Supplemental Materials for "Incentive Compatibility of Large Centralized Matching Markets"

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This supplement contains an extended model, which allows that (i) the number of firms may differ from the number workers and (ii) some workers (or firms) may not be acceptable to some firms (or workers). Accordingly, firms (or workers) may remain unmatched in a stable matching. Moreover, we allow that (iii) private values for each pair of a firm and a worker are possibly correlated. Theorem 1 in the main paper holds in this extended model as well.

1 An Extended Model.

Let F be the set of n firms and W be the set of m workers. Utilities are represented by $n \times m$ random matrices $U = [U_{f,w}]$ and $V = [V_{f,w}]$. When a firm f and a worker w match with one another, the firm f receives utility $U_{f,w}$ and the worker w receives utility $V_{f,w}$. For each pair (f, w), utilities are defined as

$$\begin{aligned} U_{f,w} &= \lambda\, U^o_w + (1-\lambda)\,\zeta_{f,w} && \text{and} \\ V_{f,w} &= \lambda\, V^o_f + (1-\lambda)\,\eta_{f,w} && (0 < \lambda \leq 1). \end{aligned}$$

We call U^o_w and V^o_f *common-values*, and $\zeta_{f,w}$ and $\eta_{f,w}$ *private-values*.[1]

Common-values are defined as random vectors

$$U^o := \langle U^o_w \rangle_{w\in W} \quad \text{and} \quad V^o := \langle V^o_f \rangle_{f\in F}.$$

$\langle U^o_w \rangle_{w\in W}$ is an i.i.d sample of size m from a distribution with a positive density function on a bounded support in $\mathbb{R}$. $\langle V^o_f \rangle_{f\in F}$ is defined similarly.

Independent private-values are defined as two $n \times m$ random matrices

$$\zeta := [\zeta_{f,w}] \quad \text{and} \quad \eta := [\eta_{f,w}].$$

Each pair $(\zeta_{f,w}, \eta_{f,w})$ is randomly drawn from a joint distribution on a bounded support in $\mathbb{R}^2$. We normalize utilities such that firms and workers remaining unmatched receive 0 utility.

A random market is defined as a tuple $\langle F, W, U, V\rangle$. We denote realized matrices of U and V by u and v. A market instance is then denoted by $\langle F, W, u, v\rangle$. With probability 1, the market has all distinct utilities,

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[1]Note that we exclude the pure private value case ($\lambda = 0$).

none of which equals to 0. As such, for each $\langle F, W, u, v\rangle$, we can derive a strict preference list $\succ_f$ as

$$\succ_f = w, w', \ldots, f, \ldots, w''$$

if and only if

$$u_{f,w} > u_{f,w'} > \cdots > 0 > \cdots > u_{f,w''}.$$

Take any $\alpha \in (0, \infty)$, and consider a sequence m_n such that $\frac{m_n}{n}$ converges to α. We study properties of stable matchings in the sequence of random markets $\langle F_n, W_{m_n}, U_{n\times m_n}, V_{n\times m_n}\rangle_{n=1}^{\infty}$. We often omit the indexes n and m_n, or simply write n and m.

2 Main Result

Given a market instance $\langle F, W, u, v\rangle$ and a matching μ, we let $u_\mu(\cdot)$ and $v_\mu(\cdot)$ denote utilities from the matching: i.e. $u_\mu(f) := u_{f,\mu(f)}$ and $v_\mu(w) := v_{\mu(w),w}$. For each $f \in F$, we define $\Delta(f; u, v)$ as the difference between utilities from firm-optimal and worker-optimal stable matchings: i.e.

$$\Delta(f; u, v) := u_{\mu_F}(f) - u_{\mu_W}(f).$$

For every $\epsilon > 0$, we have the set of firms whose utilities are within ϵ of one another for all stable matchings, which is denoted by

$$A^F(\epsilon; u, v) := \{f \in F \mid \Delta(f; u, v) < \epsilon\}.$$

Theorem 2.1. *For every $\epsilon > 0$,*

$$E\left[\frac{\left|A^F(\epsilon; U, V)\right|}{n}\right] \to 1, \quad as \quad n \to \infty.$$

We have similar notations and a theorem for workers, which are omitted here.

The intuition of Theorem 2.1 is from the fact that the set of unmatched firms and workers is the same for all stable matchings (McVitie and Wilson (1970)). Firms and workers who remain unmatched have no difference in utilities from all stable matchings. Firms and workers who are matched in stable matchings have small differences in utilities by Theorem 1 in the main paper.

3 Proof of Theorem 2.1

We prove the theorem when $0 < \lambda < 1$. If $\lambda = 1$, assortative matching forms a unique stable matching, and Theorem 1 follows immediately.

We first simplify the notations by compressing λ and $1 - \lambda$ and considering utilities defined as

$$U_{f,w} = U^o_w + \zeta_{f,w} \quad \text{and} \quad V_{f,w} = V^o_f + \eta_{f,w}.$$

We do not lose generality since we can regard common-values and private-values as the ones already multiplied by λ and $1 - \lambda$, respectively.

Let $U^o := \langle U^o_w\rangle_{w\in W}$ be an i.i.d sample of size m from a distribution G^W, and $V^o := \langle V^o_f\rangle_{f\in F}$ be an i.i.d sample of size n from a distribution G^F. G^W and G^F have strictly positive density functions on $\mathbb{R}$. Each pair $(\zeta_{f,w}, \eta_{f,w})$ is randomly drawn from a joint distribution Γ with a support bounded above by $(\bar{u}, \bar{v})$.

We define

$$B^F(\epsilon; u, v) := F \backslash A^F(\epsilon; u, v) = \{f \in F \mid \Delta(f; u, v) \geq \epsilon\}$$

and prove that $\frac{|B^F(\epsilon;U,V)|}{n}$ converges to 0 in probability, which is equivalent to proving convergence to 0 in the mean (Theorem A.2). That is, we fix $\epsilon > 0$ and $K \in \mathbb{N}$, and prove that

$$P\left(\frac{|B^F(\epsilon; U, V)|}{n} > \frac{14}{K}\right) \to 0, \quad \text{as} \quad n \to \infty.$$

3.1 Preliminary Notations

1. ξ_q^F (or ξ_q^W) : q^{th} quantile of G^F (or G^W).
2. $\hat{\xi}_{q;n}^F$: empirical q^{th} quantile of a sample of size n from G^F. We also use $\hat{\xi}_{q;n}^F$ to denote its realization.
3. $\hat{\xi}_{q;m}^W$: empirical q^{th} quantile of a sample of size m from G^W. We also use $\hat{\xi}_{q;m}^W$ to denote its realization.

Since common-values are all distinct with probability 1, we index firms and workers in the order of their common-values: i.e.

$$v_{f_i}^o > v_{f_j}^o \quad \text{and} \quad u_{w_i}^o > u_{w_j}^o, \quad \text{if} \quad i < j.$$

Then, $U_{w_i;m}^o$ (or $V_{f_i;n}^o$) represents the i^{th} highest value of m (or n) order statistics from G^W (or G^F). Note that $U_{w_i;m}^o = \hat{\xi}^W_{\left(1-\frac{i-1}{m}\right);m}$ and $V_{f_i;n}^o = \hat{\xi}^F_{\left(1-\frac{i-1}{n}\right);n}$ by the relationship between order statistics and empirical quantiles (see Appendix A).

Some firms may remain unmatched in stable matchings due to unequal populations of firms and workers, or because some firms (or workers) are not acceptable to some workers (or firms). Especially if a firm has a common value less than $\bar{u}$, all workers consider the firm unacceptable. Roughly, $G^W(-\bar{u})$ is the proportion of workers who are not acceptable to any firm, and $G^F(-\bar{v})$ is the proportion of firms which are not acceptable to any worker. Accordingly, we denote an asymptotic upper bound of the proportion of firms matched in stable matchings by

$$\beta := \min\{\alpha(1 - G^W(-\bar{u})), \quad 1 - G^F(-\bar{v})\}.$$

3.2 Tier-Grouping

We partition $\mathbb{R}$ into

$$\begin{aligned}
I_1^W &:= (\xi^W_{1-\frac{1}{\alpha K}}, \infty) \\
I_2^W &:= (\xi^W_{1-\frac{2}{\alpha K}}, \xi^W_{1-\frac{1}{\alpha K}}] \\
&\cdots \\
I_k^W &:= (\xi^W_{1-\frac{k}{\alpha K}}, \xi^W_{1-\frac{k-1}{\alpha K}}] \\
&\cdots \\
I_{K'}^W &:= (\xi^W_{1-\frac{K'}{\alpha K}}, \xi^W_{1-\frac{K'-1}{\alpha K}}] \\
I_{K'+1}^W &:= (-\infty, \xi^W_{1-\frac{K'}{\alpha K}}],
\end{aligned}$$

where $K' = \lceil \beta K \rceil$.[2]

For each $\langle F, W, u, v \rangle$, we define **the set of workers in tier-k** (with respect to *workers'* common-values) as

$$W_k(u) := \left\{ w \mid u^o_w \in I^W_k \right\} \quad \text{for} \quad k = 1, 2, \dots, K'+1$$

and define **the set of firms in tier-k** (with respect to *workers'* common-values) as

$$F_k(u) := \{ f_i \in F \mid w_i \in W_k(u) \} \quad \text{for} \quad k = 1, 2, \dots, K', \quad \text{and}$$
$$F_{K'+1}(u) := F \setminus \bigcup_{k=1}^{K'} F_k(u).$$

Note that $F_{K'+1}(u)$ may include firms with indexes larger than the number of workers.

Similarly, we partition $\mathbb{R}$ into

$$\begin{aligned}
I^F_1 &:= (\xi^F_{1-\frac{1}{K}}, \infty) \\
I^F_2 &:= (\xi^F_{1-\frac{2}{K}}, \xi^F_{1-\frac{1}{K}}] \\
&\dots \\
I^F_k &:= (\xi^F_{1-\frac{k}{K}}, \xi^F_{1-\frac{k-1}{K}}] \\
&\dots \\
I^F_{K'} &:= (\xi^F_{1-\frac{K'}{K}}, \xi^F_{1-\frac{K'-1}{K}}] \\
I^F_{K'+1} &:= (-\infty, \xi^F_{1-\frac{K'}{K}}].
\end{aligned}$$

where $K' = \lceil \beta K \rceil$.

We define **the set of firms in tier-k** (with respect to *firms'* common-values) as

$$F_k(v) := \left\{ f \mid v^o_f \in I^F_k \right\} \quad \text{for} \quad k = 1, 2, \dots, K', K'+1$$

and define **the set of workers in tier-k** (with respect to *firms'* common-values) as

$$W_k(v) := \{ w_i \in W \mid f_i \in F_k(v) \} \quad \text{for} \quad k = 1, 2, \dots, K', \quad \text{and}$$
$$W_{K'+1}(v) := W \setminus \bigcup_{k=1}^{K'} W_k(v).$$

Note that $W_{K'+1}(v)$ may include workers with indexes larger than the number of firms.

We use the following notations.

1. $u^o_k := \xi^W_{1-\frac{k}{\alpha K}}$ for $k = 1, 2, \dots, K'$: The threshold level of tier-k and tier-$(k+1)$ workers' common-values. That is, $w \in W_k(u)$ if and only if $u^o_k < u^o_w \leq u^o_{k-1}$.

2. $v^o_k := \xi^F_{1-\frac{k}{K}}$ for $k = 1, 2, \dots, K'$: The threshold level of tier-k and tier-$(k+1)$ firms' common-values. That is, $f \in F_k(v)$ if and only if $v^o_k < v^o_f \leq v^o_{k-1}$.

[2] K' is the smallest integer which is greater than or equal to βK. If $1 - \frac{K'}{\alpha K} \leq 0$, we let $\xi^W_{1-\frac{K'}{\alpha K}}$ equals to the infimum of the support of G^W.

Remark 1. 1. *The set of tier-k workers (with respect to workers' common-values) is defined with a random sample. Therefore, $W_k(U)$ is random, and so is $F_k(U)$; whereas, u_k^o is a constant. Similarly, $F_k(V)$ and $W_k(V)$ are random; whereas, v_k^o is a constant.*

2. *Tiers with respect to workers' common-values are in general not the same as tiers with respect to firms' common-values. In particular, we are most likely to have $|F_k(U)| \neq |F_k(V)|$.*

Throughout the proof, we mainly use tiers defined with respect to workers' common-values. However, we need both tier structures in the last part of the proof. We simply write "tier-k" to denote tier-k with respect to workers' common-values, and use "(w.r.t firm) tier-k" to denote tier-k with respect to firms' common-values.

3.3 High-Probability Events

We introduce three events and show that the events occur with probabilities converging to 1 as the market becomes large. We provide proofs for completeness, but the main ideas are simply from the (weak) law of large numbers. In the next section, we will leave the probability that the following events do not occur as a remainder term converging to zero, and focus on the cases where the following events all occur.

3.3.1 No vanishing tier and an equal number of firms and workers in each tier.

Event 1 ($\mathcal{E}_1$)**.** 1. *For $k = 1, 2, \dots, K'$, the sets $F_k(U)$, $W_k(U)$, $F_k(V)$, and $W_k(V)$ are all non empty.*

2. *For $k = 1, 2, \dots, K' - 1$,*

$$|F_k(U)| = |W_k(U)| \quad \textit{and} \quad |F_k(V)| = |W_k(V)|.$$

Proof. The second part immediately follows from the first part. For instance, $F_{K'}(U) \neq \emptyset$ implies that the total number of firms is larger than the number of workers in tier up to $K' - 1$. By definition of tiers with respect to workers' common-values, we have $|F_k(U)| = |W_k(U)|$ for all $k = 1, 2, \dots, K' - 1$.

We only prove that $F_{K'}(U)$ and $W_{K'}(U)$ are non empty with probability converging to one as the market becomes large. Proofs for $k = 1, 2, \dots, K' - 1$ are almost analogous, and we omit here.

Note that

$$1 - \frac{K' - 1}{\alpha K} > 1 - \frac{\beta K}{\alpha K} \geq 0,$$

which implies that for each $w \in W$,

$$P\left(u_w^o \in I_{K'}^W\right) = G^W\left(\xi^W_{1 - \frac{K'-1}{\alpha K}}\right) - G^W\left(\xi^W_{1 - \frac{K'}{\alpha K}}\right) > 0.$$

As such, $W_{K'}(U) = \emptyset$ occurs with probability converging to 0 as the market becomes large.

When $W_{K'}(U)$ is not empty, $F_{K'}(U)$ remains empty only if the total number of firms is no more than the number of workers in tiers up to $K' - 1$. That is,

$$1 \leq \frac{1}{n} \sum_{k=1}^{K'-1} |W_k(U)|. \tag{1}$$

Note that

$$\frac{1}{n}\sum_{k=1}^{K'-1}|W_k(U)| = \frac{m}{n}\cdot\frac{1}{m}\sum_{k=1}^{m}\mathbf{1}\{U^o_w \geq u^o_{K'-1}\}$$
$$\xrightarrow{p} \alpha\cdot\frac{K'-1}{\alpha K} = \frac{\lceil \beta K\rceil}{K} - \frac{1}{K} \leq 1 - \frac{1}{K}.$$

The convergence in probability is by the (weak) law of large numbers and Theorem A.3. Therefore, the inequality (1) holds with probability converging to zero, and thus $F_{K'}(U)$ is not empty with probability converging to 1. □

3.3.2 Distinct common-values of the agents in non-adjacent tiers.

Let $\tilde{\epsilon} > 0$ be such that for any $v, v' \in \mathbb{R}$,

$$|v - v'| \leq \tilde{\epsilon} \implies |G^F(v) - G^F(v')| < \frac{1}{3K},$$

and for any $u, u' \in \mathbb{R}$,

$$|u - u'| \leq \tilde{\epsilon} \implies |G^W(u) - G^W(u')| < \frac{1}{3\alpha K}.$$

There exists such an $\tilde{\epsilon}$ since G^F and G^W are continuous on their bounded supports, so uniformly continuous.

Event 2 ($\mathcal{E}_2$)**.** *For every $k = 1, 2, \ldots, K'-2$,*

$$\min_{\substack{f\in F_k(U)\\ f'\in F_{k+2}(U)}} |V^o_f - V^o_{f'}| > \tilde{\epsilon} \quad \textit{and} \quad \min_{\substack{w\in W_k(V)\\ w'\in W_{k+2}(V)}} |U^o_w - U^o_{w'}| > \tilde{\epsilon}.$$

Proof. We prove only the first part. Fix a realized matrix u such that $\mathcal{E}_1$ holds. For any $k \in 1, 2, \ldots, K'-2$ and for any $w_i \in W_k(u)$ and $w_j \in W_{k+2}(u)$,

$$u^o_{w_i} > u^o_k = \xi^W_{1-\frac{k}{\alpha K}} \quad \text{and} \quad u^o_{w_j} \leq u^o_{k+1} = \xi^W_{1-\frac{k+1}{\alpha K}}. \tag{2}$$

For any $q \in (0,1)$, $\hat{\xi}^W_{q;m} \xrightarrow{p} \xi^W_q$ (Theorem A.4), from which the following inequalities hold with probability converging to 1.

$$\xi^W_{1-\frac{k}{\alpha K}} > \hat{\xi}^W_{1-\frac{k}{\alpha K}-\frac{1}{8\alpha K}} \quad \text{and} \quad \xi^W_{1-\frac{k+1}{\alpha K}} < \hat{\xi}^W_{1-\frac{k+1}{\alpha K}+\frac{1}{8\alpha K}}. \tag{3}$$

Considering (2) and the relation between order statistics and empirical quantiles (see Appendix A), if (3) holds, we have

$$1 - \frac{k}{\alpha K} - \frac{1}{8\alpha K} < \min_{w_i\in W_k(u)}\left(1 - \frac{i-1}{m}\right) = \min_{f_i\in F_k(u)}\left(1 - \frac{i-1}{m}\right),$$

which implies that

$$1 - \frac{k}{K} - \frac{1}{8K} < \min_{f_i\in F_k(u)}\left(1 - \frac{\alpha(i-1)}{m}\right) < \min_{f_i\in F_k(u)}\left(1 - \frac{i-1}{n} + \frac{1}{8K}\right) \quad \text{with large } n.$$

In addition, we have

$$1-\frac{k+1}{K}+\frac{1}{8K}>\max_{w_j\in W_{k+2}(u)}\left(1-\frac{\alpha(j-1)}{m}\right)=\max_{f_j\in F_{k+2}(u)}\left(1-\frac{\alpha(j-1)}{m}\right),$$

which implies that

$$1-\frac{k+1}{K}+\frac{1}{4K}>\max_{f_j\in F_{k+2}(u)}\left(1-\frac{j-1}{n}\right)\quad\text{with large } n.$$

As such for every $f_i\in F_k(u)$ and $f_j\in F_{k+2}(u)$,

$$v^o_{f_i}>\hat{\xi}^F_{1-\frac{k}{K}-\frac{1}{4K}}\quad\text{and}\quad v^o_{f_j}<\hat{\xi}^F_{1-\frac{k+1}{K}+\frac{1}{4K}}.$$

Therefore,

$$\begin{aligned} P\Big(\inf_{\substack{f_i\in F_k(U)\\ f_j\in F_{k+2}(U)}}\big|V^o_{f_i}-V^o_{f_j}\big|\le\tilde{\epsilon}\Big) &\le P\left(\big|\hat{\xi}^F_{1-\frac{k}{K}-\frac{1}{4K}}-\hat{\xi}^F_{1-\frac{k+1}{K}+\frac{1}{4K}}\big|\le\tilde{\epsilon}\right)+R_n\\ &\le P\left(\big|G^F(\hat{\xi}^F_{1-\frac{k}{K}-\frac{1}{4K}})-G^F(\hat{\xi}^F_{1-\frac{k+1}{K}+\frac{1}{4K}})\big|<\frac{1}{3K}\right)+R_n, \end{aligned} \tag{4}$$

where R_n corresponds to the probability that either $\mathcal{E}_1$ does not hold or (3) is violated: i.e. $R_n\to 0$. The last inequality is by the definition of $\tilde{\epsilon}$.

Note that

$$G^F(\hat{\xi}^F_{1-\frac{k}{K}-\frac{1}{4K}})-G^F(\hat{\xi}^F_{1-\frac{k+1}{K}+\frac{1}{4K}})\xrightarrow{p}\frac{1}{2K}$$

by Theorem A.4 and continuity of G^F(Theorem A.3). As a result, the right hand side of (4) converges to 0. □

3.3.3 Similarity between tiers w.r.t workers' common-values and tiers w.r.t firms' common-values

Event 3 ($\mathcal{E}_3$)**.** *For every* $k=1,2,3,\ldots,K'+1$,

$$F_k(U)\subset\bigcup_{k'=k-1}^{k+1}F_{k'}(V)\quad\textit{and}\quad W_k(V)\subset\bigcup_{k'=k-1}^{k+1}W_{k'}(U).^3$$

Proof. We prove the first part for $k=1,\ldots,K'$ under the condition that $\mathcal{E}_1$ holds.[4]

For each realized (u,v), we have

$$\{u^o_w|w\in W_k(u)\}\subset\big(u^o_k,u^o_{k-1}\big]=\left(\xi^W_{1-\frac{k}{\alpha K}},\xi^W_{1-\frac{k-1}{\alpha K}}\right]. \tag{5}$$

Suppose

$$\left(\xi^W_{1-\frac{k}{\alpha K}},\xi^W_{1-\frac{k-1}{\alpha K}}\right]\subset\left(\hat{\xi}^W_{1-\frac{k}{\alpha K}-\frac{1}{3\alpha K}},\hat{\xi}^W_{1-\frac{k-1}{\alpha K}+\frac{1}{3\alpha K}}\right], \tag{6}$$

[3] We define $F_0(V)$, $W_0(V)$, $W_{K'+2}(U)$, and $W_{K'+2}(U)$ as empty sets.

[4] For $k=1,2$, we need to modify the proof by replacing the intervals such as $(\xi^W_{1-\frac{k}{\alpha K}},\xi^W_{1-\frac{k-1}{\alpha K}}]$ with $(\xi^W_{1-\frac{k}{\alpha K}},\infty)$ and $(\xi^F_{1-\frac{k+1}{K}},\xi^F_{1-\frac{k-2}{K}}]$ with $(\xi^F_{1-\frac{k+1}{K}},\infty)$. We omit the modifications since they are trivial and tedious.

and

$$\left(\hat{\xi}^F_{1-\frac{k}{K}-\frac{2}{3K}}, \hat{\xi}^F_{1-\frac{k-1}{K}+\frac{2}{3K}}\right] \subset \left(\xi^F_{1-\frac{k+1}{K}}, \xi^F_{1-\frac{k-2}{K}}\right]. \tag{7}$$

If (6) hold, then (5) implies that for every tier-k worker w_i, we have

$$u^o_{w_i} \in \left(\hat{\xi}^W_{1-\frac{k}{\alpha K}-\frac{1}{3\alpha K}}, \hat{\xi}^W_{1-\frac{k-1}{\alpha K}+\frac{1}{3\alpha K}}\right],$$

and thus,

$$1 - \frac{i-1}{m} \in \left(1 - \frac{k}{\alpha K} - \frac{1}{3\alpha K}, 1 - \frac{k-1}{\alpha K} + \frac{1}{3\alpha K}\right],$$

which implies that

$$1 - \frac{i-1}{n} \in \left(1 - \frac{k}{K} - \frac{2}{3K}, 1 - \frac{k-1}{K} + \frac{2}{3K}\right] \quad \text{with large } n.$$

Then for any tier-k firm f_i, we have

$$v^o_{f_i} \in \left(\hat{\xi}^F_{1-\frac{k}{K}-\frac{2}{3K}}, \hat{\xi}^F_{1-\frac{k-1}{K}+\frac{2}{3K}}\right],$$

which implies that

$$\left\{v^o_f \mid f \in F_k(u)\right\} \subset \left(\hat{\xi}^F_{1-\frac{k}{K}-\frac{2}{3K}}, \hat{\xi}^F_{1-\frac{k-1}{K}+\frac{2}{3K}}\right].$$

Consequently if both (6) and (7) hold, then

$$\begin{aligned}
\left\{v^o_f \mid f \in F_k(u)\right\} &\subset \left(\hat{\xi}^F_{1-\frac{k}{K}-\frac{2}{3K}}, \hat{\xi}^F_{1-\frac{k-1}{K}+\frac{2}{3K}}\right] \\
&\subset \left(\xi^F_{1-\frac{k+1}{K}}, \xi^F_{1-\frac{k-2}{K}}\right] \\
&= \bigcup_{k'=k-1}^{k+1} I^F_{k'}.
\end{aligned}$$

In other words,

$$F_k(u) \subset \bigcup_{k'=k-1}^{k+1} F_{k'}(v).$$

Inequalities (6) and (7), and $\mathcal{E}_1$ occur with probability converging to 1 (Theorem A.4), and thus the event $\mathcal{E}_3$ for $k = 1, 2, \ldots, K'$ also occurs with probability converging to 1.

Lastly for $k = K' + 1$,

$$F_{K'+1}(U) \subset F_{K'}(V) \cup F_{K'+1}(V)$$

occurs with probability converging to 1, since the event occurs whenever

$$F_k(V) \subset \bigcup_{k'=k-1}^{k+1} F_{k'}(U) \quad \text{for all } k = 1, 2, \ldots, K' - 1$$

holds.

□

3.4 Proof of the Theorem 1

We choose K large enough that

$$\max_{1\le k\le K'-2}\left|u_k^o-u_{k+1}^o\right| \equiv \max_{1\le k\le K'-2}\left|\xi^W_{1-\frac{k}{\alpha K}}-\xi^W_{1-\frac{k+1}{\alpha K}}\right| < \frac{\epsilon}{9}.^{5} \tag{8}$$

The proof of Theorem 1 is completed by the following inequalities.

$$\begin{aligned} P\left(\frac{|B^F(\epsilon;U,V)|}{n}>\frac{14}{K}\right) &= P\left(\sum_{1\le k\le K'+1}\frac{|B^F_k(\epsilon;U,V)|}{n}>\frac{14}{K}\right)\\ &< P\left(\sum_{7\le k\le K'-3}\frac{|B^F_k(\epsilon;U,V)|}{n}+\sum_{\substack{k=1,\dots,6,\\ K'-2,K'-1,K'}}\frac{F_k(U)}{n}+\frac{|B^F_{K'+1}(\epsilon;U,V)|}{n}>\frac{14}{K}\right). \end{aligned}$$

We show that the last term converges to 0. We first prove that for each $k=7,\dots,K'-3$, the proportion $\frac{|B^F_k(\epsilon;U,V)|}{n}$ converges to 0 in probability (Proposition 3.2). The proof identifies asymptotic upper and lower bounds of utilities from all stable matchings and shows that the two bounds are close to each other. We then prove that $\frac{|B^F_{K'+1}(\epsilon;U,V)|}{n}$ is asymptotically bounded above by $\frac{4}{K}$ (Proposition 3.3). The proof shows that most tier-$K'+1$ firms remain unmatched in stable matchings, and thus have no difference in utilities. Lastly, for each $k=1,\dots,6,K'-2,K'-1,K'$, the proportion $\frac{F_k(U)}{n}$ converges to at most $\frac{1}{K}$ in probability by the (weak) law of large numbers.

3.4.1 For $k=7,\dots,K'-3$, $\frac{|B^F_k(\epsilon;U,V)|}{n}\xrightarrow{p}0$.

We first identify an asymptotic lower bound on utilities of firms in each tier, using techniques from the theory of random bipartite graphs (Proposition 3.1). Similarly, we find an asymptotic lower bound on utilities of workers in each tier (Proposition 3.1*). The asymptotic lower bound on utilities of workers induces an asymptotic upper bound on utilities of firms in each tier. Lastly, we complete the proof by showing that the asymptotic lower and upper bounds are close to each other (Proposition 3.2).

Proposition 3.1. *For each instance $\langle F,W,u,v\rangle$ and for each $\bar{k}=1,2,\dots,K'-3$, define*

$$\hat{B}^F_{\bar{k}}(\epsilon;u,v) := \left\{f\in F_{\bar{k}}(u) : u_{\mu_W}(f)\le u^o_{\bar{k}+2}+\bar{u}-\epsilon\right\}.^{6}$$

Then for any $\epsilon>0$,

$$\frac{|\hat{B}^F_{\bar{k}}(\epsilon;U,V)|}{n}\xrightarrow{p}0 \quad \text{as} \quad n\to\infty.$$

Proof. For each instance $\langle F,W,u,v\rangle$ and for each $k=1,2,\dots,K'+1$, let $F_{\le k}(u):=\bigcup_{k'\le k}F_{k'}(u)$ and $F_{<k}(u):=\bigcup_{k'<k}F_{k'}(u)$. We similarly define $W_{\le k}(u)$ and $W_{<k}(u)$.

Take any $\bar{k}$ from $\{1,2,\dots,K'-3\}$. We construct a bipartite graph with $F_{\bar{k}}(u)\cup W_{\le\bar{k}+2}(u)$ as a bi-partitioned set of nodes. Two vertices $f\in F_{\bar{k}}(u)$ and $w\in W_{\le\bar{k}+2}(u)$ are joined by an edge if and only if

$$\zeta_{f,w}\le\bar{u}-\epsilon \quad \text{or} \quad \eta_{f,w}\le\bar{v}-\tilde{\epsilon},$$

[5] We can always satisfy the condition since G^W has a strictly positive density function on a bounded support.

[6] Note that $u^o_{\bar{k}+2}+\bar{u}$ is the maximal utility that a firm can achieve by being matched with a worker in tier-$(\bar{k}+3)$.

where $\tilde{\epsilon}$ is the value taken before, while defining $\mathcal{E}_2$.

Let $\bar{W}_{\leq \bar{k}+2}(u,v)$ be the set of workers in tiers up to $\bar{k}+2$ who are *not* matched with firms in tiers up to $\bar{k}+1$ in μ_W. That is,

$$\bar{W}_{\leq \bar{k}+2}(u,v) := \left\{ w \in W_{\leq \bar{k}+2}(u) \,|\, \mu_W(w) \notin F_{\leq \bar{k}+1}(u) \right\}.$$

We now show that if $\mathcal{E}_2$ holds, then

$$\hat{B}^F_{\bar{k}}(\epsilon; u, v) \cup \bar{W}_{\leq \bar{k}+2}(u,v)$$

is a biclique.

Suppose, towards a contradiction, that a pair of $f \in \hat{B}^F_{\bar{k}}(\epsilon; u, v)$ and $w \in \bar{W}_{\leq \bar{k}+2}(u,v)$ is *not* joined by an edge: i.e.

$$\zeta_{f,w} > \bar{u} - \epsilon \quad \text{and} \quad \eta_{f,w} > \bar{v} - \tilde{\epsilon}.$$

Then, we have

$$u_{f,w} = u^o_w + \zeta_{f,w} > u^o_{\bar{k}+2} + \zeta_{f,w} > u^o_{\bar{k}+2} + \bar{u} - \epsilon, \tag{9}$$

and

$$v_{f,w} = v^o_f + \eta_{f,w} \geq \min_{f' \in F_{\bar{k}}(u)} v^o_{f'} + \eta_{f,w} > \min_{f' \in F_{\bar{k}}(u)} v^o_{f'} + \bar{v} - \tilde{\epsilon}.^7$$

Conditioned on $\mathcal{E}_2$, we can proceed further and obtain

$$\begin{aligned} v_{f,w} &> \min_{f' \in F_{\bar{k}}(u)} v^o_{f'} + \bar{v} - \left(\min_{f' \in F_{\bar{k}}(u)} v^o_{f'} - \max_{f'' \in F_{\bar{k}+2}(u)} v^o_{f''} \right) \\ &= \max_{f'' \in F_{\bar{k}+2}(u)} v^o_{f''} + \bar{v}. \end{aligned} \tag{10}$$

On the other hand, $f \in \hat{B}^F_{\bar{k}}(\epsilon; u, v)$ implies that

$$u_{\mu_W}(f) \leq u^o_{\bar{k}+2} + \bar{u} - \epsilon,$$

and $w \in \bar{W}_{\leq \bar{k}+2}(u,v)$ implies that

$$v_{\mu_W}(w) \leq \max_{f'' \in F_{\bar{k}+2}(u)} v^o_{f''} + \bar{v},$$

since a worker can obtain utility higher than $\max_{f'' \in F_{\bar{k}+2}(u)} v^o_{f''} + \bar{v}$ only by matching with a firm in $F_{\leq \bar{k}+1}(u)$.

Equations (9) and (10) imply that (f, w) would have blocked μ_W, contradicting that μ_W is stable. Therefore,

$$\hat{B}^F_{\bar{k}}(\epsilon; u, v) \cup \bar{W}_{\leq \bar{k}+2}(u,v).$$

is a biclique, which is not necessarily balanced.

We now control the size of $\hat{B}^F_{\bar{k}}(\epsilon; U, V)$ by referencing Theorem 3. Let u^o and v^o be realized common-values such that events $\mathcal{E}_1$ and $\mathcal{E}_2$ hold. Then, the remaining randomness of U and V is from ζ and η. Consider a random bipartite graph with $F_{\bar{k}}(U) \cup W_{\leq \bar{k}+2}(U)$ as a bi-partitioned set of nodes, where each pair of $f \in F_{\bar{k}}(U)$ and $w \in W_{\leq \bar{k}+2}(U)$ is joined by an edge if and only if

$$\zeta_{f,w} \leq \bar{u} - \epsilon \quad \text{or} \quad \eta_{f,w} \leq \bar{v} - \tilde{\epsilon}.$$

[7] We should not replace $\min_{f' \in F_{\bar{k}}(u)} v^o_{f'}$ with $v^o_{\bar{k}}$. $F_{\bar{k}}(u)$ is defined with respect to workers' common-values, rather than firms' common-values.

In other words, every pair is joined by an edge independently with probability $p(\epsilon) = 1 - \Gamma(\bar{u} - \epsilon, \bar{v} - \tilde{\epsilon})$.

We write $\beta(n) := 2 \cdot \log(|W_{\leq \bar{k}+2}(U)|)/\log \frac{1}{p(\epsilon)}$ and show that

$$P\left(|\hat{B}^F_{\bar{k}}(\epsilon; U, V)| \leq \beta(n)\right) \to 1 \quad \text{as} \quad n \to \infty. \tag{11}$$

Consider that

$$P\left(|\hat{B}^F_{\bar{k}}(\epsilon; U, V)| \leq \beta(n)\right) \geq P\left(\min\{|\hat{B}^F_{\bar{k}}(\epsilon; U, V)|, |\bar{W}_{\leq \bar{k}+2}(U, V)|\} \leq \beta(n)\right) - P\left(|\bar{W}_{\leq \bar{k}+2}(U, V)| \leq \beta(n)\right).$$

We show that the two terms on the right hand side converges respectively to 1 and 0 in probability.

Let $\alpha(U, V) \times \alpha(U, V)$ be the size of a maximum balance biclique of the random graph

$$G\left(F_{\bar{k}}(U) \cup W_{\leq \bar{k}+2}(U)\ ,\ p(\epsilon)\right).$$

Since every realized $\hat{B}^F_{\bar{k}}(\epsilon; u, v) \cup \bar{W}_{\leq \bar{k}+2}(u, v)$ is a biclique, it contains a balanced biclique of the size equals to

$$\min\left\{|\hat{B}^F_{\bar{k}}(\epsilon; u, v)|\ ,\ |\bar{W}_{\leq \bar{k}+2}(u, v)|\right\}.$$

Therefore,

$$P\left(\min\left\{|\hat{B}^F_{\bar{k}}(\epsilon; U, V)|, |\bar{W}_{\leq \bar{k}+2}(U, V)|\right\} \leq \beta(n)\right) \geq P\left(\alpha(U, V) \leq \beta(n)\right) \to 1, \tag{12}$$

where the convergence is from Theorem 3.

On the other hand, observe that $\bar{W}_{\leq \bar{k}+2}(U, V)$ is the size of at least $|W_{\bar{k}+2}(U)|$. Among workers in tiers up to $\bar{k}+2$ at most $|W_{\leq \bar{k}+1}(U)|$ are matched with firms in tiers up to $\bar{k}+1$. In addition, $\frac{|W_{\bar{k}+2}(U)|}{n}$ converges to $\frac{1}{K}$ by the (weak) law of large numbers. Therefore,

$$P\left(|\bar{W}_{\leq \bar{k}+2}(U, V)| \leq \beta(n)\right) \to 0. \tag{13}$$

Equations (12) and (13) imply that (11) holds.

Lastly, we consider random utilities U and V, in which common-values are yet realized. For every $\epsilon' > 0$,

$$\begin{aligned} P\left(\frac{|\hat{B}^F_{\bar{k}}(\epsilon; U, V)|}{n} > \epsilon'\right) &= P\left(|\hat{B}^F_{\bar{k}}(\epsilon; U, V)| > \epsilon' \cdot n\right) \\ &\leq P\left(|\hat{B}^F_{\bar{k}}(\epsilon; U, V)| > \beta(n) \mid \mathcal{E}_1, \mathcal{E}_2, \beta(n) \leq \epsilon' n\right) + R_n, \quad \text{with large } n, \end{aligned}$$

where R_n is the probability that either $\mathcal{E}_1$ or $\mathcal{E}_2$ does not hold, or $\beta(n) \leq \epsilon' n$ is violated: i.e. $R_n \to 0$. We complete the proof by applying (11). □

We also obtain the counterpart proposition of Proposition 3.1 in terms of tiers defined with respect to firms' common-values.

Proposition 3.1* *For each* $\bar{k} = 1, 2, \ldots, K' - 3$, *define*

$$\hat{B}^W_{\bar{k}}(\epsilon; u, v) := \left\{w \in W_{\bar{k}}(v) | v_{\mu_F}(w) \leq v^o_{\bar{k}+2} + \bar{v} - \epsilon\right\}.$$

Then for any $\epsilon > 0$,

$$\frac{|\hat{B}^W_{\bar{k}}(\epsilon; U, V)|}{n} \xrightarrow{p} 0 \quad \textit{as} \quad n \to \infty.$$

Proof. We omit the proof since it is analogous to the proof of Proposition 3.1. □

For each instance $\langle F, W, u, v\rangle$, we define

$$B^F_{\bar{k}}(\epsilon; u, v) := \{f \in F_{\bar{k}}(u) | \Delta(f; u, v) \geq \epsilon\} \quad \text{for} \quad \bar{k} = 1, 2, \ldots, K' + 1.$$

Proposition 3.2. *If $\bar{k} = 7, 8, \ldots, K' - 3$, then for any $\epsilon > 0$,*

$$\frac{|B^F_{\bar{k}}(\epsilon; U, V)|}{n} \xrightarrow{p} 0 \quad \textit{as} \quad n \to \infty.$$

Proof. In Proposition 3.1*, for $k = 1, 2, \ldots, K' - 3$, let

$$\epsilon_k := v^o_{k+2} - v^o_{k+3},$$

and write

$$\hat{B}^W_k(\epsilon_k; u, v) = \left\{w \in W_k(v) | v_{\mu_F}(w) \leq v^o_{k+3} + \bar{v}\right\}.$$ [8]

By Proposition 3.1*,

$$\frac{|\hat{B}^W_k(\epsilon_k; U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{14}$$

Note that a worker receives utility higher than $v^o_{k+3} + \bar{v}$ only by matching with a firm in (w.r.t firm) tiers up to $k + 3$.[9] Thus for $k = 5, 6, \ldots, K' + 1$,

$$\{w \in W_{\leq k-4}(V) : \mu(w) \in F_k(V)\} \subset \bigcup_{k'=1}^{k-4} \hat{B}^W_{k'}(\epsilon_{k'}; U, V). \tag{15}$$

If event $\mathcal{E}_3$ holds, we can translate (15) into an expression with tiers w.r.t workers' common-values. That is, for $k = 7, 8, \ldots, K' + 1$,

$$\begin{aligned}
\{w \in W_{\leq k-6}(U) : \mu_F(w) \in F_k(U)\} &\subset \bigcup_{k'=k-1}^{k+1} \{w \in W_{\leq k-6}(U) : \mu_F(w) \in F_{k'}(V)\} \\
&\subset \bigcup_{k'=k-1}^{k+1} \{w \in W_{\leq k-5}(V) : \mu_F(w) \in F_{k'}(V)\} \\
&\subset \bigcup_{k'=k-1}^{k+1} \{w \in W_{\leq k'-4}(V) : \mu_F(w) \in F_{k'}(V)\}
\end{aligned}$$

where the first and second inequalities are from $\mathcal{E}_3$.

[8] Recall that v^o_k is a constant, defined as $v^o_k := \xi^F_{1-\frac{k}{K}}$.

[9] Recall that $f \in F_k(v)$ if and only if $v^o_k < v^o_f \leq v^o_{k-1}$. Thus, if $f \in F_{>k+3}(v)$ then $v^o_f \leq v^o_{k+3}$.

By applying (15), we obtain

$$\{w \in W_{\leq k-6}(U) : \mu_F(w) \in F_k(U)\} \subset \bigcup_{k'=1}^{k-3} \hat{B}^W_{k'}(\epsilon_{k'}; U, V).$$

It follows that

$$\frac{|\{f \in F_k(U) : \mu_F(f) \in W_{\leq k-6}(U)\}|}{n} \xrightarrow{p} 0, \tag{16}$$

because for every $\epsilon > 0$,

$$P\left(\frac{|\{f \in F_k(U) : \mu_F(f) \in W_{\leq k-6}(U)\}|}{n} > \epsilon\right) \leq P\left(\sum_{k'=1}^{k-3} \frac{|\hat{B}^W_{k'}(\epsilon_{k'}; U, V)|}{n} > \epsilon\right) + R_n,$$

where R_n is the probability that $\mathcal{E}_3$ does not hold: i.e. $R_n \to 0$. The right hand side converges to 0 by (14).

We complete the proof of Proposition 3.2 by proving the following claim. Proposition 3.1 and (16) show that the normalized sizes of two sets on the right hand side of (17) converge to 0 in probability.

Claim 3.1. *For* $\bar{k} = 7, 8, \ldots, K' - 3$ *and each instance* $\langle F, W, u, v\rangle$,

$$B^F_{\bar{k}}(\epsilon; u, v) \subset \hat{B}^F_{\bar{k}}(\epsilon/9; u, v) \cup \left\{f \in F_{\bar{k}}(u) | \mu_F(f) \in W_{\leq \bar{k}-6}(u)\right\}. \tag{17}$$

Proof of Claim 3.1. If a firm $f \in F_{\bar{k}}(u)$ is *not* in $\hat{B}^F_{\bar{k}}(\epsilon/9; u, v)$, then

$$u_{\mu_W}(f) > u^o_{\bar{k}+2} + \bar{u} - \epsilon/9,$$

and if the firm f is *not* in $\left\{f \in F_{\bar{k}}(u) | \mu_F(f) \in W_{\leq \bar{k}-6}(u)\right\}$, then

$$u_{\mu_F}(f) \leq u^o_{\bar{k}-6} + \bar{u}.$$

Therefore, using (8) we obtain

$$u_{\mu_F}(f) - u_{\mu_W}(f) \leq u^o_{\bar{k}-6} - u^o_{\bar{k}+2} + \epsilon/9 < \epsilon,$$

and thus f is *not* in $B^F_{\bar{k}}(\epsilon; u, v)$. □

□

3.4.2 Firms in tier $K' + 1$

We show that most firms in tier-$(K' + 1)$ remain unmatched in stable matchings. Unmatched firms' utilities from μ_F and μ_W are clearly less than ϵ difference from each other.

Proposition 3.3.

$$P\left(\frac{|B^F_{K'+1}(\epsilon; U, V)|}{n} > \frac{4}{K}\right) \to 0 \quad as \quad n \to \infty.$$

Proof. We divide the proof into two cases.

Case 1. $\beta = 1 - G^F(-\bar{v})$: only a small proportion of firms in tier $K' + 1$ are acceptable to workers.

For each $\langle F, W, u, v\rangle$, if $\mathcal{E}_3$ holds,

$$F_{K'+1}(u) \subset F_{K'}(v) \cup F_{K'+1}(v).$$

If $f \in F_{K'+1}(v)$,

$$v_f^o \leq \xi^F_{1-\frac{K'}{K}} = \xi^F_{1-\frac{\lceil \beta K \rceil}{K}} \leq \xi^F_{1-\beta} = -\bar{v}.^{10}$$

That is, if there is a firm in tier-$K'+1$, the firm is unacceptable to all workers regardless of the firm's private values to the workers. The firm remains unmatched in all stable matchings and have no difference in utilities from stable matchings. Therefore, conditioned on $\mathcal{E}_3$,

$$\frac{|B^F_{K'+1}(\epsilon; U, V)|}{n} \leq \frac{|F_{K'}(V)|}{n}.$$

Proposition 3.3 holds from the following convergence result.

$$\frac{|F_{K'}(V)|}{n} \xrightarrow{p} \frac{\beta K - (\lceil \beta K \rceil - 1)}{K} \quad \text{as} \quad n \to \infty.$$

Case 2. $\beta = \alpha(1 - G^W(-\bar{u}))$: firms in tier-$(K'+1)$ see only a small proportion of acceptable workers available.

For each market $\langle F, W, u, v\rangle$, if $w \in W_{K'+1}(u)$,

$$u_w^o \leq \xi^W_{1-\frac{\lceil \beta K \rceil}{\alpha K}} \leq \xi^W_{1-\frac{\beta}{\alpha}} = -\bar{u}.^{11}$$

That is, workers in $W_{K'+1}(u)$ are unacceptable to all firms. Therefore, the total number of matched workers in stable matchings is no more than the total number of workers in tiers up to K': i.e.

$$|\{w \in W | \mu_W(w) \in F\}| \leq \sum_{k=1}^{K'} |W_k(U)|,$$

which implies that

$$|\{f \in F | \mu_W(f) \in W\}| \leq \sum_{k=1}^{K'} |W_k(U)|.$$

As such, we have

$$\begin{aligned} |\{f \in F_{K'+1}(U) | \mu_W(f) \in W\}| &= |\{f \in F | \mu_W(f) \in W\}| - \sum_{k=1}^{K'} |\{f \in F_k(U) | \mu_W(f) \in W\}| \\ &\leq \sum_{k=1}^{K'} |W_k(U)| - \sum_{k=1}^{K'} |\{f \in F_k(U) | \mu_W(f) \in W\}|. \end{aligned}$$

[10] Note that $f \in F_{K'+1}(v)$ implies $1 - \frac{K'}{K} > 0$ and $G^F(-\bar{v}) > 0$, which we used to derive the inequalities.

[11] Note that $w \in W_{K'+1}(u)$ implies $1 - \frac{K'}{K} > 0$ and $G^W(-\bar{u}) > 0$, which we used to derive the inequalities.

Conditioned on $\mathcal{E}_1$,

$$\begin{aligned}|\{f \in F_{K'+1}(U) | \mu_W(f) \in W\}| &\leq \sum_{k=1}^{K'-3} (|F_k(U)| - |\{f \in F_k(U) | \mu_W(f) \in W\}|) + \sum_{k=K'-2}^{K'} |W_k(U)| \\ &= \sum_{k=1}^{K'-3} |\{f \in F_k(U) | \mu_W(f) \notin W\}| + \sum_{k=K'-2}^{K'} |W_k(U)|.\end{aligned}$$

With a small $\epsilon' > 0$,

$$\begin{aligned}\frac{|B^F_{K'+1}(\epsilon; U, V)|}{n} &\leq \frac{|\{f \in F_{K'+1}(U) | \mu_W(f) \in W\}|}{n} \\ &\leq \sum_{k=1}^{K'-3} \frac{|\hat{B}^F_k(\epsilon'; U, V)|}{n} + \sum_{k=K'-2}^{K'} \frac{|W_k(U)|}{n} \\ &\xrightarrow{p} 0 + \frac{3 + (\beta K - \lceil \beta K \rceil)}{K},\end{aligned}$$

where the convergence in probability is from Proposition 3.1 and the (weak) law of large numbers.

□

References

McVitie, D., and L. Wilson (1970): "Stable marriage assignment for unequal sets," *BIT Numerical Mathematics*, 10(3), 295–309.