University of Pennsylvania **Economics 705, Fall 2014**

Prelim Examination

Friday August 8, 2014 Time limit: 150 minutes

Instructions:

(i) The exam consists of two parts. The total number of points for each part is 50. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

Part I

Question 1 (13 Points): In many areas of economics we use sample averaging to approximate expectations. The samples are often generated on the computer using random number generators. Consider the following problem:

Suppose that

$$\left[\begin{array}{c} X \\ Y \end{array}\right] \sim N\left(\left[\begin{array}{c} \mu_X \\ \mu_Y \end{array}\right], \left[\begin{array}{cc} \sigma_{XX} & \sigma_{XY} \\ \sigma_{YX} & \sigma_{YY} \end{array}\right]\right). \tag{1}$$

Denote the joint density of (Y, X) by $p(x, y)$. To approximate the integral

$$\mathbb{E}[h(X,Y)] = \int\int h(x,y)p(x,y)dxdy \tag{2}$$

we could use a Monte Carlo average of the form:

$$\widehat{\mathbb{E}}[h(X,Y)] = \frac{1}{N}\sum_{i=1}^{N} h(X_i, Y_i), \tag{3}$$

where the pairs (X_i, y_i), $i = 1, \ldots, N$ are independently and identically distributed according to (1). You may assume that the function $h(x, y)$ is bounded.

(i) (2 Points) Is $\widehat{\mathbb{E}}[h(X,Y)]$ an unbiased estimator of $\mathbb{E}[h(X,Y)]$? Explain.

(ii) (3 Points) Derive the limit distribution of $\widehat{\mathbb{E}}[h(X,Y)]$? How exactly would you use the limit distribution to assess the accuracy of the Monte Carlo average? Explain.

(iii) (8 Points) Suppose you can evaluate the integral

$$\int h(x,y)p(x|y)dx = g(y) \tag{4}$$

analytically. Use this ability to construct a better estimator of $\mathbb{E}[h(X,Y)]$, denoted by $\widehat{\mathbb{E}}^*[h(X,Y)]$. Show formally that your proposed estimator $\widehat{\mathbb{E}}^*[h(X,Y)]$ is more precise than the original estimator $\widehat{\mathbb{E}}[h(X,Y)]$ in (3).

Question 2 (24 Points): In golf tournaments, players play several rounds and the final score is the average score across the rounds. Suppose that we model the score of player i in round t as

$$Y_{it} = \lambda_i + U_{it}, \quad U_{it} \sim N(0,1), \tag{5}$$

where λ_i reflects the innate skill of player i and U_{it} are some random shocks that affect the player's score in round t. We assume that the U_{it}'s are independent across i and t. There are N players in the tournament, and they play $T = 2$ rounds.

(i) (2 Points) Let's focus on player i. Suppose you observe her/his score in round $t = 1$, Y_{i1}. Derive the maximum likelihood estimator of λ_i and call this estimator $\hat{\lambda}_i^{ml}$. What is your point prediction of the score for round $t = 2$, Y_{i2}?

(ii) (4 Points) Suppose that you learned that in the cross section, the distribution of skill is

$$\lambda_i \sim N(0, \sigma^2) \tag{6}$$

and someone gave you the exact value of σ^2. Use this information to construct an alternative estimator (think Bayes!) of λ_i. Call this estimator $\hat{\lambda}_i^B$.

(iii) (4 Points) In what sense is $\hat{\lambda}_i^B$ a better estimator than $\hat{\lambda}_i^{ml}$? Explain.

(iv) (3 Points) In view of (5) and (6) what is the distribution of Y_{i1} conditional on σ^2, $p(Y_{i1}|\sigma^2)$?

(v) (4 Points) Using the cross-sectional information Y_{i1}, $i = 1, \ldots, N$, from the first round of the tournament, derive a formula for the maximum likelihood estimator of σ^2, denoted by $\hat{\sigma}^2_{ml}$.

(vi) (4 Points) Show that $\hat{\sigma}^2_{ml}$ is a consistent estimator of σ^2 and derive its limit distribution.

(vii) (3 Points) TRUE or FALSE? Because according to the model given by (5) and (6) performance in the tournament is independent across players, information about the performance of players $i = 2, \ldots, N$ in round $t = 1$ does not help predicting the performance of player $i = 1$ in round $t = 2$. Explain carefully.

Question 3: (13 Points) Linear Regression Model

Consider the linear regression model

$$y_i = x_i'\theta + u_i, \quad u_i|x_i \sim iid(0,1), \quad x_i \geq \epsilon > 0, \quad \mathbb{E}[x_i^2] = Q, \quad i = 1, \ldots, n$$

Moreover, the x_i's are also independent across i. Notice that we assumed that the conditional variance of u given x is known to be one.

(i) (3 Points) Derive the likelihood function and the maximum likelihood estimator $\hat{\theta}$ under the assumption that the u_i's are in fact normally distributed.

(ii) (10 Points) TRUE or FALSE? In this framework the Wald test of the null hypothesis $H_0: \ \theta = 0$ is more powerful than the likelihood ratio test.
Note: to answer this question, formally analyze both tests. As part of your analysis, characterize the acceptance and rejection regions for both tests for a type-I error of $\alpha = 0.10$.

Part II

Question 4: (30 Points): Take a linear regression model

$$Y = X\beta + U,$$

where X is $n \times k$ and β is $k \times 1$. Assume the data is i.i.d. and $\mathbb{E}[U_i|X_i] = 0$. Suppose the parameter β is known to satisfy the restrictions

$$\beta = Q\theta,$$

where Q is $k \times m$ and θ is $m \times 1$, $m < k$. The matrix Q is known and it is of full rank. The parameter θ is unknown.

(i) (3 Points) How to identify θ.

(ii) (3 Points) Is there a simple way to estimate θ by least squares? If so, find this estimator $\widehat{\theta}_{LS}$.

(iii) (3 Points) Let $\widehat{\beta}_{LS}$ denote the LS estimator for β. One can also estimate θ from $\widehat{\beta}_{LS}$ using the minimum distance criterion. Specifically, for some symmetric positive definite $k \times k$ matrix W, define

$$C(\theta) = \left(\widehat{\beta}_{LS} - Q\theta\right)' W \left(\widehat{\beta}_{LS} - Q\theta\right)$$

and define $\widehat{\theta}_{MD}$ as the minimizer of $C(\theta)$:

$$\widehat{\theta}_{MD} = \arg\min C(\theta).$$

Find this minimum distance estimator $\widehat{\theta}_{MD}$.

(iv) (3 Points) Is $\widehat{\theta}_{MD}$ consistent for θ?

(v) (3 Points) Find the asymptotic distribution for $\widehat{\theta}_{MD}$.

(vi) (3 Points) Is there a choice of W so that $\widehat{\theta}_{LS} = \widehat{\theta}_{MD}$?

(vii) (3 Points) What is the optimal choice of W for $\widehat{\theta}_{MD}$?

(viii) (3 Points) Suppose $\widehat{\theta}_{MD}^{opt}$ is constructed with the optimal weight matrix. Compare the efficiency of $\widehat{\theta}_{MD}^{opt}$ and $\widehat{\theta}_{LS}$.

(ix) (3 Points) How to test the hypothesis: $H_0 : R\theta = 0$ vs $H_1 : R\theta \neq 0$, where R is a $r \times m$ matrix with rank r and $r < m$. Be specific about the test statistic, the critical value, and the decision rule.

(x) (3 Points) How to construct a confidence interval for the first element of θ, denoted by θ_1?

Question 5: (7 Points) Consider the model

$$\begin{aligned} Y_i^* &= X_i\beta + U_i, \text{ where} \\ \mathbb{E}[X_iU_i] &= 0. \end{aligned}$$

We do not observe the latent variable Y_i^*. Instead we observe an i.i.d. sample of X_i and Y_i, where

$$Y_i = Y_i^* + V_i$$

for some measurement error V_i that satisfies

$$\mathbb{E}[X_iV_i] = 0 \text{ and } \mathbb{E}[U_iV_i] \neq 0.$$

Let $\widehat{\beta}_{LS}$ denote the LS estimator of regressing Y_i on X_i.

(i) (3 Points) Is $\widehat{\beta}_{LS}$ a consistent estimator of β? If not, how to find instruments for X_i?

(ii) (4 Points) Derive the asymptotic distribution of $\widehat{\beta}_{LS}$ if $\mathbb{E}[X_iV_i] = c/\sqrt{n}$ for some constant c.

Question 6 (13 Points): The Tobit model is

$$\begin{aligned} Y_i^* &= X_i'\beta + U_i, \\ U_i &\sim N(0, \sigma^2), \\ Y_i &= Y_i^* 1\left(Y_i^* \geq 0\right), \end{aligned}$$

where $1(\cdot)$ is the indicator function. Suppose X_i and U_i are independent and the data is i.i.d.

(i) (3 Points) Find $\mathbb{E}[Y_i|X_i]$. Recall that if $Z \sim N(0,1)$, $\mathbb{E}(Z|Z > c) = \lambda(-c)$, where $\lambda\left(c\right) = \phi(c)/\Phi(c)$ and Φ, ϕ are the cdf and pdf of a standard normal distribution, respectively.

(ii) (3 Points) Use the result above to suggest a LS estimator for the parameter β.

(iii) (4 Points) Construct a maximum likelihood estimator of β.

(iv) (3 Points) Show the asymptotic distribution of this maximum likelihood estimator and estimate the its standard error.

END OF THE EXAM