University of Pennsylvania **Economics 705, Spring 2015**

Prelim Examination

Friday June 5, 2015 Time limit: 150 minutes

Instructions:

(i) The exam consists of two parts. The total number of points for each part is 50. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

Part I

Question 1: TRUE or FALSE Questions (10 Points)

You need to explain your answers to get full credit.

(i) (3 Points) TRUE or FALSE? $\mathbb{E}[X] = 0$ implies that $\mathbb{E}[X|Y = y] = 0$ for each y.

(ii) (2 Points) TRUE or FALSE? Suppose that $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$, then

$$\left[\begin{array}{c} X \\ Y \end{array}\right] \sim N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{array}\right]\right).$$

(iii) (3 Points) Consider the regression $y_i = \theta x_i + u_i$, where x_i is scalar. Suppose that $u_i|x_i \sim N(0, \sigma^2)$ and (u_i, x_i) are independently and identically distributed across i.

TRUE or FALSE? The larger the variance of x_i, the more precise is the OLS estimator.

(iv) (2 Points) Suppose that U is a random variable that is uniformly distributed with on the interval $[0, 1]$. Let $F(\cdot)$ be a cumulative density function and $F^{-1}(\cdot)$ its inverse. TRUE or FALSE? The transformed random variable $X = F^{-1}(U)$ has distribution $F(\cdot)$.

Question 2: Bayesian vs. Frequentist Inference (16 Points)

(i) (8 Points) Suppose it is known that 1 in 1,000 individuals has a particular illness, called $\mathcal{I}$. Moreover, suppose that there is a test procedure that returns a "negative" with probability 99% if a patient does not have disease $\mathcal{I}$ and a "positive" with probability 99% if a patient does have the disease $\mathcal{I}$. Now consider a physician who administers the test on a patient. Suppose the test indicates "positive." What would a Bayesian physician infer from the outcome of the test? What would a frequentist physician infer from the outcome of the test?

(ii) (8 Points) Suppose that $X \sim N(\theta, 1)$ and we are interested in testing the hypothesis $H_0 : \ \theta \geq 0$ versus the alternative $H_1 : \ \theta \leq 0$. Assume that the prior distribution for θ is uniform on the interval $[-M, M]$, where M is a very large number. Moreover, assume the observed value of X is -1.5. What does a Bayesian conclude about the null hypothesis? What does a frequentist conclude about the null hypothesis?

Problem 3: Linear Regression Model (24 Points)

Consider the linear regression model

$$y_i = x_i'\theta + u_i, \quad u_i|x_i \sim iidN(0,1), \quad x_i \sim iidN(0,Q). \tag{1}$$

Notice that we assumed that the conditional variance of u given x is known to be one. Suppose that $k = 2$ and we can write $x_i = [x_{1,i}, x_{2,i}]'$ and $\theta = [\theta_1, \theta_2]'$. Our goal is to test the null hypothesis $H_0: \ \theta_2 = 0$.

(i) (4 Points) Provide a formula for the joint density of $(y_1, x_1, y_2, x_2, \ldots, y_n, x_n)$ and then factorize the joint density into a part that depends on θ and a part that does not depend on θ.

(ii) (3 Points) Derive the constrained MLE of θ_1, imposing that $\theta_2 = 0$:

$$\bar{\theta}_1 = \text{argmax}_{\theta_1 \in \mathbb{R}} \, l\big([\theta_1, 0]'|Y\big),$$

where $l(\theta|Y)$ is the log likelihood function.

(iii) (4 Points) What is the probability limit of $\bar{\theta}_1$ if $\theta_2 \neq 0$? Under what condition (if any) does $\bar{\theta}_1$ remain consistent.

(iv) (3 Points) Provide an explicit formula for the likelihood ratio statistic for the null hypothesis that $\theta_2 = 0$.

(v) (6 Points) Under the null hypothesis that $\theta_2 = 0$, derive an expression for the likelihood ratio statistic of the form

$$LR = U'\big(??\big)U,$$

where ?? does not depend on U. Hint: you might want to utilize the residual projection matrices $M = I - X(X'^{-1}X$, M_1 (defined below), and M_2 (defined based on X_2 instead of X_1).

(vi) (4 Points) Derive the limit distribution for the likelihood ratio statistic under the null hypothesis:

$$LR \Longrightarrow ???$$

and describe how the test is implemented in an application.

Note: The Frisch-Waugh-Lovell theorem implies that the residuals from the regressions

$$Y = X_1\theta_1 + X_2\theta_2 + residuals$$

and

$$M_1 Y = M_1 X_2 \theta_2 + residuals$$

are identical. Here $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$.

Part II

Question 4: Least Absolute Deviations (20 Points)

Consider the model

$$y_i = x_i\theta_0 + u_i,$$

where conditional on x_i, u_i has a median at 0, e.g.,

$$med(u_i|x_i) = 0.$$

- The observed data $\{(x_i, y_i) \in R^2 : i = 1, ..., n\}$ are i.i.d. For simplicity, suppose any finite moments of y_i, x_i, u_i are all bounded and the parameter space Θ is compact.
- In the answers below, you can make use of the following fact: $E|y_i - \beta|$ is minimized when β is the median of y_i.
- The least absolute deviation estimator $\widehat{\theta}$ minimizes

$$Q_n(\theta) = n^{-1} \sum_{i=1}^{n} |y_i - x_i\theta| \, .$$

(i) (5 Points) For any given $\theta \in \Theta$, show $Q_n(\theta) \to_p Q(\theta)$ for some non-stochastic function $Q(\theta)$ of θ. If you invoke any theorems, such as LLN or CLT, make sure to verify the regularity conditions.

(ii) (5 Points) Show the convergence in part (i) is uniform over $\theta \in \Theta$.

(iii) (5 Points) Verify that $Q(\theta)$ is minimized at θ_0.

(iv) (5 Points) To show $\widehat{\theta}$ is a consistent estimator of θ_0, what else do you need besides results in parts (ii) and (iii)? Verify these conditions.

Question 5: Linear Instrumental Variable Regression (20 Points)

Consider the model

$$y_i = x_{1,i}\theta_1 + x_{2,i}\theta_2 + u_i,$$

where $x_{1,i}$ is endogenous and $x_{2,i}$ is exogenous, i.e.,

$$E(x_{1,i}u_i) \neq 0 \text{ and } E(x_{2,i}u_i) = 0.$$

Equivalently, the model can be written as $y_i = x_i'\theta + u_i$, where $x_i = (x_{1,i}, x_{2,i})' \in R^2$ and $\theta = (\theta_1, \theta_2)' \in R^2$.

You have $k \geq 1$ instruments $z_i \in R^k$. They are uncorrelated with u_i and are correlated with $x_{1,i}$ and $x_{2,i}$.

(i) (5 Points) Propose an estimator of $\theta = (\theta_1, \theta_2)'$ and show its consistency.

(ii) (5 Points) Derive the asymptotic distribution of the estimator in part (i).

(iii) (5 Points) Propose a method to test the hypothesis $H_0 : \theta_1\theta_2 = 1$ vs $H_0 : \theta_1\theta_2 \neq 1$. Be clear with the test statistic and justify the critical value.

(iv) (5 Points) Suppose the researcher misspecifies the regression model as

$$y_i = x_{1,i}\theta_1 + u_i,$$

although $\theta_2 \neq 0$ in the true data generating process. Derive the limit of the two stage least squares (TSLS) estimator under this model misspecification. Can you think of a scenario where the TSLS is consistent despite of this misspecification?

Question 6: Efficient Minimum Distance Estimator (10 Points)

Suppose you have two independent samples that satisfy

$$y_{1i} = x_{1i}\beta_1 + u_{1i}$$

and

$$y_{2i} = x_{2i}\beta_2 + u_{2i}.$$

Both sample sizes are n and both x_{1i} and x_{2i} are scalars. You estimate β_1 and β_2 by OLS and obtain consistent estimators $\widehat{\beta}_1$ and $\widehat{\beta}_2$ with asymptotic variance estimators $\widehat{V}_{\beta_1}$ and $\widehat{V}_{\beta_2}$.

Under the additional restriction $\beta_1 = \beta_2$, consider an efficient minimum distance estimator of $\beta = \beta_1 = \beta_2$ based on $\widehat{\beta}_1$ and $\widehat{\beta}_2$.

(i) (5 Points) Find this efficient minimum distance estimator $\widetilde{\beta}$ of $\beta = \beta_1 = \beta_2$.

(ii) (5 Points) Derive the asymptotic distribution of $\widetilde{\beta}$.