Efficient Information Aggregation with Costly Voting*

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1 Introduction

There has been recent interest in the limit properties of voter turnout in costly voting models (see, for instance Taylor and Yildirim (2005)). However, the focus of this work has been on the so-called "private values" case—voters privately know which candidate they prefer and the only strategic decision is whether or not to go to the polls. A related literature pertains to the "common values" case—voter preferences are determined by the realization of an unknown "state of nature" and each voter receives a noisy signal about the state. This line of research has focused on how differences in voting rules lead to differences in informational efficiency (see Feddersen and Pesendorfer (1998)). However, this class of models does not consider costs of voting.

In this paper, we analyze the common values case under costly voting. In our basic model, voters each have privately known and independently and identically distributed voting costs. In a variation of the model, in Section 3.2, we analyze the case where voters have commonly known, identical and fixed voting costs. As we show, the distinction matters to the informational efficiency of an election when voters are strategic. In particular,

1. With private voting costs, majority-rule elections are informationally *efficient* in the limit: as the size of the electorate grows large, the expected number of voters is infinite and so the correct candidate is elected with probability one.

2. With common and fixed voting costs, majority-rule elections are informationally *inefficient* even in the limit: as the size of the electorate grows large, the

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expected number of voters converges to a finite limit and so the wrong candidate is elected with positive probability.

Our model relates to the extant strategic voting literature as follows: (a) it adds private information about voter preferences to the models of Palfrey and Rosenthal (1983, 1985); (b) it allows for voting costs in the model of Feddersen and Pesendorfer (1998); and (c) it parallels the analysis of Börgers (2004) and Taylor and Yildirim (2005) for the common values case.

2 The Model

There are two candidates named A and B who are competing in an election in which a simple majority of those voting is needed in order to be elected. Ties are resolved by a coin flip. There are two equally likely states of nature, α and β. Candidate A is the better candidate in state α while candidate B is the better candidate in state β. This means that in state α the payoff of any voter is 1 if A is elected and 0 if B is elected. In state β, the roles of A and B are reversed.

There is an electorate consisting of $N+1$ *citizens* (or potential voters). Prior to making the decision whether, and for whom, to vote, each citizen receives a private signal S_i regarding the true state of nature. The signal can take on one of two values, a or b. The probability of receiving a particular signal depends on the true state of nature. Specifically,

$$\Pr[a \mid \alpha] = \Pr[b \mid \beta] \equiv p$$

We suppose that $p > \frac{1}{2}$ so that the signals are informative and $p < 1$ so they are noisy. Conditional on the state of nature, the signals of the voters are realized independently.

Voting is costly and the costs of voting vary across voters. The cost of voting for each voter is private information and determined by a realization from a continuous probability distribution F with support $[0, \overline{c}]$. We suppose that $\overline{c} > p - \frac{1}{2}$ and that F admits a density f that is strictly positive on $(0, \overline{c})$. Voting costs are independently distributed across voters and independent of the signal as to who is the better candidate.

Thus prior to the voting decision, each voter has two pieces of private information—his cost of voting and a signal regarding the state.

3 Equilibrium with Majority Rule

We will show that under majority rule, there exists an equilibrium of the voting game with the following features. First, there is a cut-off level of the cost of voting, say c, such that a citizen with a cost realization c_i votes if and only if $c_i \leq c$. Second, all those who vote do so sincerely—that is, all those with a signal of a vote for A and those with a signal of b vote for B.

We study majority rule under the usual assumptions—whoever receives more votes wins and if there is a tie, then the outcome is determined by a coin toss.

Suppose that all N voters except 1 follow the strategy outlined above. We will argue that citizen 1 should also vote according to his signal.

Suppose citizen 1 receives a signal $S_1 = a$. Denote by $\langle k, l \rangle$ the event that out of $k + l$ voters other than 1, k vote for A and l vote for B. Since all of these $k + l$ voters vote sincerely, the probability of this event conditional on citizen 1's signal being a is

$$\begin{aligned}
\Pr[\langle k, l \rangle \mid S_1 = a] &= \Pr[\langle k, l \rangle \mid \alpha, S_1 = a] \Pr[\alpha \mid S_1 = a] \\
&\quad + \Pr[\langle k, l \rangle \mid \beta, S_1 = a] \Pr[\beta \mid S_1 = a] \\
&= \Pr[\langle k, l \rangle \mid \alpha] \Pr[\alpha \mid S_1 = a] + \Pr[\langle k, l \rangle \mid \beta] \Pr[\beta \mid S_1 = a] \\
&= p^{k+1} (1 - p)^l + (1 - p)^{k+1} p^l
\end{aligned}$$

where we have used the fact that $\Pr[\alpha \mid S_1 = a] = p$ and $\Pr[\beta \mid S_1 = a] = 1 - p$. For future reference, it is useful to note that,

$$\begin{aligned}
\Pr[\langle k, k \rangle \mid S_1 = a] &= p^{k+1} (1 - p)^k + (1 - p)^{k+1} p^k \\
&= p^k (1 - p)^k (p + (1 - p)) \\
&= p^k (1 - p)^k \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\Pr[\langle k, k + 1 \rangle \mid S_1 = a] &= p^{k+1} (1 - p)^{k+1} + (1 - p)^{k+1} p^{k+1} \\
&= 2p^{k+1} (1 - p)^{k+1} \qquad (2)
\end{aligned}$$

If the threshold cost is c, then the probability that out of the other N people exactly m come to the polls is

$$\binom{N}{m} F(c)^m (1 - F(c))^{N-m} \qquad (3)$$

The benefits of voting are positive if and only if the person is pivotal—that is, either his vote breaks a tie or his vote results in a tie. The first event is of the form $\langle k, k \rangle$ and the second event is either of the form $\langle k, k + 1 \rangle$ or $\langle k + 1, k \rangle$. In the event that the other voters are tied, it is clear that 1 should vote for A since if he does so, the probability that the right candidate is elected is $p > \frac{1}{2}$ while if he votes for B, the probability that the right candidate is elected is only $1 - p < \frac{1}{2}$. In the event that the other voters vote in a way that B is winning by one vote, then 1 is indifferent between voting for A or B. Finally, in the event that the other voters vote in a way that A is winning by one vote, it is clear that 1 should vote for A since $k + 1$ voters received a signals and only k voters received b signals. Thus, if all other voters vote sincerely, voter 1 should also do so.

Next, suppose citizen 1 has a cost of voting of c_1. Denote by $\lfloor \frac{N}{2} \rfloor$, the largest integer smaller than or equal to $\frac{N}{2}$. Using (1), (2) and (3), the difference in the expected payoff to citizen 1 from voting versus not voting is

$$\begin{aligned} \Delta &= \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k} F(c)^{2k} (1 - F(c))^{N-2k} p^{k+1} (1-p)^k \left(p - \tfrac{1}{2}\right) \\ &+ \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k+1} F(c)^{2k+1} (1 - F(c))^{N-2k-2} 2p^{k+1} (1-p)^{k+1} \left(\tfrac{1}{2} - \tfrac{1}{2}\right) - c_1 \end{aligned}$$

The kth term in the first summation represents a situation in which $2k$ other voters show up at the polls and their votes are tied. In this case, by voting for A, the probability that the correct candidate is chosen is p while if 1 does not vote, this probability is $\frac{1}{2}$. The kth term in the second summation represents a situation in which $2k+1$ other voters show up and k vote for A while $k+1$ vote for B. In this case, citizen 1's vote will result in a tie and the correct candidate is chosen with probability $\frac{1}{2}$. If citizen 1 does not vote, then B is chosen and is the correct candidate with probability $\frac{1}{2}$ since, including 1's signal, there are $k+1$ signals in favor of both A and B.

The expression above simplifies to

$$\Delta = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k} F(c)^{2k} (1 - F(c))^{N-2k} p^k (1-p)^k \left(p - \tfrac{1}{2}\right) - c_1$$

The equilibrium threshold cost c is determined by the condition that a citizen with cost c is just indifferent between voting and not voting; that is, when

$$\left(p - \tfrac{1}{2}\right) \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k} F(c)^{2k} (1 - F(c))^{N-2k} s^{2k} = c$$

where

$$s \equiv \sqrt{p(1-p)} \in \left(0, \tfrac{1}{2}\right)$$

Writing $q = F(c)$ as the *quantile* ("percentage") of those voting, the equilibrium condition becomes

$$\left(p - \tfrac{1}{2}\right) \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k} (1-q)^{N-2k} (qs)^{2k} = F^{-1}(q) \tag{4}$$

The left-hand side of (4) can be further simplified. For this we need to consider the case when N is odd separately from the case when N is even.

Case 1. N is even, so $N = 2r$ for some integer r and $\lfloor \frac{N}{2} \rfloor = r$. The binomial theorem implies that the summation on the left-hand side of (4) is

$$\sum_{k=0}^{r} \binom{2r}{2k} (1-q)^{2r-2k} (qs)^{2k} = \tfrac{1}{2}\left[(1-q-qs)^{2r} + (1-q+qs)^{2r}\right]$$
$$= \tfrac{1}{2}\left[(1-(1+s)q)^{N} + (1-(1-s)q)^{N}\right]$$

Case 2. N is odd, so $N = 2r+1$ for some integer r and again $\lfloor \frac{N}{2} \rfloor = r$. The binomial theorem again implies that the summation on the left-hand side of (4) is

$$\sum_{k=0}^{r} \binom{2r+1}{2k} (1-q)^{2r+1-2k} (qs)^{2k} = \tfrac{1}{2}\left[(1-q-qs)^{2r+1} + (1-q+qs)^{2r+1}\right]$$
$$= \tfrac{1}{2}\left[(1-(1+s)q)^{N} + (1-(1-s)q)^{N}\right]$$

Thus, for all N, the equilibrium condition can be written concisely as

$$\tfrac{1}{2}\left(p-\tfrac{1}{2}\right)\left[(1-(1+s)q)^{N} + (1-(1-s)q)^{N}\right] = F^{-1}(q) \tag{5}$$

Let $\phi(q,N)$ denote the expression on the left-hand side of (5). Observe that $\phi(q,N)$ is continuous and decreasing in q. For all N, $\phi(0,N) = p - \frac{1}{2}$. When N is odd, $\phi(1,N) = 0$ and when N is even, $\phi(1,N) = \left(p-\frac{1}{2}\right)s^{N} < p - \frac{1}{2}$. The right-hand side of (5), $F^{-1}(q)$, is continuous and increasing in q. $F^{-1}(0) = 0$ and $F^{-1}(1) = \bar{c} > p - \frac{1}{2}$.

Thus, there always exists a unique equilibrium quantile q_N satisfying $0 < q_N < 1$. This in turn determines a unique equilibrium cost threshold $c_N = F^{-1}(q_N)$. We have established

Proposition 1 *Under majority rule with private costs, there exists a symmetric equilibrium that has the following features: (i) there is a cost threshold such that every citizen votes if and only if his private cost of voting is below this threshold; (ii) voting is sincere.*

3.1 Asymptotics

We now examine how the equilibrium identified in Proposition 1 behaves as the size of the electorate N grows.

Let q_N denote the equilibrium quantile determined by (5), that is, $\phi(q_N, N) = F^{-1}(q_N)$. Since $\phi(q,N)$ is decreasing in N, the equilibrium $q_{N+1} < q_N$; that is, the *turnout percentage* decreases as N increases. This, of course, is a feature shared by other rational voting models.

Note also that the limiting turnout percentage $\lim q_N = 0$. Since q_N is a decreasing sequence, it has a limit and if $\lim q_N = q_\infty > 0$, then $\phi(q, N)$, the left-hand side of (5), is less than

$$\tfrac{1}{2}\left(p - \tfrac{1}{2}\right)\left[\left(1 - q_\infty(1+s)\right)^N + \left(1 - q_\infty(1-s)\right)^N\right] \tag{6}$$

As $N \to \infty$, the limit of (6) is zero. This is because $s < \frac{1}{2}$ implies $|1 - q_\infty(1+s)| < 1$ and clearly, $1 - q_\infty(1-s) < 1$. Thus $\lim \phi(q, N)$ is also 0. The limit of the right-hand side of (5), however, is $F^{-1}(q_\infty) > 0$. This is impossible, so $\lim q_N = 0$.

We thus have

Proposition 2 *Under majority rule with private voting costs, the equilibrium turnout percentage is decreasing in the size of the electorate and is zero in the limit.*

We now claim that, despite the fact that the turnout percentage goes to zero in the limit, the correct candidate is elected. In other words, in large electorates, information is efficiently aggregated.

Proposition 3 *Under majority rule with private voting costs, the expected number of voters is unbounded as the size of the electorate increases.*

Proof. We will show that the limit of the expected number of voters, $\lim_{N\to\infty} Nq_N = \infty$. Suppose, by way of contradiction, that the infimum limit of Nq_N is finite, that is, $\lim_{N\to\infty} \inf_{K \geq N} Kq_K < \infty$. Then there exists a $T > 0$ and a convergent subsequence q_N such that for all N large enough, $q_N < \frac{T}{N}$. This implies that, for all N along the subsequence that are large enough,

$$\begin{aligned}\phi(q_N, N) &= \tfrac{1}{2}\left(p - \tfrac{1}{2}\right)\left[\left(1 - (1+s)q_N\right)^N + \left(1 - (1-s)q_N\right)^N\right] \\ &> \tfrac{1}{2}\left(p - \tfrac{1}{2}\right)\left[\left(1 - (1+s)\tfrac{T}{N}\right)^N + \left(1 - (1-s)\tfrac{T}{N}\right)^N\right]\end{aligned}$$

and since

$$\lim\left[\left(1 - (1+s)\tfrac{T}{N}\right)^N + \left(1 - (1-s)\tfrac{T}{N}\right)^N\right] = e^{-(1+s)T} + e^{-(1-s)T}$$

we have

$$\lim \phi(q_N, N) \geq \tfrac{1}{2}\left(p - \tfrac{1}{2}\right)\left[e^{-(1+s)T} + e^{-(1-s)T}\right] > 0$$

But since $\lim q_N = 0$, the equilibrium condition (5) implies

$$\lim \phi(q_N, N) = \lim F^{-1}(q_N) = 0$$

We thus have a contradiction and $\lim Nq_N = \infty$. ■

It now follows that,

Proposition 4 *Under majority rule ith private voting costs, the probability that in each state the correct candidate is elected goes to one as the size of the electorate increases.*

3.2 Fixed Costs of Voting

In this subsection we consider a model in which the costs of voting are identical across voters and commonly known.

We look for a symmetric mixed-strategy equilibrium in which each citizen goes to the polls with probability q and, when voting, votes sincerely. The calculations regarding sincere voting conditional on going to the polls are identical to those in Section 3. Furthermore, the calculations regarding the benefit of voting over not voting are also the same. The only difference is that for each individual, this benefit is equated to the common fixed cost c of voting. We assume that $0 < c < p - \frac{1}{2}$.

In a mixed strategy equilibrium every potential voter must be indifferent between voting and not voting, that is,

$$\tfrac{1}{2}\left(p-\tfrac{1}{2}\right)\left[(1-(1+s)\,q)^{N}+(1-(1-s)\,q)^{N}\right]=c \tag{7}$$

Recall that $s = \sqrt{p(1-p)}$ and satisfies $0 < s < \frac{1}{2}$. This equation determines an equilibrium probability of voting q_N.

Once again let us denote by $\phi(q, N)$ the left-hand side of the equation above. As before, $\phi(q, N)$ is decreasing in q, $\phi(0, N) = \left(p - \frac{1}{2}\right)$, $\phi(1, N) = 0$ if N is even and $\phi(1, N) = \left(p - \frac{1}{2}\right) s^N$ if N is odd. Since $c < p - \frac{1}{2}$, there exists a unique $q_N \in (0, 1)$ satisfying equation (7) for N sufficiently large. Specifically, let N^* be the smallest N for which $\left(p - \frac{1}{2}\right) s^N < c$. For all $N > N^*$, there exists a mixed strategy equilibrium in which the probability of voting $q_N < 1$ and

$$\phi(q_N, N) = c$$

We summarize this by

Proposition 5 *Under majority rule with with common fixed costs, there exists a symmetric equilibrium that has the following features: (i) the decision of whether or not to vote is random and the probability of voting is the same for all potential voters; (ii) voting is sincere.*

It is easy to verify that once again q_N is decreasing in N and so, as in Section 3, the probability of voting decreases with the size of the electorate and furthermore $\lim q_N = 0$. To summarize:

Proposition 6 *Under majority rule with common fixed costs, the equilibrium turnout percentage is decreasing in the size of the electorate and is zero in the limit.*

But in contrast to Proposition 3 of Section 3, with fixed, common voting costs, the expected number voters is not infinite.

Proposition 7 *Under majority rule with common fixed costs, the expected number of voters reaches a finite limit as the size of the electorate increases.*

Proof. Taking limits in (7), we see that $\lim_{N\to\infty} Nq_N = T < \infty$ where T satisfies

$$\begin{aligned}\lim\left[\left(1-(1+s)\tfrac{T}{N}\right)^N + \left(1-(1-s)\tfrac{T}{N}\right)^N\right] &= \frac{2c}{p-\frac{1}{2}} \\ e^{-(1+s)T}+e^{-(1-s)T} &= \frac{2c}{p-\frac{1}{2}}\end{aligned}$$

The left-hand side has the value 2 at $T=0$, is decreasing in T and approaches 0 as $T\to\infty$. The right-hand side is a positive number less than 2. Thus there exists a $T<\infty$ which equates the two. ■

It now follows that,

Proposition 8 *With fixed and common voting costs, the probability that in each state the correct candidate is elected is bounded away from* 1 *as the size of the electorate increases.*

How large is the expected number of voters in the limit? Since $e^{-(1-s)T} > e^{-(1+s)T}$, we have

$$2e^{-(1-s)T} > e^{-(1+s)T}+e^{-(1-s)T} = \frac{2c}{p-\frac{1}{2}}$$

or equivalently,

$$T < \frac{\ln\frac{p-\frac{1}{2}}{c}}{1-s}$$

A weaker bound can be obtained by observing that since $1-s>\frac{1}{2}$ and $p-\frac{1}{2}<\frac{1}{2}$

$$T < -2\ln 2c$$

This shows that T is quite small; it is of the order of $-\ln c$.

4 Unanimity Rule

In this section, we consider the *unanimity* rule. Specifically, suppose that B is the default choice and that A must obtain the votes of all those who choose to vote in order to be elected. If no one votes, then B is winner.

Consider a particular citizen, say 1, and define the following sets of vote totals of the other N voters.

$$\begin{aligned}Piv_A &= \{\langle 0,0\rangle\} \\ Piv_B &= \{\langle k,0\rangle : 1\le k\le N\}\end{aligned}$$

Piv_A consists of those events in which citizen 1 is *pivotal* for A in the following sense. If he does not vote, then B wins and if he votes for A, then A wins. With the unanimity rule, the only event of this sort is $\langle 0,0\rangle$. Similarly, Piv_B consists of those events in which if citizen 1 does not vote, then A wins and if he votes for B, then B wins. With the unanimity rule, Piv_B consists of all events in which A receives the votes of all other N voters and so by voting for B, the citizen can "veto" A.

Fix two numbers c_a and c_b in $(0,1)$. Let every citizen i follow the strategy: if the signal $S_i = a$ and $c_i \leq c_a$ then vote for A ; if $S_i = a$ and $c_i > c_a$, then do not vote. If the signal $S_i = b$, and $c_i \leq c_b$ then vote for B ; if $S_i = b$ and $c_i > c_b$, then do not vote. Note that this strategy incorporates *sincere* voting.

We will argue that there exist c_a and c_b that sustain these strategies as an equilibrium. Suppose that all citizens except 1 follow the strategy.

Consider a citizen, say 1, with signal $S_1 = a$. The difference in payoffs between voting for A and not voting, denoted by $\emptyset$, is

$$\begin{aligned} \pi^a_{A\emptyset} &= \Pr[\alpha \mid a] \Pr[Piv_A \mid \alpha] - \Pr[\beta \mid a] \Pr[Piv_A \mid \beta] \\ &= p \Pr[Piv_A \mid \alpha] - (1-p) \Pr[Piv_A \mid \beta] \end{aligned} \tag{8}$$

since a vote for A makes a difference only if the others' vote totals lie in Piv_A.Using the definition of Piv_A, we obtain that

$$\pi^a_{A\emptyset}(c_a, c_b) = p\left(1 - pc_a - (1-p)c_b\right)^N - (1-p)\left(1 - (1-p)c_a - pc_b\right)^N$$

making explicit the dependence of $\pi^a_{A\emptyset}$ on c_a and c_b. This is because in state α, the probability that a given voter both gets a signal a and has a cost $c \leq c_a$ is pc_a. The probability that he gets a signal b and has a cost $c \leq c_b$ is $(1-p)c_b$. The probability that a given voter does not vote in state α is therefore $(1 - pc_a - (1-p)c_b)$. Similarly, the probability that a given voter does not vote in state β is $(1 - (1-p)c_a - pc_b)$.

Similarly, consider a citizen with signal b. The difference in payoffs between voting for B and not voting is

$$\begin{aligned} \Pi^b_{B\emptyset} &= \Pr[\beta \mid b] \Pr[Piv_B \mid \beta] - \Pr[\alpha \mid b] \Pr[Piv_B \mid \alpha] \\ &= p \Pr[Piv_B \mid \beta] - (1-p) \Pr[Piv_B \mid \alpha] \end{aligned} \tag{9}$$

and using the definition of Piv_B, we obtain

$$\begin{aligned} \Pi^b_{B\emptyset}(c_a, c_b) &= p \sum_{k=1}^{N} \binom{N}{k} \left(1 - (1-p)c_a - pc_b\right)^{N-k} \left((1-p)c_a\right)^k \\ &\quad - (1-p) \sum_{k=1}^{N} \binom{N}{k} \left(1 - pc_a - (1-p)c_b\right)^{N-k} (pc_a)^k \\ &= p\left(\left(1 - pc_b\right)^N - \left(1 - (1-p)c_a - pc_b\right)^N\right) \\ &\quad - (1-p)\left(\left(1 - (1-p)c_b\right)^N - \left(1 - pc_a - (1-p)c_b\right)^N\right) \end{aligned}$$

where the second equality is obtained by using the binomial theorem.

Lemma 1 *There exists a solution* (c_a^*, c_b^*) *to*

$$\Pi^a_{A\emptyset}(c_a, c_b) = c_a \tag{10}$$
$$\Pi^b_{B\emptyset}(c_a, c_b) = c_b \tag{11}$$

that satisfies $c_a^* > c_b^*$.

Proof. First, we claim that for all $c_a > 0$, $\Pi^b_{B\emptyset}(c_a, 0) > 0$. This is because

$$\Pi^b_{B\emptyset}(c_a, 0) = p\left(1 - (1 - (1-p)\,c_a)^N\right) - (1-p)\left(1 - (1 - pc_a)^N\right)$$

implies that

$$\frac{\partial}{\partial c_a}\Pi^b_{B\emptyset}(c_a, 0) = p(1-p)\,N\left((1-(1-p)\,c_a)^{N-1} - (1-pc_a)^{N-1}\right) > 0$$

and together with $\Pi^b_{B\emptyset}(0,0) = 0$, this establishes the claim. Second, we claim that for all $c \in (0,1)$, $\Pi^b_{B\emptyset}(c,c) < 0$. This is because

$$\begin{aligned}
\Pi^b_{B\emptyset}(c,c) &= \sum_{k=1}^{N}\binom{N}{k}\left(p\,(1-c)^{N-k}\,((1-p)\,c)^k - (1-p)\,(1-c)^{N-k}\,(pc)^k\right) \\
&= \sum_{k=1}^{N}\binom{N}{k}p\,(1-p)\,(1-c)^{N-k}\,c^k\left((1-p)^{k-1} - p^{k-1}\right) \\
&< 0
\end{aligned}$$

This in turn implies that for all $c \in (0,1)$,

$$\Pi^b_{B\emptyset}(c,c) < c$$

We have thus argued that the curve implicitly defined by

$$\Pi^b_{B\emptyset}(c_a, c_b) = c_b$$

starts at zero and stays in the region of the unit square below the 45° line.

Next, we claim that for all $c_b \in [0,1]$, $\Pi^a_{A\emptyset}(0, c_b) > 0$. To see this, note that

$$\Pi^a_{A\emptyset}(0, c_b) = p\,(1-(1-p)\,c_b)^N - (1-p)\,(1-pc_b)^N$$

and so

$$\frac{\partial}{\partial c_b}\Pi^a_{A\emptyset}(0, c_b) = -Np\,(1-p)\,(1-(1-p)\,c_b)^{N-1} + Np\,(1-p)\,(1-pc_b)^{N-1} < 0$$

and together with the fact that $\Pi^a_{A\emptyset}(0,1) = p^{N+1} - (1-p)^{N+1} > 0$, this establishes the claim. Finally, since payoffs are bounded above by 1, we have that $\Pi^a_{A\emptyset}(c_a, c_b) < 1$.

Thus there is a solution (c_a^*, c_b^*) to the system of equations (10) and (11) that satisfies $c_a^* > c_b^*$. ■

We now argue that if all other voters follow the strategies outlined above for the thresholds (c_a^*, c_b^*), then sincere voting is optimal for citizen 1.

Let

$$Piv = Piv_A \cup Piv_B$$

be the set of events in which by changing his vote, a voter can change the outcome.

Suppose that citizen 1's signal $S_1 = a$. The difference in payoffs from voting for A versus voting for B is

$$\Pi_{AB}^a = p \Pr[Piv \mid \alpha] - (1-p) \Pr[Piv \mid \beta]$$

Lemma 2 *If* $c_a^* > c_b^*$, *then* $\Pi_{AB}^a(c_a^*, c_b^*) > 0$.

Proof. Note that since Piv_A and Piv_B are disjoint,

$$\begin{aligned}
\Pi_{AB}^a(c_a^*, c_b^*) &= p \Pr[Piv \mid \alpha] - (1-p) \Pr[Piv \mid \beta] \\
&= p \Pr[Piv_A \mid \alpha] - (1-p) \Pr[Piv_A \mid \beta] \\
&\quad + p \Pr[Piv_B \mid \alpha] - (1-p) \Pr[Piv_B \mid \beta] \\
&= c_a^* - ((1-p) \Pr[Piv_B \mid \beta] - p \Pr[Piv_B \mid \alpha])
\end{aligned}$$

using (9) and (10). But since

$$\begin{aligned}
(1-p) \Pr[Piv_B \mid \beta] - p \Pr[Piv_B \mid \alpha] &< p \Pr[Piv_B \mid \beta] - (1-p) \Pr[Piv_B \mid \alpha] \\
&= c_b^*
\end{aligned}$$

because of (11), we have that

$$\Pi_{AB}^a(c_a^*, c_b^*) > c_a^* - c_b^* > 0$$

■

We have thus argued that given a signal of a, it is optimal to vote for A. It remains to show that given a signal of b, it is optimal to vote for B.

Suppose that citizen 1's signal $S_1 = b$. The difference in payoffs from voting for A versus voting for B is

$$\Pi_{BA}^b = p \Pr[Piv \mid \beta] - (1-p) \Pr[Piv \mid \alpha]$$

Lemma 3 *If* $c_a^* > c_b^*$, *then* $\Pi_{BA}^b(c_a^*, c_b^*) > 0$.

Proof. Again, since Piv_A and Piv_B are disjoint,

$$\begin{aligned}
\Pi_{BA}^b(c_a^*, c_b^*) &= p \Pr[Piv \mid \beta] - (1-p) \Pr[Piv \mid \alpha] \\
&= p \Pr[Piv_B \mid \beta] - (1-p) \Pr[Piv_B \mid \alpha] \\
&\quad + p \Pr[Piv_A \mid \beta] - (1-p) \Pr[Piv_A \mid \alpha] \\
&= c_b^* + p(1 - (1-p) c_a^* - p c_b^*)^N - (1-p)(1 - p c_a^* - (1-p) c_b^*)^N
\end{aligned}$$

But since $c_a^* > c_b^*$ and $p > \frac{1}{2}$, we have $(1 - (1-p)\, c_a^* - p c_b^*) > (1 - p c_a^* - (1-p)\, c_b^*)$ and so $\Pi_{BA}^b (c_a^*, c_b^*) > 0$. ■

We have thus established

Proposition 9 *Under the unanimity rule with private costs, there exists an equilibrium that has the following features: (i) there is a pair of cost thresholds* (c_a^*, c_b^*) *such that every citizen with signal a (resp. b) votes if and only if his private cost of voting is below* c_a^* *(resp.* c_b^**); (ii) voting is sincere.*

4.1 Asymptotics

How well does the unanimity rule do at information aggregation? For information to aggregate, it must be that the correct candidate is chosen with probability one in the limit. Let $c_i(N)$ denote the equilibrium cost threshold for a voter with signal i when there are N other voters.

Thus, in state α, the correct candidate is A and the expected fraction of the vote obtained by candidate A in state α is $\frac{(N+1) p c_a(N)}{(N+1)(p c_a(N) + (1-p) c_b(N))}$. Simplifying this expression and noting that A must receive a share λ of the vote to win, we then have that a necessary condition for information aggregation that:

$$\lim_{N \to \infty} \frac{c_a(N)}{c_b(N)} \geq \frac{1-p}{p} \frac{\lambda}{1-\lambda}$$

Liekwise, in state β, the share of A votes must fall short of the threshold, λ; hence

$$\lim_{N \to \infty} \frac{c_a(N)}{c_b(N)} < \frac{p}{(1-p)} \frac{\lambda}{1-\lambda}$$

Hence, for information to aggregate, the following inequality must be satisfied:

$$\frac{1-p}{p} \frac{\lambda}{1-\lambda} \leq \lim_{N \to \infty} \frac{c_a(N)}{c_b(N)} < \frac{p}{(1-p)} \frac{\lambda}{1-\lambda} \tag{12}$$

We are now in a position to evaluate information aggregation under the unanimity rule.

Proposition 10 *Information does not aggregate under the unanimity rule with private costs.*

Proof. Notice however, that under the unanimity rule, $\lambda = 1$. Hence, equation (**??**) becomes

$$\infty \leq \lim_{N \to \infty} \frac{c_a(N)}{c_b(N)} < \infty$$

which is obviously impossible to satisfy. ■

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