be used as a welfare benchmark.

We portray our results on these quantities for each of the eleven search phrases in Table 2

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	$\frac{1}{EPoA^1}$	$\frac{\bar{\mathbf{T}}^1}{\text{REV}}$	$\lambda^1$	$\frac{LB-\mathbf{T}}{\text{REV}}$	$\frac{1}{LB-EPoA}$	$\frac{\mathbf{T}_{\text{avg}}}{\text{REV}}$	$\frac{1}{FA-EPoA}$
phrase1	.567	1.4164	.803	.562	.788	.511	.783
phrase2	.606	1.2848	.779	.553	.792	.509	.784
phrase3	.279	4.182	1.167	3.364	.325	2.966	.320
phrase4	.284	3.860	1.098	2.298	.401	1.556	.507
phrase5	.673	1.099	.740	.437	.828	.386	.829
phrase6	.628	1.183	.743	.495	.832	.488	.791
phrase7	.672	1.031	.693	.503	.824	.459	.802
phrase8	.645	1.036	.669	.520	.795	.419	.817
phrase9	.622	1.169	.726	.581	.759	.441	.809
phrase10	.597	1.138	.680	.545	.771	.377	.833
phrase11	.573	1.431	.820	.631	.780	.502	.786

Table 2: Empirical Price of Anarchy analysis for a set of eleven search phrases on the BingAds system. Phrases are grouped together according to the thematic category of the search phrase. The columns have the following interpretation: 1)  $EPoA^1$  is the upper bound on the empirical price of anarchy, i.e. if  $1/EPoA^1$  is  $x$  it means that the welfare of the auction is at least  $x \cdot 100\%$  efficient. This lower bound is computed by using the polynomially computable upper bound  $\bar{\mathbf{T}}^1$  of  $\mathbf{T}$  and then also optimizing over  $\lambda$ . 2)  $\mu^1 = \bar{\mathbf{T}}^1/\text{REV}$  is the ratio of the upper bound on the maximum sum of thresholds over the revenue of the auction. 3)  $\lambda^1$  is the minimum lambda across advertisers after running the optimization problem presented in Lemma 9 for the allocation curve of each advertiser, assuming some upper bound on the value. Then based on Lemma 8,  $EPoA^1 = \lambda^1/\mu^1$ . 4)  $LB - \mathbf{T}/\text{REV}$ : we use the optimal allocation computed by assuming the linear form of thresholds used for  $\bar{\mathbf{T}}^1$ . Then we evaluate the true thresholds on this allocation. This is a feasible allocation and hence the value of the thresholds on this allocation, denoted  $LB - \mathbf{T}$  is a lower bound on the value of  $\mathbf{T}$ . Thus this ratio is a lower bound on how well the auction is revenue covered. 5)  $LB - EPoA$ , this is simply the empirical price of anarchy bound that would have been implied if  $\mathbf{T} = LB - \mathbf{T}$  and even if we optimized over  $\lambda$ . Thus  $\frac{1}{LB-EPoA}$  is an upper bound on how good our efficiency bound could have been even if we solved the hard problem of computing  $\mathbf{T}$ . 6)  $\mathbf{T}_{\text{avg}}$ , this corresponds to the optimal thresholds with respect to any quality score independent feasible allocation as defined in Section 3. 7)  $FA - EPoA$  a bound on the empirical price of anarchy with respect to a quality score independent allocation rule. For this price of anarchy we did not optimize over  $\lambda$ , hence  $FA - EPoA = \frac{\mu}{1 - \exp(-\mu)}$ .

## A Appendix for Price of Anarchy from Data

**Lemma 4 (Restatement).** *For any bidder  $i$  with value  $v_i$  and allocation amount  $x'_i$ ,*

$$u_i(v_i) + \frac{1}{\mu} T_i(x'_i) \geq \frac{1 - e^{-\mu}}{\mu} x'_i v_i.$$

(*Proof sketch*). The proof proceeds analogously to the proof of value covering in Hartline et al. [2014], first defining a lower bound  $\underline{T}(x) = \int_0^x \underline{\tau}(z) dz$  s.t.  $\underline{\tau}(z) \leq \tau(z)$  and hence  $\underline{T}(x) \leq T_i(x)$ .

$$\begin{aligned} \underline{T}(x'_i) &= \int_0^{x'_i} \underline{\tau}(z) dz \\ &= \int_0^{x'_i} \max(0, v - u_i(v_i)/z) dz \end{aligned}$$

Evaluating the integral gives  $\underline{T}(x'_i) = (v_i x'_i - u_i(v_i)) - u_i(v_i) \left( \log x'_i - \log \frac{u_i(v_i)}{v_i} \right)$ , thus

$$u_i(v_i) + \frac{1}{\mu} \underline{T}(x'_i) = u_i(v_i) + \frac{1}{\mu} \left( v_i x'_i - u_i(v_i) \left( 1 + \log x'_i - \log \frac{u_i(v_i)}{v_i} \right) \right)$$

and

$$\frac{u_i(v_i) + \frac{1}{\mu} \underline{T}(x'_i)}{v_i} = \frac{u_i(v_i)}{v_i} + \frac{1}{\mu} \left( x'_i - \frac{u_i(v_i)}{v_i} \left( 1 + \log x'_i - \log \frac{u_i(v_i)}{v_i} \right) \right) \quad (33)$$

The right side of Equation (33) is convex in  $\frac{u_i(v_i)}{v_i}$ , so we can minimize it by taking first-order conditions in  $\frac{u_i}{v_i}$ , giving

$$0 = 1 - \frac{1}{\mu} \left( \log x'_i - \log \frac{u_i(v_i)}{v_i} \right).$$

Thus the right side of Equation (33) is minimized with  $u_i(v_i)/v_i = x'_i e^{-\mu}$ , giving our desired result,

$$\frac{u_i(v_i) + \frac{1}{\mu} \underline{T}(x'_i)}{v_i} \geq \frac{1 - e^{-\mu}}{\mu} x'_i.$$

□

**Lemma 10 (Restatement).** *For any  $\mu$ -revenue covered mechanism  $M$  and strategy profile  $\sigma$  with  $\mu \geq 1$ , if  $\tau(\epsilon) \geq (1 - 1/k)\tau(\mathbf{x}')$  for any feasible allocation amount  $\mathbf{x}'$  and  $\epsilon > 0$ ,  $M$  and  $\sigma$  are empirically  $(1 - 1/k)$ -value covered.*

(*Proof sketch*). First, for bidders with values  $v_i < \tau(1)$ , the bound holds even without the  $u_i$  term, as

$$T_i(x'_i) = \int_0^{x'_i} \tau(z) dz \quad (34)$$

$$\geq \int_0^{x'_i} \tau(0) \quad (35)$$

$$\geq x'_i(1 - 1/k)\tau(1) \quad (36)$$

$$\geq x'_i(1 - 1/k)v_i \quad (37)$$

Consider bidders with values  $v_i \geq \tau(1)$ . As such a bidder can always choose the bid with price-per-click  $\tau(1)$  and get utility  $v_i - \tau(1)$ , we know  $u_i(v_i) \geq v_i - \tau(1)$ . For any allocation they choose, we then have

$$u_i(v_i) + \frac{1}{\mu}T_i(x'_i) \geq v_i - \tau(1) + \frac{1}{\mu} \int_0^{x'_i} \tau(z) dz \quad (38)$$

$$\geq v_i - \tau(1) + \frac{1}{\mu}x'_i(1 - 1/k)\tau(1) \quad (39)$$

$$\geq v_i - \tau(1) + \frac{x'_i(1 - 1/k)\tau(1)}{\max(1, \mu)} \quad (40)$$

$$\geq (v_i - \tau) \left(1 - \frac{x'_i(1 - 1/k)}{\max(1, \mu)}\right) + \frac{x'_i(1 - 1/k)v_i}{\max(1, \mu)} \quad (41)$$

$$\geq \frac{x'_i(1 - 1/k)v_i}{\max(1, \mu)} \quad (42)$$

We can improve on the bound by considering the worst-case price-per-click allocation rule that satisfies  $\tau(1) = 1$  and  $\tau(0) = 1 - \frac{1}{k}$ , much like in the proof of value covering.

The worst case price-per-click allocation rule  $\tilde{x}$ , for agents with value  $v = u + 1$  is

$$\tilde{x}(z) = \begin{cases} 1 & \text{if } 1 \leq z \\ \frac{u}{v-z} & \text{if } 1 - \frac{1}{k} \leq z \leq 1 \\ 0 & \text{if } z \leq 1 - \frac{1}{k} \end{cases} \quad (43)$$

Note that this is exactly the price-per-click allocation rule that results in the bidder being indifferent over all bids in  $[1 - \frac{1}{k}, 1]$ , as opposed to the indifference over  $[0, 1]$  for the normal value covering proof (with a little more normalization).

We can again define  $\underline{T}(x'_i)$  to be the threshold based on  $\tilde{x}$ . We will solve numerically

for the case that  $x'_i = 1$  as every other case is strictly worse. So,

$$\underline{T}(1) = \int_0^1 \tau(z) dz \quad (44)$$

$$= 1 - \int_{1-1/k}^1 \tilde{x}(y) dy \quad (45)$$

$$= 1 - \int_{1-1/k}^1 \frac{u}{v-y} dy \quad (46)$$

$$= 1 + u \left( \log(v-1) - \log\left(v - \left(1 - \frac{1}{k}\right)\right) \right) \quad (47)$$

$$= 1 + u \log \frac{v-1}{v - \left(1 - \frac{1}{k}\right)} \quad (48)$$

Thus,  $u_i + \frac{1}{\mu} \underline{T}(1) = u + \frac{1}{\mu} \left( 1 + u \log \frac{v-1}{v - \left(1 - \frac{1}{k}\right)} \right)$ , and

$$\frac{v}{u_i + \frac{1}{\mu} \underline{T}(1)} = \frac{v}{u + \frac{1}{\mu} \left( 1 + u \log \frac{v-1}{v - \left(1 - \frac{1}{k}\right)} \right)} \quad (49)$$

In the worst case,  $u = v - 1$ , so

$$\frac{v}{u_i + \frac{1}{\mu} \underline{T}(1)} = \frac{v}{v-1 + \frac{1}{\mu} \left( 1 + (v-1) \log \frac{v-1}{v - \left(1 - \frac{1}{k}\right)} \right)} \quad (50)$$

Numerically minimizing for a variety of  $\mu$  and  $k$  values give the results in Table 1.  $\square$

## B Uniform inference for price and allocation functions

**Theorem 15 (Restatement).** *For estimators (31) and (32) there exist universal constants  $C_1, C_2$  and  $\phi$  such that*

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{p}_i(a) - p_i(a)| \right] \leq C_1 B \alpha_1 \Gamma n^\phi$$

and

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{x}_i(b) - x_i(b)| \right] \leq C_2 \alpha_1 n^\phi.$$

*Proof.* First of all, we notice that estimators (31) and (32) can be expressed in terms of function  $\rho(\cdot)$ :

$$\hat{x}_i(a) = \frac{1}{T} \sum_{t=1}^T \sum_{j=1}^J \alpha_j \gamma_{i,t} \mathbf{1}\{\rho(a, b_{t,-i}, s_t, j) = i\}$$

and

$$\hat{p}_i(a) = \frac{1}{T} \sum_{t=1}^T \sum_{j=1}^J \alpha_j \gamma_i \frac{s_{j+1} b_{j+1}}{s_i} \mathbf{1}\{\rho(a, b_{t,-i}, \gamma_t, j) = i\}$$

Consider the class of functions  $\mathcal{R}_j = \{\rho(a, \cdot, \cdot, j), a \in [0, B]\}$  which defined as the index of the bidder in position  $j$  that results from applying the allocation rule to the bid profile where the bid of bidder  $i$  is fixed at  $a$ . Provided that  $\rho(\cdot)$  is based on linear functions of  $b_t$ ,  $s_t$  and  $b_t * s_t$ . This means that  $\rho(a, \cdot, \cdot, j)$  is constructed from a linear space of functions to the linear span of  $(a, \cdot * a)$ . Thus the VC dimension of such function is at most 4 by Lemma 2.6.15 in Van Der Vaart and Wellner [1996]. Next, the comparison indicator for elements of  $s_t$ ,  $\tilde{b}_t$  and  $s_t * \tilde{b}_t$  leads to at most  $3n$  functions of the form  $\mathbf{1}\{a > \cdot\}$  or  $\mathbf{1}\{\cdot \times a > \cdot\}$ . By Lemma 2.6.18 in Van Der Vaart and Wellner [1996] and the previous finding, each of these  $3n$  classes is a VC class. The application of weighted sums and differences to these classes form a linear space of dimension  $3n$ . Thus the resulting class  $\mathcal{R}_j$  is a VC class where the VC dimension is bounded by low order polynomial in  $n$ .

Now consider the class of functions

$$\mathcal{X}_i = \left\{ \sum_{j=1}^J \alpha_j \times \cdot \times \mathbf{1}\{f(\cdot) = i\}, f \in \mathcal{R}_i \right\},$$

and

$$\mathcal{P}_i = \left\{ \sum_{j=1}^J \alpha_j \times \cdot \times \cdot \times \mathbf{1}\{f(\cdot) = i\}, f \in \mathcal{R}_i \right\},$$

These classes are VC classes by Lemma 2.6.18 in Van Der Vaart and Wellner [1996]. Provided our previous result, we can bound the VC dimension of these classes by a constant multiple of  $n^\phi$  for some  $\phi < \infty$ . Now denote  $F_{\mathcal{X}_i}$  the envelope function for the class  $\mathcal{X}_i$  and  $F_{\mathcal{P}_i}$  the envelope function for  $\mathcal{P}_i$ . By Theorem 2.6.7. in Van Der Vaart and Wellner [1996] the covering numbers for classes of functions  $\mathcal{X}_i$  and  $\mathcal{P}_i$  for a given probability measure  $Q$  can be bounded by

$$\log N(\epsilon \|F_{\mathcal{X}_i}\|_{Q,2}, \mathcal{X}_i, L_2(Q)) \leq A_1 n^\phi \log \left( \frac{1}{\epsilon} \right)$$

and

$$\log N(\epsilon \|F_{\mathcal{P}_i}\|_{Q,2}, \mathcal{P}_i, L_2(Q)) \leq A_2 n^\phi \log \left( \frac{1}{\epsilon} \right),$$

where  $\|f\|_{Q,2} = (\int f(x)^2 dQ(x))^{1/2}$ .

Define the uniform covering integral as

$$J(t, \mathcal{X}_i) = \sup_Q \int_0^t \sqrt{1 + \log N(\epsilon \|F_{\mathcal{X}_i}\|_{Q,2}, \mathcal{X}_i, L_2(Q))} d\epsilon,$$

and similarly define  $J(t, \mathcal{P}_i)$ . Provided that our bounds on both covering numbers do not contain the measure  $Q$ , we note that

$$\int_0^t \sqrt{1 + \log N(\epsilon \|F_{\mathcal{X}_i}\|_{Q,2}, \mathcal{X}_i, L_2(Q))} d\epsilon \propto n^\phi t \log\left(\frac{1}{t}\right) \leq e n^\phi.$$

Next notice that by construction of  $\mathcal{X}_i$  and  $\mathcal{P}_i$ , the corresponding envelopes are bounded, which implies that

$$\|F_{\mathcal{X}_i}\|_{Q,2} \leq \alpha_1, \quad \text{and} \quad \|F_{\mathcal{P}_i}\|_{Q,2} \leq \alpha_1 B \Gamma.$$

Next, the application of Theorem 2.14.1 in Van Der Vaart and Wellner [1996] allows us to evaluate

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{p}_i(b) - p_i(b)| \right] \leq O(J(1, \mathcal{P}_i) \|F_{\mathcal{P}_i}\|_{Q,2})$$

and

$$E \left[ \sup_{b \in [0, B]} \sqrt{T} |\hat{x}_i(b) - x_i(b)| \right] \leq O(J(1, \mathcal{X}_i) \|F_{\mathcal{X}_i}\|_{Q,2}).$$

That yields the evaluation in the statement of the theorem.  $\square$

**Theorem 16 (Restatement).**

$$E \left[ \sup_{x \in [0, 1]} \sqrt{T} \left( \hat{T}_i(x) - T_i(x) \right) \right] \leq O(n^\phi)$$

*Proof.* Consider the following chain of evaluations:

$$\begin{aligned} \hat{p}_i(\hat{x}_i^{-1}(z)) - p_i(x_i^{-1}(z)) &= \hat{p}_i(\hat{x}_i^{-1}(z)) - p_i(\hat{x}_i^{-1}(z)) \\ &+ p_i(\hat{x}_i^{-1}(z)) - p_i(x_i^{-1}(z)) \end{aligned}$$

Then consider the integral of the first component of this evaluation via

$$\int_0^1 \frac{\hat{p}_i(\hat{x}_i^{-1}(z)) - p_i(\hat{x}_i^{-1}(z))}{z} dz = \int_0^\Delta \frac{\hat{p}_i(\hat{x}_i^{-1}(z)) - p_i(\hat{x}_i^{-1}(z))}{z} dz + \int_\Delta^1 \frac{\hat{p}_i(\hat{x}_i^{-1}(z)) - p_i(\hat{x}_i^{-1}(z))}{z} dz.$$

The first integral in the sum evaluates as  $\kappa \Delta / \zeta$ . For the second integral we can evaluate

$$\left| \int_{\Delta}^1 \frac{\widehat{p}_i(\widehat{x}_i^{-1}(z)) - p_i(\widehat{x}_i^{-1}(z))}{z} dz \right| \leq \log \left( \frac{1}{\Delta} \right) \sup_b |\widehat{p}_i(b) - p_i(b)|$$

The second term will dominate, moreover from Theorem 15

$$E \left[ \left| \sqrt{T} \int_0^1 \frac{\widehat{p}_i(\widehat{x}_i^{-1}(z)) - p_i(\widehat{x}_i^{-1}(z))}{z} dz \right| \right] \leq C_1 n^\phi B \alpha_1 \Gamma$$

Now consider the term

$$|p_i(\widehat{x}_i^{-1}(z)) - p_i(x_i^{-1}(z))| \leq L_p |\widehat{x}_i^{-1}(z) - x_i^{-1}(z)|,$$

where  $L_p$  is the Lipschitz constant. Next, consider an identity

$$\widehat{x}_i(\widehat{x}_i^{-1}(z)) - x_i(x_i^{-1}(z)) = 0 = \widehat{x}_i(\widehat{x}_i^{-1}(z)) - x_i(\widehat{x}_i^{-1}(z)) + x_i(\widehat{x}_i^{-1}(z)) - x_i(x_i^{-1}(z)).$$

By our assumption

$$|x_i(\widehat{x}_i^{-1}(z)) - x_i(x_i^{-1}(z))| \geq L_x |\widehat{x}_i^{-1}(z) - x_i^{-1}(z)|.$$

Thus

$$\sup_z |\widehat{x}_i^{-1}(z) - x_i^{-1}(z)| \leq \frac{1}{L_x} \sup_z |\widehat{x}_i(\widehat{x}_i^{-1}(z)) - x_i(\widehat{x}_i^{-1}(z))| \leq \frac{1}{L_x} \sup_b |\widehat{x}_i(b) - x_i(b)|.$$

This means that the application of Theorem 15 leads to

$$E \left[ \sqrt{T} \sup_z |p_i(\widehat{x}_i^{-1}(z)) - p_i(x_i^{-1}(z))| \right] \leq C_2 \frac{L_p}{L_x} n^\phi.$$

Splitting the integral into the two integrals from 0 to  $\Delta$  and from  $\Delta$  to 1, allows us to bound the first integral by  $L_p \Delta / \zeta$  and the second integral is bounded by

$$\log \left( \frac{1}{\Delta} \right) \sup_z |p_i(\widehat{x}_i^{-1}(z)) - p_i(x_i^{-1}(z))|.$$

Combining this with our previous result, yields the statement of the theorem.  $\square$