1. Bubbly Liquidity

Consider an overlapping generations economy in which one household is born at each period t, $t = 0, 1, 2, \ldots$. The household lives for three periods: t, $t+1$, and $t+2$. We will call the household in her first period of life "young", in her second "middle-age", and in her last "old." The household consumes the only good in the economy when she is old, c_{t+2}^{t}, and orders consumption allocations according to a linear utility function:

$$u\left(c_{t+2}^{t}\right) = c_{t+2}^{t}.$$

Also, at $t = 0$, there are one initial old (-2) and one initial middle-age (-1) households alive.

The household is born with an amount A of the good, a wealth which she can invest when young in three different assets:

1. Lucas trees: the household can rent, for one period, Lucas trees owned by an agent outside the economy. In exchange for a rental fee per-tree p_t^l paid at time t, the household will get the fruit of the tree: 1 unit of the good in time $t+1$, also per-tree. Let us call A_t^l the amount of wealth invested in renting Lucas trees.

2. Securities issued by the middle-age household. The securities s_t with price p_t^s pay 1 unit of good at time $t+1$. Let us call A_t^s the amount of wealth invested in securities.

3. A bubble: a bubble is an asset with price $b_t \geq 0$ that does not yield any dividend and that is purchased only because it will have a positive value $b_{t+1} \geq 0$ in the next period. Let us call A_t^b the amount of wealth invested in the bubble.

Therefore:

$$A = A_t^l + A_t^s + A_t^b.$$

A middle-age household born at time t can invest i_{t+1} units of good in period $t+1$ in a project that delivers $\rho_1 i_{t+1}$ units of good in period $t+2$. To finance i_{t+1}, the household can use the current value of the portfolio invested in period t plus an amount of securities s_{t+1} pledged against the payoffs from the project. Because of financial frictions, the household can only pledge a total amount of $\rho_0 i_{t+1}$ to pay those securities. We assume that $\rho_1 > \rho_0 > 0$.

When old, the household receives the return of the project $\rho_1 i_{t+1}$, pays off s_{t+1}, and consumes the remanent c_{t+2}^{t}.

The interest rate between period t and period $t+1$ is $1+r_{t+1}$. While $1+r_{t+1}$ is endogenous, you can assume that in all equilibria, $\rho_1 > 1 + r_{t+1} > \rho_0$ for all t.

There is a total of $l > 0$ Lucas trees in the economy. The initial old invested at scale i_{-1} and issued s_{-1} securities. The initial middle-age rented l Lucas trees and owns b_0 and s_{-1}. All agents behave competitively with respect to prices and all markets clear.

1. Show that a middle-age household will invest in the production technology the maximum amount she can finance. Once you have shown this, you can use the result for the next questions.

2. Find p_t^l, p_t^s, and the relation between b_{t+1} and b_t, all as a function of r_{t+1}.

3. Use the results in 2. to describe the non-arbitrage condition existing between the three assets that a young household can invest in. Once you have shown this, you can use the result for the next questions.

4. Define a sequential markets equilibrium for this economy.

5. Characterize the sequential markets equilibrium. In particular you need to:

    1. From the problem of the young household, find an expression for i_t that depends on A, l, b_t, and r_{t+1} (hint: think about this equation as an asset demand equation).

    2. From the problem of the middle-age household, find an expression for i_t that depends on i_{t-1}, l, b_t, and r_{t+1} (hint: think about this equation as an asset supply equation).

    3. Use the non-arbitrage condition for the bubble found in 2.

    4. Argue that the asset demand equation, the asset supply equation, and the non-arbitrage condition for the bubble fully describe the dynamic behavior of the economy.

6. Assume that we are in an equilibrium where the bubble asset is not valued: $b_0 = 0$. Find the steady state of the economy.

7. Bonus question: show that when $b_0 = 0$, the economy converges monotonically to the steady state you just found.