

Structural Estimation and Counterfactual Experiments in Games when the Data Come from Multiple Equilibria

Victor Aguirregabiria*
University of Toronto

Pedro Mira*
CEMFI, Madrid

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Abstract

This paper deals with identification, estimation, and statistical inference in discrete games of incomplete information when the data come from multiple equilibria. Most recent work in the econometrics of games has advocated two-step estimation methods which rely on the assumption that, even if the model has multiple equilibria, a single equilibrium is played in the data. More precisely, the assumption is that the data can be partitioned into a small number of subsamples, according to the value of a discrete exogenous variable, such that the same equilibrium is played within each subsample. We relax this assumption and study the identification and estimation of games when we allow for two types of unobserved heterogeneity from the point of view of the researcher: heterogeneity that is payoff relevant, and heterogeneity non-payoff relevant but that affects the equilibrium selection, what we call, "sunspots". First, we show that the parameters in the payoff function are identified under the same type of exclusion restrictions that we need to identify the model without sunspots. The separate identification of the distributions of payoff relevant unobservables and sunspots requires stronger assumptions. Second, we investigate how the estimated model, including the estimated equilibrium selection mechanism, can be used to predict the equilibrium outcome associated with a counterfactual change in structural parameters. And third, we propose an estimation procedure that extends the NPL method in Aguirregabiria and Mira (2002, 2007) to deal with multiple equilibria in the data.

Keywords: Discrete games of incomplete information; Multiple equilibria in the data; Pseudo maximum likelihood estimation; Genetic algorithms.

JEL codes: C13, C35.

Corresponding Author: Victor Aguirregabiria. Address: 150 St. George Street. Toronto, ON, M5S 3G7. Phone: (416) 978-4358. E-mail: victor.aguirregabiria@utoronto.ca

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1 Introduction

Multiplicity of equilibria is a prevalent feature in games. In the econometric analysis of games, it is relevant to distinguish two different issues related to multiple equilibria. A first issue is that, for a given value of the structural parameters, the model implies multiple predictions on the probability distribution or moment conditions of the endogenous variables. The mapping that maps structural parameters into data is no longer a function but a correspondence. A second difficulty comes from the possibility that the observations in the data come from more than one equilibrium associated to the same value of the structural parameters. Most of the recent research on the econometrics of games of incomplete information has dealt with the problem but assuming out the possibility of the second problem. More precisely, the assumption is that the data can be partitioned into a small number of subsamples, according to the value of a discrete exogenous variable, such that the same equilibrium is played within each subsample. Given this assumption, structural parameters in these models are identified under exactly the same conditions (exclusion restrictions) as those in games with equilibrium uniqueness (see Aguirregabiria and Mira, 2002a, Pesendorfer and Schmidt-Dengler, 2003, and Bajari et al., 2010). Papers in this literature also propose simple two-step estimators that simplify significantly the estimation of games with multiple equilibria (see Aguirregabiria and Mira, 2007, and Bajari, Benkard and Levin, 2007, among others).¹

However, much less is known about the identification and estimation of games when the data come from more than one equilibrium. Authors in different areas of economics consider that multiplicity of equilibria can be a necessary feature of models to explain important aspects of economic data. This type of arguments have been used to explain macroeconomic fluctuations (Farmer and Guo, 1995), regional variation in the density of economic activity (Krugman, 1991, and Bayer and Timmins, 2005, 2007) or local market variation in firms' strategic behavior (Sweeting, 2008, and Ellickson and Misra, 2008). Furthermore, contrary to the view that models with multiple equilibria cannot be falsified, some studies have found that multiple equilibria in the data cannot explain empirical evidence such as the decline in wage inequality in the US during the last three decades (Moro, 2003), or the variation of industrial composition across 114 Japanese cities (Davis and Weinstein, 2004).

In this paper, we study the identification and estimation of games when we allow for two types of unobserved heterogeneity from the point of view of the researcher: heterogeneity that is payoff

¹Other recent contributions to this topic in the context of games of incomplete information are Berry and Tamer (2006); Sweeting (2008); and Bajari, Hahn, Hong and Ridder (2008).

relevant, and heterogeneity non-payoff relevant but that affects the equilibrium selection, what we call, "sunspots". First, we show that the parameters in the payoff function are identified under the same type of exclusion restrictions that we need to identify the model without sunspots. The separate identification of the distributions of payoff relevant unobservables and sunspots requires stronger assumptions. Second, we investigate how the estimated model, including the estimated equilibrium selection mechanism, can be used to predict the equilibrium outcome associated with a counterfactual change in structural parameters. And third, we propose an estimation procedure that extends the NPL method in Aguirregabiria and Mira (2002, 2007) to deal with multiple equilibria in the data.

An important computational issue associated the estimation of models with multiple equilibria in the data is that one should maximize the sample criterion function not only with respect to the structural parameters but also with respect to the *equilibrium types* that generate the observations in the data. If no structure is imposed on equilibrium selection, the number of possible combinations of equilibria in the data increases exponentially with sample size and becomes huge even for the simplest problems. We propose a hybrid algorithm which combines the NPL procedure in Aguirregabiria and Mira (2002, 2007) with a genetic algorithm. We show that this hybrid algorithm can be used to recover the equilibrium selection mechanism non-parametrically, as well as the structural parameters. The NPL algorithm may have more than one fixed point and in that case the researcher needs to select the fixed point which yields the highest value of the pseudo likelihood in order to guarantee consistency of the estimator. We show that multiplicity of NPL fixed points is more likely to occur in the class of applications we consider in this paper; i.e., if multiple equilibria can be present in the data and the researcher imposes little structure on equilibrium selection. This motivates the combination of NPL with an stochastic global search procedure such as a genetic algorithm. The use of NPL avoids repeated computation of equilibria for each trial value of the structural parameters. The genetic algorithm searches efficiently over the huge space of possible combinations of equilibria in the data and deals with potential multiplicity of NPL fixed points.² We test the ability of this procedure to obtain consistent estimates using a Monte Carlo experiment in the context of the game of price competition and collusion.

²GAs were first proposed by Holland (1975). For an excellent survey on the theory and application of GAs see Mitchell (1996). Among their many applications, GAs have been successfully used to search for the global optimum of discrete and step functions with very large search spaces (see chapter 4 in Mitchell, 1996, Mitchell, Holland and Forrest, 1994, and section 8.3 in Judd, 1998). The problem of maximum pseudo likelihood estimation of models with multiple equilibria has similar features. Although GAs have been extensively used in experimental and evolutionary economics, the application of GAs in econometrics has been rare. Some important exceptions are Dorsey and Mayer (1995) and Beenstock and Szpirob (2002).

The rest of the paper is organized as follows. Section 2 introduces the class of models and illustrates it with a game of collusion. Section 3 presents our identification results. The estimation problem and the algorithm are described in section 4. Section 5 describes an approach to implement counterfactual experiment using the estimated model. Section 6 presents our Monte Carlo experiment. We summarize and conclude in section 7.

2 Model

2.1 Basic framework and assumptions

Consider a game that is played by N players which are indexed by $i \in I = \{1, 2, \dots, N\}$. Each player has to choose an action from a discrete set of alternatives $A = \{0, 1, \dots, J\}$. The decision of player i is represented by the variable $a_i \in A$. The utility or payoff function of player i is $\Pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}, \boldsymbol{\varepsilon}_i)$. The vector $\mathbf{a}_{-i} \in A^{N-1}$ contains the decisions of players other than i . The vector $\mathbf{x} \in X$ contains exogenous characteristics that describe players and the environment in which the game is played. The variables in \mathbf{x} affect players' utilities and they are common knowledge for all players. The vector $\boldsymbol{\varepsilon}_i$ represents characteristics that are private information of player i .

ASSUMPTION 1: For any player $i \in I$, the vector $\boldsymbol{\varepsilon}_i$ is: (A) a vector of q real valued random variables, $\boldsymbol{\varepsilon}_i \in \mathbb{R}^q$; (B) independent of common knowledge variables \mathbf{x} ; and (C) independently distributed across players with a distribution function $G_i(\cdot)$ that is absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^q .

We assume that a player's strategy is a function of payoff-relevant variables which are known by the player: i.e., a function of $(\mathbf{x}, \boldsymbol{\varepsilon}_i)$. Let $\boldsymbol{\sigma} = \{\sigma_i(\mathbf{x}, \boldsymbol{\varepsilon}_i) : i \in I\}$ be a set of strategy functions where $\sigma_i : X \times \mathbb{R}^q \rightarrow A$. Associated with a set of strategy functions we can define a vector of *choice probabilities* $\mathbf{P}^\boldsymbol{\sigma}(\mathbf{x}) \equiv \{P_i^\boldsymbol{\sigma}(a_i|\mathbf{x}) : (a_i, i) \in A \times I\}$ such that:

$$P_i^\boldsymbol{\sigma}(a_i|\mathbf{x}) \equiv \int I\{\sigma_i(\mathbf{x}, \boldsymbol{\varepsilon}_i) = a_i\} dG_i(\boldsymbol{\varepsilon}_i) \quad (1)$$

where $I\{\cdot\}$ is the indicator function. These probabilities represent the expected behavior of player i from the point of view of the other players (who do not know $\boldsymbol{\varepsilon}_i$) when he follows his strategy in $\boldsymbol{\sigma}$. Note that, by Assumption 1(C), players' actions are independent once we condition on \mathbf{x} . That is, $\Pr(a_1, a_2, \dots, a_N|\mathbf{x}, \boldsymbol{\sigma}) = \prod_{i=1}^N P_i^\boldsymbol{\sigma}(a_i|\mathbf{x})$.

Given beliefs about the behavior of other players, each player maximizes his expected utility. Let $\pi_i^\boldsymbol{\sigma}(a_i, \mathbf{x}, \boldsymbol{\varepsilon}_i)$ be player i 's expected utility if he chooses alternative a_i (not necessarily an optimal choice) and the other players behave according to their respective strategies in $\boldsymbol{\sigma}$. By the

independence of private information in Assumption 1, we have that:

$$\pi_i^\sigma(a_i, \mathbf{x}, \boldsymbol{\varepsilon}_i) = \sum_{\mathbf{a}_{-i} \in A^{N-1}} \left(\prod_{j \neq i} P_j^\sigma(a_j | x) \right) \Pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}, \boldsymbol{\varepsilon}_i) \quad (2)$$

DEFINITION: A Bayesian Nash equilibrium (BNE) in this game is a set of strategy functions $\boldsymbol{\sigma}^*$ such that for any player i and for any $(\mathbf{x}, \boldsymbol{\varepsilon}_i) \in X \times \mathbb{R}^q$,

$$\sigma_i^*(\mathbf{x}, \boldsymbol{\varepsilon}_i) = \arg \max_{a_i \in A} \pi_i^{\boldsymbol{\sigma}^*}(a_i, \mathbf{x}, \boldsymbol{\varepsilon}_i) \quad (3)$$

We can represent this BNE in the space of players' choice probabilities. This representation is convenient for the econometric analysis of this class of models. Let $\boldsymbol{\sigma}^*$ be a set of BNE strategies, and let $\mathbf{P}^*(\mathbf{x})$ be the vector of choice probabilities associated with these strategies. By definition, $P_i^*(a_i | \mathbf{x}) = \int I \{ \sigma_i^*(\mathbf{x}, \boldsymbol{\varepsilon}_i) = a_i \} dG_i(\boldsymbol{\varepsilon}_i)$. Solving the equilibrium condition (3) in this expression we get that for any $(a_i, i) \in A \times I$:

$$P_i^*(a_i | \mathbf{x}) = \int I \left\{ a_i = \arg \max_{a \in A} \pi_i^{\boldsymbol{\sigma}^*}(a, \mathbf{x}, \boldsymbol{\varepsilon}_i) \right\} dG_i(\boldsymbol{\varepsilon}_i) \quad (4)$$

As shown in equation (2), the function π_i^σ depends on other players' strategies only through their choice probabilities associated with σ . Therefore, the right hand side in equation (4) defines a function $\Psi_i(a_i | \mathbf{x}, \mathbf{P}_{-i}^*(\mathbf{x}))$, where $\mathbf{P}_{-i}^*(\mathbf{x}) \equiv \{ P_j^\sigma(a_j | \mathbf{x}) : (a_j, j) \in A \times I, j \neq i \}$. This function can be evaluated at any set of choice probabilities $\mathbf{P}_{-i}(\mathbf{x})$, not just equilibrium probabilities. For arbitrary $\mathbf{P}_{-i}(\mathbf{x})$, we have that the function $\Psi_i(a_i | \mathbf{x}, \mathbf{P}_{-i}(\mathbf{x}))$ is defined as:

$$\Psi_i(a_i | \mathbf{x}, \mathbf{P}_{-i}(\mathbf{x})) \equiv \int I \left\{ a_i = \arg \max_{a \in A} \left(\sum_{a_{-i}} \left(\prod_{j \neq i} P_j(a_j | x) \right) \Pi_i(a, \mathbf{a}_{-i}, \mathbf{x}, \boldsymbol{\varepsilon}_i) \right) \right\} dG_i(\boldsymbol{\varepsilon}_i) \quad (5)$$

We call the functions Ψ_i *best response probability functions* because they provide the probability that an action is optimal for player i given that he believes that his opponents behave according to the probabilities in $\mathbf{P}_{-i}(\mathbf{x})$. Therefore, the vector of equilibrium probabilities $\mathbf{P}^*(\mathbf{x})$ is such that every player's choice probabilities are best responses to the other players' probabilities. That is, $\mathbf{P}^*(\mathbf{x})$ is a fixed point of the best response mapping $\boldsymbol{\Psi}(\mathbf{x}, \mathbf{P}) \equiv \{ \Psi_i(a_i | \mathbf{x}, \mathbf{P}_{-i}(\mathbf{x})) : (a_i, i) \in A \times I \}$:

$$\mathbf{P}^*(\mathbf{x}) = \boldsymbol{\Psi}(\mathbf{x}, \mathbf{P}^*(\mathbf{x})) \quad (6)$$

ASSUMPTION 2: The payoff functions $\{ \Pi_i : i \in I \}$ and the probability distributions $\{ G_i : i \in I \}$ are such that the best response probability mapping $\boldsymbol{\Psi}(\mathbf{x}, \mathbf{P})$ has the following properties: (A) it is continuously differentiable in \mathbf{P} ; and (B) for any (i, a_i, \mathbf{x}) and for any $\mathbf{P}(\mathbf{x})$, the probabilities $\Psi_i(a_i | \mathbf{x}, \mathbf{P}(\mathbf{x}))$ are bounded away from zero and one.

Assumption 2(A) implies that, by Brower's fixed point theorem, the mapping $\Psi(\mathbf{x}, \cdot)$ has at least one equilibrium. That equilibrium may not be unique. Assumption 2(B) is needed for the uniform convergence of the log-likelihood function and for consistency and asymptotic normality of the maximum likelihood estimator. A sufficient condition for Assumption 2 to hold is that $\Pi_i = \bar{\Pi}_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) + \varepsilon_i(a_i)$ (i.e., additive separability of ε_i) and $\varepsilon_i = (\varepsilon_i(0), \varepsilon_i(1), \dots, \varepsilon_i(J))$ has a continuous distribution in the Euclidean space \mathbb{R}^{J+1} . This is a very standard specification in econometric discrete choice models.

ASSUMPTION 3: The primitives of the model $\{\Pi_i, G_i : i \in I\}$ are known to the researcher up to a finite vector of parameters $\theta \in \Theta \subseteq \mathbb{R}^K$. Primitives are continuously differentiable in θ .

We use $\Psi(\mathbf{x}, \theta, \mathbf{P})$ to denote the best response mapping associated with (\mathbf{x}, θ) , and $\mathbf{P}^*(\mathbf{x}, \theta) \equiv \{P_i^*(a|\mathbf{x}, \theta) : (a, i) \in A \times I\}$ to represent an equilibrium associated with (\mathbf{x}, θ) such that $\mathbf{P}^*(\mathbf{x}, \theta) = \Psi(\mathbf{x}, \theta, \mathbf{P}^*(\mathbf{x}, \theta))$. For some values of (\mathbf{x}, θ) the model has multiple equilibria. The set of BNE associated with (\mathbf{x}, θ) is:

$$\Gamma(\mathbf{x}, \theta) \equiv \{ \mathbf{P} : \mathbf{P} = \Psi(\mathbf{x}, \theta, \mathbf{P}) \} \quad (7)$$

We assume that $\Gamma(\mathbf{x}, \theta)$ is a finite set, and use the variable $\tau \in \{1, 2, \dots\}$ to index the equilibria in this set.

2.2 A model of collusion with asymmetric information

In the market of a differentiated product, two firms produce their respective varieties of the product. We call them firm 1 and firm 2. Following Dixit (1979) and Singh and Vives (1984) we consider a linear demand system. The demand for firm i is $q_i = v - \alpha(p_i - p_j) - p_i$, where: q_i represents the quantity sold by firm i ; p_i is the price of firm i ; the variable $v > 0$ is the size of the market; and the parameter $\alpha \geq 0$ represents the degree of substitutability between the two varieties of the product. Production costs are linear and the same for the two firms. The profit of firm i is $\pi_i = (p_i - c)q_i$, where c is the marginal production cost.

If these firms compete in prices 'a la Nash-Bertrand', it is straightforward to obtain that equilibrium prices are $p_1^{BB} = p_2^{BB} = c + (v - c)/(2 + \alpha)$, and equilibrium profits of both firms are $\pi_{BB} \equiv (1 - \delta) \left(\frac{v-c}{2}\right)^2$, where $\delta \in (0, 1)$ is a parameter equal to $\alpha^2/(2 + \alpha)^2$. Suppose that these firms make a collusive agreement. They decide to choose the same price p to maximize joint profits $\pi_1 + \pi_2 = (p - c)Q$, where $Q = q_1 + q_2$. Collusive profits are evenly split between the two firms. It is simple to verify that the optimal price under collusion is $p^{CC} = (v + c)/2$, and profits of an individual firm are $\pi_{CC} \equiv \left(\frac{v-c}{2}\right)^2$. Note that $\pi_{BB} = (1 - \delta)\pi_{CC}$ where $\delta \in (0, 1)$, i.e., for any

$\alpha > 0$ profits under collusion are larger than profits under Bertrand competition. Bertrand profits are a decreasing function of the parameter α that measures the degree of product substitutability. Now, suppose that there is not an explicit collusive agreement between these firms. Instead, each firm decides independently whether to charge the Bertrand price or the collusive price. If firm i chooses the Bertrand price and firm j chooses the collusive price, then it is simple to show that their profits are $\pi_{BC} \equiv (1 + \delta) \pi_{CC}$ for firm i , and $\pi_{CB} \equiv (1 - 2(1 + \alpha) \delta) \pi_{CC}$ for firm j . For any $\alpha > 0$ the following inequalities hold: $\pi_{BC} > \pi_{CC}$ and $\pi_{BB} > \pi_{CB}$. Therefore, choosing the Bertrand price is a dominant strategy, and the Bertrand outcome is the only Nash equilibrium of this binary choice game.

Consider that we modify this game to incorporate private information.³ Let $a_i \in \{0, 1\}$ be the indicator of the event "firm i chooses the collusive price at period t ". Suppose that firms have inventory holding costs. Let $\varepsilon_i(0)$ and $\varepsilon_i(1)$ be firm i 's inventory costs with the Bertrand price and with the collusive price, respectively. These variables are private information of firm i and they are independently distributed over firms. A firm's (expected) demand if charging the Bertrand price is larger than with the collusive price. Therefore, on average, a firm needs more inventories when it charges the Bertrand price. We assume that the random variable $\varepsilon_i(0) - \varepsilon_i(1)$ is normally distributed with mean μ and variance σ_ε^2 . Both μ and σ_ε^2 are common knowledge parameters. and μ is greater than zero because, on average, inventory costs are higher when charging the Bertrand price. The profit function of firm i is:

$$\Pi_i = \begin{cases} a_j \pi_{CC} + (1 - a_j) \pi_{CB} - \varepsilon_i(1) & \text{if } a_i = 1 \\ a_j \pi_{BC} + (1 - a_j) \pi_{BB} - \varepsilon_i(0) & \text{if } a_i = 0 \end{cases} \quad (8)$$

Let P_j represents firm i 's belief on the probability that firm $j \neq i$ chooses the collusive price. Then, the expected profit of firm i is: $\pi_i^P = P_j \pi_{CC} + (1 - P_j) \pi_{CB} - \varepsilon_i(1)$ if $a_i = 1$; and $\pi_i^P = P_j \pi_{BC} + (1 - P_j) \pi_{BB} - \varepsilon_i(0)$ if $a_i = 0$. Given these expected profits, the best response function for firm i is:

$$\{a_i = 1\} \Leftrightarrow \tilde{\varepsilon}_i \leq \{\tilde{\pi}_0 + \tilde{\pi}_1 P_j\} \quad (9)$$

where $\tilde{\varepsilon}_i \equiv (\varepsilon_i(1) - \varepsilon_i(0) + \mu) / \sigma_\varepsilon$, which is a standard normal random variable; $\tilde{\pi}_0 \equiv (\mu + \pi_{CB} - \pi_{BB}) / \sigma_\varepsilon$; and $\tilde{\pi}_1 \equiv (\pi_{CC} - \pi_{BC} - \pi_{CB} + \pi_{BB}) / \sigma_\varepsilon$. Following our description of a BNE in section 2.1, it follows that the equilibrium probability mapping of this game is:

$$\Psi(\mathbf{P}) = \begin{pmatrix} \Psi_1(P_2) \\ \Psi_2(P_1) \end{pmatrix} = \begin{pmatrix} \Phi(\tilde{\pi}_0 + \tilde{\pi}_1 P_2) \\ \Phi(\tilde{\pi}_0 + \tilde{\pi}_1 P_1) \end{pmatrix} \quad (10)$$

³The model here is in the spirit of previous work that shows how a collusive outcome can be a Nash equilibrium in a model with asymmetric information. See Cramton and Palfrey (1990), and Laffont and Martimort (1997).

where $\Phi(\cdot)$ is the CDF of the standard normal. Note that the model is symmetric and players have identical best response functions; i.e., $\Psi_1(P) = \Psi_2(P) = \Phi(\tilde{\pi}_0 + \tilde{\pi}_1 P)$. In our numerical examples using this model, we concentrate on symmetric equilibria. It is simple to verify that $\tilde{\pi}_1 \equiv (\pi_{CC} - \pi_{BC} - \pi_{CB} + \pi_{BB})/\sigma_\varepsilon = 2\alpha\delta/\sigma_\varepsilon > 0$. Therefore, there is strategic complementarity between the pricing decisions of the firms. In the complete information version of the model, the unique Nash equilibrium is the non-collusive outcome. In contrast, with asymmetric information the model can have multiple equilibria. Furthermore, in one of the equilibria the probability of charging a collusive price can be high. We illustrate this case in figure 1. The figure represents the best response mapping $\Phi(\tilde{\pi}_0 + \tilde{\pi}_1 P)$ with $\tilde{\pi}_0 = -1.80$ and $\tilde{\pi}_1 = 3.55$. These values of $\tilde{\pi}_0$ and $\tilde{\pi}_1$ correspond to the following values of the "structural" parameters: $v - c = 2$, $\sigma_\varepsilon = 1$, $\mu = 2.2$, and $\alpha = 4.0$. For these values of the parameters, the model has three symmetric equilibria: a "non-collusive" equilibrium, with $P^* = 0.054$; a "collusive" equilibrium, with $P^* = 0.937$; and a "mixed" equilibrium with $P^* = 0.521$. Both the non-collusive and the collusive equilibria are stable, but the "mixed" equilibrium is unstable.⁴

To complete the description of the model, we consider that the common knowledge primitives of the model $\{v, c, \alpha, \mu, \sigma_\varepsilon\}$, and therefore $\tilde{\pi}_0$ and $\tilde{\pi}_1$, can be described in terms of two vectors, \mathbf{x} and $\boldsymbol{\theta}$. As explained in section 2.1, the difference between these two vectors is that the variables in \mathbf{x} are observable to the researcher, and $\boldsymbol{\theta}$ is a vector of unknown parameters for the researcher. In this model of collusion, \mathbf{x} may contain socioeconomic variables that affect demand or costs, such as market population, average income, input prices, etc.

3 Identification

3.1 Data and data generating process

Suppose that the researcher observes T different realizations of the game; e.g., T different markets, or T periods of time. We index by t each realization of the game. For the sake of concreteness in our discussion, we consider that these multiple realizations of the game represent the same firms playing the game at T different local markets. For every market t , the researcher observes the

⁴In this model, neither incomplete information (i.e., $\sigma_\varepsilon > 0$) nor the difference in inventory costs (i.e., $\mu > 0$) alone are sufficient to generate a collusive equilibrium. With $\mu = 0$ the model has always a unique BNE equilibrium, regardless the values of $v - c$, α and σ_ε . In all these equilibria the probability of charging the collusive price is close to zero. With complete information (i.e., $\sigma_\varepsilon = 0$), the Bertrand outcome is still the only Nash equilibrium unless μ is very high. It is the combination of incomplete information and inventory holding costs ($\mu > 0$) what generates multiple equilibria with one of them being a "collusive" equilibrium.

vector \mathbf{x}_t and players' actions.

$$Data = \{ a_{1t}, a_{2t}, \dots, a_{Nt}, \mathbf{x}_t : t = 1, 2, \dots, T \} \quad (11)$$

We consider the case where the number of players N is small (e.g., $N = 2$), and the number of markets is relatively large (e.g., $T = 300$) - which is the most common sampling framework in IO applications.

Now, we describe a data generating process (DGP) that allows for different equilibria to be played in markets with similar, and even identical, values of \mathbf{x}_t . Let $\boldsymbol{\theta}^0 \in \Theta$ be the true value of the vector of structural parameters $\boldsymbol{\theta}$ in the population under study. We are interested in the estimation of $\boldsymbol{\theta}^0$. Let $P_{it}^0(a|\mathbf{x}_t)$ be the conditional choice probability (CCP) $\Pr(a_{it} = a|i, t, \mathbf{x}_t)$ in the population under study, and define the vector $\mathbf{P}_t^0(\mathbf{x}_t) \equiv \{P_{it}^0(a|\mathbf{x}_t) : (a, i) \in A \times I\}$ that contains the population CCPs of every player in market t . We assume that $\mathbf{P}_t^0(\mathbf{x}_t)$ is an equilibrium of the mapping $\boldsymbol{\Psi}(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P})$. However, it is possible that, for some values \mathbf{x}_t in the sample, the equilibrium mapping $\boldsymbol{\Psi}(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P})$ has multiple equilibria. Because the structural model is incomplete, to describe the DGP we have to introduce the concept of *equilibrium selection mechanism*, i.e., a function that describes how the equilibrium in a market, say t , is selected from the set of equilibria $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)$.

The set of equilibria $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)$ is discrete, and $|\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)|$ represents the number of elements in that set. Let $\tau \in \Upsilon(\mathbf{x}_t, \boldsymbol{\theta}^0) \equiv \{1, 2, \dots, |\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)|\}$ be an integer that indexes the elements in the set $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)$, such that $\mathbf{P}^\tau(\mathbf{x}_t, \boldsymbol{\theta}^0)$ is the τ -th element of the set $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)$.⁵ The random variable $\tau_t \in \Upsilon(\mathbf{x}_t, \boldsymbol{\theta}^0)$ is the *equilibrium selection index* and it represents the equilibrium selected in market t . This variable is unobservable to the researcher. More formally:

ASSUMPTION 4: (A) The population $\mathbf{P}_t^0(\mathbf{x}_t)$ is an equilibrium of the mapping $\boldsymbol{\Psi}(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P})$. That is, $\mathbf{P}_t^0(\mathbf{x}_t) = \mathbf{P}^{\tau_t}(\mathbf{x}_t, \boldsymbol{\theta}^0)$ for $\tau_t \in \Upsilon(\mathbf{x}_t, \boldsymbol{\theta}^0)$. (B) the equilibrium selection index τ_t is a random variable unobservable to the researcher, independent of $\boldsymbol{\varepsilon}_t$ and independently distributed over t with probability distribution $\lambda^0(\tau|x_t)$ on $\Upsilon(x_t, \boldsymbol{\theta}^0)$, where $\lambda^0(\tau|x_t) \equiv \Pr(\tau_t = \tau|x_t)$. (C) Only stable equilibria are selected.

Assumption 4(C), on the stability of equilibria in the population, is not needed for our identification results or for the consistency of the NPL estimator. However, it is a condition for our algorithm to

⁵For instance, in the model of collusion in section 2.2, figure 1 shows a set of equilibria with three elements. We can use $\tau = 1$ to represent the equilibrium with the smallest probability of collusion, $\tau = 2$ for the intermediate probability of collusion, and $\tau = 3$ for the largest.

find the consistent NPL fixed point (see Aguirregabiria and Mira, 2008). An equilibrium $\mathbf{P}_t^0(\mathbf{x}_t)$ is stable if the Jacobian matrix $\partial\Psi(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P}_t^0(\mathbf{x}_t))/\partial\mathbf{P}'$ has all its eigenvalues in the unit circle.

ASSUMPTION 5: Suppose the researcher knows the population probabilities $\mathbf{P}^\tau(\mathbf{x}, \boldsymbol{\theta}^0)$ for every value of x with positive probability and for all τ with $\lambda^0(\tau|x_t) > 0$. Then, the set of equilibrium conditions $\mathbf{P}^\tau(\mathbf{x}) = \Psi(\mathbf{x}, \boldsymbol{\theta}^0, \mathbf{P}^\tau(\mathbf{x}))$ uniquely identify the vector of structural parameters $\boldsymbol{\theta}^0$.

In a model without multiple equilibria in the data, assumption 5 is sufficient for the identification of $\boldsymbol{\theta}^0$. This is because in that class of models the population probabilities $\mathbf{P}_t^0(\mathbf{x}_t)$ are nonparametrically identified under standard regularity conditions. However, as we discuss in section 3.3, that is not necessarily the case in models with multiple equilibria in the data.

3.2 Likelihood and pseudo-likelihood functions

Before we discuss the identification of the model, it is useful to define the conditional (on \mathbf{x}) likelihood function and a pseudo-likelihood function for this model and data. The parameters of interest in these likelihood functions consist of the vector of structural parameters $\boldsymbol{\theta}$ and the vector of probability functions $\boldsymbol{\lambda} \equiv \{\lambda(\cdot|\mathbf{x}_t) : \text{for any } \mathbf{x}_t \text{ in the sample}\}$, where $\lambda(\cdot|\mathbf{x}_t)$ is well-defined probability distribution on the discrete set $\Gamma(\mathbf{x}_t, \boldsymbol{\theta})$. Given assumptions 1-4, the conditional log-likelihood function is:

$$l_T(\boldsymbol{\lambda}, \boldsymbol{\theta}) = T^{-1} \sum_{t=1}^T \log \left[\sum_{\tau \in \Gamma(\mathbf{x}_t, \boldsymbol{\theta})} \lambda(\tau|\mathbf{x}_t) \left(\prod_{i=1}^N P_i^\tau(a_{it}|\mathbf{x}_t, \boldsymbol{\theta}) \right) \right] \quad (12)$$

For $T = \infty$, the population log-likelihood function $l_0(\boldsymbol{\lambda}, \boldsymbol{\theta}) \equiv \mathbb{E}\{\log[\sum_{\tau \in \Gamma(\mathbf{x}_t, \boldsymbol{\theta})} \lambda(\tau|\mathbf{x}_t) \prod_{i=1}^N P_i^\tau(a_{it}|\mathbf{x}_t, \boldsymbol{\theta})]\}$.

For the definition of the pseudo-likelihood function, consider first a data generating process where a single equilibrium is played conditional on the value of \mathbf{x}_t ; i.e., the distribution $\lambda^0(\tau|\mathbf{x}_t)$ is degenerate. Let $\tilde{\mathbf{P}}$ be an arbitrary vector of conditional choice probabilities (CCPs) for every action, player, and value of \mathbf{x} that may occur in the data; i.e., $\tilde{\mathbf{P}} \equiv \{\mathbf{P}(\mathbf{x}) : \text{for all } x \in X\}$. The *pseudo log-likelihood function* is defined as:

$$Q_T(\boldsymbol{\theta}, \tilde{\mathbf{P}}) = \sum_{t=1}^T \sum_{i=1}^N \log \Psi_i(a_{it} | \mathbf{x}_t, \boldsymbol{\theta}, \mathbf{P}_t(\mathbf{x}_t)) \quad (13)$$

We call this function a pseudo likelihood because the probabilities $\Psi_i(\cdot)$ are not necessarily equilibrium probabilities associated with $\boldsymbol{\theta}$ but just best responses to an arbitrary vector of beliefs $\tilde{\mathbf{P}}$. Consider next the general case where the equilibrium selection index τ_t is a random variable

conditional on \mathbf{x}_t . The vector of arbitrary beliefs $\tilde{\mathbf{P}}$ needs to be augmented to include CCPs for every equilibrium type that might be selected given each \mathbf{x} . The pseudo-likelihood is a mixture and has the mixing weights λ as an additional argument:

$$Q_T(\boldsymbol{\lambda}, \boldsymbol{\theta}, \tilde{\mathbf{P}}) = \sum_{t=1}^T \log \left[\sum_{\tau \in \Upsilon(\mathbf{x}_t)} \lambda(\tau|\mathbf{x}_t) \left(\prod_{i=1}^N \Psi_i(a_{it}|\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{P}^\tau(\mathbf{x}_t)) \right) \right] \quad (14)$$

where embedded in $(\boldsymbol{\lambda}, \tilde{\mathbf{P}})$ is a conjecture on the number of equilibria that are played in markets with each particular value of \mathbf{x} , and this conjecture is invariant with respect to the structural parameters $\boldsymbol{\theta}$. Note that, conditional on this conjecture, the likelihood (12) can be written as a constrained version of the pseudolikelihood:

$$l_T(\boldsymbol{\lambda}, \boldsymbol{\theta}) = Q_T(\boldsymbol{\lambda}, \boldsymbol{\theta}, \tilde{\mathbf{P}}) \text{ s.t. } \tilde{\mathbf{P}} = \boldsymbol{\Psi}(\boldsymbol{\theta}, \tilde{\mathbf{P}})$$

Finally, the population pseudo log-likelihood function is $Q_0(\boldsymbol{\lambda}, \boldsymbol{\theta}, \tilde{\mathbf{P}}) \equiv \mathbb{E}\{\log[\sum_{\tau \in \Upsilon(\mathbf{x}_t)} \lambda(\tau|\mathbf{x}_t) \prod_{i=1}^N \Psi_i(a_{it}|\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{P}^\tau(\mathbf{x}_t))]\}$ where the expectation is taken over the true population distribution of actions and \mathbf{x} values.

3.3 Identification

Consider identification of the pair (λ^0, θ^0) which pins down the data generating process within the class of discrete games described above. Our framework is semiparametric in that it combines a fully parametric model of payoffs and minimal assumptions on the equilibrium selection mechanism (ESM). However, note that the ESM is of capital interest to any researcher intending to conduct counterfactual or policy analysis. We discuss identification for three different classes of models: *Case 1*: deterministic equilibrium selection (i.e., τ_t is constant) and \mathbf{x}_t has discrete support; *Case 2*: deterministic equilibrium selection and \mathbf{x}_t has continuous support; and *Case 3*: stochastic equilibrium selection, i.e., τ_t has a non-degenerate probability distribution. This third case is the more general, and in order to help fix ideas we simplify its discussion by considering a discrete x_t . We focus on two questions. First, can the semiparametric model be identified (and estimated) under a two-step approach? Second, when the data are generated by mixtures of equilibrium strategies how can the researcher distinguish between 'common knowledge unobservables' and 'sunspots' (i.e. multiple equilibria) as competing explanations?

3.3.1 Case 1: Deterministic equilibrium selection and discrete \mathbf{x}

In this case, the equilibrium selection function is $\tau_t = \tau^0(\mathbf{x}_t)$. Conditional on the value of \mathbf{x}_t only one equilibrium is selected. This condition is weaker than the assumption "only one equilibrium

is played in the data" that has been one of the cornerstones of the recent literature exploiting the two-step approach.⁶

Remark 1: Given that $\tau_t = \tau^0(\mathbf{x}_t)$, we have that $\mathbf{P}_t^0(\mathbf{x}_t) = \mathbf{P}^{\tau^0(\mathbf{x}_t)}(\mathbf{x}_t, \boldsymbol{\theta}^0)$. This implies that once we condition on \mathbf{x} players' CCPs do not vary over markets. That is, for any market t , the probability $P_{it}^0(a|i, t, \mathbf{x}) \equiv \Pr(a_{it} = a|i, t, \mathbf{x}_t = \mathbf{x})$ is equal to $P_i^0(a|\mathbf{x}) \equiv \Pr(a_{it} = a|i, \mathbf{x}_t = \mathbf{x})$. Therefore, the probabilities in $\mathbf{P}^0(\mathbf{x})$ are nonparametrically identified under very mild regularity conditions. These probabilities can be consistently estimated using simple frequency estimators. For instance, the frequency estimator $\sum_{t=1}^T I\{a_{it} = a; \mathbf{x}_t = \mathbf{x}\} / \sum_{t=1}^T I\{\mathbf{x}_t = \mathbf{x}\}$ is a consistent estimator of $P_i^0(a|\mathbf{x})$.

Remark 2: Given the point-wise identification of $\mathbf{P}^0(\mathbf{x}_t)$ for every value of \mathbf{x}_t in the sample, Assumption 5 implies that the equilibrium conditions $\mathbf{P}^0(\mathbf{x}_t) = \Psi(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P}^0(\mathbf{x}_t))$ identify $\boldsymbol{\theta}^0$. Furthermore, given $\boldsymbol{\theta}^0$, we can obtain all the equilibria from the mapping $\Psi(\mathbf{x}_t, \boldsymbol{\theta}^0, \cdot)$ and identify the equilibrium type that $\mathbf{P}^0(\mathbf{x}_t)$ represents. That is, the equilibrium selection function $\tau^0(\mathbf{x}_t)$ is identified.

Remark 3: The identification result in Remark 2 shows that we can use a simple two-step method to estimate this class of models with multiple equilibria. In the first-step, the probabilities $\{\mathbf{P}^0(\mathbf{x}_t)\}$ are estimated nonparametrically using a frequency estimator. Let $\hat{\mathbf{P}} \equiv \{\hat{\mathbf{P}}(\mathbf{x}_t) : t = 1, 2, \dots, T\}$ be the vector of nonparametric estimates. In the second step, the estimator of $\boldsymbol{\theta}^0$ is defined as the value of $\boldsymbol{\theta}$ that maximizes the pseudo-likelihood function $Q_T(\boldsymbol{\theta}, \hat{\mathbf{P}})$. This estimator is root-T consistent and asymptotically normal (see for instance Aguirregabiria, 2004). This estimator does not impose the equilibrium conditions, and therefore in general the first-step estimate $\hat{\mathbf{P}}(\mathbf{x}_t)$ is not equal to $\Psi(\mathbf{x}_t, \hat{\boldsymbol{\theta}}, \hat{\mathbf{P}}(\mathbf{x}_t))$ which is the vector of estimated CCPs that is obtained from estimation of the structural model in the second step. This implies that it is not trivial to recognize the equilibrium type associated with \mathbf{x}_t , i.e. estimating $\tau^0(\mathbf{x}_t)$ will require a 'third step'. Several solutions come to mind. For instance, we can consider the following consistent estimator of $\tau^0(\mathbf{x}_t)$:

$$\hat{\tau}^0(\mathbf{x}_t) = \arg \min_{\tau \in \Upsilon(\mathbf{x}_t, \hat{\boldsymbol{\theta}})} \left\| \hat{\mathbf{P}}(\mathbf{x}_t) - \mathbf{P}^\tau(\mathbf{x}_t, \hat{\boldsymbol{\theta}}) \right\| \quad (15)$$

If we assume that all the selected equilibria are stable (i.e., Assumption 4(C)), then we can also use the following estimator: $\hat{\tau}^0(\mathbf{x}_t)$ is such that $\mathbf{P}^{\hat{\tau}^0(\mathbf{x}_t)}(\mathbf{x}_t, \hat{\boldsymbol{\theta}})$ is the limit of the best response iterations $\hat{\mathbf{P}}^k(\mathbf{x}_t) = \Psi(\mathbf{x}_t, \hat{\boldsymbol{\theta}}, \hat{\mathbf{P}}^{k-1}(\mathbf{x}_t))$ initiated with $\hat{\mathbf{P}}(\mathbf{x}_t)$.

⁶In most papers using the two-step approach it has been assumed, more or less explicitly, that only one equilibrium is present in the DGP. Aguirregabiria and Mira (2007) refer to the possibility that different equilibria might be selected across subsamples defined by the value of common knowledge variables, as long as the sample partition is known by the researcher.

Remark 4: In a model without multiple equilibria in the data, the probabilities in $\mathbf{P}^0(\mathbf{x}_t)$ are smooth functions of \mathbf{x}_t . This is because the equilibrium mapping $\Psi(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{P})$ is continuously differentiable in all its arguments. However, without further assumptions, that property is not guaranteed in a model with multiple equilibria in the data. $\mathbf{P}^0(\mathbf{x}_t)$ can be discontinuous in \mathbf{x}_t because $\mathbf{P}^0(\mathbf{x}_t)$ is equal to $\mathbf{P}^{\tau^0(\mathbf{x}_t)}(\mathbf{x}_t, \boldsymbol{\theta}^0)$ and $\tau^0(\mathbf{x}_t)$ can be discontinuous with respect to \mathbf{x}_t . This point has implications for the two-step estimation of the model. Without any further restrictions on $\tau^0(\mathbf{x}_t)$, the probability function $\mathbf{P}^0(\mathbf{x}_t)$ cannot be assumed to be smooth. Therefore, the first step has to use either a raw frequency estimator or another estimator which does not rely on smoothness as explained for case 2 below. This has implications for the finite sample properties of the estimator. Therefore, multiple equilibria in the data introduces additional limitations in the two-step estimation of games.

3.3.2 Case 2: Deterministic equilibrium selection and continuous \mathbf{x}

Again, we have that $\tau_t = \tau^0(\mathbf{x}_t)$ such that conditional on \mathbf{x}_t only one equilibrium is selected. Similarly to Case 1, we have that for any market t , $\mathbf{P}_t^0(\mathbf{x}_t) = \mathbf{P}^0(\mathbf{x}_t) = \mathbf{P}^{\tau^0(\mathbf{x}_t)}(\mathbf{x}_t, \boldsymbol{\theta}^0)$. However, without further restrictions on $\tau^0(\cdot)$, the probability function $\mathbf{P}^0(\cdot)$ may not be continuous and differentiable with respect to \mathbf{x} . In fact, if the model has multiple equilibria the function $\mathbf{P}^0(\cdot)$ may be discontinuous if only because some equilibria can appear and disappear when we move continuously along the space of \mathbf{x}_t . This point is illustrated in Figures 3a-3c. These figures represent the equilibrium mapping $\Phi(2.0 - 7.31 x_t + 6.75 x_t P)$ for four different values of x_t : 0.47, 0.50, 0.55, and 0.66. For any value of x_t in the $[0.47, 0.66]$, the model has multiple equilibria. However, the model has a unique equilibrium for values $x_t < 0.47$ or $x_t > 0.66$. Suppose that $\tau^0(\cdot)$ is such that it always select the equilibrium with the highest value of P . Then, for values of x_t in the interval $(-\infty, 0.66)$ the function $\mathbf{P}^0(\cdot)$ is continuous. However, at $x_t = 0.66$ the equilibrium with the high probability disappears. Therefore, the function $\mathbf{P}^0(\cdot)$ has a discontinuity at $x_t = 0.66$. Note that something similar occurs when $\tau^0(\cdot)$ selects the equilibrium with the lowest value of P . In that case, for $\mathbf{P}^0(\cdot)$ has a discontinuity at $x_t = 0.47$.

The discontinuity of the population probability functions \mathbf{P}^0 does not imply that the model is not identified. Müller (1992) studies the nonparametric (kernel) estimation of a single-variable regression function with 'change-points' or discontinuities. Delgado and Hidalgo (2000) extend the analysis to multivariate nonparametric regression models. The methods proposed in these papers use one-sided kernels to estimate the limits of the regression function from the left and from the right. These papers show the consistency and asymptotic normality of kernel estimators

of regression functions with unknown discontinuity points. Given the identification of the CCP function \mathbf{P}^0 , Assumption 5 implies the identification of $\boldsymbol{\theta}^0$ and τ^0 . Furthermore, the two-step method is consistent and asymptotically normal.

Though the two-step method is still consistent in this case, it has some additional limitations. A first limitation is computational. The methods proposed by Müller (1992) and Delgado and Hidalgo (2000), among others, require one to check every point in the space of \mathbf{x}_t as a potential discontinuity point. Therefore, the method can be computationally challenging when \mathbf{x}_t includes several continuous variables. In this case simplicity, which in other contexts is the main advantage of the two-step approach, is no longer a feature of this method. A second limitation, is that the finite sample bias of the two-step estimator can be more serious in this case.

Remark 6: The recent paper by Ellikson and Misra (2008) provides an example of an empirical application where the exogenous explanatory variables in \mathbf{x} explain players' actions in two different ways: as payoff relevant variables, and as variables that affect the equilibrium selection mechanism in a deterministic fashion. The authors study supermarket pricing strategies across 8000 local markets in the US. In line with the marketing literature they observe that most supermarkets position themselves by choosing either "Every Day Low Prices" (EDLP) across many items, or offering temporary price reductions (PROMotions) on a limited range of items. They note that there is a lot of heterogeneity in pricing strategies within supermarket chains and across geographically close areas. Using an empirical discrete game to analyze their data, they find that stores competing in a given market have incentives to coordinate their actions, and that consumer demographics play a significant role in the choice of local pricing strategies. An interpretation is that, given incentives to coordinate, both 'EDLP' and 'PROMO' equilibria can occur but one or the other equilibrium type is selected on the basis of market demographics.⁷

3.3.3 Case 3: Stochastic equilibrium selection with discrete \mathbf{x}_t

When the equilibrium selection function includes unobservable variables for the econometrician, the vector of observable probabilities $\mathbf{P}_t^0(\mathbf{x}_t)$ is equal to $\mathbf{P}^{\tau^0(\mathbf{x}_t, \xi_t)}(\mathbf{x}_t, \boldsymbol{\theta}^0)$ which is a mixture. In some cases the equilibrium probabilities which are the mixture components can be nonparametrically identified, and in such cases a two-step method may be feasible. However, we show that sufficient

⁷As stated in the Introduction of that paper: "... a key implication of our analysis is that these demographic factors act as a coordinating device for rival firms, helping shape the pricing landscape by defining an equilibrium correspondence". The structural empirical analysis in the paper seems based on an alternative interpretation, which is that different but unique equilibria emerge across markets with different demographics. However, in their conclusion the authors mention "the possibility that games such as this might support multiple equilibria", adding that "Developing methods that are robust to such possibilities remains an important area for future research".

conditions for 2-step identification are quite stringent and exclude interesting models which can still be identified from the likelihood or from recursive extensions of the 2-step approach.

The model implies the following restrictions on the probabilities of outcomes. For every outcome of the game a_t , and every value of \mathbf{x}_t , we have that:

$$\Pr(a_t|\mathbf{x}_t) = \left(P^1(a_t|\mathbf{x}_t, \boldsymbol{\theta}^0), P^2(a_t|\mathbf{x}_t, \boldsymbol{\theta}^0), \dots, P^{|\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)|}(a_t|\mathbf{x}_t, \boldsymbol{\theta}^0) \right) \boldsymbol{\lambda}^0(\mathbf{x}_t) \quad (16)$$

where $\Pr(a_t|\mathbf{x}_t)$ represents the population probability which is non-parametrically identified in the data, with $\Pr(a_t|\mathbf{x}_t) = \prod_{i=1}^N \Pr(a_{it}|\mathbf{x}_t)$, and $\boldsymbol{\lambda}^0(\mathbf{x}_t)$ is the $|\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)| \times 1$ vector $\{\lambda^0(\tau|\mathbf{x}_t) : \tau \in \Upsilon(\mathbf{x}_t, \boldsymbol{\theta}^0)\}$. This system of restrictions can be represented in matrix form. Let $\tilde{\Pi}^0(\mathbf{x}_t)$ be the $(|A|^N - 1) \times 1$ vector with the population probabilities $\Pr(a_t|\mathbf{x}_t)$ for every outcome except for one.⁸ With a certain abuse of notation, let the set of distributions induced by all equilibria be represented by $\tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0)$, a $(|A|^N - 1) \times |\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^0)|$ matrix where column τ contains the outcome probabilities $P^\tau(a_t|\mathbf{x}_t, \boldsymbol{\theta}^0)$ for every outcome (except for one) in the τ -th equilibrium associated with $(\mathbf{x}_t, \boldsymbol{\theta}^0)$. Therefore, we have that for every \mathbf{x}_t :

$$\tilde{\Pi}^0(\mathbf{x}_t) = \tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0) \boldsymbol{\lambda}^0(\mathbf{x}_t) \quad (17)$$

and, in matrix form,

$$\tilde{\Pi}^0 = \tilde{\Gamma}(\boldsymbol{\theta}^0) \boldsymbol{\lambda}^0 \quad (18)$$

where $\tilde{\Pi}^0$ is a column vector stacking $\tilde{\Pi}^0(\mathbf{x}_t)$ for all x in X ; $\tilde{\Gamma}(\boldsymbol{\theta}^0)$ is a block-diagonal matrix with blocks $\tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0)$; and $\boldsymbol{\lambda}^0$ is the column vector which stacks $\boldsymbol{\lambda}^0(\mathbf{x}_t)$ for all x .

Consider two alternative representations of the restrictions in (17). In the first one, replace the parametric mapping $\tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0)$ by the (non-parametric) matrix $\tilde{\mathbf{P}}^0(\mathbf{x}_t, \mathbf{P}^0(\mathbf{x}_t))$ containing game outcome probabilities induced by a set of equilibrium CCPs $\mathbf{P}^0(\mathbf{x}_t)$. In the second one, replace $\tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0)$ by the matrix $\tilde{\Psi}(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P}^0(\mathbf{x}_t))$ which contains probabilities of game outcomes obtained from the best responses to equilibrium CCP's, for each equilibrium. Note that $\tilde{\Gamma}(\mathbf{x}_t, \boldsymbol{\theta}^0)$, $\tilde{\mathbf{P}}^0(\mathbf{x}_t)$ and $\tilde{\Psi}(\mathbf{x}_t, \boldsymbol{\theta}^0, \mathbf{P}^0(\mathbf{x}_t))$ have exactly the same dimensions and the same values because $\mathbf{P}^0(\mathbf{x}_t)$ are the equilibrium CCPs consistent with the population structural parameters. In matrix form, the alternative representations of (18) are:

$$\tilde{\Pi}^0 = \tilde{\mathbf{P}}^0(\mathbf{P}^0) \boldsymbol{\lambda}^0 \quad (19)$$

⁸We are using $\tilde{\cdot}$ to distinguish probabilities of outcomes of the game from the probabilities of actions of individual players.

and

$$\tilde{\Pi}^0 = \tilde{\Psi}(\theta^0, \mathbf{P}^0) \lambda^0 \quad (20)$$

Based on these restrictions, we give the following definitions.

DEFINITION: (1) (λ^0, θ^0) are jointly identified in the likelihood iff, given the population $\tilde{\Pi}^0$, there is a unique pair (λ^0, θ^0) which satisfies conditions (18). (2) (λ^0, θ^0) are jointly identified in the pseudo-likelihood iff, given the population mixture $\tilde{\Pi}^0$, there is a unique triple $(\lambda^0, \theta^0, \mathbf{P}^0)$ such that: (a) the pair (λ^0, θ^0) uniquely satisfies conditions (20) given $(\tilde{\Pi}^0, \mathbf{P}^0)$; and (b) \mathbf{P}^0 are equilibrium CCPs for θ^0 . (3) (λ^0, θ^0) are two-step identified iff: (a) (step 1 identification): given the population $\tilde{\Pi}^0$, there is a unique pair $(\lambda^0, \mathbf{P}^0)$ that satisfies conditions (19); and (b) (step 2 identification): given $(\lambda^0, \mathbf{P}^0)$, θ^0 satisfies Assumption 5.

Bajari, Hahn, Hong and Ridder (2008) describe 'order' and 'rank' conditions for joint identification in the likelihood. If $|\Gamma|$ equilibria are selected for all values of x , counting equations and parameters gives the following 'order' condition for identification in the likelihood and the pseudo-likelihood: $(|A|^N - 1)|X| > |\theta| + (|\Gamma| - 1)|X|$ where $|\theta|$ is the number of structural parameters. Therefore, joint identification will generally follow if the number of outcomes of the game ($|A|^N$) is sufficiently large relative to the number of equilibria that are selected.⁹

The following Lemma establishes that the conditions for each of the three forms of identification, in the order we have listed them, are increasingly strong.

LEMMA: (A) Two-step identification implies joint identification in the pseudo-likelihood and joint identification in the likelihood. (B) Joint identification in the pseudo-likelihood implies joint identification in the likelihood. However, the converse of (A) is not true in general. In particular, (C) joint identification in the pseudo-likelihood implies step 2 identification but it does not imply step 1 identification.

Remark 7: The identification Lemma implies that the applicability of two-step methods in this context is limited by the strength of the conditions that are required for step 1 identification. In a different context, Hall and Zhou (2003) have studied the identification of the class of non-parametric finite mixture models in (19), i.e., for a given \mathbf{x} , $\Pi(a_1, a_2, \dots, a_N) = \lambda_1(\prod_{i=1}^N P_i^1(a_i)) + \lambda_2(\prod_{i=1}^N P_i^2(a_i)) + \dots + \lambda_R(\prod_{i=1}^N P_i^{|\Gamma|}(a_i))$. They present sufficient conditions of identification for the case of mixing between two distributions or equilibria, i.e., $|\Gamma| = 2$. The most important conditions

⁹ Although the order conditions are the same, the rank conditions for joint identification in the likelihood and the pseudolikelihood will be different.

are (see Theorem 4.3 in their paper): (1) N is greater or equal than 3; and (2) (*irreducibility condition*) for any "player" i and any action a_i , $P_i^1(a_i) \neq P_i^2(a_i)$. They also note that these conditions are not sufficient in finite mixture models with more than two mixing distributions (i.e., with $R > 2$). Kasahara and Shimotsu (2008) extend the study of this identification problem to models with $|\Gamma| > 2$ and show that significantly stronger (order and rank) conditions are needed. These conditions substantially restrict the class of models that are step-1 identified. In addition to having 3 or more players, we also need the number of actions per player to be at least as large as the number of equilibria that are present in the data. If the game and the equilibria are symmetric, step-1 identification requires that $N \geq 2|\Gamma| - 1$. The following is an example of a DGP for which the sufficient conditions for step-1 identification fail, whereas joint identification in the pseudo-likelihood will in general obtain.

EXAMPLE: Consider a two-players binary choice game with the following best response probability functions: $\Phi(\theta_{01} + \theta_1 x_t + \theta_2 P_2(x_t))$ for player 1, and $\Phi(\theta_{02} + \theta_1 x_t + \theta_2 P_1(x_t))$ for player 2. The vector of structural parameters is $(\theta_{01}, \theta_{02}, \theta_1, \theta_2)$. x_t is a discrete random variable with M points of support, with $M \geq 3$. The equilibrium selection mechanism is such that when multiple (i.e., three) equilibria exist, the two stable equilibria are selected with probability 1/2 each. The researcher considers an unrestricted nonparametric specification for $\lambda(x)$. For this model and DGP, \mathbf{P}^0 and $\boldsymbol{\lambda}^0$ are not step-1-identified. However, $(\boldsymbol{\lambda}^0, \boldsymbol{\theta}^0)$ are jointly identified in the pseudo-likelihood. [PROOF].

Remark 8: In contrast with results for games of complete information, point identification is possible in games of incomplete information under conditions that are "not too stringent for relevant applications". [also see discussion in Bajari, Hahn, Hong and Ridder 2008.]

Remark 9: It is also important to note that, when step-1-identification holds, the procedure or algorithm to estimate nonparametrically $(\mathbf{P}^0, \boldsymbol{\lambda}^0)$ is quite complicated. In particular, the estimation method proposed by Hall and Zhou (pages 204-207 of their paper) consists of the minimization (for every value of \mathbf{x}) of a sample criterion function with multiple local minima and maxima. Therefore, the algorithm should perform a global search to guarantee that the global minimum has been reached. This global search is always computationally intensive. Therefore it would seem that the computational simplicity of two-step methods, which is one of their main advantages, disappears in Case 3.

Remark 10: Sweeting (2008) studies 2- step identification in his binary choice model of the tim-

ing of radio commercials. He argues that, if 2 symmetric equilibria are selected, 'excess dispersion' of the binomial distribution of outcomes will identify the mixture if the number of homogenous players is sufficiently large ($N \geq 3$), and he obtains 'order' conditions for versions of his model with heterogenous players. Sweeting correctly notes that multiple equilibria in the data can aid in step 2 identification. However, we believe he is wrong in suggesting that two-step identification is made easier because step 1 identification is also made more difficult when multiple equilibria are present in the data.

Remark 11: Our definition and discussion of step 1 identification implicitly took the number of equilibria present in the data for each x as known (and fixed). The definition of step 1 identification can be extended by adding the proviso that there is a unique \mathbf{P}^0 "allowing for any number of equilibria" ... Kasahara and Simotsu (2008) explicitly study identification and estimation of the number of mixture components, as opposed to their identities and mixing weights. In our context, their analysis applies as follows. Let Z_1 and Z_2 be two discrete random variables which summarize outcomes from one realization of our game, such that Z_1 and Z_2 are independent conditional on observed x and conditional on equilibrium play. Let Π_z be a matrix describing the (population) joint distribution of (Z_1, Z_2) . For instance, if the game has two players and three actions we may choose $Z_i = a_i$ and Π_z is a 3×3 matrix. If the game has 4 players, Z_1 could be any function of outcomes for players 1-2 and Z_2 a function of outcomes for players 3-4, respectively. KS's main result (Proposition 1) is that the rank of Π_z is a lower bound of the number of mixture components. They propose a test statistic and a sequential procedure to estimate this lower bound, based on a non-parametric estimate of the matrix Π_z . This is interesting for our purposes, because estimating the joint distribution of Z 's and the rank of Π_z can be considerably less costly than estimating the weights and mixture components, and prior knowledge of the number of components will facilitate the implementation of our NPL algorithm. [NOTE: There is clearly some dependence on the particular choice of Z 's - reduction of data can lead to loss of information and identifying power. The researcher should seek a compromise between 'reduction' which improves precision, and the ability to detect mixing.]

Remark 12: If x is continuous, the point we made about the implications of 'jumps' in $\tau^0(x)$ in our discussion of Case 2 can now be extended to the selection probability function $\lambda^0(x)$. Without further restrictions, non-parametric methods which do not rely on smoothness should be used to estimate the discrete probability distributions $\tilde{\Pi}^0(\mathbf{x})$ or $\Pi_z(x)$, and the discussion about identification of stochastic selection for discrete x should still apply.

Remark 13: Identification of a stochastic ESM relies on a key implication of the structure of this class of games of incomplete information, i.e., conditional on common knowledge unobservables x the actions of players are independent of each other because the private information components of payoffs are iid across players. Independence of actions across players can be tested directly in the data, and rejection of the hypothesis is taken as evidence of multiplicity of equilibria, or 'sunspots'. However, it should be stressed that an alternative interpretation is that some payoff-relevant common knowledge variables are not observed by the econometrician. Can the researcher distinguish between these two alternative explanations? This question is important because inferences about structural parameters, as well as counterfactual predictions, are likely to be different under each of the competing hypothesis. [NOTE: a) Under reasonable assumptions (smoothness, support) payoff-relevant common knowledge unobservables would NOT lead to discontinuities in the distributions $\tilde{\Pi}^0(\mathbf{x})$ or $\Pi_z(x)$. Can we test for this for x both continuous (probably) and discrete (doubtful)?; b) Likewise, changes in the number of components of the mixture with x would point to multiplicity of equilibria; c) Suppose we cannot reject that $\Pi_z(x)$ is both smooth and a mixture with a fixed number of components. Then, can we still distinguish between the two models? d) The finite mixture model is the natural one under sunspots, whereas in the case of payoff-relevant unobservables the finite mixture model is likely an approximation.]

To conclude: As seen from our Identification Lemma and our Example, the parameters (λ^0, θ^0) are identified in models where the equilibrium probabilities $\{\mathbf{P}_t^0(\mathbf{x}_t)\}$ are not non-parametrically identified and two-step estimation is not feasible. In some other cases the equilibrium probabilities and the selection mechanism are non-parametrically identified but 'step 1' non-parametric estimation becomes a much more complex task, and the 'curse of dimensionality' and finite sample biases much more serious problems, than in models with only one equilibrium in the data. Faced with these issues, a researcher may want to combine NPL with a non-parametric 'step 0' focused on the estimation of the number of equilibria that are present in the data and/ or the search for break points. An alternative (or complementary) strategy is to impose some additional structure on the ESM [as required for counterfactuals; see Section 5.] see discussion on Examples: smoothness of $\tilde{\Pi}^0(\mathbf{x})$ or $\Pi_z(x)$ functions. In Case 3 this is implied by smoothness of $\lambda(\cdot)$. In Cases 1-2 this is compatible with multiple equilibria only if τ were to jump at values of x where the CCP functions intersect.]. In the next section we discuss the difficulties involved in likelihood estimation and we propose a simpler method and algorithm for the estimation of (λ^0, θ^0) .

4 Estimation

4.1 Nested Pseudo Likelihood (NPL) estimator

Aguirregabiria and Mira (2007) have proposed and applied an estimator of games of incomplete information, the so-called Nested Pseudo Likelihood (NPL) estimator. The method has several advantages with respect to maximum likelihood and two-step estimation methods. The NPL method is computationally much simpler than direct maximization of the likelihood in (12) because it avoids the search for - and exact calculation of - every equilibrium type for every value of the structural parameters.¹⁰ The NPL method is statistically more efficient than the two-step method, both asymptotically and in small samples. Furthermore, the NPL method identifies structural parameters in situations where the two-step method cannot, such as models where the econometrician does not observe some common knowledge state variables in \mathbf{x}_t (see Aguirregabiria and Mira, 2007), or in models with multiple equilibria in the data and stochastic equilibrium selection, such as the models that we consider in this paper.

To define the NPL estimator it is useful to start with the definition of the *NPL mapping* for models with deterministic equilibrium selection mechanisms (cases 1 and 2). Let \mathbf{P} be an arbitrary vector of conditional choice probabilities (CCPs) for every action, player, and market in the data; i.e., $\mathbf{P} \equiv \{\mathbf{P}_t(\mathbf{x}_t) : t = 1, 2, \dots, T\}$.¹¹ Suppose that the pseudo likelihood function $Q_T(\boldsymbol{\theta}, \mathbf{P})$ is such that, for any vector \mathbf{P} there is a unique value of $\boldsymbol{\theta} \in \Theta$ that maximizes $Q_T(\boldsymbol{\theta}, \mathbf{P})$. Let $\tilde{\boldsymbol{\theta}}_T(\mathbf{P})$ be that value. The *NPL mapping* is a fixed point mapping from the space of \mathbf{P} into itself. It is defined as $\phi_T(\mathbf{P}) \equiv \{\phi_T(t, \mathbf{P}) : t = 1, 2, \dots, T\}$ such that:

$$\phi_T(t, \mathbf{P}) \equiv \Psi \left(\mathbf{x}_t, \tilde{\boldsymbol{\theta}}_T(\mathbf{P}), \mathbf{P}_t(\mathbf{x}_t) \right) \quad (21)$$

That is, the NPL mapping $\phi_T(\mathbf{P})$ returns the CCPs which are the best response to \mathbf{P} when the vector of structural parameters is the pseudo maximum likelihood estimator $\tilde{\boldsymbol{\theta}}_T(\mathbf{P}) = \arg \max_{\boldsymbol{\theta}} Q_T(\boldsymbol{\theta}, \mathbf{P})$. An *NPL fixed point* is a vector of probabilities \mathbf{P} that solves the fixed point problem $\mathbf{P} = \phi_T(\mathbf{P})$. For estimation purposes, it is convenient to consider an alternative (equivalent) definition of an

¹⁰In the spirit of our NPL algorithm, another alternative to direct "full solution" maximization of the likelihood would be to exploit the pseudolikelihood representation of the likelihood in (??). The ML estimator can then be defined as a constrained pseudolikelihood estimator of (λ, θ, P) , and an off-the-shelf constrained maximization algorithm can be used to compute it as advocated in Judd and Su (2007). This approach is worth exploring, but it remains to be seen that it will be practical in this context. Note that the dimensionality of the 'incidental' P is much larger than the dimensionality of λ and θ . Furthermore, in order to obtain the ML estimator the constrained PML approach would need to be adapted to take into account the dependence on θ of the number of equilibria in P .

¹¹In case 1 the vector of arbitrary beliefs \mathbf{P} is constrained by the condition $\mathbf{P}_t(\mathbf{x}_t) = \mathbf{P}_s(\mathbf{x}_s)$ if $x_t = x_s$.

NPL fixed point as a pair $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{P}})$ that satisfies the following conditions:

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \arg \max_{\boldsymbol{\theta} \in \Theta} Q_T(\boldsymbol{\theta}, \hat{\mathbf{P}}) \\ \text{and} & \\ \hat{\mathbf{P}}_t(\mathbf{x}_t) &= \boldsymbol{\Psi}(\mathbf{x}_t, \hat{\boldsymbol{\theta}}, \hat{\mathbf{P}}_t(\mathbf{x}_t)) \quad \text{for any } t = 1, 2, \dots, T\end{aligned}\tag{22}$$

That is, $\hat{\boldsymbol{\theta}}$ is a pseudo maximum likelihood estimator associated with $\hat{\mathbf{P}}$, and for any observation t in the sample $\hat{\mathbf{P}}_t(\mathbf{x}_t)$ is an equilibrium of the mapping $\boldsymbol{\Psi}(\mathbf{x}_t, \hat{\boldsymbol{\theta}}, \cdot)$. Under Assumptions 1 to 3, the NPL mapping is continuous and differentiable in the compact set $\Theta \times [0, 1]^{NT|A|}$. Therefore, Brower's fixed-point theorem guarantees that for a given sample there exists at least one NPL fixed-point. Uniqueness is not guaranteed. The NPL estimator $(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})$ is the NPL fixed point that maximizes the pseudo likelihood Q_T .

[EXTENSION: Stochastic selection mechanism: Following Aguirregabiria and Mira (2007), the vector \mathbf{P} is augmented to include CCPs for every equilibrium type that might be selected in every sampled value of x . The pseudo-likelihood is the mixture $Q_T(\boldsymbol{\theta}, \lambda, \mathbf{P})$ defined in section 3.2, and it is maximized in $(\boldsymbol{\theta}, \lambda)$ given P . Everything else is the same.]

Proposition 2 establishes the consistency and asymptotic normality of the NPL estimator.

PROPOSITION 2: Consider the following population counterparts of the sample functions Q_T , $\tilde{\boldsymbol{\theta}}_T$, and ϕ_T : $Q_0(\boldsymbol{\theta}, \mathbf{P}) \equiv E(Q_T(\boldsymbol{\theta}, \mathbf{P}))$; $\tilde{\boldsymbol{\theta}}_T(\mathbf{P}) \equiv \arg \max_{\boldsymbol{\theta} \in \Theta} Q_0(\boldsymbol{\theta}, \mathbf{P})$; and $\phi_0(\mathbf{P}) \equiv \boldsymbol{\Psi}(\tilde{\boldsymbol{\theta}}_0(\mathbf{P}), \mathbf{P})$. The set of population NPL fixed points is $\Upsilon_0 \equiv \{(\boldsymbol{\theta}, \mathbf{P}) \in \Theta \times [0, 1]^{N|A||X|} : \boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}_0(\mathbf{P}) \text{ and } \mathbf{P} = \phi_0(\mathbf{P})\}$. Suppose that: (i) Assumptions 1 to 5 hold; (ii) $\boldsymbol{\Psi}$ is twice continuously differentiable; (iii) Θ is a compact set; (iv) $\boldsymbol{\theta}^0 \in \text{int}(\Theta)$; (v) $(\boldsymbol{\theta}^0, \mathbf{P}^0)$ is an isolated population NPL fixed point; i.e., there is an open ball around it that does not contain any other element of Υ_0 ; (vi) there exists a closed neighborhood of \mathbf{P}^0 , $\mathbb{N}(\mathbf{P}^0)$, such that, for all \mathbf{P} in $\mathbb{N}(\mathbf{P}^0)$, Q_0 is globally concave in $\boldsymbol{\theta}$ and $\partial^2 Q_0(\boldsymbol{\theta}, \mathbf{P}^0)/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'$ is a nonsingular matrix; and (vii) the operator $\boldsymbol{\varphi}_0(\mathbf{P}) - \mathbf{P}$ has a nonsingular Jacobian matrix at \mathbf{P}^0 . Then, $\hat{\boldsymbol{\theta}}_{NPL}$ is a consistent estimator and $\sqrt{M}(\hat{\boldsymbol{\theta}}_{NPL} - \boldsymbol{\theta}^0) \rightarrow_d N(0, V_{NPL})$, with:

$$V_{NPL} = \left[\Omega_{\theta\theta} + \Omega_{\theta P} (I - \boldsymbol{\Psi}'_P)^{-1} \boldsymbol{\Psi}'_{\theta} \right]^{-1} \Omega_{\theta\theta} \left[\Omega_{\theta\theta} + \boldsymbol{\Psi}'_{\theta} (I - \boldsymbol{\Psi}_P)^{-1} \Omega'_{\theta P} \right]^{-1}$$

where: $\Omega_{\theta\theta}$ is the variance of the score $s_{\theta t} \equiv \sum_{i=1}^N \partial \ln \Psi_i(a_{it}|x_t; P_t^0, \boldsymbol{\theta}^0)/\partial\boldsymbol{\theta}$, i.e., $\Omega_{\theta\theta} \equiv E(s_{\theta t} s'_{\theta t})$; $\Omega_{\theta P}$ is the covariance between the score $s_{\theta t}$ and the score $s_{P t} \equiv \sum_{i=1}^N \partial \ln \Psi_i(a_{it}|x_t; P_t^0, \boldsymbol{\theta}^0)/\partial P_t$, i.e., $\Omega_{\theta P} \equiv E(s_{\theta t} s'_{P t})$; and $\boldsymbol{\Psi}_P$ and $\boldsymbol{\Psi}_{\theta}$ are the Jacobian matrices of $\boldsymbol{\Psi}(P^0, \boldsymbol{\theta}^0)$ with respect to P and $\boldsymbol{\theta}$, respectively.

How do we compute the NPL estimator? Obtaining an NPL fixed point is computationally quite simple because we only have to iterate in the NPL mapping. Let $\hat{\mathbf{P}}_0$ be an initial vector of CCPs. We can generate a sequence of estimators $\{(\hat{\boldsymbol{\theta}}_K, \hat{\mathbf{P}}_K) : K \geq 1\}$ by iterating in the NPL mapping. More precisely, the K -step estimator is defined as:

$$\hat{\boldsymbol{\theta}}_K = \arg \max_{\theta \in \Theta} Q_T \left(\theta, \hat{\mathbf{P}}_{K-1} \right) \quad (23)$$

and the vector of probabilities $\hat{\mathbf{P}}_K = \{\hat{\mathbf{P}}_{K,t}(\mathbf{x}_t) : t = 1, 2, \dots, T\}$ is obtained as

$$\hat{\mathbf{P}}_K(\mathbf{x}_t) = \Psi(\mathbf{x}_t, \hat{\boldsymbol{\theta}}_K, \hat{\mathbf{P}}_{K-1,t}(\mathbf{x}_t)) \quad (24)$$

Both the pseudo maximum likelihood iteration in (23) and the best response iteration in (24) are typically very simple computations, especially in static games. Upon convergence, this iterative procedure returns an NPL fixed point.

Remark 14: If the initial $\hat{\mathbf{P}}_0$ is a consistent estimator of the vector of population equilibrium probabilities, then all the estimators in the sequence $\{(\hat{\boldsymbol{\theta}}_K, \hat{\mathbf{P}}_K) : K \geq 1\}$ are consistent and asymptotically normal. Furthermore, if the matrix $\{I - \Psi_\theta[\Psi'_\theta \text{diag}(\mathbf{P}^0)^{-1} \Psi_\theta]^{-1} \Psi'_\theta \text{diag}(\mathbf{P}^0)^{-1}\} \Psi_P$ has all its eigenvalues in the unit circle, then the asymptotic variance and the finite sample bias of these estimators declines with the number of iterations K (see Aguirregabiria 2004, Aguirregabiria and Mira 2008). Therefore, for those models where the equilibrium selection mechanism is not stochastic (Cases 1 and 2 above) such that we can obtain consistent nonparametric estimates of \mathbf{P}^0 , this recursive algorithm is a simple procedure to improve the statistical properties of the two-step method.

Remark 15: In models where the equilibrium selection mechanism is stochastic, from the point of view of the econometrician (Case 3 above), the initial $\hat{\mathbf{P}}_0$ may not be a consistent estimator of \mathbf{P}^0 . We need to iterate until convergence to obtain an NPL fixed point. If the model is such that the NPL mapping has only one fixed point, then it is the consistent NPL estimator. For instance, that is the case in single-agent dynamic programming models (see Aguirregabiria and Mira 2002, 2008).¹² However, in static and dynamic games we can have multiple NPL fixed points. Although Aguirregabiria and Mira did not find multiple NPL fixed points in their dynamic entry/exit game, we believe this situation is likely to arise when multiple equilibria are actually present in the data generating process, rather than just a feature of the model. In section 5, we present some numerical experiments to illustrate the 'conjecture' that multiple equilibria in the data seems to make multiple

¹²In fact, in the context of single-agent dynamic discrete choice models, the NPL estimator is the same as the maximum likelihood estimator.

NPL fixed points more 'likely'.¹³ Therefore, to obtain the NPL estimator, it is necessary to search for the multiple NPL fixed points and select the one with the largest value of the pseudo likelihood. Though getting an NPL fixed point is a simple task, searching for multiple NPL fixed points is more complicated. The stochastic algorithm that we propose in the next sub-section addresses this problem. The algorithm combines the NPL procedure with a Genetic Algorithm (GA). The NPL algorithm avoids the repeated computation of equilibria for each trial value of θ . The GA performs a global search over the space of NPL fixed points.

Remark 16: By definition of the NPL estimator $(\hat{\theta}_{NPL}, \hat{\mathbf{P}}_{NPL})$, the probabilities in $\hat{\mathbf{P}}_{NPL}$ are equilibrium probabilities associated with $\hat{\theta}_{NPL}$. Therefore, the NPL estimator for cases 1 and 2 provides a direct consistent estimator of the equilibrium selection function $\tau^0(\mathbf{x}_t)$. That is:

$$\hat{\tau}^0(\mathbf{x}_t) = \sum_{\tau \in \Upsilon(\mathbf{x}_t, \hat{\theta}_{NPL})} \tau I \left\{ \hat{\mathbf{P}}_{NPL,t}(\mathbf{x}_t) = \mathbf{P}^\tau(\mathbf{x}_t, \hat{\theta}) \right\} \quad (25)$$

We illustrate this in our Monte Carlo examples in section 5.

4.2 Hybrid genetic algorithm

Computing a NPL fixed-point is a computationally simple task even in relatively complex models like dynamic games (see Aguirregabiria and Mira, 2007). However, when there is no unique NPL fixed-point, computing the NPL estimator can be a more demanding problem. A possible procedure to obtain the NPL estimator is a *parallel NPL* method. That is, we can obtain M fixed points of the NPL procedure by applying this method with M different initial values for the choice probabilities. These M initial values can be obtained by drawing M bootstrap samples from the original sample and then calculating a nonparametric estimator of the probabilities for each bootstrap sample. Given these NPL fixed-points, we choose as estimator of θ^0 the fixed point with the highest value of the pseudo likelihood. A limitation of this approach is that we may need a large number of parallel processors to guarantee that this estimator is the NPL estimator. To deal with this problem we combine the *parallel NPL* method with a genetic algorithm.

The objective is to obtain the NPL fixed point $(\hat{\theta}, \hat{\mathbf{P}})$ that maximizes the pseudo likelihood $Q_T(\theta, \mathbf{P})$. Following the terminology in the GA literature, the objective function $Q_T(\theta, \mathbf{P})$ is called also *fitness function*. We initialized the algorithm with M different vectors $\hat{\mathbf{P}}$ (each of dimension $N|A|T \times 1$). At every iteration of the algorithm we perform four operations on the M vectors of

¹³Consider two models with the same value of the structural parameters, θ^0 , and the same sample values of the exogenous variables $\{\mathbf{x}_t\}$. The only difference between these models is in the equilibrium selection mechanism: one always selects the same equilibrium, while the other selects multiple equilibria. We have found examples where the first model has a unique NPL fixed point but the second model has multiple NPL fixed points.

probabilities: NPL fixed-points; selection; crossover; and mutation. The last three operations are characteristic of GAs. These operations allow for a more global search over the space of NPL fixed points than a simple parallel NPL method. We first describe the algorithm. Then, we discuss the criterion to select the three parameters that characterize the GA: *population size*, *cross-over rate*, and *mutation rate*.

4.2.1 Description of the algorithm

(0) *Initial population*. The initial "population" of probability vectors is $\Pi^1 = \{\hat{\mathbf{P}}_m^1 \in [0, 1]^{N|A|T} : m = 1, 2, \dots, M\}$, where M is the *population size*. This initial population may be arbitrarily chosen, or it may come from nonparametric estimates. For instance, the probabilities could be obtained as M bootstrap nonparametric estimates of players' choice probabilities.

The GA generates a sequence of populations of size M that we denote by $\{\Pi^K : K \geq 1\}$. Associated with this sequence of probabilities the GA also generates a sequence of sets of parameter estimates $\{\Theta^K : K \geq 1\}$. with $\Theta^K = \{\hat{\theta}_m^K : m = 1, \dots, M\}$. An iteration of the algorithm consists in the creation of a new generation with the offsprings of the existing generation. An iteration can be described in terms of three processes or steps that are followed sequentially: (1) NPL fixed points; (2) mating or selection of parents; and (3) crossover and mutation.

(1) *NPL fixed points*. Let $\hat{\mathbf{P}}_m^K \in [0, 1]^{N|A|T}$ be one the vectors of probabilities in the population Π^K . Using $\hat{\mathbf{P}}_m^K$ as the starting value, we iterate in the NPL algorithm (i.e., equations (23) and (24)) until convergence. Let $\{\hat{\theta}_m^K, \hat{\mathbf{P}}_m^{K*}\}$ be the NPL fixed point that we get upon convergence. We apply the same procedure to all the vectors of probabilities in the population Π^K . The new set of NPL fixed points is $\{\hat{\theta}_m^K, \hat{\mathbf{P}}_m^{K*} : m = 1, \dots, M\}$. This is the population of NPL fixed points at iteration K .

(2) *Selection of parents*. We draw, with replacement, M pairs (θ, \mathbf{P}) from the population of NPL fixed points at iteration K . The probability that a vector is chosen depends on its *fitness* or pseudo likelihood $Q(\hat{\theta}_m^K, \hat{\mathbf{P}}_m^{K*})$. More precisely, the probability that the m -th element is selected at iteration K is:

$$\frac{\exp \left\{ \varphi Q_T \left(\hat{\theta}_m^K, \hat{\mathbf{P}}_m^{K*} \right) \right\}}{\sum_{j=1}^M \exp \left\{ \varphi Q_T \left(\hat{\theta}_j^K, \hat{\mathbf{P}}_j^{K*} \right) \right\}} \quad (26)$$

where $\varphi \geq 0$ is a parameter that measures the strength of the dependence of selection on fitness. If $\varphi = 0$, every individual has the same probability $1/M$ of being selected. If $\varphi = \infty$, only the fittest individual is selected in the M draws.

(3) *Crossover and mutation.* Each couple generates one offspring. An offspring inherits "chromosomes" from its parents, but there maybe mutation as well. Let $\hat{\mathbf{P}}_m$ and $\hat{\mathbf{P}}_n$ be a couple of parents selected in step (2). Then, the offspring from this couple is $\hat{\mathbf{P}}'$ such that:

$$\hat{\mathbf{P}}' = \mathbf{D} * \left\{ \hat{\mathbf{P}}_m + \delta \mathbf{Z} * \left(\hat{\mathbf{P}}_m - \mathbf{U} \right) \right\} + (\mathbf{1} - \mathbf{D}) * \left\{ \hat{\mathbf{P}}_n + \delta \mathbf{Z} * \left(\hat{\mathbf{P}}_n - \mathbf{U} \right) \right\} \quad (27)$$

where \mathbf{D} , \mathbf{Z} , and \mathbf{U} are vectors with the same dimension as $\hat{\mathbf{P}}$; δ is a parameter that represents the magnitude of mutation; $*$ is the element-by-element product. \mathbf{D} is a vector of 0's and 1's that represents the identity of the parent who transmits each chromosome. These 0's and 1's are random draws from a Bernoulli random variable with probability 1/2. \mathbf{Z} is also a vector of 0's and 1's that represents the indicator of a mutation for each chromosome. These 0's and 1's are random draws from a Bernoulli random variable with probability equal to γ . The parameter $\gamma \in (0, 1)$ measures the amount of mutation. Finally, \mathbf{U} is a vector with independent random draws from a $U(0, 1)$.

We perform steps (2) and (3) M times to obtain a new population of probabilities $\Pi^{K+1} = \{\hat{\mathbf{P}}_m^{K+1} : m = 1, 2, \dots, M\}$. The procedure iterates several times (not necessarily until convergence). Through all the iterations, we always keep track the NPL fixed point $(\hat{\boldsymbol{\theta}}_m^K, \hat{\mathbf{P}}_m^{K*})$ that has generated the largest value of the pseudo-likelihood over all the considered populations. After the last iteration, that is our selected NPL estimator.

4.2.2 Selection of parameters of the hybrid GA

Our GA depends on the population size (M), the sensitivity of selection to fitness (φ), and the mutation rate (δ). The trade-off in the choice of the population size is simple. The larger the population size, the more likely is that the algorithm finds the global maximum of the problem. However, the computational cost of the algorithm increases linearly with the population size.

5 Prediction and counterfactual experiments

Suppose that, after the estimation of $(\boldsymbol{\theta}^0, \mathbf{P}^0)$ we would like to predict the effects on players' behavior (choice probabilities) of a counterfactual change in the economic environment or the primitives of the model. We consider three different types of prediction exercises in a given market t :

- (a) a change from \mathbf{x}_t to a counterfactual \mathbf{x}_t^* in X , keeping $(\lambda^0, \boldsymbol{\theta}^0)$ constant;
- (b) a change from \mathbf{x}_t to a counterfactual \mathbf{x}_t^* outside X , keeping $(\lambda^0, \boldsymbol{\theta}^0)$ constant.
- (c) a change from $\boldsymbol{\theta}^0$ to a counterfactual $\boldsymbol{\theta}^*$, keeping \mathbf{x}_t and λ^0 constant;

Of course, the interesting case is when the sets $\Gamma(\mathbf{x}_t^*, \boldsymbol{\theta}^0)$ (for prediction exercises (a) and (b)) and $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^*)$ (in exercise (c)) contain multiple equilibria. Otherwise, we know that the single equilibrium is selected with probability one.

For these prediction exercises, we need to interpolate (or extrapolate) the selection function for values of \mathbf{x}_t which are not in the sample and for values of θ for which no data whatsoever are available. Of course, interpolation generates statistically consistent predictions only if the function λ^0 satisfies certain smoothness properties with respect to \mathbf{x} . We know that $\lambda^0(\tau|\mathbf{x}_t)$ may have discontinuity points. For instance, in prediction exercise (a) some equilibria appear and disappear when we move along the support of \mathbf{x}_t and this may generate jumps in the function λ^0 . The extrapolations in (b) and (c) are more difficult - note that, in general, the structural parameter θ should appear as an argument of the equilibrium selection function $\lambda^0(\tau|\mathbf{x}, \theta)$. In line with our approach in the rest of the paper, we consider a researcher who is as agnostic as possible about the equilibrium selection mechanism. However, it is clear that there is a tension between the incompleteness of the model and the ability to predict counterfactuals. Counterfactuals are not possible unless the researcher is willing to impose some structure on $\lambda^0(\tau|\mathbf{x}, \theta)$. Some of this additional structure may be testable. We establish this more formally in the following definitions and assumptions.

DEFINITION. We say that the sets of equilibria $\Gamma(\mathbf{x}, \boldsymbol{\theta})$ and $\Gamma(\mathbf{x}', \boldsymbol{\theta}')$ have the same 'type' if we can define a continuous sequence from $(\mathbf{x}, \boldsymbol{\theta})$ to $(\mathbf{x}', \boldsymbol{\theta}')$ such that none of the equilibria disappears and no new equilibria appear.

DEFINITION. Given a pair $(\mathbf{x}^*, \boldsymbol{\theta})$, we say that a vector $\mathbf{x} \in X$ belongs to the set $X^e(\mathbf{x}^*, \boldsymbol{\theta})$ if and only if the set of equilibria $\Gamma(\mathbf{x}, \boldsymbol{\theta})$ is of the same type as the set of equilibria $\Gamma(\mathbf{x}^*, \boldsymbol{\theta})$.

ASSUMPTION 6A. For any $\mathbf{x} \in X$, the function λ^0 is continuously differentiable in the interior of the set $X^e(\mathbf{x}, \boldsymbol{\theta})$. [INTUITION: discontinuities in λ^0 exist only when equilibria appear or disappear - ESM does not jump within 'same type' sets.][The assumption is even stronger for discrete x , but this is needed. If x is discrete, the support of $\lambda()$ does not change within 'same type' sets.

ASSUMPTION 6B. Assumption 6A holds. Furthermore, the function $\lambda^0()$ is continuous for all (x, θ) . [INTUITION: In this stronger version, if an equilibrium type disappears as x approaches x^* , then its selection probability converges to zero. An implication of this is that the distribution function $\tilde{\Pi}^0()$ is continuous. The converse is not necessarily true.]

ASSUMPTION 6C. Assumptions 6A and 6B hold.

Remark 17: Given estimates of the model, Assumptions 6ABC can be falsified for $\theta = \theta^0$ and sampled values of x . If not rejected, the model could be reestimated ...]

Given the additional structure imposed by Assumption 6A we can define the following (new) estimator of $\lambda^0(\tau|\mathbf{x}, \theta^0)$. Let \mathbf{x}^* be a value in X , in the sample or not. And let $\hat{\boldsymbol{\theta}}^0$ be our consistent estimator of $\boldsymbol{\theta}^0$. Define X_T^* as the set of sample values of \mathbf{x} that have the same type of equilibria as \mathbf{x}^* given $\hat{\boldsymbol{\theta}}^0$: i.e., $X_T^* \equiv \{\mathbf{x}_t \text{ is sample: } \mathbf{x}_t \in X^e(\mathbf{x}^*, \hat{\boldsymbol{\theta}}^0)\}$. Given Assumption 6A, we can use the sub-sample $\{\hat{\tau}_t, \mathbf{x}_t : \mathbf{x}_t \in X_T^*\}$ to estimate consistently $\lambda^0(\cdot|\mathbf{x}^*, \hat{\boldsymbol{\theta}}^0)$ for models belonging to Case 3 using a smooth nonparametric estimator. For instance, we can use the following kernel estimator:

$$\tilde{\lambda}^0(\tau|\mathbf{x}^*, \hat{\boldsymbol{\theta}}^0) = \frac{\sum_{t:\mathbf{x}_t \in X_T^*} \hat{\lambda}^0(\tau|\mathbf{x}_t) K\left(\frac{\mathbf{x}_t - \mathbf{x}^*}{b_T}\right)}{\sum_{t:\mathbf{x}_t \in X_T^*} K\left(\frac{\mathbf{x}_t - \mathbf{x}^*}{b_T}\right)} \quad (28)$$

(a) *Prediction of a change in \mathbf{x} .* Let $\mathbf{P}_t(\mathbf{x}_t^*)$ be the vector of players' choice probabilities under the counterfactual scenario. We know that $\mathbf{P}_t(\mathbf{x}_t^*)$ is a vector of random variables with support $\Gamma(\mathbf{x}_t^*, \boldsymbol{\theta}^0)$ (i.e., the space of equilibria associated with $(\mathbf{x}_t^*, \boldsymbol{\theta}^0)$) and probability distribution $\lambda^0(\cdot|\mathbf{x}_t^*)$. Given our consistent estimator $\hat{\boldsymbol{\theta}}^0$, we can obtain the set of equilibria $\Gamma(\mathbf{x}_t^*, \hat{\boldsymbol{\theta}}^0)$, the sub-sample X_T^* , and the estimator $\tilde{\lambda}^0(\cdot|\mathbf{x}_t^*)$ that we have defined above.

In cases (b) and (c) we will need an additional assumption that the only effect of changes in θ (or changes in x outside X) on equilibrium selection is through the set of equilibria that exist.]

(c) *Prediction of a change in $\boldsymbol{\theta}^0$.* Let $\mathbf{P}_t^*(\mathbf{x}_t)$ be the vector of players' choice probabilities under the counterfactual scenario $(\mathbf{x}_t, \boldsymbol{\theta}^*)$. $\mathbf{P}_t^*(\mathbf{x}_t)$ is a vector of random variables with support $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^*)$ and probability distribution $\lambda^*(\cdot|\mathbf{x}_t)$. We can obtain the set of equilibria $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^*)$. Define $X_T^{(\mathbf{x}_t, \boldsymbol{\theta}^*)}$ as the set of sample values of \mathbf{x} that have the same type of equilibria as $(\mathbf{x}_t, \boldsymbol{\theta}^*)$. Suppose that we assume that $\boldsymbol{\theta}^*$ affects the distribution of $\mathbf{P}_t^*(\mathbf{x}_t)$ only through the support $\Gamma(\mathbf{x}_t, \boldsymbol{\theta}^*)$. Under this assumption we can estimate consistently $\lambda^*(\cdot|\mathbf{x}_t)$ using the kernel estimator:

$$\tilde{\lambda}^*(\tau|\mathbf{x}_t) = \frac{\sum_{s:\mathbf{x}_s \in X_T^{(\mathbf{x}_t, \boldsymbol{\theta}^*)}} \hat{\lambda}^0(\tau|\mathbf{x}_s) K\left(\frac{\mathbf{x}_s - \mathbf{x}_t}{b_T}\right)}{\sum_{s:\mathbf{x}_s \in X_T^{(\mathbf{x}_t, \boldsymbol{\theta}^*)}} K\left(\frac{\mathbf{x}_s - \mathbf{x}_t}{b_T}\right)}$$

6 Monte Carlo experiment

6.1 Specification of the experiment

In this section we test the performance of the previous GA using a Monte Carlo experiment. The experiment is based on the model of collusion that we presented in section 2.2. At market t , a (symmetric) equilibrium probability of charging the collusive price is a fixed point of the mapping $P(\mathbf{x}_t) = \Phi(\tilde{\pi}_{0t} + \tilde{\pi}_{1t} P(\mathbf{x}_t))$, where \mathbf{x}_t is a vector of observable characteristics affecting $\tilde{\pi}_{0t}$, $\tilde{\pi}_{1t}$, and possibly the equilibrium selection mechanism. For simplicity, we consider that \mathbf{x}_t contains only variable, say market size, that we represent by x_t . Furthermore, $\tilde{\pi}_{0t}$ and $\tilde{\pi}_{1t}$ are linear functions of market size: $\tilde{\pi}_{0t} = \theta_{00} + \theta_{01}x_t$ and $\tilde{\pi}_{1t} = \theta_{10} + \theta_{11}x_t$. Therefore, the equilibrium mapping is:

$$\Psi(x_t, \boldsymbol{\theta}, P) = \Phi (\theta_{00} + \theta_{01} x_t + \theta_{10} P(x_t) + \theta_{11} x_t P(x_t)) \quad (29)$$

Our choice of the parameter values for the Monte Carlo experiment is based on the following criteria.

First, we choose the parameters α and \bar{v}/\bar{c} to match the average markups reported in empirical papers that study "high-low" pricing in retail markets. Aguirregabiria (1999, Table 2 in page 290) reports average markups of 20.5% under the regular price (i.e., high price) and 5.1% under the promotion price (i.e., low price). The values of the parameters α and \bar{v}/\bar{c} that match these statistics are $\alpha = 6.0$ and $\bar{v}/\bar{c} = 1.4$.¹⁴ Given these parameters and the normalization $\sigma_\varepsilon = 1$, we have that $\theta_2 = -7.31$, $\theta_3 = 6.75$, and the equilibrium mapping is $\Phi(\mu - 7.31 x_t + 6.75 x_t P(x_t))$.

Second, the parameter μ and the distribution of x_t are chosen such that the model has multiple equilibria for any possible value of x_t . There are many parameterizations that satisfy this condition. We have arbitrarily chosen the following parameters: $\mu = 2.0$, and x_t has a uniform distribution in the interval $[0.5, 0.6]$.

Therefore, the DGP in the Monte Carlo experiment can be described in terms of the following parameters

$$\theta_1 = 2.0 \quad ; \quad \theta_2 = -7.31 \quad ; \quad \theta_3 = 6.75 \quad ; \quad x_t \sim U(0.5, 0.6)$$

For different values of x_t in the interval $(0.5, 0.6)$, we have generated the graph of the best response function $\Phi(2.0 - 7.31x_t + 6.75x_tP)$ as a function P . Figures 3A, 3B and 3C present the best response function and the corresponding equilibria for three different values of x_t : 0.50, 0.55, and 0.60.

¹⁴In our model, the formula for the markups under Bertrand and under collusion are $(v_t/c_t - 1)/(2 + \alpha)$ and $(v_t/c_t - 1)/2$, respectively. Therefore, we choose the value of α and \bar{v}/\bar{c} that solves the system of equations $(\bar{v}/\bar{c} - 1)/(2 + \alpha) = 0.051$ and $(\bar{v}/\bar{c} - 1)/2 = 0.205$. The solution is $\alpha = 6.04$ and $\bar{v}/\bar{c} = 1.41$

To complete the specification of the model we have to define an equilibrium selection mechanism. Define the three equilibrium types as "non-collusive" (i.e., the equilibrium with the lowest value of P^*), "collusive" (i.e., the equilibrium with the highest value of P^*), and "mixed". It is clear that the "mixed" equilibrium is always an unstable while the "collusive" and the "non-collusive" equilibria are stable. The equilibrium selection mechanism is such that: (1) the unstable equilibrium is never selected; and (2) when $x_t \leq 0.55$ (i.e., when variable profits under collusion are small), the non-collusive equilibrium is chosen, and the collusive equilibrium is selected when $x_t > 0.55$. Therefore, the probability of collusion conditional on x_t in the population, $P^0(x_t) \equiv \Pr(a_{it} = 1|x_t)$, is:

$$P^0(x_t) = \begin{cases} P_{BB}^*(x_t, \theta^0) & \text{if } x_t \leq 0.55 \\ P_{CC}^*(x_t, \theta^0) & \text{if } x_t > 0.55 \end{cases} \quad (30)$$

where $P_{BB}^*(x_t, \theta^0)$ and $P_{CC}^*(x_t, \theta^0)$ are the non-collusive and collusive equilibrium probabilities, respectively. Figure 4 presents the graph $P^0(x_t)$ as a function of x_t .

In the Monte Carlo experiment we have generated 1,000 random samples with sample size $T = 500$. Let $\{a_{1t}, a_{2t}, x_t : t = 1, 2, \dots, T\}$ be one of these Monte Carlo samples. The values of $\{x_t\}$ are random draws from the distribution of a $U(0.5, 0.6)$ random variable. Given x_t , the binary values a_{1t} and a_{2t} are two independent random draws from a Bernoulli random variable with probability $P^0(x_t)$. For each sample we have obtained two estimators: the Maximum Likelihood estimator (MLE) using a nested fixed-point algorithm in the spirit of Rust (1987); and the NPL-Genetic Algorithm (NPL-GA) estimator.

The MLE estimator is a benchmark to evaluate the performance of other NPL-Genetic Algorithm. There are two reasons why we can implement the MLE in this experiment. First, we know the equilibrium selection mechanism in the DGP and therefore we know which equilibrium we have to select for each trial value of θ and for each value of x_t in the sample.¹⁵ And second, in this simple model with strategic complementarity and symmetric firms it is easy to compute the equilibrium with the lowest probability of collusion (i.e., non-collusive equilibrium) and the equilibrium with the highest probability of collusion (i.e., collusive equilibrium). As we have discussed in section 3.1, this MLE is not feasible in most actual applications because we do not know the equilibrium selection mechanism and because it is not easy to compute all the relevant equilibria. We use a nested fixed-point algorithm based on the BHHH iteration that we have described in equation (??).

¹⁵Strictly speaking an MLE would be implemented with no knowledge of the selection mechanism, and the researcher would need to consider both the collusive and the Bertrand solutions for every observation. As explained earlier this would be much more difficult and in practice it is unlikely that it would make much of a difference in this particular example because the conditional choice probabilities corresponding to both equilibria are very different.

For each trial value of θ and each sample observation x_t we calculate an equilibrium by iterating in the best response probability mapping. If $x_t \leq 0.55$, we initialize the iterations with a probability equal to zero and therefore we converge to the non-collusive equilibrium associated with (x_t, θ) . If $x_t > 0.55$, we initialize the iterations with a probability equal to one and therefore we converge to the collusive equilibrium associated with (x_t, θ) . For some values of (x_t, θ) the equilibrium is unique and therefore the collusive and the non-collusive equilibria are the same.

For the *NPL-Genetic Algorithm* we have considered the following parameters. The initial population has $M = 50$ vectors of probabilities $\{P(x_t) : t = 1, 2, \dots, 500\}$. These initial probabilities are nonparametric kernel estimates of $\Pr(a_{it} = 1|x_t)$ based on 50 different bootstrap samples. The parameter that measures the dependence of selection on fitness is $\varphi = 1$. The probability of mutation is $\gamma = 0.1$. And the parameter that represents the magnitude of the mutation, δ , is equal to 0.05.

The computer program that implements this Monte Carlo experiment has been written in GAUSS language and it can be downloaded from Victor Aguirregabiria's web page.

6.2 Results

Tables 1 and 2 present the results of the Monte Carlo experiment. First, it is important to notice that for all 7,000 applications of the NPL method in this experiment, as well as for others not reported here, the NPL always converged. Furthermore, in a large proportion of cases the NPL estimator was numerically equivalent to the MLE. However, in contrast with results reported by Aguirregabiria and Mira (2002 and 2007) for dynamic models, we often converged to different NPL fixed points if we initialized the algorithm with different probabilities.

Table 1 compares the ability of the three NPL methods to obtain estimates which coincided with the MLE. The Single-NPL reaches the MLE in 764 of the 1,000 Monte Carlo replications. The Best-of-Five-NPL performs only slightly better than the Single-NPL. It reaches the MLE only 23 times more than the Single-NPL. However, the NPL-GA performs much better and provides the MLE for almost every sample, i.e., 97.1% of the times. This result confirms that the GA is an efficient method to obtain the global optimum over the NPL fixed-points.

Table 2 reports finite sample biases, standard deviations and mean square errors of the different estimators. These statistics have been calculated using the 1,000 Monte Carlo replications. According to these results, the finite sample distribution of the NPL-GA estimator is very close to the of the MLE and much closer to the true parameters than the one the Single-NPL or the Best-of-Five-NPL. Notice also that the Best-of-Five-NPL estimator is very similar in terms of bias and

variance to the Single-NPL. The experiment also confirms the very poor finite sample properties of the two-stage PML, a result that has been reported in previous Monte Carlo studies by Hotz, Miller, Sanders and Smith (1994), and Aguirregabiria and Mira (2002 and 2006).

7 Conclusions

This paper deals with the estimation of discrete games of incomplete information when the data come from multiple equilibria. We propose a new algorithm to obtain consistent estimates of structural parameters in this class of models. The algorithm is based on the nested pseudo likelihood (NPL) method proposed by Aguirregabiria and Mira (2002 and 2007). The NPL avoids the repeated computation of all the equilibria of the game for each trial value of the structural parameters. However, the NPL may have several fixed-points and in that case only the fixed point with the highest value of the likelihood function gives a consistent estimator. We conjecture that multiplicity of NPL fixed points is likely to occur when the strategic game has multiple equilibria and different equilibria are present in the data. A possible approach to obtain the consistent estimator to initialize the NPL algorithm with a number $M > 1$ of different vectors of probabilities, obtain M fixed-points of the NPL, and then choose the one with the highest value of the likelihood function. The main limitation of this approach is that we may need a large number of initial values to guarantee that this estimator is consistent. Instead, we propose here to combine the NPL with a genetic algorithm (GA). This hybrid algorithm starts also with a set of M different probabilities but it does not apply the NPL separately to each of these M probabilities. Instead, at each iteration the GA combines the information of M candidates and applies the concept of *selection of the fittest* to generate a new set of M candidates. This feature of the algorithm implies a more global search for the consistent NPL fixed point.

We have illustrated the application of this algorithm in the context of a game of price competition and collusion. Though the model is relatively simple, with only two players, one exogenous explanatory variables and three equilibria, the standard approach to compute the MLE is practically unfeasible. We show that NPL, without GA, does reach the consistent NPL fixed point when we initialize it with probabilities which are not too far away from the MLE estimates. However, when the initial probabilities are not good enough, the NPL converges to an NPL fixed-point which is quite far from the MLE. In contrast, the NPL-GA works extremely well in this example.

For the sake of simplicity, we have presented our algorithm in the context of a relatively simple model: a static game where all common knowledge variables are observable to the researcher.

However, it is possible to use the results in Aguirregabiria and Mira (2007) to apply the NPL-GA to static and dynamic games with state variables that are unobservable to the econometrician.

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Table 1
Proportion of Monte Carlo Simulations in which
NPL algorithms reach the MLE

<i>Estimator / Algorithm</i>	<i>Percentage</i>
<i>Single NPL</i>	76.4 %
<i>Best of 5 NPLs</i>	78.7 %
<i>NPL-Genetic Algorithm</i>	97.1 %

Table 2
Bias, Std. Deviation, and Mean Square Error

BIAS				
<i>Estimator / Algorithm</i>	<i>Parameters</i>			
	θ_{00}	θ_{01}	θ_{10}	θ_{11}
<i>Two-stage PML</i>	0.190	-6.110	5.736	0.319
<i>Single NPL</i>	-0.107	0.424	-0.002	-0.316
<i>Best of 5 NPLs</i>	-0.085	0.366	-0.002	-0.311
<i>NPL-Genetic Algorithm</i>	-0.046	0.107	0.103	-0.169
<i>MLE</i>	-0.044	0.102	0.104	-0.156
STANDARD DEVIATION				
<i>Estimator / Algorithm</i>	<i>Parameters</i>			
	θ_{00}	θ_{01}	θ_{10}	θ_{11}
<i>Two-stage PML</i>	0.234	0.981	0.638	0.578
<i>Single NPL</i>	0.240	0.579	0.319	0.592
<i>Best of 5 NPLs</i>	0.238	0.576	0.319	0.590
<i>NPL-Genetic Algorithm</i>	0.201	0.506	0.301	0.536
<i>MLE</i>	0.201	0.506	0.301	0.536
SQUARE ROOT OF MSE				
<i>Estimator / Algorithm</i>	<i>Parameters</i>			
	θ_{00}	θ_{01}	θ_{10}	θ_{11}
<i>Two-stage PML</i>	0.301	6.188	5.771	0.660
<i>Single NPL</i>	0.263	0.718	0.319	0.671
<i>Best of 5 NPLs</i>	0.253	0.682	0.319	0.676
<i>NPL-Genetic Algorithm</i>	0.206	0.517	0.318	0.562
<i>MLE</i>	0.206	0.516	0.318	0.558

Figure 1. Best Response Probability Function and Multiple Equilibria

Best response function: $\Psi(P) = \Phi(-1.80 + 3.55P)$

Equilibria: Non collusive ($P^* = 0.054$); Collusive ($P^* = 0.937$); Mixed ($P^* = 0.521$)

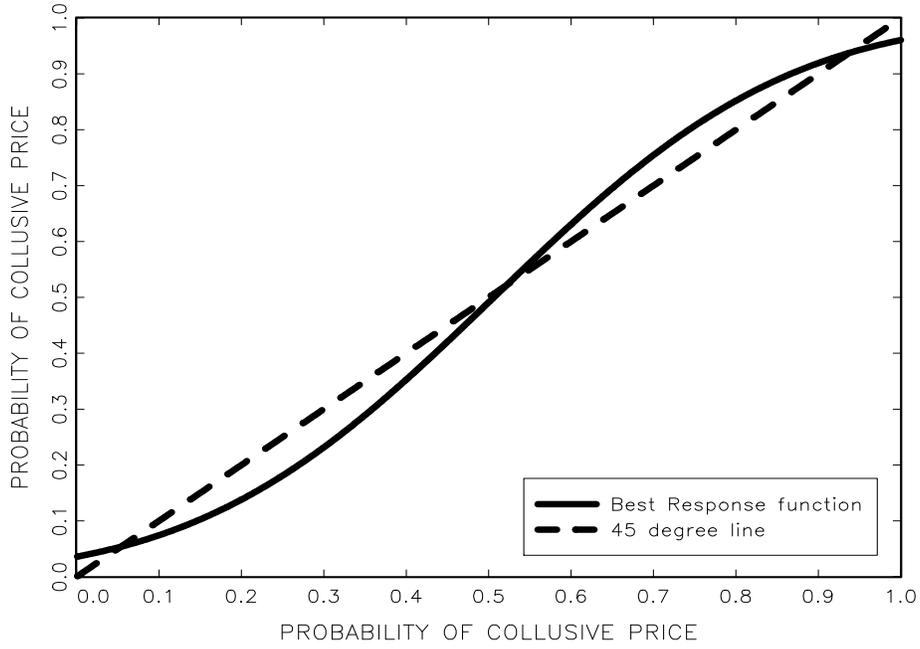
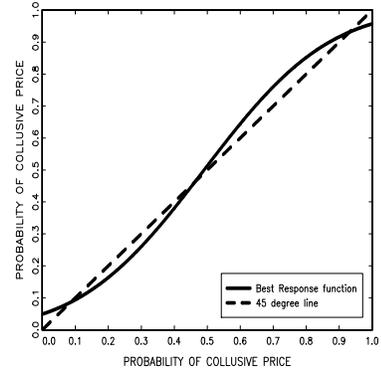
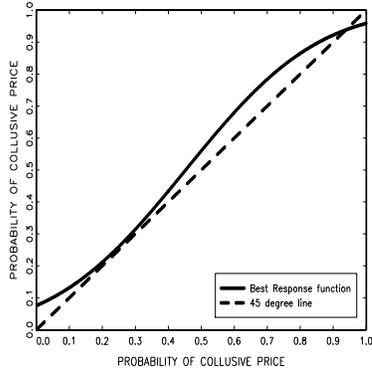


FIGURE 2. Best Response Function $\Phi(2.0 - 7.31 x_t + 6.75 x_t P)$ for different values of x_t

(A) $x_t = 0.47$. Equilibrium: 0.938

(B) $x_t = 0.50$. Equilibria: 0.086; 0.462; 0.932



(C) $x_t = 0.55$. Equilibria: 0.028; 0.643; 0.917

(D) $x_t = 0.66$. Equilibrium: 0.001

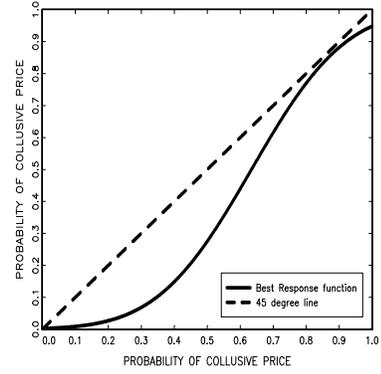
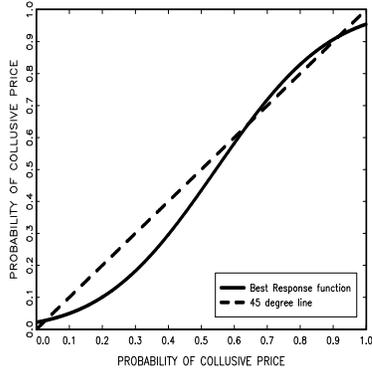


FIGURE 3. Equilibrium Probability in the DGP of the Monte Carlo Experiment

