A Simple Scheme to Improve the Efficiency of Referenda*

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Abstract

This paper proposes a simple scheme designed to elicit and reward intensity of preferences in referenda: voters faced with a number of binary proposals are given one regular vote for each proposal plus an additional number of bonus votes to cast as desired. Decisions are taken according to the majority of votes cast. In our base case, where there is no systematic difference between proposals' supporters and opponents, there is always a positive number of bonus votes such that ex ante utility is increased by the scheme, relative to simple majority voting. When the distributions of valuations of supporters and opponents differ, the improvement in efficiency is guaranteed if the distributions can be ranked according to first order stochastic dominance. If they are, the existence of welfare gains is independent of the exact number of bonus votes.

1 Introduction

In binary decisions—when a proposal can either pass or fail—majority voting has a number of important qualities: it treats all voters symmetrically, it is neutral towards the two alternatives, it reflects accurately changes in preferences in either direction, and it guarantees that voters cannot gain by lying about their preferences. It has, however, one drawback: it fails to account for the intensity of these preferences. Far from being a detail, this one weakness contributes to important practical problems: first of all, the possibility to inflict great harm to the minority; more generally, the blocking of proposals that would increase

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conventional measures of social welfare, the temptation to recur to log-rolling in committees, the common lack of transparency of political deliberations. In all democratic systems, sophisticated institutions are designed to counter these difficulties. In some cases, however, it may be useful to approach the problem more directly, and ask whether a voting system as simple as majority voting but rewarding intense preferences could be designed for binary decisions.

The functioning of prices in a market offers some inspiration: prices elicit consumers' intensity of preferences by differentiating across goods and functioning in tandem with a budget constraint. The budget constraint plays a central role and suggests an immediate idea: suppose voters were given a stock of votes, and asked to allocate them as they see fit over a series of binary proposals, each of which would then be decided on the basis of the majority of votes cast. Would voters be led to cast more votes over those issues to which they attach more importance? And would the final result then be an expected welfare gain, relative to simple majority voting, as the probability of winning a vote shifts for each voter from issues of relatively less importance towards issues of relatively more importance? We have proposed a voting system of this type—*storable votes*—in two recent papers that study voting behavior in committees (Casella, 2005, and Casella, Gelman and Palfrey, forthcoming). The simple intuition proves correct: both in theory and in experiments subjects cast more votes when the intensity of their preferences is higher. The efficiency gains are also borne out: both in theory and in the experiments, ex ante utility is typically higher with storable votes (although some counterexamples exist). (See also, independently, Hortala-Vallve (2003)).

In this paper, we propose to apply this idea to referenda[1]. There are several reasons to do so. First, as tools for policy-making, referenda are becoming both more common and more important, a point made abundantly clear in Europe by the recent derailing of the European Union Constitution[2]. Second, from a theoretical point of view, they are relatively easy to study because the large population of voters eliminates most of the strategic considerations that complicate the analysis of voting choices in committees. Finally, referenda are often submitted to voters in bundles—think of the sets of propositions on which voters vote contemporaneously in many US states and European countries.[3] Consider then a voting mechanism where voters are faced with a number of contemporaneous, unrelated referenda, and are asked to cast one vote on each

[1] We use the term "referenda" to indicate any proposition decided by popular majority voting, whether initiated by the government (referenda in the proper sense) or by the people (initiatives).

[2] Gerber (1999), Matsusaka (2004), the Inititative and Referendum Institute at www.iandrinstitute.org, and the Direct Democracy Institute at www.c2d.unige.ch provide a wealth of information on the history and practice of direct democracy around the world. Referenda are now used in many democracies (in Switzerland, of course, but also in the U.S., the European Union, Australia, and other countries), and their number is rising (in US states, for example, the number of referenda has increased in every decade since 1970, at an average rate of seventy per cent per decade).

[3] In many European countries, the practice of bundling referenda is less common when the stakes are high - a mistake, according to our analysis.

referendum but in addition are given a number of "bonus votes" to cast as desired over the different referenda. Each referendum is then decided according to the majority of the votes. Does the addition of the bonus votes allow voters to express the intensity of their preferences and increase their ex ante welfare, relative to simple majority voting? This is the question studied in this paper.

We begin by addressing the problem with a simple model where individual valuations are drawn independently from a known distribution, identical across both voters and referenda and symmetrical with respect to the direction of preferences. We find that the answer is positive if the number of bonus votes is not too large. Intuitively, the bonus votes give voters the possibility to target the single issue that is most important to them (in equilibrium the bonus votes are never split), but at the cost of more uncertainty over the other proposals. The trade-off between the two effects implies that the optimal number of bonus votes should be related to the expected edge between the representative voter's highest expected valuation and his or her mean valuation over all proposals. If such an edge is small, the number of bonus votes should be correspondingly small. But the number should not be zero: for all distributions of valuations there is a positive number of bonus votes such that ex ante welfare rises, relative to simple majority voting.

After presenting our analysis in the simplest setting, we devote the rest of the paper to relaxing different assumptions and checking the robustness of the first result. We verify that the result continues to hold if the distributions of valuations differ across referenda, as seems plausible, and that allowing voters to fine-tune the ranking of different referenda through the assignment of a perfectly divisible vote, as opposed to a number of discrete bonus votes, is not a significant improvement. We study whether the result holds when the probability of approval of each referendum is not known, but is a draw from a probability distribution - i.e. when the median of the distribution of valuations is itself unknown. Indeed we find that in this case our conclusion is strengthened: granting bonus votes increases ex ante welfare, relative to majority voting, regardless of the number of votes.

In all of these cases, for simplicity we rule out systematic asymmetries in the distributions of valuations. But by doing so, we also limit by assumption the role of the bonus votes. The intuition behind the idea discussed in this paper is extremely simple: bonus votes are valuable when the preferences of the minority are particularly intense, relative to the majority - in other words, their role is to recognize and give weight to possible asymmetries in valuation draws. When the distributions are assumed to be symmetric, asymmetries can only be occasional sample deviations from the theoretical distributions, bound to disappear in large electorates. Indeed, although bonus votes can improve ex ante welfare in all the models discussed above, the quantitative improvement over simple majority voting becomes vanishingly small in the limit, as the population approaches infinity. (The same can be said of majority voting over random decision making.) Introducing asymmetries is then important, but their choice can be arbitrary. In a thorough empirical analysis of more than 800 ballot propositions in California from 1912 through 1989, Matsusaka

(1992) identifies the expectation of an equally split electorate as characteristic of propositions submitted to popular vote (as opposed to being decided by the legislature). Anchoring our model with this observation, we assume that the population is equally split on all proposals, but mean intensity is higher on one side. In this case, bonus votes are guaranteed to increase ex ante utility if the distribution of valuations on the side with higher mean first-order stochastically dominates the distribution on the opposite side; that is, if the mass of voters with more intense preferences is larger on the side with higher mean. When this sufficient condition is satisfied, the superiority of bonus votes over majority voting holds independently of the exact number of bonus votes and remains true asymptotically (whereas, with equally split electorates, majority voting again converges to random decision-making). First order stochastic dominance is a sufficient condition for robust welfare gains, but our numerical exercises suggest that the result is more general: especially when the number of referenda is not large, counterexamples where simple majority voting is superior are not easy to construct.

It is this more general case of asymmetric distributions that better captures the basic intuition for bonus votes.[4] If voters are equally split on a proposal, efficiency demands that the side with the higher intensity of preferences prevails; and if the voters are not equally split, a strongly affected minority should at time prevail over a less affected majority. This is the outcome that bonus votes can deliver. The conclusion need not involve interpersonal comparisons of utility: in the ex ante evaluation, at a constitutional stage taking place before specific ballots are realized, all voters are identical and the representative voter weighs the probabilities of his or her yet unrevealed valuations.

But is the need for stronger minority representation a real need in practice? Anecdotal reports abound on the distorting effects of money in direct democracy, and more precisely on the disproportionate power of narrow business interests.[5] Is there room for a voting scheme that is designed to increase further the power of minorities? Perhaps surprisingly, the informed answer seems to be yes. Gerber (1999) and Matsusaka (2004) provide exhaustive empirical analyses of direct democracy in US states, where money spent in referenda campaigns is largest and unlimited. Although their emphasis differs, they both conclude that there is no evidence that business interests are succeeding at manipulating the process in their favor any more than grass-root citizens' groups (or, according to Matsusaka, away from the wishes of the majority). In fact both books isolate the need to protect minorities, stripped of the checks and balances of representative democracy and of the pragmatic recourse to log-rolling, as the most urgent task

[4] The asymmetry of the distribution seems natural when talking informally, but is difficult to justify in analyses based on a single referendum. The approach posits cardinal valuations, but on what basis can one side claim a larger mean valuation than the other? A normalization, a reference criterion, is required. Studying multiple proposals contemporaneously provides such a reference.

[5] See, for example, Broder (2000), with the expressive title *Democracy Derailed*. Opposite views on the promise of direct democracy, held with equal strength, are also common: see for example *The Economist*, Dec 21, 1996 ("The idea that people should govern themselves can at last mean just that") or *The Economist*, Jan 23, 2003.

in improving the process.[6]

The protection of minorities is the heart of the existing voting system that most closely resembles the mechanism described here. "Cumulative voting" applies to a single multi-candidate election and grants each voter a number of votes equal to the number of positions to be filled, with the proviso that the votes can either be spread or cumulated on as few of the candidates as desired. The system has been advocated as an effective protection of minority rights (Guinier, 1994) and has been recommended by the courts as redress to violations of fair representation in local elections (Issacharoff, Karlan and Pildes, 2001). There is some evidence, theoretical (Cox, 1990), empirical (Bowler, Donovan and Brockington, 2003), and experimental (Gerber, Morton and Rietz, 1998) that cumulative voting does indeed work in the direction intended. The bonus votes scheme discussed in this paper differs because it applies to a series of independent decisions, each of which can either pass or fail, but the intuition inspiring it is similar.

The idea of eliciting preferences by linking independent decisions through a common budget constraint can be exploited quite generally, as shown by Jackson and Sonnenschein (forthcoming). Their paper proposes a specific mechanism that allows individuals to assign different priority to different outcomes while constraining their choices in a tightly specified manner. The mechanism converges to the first best allocation as the number of decisions grows large, but the design of the correct menu of choices offered to the agents is complex, and the informational demands on the planner severe—the first best result comes at the cost of the mechanism's complexity. The recourse to bonus votes in referenda that we discuss in the present paper builds on the same principle but with a somewhat different goal: a mechanism with desirable if not fully optimal properties that is simple enough to be put in practice. It is this simplicity that we particularly value: the reader should keep in mind that we propose and study a minor, plausible modification to existing voting practices.

The paper proceeds as follows. Section 2 describes the model; sections 3 and 4 establish the first result and discuss its intuition in the simplest setting, when the distributions of valuations are identical across individuals and proposals and are known to be symmetrical between opponents and supporters of each proposal. Section 5 extends the model to the case where distributions remain symmetrical but differ across proposals. Sections 6 justifies the choice of granting a discrete number of bonus votes, as opposed to a perfectly divisible vote. Section 7 studies the bonus votes mechanism when the probability of approval of each proposal is stochastic. Section 8 addresses the case of asymmetric distributions. Section 9 discusses briefly two final points: the possibility of correlated referenda, and the effect of non-cumulative bonus votes. Section 10 concludes. The Appendix contains some of the proofs.

[6]Even if their conclusion were incorrect, a priori it is unclear how granting bonus votes would interact with the existence of special interests and their differential access to campaign money.

2 The Basic model

A large number n of voters are asked to vote, contemporaneously, on a set of k unrelated proposals (with $k > 1$). Each proposal can either pass or fail, and we will refer to each vote as an unrelated referendum. Each voter is asked to cast one vote in each referendum, but in addition is given a set of m bonus votes. It is natural to think of each bonus vote as equivalent to one regular vote, but we can suppose, more generally, that each bonus vote is worth ϑ regular votes, with $\vartheta > 0$. For example, imagine regular votes as green, and bonus votes as blue; if $\vartheta = 1/2$, it takes 2 blue votes to counter 1 green vote, and viceversa if $\vartheta = 2$. The parameter ϑ can take any value between $1/C$ and C, where C is an integer, small relative to n but otherwise arbitrary. We denote by θ the aggregate value of all bonus votes: $\theta \equiv m\vartheta$.[7]

The valuation that voter i attaches to proposal r is summarized by v_{ir}. A negative valuation indicates that an individual is against the proposal, while a positive valuation indicates that he or she is in favor, and the valuation's absolute value, which we denote by v_{ir}, summarizes the intensity of i's preferences: voter i's payoff from proposal r is v_{ir} if the referendum is resolved in the preferred direction, and 0 otherwise. Individual valuations are drawn, independently across individuals and across proposals from probability distributions $\mathbf{F} \equiv \{F_r(\mathrm{v}), r = 1, .., k\}$ that can vary across proposals but are common knowledge. The distributions $\mathbf{F}$ are symmetrical around 0 (there is no systematic difference between voters who oppose and voters who favor any proposal) and have full support normalized to $[-1, 1]$. Each individual knows his or her own valuation over each proposal, but only the probability distribution of the others' valuations. There is no cost of voting.

We restrict attention to Bayesian equilibria in undominated strategies where, conditional on their set of valuations, all voters select the same optimal strategy. Since there can be no gain from voting against one's preferences, in these equilibria voters vote sincerely. The only decision is the number of votes x_r to cast in each referendum r, where we use the convention that negative votes are votes cast against a proposal and positive votes are votes cast in favor. Calling $\mathbf{v}_i$ the vector of valuations of voter i, voter i's strategy is then indicated by $x_{ir}(\mathbf{v}_i, m, \vartheta, \mathbf{F})$.

Before characterizing the equilibrium of the game, it is helpful to establish some preliminary results.

3 Three preliminary results

In any referendum, the votes cast by an individual voter can take any value in $\{1, 1 + \vartheta, .., 1 + m\vartheta\}$. With arbitrary ϑ, deriving the asymptotic distribution of the votes requires some care, and we discuss the appropriate limit theorem

[7] The set-up generalizes trivially to the case where different bonus votes are allowed to have different values, relative to regular votes. As we shall see, the only important parameter is the aggregate value θ.

in the Appendix. Once the distribution is pinned down, though, it becomes clear that the number of candidate equilibria can be reduced drastically. In this section, we present the intuitive arguments behind the simplification. We summarize the arguments as lemmas, and prove them formally in the Appendix.

Define as $\mathbf{v}_i$ the vector of voter i's absolute valuations. Then:

Lemma 1. *In equilibrium,* $|x_{ir}(\mathbf{v}_i, m, \vartheta, \mathbf{F})| = x_{ir}(\mathbf{v}_i, m, \vartheta, \mathbf{F})\ \forall i,\ \forall r$*: for all voters and in all referenda the absolute number of votes cast is independent of the signs of the voter's valuations.*

In deciding how to distribute the bonus votes over any two referenda, a voter must compare the relative return from obtaining the desired outcome (thus the ratio of the two absolute valuations) to the relative probability of being pivotal. But in a large electorate the probability of being pivotal never depends on the direction of the vote: either the referendum is expected to be won by one side with probability close to 1, in which case the probability of being pivotal is always negligible; or the rest of the electorate is expected to be equally split, in which case the probability of being pivotal is again independent of whether the voter favors the proposal or opposes it. In either case, the voter's best response strategy cannot depend on the sign of the valuations.[8]

With distributions $\mathbf{F}$ symmetrical around 0 and valuations draws independent across voters, one implication of Lemma 1 is that in equilibrium the outcome of all referenda must be uncertain. A second implication is that we can simplify our notation. As long as the distributions $\mathbf{F}$ are symmetrical, we can work with distributions $\mathbf{G} \equiv \{G_r(v),\ r = 1, .., k\}$ defined over absolute valuations and support $[0, 1]$, and we will understand the strategies $x_{ir}(\mathbf{v}_i, m, \vartheta, \mathbf{G})$ to refer to the absolute number of votes cast.

Lemma 2. *In all equilibria, each voter cumulates all his bonus votes on one referendum. All equilibria are equilibria of the simpler game where a single bonus vote of value* $m\vartheta \equiv \theta$ *is granted to all voters.*

In all referenda, the asymptotic distribution of votes is such that the probability of being pivotal is approximately proportional to the number of votes cast (see the Appendix). The implication is that the problem faced by a voter when choosing how to allocate bonus votes is linear in the number of votes and has a corner solution: all bonus votes should be cumulated on one proposal.

Lemma 2 allows us to simplify the problem drastically. We can model the menu of bonus votes as a single bonus vote of value θ and reduce the voters'

[8] This observation rules out strategies that seem counterintuitive but not a priori impossible. Consider for example the strategy: $|x_{ir}| = (1 + m\vartheta)$ if $\mathrm{v}_{ir} < \mathrm{v}_{is}$ for all s, and $|x_{ir}| = 1$ otherwise (i.e. cast all bonus votes on the referendum that is most opposed, or on the least favored if none is opposed), a strategy where bonus votes are used disproportionately against all referenda. The strategy does not violate sincere voting. With symmetric distributions $\mathbf{F}$ and independent draws across voters, all referenda are expected to fail with probability close to 1, but equal across referenda. The probability of being pivotal is then negligible but identical across referenda, and the best response strategy for each voter is to cast all bonus votes on the referendum with highest valuation, voting sincerely and regardless of the valuation's sign. The suggested strategy cannot be an equilibrium.

problem to the choice of the single referendum in which the bonus vote will be cast. From a practical point of view, granting a single bonus vote seems a preferable design: it simplifies the voters' problem and has no effect in equilibrium. For the remainder of the paper, we will then refer to a single bonus vote.

Call ϕ_r a voter's ex ante probability of casting the bonus vote on referendum r (before observing his or her valuations), where $\sum_{r=1}^{k} \phi_r = 1$. We can establish:

Lemma 3. *Call p_r the probability that voter i obtains the desired outcome in referendum r when casting a regular vote only, and $p_{\theta r}$ the corresponding probability when adding the bonus vote. Then:*

$$\begin{aligned} p_r &\simeq \frac{1}{2} + \frac{1}{\sqrt{2\pi n\left[1+\phi_r(\theta^2+2\theta)\right]}} \\ p_{\theta r} &\simeq \frac{1}{2} + \frac{1+\theta}{\sqrt{2\pi n\left[1+\phi_r(\theta^2+2\theta)\right]}} \end{aligned} \tag{1}$$

Voter i's optimal strategy is to cast a bonus vote in referendum s if and only if :

$$\frac{v_{is}}{v_{ir}} > \sqrt{\frac{1+\phi_s(\theta^2+2\theta)}{1+\phi_r(\theta^2+2\theta)}} \qquad \forall r \neq s \tag{2}$$

The probabilities are valid up to an approximation of order $O(n^{-3/2})$ (see the Appendix). Voter i will choose to cast the bonus vote in referendum s over referendum r if and only if $v_{is}p_{\theta s} + v_{ir}p_r > v_{ir}p_{\theta r} + v_{is}p_s$. Substituting (1), we obtain (2). If (2) holds for all r's different from s, then the voter will cast the extra vote on referendum s.

We are now ready to characterize the equilibria, and we begin by the simpler case where $G_r(v) = G(v)$ for all r.

4 Identical distributions

When the distributions of valuations are identical across proposals, intuition suggests a simple strategy: let each voter cast the bonus vote in the referendum to which he or she attaches the highest valuation. Indeed we can show:

Proposition 1. *If $G_r(v) = G(v)$ $\forall r$, then there exists a unique equilibrium where each voter casts the bonus vote in referendum s if and only if $v_{is} \geq v_{ir}$ $\forall r$.*

Proof of Proposition 1. Given Lemma 3, the proof of proposition 1 is straightforward. (i) To see that the candidate strategy is indeed an equilibrium strategy, suppose all voters but i cast their bonus vote in the referendum with highest absolute valuation. Since all valuations are drawn from the same probability distribution, with k draws each has probability $1/k$ of being the highest, implying $\phi_s = \phi_r = 1/k\ \forall r$. Thus the square root in (2) equals 1 and by Lemma 3 voter i should follow the same strategy, establishing that it is indeed an equilibrium. (ii) To see that the equilibrium is unique, suppose to the contrary that there is an equilibrium where not all ϕ_r's are equal, and call s the referendum such that $\phi_s = \max\{\phi_r\}$. Then $\sqrt{[1+\phi_s(\theta^2+2\theta)]/[1+\phi_r(\theta^2+2\theta)]} \equiv \alpha(s,r) \geq 1\ \forall r$, with at least one strict inequality. Call r' one of the referenda for which the strict inequality holds (at least one r' must exist). Then, by (1), in the bilateral comparison between s and r' the expected share of voters casting their bonus vote on s is lower than on r'. We can have $\phi_s = \max\{\phi_r\}$ only if there exists at least one r'' such that $\alpha(s,r'') < \alpha(r',r'')$. But this requires $\phi_{r'} > \phi_s$, contradicting $\phi_s = \max\{\phi_r\}$. □

In our opinion, both the uniqueness of the equilibrium strategy and its simplicity are strong assets of the mechanism. The immediate response to being allowed to cast a bonus vote is to cast it over the issue that matters most. It seems to us important in practice that the best strategy associated with a mechanism be both sincere and simple, and that voters' concerns with strategic calculations be limited to a minimum.

To evaluate the potential for welfare gains, we use as criterion ex ante efficiency: the expected utility of a voter before having drawn his or her valuations (or equivalently before being informed of the exact slate of proposals on the ballot). By Proposition 1, the expected share of voters casting their bonus vote is equal in all referenda ($\phi_r = 1/k\ \forall r$), implying, by (1), that the probability of obtaining the desired outcome depends on whether the bonus vote is cast, but not on the specific referendum: $p_r = p$, $p_{\theta r} = p_\theta\ \forall r$. Denote by Ev the expected absolute valuation over any proposal, and by $Ev_{(j)}$ the expected jth order statistics among each individual's k absolute valuations (where therefore $Ev_{(k)}$ is the expected value of each voter's highest absolute valuation). Since voters cast their bonus vote in the referendum associated with the highest valuation, expected ex ante utility EU is given by:

$$EU = Ev_{(k)}p_\theta + \sum_{j=1}^{k-1} Ev_{(j)}p = k(Ev)p + Ev_{(k)}(p_\theta - p). \tag{3}$$

Substituting (1) and $\phi_r = 1/k\ \forall r$, we can write:

$$EU \simeq kEv\left(\frac{1}{2} + \frac{\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}}\right) + Ev_{(k)}\left(\frac{\theta\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}}\right). \tag{4}$$

Our reference is expected ex ante utility with a series of simple majority referenda, which we denote EW, where, as established in Lemma A.1 in the Appendix:

$$EW \simeq kEv\left(\frac{1}{2} + \frac{1}{\sqrt{2\pi n}}\right), \tag{5}$$

again ignoring terms of order $O(n^{-3/2})$. Comparing (4) and (5), we see that both mechanisms dominate randomness (where each proposal is resolved in either direction with probability 1/2), although both converge to randomness, and to each other, as the population approaches infinity (a point we will discuss in more detail later). Thus a plausible scaling of efficiency is the relative improvement of the two mechanisms over randomness. Calling ER expected utility with random decision-making, we define our measure of welfare improvement as ω, where

$$\omega \equiv \left(\frac{EU - ER}{EW - ER} \right). \tag{6}$$

We will state that the voting mechanism improves efficiency over a series of simple majority referenda if $\omega > 1$.

Substituting (4) and (5) and $ER = kEv(1/2)$, we derive immediately[9]

$$\omega = \frac{kEv + \theta Ev_{(k)}}{Ev\sqrt{k^2 + k\theta^2 + 2k\theta)}}, \tag{7}$$

which then implies:

Proposition 2. *For any distribution $G(v)$ and any number of referenda $k > 1$, there exists a $\overline{\theta}(G, k) > 0$ such that $\omega > 1$ for all $\theta < \overline{\theta}$.*

The proposition follows immediately from (7). Indeed, a simple manipulation shows,

$$\omega > 1 \quad \begin{cases} \forall\theta > 0 & \text{if } (Ev_{(k)})^2 \geq k(Ev)^2 \\ \forall\theta < \frac{2kEv(Ev_{(k)} - Ev)}{k(Ev)^2 - (Ev_{(k)})^2} & \text{if } (Ev_{(k)})^2 < k(Ev)^2. \end{cases} \tag{8}$$

Given a specific distribution, the admissible range of θ values is easily pinned down. Suppose for example that $G(v)$ is the uniform distribution; then $Ev = 1/2$ and $Ev_{(k)} = k/(k+1)$, implying that efficiency improves for all $\theta < 2(k+1)/(k-1)$. If $k = 2$, the constraint is $\theta < 6$—the bonus vote cannot count more than 6 regular votes; if $k = 5$, the constraint is $\theta < 3$, and so forth. Because the ceiling on θ is declining in k, its limit as k approaches infinity provides a sufficient condition for efficiency gains: for any number of referenda, $\theta < 2$ guarantees $\omega > 1$.

In fact we can do more: from (7) we can derive the optimal θ, the value of the bonus vote that maximizes the efficiency gains, which we denote by θ^*. For arbitrary $G(v)$,[10]

$$\theta^* = \frac{k(Ev_{(k)} - Ev)}{kEv - Ev_{(k)}}. \tag{9}$$

[9] Equation (7) holds for any large n, including in the limit as n approaches infinity.

[10] For arbitrary $G(v)$,

$$Ev_{(k)} = k\int_0^1 v[G(v)]^{k-1} dG(v),$$

ensuring that the denominator in (9) is always positive.

If $G(v)$ is a uniform distribution, then $\theta^* = 1$ for any value of k: regardless of the number of referenda, the optimal value of the bonus vote is 1—that is, the bonus vote should be equivalent to a regular vote. At $\theta = 1$ and for a uniform $G(v)$, $\omega = \sqrt{k(3+k)}/(1+k)$, always larger than 1, but maximal at $k^* = 3$: given the optimal choice of θ, the number of contemporaneous referenda that maximizes efficiency gains is 3. At these parameter values, the welfare gain relative to simple majority, as defined by ω, is 6 percent.

There results, so surprisingly clean, extend easily to a general power distribution, and we summarize them in the following example:

Example 1. *Suppose that $G(v)$ can be parameterized as a power distribution: $G(v) = v^b$, $b > 0$. Then, ignoring integer constraints: (i) For all k, $\omega > 1$ if $\theta < 2/b$. (ii) For all k, $\theta^* = 1/b$. (iii) If $\theta = \theta^*$, $k^* = 2 + 1/b$.*

The parameter b determines the shape of the distribution, reducing to the uniform if $b = 1$. If $b < 1$, $G(v)$ is unimodal at 0, and the mass of voters declines monotonically as the valuations become more extreme; with $b > 1$, on the contrary, the distribution is unimodal at 1, the upper boundary of the support, and the mass of voters increases with the intensity of the valuations. For a more intuitive understanding of what the distribution implies, suppose for example that voters were asked to rank an issue as "not important," "somewhat important," "important," or "very important," and that these labels corresponded to a partition of the range of possible intensities into 4 intervals of equal size, from $[0, 0.25]$ to $[0.75, 1]$. For a uniform distribution of valuations, a quarter of the voters would choose each interval; with $b = 1/2$, half of the voters would classify the issue as "not important" and about 13 percent as "very important"; with $b = 2$, the percentages become 6 percent for "not important" and close to 45 percent for "very important." The parameter b is thus a measure of the saliency of the issue, and the more salient the set of issues, the smaller is the optimal value of the bonus vote: with $b = 1/2$, the bonus vote should count as 2 regular votes; with $b = 2$ as half, and with $b = 3$ as a third.

The sufficient condition (i) above is important. Without precise knowledge of the distribution, a policy-maker cannot set the optimal value of the bonus vote, but if the more modest goal of *some* improvement over simple majority is acceptable, this can be achieved by choosing a conservatively small θ. Consider for example setting $\theta = 1/2$—then, for all k, efficiency gains are achieved as long as $b < 4$. With $b = 4$, almost 70 percent of the voters consider the issue "very important," more than 90 percent either "important" or "very important" and less than 1 percent "not important." As long as saliency is not higher, welfare is improved by the bonus vote.

Why is there a ceiling on the acceptable values of the bonus vote? And why does this ceiling depend on the shape of the distribution? Taking θ as given, we can rewrite the necessary condition for efficiency gains as:

$$\omega > 1 \Leftrightarrow \frac{Ev_{(k)}}{Ev} > \frac{k}{\theta}\left(\sqrt{1 + \frac{\theta^2 + 2\theta}{k}} - 1\right) > 1 \quad \forall \theta > 0. \tag{10}$$

Condition (10) makes clear that an improvement in efficiency requires a sufficient wedge between the mean valuation and the highest expected valuation draw. The problem is that the introduction of the bonus vote creates noise and redistributes the probability of winning towards the referendum where the bonus vote is utilized but away from the others. Efficiency can increase only if the higher probability of being on the winning side is enjoyed over a decision that really matters to the voter, a decision that matters enough to compensate for the decline in influence in the other referenda. Predictably, the required wedge is increasing in θ: the higher the value of the bonus vote, the larger the noise in the votes distribution and the larger the shift in the probability of winning towards the referendum judged most important. Equations (1) show this effect clearly. Similarly, the wedge is increasing in k: the larger is k, the more issues over which the probability of winning declines $(k-1)$, and thus again the larger must be the valuation attached to the referendum over which the bonus vote is spent.[11]

For our purposes, the ratio $Ev_{(k)}/Ev$ summarizes all that matters about the distribution of valuations. With a power distribution $Ev = b/(1+b)$, $Ev_{(k)} = bk/(bk+1)$, and $Ev_{(k)}/Ev = (k+bk)/(1+bk)$, an expression that is declining in b. The more salient the issues—the higher b—the smaller the expected difference between the highest draw and the mean valuation, and the smaller must then θ be if (10) is to be satisfied. Hence the result described above. More generally, given $Ev_{(k)}/Ev$ and k, condition (8) specifies the constraint on θ and (9) θ's optimal value.[12]

Summarizing, the voting scheme exploits the variation in valuations to ensure that the added noise created by the bonus vote is compensated by a higher probability of winning a decision that really matters. The more intense the average valuations—the more polarized the society—the higher the variance must be for a given value of the bonus vote, or equivalently, the smaller must be the value of the bonus vote; the less intense the average valuations, the lower the required variance or equivalently the higher the optimal value of the bonus vote.[13]

[11] But $Ev_{(k)}$ is also increasing in k. Whether fulfilling (10) becomes more or less difficult as k increases depends on the distribution.

[12] It was tempting to conjecture a link between the ordering of distributions in terms of the ratio $Ev_{(k)}/Ev$ and first-order stochastic dominance—until Russell Davidson provided a counterexample.

[13] The ratio $Ev_{(k)}/Ev$ depends both on the variance of $G(v)$ and on the mean. A power distribution conflates the two, since both depend on b. (The variance equals $b/[(1+b)^2(2+b)]$ with a maximum at $b = 0.62$). A beta distribution is more flexible and isolates the two effects, but does not provide a closed form solution for the kth order statistics. We can nevertheless check conditions (8) or (10) numerically. Suppose for example $\theta = 1/2$. Then if $E(v) = 1/2$, (10) is satisfied for all k as long as the variance is larger than 0.008 (or equivalently as long as not more than 3/4 of the population are concentrated in the two deciles around the mean). But if the mean is 3/4, the minimum variance rises to 0.02 (or not more than 50 percent of the population in the two deciles around the mean); if instead the mean is 1/4, the minimum variance falls to 0.002 (or not more than 98 percent of the population in the two deciles around the mean). The necessary floor on the variance rises as the mean increases.

5 Heterogeneous distributions

The assumption that valuations are identically distributed over all proposals is, in general, unrealistic: many issues put to referendum are typically of interest only to a small minority—the calendar of the hunting season, the decision to grant landmark status to a building, the details of government procedures—while some on the contrary evoke strong feelings from most voters—divorce in Italy, affirmative action and taxation in California, equal rights for women in Switzerland.[14] Allowing for different distributions makes the problem less transparent, but does not change its logic and in fact increases the expected dispersion in valuations that makes the voting scheme valuable.

The first step is the characterization of the equilibrium—the choice of the referendum on which to cast the bonus vote. Lemmas 1 to 3 continue to apply, but now voters' bonus votes will not be spread equally over all referenda—the more salient issues will receive a larger share of bonus votes. In equilibrium, ϕ_r, the expected share of voters casting their bonus vote in referendum r, must satisfy

$$\phi_r = \int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr} v, 1)) g_r(v) dv, \tag{11}$$

where

$$\alpha_{sr} \equiv \sqrt{\frac{1+\phi_s(\theta^2+2\theta)}{1+\phi_r(\theta^2+2\theta)}}. \tag{12}$$

When $G_r(v) = G_s(v)\ \forall r, s$, as in the previous section, (11) and (12) simplify to $\phi_r = 1/k$ and $\alpha_{sr} = 1$. This is not the case now.

The equilibrium remains unique[15] but is less intuitive than in the case of identical distributions: if a referendum evokes more intense preferences and more voters are expected to cast their bonus vote on that issue, then the impact of the bonus vote will be higher elsewhere. It may be referable to cast one's bonus vote in a different referendum, even if the valuation is slightly lower. For example, in the case of 2 referenda and power distributions, suppose $b_1 = 1$ and $b_2 = 2$. Then $\alpha_{12} = 0.89$—a voter casts the bonus vote on issue 1 as long as $v_1 \geq 0.89\ v_2$—and the expected shares of bonus votes cast on the two referenda are $\phi_1 = 0.41$ and $\phi_2 = 0.59$. If $b_2 = 4$, the numbers become $\alpha_{12} = 0.82$, $\phi_1 = 0.34$ and $\phi_2 = 0.66$.

The condition for efficiency gains over simple majority again follows the logic described earlier, but is made less transparent by the need to account for the

[14] The distinction is equivalent to Matsusaka's (1992) empirical classification of initiatives into "efficiency" (low salience) and "distributional" (high salience).

[15] Consider an equilibrium $\{\phi_r^{'}\}$. Posit a second equilibrium where $\phi_s = \phi_s^{''} > \phi_s^{'}$. Then, given (11) and (12) there must exist at least one issue r such that $\phi_r^{''}/\phi_r^{'} > \phi_s^{''}/\phi_s^{'}$. Call z the issue such that $\phi_z^{''}/\phi_z^{'}$ is maximum. Then $\alpha_{rz}(\phi_z'', \phi_r'') < \alpha_{rz}(\phi_z', \phi_r')\ \forall r \neq z$, and by (11) $\phi_z^{''} < \phi_z^{'}$ establishing a contradiction. Reversing the signs, the identical argument can be used to show that there cannot be an equilibrium with $\phi_s^{''} < \phi_s^{'}$.

different distributions and for the factors of proportionality α_{rs}:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k} \left[\left(\int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr} v, 1)) v g_r(v) dv \right) \theta \beta_r \right] > \sum_{r=1}^{k} E_r(v) \left(1 - \beta_r\right), \tag{13}$$

where

$$\beta_r \equiv \frac{1}{\sqrt{1 + \phi_r(\theta^2 + 2\theta)}}.$$

Condition (13) is analogous to (10), but because the parameters β_r and α_{sr} differ across distributions and α_{sr} in general differs from 1, it does not reduce to a simple condition on the ratio of the expected highest valuation draw to the mean valuation. Nevertheless, it remains possible to state:

Proposition 3. *For any set of distributions* $\mathbf{G}$ *symmetric around zero and with full support and for any number of referenda* $k > 1$*, there exists a* $\widetilde{\theta}(\mathbf{G}, k) > 0$ *such that* $\omega > 1$ *for all* $\theta < \widetilde{\theta}$*.*

The proposition is proved in the Appendix. It states that the result we had previously established in the case of identical distributions is in fact more general, and continues to apply with heterogeneous distributions.

In practical applications, two concerns remain. The first is that calculating the correct equilibrium factors of proportionality α_{rs} is not easy. How well would voters fare if they followed the plausible rule of thumb of casting the bonus vote on the highest valuation proposal? It seems wise to make sure that the desirable properties of the mechanism are robust to the most likely off-equilibrium behavior. In fact, Proposition 3 extends immediately to this case:

Proposition 3b. *Suppose voters set* $\alpha_{sr} = 1\ \forall s, r$*. Proposition 3 continues to hold.* (See the Appendix).

The second concern was voiced earlier. If the planner is not fully informed on the shape of the distributions, or if the value of θ is to be chosen once and for all, for example in a constitutional setting, can we identify sufficient conditions on θ that ensure efficiency gains for a large range of distributions? The answer is complicated by the factors α_{rs} and thus by the lack of a simple closed-form solution even when we specialize the distributions to simple functional forms. However, in our reference example of power distributions and in the "rule-of-thumb" case where voters cast the bonus vote on the highest valuation proposal, we obtain an interesting result:

Example 2. *Suppose* $G_r(v) = v^{b_r}$*,* $b_r > 0\ \forall r$*, and set* $\alpha_{sr} = 1\ \forall s, r$*. Call* $b_k \equiv \max\{b_r\}$*. Then for all* $k > 1$*,* $\omega > 1$ *if* $\theta \leq 1/b_k$*.*

The example is proved in the Appendix. As in the case of identical distributions, we can derive a simple sufficient condition ensuring welfare gains: the value of the bonus vote can be safely set on the basis of the distribution of valuations in the most strongly felt of the issues under consideration. If we return to our previous discussion and partition the support of the valuations into four equal size intervals, setting $\theta = 1/4$ or $1/5$ would seem a prudent policy.[16] Intuitively, we expect the condition to be stronger than needed: the heterogeneity of the distributions should help in providing the spread in expected valuations that underlies the voting scheme's efficiency gains. Indeed, in all our numerical exercises with power distributions we achieved welfare gains by setting $\theta \leq k/\sum_{r=1}^{k} b_r$, the inverse of the mean b parameter, a looser constraint than $1/b_k$.[17]

This section allows us to conclude that the properties of the voting scheme, so transparent in the simple case of identical distributions, extend to the more plausible scenario of heterogeneous distributions. Having established the result in our basic model, we can now study its robustness when we relax the model's most restrictive assumptions.

6 Why discrete bonus votes?

The equivalence between granting voters a single bonus vote or multiple extra votes is driven by the discreteness we have attributed to the votes. Why not consider instead a continuous bonus vote that voters can split as they see fit between the different issues? The scheme is intuitive and more general than the one we have considered so far; indeed, from a theoretical point of view, it is a more natural starting point. There is also reason to expect that the generalization could help. We saw that in the discrete scheme the value of the bonus vote has to be tuned correctly: when the dispersion in valuations is small, the bonus vote runs the risk of being too blunt an instrument to differentiate finely between them. Why not let voters do the fine-tuning themselves, choosing the extent to which they want to divide their extra vote over the different issues? We show in this section that although some theoretical improvement over the discrete scheme is possible, some complications arise. In our opinion, the balance of the arguments comes down in favor of the discrete bonus vote we have described so far.

The main points will be clearer in the simplest setting, and in what follows we assume that there are only two proposals ($k = 2$), the distributions of valuations are identical over both proposals ($G_r(v) = G(v)$, $r = 1, 2$), and the value of the continuous bonus vote θ is set to 1.[18] Call $s_i \in [0, 1]$ the fraction of the bonus vote cast by voter i on the issue with highest valuation. Thus i will cast $(1+s_i)$ votes in one referendum and $(2-s_i)$ in the other. All previous arguments continue to apply and we can restrict candidate equilibria to symmetrical scenarios where strategies are contingent on absolute valuations. The distribution

[16] With $b = 5$, more than 3/4 of all voters consider the issue "very important," 97 percent consider it either "important" or "very important," and less than 1 in a thousand "not important."

[17] This was true whether we looked at the equilibrium or at the $\alpha_{sr} = 1$ case. With $k = 2$, efficiency gains in the "rule-of-thumb" scenario are sufficient for efficiency gains in equilibrium, but not with $k > 2$.

[18] The analysis extends immediately to $\theta < 1$. If $\theta > 1$, the logic is unchanged but the equations need to be amended.

of the vote differential faced by voter i - the net sum of all votes cast by the other voters, where votes against the proposal are counted as negative votes - is identical in both referenda. Denoting by $-i$ the choices of the other voters, such a distribution must be normal with mean 0 and variance:[19]

$$\sigma^2 = n\left[5/2 + Es_{-i}^2 - Es_{-i})\right].$$

Labeling v_{1i} the higher valuation, the expected utility of voter i after valuations are drawn is given by

$$Eu_i = v_{1i}\Phi\left(\frac{1+s_i}{\sqrt{\sigma^2}}\right) + v_{2i}\Phi\left(\frac{2-s_i}{\sqrt{\sigma^2}}\right), \tag{14}$$

where $\Phi(\cdot)$ is the normal cumulative distribution function—hence $\Phi\left(\frac{1+s_i}{\sqrt{\sigma^2}}\right)$, for example, is the probability that i's preferred outcome in referendum 1 is winning or tied (with probability 1/2) or is losing by not more than $1 + s_i$ (with probability $\Phi\left(\frac{1+s_i}{\sqrt{\sigma^2}}\right) - 1/2$). Before proceeding further, there is always an equilibrium where no one splits the bonus vote. If none of the other voters splits the bonus vote, the distribution of the vote differential faced by voter i will have steps at all discrete number of votes, $\Phi\left(\frac{1+s_i}{\sqrt{\sigma^2}}\right) = \Phi\left(\frac{2-s_i}{\sqrt{\sigma^2}}\right)\ \forall s_i \in (0,1)$, and the only relevant choices are $s_i = 0$ or $s_i = 1$. The analysis in the first part of this paper remains the correct analysis here, and the equilibrium with discrete voting identified there remains an equilibrium here.[20] Thus the first observation is that allowing the bonus vote to be divisible must increase the number of equilibria.

Consider now a candidate equilibrium where s_{-i} is continuous over the whole interval $[0,1]$. The distribution of the vote differential is then continuous and equation (14) can be differentiated to with respect to s_i, yielding,

$$v_{1i}\varphi\left(\frac{1+s_i^*}{\sqrt{\sigma^2}}\right) = v_{2i}\varphi\left(\frac{2-s_i^*}{\sqrt{\sigma^2}}\right) \quad \text{if } s_i^* \in [0,1], \tag{15}$$

where the star indicates s_i's optimal value and $\varphi(\cdot)$ is the normal density function. We can rewrite (15) as:

$$\log v_{1i} - \frac{(1+s_i^*)^2}{2\sigma^2} = \log v_{2i} - \frac{(2-s_i^*)^2}{2\sigma^2} \quad \text{if } s_i^* \in [0,1].$$

[19] In equilibrium, in each referendum voter i expects half of the other voters to cast $\pm(1+s_{-i})$ votes, and half to cast $\pm(2-s_{-i})$ votes. Hence:

$$\sigma^2 = n[(1/2)E(1+s_{-i})^2 + (1/2)E(2-s_{-i})^2]$$

[20] The logic extends immediately to all other possible discrete jumps in the proportion of the bonus vote cast in the two referenda. But then we revert to the case of discrete votes, and to the result reached earlier: the bonus vote should be cumulated on the one most important issue. In equilibrium the only relevant case is then the $0-1$ split.

Taking into account $s_i^* \in [0,1]$ and substituting σ^2 from above:

$$s_i^*(v_{1i}, v_{2i}) = \min\left[1, \frac{1}{2} + n \log\left(\frac{v_{1i}}{v_{2i}}\right)\left(\frac{5}{6} + \frac{1}{3}(Es_{-i}^2 - Es_{-i})\right)\right],$$

or, in equilibrium,

$$s_i^* = \begin{cases} \frac{1}{2} + \frac{1}{2t(n, s_{-i}^*)} \log(v_{1i}/v_{2i}) & \text{if } \log(v_{1i}/v_{2i}) \in [0, t(n, s_{-i}^*)] \\ 1 & \text{if } \log(v_{1i}/v_{2i}) > t(n, s_{-i}^*), \end{cases} \tag{16}$$

where

$$t(n, s_{-i}^*) \equiv \frac{3}{n[5 + 2E(s_{-i}^*)^2 - 2Es_{-i}^*]}.$$

The main observation can be made without an explicit solution for x^*: $t(n, s_{-i}^*)$, the upper boundary on the logarithm of relative valuations consistent with splitting the bonus vote, approaches zero at rate n. Since $v_{1i} > v_{2i}$ by definition, the probability of splitting the bonus vote approaches zero at rate n. For large n, the option of splitting the bonus vote cannot be important.

To gain a more precise sense of what this means, suppose that $G(v)$ is a uniform distribution. In the interval where the bonus vote is split, $\log(v_1/v_2)$ is of order $O(n^{-1})$ and thus, ignoring terms of order $O(n^{-2})$, can be approximated by $(v_1/v_2) - 1$. Call η the probability of splitting one's vote, or equivalently, for large n, the share of the population that splits the bonus vote (where, with a uniform distribution, $\eta = t$). In a symmetrical equilibrium with $n = 100$, $\eta = 0.006$; with $n = 1,000$, $\eta = 0.0006$—as we increase the order of magnitude of the population, the number of voters expected to split their vote remains less than a single one.[21] The same result can be stated in terms of welfare: with $n = 100$, the equilibrium with continuous voting slightly improves our measure of welfare; but for all $n \geq 1,000$ the precision of our numerical simulations is not sufficient to detect any difference.[22]

Summarizing, we have reached two conclusions. First, the equilibrium where the bonus vote is not split continues to exist when the bonus vote is perfectly divisible—moving from discrete bonus votes to a continuum increases the number of equilibria. Second, the distinction becomes irrelevant in large populations, both in terms of the proportion of voters who exploit it in equilibrium and in terms of its welfare consequences.[23]

[21] The product ηn is approximately constant because t approaches $3/(5n)$; since η equals t, ηn must be approximately $3/5 = 0.6$.

[22] With $G(v)$ uniform, $k = 2$, $\theta = 1$ and one indivisible bonus vote, $\omega = 1.054$ $\forall n$. In the symmetrical equilibrium with continuous splitting, $\omega^C = 1.055$ if $n = 100$, but $\omega^C = 1.054$ $\forall n \geq 1,000$.

[23] Notice a corollary to the last observation: in large populations, a continuous bonus vote cannot be used to guarantee welfare gains, relative to simple majority referenda, if the value of the bonus vote is not chosen correctly.

7 Stochastic probability of approval

The assumption that the distributions $\mathbf{F}$ are symmetric around 0 implies that the medians of the distributions are known and equal to 0. Because voters' valuations are independent draws from these distributions, the model is equivalent to one where each voter supports each proposal with known probability equal to 1/2. Independently of whether a voter casts his bonus vote or not, in large elections the model yields a probability of being pivotal of order $1/\sqrt{n}$. However, if the probability of approval, or equivalently the median of the $\mathbf{F}$ distributions, differs from 1/2, the probability of being pivotal becomes negligible (of order $1/e^n$). Empirically, the assumption of an equally split electorate in referenda is not implausible[24], but how important is it for our results?

Suppose that ex ante each voter had a probability ψ_r of being in favor of proposal r, and $1 - \psi_r$ of being against. Conditional on being in favor or against, the distribution of absolute valuations continues to be described by $G_r(v)$ defined over support $[0, 1]$, and thus equal for voters in favor and voters against the proposal. It seems correct to assume that the popularity of a proposal has no implication for the relative intensity of preferences of supporters and opponents: there is no systematic bias in the intensity of preferences of the minority, relative to the majority.

What matters in our model is not the absolute magnitude of the probability of being pivotal, but the *relative* magnitude, comparing one referendum to another (in choosing the optimal strategy), and comparing our voting scheme to simple majority (when evaluating welfare implications). This point is made most clearly in the simplest model, where ψ_r is known and equal to ψ for all r (and $G_r(v) = G(v)$)[25]. For all $\psi \neq 1/2$, in equilibrium the probability of being pivotal is negligible but equal in all referenda, and all voters cast their bonus vote in the referendum with highest absolute valuation.[26]

The equilibrium strategies differentiate the voting scheme from simple majority. However, in this case simple majority is efficient: with probability that approaches 1 up to a factor of order $O(e^{-n})$ the mean and the median valuation in each referendum are on the same side and dictate the same decision. We would not want the bonus vote scheme to yield a different outcome, and indeed it does not: if $\psi > 1/2$, for example, all referenda are expected to pass with probability approximating 1 with both simple majority and the bonus vote. Ignoring terms of order $O(e^{-n})$, the welfare criterion ω equals 1: the existence of the bonus vote affects equilibrium strategies but yields no differences in welfare.[27]

[24] Studying all ballot propositions in California in the period 1912-89, Matsusaka (1992) concludes that the legislature consistently defers to popular vote issues that ex ante appear particularly contested, i.e. where the electorate is approximately equally split.

[25] The assumption is that the distance $|\psi_r - 1/2|$ is constant across referenda; whether ψ_r is larger or smaller than $1/2$ in any individual referendum is irrelevant.

[26] The equilibrium is unique and pinned down by the requirement that the impact of the bonus vote is equalized across referenda. In the absence of systematic differences across referenda, the strategy is natural. In practice, its simplicity may be more important than the infinitesimal loss that a deviating voter would incur.

[27] The expected outcome remains identical to simple majority voting and efficient if the ex

Majority voting cannot be improved upon for two reasons. First, we are maintaining the assumption that, conditional on the direction of preferences, intensities are distributed symmetrically for voters who favor or oppose each referendum - an assumption we will drop in the next section. Second, we are assuming that the probability of approval is known. But, as remarked in the literature, a more general and realistic assumption is that ψ_r is stochastic: in any referendum the probability of approval is not known ex ante.[28] We assume in this section that ψ_r is distributed according to some probability distribution H_ψ defined over the support $[0,1]$. Each realized ψ_r is an independent draw from H_ψ. We constrain H_ψ to be symmetric around 1/2, and we maintain the assumption that $G_r(v) = G(v)$ for all r.

In the absence of systematic differences across referenda, in equilibrium voters continue to cast their bonus vote in the referendum to which they attach the highest valuation: the stochastic probability of approval does not affect the equilibrium strategy. But it does affect the welfare comparison. Defining $\omega_s \equiv (EU_s - ER)/(EW_s - ER)$, where the subscript identifies the model with stochastic approval, we can show:

Proposition 4. *For all distributions $H_\psi(\psi)$ and $G(v)$, and for all $k > 1$ and $\theta > 0$, $\omega_s > 1$.*

Proof of Proposition 4. Call p_{sr} the probability of obtaining one's desired outcome in referendum r when the probability of approval ψ_r is stochastic and the voter does not cast the bonus vote, and $p_{s\theta r}$ the corresponding probability when the voter does cast the bonus vote in referendum r. The notation will be simplified by writing $\psi_r \equiv 1/2 + \delta_r$ where δ_r is distributed according to H_δ defined over the support $[-1/2, 1/2]$ and symmetric around 0, and where each realized δ_r is an independent draw from H_δ. Given the equilibrium strategy and $G_r(v) = G(v)$ for all r, it follows that $p_{sr} = p_s$ and $p_{s\theta r} = p_{s\theta}$ for all r. We show in the Appendix that, for given δ, these probabilities can be approximated

ante probability of support ψ_r differs across referenda. The equilibrium strategy depends on the differences across ψ_r's, relative to the variance of the valuations' distribution, and on the number of referenda k. If each ψ_r is drawn independently from a given distribution with variance comparable to the variance of absolute valuations, and k is not large, in equilibrium all voters cast the bonus vote in referendum r with ψ_r closest to 1/2 (because the difference on the probability of being pivotal, across referenda, overrides the difference in valuations). In this case, the voting scheme is identical to majority voting. But even when absolute valuations continue to influence equilibrium strategies, there are no differences in expected welfare.

[28] See for example Good and Mayer (1975), Margolis (1977) and Chamberlain and Rothschild (1981). Gelman, Katz and Tuerlinckx (2002) discuss the implications of a number of alternative models.

by:

$$
\begin{aligned}
p_s(\delta) &\simeq \widetilde{\Phi}(0) + \frac{\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}} \exp[-\frac{n4\delta^2 k(1+\theta/k)^2}{2(k+\theta^2+2\theta)}] \\
p_{\theta s}(\delta) &\simeq \widetilde{\Phi}(0) + \frac{(1+\theta)\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}} \exp[-\frac{n4\delta^2 k(1+\theta/k)^2}{2(k+\theta^2+2\theta)}]
\end{aligned} \tag{17}
$$

where $\widetilde{\Phi}(0)$ is the cumulative function at 0 of a Normal distribution with mean $2\delta n(1+\theta/k)$ and variance $n(k+\theta^2+2\theta)/k$. (The probabilities simplify to the values in (1) above for $\delta = 0$).

Taking into account that the expected absolute valuation in each referendum is independent of the direction of the preferences, we can write ex ante expected utility with stochastic approval, EU_s, as:

$$
EU_s = \int_{-1/2}^{1/2} [k(Ev)p_s(\delta) + Ev_{(k)}(p_{s\theta}(\delta) - p_s(\delta))]dH_\delta(\delta) \tag{18}
$$

With H_ψ symmetric around 1/2, $\int_{-1/2}^{1/2} \widetilde{\Phi}(0)dH_\delta(\delta) = 1/2$. Thus:

$$
EU_s - ER =
$$

$$
[k(Ev) + \theta Ev_{(k)}] \int_{-1/2}^{1/2} \frac{\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}} \exp[-\frac{n4\delta^2 k(1+\theta/k)^2}{2(k+\theta^2+2\theta)}]dH_\delta(\delta)
$$

At large n, only realizations of δ close to 0 yield positive probabilities. The integral term can be solved as:[29]

$$
\int_{-1/2}^{1/2} \frac{\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}} \exp[-\frac{n4\delta^2 k(1+\theta/k)^2}{2(k+\theta^2+2\theta)}]dH_\delta(\delta) =
$$

$$
= \frac{\sqrt{k}}{\sqrt{2\pi n(k+\theta^2+2\theta)}} \sqrt{2\pi} \frac{\sqrt{(k+\theta^2+2\theta)}}{\sqrt{kn}2(1+\theta/k)} h_\delta(0) = \frac{1}{2(1+\theta/k)} \frac{1}{n} h_\delta(0)
$$

Therefore:

$$
EU_s - ER = [k(Ev) + \theta Ev_{(k)}] \left[\frac{1}{2(1+\theta/k)} \frac{1}{n} h_\delta(0) \right] \tag{19}
$$

[29] Exploiting the Gaussian integral:

$$
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}
$$

with large n:

$$
\int_{-c}^{c} g(x) e^{-\frac{nx^2}{2q}} dx = g(0)\sqrt{2q\pi/n}
$$

At $\theta = 0$, equation (19) reduces to the expected improvement over randomness with simple majority, or:

$$EW_s - ER = k(Ev)\left[\frac{1}{2}\frac{1}{n}h_\delta(0)\right] \tag{20}$$

We obtain:

$$\omega_s = \frac{k(Ev) + \theta Ev_{(k)}}{(Ev)(k+\theta)} \tag{21}$$

It is then immediate that $\omega_s > 1 \iff Ev_{(k)} > Ev$, a condition that is always satisfied.□

Introducing uncertainty in the probability of approval of each referendum improves the performance of the bonus vote scheme. The intuitive reason follows from our previous discussion. As we saw, bonus votes increase the variability of the total votes cast in each referendum, reducing the probability of being pivotal, relative to simple majority voting, unless the voter casts his or her own bonus vote. This effect continues to exist when the probability of approval of each referendum is stochastic, but now has a second, positive implication: the increase in variability works to reduce the impact of non-balanced expected total votes on the probability of being pivotal. The net result is that the decline in the probability of being pivotal when the bonus vote is not cast is reduced, and reduced sufficiently to guarantee that the overall effect of the bonus vote is an increase in expected welfare, relative to simple majority.

A number of observations follow. First, as established in the literature, the probability of being pivotal is of order $1/n$, a result that holds true both with majority voting and with the bonus vote scheme.[30] Second, what complicates the analysis with the bonus vote is not the stochastic nature of ψ_r but the feedback between the expected distribution of the votes in each referendum and the voters' best response strategy. The modelling assumptions made in this section, and in particular the lack of systematic differences across referenda ($G_r(v) = G(v)$ for all r), and the symmetry of H_ψ (or H_δ) allow us to solve the problem as simply as possible by pinning down the equilibrium strategy in a tractable manner. More general formulations would be more difficult to solve, but in line with the previous section, we see no obvious reason why the conclusions should change. Third, the sharp welfare result does depend on one assumption: the lack of positive correlation between the volume of approval ψ_r and the expected intensity of preferences Ev. Alternative models are possible. For example, we can think of the stochastic probability of approval as a stochastic shift in the centers of the distributions $\mathbf{F}$: in this case, $F_r(\text{v})$ has support $[-(1-c_r), 1+c_r]$ with c_r distributed according to some H_c over $[-1, 1]$ and $\psi_r = (1+c_r)/2$. Now the expected intensity of preferences depends on their direction, and because the whole distribution of valuations moves to the right

[30] With the bonus vote scheme, the probability of being pivotal is $\frac{1}{2(1+\theta/k)}\frac{1}{n}h_\delta(0)$ when the bonus vote is not cast, and $\frac{1+\theta}{2(1+\theta/k)}\frac{1}{n}h_\delta(0)$ when it is. If $\theta = 0$, we reproduce the standard result for simple majority voting.

when approval for the referendum is higher (and to the left when it is lower), the minority is constrained to have less intense preferences than the majority. Indeed, the smaller the minority the weaker its preferences. Bonus votes, meant to differentiate between popular support and intensity of preferences, would be less valuable in this model. We conjecture that they would improve expected welfare over simple majority only if valuations are sufficiently concentrated around the center of the distribution, de facto reducing the correlation between volume and intensity of support. This observation can be important in specific practical applications, but on the whole we see no a priori reason why the minority should systematically have weaker preferences than the majority.

8 Asymmetrical distributions of valuations

For all the subtleties of the different models, the intuitive reason why bonus votes can be valuable is straightforward: they give some voice to minority preferences when these are particularly intense. In other words, bonus votes recognize possible asymmetries in valuations that majority voting ignores.

When we work with symmetrical distributions of preferences, as we have done so far, we make that task particularly hard: bonus votes can then only reward occasional empirical asymmetries, sample deviations from the theoretical distribution whose importance must disappear as the population becomes large. This is why the absolute welfare improvement over simple majority disappears in the limit, as we remarked earlier. The observation is almost obvious: if we constrain the mean and the median of the distribution of valuations to coincide (or more generally to have the same sign), simple majority must be asymptotically efficient; it is only when we allow the mean and the median to differ that bonus votes can play a more substantial role.[31]

To study the problem in the simplest setting, suppose that the distributions of valuations are identical over all proposals, but now for each proposal call $P(v)$ the distribution of valuations of voters in favor, and $C(v)$ the distribution of valuations of voters against the proposal (where both distributions can be stated in terms of absolute valuations). The two distributions have different means: for concreteness, suppose $E_P(v) > E_C(v)$, implying that in each referendum the mean valuation over the whole electorate is positive. We assign the higher mean valuation to the "pro" side with no loss of generality—which side has higher mean is irrelevant and we could trivially generalize the model to allow the side with higher mean to change across proposals.

We go back to our original assumption that the median valuation over the whole electorate is fixed: this section studies the scope for welfare gains whose

[31] Ledyard and Palfrey (2002) exploit this logic to design an asymptotically efficient voting referendum: the critical threshold for approval must be fixed at that level where the sample mean valuation, when the referendum just passes, equals the theoretical mean. With a distribution symmetrical around 0, the threshold corresponds to 50 percent. (This also implies the asymptotic efficiency of random decision-making). More generally, the asymptotically efficient threshold depends on the distribution. The idea is simple and clever, but setting different thresholds for different decisions seems delicate politically.

absolute size does not disappear asymptotically, and the welfare comparison to simple majority will not depend on relative probabilities of being pivotal.[32] Where the median is determines the asymptotic welfare properties of majority voting: in this example, majority voting is asymptotically efficient if the median is positive, inefficient if it is negative, and equivalent to randomness if the median is at 0.[33] As we remarked earlier, the literature has found empirical support for the hypothesis that legislatures defer to popular vote decisions that are politically riskier, and more precisely decisions over which the electorate is equally split (Matsusaka, 1992). Let us suppose then that the median is at 0: both $P(v)$ and $C(v)$ have full support $[0,1]$, and $P(1) = C(1) = 1$. All valuations are independent, across voters and proposals.

The asymptotic properties of the bonus votes scheme depend on the shapes of the $P(v)$ and $C(v)$ distributions, mediated by the equilibrium strategy. Once again, the equilibrium strategy is pinned down by the requirement that the impact of the bonus vote must be equalized across referenda and requires voters to cast their bonus vote on the referendum felt with highest intensity.[34] The welfare properties then depend on the probability that the sign of a voter's highest valuation draw equals the sign of the mean valuation in the population, i.e. is positive in this example. Call ϕ_{rP} the probability of casting the bonus vote in favor of referendum r, where $\phi_{rP} = \phi_P = \left(\frac{1}{2}\right)^{k-1} \left[\sum_{s=0}^{k-1} (\binom{k-1}{s} \int_0^1 C(v)^{k-1-s} P(v)^s p(v) dv\right]$ for all r, and ϕ_{rC} the probability of casting the bonus vote against referendum r, where $\phi_{rC} = \phi_C = \left(\frac{1}{2}\right)^{k-1} \left[\sum_{s=0}^{k-1} (\binom{k-1}{s} \int_0^1 C(v)^{k-1-s} P(v)^s c(v) dv\right]$ for all r. Consider a sequence of bonus votes referenda indexed by the size of the population n, and similarly index our welfare criteria. Then:

Proposition 5. *For any* $\theta > 0$ *and* $k > 1$, *if* $\phi_P > \phi_C$ then *as* $n \to \infty$, $EU_n/EW_n \to 1 + [E_P(v) - E_C(v)]/[E_P(v) + E_C(v)] > 1$, *and* $\omega_n \to \infty$. (The

[32] As we saw, with asymmetries and known probability of approval for each referendum, a voter's probability of being pivotal approaches zero at rate e^{-n}, both with simple majority and with bonus votes. We take the willingness to vote as a given.

[33] If the probability of approval ψ_r is random, the welfare results can be rephrased in terms of the median of the H_ψ distribution.

[34] Even in the presence of asymmetries between opponents and supporters of each referendum, the equilibrium is generically unique - see the proof in Casella and Gelman (2005). The intuition is straightforward: in any referendum, for arbitrary $P(v)$ and $C(v)$, there cannot be equilibria where the probability of being pivotal is not negligible and equal whether voting in favor or against (because in such a case, voters would allocate the bonus vote on the basis of absolute valuations, and for arbitrary $P(v)$ and $C(v)$ the expected volume of bonus voters cast on the two sides of the issue would differ). Nor can there be equilibria where the probability of being pivotal is not negligible on one side of the issue only, because no voter would cast a bonus vote on the opposite side, and with independent valuation draws the outcome of the referendum would then be certain. Thus, generically, the probability of being pivotal must be equal and negligible in all referenda and on both sides of each proposal. Because the equilibrium strategy is pinned down by the relative probability of being pivotal across two referenda, voters cast their bonus vote on the referendum with highest valuation. But in practice, it is the simplicity of the strategy, more than the infinitesimal loss that a deviating voter would incur, that recommends focussing on it.

proof is in the Appendix).

As long as $\phi_P > \phi_C$, the probability that a referendum passes converges asymptotically to 1 for any positive θ, as opposed to approaching 1/2 in the case of simple majority. Bonus votes shift the outcome in the direction of the mean, and hence increase efficiency. As the size of the population approaches infinity, majority voting approaches randomness, but bonus votes do not, and the absolute difference in ex ante utility between the two voting mechanisms does not disappear: relative to randomness, the welfare gain associated with bonus votes grows arbitrarily large.

How likely is the condition $\phi_P > \phi_C$? Given on average higher positive valuations, the condition seems plausible, but guaranteeing it requires imposing further restrictions on the distributions. For example, the definitions of ϕ_P and ϕ_C imply immediately that a sufficient condition is first order stochastic dominance: if $P(v)$ first-order stochastically dominates $C(v)$, then $\phi_P > \phi_C$, and Proposition 5 follows. First-order stochastic dominance is satisfied by the power distribution we have used as recurring example:

Example 3. *Suppose that both $P(v)$ and $C(v)$ are power distributions with parameters b_p and b_c, where $b_p > b_c$. Then for any $\theta > 0$ and $k > 1$, as $n \to \infty$, $EU_n/EW_n \to 1 + (b_p - b_c)/(b_p + b_c + 2b_p b_c) > 1$ and $\omega_n \to \infty$.*

With first-order stochastic dominance, the probability mass of favorable valuations is concentrated towards higher values than is the case for negative valuations, and bonus votes are correspondingly concentrated on favorable votes. To see what first-order stochastic dominance implies in practice, suppose once again that the public's intensity of preferences at best can be identified through a partition of the support of (absolute) valuations into four equally sized intervals. Consider a referendum where proponents on average have more intense preferences than opponents. First-order stochastic dominance requires some monotonicity in the manner in which voters on the two sides are distributed in the four intervals. Among those judging the proposal "very important" the majority should be proponents, and similarly among those considering it either "very important" or "important"; among those judging the proposal "not important" the majority should be opponents, and similarly among those considering it either "not important" or "somewhat important."

But first-order stochastic dominance is stronger than needed. Consider two beta distributions: $P(v) = \int_0^v x^{a_P - 1}(1-x)^{b_P - 1}/\text{Beta}[a_P, b_P]$, $C(v) = \int_0^v x^{a_C - 1}(1-x)^{b_C - 1}/\text{Beta}[a_C, b_C]$, and suppose $E_P(v) = a_P/(a_P + b_P) > E_C(v) = a_C/(a_C + b_C)$. In numerical simulations, we can constrain the ratios a_P/b_P and a_C/b_C to ensure $E_P(v) > E_C(v)$, while changing the values of b_P and b_C to change the shapes of the distributions: the higher the parameter b, the larger the probability mass concentrated around the mean. Figure 1 shows the ratio ϕ_P/ϕ_C when $E_P(v) = 2/3$ and $E_C(v) = 1/2$, and $b_P \in [1/2, 5]$, $b_C \in [1/2, 5]$. We are interested in the range of parameter values for which ϕ_P/ϕ_C is larger than 1, and thus the bonus votes are asymptotically efficient. In Figure 1.a $k = 2$, and ϕ_P/ϕ_C is always larger than 1, although $P(v)$ and $C(v)$ cannot in general be ranked in terms of first-order stochastic dominance. But the result is not guaranteed: in Figure 1.b, $k = 4$, and there is a small area within our range of parameter values where ϕ_P/ϕ_C falls below 1 (the lower left corner of the figure). The problematic cases are those where the side with higher mean is concentrated in its valuation (b_P is high), while the opposite side is dispersed and bimodal at 0 and 1 ($b_C < 1$ and small). With probability increasing in k, the number of referenda, it is then possible for the bonus votes to be used predominantly by the side with lower mean valuation (the larger the number of draws, the higher the probability that the highest draw will come from the more dispersed distribution). But as the figure shows, the range of parameter values for which this occurs is small.[35] Intuition suggests that it should be smaller still if the distributions differ across referenda.

Figure 1
ϕ_P/ϕ_C.
Beta distributions: $E_P(v) = 2/3$; $E_C(v) = 1/2$;
$b_P \in [0.5, 5]$, $b_C \in [0.5, 5]$;

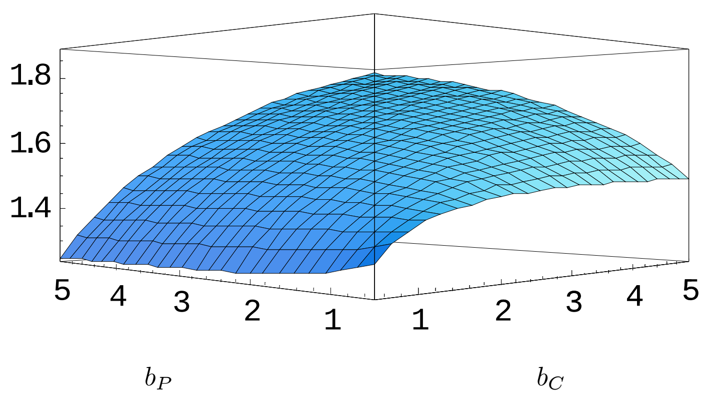

Figure 1a: $k = 2$

[35] In figure 1.b, $\phi_P/\phi_C < 1$ requires $b_C < 0.7$ and $b_P > 2$. There is a trade-off involved in the choice of k: the higher is k the larger is $(EU - EW)$ if $\phi_P/\phi_C > 1$; but if $P(v)$ does not first-order stochastically dominate $C(v)$, the lower is k, the smaller the range of distributions for which $\phi_P/\phi_C < 1$. Thus the optimal k depends on the precision of the information on the shape of the distributions.

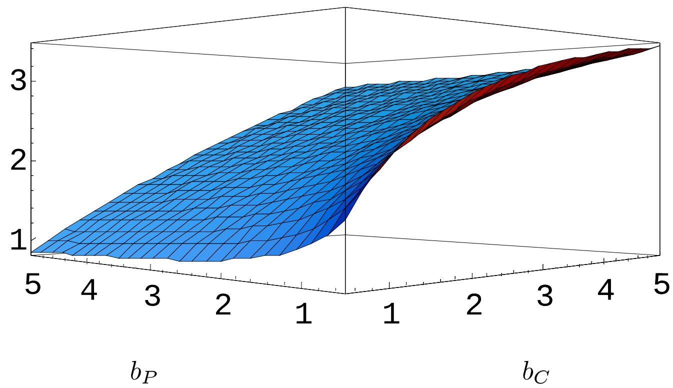

Figure 1b: $k = 4$

9 Additional discussion

The analysis can be extended further in a number of directions. In this section, we discuss two that seem of particular interest, but for which an exhaustive treatment is beyond the scope of this paper.

9.1 Related referenda

In all of the analyses of this paper, we have assumed each voter's valuations to be independent across referenda. Strictly speaking, the menu of referenda is part of the design of the mechanism, and we could demand of the planner that the referenda be unrelated. In practice, however, the assumption is likely to be violated. Does it matter? The answer depends on what we mean by "dependent valuations." If a voter's utility is not separable in the referenda's valuations, for example if preferences on a specific referendum depend on the outcome of a different one, then the correct model is not one of k binary decisions, but of a single k-dimensional choice, among 2^k possible alternatives. This is a more difficult problem, lending itself to the possible pathologies identified by voting theory[36], and is beyond the scope of this paper. If the assumption of separable utility can be maintained, however, we can address the question within the model we have used so far.

Suppose that each voter's valuations over the k referenda are drawn from a multivariate distribution $F_i(\mathrm{v}_1, ..., \mathrm{v}_k)$ which we assume identical across voters: $F_i = F$. Valuations are statistically dependent across referenda, but are independent across voters, and the referenda are held simultaneously. The main result in this case is that the previous analysis continues to apply but needs to be rephrased in terms of the marginal (unconditional) distributions of the

[36] See for example Brams, Kilgour and Zwicker (1998).

valuation in each referendum. The difficulty is not the dependence among each voter's valuations per se, but the possible heterogeneity of the marginal distributions. Without more structure on the pattern of dependence, and thus on the distributions' heterogeneity, characterizing the equilibrium strategy is very difficult.

If the marginal distributions are not heterogeneous, however, the analysis is unchanged. More precisely, suppose that the distribution $F(\mathrm{v}_1, ..., \mathrm{v}_k)$ is exchangeable, i.e. is invariant to permutations of the indexes. Then, although the valuations are statistically dependent, the model is fundamentally symmetric: ex ante there is nothing to distinguish one referendum from the others. The condition is restrictive, but it is not hard to think of scenarios that satisfy it. Consider the following example, where we make a distinction between dependence in the *direction* of preferences and dependence in the *intensity* of preferences. Suppose that the k referenda concern a single general topic and the direction of each voter's preferences is perfectly correlated among them - if the voter is in favor of one, then he or she is in favor of all.[37] Absolute valuations, however, are drawn independently across referenda according to some distribution $G_r(v)$, regardless of the sign of the preferences - each individual referendum may be considered by the voter trivial or important, and knowing the intensity over one of them does not help predict the voter's intensity over a different one. If $G_r(v) = G(v)$, then the distribution F is exchangeable. In this example we have assumed perfect correlation in the direction of preferences and zero correlation in the intensity of preferences, but all that is required is that both types of correlation be symmetrical across referenda.

If the distribution $F(\mathrm{v}_1, ..., \mathrm{v}_k)$ is exchangeable, the marginal distributions of valuations are identical across referenda, and in equilibrium each voter casts the bonus vote on the referendum with highest absolute valuation. The welfare conclusions depend of the shapes of these distributions, but follow the same logic discussed in the rest of the paper. There are two caveats: first, we are ruling out Bayesian updating on the part of the individual voter that, on the basis of the voter's own valuations, may result in heterogeneous posterior distributions. Second, if the absolute valuations are not independent, the hedge between the mean and the expected highest absolute valuation will be reduced, if the correlation is positive, or increased, if it is negative, affecting the potential for welfare gains.

9.2 Multiple non-cumulative bonus votes

We have shown that granting multiple bonus votes is in fact equivalent to granting a single one when the bonus votes can be cumulated (and the votes are discrete). But can efficiency be raised by granting several bonus votes with the constraint that they cannot be cumulated? This amounts to an increase in the number of instruments at the planner's disposal, and if the value of the bonus votes is chosen optimally (including the option of setting the values to zero) ef-

[37] More generally, all signs are perfectly predictable, once one of them is known.

ficiency must indeed rise, at least weakly. What makes the question interesting is the link it establishes between the scheme discussed in this paper and the first best mechanism proposed by Jackson and Sonnenschein (forthcoming). As described earlier, Jackson and Sonnenschein design a mechanism that asks each individual to announce the valuations attached to a series of unrelated decisions, but where the announcement is constrained to mimic the actual distribution of valuations. As the number of decisions becomes large, the distribution of valuations can be reproduced more and more finely, and the mechanism approaches first best efficiency.[38] Suppose for example that the distribution of valuations is Uniform. Then agents are simply asked to rank the decisions: the ranking can be read as fitting the same percentage of decisions into any equal subset of the distribution. In our setting, with k binary proposals the mechanism can be implemented through k referenda where each voter is endowed with a series of $k-1$ non-cumulative bonus votes of values $1, 2, .., k-1$. As k becomes large, the mechanism approaches full efficiency. If the distribution is not Uniform, the mechanism does not reduce to a simple ordinal ranking, and the optimal values of the bonus votes will in general be less intuitive.

The theoretical result established by Jackson and Sonnenschein is a limit result and requires the number of decisions to be large. It is natural to ask how well their mechanism performs when the number of decisions is finite and possibly small, and in particular how it compares to granting a single bonus vote. A full answer to this question is difficult and beyond the ambitions of this paper, but we can begin to address it in special cases. As in Jackson and Sonnenschein, suppose that the distribution of valuations is identical for all proposals, and, as in our basic model, symmetrical around 0. Each voter is endowed with k bonus votes of values ϑ_s, with $s \in \{1, ., k\}$ and $\vartheta_1 < \vartheta_2 < .. < \vartheta_k$, that cannot be cumulated. Valuations draws are independent both across proposals and across voters. We want to compare ex ante utility when all values ϑ_s are chosen optimally and when they are constrained, so that only a subset of the bonus votes have non-zero values.

In the setting we are describing, there is an equilibrium where the highest bonus vote is cast on the referendum with highest valuation, and so on in declining order. The gain in ex ante utility relative to random decision-making is given by:

$$EV_{nc} - ER = \frac{kEv + \sum_{s=1}^{k} \vartheta_s Ev_{(s)}}{\sqrt{2\pi n[1 + 1/k(\sum_{s=1}^{k} \vartheta_s^2 + 2\vartheta_s)}} \tag{22}$$

where the subscript nc indicates the non-cumulative bonus vote case, and $Ev_{(s)}$ denotes the sth order statistics. Because only relative values of the votes matter, we can set $\vartheta_1 = 0$: the vote cast on the referendum with lowest valuation - the regular vote alone - is the numeraire. The optimal values of the remaining $k-1$ bonus votes are chosen optimally by the planner and depend on the distribution

[38] The result requires that the valuations draws be independent across individuals and decisions, and that the distributions of valuations be identical over all decisions. We return to this latter point below.

of valuations.

Example 4. *Suppose that $G(v)$ is Uniform. Then $\vartheta_s^* = s - 1$, $s \in \{1, ., k\}$, for all k. If the values of the bonus votes are set optimally, the welfare criterion ω_{nc} is monotonically increasing in k. It equals* 1.08 *at* $k = 3$, 1.11 *at* $k = 5$, *and converges to* $2/\sqrt{3} = 1.15$ *as k approaches infinity. The comparison to a single bonus vote depends on k. Recall that with a single bonus vote $\theta^* = 1$ for all k. If $\theta = 1$, $\omega = 1.06$ at $k = 3$ (where it is maximized), and falls monotonically for larger k*, reaching $\omega = 1.05$ *at* $k = 5$, *and* $\omega = 1.01$ *at* $k = 50$.

In this example, multiple non-cumulative bonus votes are valuable, and their welfare improving potential is confirmed even when the number of decisions is small. In fact, a majority of the welfare gains can be reaped with little bundling - i.e. at values of k equal to 3 or 4, an observation also made by Jackson and Sonnenschein in their numerical simulations. The implication is that a single bonus vote is inferior, but can be quite effective if practical considerations constrain k to be small, or if a larger number of referenda can be costlessly divided into several bundles.

We leave more systematic comparisons for future work, but conclude this discussion with two final observations. First, in more general cases, identifying the optimal values of the non-cumulative bonus votes is less straightforward and the values themselves are less intuitive. For example, if $G(v) = v^b$, $b = 4$ and $k = 4$, $\vartheta_2^* = 1/4$, $\vartheta_3^* = 13/32$ and $\vartheta_4^* = 67/128$ (we have been unable to find a simple closed-form solution for ϑ_s^* with arbitrary k). From a practical point of view, we worry about assigning bonus votes with values that may appear arbitrary and, more importantly, that would change across different elections depending on the distribution of valuations. This was the rationale for focusing on sufficient conditions for welfare gains in the case of a single bonus vote (as opposed to insisting on optimal values). Similar sufficient conditions should be derived in the case of non-cumulative bonus votes, to see whether prudent but consistent values of the bonus votes can be identified. Second, if the distributions of valuations are less well-behaved - if the probability of approval of different proposals is stochastic, or if the distributions are asymmetrical or heterogenous - characterizing the equilibrium strategies becomes much more challenging. It may well be that in such cases referenda with multiple non-cumulative bonus votes cannot implement the Jackson and Sonnenschein mechanism. In fact, if the distributions of valuations are heterogenous, we are outside the scope of the Jackson and Sonnenschein mechanism and cannot invoke the mechanism's efficiency results, even when equilibrium play can be pinned down.

10 Conclusions

This paper has discussed an easy scheme to improve the efficiency of referenda: hold several referenda together and grant voters, in addition to their regular votes, a stock of special votes—or even more simply a single special vote—that can be allocated freely among the different referenda. By concentrating these bonus votes on the one issue to which each voter attaches most importance, voters can shift the probability of obtaining the outcome they prefer towards the issue they care most about. The mechanism is not fully efficient in general, but under plausible scenarios it leads to expected efficiency gains relative to simple majority. In some of the scenarios we have studied, the conclusion holds regardless of the value of the bonus votes; in other cases, the value of the bonus votes should not be too large, an invitation to caution that matches well what common sense also recommends. Indeed, the main virtue of the scheme seems to us its intuitive nature: the equilibrium strategies are simple, the reason for the efficiency gains transparent, and the modification with respect to existing voting on referenda minor.

Bonus votes are a simple mechanism allowing some expression of voters' intensity of preferences. They recall cumulative voting—an existing voting scheme employed in multi-candidate elections with the expressed goal of protecting minority interests. The protection of minorities built into these mechanisms is a particularly important objective as recourse to direct democracy increases. In fact the need to safeguard minorities, and in particular minorities with little access to financial resources, is the single point of agreement in the often heated debate on initiatives and referenda (for example Matsusaka (2004) and Gerber (1999)).[39] The important objective is designing voting mechanisms that increase minority representation without aggregate efficiency losses, and this is why in this paper we have insisted on the pure efficiency properties of bonus votes.[40]

The paper suggests several directions for future research. In addition to the points raised in the text, one important question the paper ignores is the composition of the agenda. In the model we have studied, the slate of referenda is exogenous. We believe this is the correct starting point: modeling agenda-formation processes is famously controversial, and in our case requires identifying groups with common interests, taking a stance on the correlations of the group members' valuations across different issues and on the forces holding the groups together, in an environment where voting is completely anonymous. From a technical point of view, it implies renouncing the assumption of independence across voters and thus the power of the limit theorems we have exploited repeatedly. Intuitively, the final outcome seems difficult to predict: bonus votes may increase the incentive to manipulate the agenda, but also the ability to nullify the manipulation. We leave serious work on this issue for the future.

A second question left unaddressed is the possible role of voting costs. If voting costs are explicitly considered, do bonus votes still lead to improvements

[39] According to Gerber (1999), narrow business interests have a comparative advantage in influencing popular votes through financial resources, while grass-root movements have a comparative advantage in gathering votes. If this logic is correct, bonus votes would both help to protect resource-less minorities against the power of the majority and reduce the power of money in direct democracy.

[40] In the legal literature, Cooter (2002) compares "median democracy" (direct democracy) to "bargain democracy" (legislatures) and argues for the practical superiority of the former, while admitting that the latter is "ideally" more efficient. Increasing the representation of intense preferences in a direct democracy is a step towards higher efficiency.

in welfare? The question is relevant because in the presence of voting costs individuals close to indifference should prefer to abstain, and thus voting costs also work to increase the representation of voters with higher intensity of preferences. Voluntary voting in the presence of voting costs has been shown to improve efficiency over compulsory voting when the distribution of valuations is symmetrical around 0 (Börgers, 2004). But in large elections the conclusion is typically reversed when the electorate is not equally split (Krasa and Polborn, 2005, Taylor and Yildirim, 2005), or when the distribution of the net benefit from voting is correlated with the direction of preferences (Campbell, 1999). In the presence of voting costs, both simple majority voting and the bonus vote scheme would differ because of the additional option to abstain, and a formal comparison between the two voting rules demands a full analysis. One obvious preliminary observation is that if voting concerns several referenda, as in fact it often does in practice, the importance of voting costs in selecting between high and low valuation individuals must be reduced. Separating the referenda would improve voters' selection but increase all costs. But the main difficulty with voting costs, and the reason we have abstracted from these costs so far, is their poor empirical record. Given the difficulty in explaining observed turn-outs, in large elections normative recommendations that rely on rational self-selection in the presence of voting costs seem particularly courageous (and indeed none of the authors cited above takes this route).

If the test is finally empirical, then the bonus vote schemes should also be subjected to empirical validation, or more precisely, given that it does not exist, to experimental testing. It is this direction that we are pursuing currently.

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12 Appendix

Before proving the lemmas in the text, we begin by a preliminary result that will be used repeatedly. Define votes in favor as positive votes and votes against as negative votes, and the vote differential, the sum of all votes cast in referendum r, as V_r.

Lemma A.1. *Consider the voting problem in the absence of bonus votes when everybody votes sincerely. Call p_r a voter's probability of obtaining the desired outcome in referendum r. Then if $F_r(\text{v})$ is symmetric around 0, $V_r \sim N(0,n)$ and $p_r \simeq 1/2 + 1/\sqrt{2\pi n}$.*

Proof of Lemma A.1. The derivation of the asymptotic distribution of the vote differential is standard (see for example Feller, 1968, pp. 179 -182). The distribution is normal with mean given by the sample mean $(1/2(-1) + 1/2(1) = 0)$ and variance given by the sum of the variances of the summands: $n[(1/2)(-1)^2 + (1/2)(1)^2] = n$. Because the distribution does not depend on $F_r(v)$, we can ignore the subscript r. Taking into account possible ties, the probability of obtaining one's desired outcome is:

$$p = prob(V_i \leq 0) + (1/2)prob(V_i = 1)$$

where V_i is the voting differential excluding voter i. Given the discreteness of the votes:

$$prob(V_i \le 0) \simeq \frac{1}{2}\left(1+\frac{1}{\sqrt{2\pi(n-1)}}e^{-0}\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{2\pi(n-1)}}\right)$$

$$(1/2)prob(V_i = 1) \simeq \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}}e^{-\frac{(1)^2}{2(n-1)}}\right)$$

$$\simeq \frac{1}{2}\left[\frac{1}{\sqrt{2\pi(n-1)}}\left(1-\frac{1}{2(n-1)}\right)\right] = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}} - O(n^{-3/2})\right)$$

Given n large and ignoring terms of order $O(n^{-3/2})$, $p \sim 1/2 + 1/\sqrt{2\pi n}$. □

Gnedenko local limit theorem. When we add to the problem bonus votes of arbitrary value, we need to be more careful about the discreteness of the asymptotic distribution of the vote differential. The subtlety is in sizing correctly the steps of the distribution. We begin by presenting the local limit theorem relevant to our problem.

Consider the problem facing voter i in referendum r when bonus votes are available. The voter has to evaluate the probability of obtaining the desired outcome when casting x_i votes, where x_i equals ± 1 if i casts no bonus votes and $\pm(1 + z\vartheta)$ if he or she casts z bonus votes. All voters have the same set of feasible options $\mathbf{X} = \{x_j\} = \{\pm(1 + z\vartheta),\ z = 0, 1, \ldots, m\}$ (where j indexes any point in the support that has positive probability). If voters use symmetric strategies, x_i is iid for all i's, we can drop the subscript i and use local limit theorems to characterize the asymptotic distribution of the votes differential in any referendum. The random variable x is distributed according to a lattice distribution—a discrete distribution such that all possible $x_j \in \mathbf{X}$ can be expressed as $a + s_j h$ with $h > 0$ and s_j integer $\forall j$. We need to impose the correct normalization. Following Gnedenko local limit theorem (Gnedenko and Kolmogorov, 1968, ch.9), select the representation $x_j = a + s_j h^0$ such that h^0 is the largest common divisor of all possible pairwise differences $x_j - x_{j'}$, and consider the normalized random variable $x' \equiv (x - a)/h^0$ and the normalized sum $V' = \sum_{i=1}^{n} x'_i$. If x' has finite mean $E(x')$ and non-zero variance $\sigma^2_{x'}$, then:

$$prob\{V' = y\} \to \frac{1}{\sigma_{x'}\sqrt{2\pi n}} Exp[\frac{-(y - nE(x'))^2}{2n\sigma^2_{x'}}] \text{ as } n \to \infty$$

For our purposes, we need to consider two cases. If $\vartheta \ge 1$, no normalization is required; if $\vartheta = 1/C$ with C discrete and larger than 1, normalize the problem so that all x are set in terms of the smallest possible integers: set $h^0 = 1/C$ and $a = 0$, and thus $\mathbf{X}' = \{-(C+m), \ldots, -(C+1), -C, C, (C+1), \ldots, (C+m)\}$. In both cases, call $\pm\rho$ the normalized value of the regular vote (where $\rho = 1$ if $\vartheta \ge 1$ and $\rho = C$ if $\vartheta = 1/C$), and $\pm\xi$ the normalized value of one bonus vote (where $\xi = \vartheta$ if $\vartheta \geq 1$ and $\xi = 1$ if $\vartheta = 1/C$). In addition, define as $\phi_{x'r}$ the probability that any voter casts x' votes in referendum r, again distinguishing between positive and negative votes. Then: $E_r(x') \equiv \mu_r = \sum_{x' \in X'} \phi_{x'r} x'$, and $\sigma_r^2 = \sum_{x' \in X'} \phi_{x'r} (x' - \mu_r)^2$.

Proof of Lemma 1. Consider two referenda, r and s. Voter i is choosing how to allocate a given number of bonus votes between r and s, and in particular is choosing between strategies x'_r, x'_s, and strategies x''_r, x''_s. Calling $p_{x'_r}$ the probability of obtaining the desired outcome in referendum r when casting x'_r votes (and similarly in the other cases), the second strategy is superior if $v_r p_{x''_r} + v_s p_{x''_s} > v_r p_{x'_r} + v_s p_{x'_s}$. To keep the notation as simple as possible, suppose that i is considering reallocating a single bonus vote: $|x''_r| = |x'_r| + \xi$, $|x''_s| = |x'_s| - \xi$, and that $\xi = 1$ (which is then the step size in the votes' distribution). Then the second strategy is superior if:

$$\frac{v_r}{v_s} > \frac{p_{x''_s} - p_{x'_s}}{p_{x'_r} - p_{x''_r}} = \frac{1/2[prob(V'_s = -x''_s) + prob(V'_s = -x'_s)]}{1/2[prob(V'_r = -x''_r) + prob(V'_r = -x'_r)]}$$

(If i is in favor of a proposal, his or her vote is positive and it is pivotal if it counters a negative vote differential of the same magnitude; the signs are reversed if i is against the proposal). Using the limit theorem:

$$\frac{v_r}{v_s} > \frac{\sigma_s}{\sigma_r} \left(\frac{Exp[-\frac{(-x''_s - n\mu_s)^2}{2n\sigma_s^2}] + Exp[-\frac{(-x'_s - n\mu_s)^2}{2n\sigma_s^2}]}{Exp[-\frac{(-x''_r - n\mu_r)^2}{2n\sigma_r^2}] + Exp[-\frac{(-x'_r - n\mu_r)^2}{2n\sigma_r^2}]} \right) = \frac{\sigma_s}{\sigma_r} O(e^{-n(\mu_s^2/\sigma_s^2 - \mu_r^2/\sigma_r^2)})$$

There are several possibilities. If $\mu_r = \mu_s = 0$, the condition simplifies to $v_r/v_s > \sigma_s/\sigma_r$. (More generally if $(\mu_s^2/\sigma_s^2 - \mu_r^2/\sigma_r^2)$ is of order smaller or equal to $1/n$, the condition becomes $v_r/v_s > a(\sigma_s/\sigma_r)$ where a is some positive constant). If $\mu_r \neq 0$ and $\mu_s = 0$ (or more generally if $(\mu_s^2/\sigma_s^2 - \mu_r^2/\sigma_r^2)$ is negative and of order larger than $1/n$), the condition is never satisfied, and no bonus vote should be spent on referendum r; finally, if $\mu_r = 0$ and $\mu_s \neq 0$ (or more generally if $(\mu_s^2/\sigma_s^2 - \mu_r^2/\sigma_r^2)$ is positive and of order larger than $1/n$), the condition is always satisfied, and no bonus vote should be spent on referendum s. (Neither of the two last two scenarios can be an equilibrium: if it is not individually optimal to cast bonus votes on referendum r (s), with symmetric distributions $\mathbf{F}$ and independent individual draws, $E(V'_r) = n\mu_r = 0$ ($E(V'_s) = n\mu_s = 0$), contradicting $\mu_r \neq 0$ ($\mu_s \neq 0$)). In none of the cases is the choice of strategy affected by the sign of i's valuations or the direction of his or her votes. Finally, since $|x'_r|$ is just a normalization of $|x_r|$, the conclusion applies to $|x_r|$ too and Lemma 1 is established. □ As remarked in the text, the following corollary follows immediately:

Corollary to Lemma 1. *If the distributions* $\mathbf{F}$ *are symmetrical around* 0*, in all equilibria* $E(V'_r) = 0 \; \forall r$.

Proof of Lemma 2. Recall that we now understand the strategies $x'_{ir}(\mathbf{v}_i, m, \vartheta, \mathbf{G})$ to refer to the absolute number of votes cast. Given $E(V'_r) = 0 \; \forall r$, by Gnedenko's theorem, and exploiting the approximation $Exp[y/n] \simeq 1 - y/n$, the probability of obtaining one's desired outcome when casting x' votes in referendum r, can be approximated by:

$$p_{x'r} \sim \frac{1}{2} + \frac{x'}{\sqrt{2\pi n\sigma_r^2}} \tag{A1}$$

up to terms of order $O(n^{-3/2})$, whether the votes are for or against. Consider the problem faced by voter i, deciding how to allocate bonus votes so as to maximize expected utility:

$$\max_{\{x'_r\}} \sum_{r=1}^{k} \left(\frac{1}{2} + \frac{x'_r}{\sqrt{2\pi n\sigma_r^2}} \right) v_{ir} \tag{A2}$$

$$\text{s.t. } x'_r \;=\; 1 + z_r\xi \text{ where } z_r \in \{0, 1, \ldots, m\} \text{ and } \sum_{r=1}^{k} z_r = m$$

Problem (A2) is linear in z_r, and the solution must be at a corner: voter i will cast all bonus votes on referendum s such that $v_{is}/\sqrt{2\pi n\sigma_s^2} > v_{ir}/\sqrt{2\pi n\sigma_r^2}$ $\forall r \neq s$. (The variance of the vote differential facing voter i in referendum r is $n\sigma_r^2 = n \sum_{x' \in X'} \phi_{x'r}(x' - \mu_r)^2$ and is taken as given by i.) $\Box$

Proof of Lemma 3. By lemmas 1and 2, $E(V'_r) = 0$ $\forall r$ and $n\sigma_r^2 = n[\phi_r(\rho + m\xi)^2 + (1 - \phi_r)\rho^2] = n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]$. Thus:

$$p_{\rho r} \sim \frac{1}{2} + \frac{\rho}{\sqrt{2\pi n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]}}$$

$$p_{m\xi r} \sim \frac{1}{2} + \frac{\rho + m\xi}{\sqrt{2\pi n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]}}$$

up to terms of order $O(n^{-3/2})$. If we renormalize $\rho \equiv 1$ and $\theta \equiv m\xi/\rho$, we can write:

$$p_r \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi n[1 + \phi_r(\theta^2 + 2\theta)]}}$$

$$p_{\theta r} \sim \frac{1}{2} + \frac{1 + \theta}{\sqrt{2\pi n[1 + \phi_r(\theta^2 + 2\theta)]}}$$

The second part of the Lemma is proved in the text. $\Box$.

Proof of Proposition 3. At $\theta = 0$, $\omega = 1$—as must be the case by the definition of ω and as can be verified by setting $\theta = 0$ in (13). Proposition 3 must hold if ω is increasing in θ at $\theta = 0$: although θ must be a rational number larger than $1/\sqrt{n}$, there is always a value of n such that θ can take values arbitrarily close to 0. Because in addition ω is continuous in θ in the neighborhood of $\theta = 0$, in this neighborhood we can treat θ as a continuous variable. Showing

$d\omega/d\theta > 0$ at $\theta = 0$ amounts to differentiating (13), taking into account (11) and (12). The derivative is greatly simplified by being evaluated at $\theta = 0$. In fact it is not difficult to show that it reduces to:

$$\left.\frac{d\omega}{d\theta}\right|_{\theta=0} = \sum_{r=1}^{k}\left[\left(\int_0^1 \prod_{s\neq r} G_s(v) v g_r(v) dv\right)\right] - \sum_{r=1}^{k}\left[Ev_r(v)\left(\int_0^1 \prod_{s\neq r} G_s(v) g_r(v) dv\right)\right] \tag{A3}$$

Integrating by part the first summation, we obtain:

$$\sum_{r=1}^{k}\left[\left(\int_0^1 \prod_{s\neq r} G_s(v) v g_r(v) dv\right)\right] = \int_0^1 \left(1 - \prod_{r=1}^{k} G_r(v)\right) dv = Ev_{(k)}$$

where $Ev_{(k)}$ now stands for the expected highest draw over all distributions. Thus (A3) can be rewritten more simply as:

$$\left.\frac{d\omega}{d\theta}\right|_{\theta=0} = Ev_{(k)} - \sum_{r=1}^{k} Ev_r(v)\ \phi_r|_{\theta=0}$$

But since $\phi_r|_{\theta=0} \in (0,1)\forall r$, the expression must be strictly positive, and the proposition is established.□

Proof of Proposition 3b. The proof proceed identically to the proof of Proposition 3. Indeed, $\left.\frac{d\omega}{d\theta}\right|_{\theta=0} = \left.\frac{d\omega^R}{d\theta}\right|_{\theta=0}$, where $\omega^R \equiv \omega(\alpha_{sr} = 1 \forall s, r)$, (A3) continues to hold, and the argument is unchanged.□

Proof of Example 2. Here it turns out to be easier to work with $\tau \equiv 1/\theta$, the value of the regular votes relative to the bonus vote. With power distributions, the condition $\omega > 1$ then corresponds to:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k}\left(\frac{b_r c_r}{\sum_{r=1}^{k} b_r + 1}\right) + \tau \sum_{r=1}^{k}\left(\frac{b_r c_r}{1 + b_r}\right) > \sum_{r=1}^{k}\left(\frac{b_r}{1 + b_r}\right) \tag{A4}$$

where:

$$c_r \equiv \sqrt{\frac{\sum_{r=1}^{k} b_r}{b_r(1 + 2\tau) + \tau^2}}$$

The proof proceeds in three steps. First, we know from the proof of Proposition 3 that as τ approaches infinity, or equivalently θ approaches 0, ω approaches 1 from above. This immediately establishes that either $\omega > 1\ \forall\tau$, or there exists at least one internal maximum at a finite value of τ. Second, we can derive the first-order condition that an internal maximum, if it exists, must satisfy. Differentiating the left-hand side of (A4) with respect to τ, we find that the first derivative equals zero at τ^* defined by the implicit equation:

$$\tau^* = \sum_{r=1}^{k}\left(\frac{\gamma_r(\tau^*)}{\sum_{r=1}^{k} \gamma_r(\tau^*)}\right) b_r, \tag{A5}$$

where

$$\gamma_r(\tau^*) \equiv \frac{b_r}{1+b_r}\left[\sum_{s\neq r} b_s\left(\frac{1}{\sum_{j=1}^k b_j\tau^{*2}+b_r(1+2\tau^*)}\right)^{3/2}\right].$$

For our purposes, the important point is that any and all τ^*must be a weighted average of the distribution parameters $\{b_1,\ldots,b_k\}$, with weights $w_r(\tau^*) \equiv \left(\gamma_r(\tau^*)/\sum_{r=1}^k \gamma_r(\tau^*)\right) \forall\ r=1,\ldots,k$ and such that $\sum_{r=1}^k w_r(\tau^*)=1$. In particular, each weight is strictly between 0 and 1 for all positive finite b_r and τ^* (including the limit as τ^* approaches 0). Thus any and all τ^* must satisfy $\tau^* < b_k$ where $b_k \equiv \max\{b_r\}$. Third, consider the limit of ω as τ approaches 0:

$$\lim_{\tau->0}\omega = \left(\frac{\sqrt{\sum_{r=1}^k b_r}(\sum_{r=1}^k\sqrt{b_r})}{1+\sum_{r=1}^k b_r}\right)/\left(\sum_{r=1}^k\left(\frac{b_r}{1+b_r}\right)\right) \tag{A6}$$

The limit is positive and finite. There are two possibilities. If (A6) is smaller than 1, then by step 1 above an internal maximum must exist. Call $\tau^{*\prime}$ the largest value of τ that satisfies (A5), and $\omega(\tau^{*\prime})$ must be a maximum: then $\omega > 1\forall\tau > \tau^{*\prime}$. And since $\tau^{*\prime} < b_k$, $\omega > 1\ \forall\tau \geq b_k$. If (A6) is larger than 1, either no internal maximum exists and ω is larger than 1 for all τ—in which case, $\omega > 1\ \forall\tau \geq b_k$ is trivially satisfied. Or an internal maximum exists, and the argument above continues to hold: $\omega > 1\forall\tau > \tau^{*\prime}$, and since $\tau^{*\prime} < b_k$, $\omega > 1$ $\forall\tau \geq b_k$. Thus in all cases, $\omega > 1\ \forall\tau \geq b_k$ or, equivalently, $\omega > 1\ \forall\theta \leq 1/b_k$.□

Derivation of p_s and $p_{\theta s}$. For given δ_r, the votes differential in referendum r continues to be distributed according to a Normal distribution with mean given by the sample mean, and variance given by the sample variance. Given the equilibrium strategy, $E(V(\delta_r)) = n[\psi_r(1/k)(1+\theta)+\psi_r(1-1/k)(1)+(1-\psi_r)/k(-1-\theta)+(1-\psi_r)(1-1/k)(-1)] = n[(2\psi_r-1)(1+\theta/k)]$, where $\psi_r = prob(\text{v}_{ir} > 0)$ for all i,r and $1/k = prob(v_{ir} = \max\{v_i\})$ over all r, and thus $1/k = prob(|x_{ir}| = 1+\theta)$. Using $\psi_r = 1/2+\delta_r$, we can write $EV(\delta_r) = 2n\delta_r(1+\theta/k)$. Similarly, the variance of the votes differential $\sigma_r^2(\delta_r)$ is given by $n[(\psi_r+1-\psi_r)(1/k)(1+\theta)^2+(\psi_r+1-\psi_r)(1-1/k)(1)+O(\delta_r^2)] = n[(\theta^2+2\theta+k)/k+O(\delta_r^2)]$. Taking into account the discreteness of the distribution and ignoring terms of order δ_r^2 and higher, the expressions in (17) follow.

Proof of Proposition 5. The proof proceeds by showing that if $P(v)$ first-order stochastically dominates $C(v)$ for all r, each referendum passes with probability approaching 1. With a large population this outcome is ex ante efficient and dominates the outcome of simple majority voting. Recall:

$$\phi_P = \left(\frac{1}{2}\right)^{k-1}\left[\sum_{s=0}^{k-1}(\binom{k-1}{s}\int_0^1 C(v)^{k-1-s}P(v)^s p(v)dv\right]$$

$$\phi_C = \left(\frac{1}{2}\right)^{k-1}\left[\sum_{s=0}^{k-1}(\binom{k-1}{s}\int_0^1 P(v)^{k-1-s}C(v)^s c(v)dv\right]$$

or, identically, using the index $j \equiv k-1-s$:

$$\phi_C = \left(\frac{1}{2}\right)^{k-1} \left[\sum_{j=0}^{k-1} \binom{k-1}{j} \int_0^1 C(v)^{k-1-j} P(v)^j c(v) dv \right]$$

Because both $P(v)$ and $C(v)$ are strictly increasing in v, and $P(v)$ first-order stochastically dominates $C(v)$, each term summed in ϕ_P is larger than its corresponding term in ϕ_C, and thus $\phi_P > \phi_C$. The vote differential in each referendum is normally distributed with mean $EV = (n/2)\theta(\phi_P - \phi_C) > 0$ and variance $\sigma_V^2 = (n/2)[(\phi_P + \phi_C)(2\theta + \theta^2) - \theta^2/2(\phi_P - \phi_C)^2 + 2]$. Recall that $\Phi(x) \simeq 1 - x^{-1}\frac{1}{\sqrt{2\pi}} Exp[\frac{-x^2}{2}]$ when x is large and $\Phi(\cdot)$ is the standard normal distribution function (see for example Feller, 1968, chapter 7). Hence:

$$\begin{aligned} prob(V &> 0) = prob\left[(\frac{V-EV}{\sigma_V}) > -\frac{EV}{\sigma_V} \right] = \\ &= \Phi\left(\frac{EV}{\sigma_V}\right) \simeq 1 - \frac{1}{\sqrt{2\pi}} \frac{\sigma_V}{EV} e^{-EV^2/(2\sigma_V^2)} = 1 - \frac{1}{\sqrt{2\pi n}} O(e^{-n}) \end{aligned}$$

and the probability that proposal r passes equals

$$prob(V > 0) + \frac{1}{2} prob(V = 0) \simeq 1 - \frac{1}{2}\frac{1}{\sqrt{2\pi n}} O(e^{-n}) \simeq 1$$

Thus a proposal passes with probability approaching 1.

We can then write ex ante utility as:

$$EU \simeq \sum_r \frac{1}{2} E_{Pr}(v)$$

where 1/2 is the ex ante probability of being in favor of any proposal (given the 0 median). With simple majority voting, on the other hand:

$$EW \simeq \sum_r \left(\frac{1}{2} + \frac{1}{\sqrt{2\pi n}}\right) \left(\frac{E_{Pr}(v) + E_{Cr}(v)}{2}\right)$$

Because $E_{Pr}(v) > E_{Cr}(v)$, we can conclude that there always exists a large but finite $\widetilde{n}$ such that for all $n > \widetilde{n}$, $EU > EW$. The result holds for any positive θ, independently of its precise value. In addition, if we consider a sequence of referenda with increasing n, as $n \to \infty$, $EW_n \to ER = 1/2 \sum_r \left(\frac{E_{Pr}(v)+E_{Cr}(v)}{2}\right)$, while $EU_n \to \sum_r \frac{1}{2} E_{Pr}(v)$ yielding the Corollary to Proposition 5 in the text. □