NBER WORKING PAPER SERIES

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Working Paper 22157
http://www.nber.org/papers/w22157

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 2016

We thank Todd Keister and Guillermo Ordoñez for useful comments. The views expressed in this paper are those of the authors and do not necessarily reflect those of the Federal Reserve Bank of Philadelphia, the Federal Reserve System, or the National Bureau of Economic Research. Lastly, we also thank the NSF for financial support.

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NBER Working Paper No. 22157
April 2016
JEL No. E40,E42,E52

ABSTRACT

Can competition among privately issued fiat currencies such as Bitcoin or Ethereum work? Only sometimes. To show this, we build a model of competition among privately issued fiat currencies. We modify the current workhorse of monetary economics, the Lagos-Wright environment, by including entrepreneurs who can issue their own fiat currencies in order to maximize their utility. Otherwise, the model is standard. We show that there exists an equilibrium in which price stability is consistent with competing private monies, but also that there exists a continuum of equilibrium trajectories with the property that the value of private currencies monotonically converges to zero. These latter equilibria disappear, however, when we introduce productive capital. We also investigate the properties of hybrid monetary arrangements with private and government monies, of automata issuing money, and the role of network effects.

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1 Introduction

Can competition among privately issued fiat currencies work? The sudden appearance of Bitcoin, Ethereum, and other cryptocurrencies has triggered a wave of public interest in privately issued monies.[1] A similar interest in the topic has not been seen, perhaps, since the vivid polemics associated with the demise of free banking in the English-speaking world in the middle of the 19th century (White, 1995). Somewhat surprisingly, this interest has not translated, so far, into much research within monetary economics. Most papers analyzing the cryptocurrency phenomenon have either been descriptive (Böhme, Christin, Edelman, and Moore, 2015) or have dealt with governance and regulatory concerns from a legal perspective (Chuen, 2015).[2] In comparison, there has been much research related to the computer science aspects of the phenomenon (Narayanan, Bonneau, Felten, Miller, and Goldfeder, 2016).

This situation is unfortunate. Without a theoretical understanding of how currency competition works, we cannot answer a long list of positive and normative questions. Among the positive questions: Will a system of private money deliver price stability? Will one currency drive all others from the market? Or will several of these currencies coexist along the equilibrium path? Will the market provide the socially optimum amount of money? Can private monies and a government-issued money compete? Do private monies require a commodity backing? Can a unit of account be separated from a medium of exchange? Among the normative questions: Should governments prevent the circulation of private monies? Should governments treat private monies as currencies or as any other regular property? Should the private monies be taxed? Even more radically, now that cryptocurrencies are technically feasible, should we revisit Friedman and Schwartz's (1986) celebrated arguments justifying the role of governments as money issuers? There are even questions relevant for entrepreneurs: What is the best strategy to jump start the circulation of a currency? How do you maximize the seigniorage that comes from it?

To address some of these questions, we build a model of competition among privately issued fiat currencies. We modify the workhorse of monetary economics, the Lagos and Wright (2005) (LW) environment, by including entrepreneurs who can issue their own fiat currencies in order to maximize their utility. Otherwise, the model is standard. Following LW has two important advantages. First, since the model is particularly amenable to analysis, we can derive many insights about currency competition. Second, the use of the LW framework

[1] As of April 3, 2016, besides Bitcoin, three other cryptocurrencies (Etherum, Ripple, and Litecoin) have market capitalizations over \$100 million and another eleven between \$10 and \$100 million. Updated numbers are reported by `https://coinmarketcap.com/`. Following convention, we will use Bitcoin, with a capital B, to refer to the whole payment environment, and bitcoin, with a lower case b, to denote the currency units of the payment system. See Antonopoulos (2015) for a technical introduction to Bitcoin.

[2] Some exceptions are Chiu and Wong (2014) and Hendrickson, Hogan, and Luther (2016).

makes our many new results easy to compare with previous findings in the literature.

We highlight six of our results. First, we show the existence of a stationary equilibrium with the property that the value of all privately issued currencies is constant over time. In other words, there exists an equilibrium in which price stability is consistent with competing private monies. This equilibrium captures Hayek's (1999) vision of a system of private monies competing among themselves to provide a stable means of exchange.

Second, there exists a continuum of equilibrium trajectories with the property that the value of private currencies monotonically converges to zero. This result is intriguing because it shows that the self-fulfilling inflationary episodes highlighted by Obstfeld and Rogoff (1983) and Lagos and Wright (2003) in economies with government-issued money and a money-growth rule are not an inherent feature of public monies. Private monies are also subject to self-fulfilling inflationary episodes, even when they are issued by profit-maximizing, long-lived entrepreneurs who care about the future value of their monies.[3]

Third, we show that although the equilibrium with stable currencies Pareto dominates all other equilibria in which the value of private currencies declines over time, a purely private monetary system does not provide the socially optimum quantity of money. Private money does not solve the trading frictions at the core of LW and, more generally, of essential models of money (Wallace, 2001). In a well-defined sense, the market fails at providing the right amount of money in ways that it does not fail at providing the right amount of other goods and services.

Fourth, we characterize asymmetric equilibria in which one private currency drives the other currencies out of the market. In these equilibria, a single entrepreneur becomes the sole issuer of currency in the economy (a possibility conjectured by Hayek, 1999). Which currency dominates is indeterminate. However, the threat of entry constrains the behavior of this surviving entrepreneur. Market participants understand the discipline imposed by free entry, even though everybody sees a single private agent supplying all currency in the economy. As in the symmetric class, these equilibria may imply a stable or a declining value of money.

Fifth, when we introduce a government competing with private money, we recover the set of equilibrium allocations characterized by LW as a particular case in our analysis. Also, we show that a hybrid monetary arrangement with constant prices requires the government to follow a constant money supply policy. Because the problem of achieving a unique efficient equilibrium remains unresolved under a money-growth rule, we investigate the extent to

[3] Tullock (1975) suggested that competition among monies would prevent inflation (although he dismissed this possibility due to the short planning horizon of most governments, which prevents them from valuing the future income streams from maintaining a stable currency). Our analysis is a counterexample to Tullock's suggestion.

which the presence of government money can simultaneously promote stability and efficiency under an alternative policy rule. In particular, we study the properties of a policy rule that pegs the real value of government money. Under this alternative regime, the properties of the dynamic system substantially change as we vary the policy parameter determining the target for the real value of government money. We show that it is possible to implement an efficient allocation as the unique global equilibrium, but this requires driving private money out of the economy.

Sixth, we study the situation where the entrepreneurs use productive capital. We can think about this case as an entrepreneur, for example, issuing money to be used for purchases of goods on her internet platform. The presence of productive capital fundamentally changes the properties of the dynamic system describing the evolution of the real value of private currencies. Autarky is no longer a steady state and, as a result, there can be no equilibrium with the property that the value of private currencies converges to zero. Furthermore, it is possible to obtain an allocation that is arbitrarily close to the efficient one as the unique equilibrium provided capital is sufficiently productive. This allocation vindicates Hayek's proposal. It also links our research to the literature on the provision of liquidity by productive firms (Holmström and Tirole, 2011, and Dang, Gorton, Holmström, and Ordoñez, 2014).

We have further results involving automata issuing currency (inspired by the software protocol behind Bitcoin and other cryptocurrencies) and the possible role of network effects on currency circulation, but in the interest of space, we delay their discussion until later in the paper.

The careful reader might have noticed that we have used the word "entrepreneur" and not the more common "banker" to denote the issuers of private money. We believe this linguistic turn is important. Our model highlights how the issuing of a private currency is logically separated from banking. Both tasks were historically linked for logistical reasons: banks had a central location in the network of payments that made it easy for them to introduce currency in circulation. We will argue that the internet has broken the logistical barrier. The issuing of bitcoins, for instance, is done through a proof-of-work system that is independent of any banking activity (or at least of banking understood as the issuing and handling of deposits and credit).[4]

This previous explanation may also address a second concern of this enlightened reader: What are the differences between private monies issued in the past by banks (such as during the Scottish free banking experience between 1716 and 1845) and modern cryptocurrencies? As we mentioned, a first difference is the distribution process, which is now much wider and

[4]Similarly, some of the community currencies that have achieved a degree of success do not depend on banks backing or issuing them (see Greco, 2001).

more dispersed than before. A second difference is the possibility, through the protocols embodied in the software, of having quasi-commitment devices regarding how much money will be issued. The most famous of these devices is the 21 million bitcoins that will eventually be released.[5] We will discuss how to incorporate an automaton issuer of private money into our model to analyze this property of cryptocurrencies. Third, cryptographic techniques, such as those described in von zur Gathen (2015), make it harder to counterfeit digital currencies than traditional physical monies, minimizing a historical obstacle that private monies faced (Gorton, 1989). Fourth, most (but not all) historical cases of private money were of commodity-backed currencies, while most cryptocurrencies are fully fiduciary.

Note that we ignore all issues related to the payment structure of cryptocurrencies, such as the blockchain, the emergence of consensus on a network, or the possibilities of Goldfinger attacks (see Narayanan, Bonneau, Felten, Miller, and Goldfeder 2016 for more details). While these topics are of foremost importance, they require a specific modeling strategy that falls far from the one we follow in this paper and that we feel is more suited to the macroeconomic questions we focus on.

We are not the first to study private money. The literature is large and has approached the topic from many angles. At the risk of being highly selective, we build on the tradition of Cavalcanti, Erosa, and Temzelides (1999, 2005), Cavalcanti and Wallace (1999), Williamson (1999), Berentsen (2006), and Monnet (2006).[6] As we mentioned above, our emphasis is different from that in these previous papers, as we depart from modeling banks and their reserve management problem. Our entrepreneurs issue fiat money that cannot be redeemed for any other asset. Our characterization captures the technical features of most cryptocurrencies, which are purely fiat (in fact, since the cryptocurrencies cannot be used to pay taxes in most sovereigns, they are more "fiat" than public monies). Our partial vindication of Hayek shares many commonalities with Martin and Schreft (2006), who were the first to prove the existence of equilibria for environments in which outside money is issued competitively. Lastly, we cannot forget Klein (1974) and his application of industrial organization insights to competition among monies.

The rest of the paper is organized as follows. Section 2 presents our model. Sections 3 and 4 analyze the existence and properties of the equilibria of the model. Section 5 introduces a government issuing public money. Section 6 considers digital currencies issued by automata. Section 7 extends the model with productive capital. Section 8 incorporates network effects. Section 9 offers some concluding remarks.

[5] We use the term "quasi-commitment" because the software code can be changed by sufficient consensus in the network. This possibility is not appreciated enough in the discussion about open-source cryptocurrencies. For the importance of commitment, see Araujo and Camargo (2008).

[6] See, as well, from a very different methodological perspective, Selgin and White (1994).

2 Model

The economy consists of a large number of buyers and sellers, with a $[0,1]$-continuum of each type, and a countable infinity of entrepreneurs. Each period is divided into two subperiods. In the first subperiod, all types interact in a centralized market (CM) where a perishable good, referred to as the CM good, is produced and consumed. Buyers and sellers can produce the CM good by using a linear technology that requires effort as input. All agents want to consume the CM good.

In the second subperiod, buyers and sellers interact in a decentralized market (DM) characterized by pairwise meetings, with entrepreneurs remaining idle. In particular, a buyer is randomly matched with a seller with probability $\sigma \in (0,1)$ and vice versa. In the DM, buyers want to consume, but cannot produce, whereas sellers are able to produce, but do not want to consume. A seller is able to produce a perishable good, referred to as the DM good, using a divisible technology that delivers one unit of the good for each unit of effort he exerts. An entrepreneur is neither a producer nor a consumer of the DM good.

Entrepreneurs are endowed with a technology to create tokens, which can take either a physical or an electronic form. The essential feature of the tokens is that their authenticity can be publicly verified at zero cost (for example, thanks to the application of cryptography techniques), so that counterfeiting will not be an issue. As we will see, this technology will permit entrepreneurs to issue tokens that can circulate as a medium of exchange.

Let us now explicitly describe preferences. Start with a typical buyer. Let $x_t^b \in \mathbb{R}$ denote net consumption of the CM good, and let $q_t \in \mathbb{R}_+$ denote consumption of the DM good. A buyer's preferences are represented by

$$U^b\left(x_t^b, q_t\right) = x_t^b + u\left(q_t\right).$$

The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, increasing, and strictly concave, with $u'(0) = \infty$ and $u(0) = 0$. Consider now a typical seller. Let $x_t^s \in \mathbb{R}$ denote net consumption of the CM good, and let $n_t \in \mathbb{R}_+$ denote the seller's effort level to produce the DM good. A seller's preferences are represented by

$$U^s\left(x_t^s, n_t\right) = x_t^s - w\left(n_t\right).$$

Assume that $w : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, increasing, and weakly convex, with $w(0) = 0$. Finally, let $x_t^i \in \mathbb{R}_+$ denote entrepreneur i's consumption of the CM good, with

$i \in \{1, 2, 3, ...\}$. Entrepreneur i has preferences represented by

$$\sum_{t=0}^{\infty} \beta^t x_t^i.$$

The discount factor $\beta \in (0, 1)$ is common across all types.

Throughout the analysis, we assume that buyers and sellers are anonymous (i.e., their identities are unknown and their trading histories are privately observable), which precludes credit in the decentralized market. In contrast, an entrepreneur's trading history can be publicly observable (i.e., an entrepreneur is endowed with a record-keeping technology that permits him to reveal costlessly his trading history). As we will argue below, public knowledge of an entrepreneur's trading history will permit the circulation of private currencies.

3 Perfect Competition

Because the meetings in the DM are anonymous, there is no scope for trading future promises in this market. As a result, a medium of exchange is essential to achieve allocations that we could not achieve without it. In a typical monetary model, a medium of exchange is supplied in the form of a government issued fiat currency, with the government following a monetary policy rule (e.g., money-growth rule). Also, all agents in the economy can observe the money supply at each date. These features allow agents to form beliefs about the exchange value of money in the current and future periods.

We start our analysis by considering the possibility of a purely private monetary arrangement. In particular, we focus on a system in which entrepreneurs issue intrinsically worthless tokens that circulate as a medium of exchange, attaining a strictly positive value. These privately issued currencies are not associated with any promise to exchange them for goods at some future date. Because the currency issued by an entrepreneur is nonfalsifiable and an entrepreneur's trading history is publicly observable, all agents can verify the total amount put into circulation by any individual issuer.

Profit maximization will determine the money supply in the economy. Since all agents know that an entrepreneur enters the currency-issuing business to maximize profits, one can describe individual behavior by solving the entrepreneur's optimization problem in the market for private currencies. These predictions about individual behavior allow agents to form beliefs regarding the exchange value of private currencies, given the observability of individual issuances. In other words, profit maximization in a private money arrangement serves the same purpose as the monetary policy rule in the case of a government monopoly on currency issue. As a result, it is possible to conceive a system in which privately issued currencies can attain a positive exchange value.

In our analysis, a monetary unit comes in several brands issued by competing entrepreneurs. Let $\phi_t^i \in \mathbb{R}_+$ denote the value of a "dollar" issued by entrepreneur $i \in \{1, 2, 3, ...\}$ in terms of the CM good. Because there is free entry in the currency-issuing business and the operational cost for an entrepreneur is zero, the number of entrepreneurs in the market will be indeterminate. Throughout the paper, we restrict attention to equilibria in which a fixed number $N \in \{1, 2, 3, ...\}$ of entrepreneurs enter the market for private currencies, with $\boldsymbol{\phi}_t = \left(\phi_t^1, ..., \phi_t^N\right) \in \mathbb{R}_+^N$ denoting the vector of real prices. We refer to those who enter the currency-issuing business as active entrepreneurs.[7]

3.1 Buyer

Let us start by describing the portfolio problem of a typical buyer. Let $W^b(\mathbf{m}, t)$ denote the value function for a buyer holding a portfolio $\mathbf{m} = \left(m^1, ..., m^N\right) \in \mathbb{R}_+^N$ of privately issued currencies in the CM, and let $V^b(\mathbf{m}, t)$ denote the value function in the DM. The Bellman equation can be written as

$$W^b(\mathbf{m}, t) = \max_{(x, \hat{\mathbf{m}}) \in \mathbb{R} \times \mathbb{R}_+^N} \left[x + V^b(\hat{\mathbf{m}}, t)\right]$$

subject to the budget constraint

$$\boldsymbol{\phi}_t \cdot \hat{\mathbf{m}} + x = \boldsymbol{\phi}_t \cdot \mathbf{m}.$$

The vector $\hat{\mathbf{m}} \in \mathbb{R}_+^N$ describes the buyer's monetary portfolio after trading in the CM, and $x \in \mathbb{R}$ denotes net consumption of the CM good. The value function $W^b(\mathbf{m}, t)$ can be written as

$$W^b(\mathbf{m}, t) = \boldsymbol{\phi}_t \cdot \mathbf{m} + W^b(\mathbf{0}, t),$$

with the intercept given by

$$W^b(\mathbf{0}, t) = \max_{\hat{\mathbf{m}} \in \mathbb{R}_+^N} \left[-\boldsymbol{\phi}_t \cdot \hat{\mathbf{m}} + V^b(\hat{\mathbf{m}}, t)\right]. \tag{1}$$

The value for a buyer holding a portfolio $\mathbf{m}$ in the DM is given by

$$V^b(\mathbf{m}, t) = \sigma\left[u(q(\mathbf{m}, t)) + \beta W^b(\mathbf{m} - \mathbf{d}(\mathbf{m}, t), t+1)\right] + (1 - \sigma)\beta W^b(\mathbf{m}, t+1), \tag{2}$$

[7]It would be easy to extend the model to situations where the operational costs of running a currency are positive, which will pin down the number of entrepreneurs in equilibrium.

with $\{q(\mathbf{m},t), \mathbf{d}(\mathbf{m},t)\}$ representing the terms of trade. Specifically, $q(\mathbf{m},t) \in \mathbb{R}_+$ denotes production of the DM good and $\mathbf{d}(\mathbf{m},t) = \left(d^1(\mathbf{m},t), ..., d^N(\mathbf{m},t)\right) \in \mathbb{R}_+^N$ denotes the vector of currencies the buyer transfers to the seller. Because $W^b(\mathbf{m},t) = \boldsymbol{\phi}_t \cdot \mathbf{m} + W^b(\mathbf{0},t)$, we can rewrite (2) as

$$V^b(\mathbf{m},t) = \sigma\left[u(q(\mathbf{m},t)) - \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{d}(\mathbf{m},t)\right] + \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} + \beta W^b(\mathbf{0},t+1).$$

The standard approach in the search-theoretic literature determines the terms of trade by Nash bargaining. Lagos and Wright (2005) demonstrate that Nash bargaining can result in a holdup problem, leading to inefficient trading activity in the DM. Aruoba, Rocheteau, and Waller (2007) consider alternative axiomatic bargaining solutions and conclude that the properties of these solutions matter for the efficiency of monetary equilibrium. To avoid inefficiencies arising from the choice of the bargaining protocol, which may complicate the interpretation of the main results in the paper, we assume the buyer makes a take-it-or-leave-it offer to the seller.[8]

With take-it-or-leave-it offers, the buyer chooses the amount of the DM good, represented by $q \in \mathbb{R}_+$, the seller is supposed to produce and the vector of currencies, represented by $\mathbf{d} = \left(d^1, ..., d^N\right) \in \mathbb{R}_+^N$, to be transferred to the seller to maximize expected surplus in the DM. Formally, the terms of trade $(q, \mathbf{d})$ are determined by solving

$$\max_{(q,\mathbf{d}) \in \mathbb{R}_+^{N+1}} \left[u(q) - \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{d}\right]$$

subject to the seller's participation constraint

$$-w(q) + \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{d} \geq 0$$

and the buyer's liquidity constraint

$$\mathbf{d} \leq \mathbf{m}. \tag{3}$$

Let $q^* \in \mathbb{R}_+$ denote the quantity satisfying $u'(q^*) = w'(q^*)$ (i.e., q^* is the surplus-maximizing quantity). The solution to the buyer's problem is given by

$$q(\mathbf{m},t) = \begin{cases} w^{-1}\left(\beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m}\right) \text{ if } \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} < \beta^{-1} w(q^*), \\ q^* \text{ if } \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \geq \beta^{-1} w(q^*), \end{cases}$$

[8] Lagos and Wright (2005) show that, with take-it-or-leave-it offers by the buyer, it is possible to achieve the socially efficient allocation provided the government implements the Friedman rule.

$$\boldsymbol{\phi}_{t+1} \cdot \mathbf{d}(\mathbf{m},t) = \begin{cases} \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \text{ if } \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} < \beta^{-1} w(q^*), \\ \beta^{-1} w(q^*) \text{ if } \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \geq \beta^{-1} w(q^*). \end{cases}$$

Note that the solution to the bargaining problem allows us to characterize real expenditures in the DM, given by $\boldsymbol{\phi}_{t+1} \cdot \mathbf{d}(\mathbf{m},t)$, as a function of the buyer's portfolio, with the composition of the basket of currencies transferred to the seller remaining indeterminate.

The indeterminacy of the portfolio of currencies transferred to the seller in the DM is reminiscent of Kareken and Wallace (1981). These authors have established that, in the absence of portfolio restrictions and barriers to trade, the exchange rate between two currencies is indeterminate in a flexible-price economy. In our framework, a similar result holds with respect to privately issued currencies, given the absence of transaction costs when dealing with different currencies in the CM.

Given the previously derived solution to the bargaining problem, the value function $V(\mathbf{m}, t)$ takes the form

$$V^b(\mathbf{m}, t) = \sigma \left[u\left(w^{-1}\left(\beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \right) \right) - \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \right] + \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} + \beta W^b(\mathbf{0}, t+1)$$

if $\boldsymbol{\phi}_{t+1} \cdot \mathbf{m} < \beta^{-1} w(q^*)$ and the form

$$V^b(\mathbf{m}, t) = \sigma \left[u(q^*) - w(q^*) \right] + \beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} + \beta W^b(\mathbf{0}, t+1)$$

if $\boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \geq \beta^{-1} w(q^*)$. Then, the first-order conditions for the optimal portfolio choice are

$$-\phi_t^i + V_i^b(\hat{\mathbf{m}}, t) \leq 0,$$

with equality if $\hat{m}^i > 0$. If $\phi_{t+1}^i / \phi_t^i < \beta^{-1}$ for all i, then the optimal portfolio choice satisfies

$$\sigma \frac{u'(q(\mathbf{m}, t))}{w'(q(\mathbf{m}, t))} + 1 - \sigma = \frac{\phi_t^i}{\beta \phi_{t+1}^i} \tag{4}$$

for each brand i, with $q(\mathbf{m}, t) = w^{-1}\left(\beta \times \boldsymbol{\phi}_{t+1} \cdot \mathbf{m} \right)$. These conditions imply that, in an equilibrium with multiple currencies, the expected return on money must be equalized across all valued currencies (i.e., all currencies for which $\hat{m}^i > 0$). In the absence of portfolio restrictions, an agent is willing to hold in portfolio two alternative currencies only if they yield the same rate of return, given that these assets are equally useful in facilitating exchange in the DM.

3.2 Seller

Let $W^s(\mathbf{m}, t)$ denote the value function for a seller holding a portfolio $\mathbf{m} \in \mathbb{R}_+^N$ of privately issued currencies in the CM, and let $V^s(\mathbf{m}, t)$ denote the value function in the DM. The Bellman equation can be written as

$$W^s(\mathbf{m}, t) = \max_{(x, \hat{\mathbf{m}}) \in \mathbb{R} \times \mathbb{R}_+^N} \left[x + V^s(\hat{\mathbf{m}}, t) \right]$$

subject to the budget constraint

$$\boldsymbol{\phi}_t \cdot \hat{\mathbf{m}} + x = \boldsymbol{\phi}_t \cdot \mathbf{m}.$$

The value $V^s(\mathbf{m}, t)$ satisfies

$$V^s(\mathbf{m}, t) = \sigma \left[-w(q(\tilde{\mathbf{m}}, t)) + \beta W^s(\mathbf{m} + \mathbf{d}(\tilde{\mathbf{m}}, t), t+1) \right] + (1-\sigma) \beta W^s(\mathbf{m}, t+1).$$

Here the vector $\tilde{\mathbf{m}} \in \mathbb{R}_+^N$ denotes the buyer's monetary portfolio (i.e., the seller's trading partner in the DM). In the LW framework, the terms of trade in the decentralized market only depend on the buyer's monetary portfolio, which implies that monetary assets do not bring any additional benefit to the seller in the decentralized market. As a result, the seller optimally chooses not to hold monetary assets across periods when $\phi_{t+1}^i / \phi_t^i < \beta^{-1}$ for all i.

3.3 Entrepreneur

Now we describe the entrepreneur's portfolio problem. The entrepreneur's budget constraint is given by

$$x_t^i + \phi_t^i M_{t-1}^i + \sum\nolimits_{j \neq i} \phi_t^j M_t^j = \phi_t^i M_t^i + \sum\nolimits_{j \neq i} \phi_t^j M_{t-1}^j$$

at each date $t \geq 0$. Here $M_t^i \in \mathbb{R}_+$ denotes entrepreneur i's circulation in the current period and $M_t^j \in \mathbb{R}_+$ denotes entrepreneur i's holdings of currency issued by entrepreneur $j \neq i$. Note that $M_t^i - M_{t-1}^i$ gives entrepreneur i's net circulation. The privately issued currencies are not associated with an explicit promise by the issuers to exchange them for goods at a future date. But an entrepreneur can make purchases and sales in the CM to adjust the circulation of his own brand if this is consistent with profit maximization.

If $\phi_{t+1}^j / \phi_t^j < \beta^{-1}$ for all $j \neq i$, the entrepreneur i chooses not to hold other currencies across periods so that $M_t^j = 0$ for all $j \neq i$. Thus, we can rewrite the budget constraint as

$$x_t^i + \phi_t^i M_{t-1}^i = \phi_t^i M_t^i$$

at each date $t \geq 0$. In this case, the entrepreneur's lifetime utility is given by

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \phi_\tau^i \left(M_\tau^i - M_{\tau-1}^i \right).$$

The entrepreneur's lifetime utility derives from the currency-issuing business, given that seigniorage is the only source of income (a privilege derived from access to the record-keeping technologies). Consider a sequence of real prices $\left\{ \phi_t^i \right\}_{t=0}^{\infty}$ with $\phi_t^i > 0$ for all $t \geq 0$. Then, we must have

$$M_t^i \geq M_{t-1}^i$$

in every period t because consumption must be nonnegative. When $\phi_t^i > 0$ for all $t \geq 0$, it follows that entrepreneur i's money supply must be weakly increasing.

As previously mentioned, we assume free entry in the market for private currencies. In the absence of operational costs, the free-entry condition implies

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \phi_\tau^i \left(M_\tau^i - M_{\tau-1}^i \right) = 0$$

at all dates $t \geq 0$. This condition, together with nonnegative consumption $x_t^i \geq 0$ at all dates, implies

$$\phi_t^i \left(M_t^i - M_{t-1}^i \right) = 0$$

at all dates $t \geq 0$. If $\phi_t^i > 0$ holds at all dates $t \geq 0$, then the only feasible choice for entrepreneur i is $M_t^i = M_{t-1}^i$ in every period. In other words, entrepreneur i must choose $M_t^i = M^i$ at all dates $t \geq 0$, given some initial state $M_{-1}^i = M^i > 0$. Along an equilibrium trajectory, the nominal supply of each brand remains constant over time to make it consistent with free entry. The market discipline imposed by free entry takes the form of a constant nominal supply in the absence of operational costs. However, as we will see, the real value of a unit of money issued by entrepreneur i can vary over time.

3.4 Equilibrium

We say that currency i is valued in equilibrium if $\phi_t^i > 0$ for all $t \geq 0$. If currency i is valued, then we denote its real return by $\gamma_{t+1}^i \equiv \phi_{t+1}^i / \phi_t^i$. In an equilibrium with the property that currencies i and j coexist as a medium of exchange, condition (4) implies $\gamma_{t+1}^{i}=\gamma_{t+1}^{j}$ at all dates. Let γ_{t+1} denote the common rate of return in an equilibrium with multiple currencies (i.e., the common return across all currencies that are valued in equilibrium). If $\gamma_{t+1}<\beta^{-1}$, then the liquidity constraint (3) is binding. Let $q_t\in\mathbb{R}_+$ denote the quantity traded in the DM in period t. We can rewrite the first-order conditions (4) as a single condition

$$\sigma\frac{u'\left(q_t\right)}{w'\left(q_t\right)}+1-\sigma=\frac{1}{\beta\gamma_{t+1}}. \tag{5}$$

This condition determines production in a bilateral meeting as a function of the expected return on money. Thus, we can use (5) to implicitly define $q_t=q\left(\gamma_{t+1}\right)$, with $q'\left(\gamma\right)>0$ for all $\gamma>0$. Thus, a higher return on money results in a larger amount produced in the DM. The demand for real balances is given by

$$z\left(\gamma_{t+1}\right)\equiv\frac{w\left(q\left(\gamma_{t+1}\right)\right)}{\beta\gamma_{t+1}}. \tag{6}$$

As we will see, the demand for real balances can be either increasing or decreasing in the rate of return γ_{t+1}, depending on the specification of preferences and technologies.

In equilibrium, the market-clearing condition

$$\sum_{i=1}^{N}\phi_t^iM_t^i=z\left(\gamma_{t+1}\right) \tag{7}$$

must hold in all periods $t\geq 0$. The only asset market in the economy is the market for privately issued currencies. A buyer rebalances his portfolio in the centralized market and a seller exits the centralized market holding no assets. Because quasi-linear preferences imply that all buyers have the same demand function, condition (6) provides the aggregate demand for real balances. The real value of the money supply is obtained by summing up the real value of each private currency.

Let $b_t^i\equiv\phi_t^iM_t^i$ denote the real value of the total supply of currency i in period t and let $\mathbf{b}_t\in\mathbb{R}_+^N$ denote the vector of real values in period t. Free entry in the market for private currencies requires

$$b_t^i-\gamma_t b_{t-1}^i=0 \tag{8}$$

for each i at all dates $t\geq 0$. Finally, the market-clearing condition can be written as

$$\sum_{i=1}^{N}b_t^i=z\left(\gamma_{t+1}\right) \tag{9}$$

at all dates $t \geq 0$. Given these conditions, we can formally define an equilibrium in terms of the sequence $\{\mathbf{b}_t, \gamma_t\}_{t=0}^{\infty}$.

Definition 1 *A perfect-foresight equilibrium is a sequence $\{\mathbf{b}_t, \gamma_t\}_{t=0}^{\infty}$ satisfying (8), (9), $b_t^i \geq 0$, and $0 \leq \gamma_t \leq \beta^{-1}$ for all $t \geq 0$ and $i \in \{1, ..., N\}$.*

It is a property of any equilibrium allocation that the exchange rate among private currencies remains indeterminate: the solution to the bargaining problem allows us to derive total real expenditures in the DM, but not the composition of the monetary portfolio transferred to the seller. As previously mentioned, this result is a direct consequence of the absence of portfolio restrictions and transaction costs.

4 Existence

We start our analysis by characterizing symmetric equilibria with the property $b_t^i = b_t^j$ for all i and j. In these equilibria, all active entrepreneurs issue currencies that circulate as a medium of exchange, with the market for privately issued currencies equally divided among them. The following proposition formally establishes the existence of symmetric equilibria.

Proposition 1 *There exists a continuum of equilibrium trajectories with the property $b_t^i = b_t^j$ for all i and j at all dates $t \geq 0$. These equilibria can be characterized by a sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfying*

$$z\left(\gamma_{t+1}\right) = \gamma_t z\left(\gamma_t\right) \tag{10}$$

and $0 \leq \gamma_t \leq \beta^{-1}$ at all dates $t \geq 0$.

Proof. Because $b_t^i = b_t^j$ for all i and j, the market-clearing condition reduces to

$$N b_t = z\left(\gamma_{t+1}\right)$$

for some common value $b_t \geq 0$. Free entry implies $b_t = \gamma_t b_{t-1}$ at each date $t \geq 0$. Thus, a symmetric equilibrium can be defined as a sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfying (10). This equation implicitly defines a mapping $\gamma_{t+1} = f\left(\gamma_t\right)$ with at least two fixed points: the origin $\left(\gamma_t, \gamma_{t+1}\right) = (0, 0)$ and the interior steady state $\left(\gamma_t, \gamma_{t+1}\right) = (1, 1)$. As a result, there exists a continuum of equilibrium trajectories that can be constructed starting from any arbitrary point $\gamma_0 \in (0, 1)$. ■

An immediate consequence of the previous proposition is the existence of a stationary equilibrium with the property that the value of all privately issued currencies is constant over time. Or more plainly: there exists an equilibrium in which price stability is consistent with competing private monies.

Corollary 2 *There exists an interior stationary equilibrium with* $\gamma_t = 1$ *for all* $t \geq 0$*.*

Proof. As we have seen, equation (10) implicitly defines a mapping $\gamma_{t+1} = f(\gamma_t)$ for which $(\gamma_t, \gamma_{t+1}) = (1, 1)$ is a fixed point. Thus, setting $\gamma_t = 1$ for all $t \geq 0$ is a solution satisfying the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$. ■

In this equilibrium, the exchange value of private currencies remains stable over time. Agents do not expect monetary conditions to vary over time so that the real value of private currencies, as well as their expected return, remains constant. Thus, it is possible to have an equilibrium with stable currencies, even though these privately issued tokens are not associated with any explicit promise to exchange them for goods at some future date. Our analysis finds that the type of purely private arrangement initially envisaged by Hayek (1999) is feasible. Hayek argued that private agents through markets can achieve desirable outcomes, even in the field of money and banking. According to his view, government intervention is not necessary for the establishment of a stable monetary system.

However, we will show momentarily that other allocations with undesirable properties can also be consistent with the same equilibrium conditions. These equilibria are characterized by the persistently declining purchasing power of private currencies and falling trading activity. There is no reason to forecast that the equilibrium with stable value will prevail over these different equilibria.

Furthermore, even the equilibrium with a stable value of currencies is socially inefficient. In this equilibrium, the quantity traded in the DM $\hat{q}$ satisfies

$$\sigma \frac{u'(\hat{q})}{w'(\hat{q})} + 1 - \sigma = \frac{1}{\beta},$$

which is below the socially efficient quantity (i.e., $\hat{q} < q^*$). Although the equilibrium with stable currencies Pareto dominates all other equilibria in which the value of private currencies declines over time, a purely private monetary system does not provide the socially optimum quantity of money, as defined in Friedman (1969).

Our next step is to characterize nonstationary equilibria with the property that the value of privately issued currencies collapses along the equilibrium path. At this point, it makes sense to restrict attention to preferences and technologies that imply an empirically plausible money demand function satisfying the property that the demand for real balances is decreasing in the inflation rate (i.e., increasing in the real return on money). In particular, it is helpful to assume that $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. In this case, the demand for real balances satisfies

$$z\left(\gamma_{t+1}\right) = \frac{\left(\beta\gamma_{t+1}\right)^{\frac{1+\alpha}{\eta+\alpha}-1}}{1+\alpha}\left[\frac{\sigma}{1-(1-\sigma)\beta\gamma_{t+1}}\right]^{\frac{1+\alpha}{\eta+\alpha}}.$$

Note that $z'(\gamma) > 0$ for all $\gamma > 0$ so that the demand for real balances is increasing in the real return on money. The dynamic system describing the equilibrium evolution of γ_t is given by

$$\frac{\gamma_{t+1}^{\frac{1+\alpha}{\eta+\alpha}-1}}{\left[1-(1-\sigma)\beta\gamma_{t+1}\right]^{\frac{1+\alpha}{\eta+\alpha}}} = \frac{\gamma_t^{\frac{1+\alpha}{\eta+\alpha}}}{\left[1-(1-\sigma)\beta\gamma_t\right]^{\frac{1+\alpha}{\eta+\alpha}}}. \tag{11}$$

Because the initial choice γ_0 is arbitrary, there exist multiple equilibrium trajectories that converge monotonically to autarky.[9]

Proposition 2 *Suppose* $u(q) = (1-\eta)^{-1} q^{1-\eta}$ *and* $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, *with* $0 < \eta < 1$ *and* $\alpha \geq 0$. *Then, there exists a continuum of equilibrium trajectories starting from any* $\gamma_0 \in (0,1)$ *with the property that the value of private money monotonically converges to zero.*

Proof. Note that $\gamma_t = 0$ when $\gamma_{t+1} = 0$ because the demand function $z(\gamma)$ goes to zero as γ converges to zero from above. When $\gamma_{t+1} = \beta^{-1}$, it follows that $\gamma_t < \beta^{-1}$. Using the *Implicit Function Theorem*, we have

$$\frac{d\gamma_t}{d\gamma_{t+1}} = \frac{z'\left(\gamma_{t+1}\right)}{\gamma_t z'\left(\gamma_t\right) + z\left(\gamma_t\right)} > 0$$

for any $\gamma_{t+1} \in \left[0, \beta^{-1}\right]$. The *Inverse Function Theorem* implies

$$\frac{d\gamma_{t+1}}{d\gamma_t} = \frac{\gamma_t z'\left(\gamma_t\right) + z\left(\gamma_t\right)}{z'\left(\gamma_{t+1}\right)} > 0$$

for any $\gamma_t \in \left[0, f^{-1}\left(\beta^{-1}\right)\right]$, so the implicitly defined mapping $\gamma_{t+1} = f\left(\gamma_t\right)$ is strictly increasing. In particular, we have

$$\left.\frac{d\gamma_{t+1}}{d\gamma_t}\right|_{\gamma_t=\gamma_{t+1}=1} = \frac{z'(1) + z(1)}{z'(1)} > 1,$$

which means that the mapping $\gamma_{t+1} = f\left(\gamma_t\right)$ crosses the 45-degree line from below at the point $\left(\gamma_t, \gamma_{t+1}\right) = (1,1)$. Thus, the unique interior stationary solution is given by $\gamma_t = 1$ for

[9] Sanches (2016) has derived a similar result in an economy with the property that entrepreneurs issue debt claims that circulate as a medium of exchange.

all $t \geq 0$. In this case, a seller is willing to produce the quantity $\hat{q}$ satisfying

$$\sigma \frac{u'(\hat{q})}{w'(\hat{q})} + 1 - \sigma = \beta^{-1}.$$

For any initial choice $\gamma_0 \in (0,1)$, the mapping $\gamma_{t+1} = f(\gamma_t)$ yields a strictly decreasing sequence $\{\gamma_t\}_{t=0}^{\infty}$ with the property $\lim_{t\to\infty} \gamma_t = 0$. ■

Figure 1 plots the dynamic system (11) with $\alpha = .5$, $\beta = .9$, $\eta = .5$, and $\sigma = .9$. For any initial condition $\gamma_0 \in (0,1)$, there exists an associated equilibrium trajectory that is monotonically decreasing. Along this equilibrium path, real money balances decrease monotonically over time and converge to zero, so the equilibrium allocation approaches autarky as $t \to \infty$. The decline in the desired amount of real balances follows from the agent's optimization problem when the value of privately issued currencies persistently depreciates over time (i.e., the anticipated decline in the purchasing power of private money leads agents to reduce their real money balances over time). As a result, trading activity in the decentralized market monotonically declines along the equilibrium trajectory. This property of equilibrium allocations with privately issued currencies implies that private money is inherently unstable in that changes in beliefs can lead to undesirable self-fulfilling inflationary episodes.

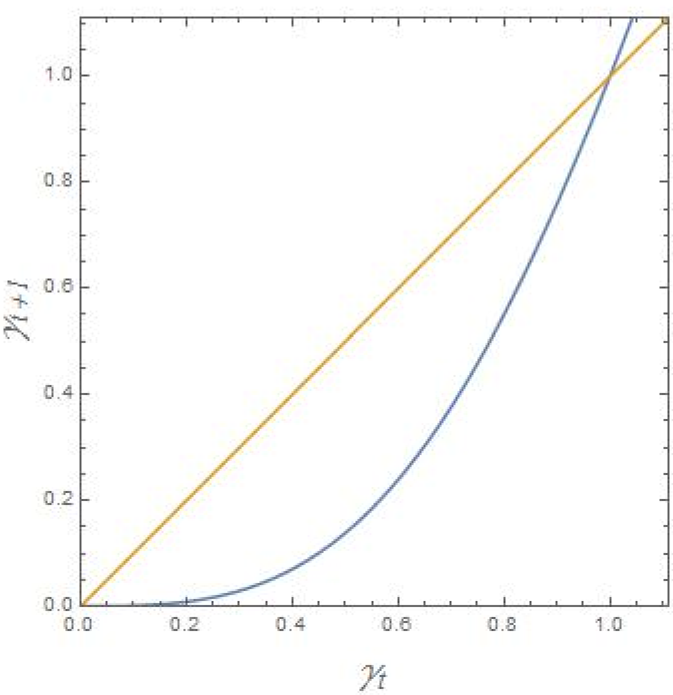


Figure 1: Dynamic system (11)

The existence of these inflationary equilibrium trajectories in a purely private monetary arrangement implies that hyperinflationary episodes are not an exclusive property of government-issued money. Obstfeld and Rogoff (1983) build economies that can display self-fulfilling inflationary episodes when the government is the sole issuer of currency and follows a money-growth rule. Lagos and Wright (2003) show that search-theoretic monetary models with government-supplied currency can also have self-fulfilling inflationary episodes under a money-growth rule. Our analysis of privately issued currencies shows that self-fulfilling inflationary equilibria can occur in the absence of government currency when private agents enter the currency-issuing business to maximize profits. Thus, replacing government monopoly under a money-growth rule with free entry and profit maximization does not overcome the fundamental fragility associated with fiduciary regimes, public or private.

In the following section, we introduce government currency and study the possibility of stabilization policies. As we will see, the set of equilibrium allocations characterized by LW arises as a special case in our analysis. In addition, we will study the equilibrium interaction between private and government monies and derive some interesting properties.

As previously mentioned, the stationary equilibrium with a stable value of private currencies Pareto dominates any nonstationary equilibria, even though that equilibrium does not achieve the first best. To verify this claim, note that the quantity traded in the DM starts from a value below $\hat{q}$ and decreases monotonically in a nonstationary equilibrium. As a result, the allocation in the equilibrium with a stable value of money Pareto dominates the allocation associated with any inflationary equilibrium.

Now we turn to asymmetric equilibria. Given that utility is linear and transferable in the CM, the allocation in the DM is what matters for the welfare analysis. However, we are still interested in studying the CM allocation in our positive analysis of currency competition. Given that the production of tokens costs nothing, there is no condition to pin down the market share of each active entrepreneur, represented by the real values b_t^i, in the market for private currencies. Although the indeterminacy of the market share has no welfare consequences, we want to study some asymmetric equilibria in our positive analysis.

An asymmetric equilibrium of interest involves a unique private currency circulating in the economy, despite free entry. The following proposition shows that any equilibrium allocation can be characterized by the dynamic system (10) and that it is possible to construct an equilibrium in which a unique private currency dominates other currencies as a medium of exchange, even though there is free entry in the currency-issuing business. Again, this occurs because the market shares across different types of money are indeterminate.

Proposition 3 *An equilibrium allocation can be characterized by a sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfying (10) and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$ at all dates. In particular, there exists* *an equilibrium with valued money satisfying $b_t^1 = b_t > 0$ and $b_t^i = 0$ for all $i \geq 2$ at all dates $t \geq 0$.*

Proof. Because (8) must hold for all i, we have

$$\sum_{i=1}^{N} b_t^i = \gamma_t \sum_{i=1}^{N} b_{t-1}^i.$$

Condition (9) implies

$$\sum_{i=1}^{N} b_t^i = z\left(\gamma_{t+1}\right).$$

Thus, the dynamic system (10), together with the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$, describes the evolution of the real return on money γ_t in any equilibrium. As we have seen, condition (5) implies that the sequence $\{\gamma_t\}_{t=0}^{\infty}$ determines the equilibrium allocation in the DM.

Note that $b_t^j = 0$ implies either $\phi_t^j = 0$ or $M_t^j = 0$, or both. Because $b_t^i = 0$ for all $i \geq 2$, the market-clearing condition implies $b_t^1 = z\left(\gamma_{t+1}\right)$. Free entry requires that the sequence of returns $\{\gamma_t\}_{t=0}^{\infty}$ satisfy (10). Hence, there exists an equilibrium with $b_t^1 = z(1) > 0$ and $b_t^i = 0$ for all $i \geq 2$ at all dates $t \geq 0$. In addition, there exists a continuum of equilibria with $b_t^1 = z\left(\gamma_{t+1}\right) > 0$ and $b_t^i = 0$ for all $i \geq 2$ at all dates $t \geq 0$, with $\{\gamma_t\}_{t=0}^{\infty}$ representing any sequence that solves (10) starting from an arbitrary point $\gamma_0 \in (0,1)$. ■

In these equilibria, a single entrepreneur becomes the sole issuer of currency in the economy. The threat of entry constrains his behavior and market participants understand the discipline imposed by free entry, even though all agents see a single private agent supplying all currency in the economy. As in the symmetric class, an equilibrium with a stable value of money is as likely to occur as an equilibrium with a declining value of money.

5 Government

In this section, we study the interaction between private and government monies and the possibility of monetary stabilization through public policy. Suppose the government enters the currency-issuing business by creating its own brand. The government budget constraint is given by

$$\phi_t \bar{M}_t + \tau_t = \phi_t \bar{M}_{t-1},$$

where $\tau_t \in \mathbb{R}$ is the real value of lump-sum taxes, $\phi_t \in \mathbb{R}_+$ is the real value of government-issued currency, and $\bar{M}_t \in \mathbb{R}_+$ is the amount supplied at date t. What makes government money fundamentally different from private money is that behind the government brand there is a fiscal authority with the power to tax agents in the economy.

5.1 Money-growth rule

We start by assuming that the government follows a money-growth rule of the form

$$\bar{M}_t = (1+\omega)\,\bar{M}_{t-1},$$

with $\omega \geq \beta - 1$. In any equilibrium with valued government money, we must have

$$\frac{\phi_{t+1}}{\phi_t} = \gamma_{t+1} \tag{12}$$

at all dates $t \geq 0$. Here γ_{t+1} continues to represent the common rate of return across all currencies that are valued in equilibrium. In the absence of portfolio restrictions, government money must yield the same rate of return as other monetary assets for it to be valued in equilibrium.

Let $m_t \equiv \phi_t \bar{M}_t$ denote the real value of government-issued currency. Then, condition (12) implies

$$m_t = (1+\omega)\,m_{t-1}\gamma_t \tag{13}$$

at all dates. The market-clearing condition in the asset market becomes

$$m_t + \sum_{i=1}^{N} b_t^i = z\left(\gamma_{t+1}\right) \tag{14}$$

for all $t \geq 0$. Given these changes in the equilibrium conditions, it is now possible to provide a formal definition of equilibrium in the presence of government intervention.

Definition 3 *Given the policy parameter ω, a perfect-foresight equilibrium is a sequence $\{\mathbf{b}_t, m_t, \gamma_t\}_{t=0}^{\infty}$ satisfying (8), (13), (14), $b_t^i \geq 0$, $m_t \geq 0$, and $0 \leq \gamma_t \leq \beta^{-1}$ for all $t \geq 0$ and $i \in \{1, ..., N\}$.*

We start our analysis of a hybrid monetary arrangement by characterizing equilibria with the property $b_t^i = b_t^j = m_t$ for all i and j. In these symmetric equilibria, private and government monies attain the same exchange value. Our next result shows that an equilibrium with this property exists if and only if the government follows the policy $\omega = 0$ (i.e., constant government supply).

Proposition 4 *A symmetric equilibrium with the property that private and government monies attain the same value exists if and only if* $\omega = 0$. *In this case, there exist multiple equilibrium trajectories characterized by a sequence* $\{\gamma_t\}_{t=0}^{\infty}$ *satisfying (10) and* $0 \leq \gamma_t \leq \beta^{-1}$ *at all dates* $t \geq 0$.

Proof. Symmetry requires $b_t^i = m_t$ for all i at all dates. Let b_t denote the common value. Thus, the market-clearing condition implies

$$b_t = \frac{z\left(\gamma_{t+1}\right)}{N+1}.$$

Since the free-entry condition, $b_t - \gamma_t b_{t-1} = 0$, must hold at all dates, the equilibrium evolution of the real return on money γ_t must satisfy (10).

As previously mentioned, condition (12) implies $m_t = (1+\omega) m_{t-1}\gamma_t$. Because $m_t = b_t$ must hold in a symmetric equilibrium, we have

$$z\left(\gamma_{t+1}\right) = (1+\omega) z\left(\gamma_t\right)\gamma_t.$$

This means that an equilibrium with the property that private and government monies attain the same value exists if and only if $\omega = 0$. As a result, an equilibrium sequence $\{\gamma_t\}_{t=0}^{\infty}$ must satisfy (10) and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$ at all dates. ■

The previous result implies that price stability with both private and government monies circulating as competing media of exchange can be attained under a money-growth rule only if the government maintains a constant supply of its own brand. As in the purely private monetary economy, there exist multiple equilibrium trajectories with the property that the value of money monotonically declines over time, converging to zero. Thus, the introduction of the government brand under a money-growth rule does not resolve the instability of fiduciary-currency regimes.

A natural question to ask is whether it is possible to have an equilibrium with both private and government monies circulating in the economy when government policy deviates from $\omega = 0$. If $\omega \neq 0$, then we have to consider the possibility of asymmetric equilibria with the property that government money and private money have different values, even though both monies yield the same rate of return. These equilibria can be characterized by a sequence $\{b_t, m_t\}_{t=0}^{\infty}$ satisfying

$$Nb_t + m_t = z\left(\gamma_{t+1}\right), \tag{15}$$

$$b_t = \gamma_t b_{t-1}, \tag{16}$$

$$m_t = (1+\omega) m_{t-1}\gamma_t, \tag{17}$$

together with the boundary conditions $b_t \geq 0$, $m_t \geq 0$, and $0 \leq \gamma_t \leq \beta^{-1}$. The following proposition establishes that a hybrid monetary arrangement with a constant return on money does not exist when the government pursues either an expansionary or a contractionary policy.

Proposition 5 *There is no equilibrium with the properties that the return on monetary assets remains constant over time and private and government monies coexist as media of exchange when $\omega \neq 0$.*

Proof. Suppose that $\gamma_t = \bar{\gamma}$ for all $t \geq 0$. Then, (16) implies, at all dates,

$$\frac{b_t}{b_{t-1}} = \bar{\gamma}.$$

Condition (17) implies, at all dates,

$$\frac{m_t}{m_{t-1}} = (1+\omega)\,\bar{\gamma}.$$

Finally, condition (15) requires, at all dates,

$$Nb_t + m_t = Nb_{t-1} + m_{t-1}.$$

These conditions can be simultaneously satisfied if and only if $\omega = 0$ and $\bar{\gamma} = 1$. ∎

The previous result implies that expansionary and contractionary policies are not consistent with the coexistence of stable private and government monies (i.e., monetary assets with the property that their returns remain constant over time). Thus, these assets cannot jointly serve as stable media of exchange when the government deviates from the constant money supply rule. Under a money-growth regime, the existence of a stable hybrid monetary arrangement requires the government to follow a constant money supply rule.

As should be expected, it is possible to have equilibria with valued government money only. In particular, we can construct asymmetric equilibria with the property $b_t^i = 0$ for all i and $m_t \geq 0$ at all dates $t \geq 0$ (i.e., only government-issued currency circulates as a medium of exchange). In these equilibria, the sequence of returns satisfies, for all dates

$$z\left(\gamma_{t+1}\right) = (1+\omega)\, z\left(\gamma_t\right)\gamma_t. \tag{18}$$

The dynamic properties of the system (18) are exactly the same as those derived in Lagos and Wright (2003) when preferences and technologies imply a demand function for real balances that is strictly decreasing in the inflation rate. For instance, selecting the functional forms

$u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$ results in a dynamic system of the form

$$\frac{\gamma_{t+1}^{\frac{1+\alpha}{\eta+\alpha}-1}}{\left[1-(1-\sigma)\beta\gamma_{t+1}\right]^{\frac{1+\alpha}{\eta+\alpha}}} = (1+\omega)\frac{\gamma_t^{\frac{1+\alpha}{\eta+\alpha}}}{\left[1-(1-\sigma)\beta\gamma_t\right]^{\frac{1+\alpha}{\eta+\alpha}}} \tag{19}$$

with $0 < \eta < 1$ and $\alpha \geq 0$.

A policy choice ω in the range $(\beta - 1, 0)$ is associated with a steady state characterized by deflation and a strictly positive real return on money. In particular, $\gamma_t = (1+\omega)^{-1}$ for all $t \geq 0$. In this stationary equilibrium, the quantity traded in the DM, represented by $q(\omega)$, satisfies

$$\sigma\frac{u'(q(\omega))}{w'(q(\omega))} + 1 - \sigma = \frac{1+\omega}{\beta}.$$

Figure 2 plots the dynamic system (19) for several values of the money growth rate ω with $\alpha = .5$, $\beta = .9$, $\eta = .5$, and $\sigma = .9$.

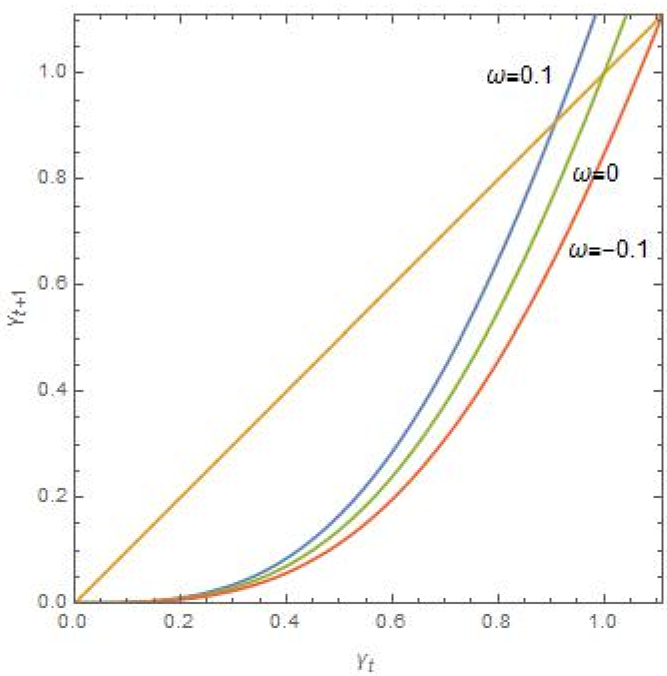


Figure 2: Dynamic system (19)

If we let $\omega \to \beta - 1$, the associated steady state delivers an efficient allocation (i.e., $q(\omega) \to q^*$ as $\omega \to \beta - 1$). This policy prescription is the celebrated Friedman rule, which eliminates the opportunity cost of holding money balances in private portfolios for transaction purposes. The problem with this arrangement is that the Friedman rule is not *uniquely* associated with an efficient allocation. There exists a continuum of suboptimal equilibrium trajectories that are also associated with the Friedman rule. These trajectories involve a persistently declining value of money.

5.2 Pegging the real value of government liabilities

In this section, we develop an alternative monetary policy rule that can implement a unique efficient equilibrium allocation. This outcome will be feasible only if the government can tax private agents to finance policy implementation. Also, it will require government money to drive private money out of the economy.

Suppose the government decides to peg the real value of its monetary liabilities. Specifically, assume the government issues currency to satisfy

$$\phi_t \bar{M}_t = m \tag{20}$$

at all dates for some target value $m > 0$. This means that the government adjusts the money supply sequence $\left\{\bar{M}_t\right\}_{t=0}^{\infty}$ to satisfy (20) in each period. In this case, the market-clearing condition in the asset market becomes

$$m + \sum_{i=1}^{N} b_t^i = z\left(\gamma_{t+1}\right) \tag{21}$$

at all dates $t \geq 0$.

Furthermore, we need to impose the boundary conditions $b_t^i \geq 0$ at all dates $t \geq 0$ for each i. These conditions imply

$$z\left(\gamma_t\right) \geq m \tag{22}$$

at all dates $t \geq 0$. In other words, the policy of pegging the real value of government money results in a lower bound for the real return on money in equilibrium. Given these changes in the equilibrium conditions, we can now provide a definition of equilibrium when the government pegs the real value of its monetary liabilities.

Definition 4 *Given the policy parameter $m > 0$, a perfect-foresight equilibrium is a sequence $\{\mathbf{b}_t, \gamma_t\}_{t=0}^{\infty}$ satisfying (8), (21), (22), $b_t^i \geq 0$, and $0 \leq \gamma_t \leq \beta^{-1}$ for all $t \geq 0$ and $i \in \{1, ..., N\}$.*

Although the government supply is an endogenous variable, we do not need to include it in our definition of equilibrium as it adjusts to satisfy (20) at each date. In particular, the money supply sequence $\left\{\bar{M}_t\right\}_{t=0}^{\infty}$ must satisfy

$$\bar{M}_{t+1} = \frac{1}{\gamma_{t+1}} \bar{M}_t$$

at all dates $t \geq 0$. Because the monetary authority pegs the real value of its monetary liabilities, the government money supply $\bar{M}_t$ varies over time to follow the equilibrium evolution of the return on monetary assets, given the target value m.

The following proposition provides a useful characterization of equilibrium when the government pegs the real value of its liabilities. It also shows that the equilibrium with constant prices is a steady state regardless of the choice of the policy parameter m.

Proposition 6 *An equilibrium allocation can be characterized by a sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfying*

$$z\left(\gamma_{t+1}\right) = \gamma_t z\left(\gamma_t\right) - \gamma_t m + m \tag{23}$$

and the boundary conditions $0 \leq \gamma_t \leq \beta^{-1}$ and $z\left(\gamma_t\right) \geq m$ at all dates. In particular, the stationary solution $\gamma_t = 1$ for all $t \geq 0$ is an equilibrium trajectory for any value of the policy parameter m.

Proof. Because (8) must hold for all i, we have

$$\sum_{i=1}^{N} b_t^i = \gamma_t \sum_{i=1}^{N} b_{t-1}^i.$$

Condition (21) implies

$$\sum_{i=1}^{N} b_t^i = z\left(\gamma_{t+1}\right) - m.$$

Thus, the dynamic system (23), together with the boundary conditions $0 \leq \gamma_t \leq \beta^{-1}$ and $z\left(\gamma_t\right) \geq m$, describes the evolution of the return on money γ_t along the equilibrium path. Finally, it is straightforward to show that $\left(\gamma_t, \gamma_{t+1}\right) = (1, 1)$ is a fixed point of the implicitly defined mapping $\gamma_{t+1} = f\left(\gamma_t\right)$. ■

Note that autarky is no longer a steady state when $m > 0$, so the policy of pegging the real value of government liabilities fundamentally changes the properties of the dynamic system. To demonstrate the main result of this section, it is helpful to assume the functional forms $u\left(q\right) = \left(1-\eta\right)^{-1} q^{1-\eta}$ and $w\left(q\right) = \left(1+\alpha\right)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. In this case, the dynamic system (23) reduces to

$$\frac{\sigma^{\frac{1+\alpha}{\eta+\alpha}} \left(\beta\gamma_{t+1}\right)^{\frac{1+\alpha}{\eta+\alpha}-1}}{\left[1-\left(1-\sigma\right)\beta\gamma_{t+1}\right]^{\frac{1+\alpha}{\eta+\alpha}}} = \frac{\beta^{\frac{1+\alpha}{\eta+\alpha}-1} \left(\sigma\gamma_t\right)^{\frac{1+\alpha}{\eta+\alpha}}}{\left[1-\left(1-\sigma\right)\beta\gamma_t\right]^{\frac{1+\alpha}{\eta+\alpha}}} - m\gamma_t + m \tag{24}$$

with

$$\frac{\left(\beta\gamma_t\right)^{\frac{1+\alpha}{\eta+\alpha}-1}}{1+\alpha} \left[\frac{\sigma}{1-\left(1-\sigma\right)\beta\gamma_t}\right]^{\frac{1+\alpha}{\eta+\alpha}} \geq m \tag{25}$$

at all dates $t \geq 0$. Condition (25) imposes a lower bound on the equilibrium return on money, which can result in the existence of a steady state at the lower bound.

The properties of equations (24)-(25) depend on the value of the policy parameter m. In particular the dynamic system (24)-(25) is a *transcritical bifurcation*.[10] To illustrate this property, it is helpful to simplify the dynamic system by assuming that $\alpha = 0$ (linear disutility of production) and $\sigma \to 1$ (no matching friction in the decentralized market). In this case, the equilibrium evolution of the return on money γ_t satisfies the law of motion

$$\gamma_{t+1} = \gamma_t^2 - \frac{m}{\beta}\gamma_t + \frac{m}{\beta} \tag{26}$$

and the boundary condition

$$\frac{m}{\beta} \leq \gamma_t \leq \frac{1}{\beta}. \tag{27}$$

The policy parameter can take on any value in the interval $0 \leq m \leq 1$. Also, the value of money remains above the lower bound m at all dates. Given that the government provides a credible lower bound for the value of money due to its taxation power, the return on money is bounded below by a strictly positive constant $\beta^{-1}m$ along the equilibrium path.

We can obtain a steady state by solving the polynomial equation

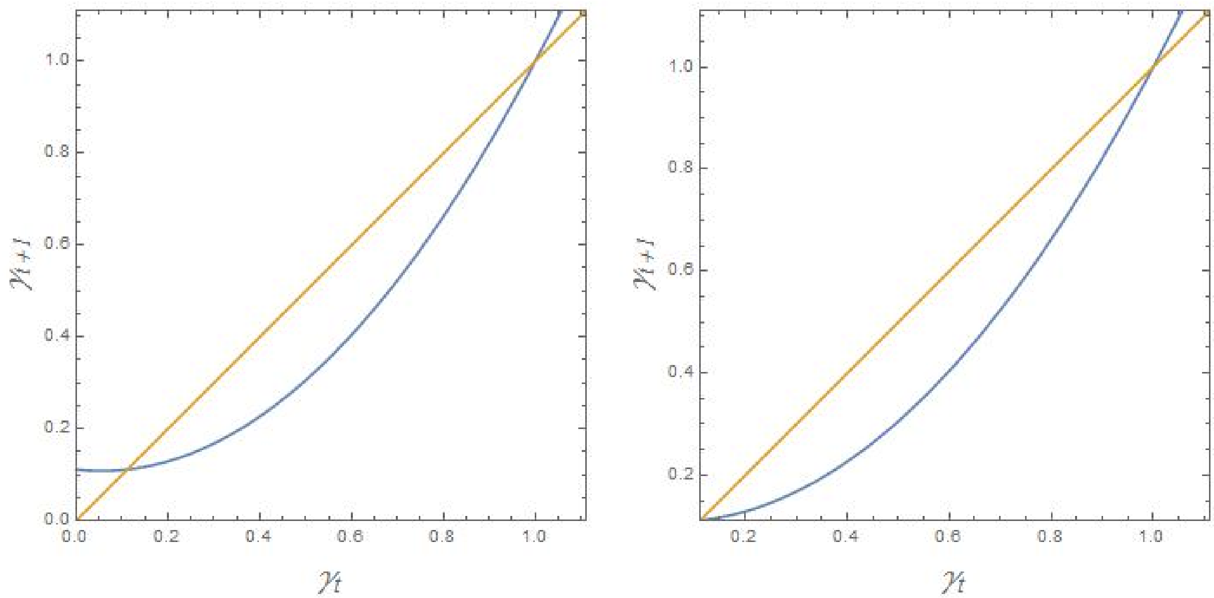
$$\gamma^2 - \left(\frac{m}{\beta} + 1\right)\gamma + \frac{m}{\beta} = 0.$$

If $m \neq \beta$, the roots are 1 and $\beta^{-1}m$. If $m = \beta$, the unique solution is 1. As we will see, the properties of the dynamic system differ considerably depending on the value of the policy parameter m.

If $0 < m < \beta$, then there exist two steady states: $\gamma_t = \beta^{-1}m$ and $\gamma_t = 1$ for all $t \geq 0$. The steady state $\gamma_t = 1$ for all $t \geq 0$ corresponds to the previously described stationary equilibrium with constant prices. The steady state $\gamma_t = \beta^{-1}m$ for all $t \geq 0$ is an equilibrium with the property that only government money is valued. To see why this occurs, note that $z(\gamma_t) = z(\beta^{-1}m) = m$ holds in every period t so that we must have $b_t^i = 0$ for each i at all dates $t \geq 0$. Figure 3 shows the dynamic system when $0 < m < \beta$ (panel (a) for $0 < \gamma_t < (1/\beta)$ and panel (b) for $(m/\beta) < \gamma_t < (1/\beta)$). In this case, the steady state with a low value of money $(\beta^{-1}m)$ is globally stable. As we can see, there exists a continuum of equilibrium trajectories starting from any point $\gamma_0 \in (\beta^{-1}m, 1)$ with the property that the

[10] In bifurcation theory, a transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. Both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide, so the unstable fixed point becomes stable and vice versa.

return on money converges to $\beta^{-1}m$. Along these trajectories, the value of money declines monotonically to the lower bound m and government money drives private money out of the economy.



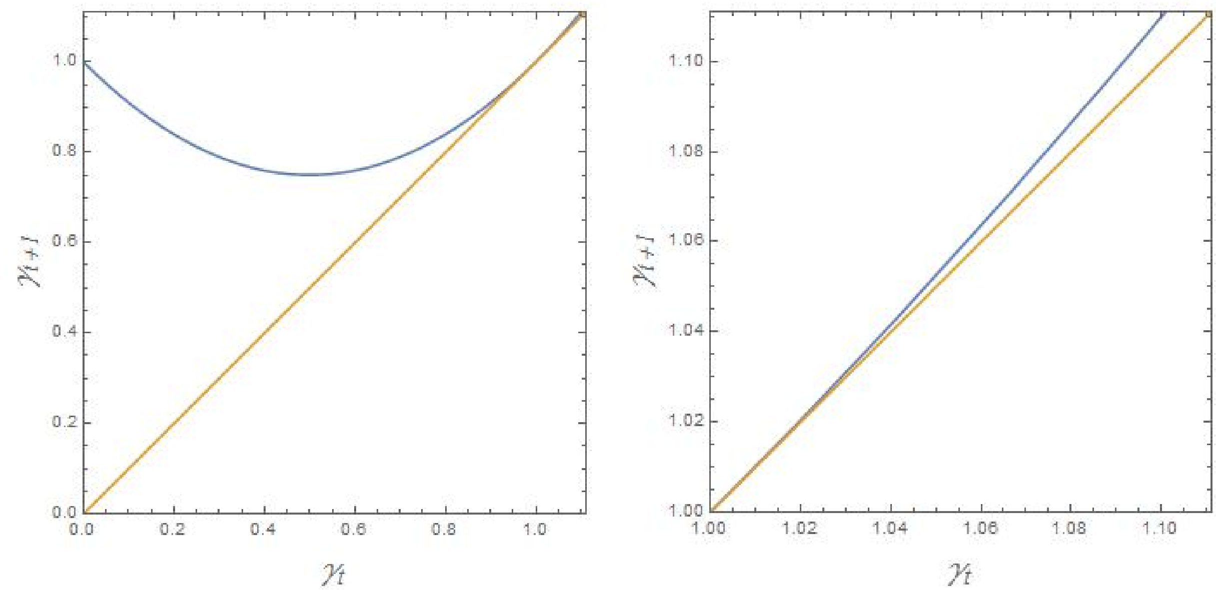
(a) $0 < \gamma_t < (1/\beta)$

(b) $(m/\beta) < \gamma_t < (1/\beta)$

Figure 3: Dynamic system for government money, $0 < m < \beta$

If $m = \beta$, the unique steady state is $\gamma_t = 1$ for all $t \geq 0$. This case is represented in Figure 4 (again with panel (a) for $0 < \gamma_t < (1/\beta)$ and panel (b) for $(m/\beta) < \gamma_t < (1/\beta)$). Note that the 45-degree line is the tangent line to the graph of (26) at the point $(1, 1)$, so the dynamic system remains above the 45-degree line. When we introduce the boundary restriction (27), we find that $\gamma_t = 1$ for all $t \geq 0$ is the unique equilibrium trajectory. Thus, the policy choice $m = \beta$ results in global determinacy, with the unique equilibrium outcome characterized by price stability.

If $\beta < m < 1$ (Figure 5), the unique steady state is $\gamma_t = \beta^{-1}m$ for all $t \geq 0$. When we introduce the boundary restriction (27), $\gamma_t = \beta^{-1}m$ for all $t \geq 0$ is the unique equilibrium trajectory. Setting the target for the value of government money in the interval $\beta < m < 1$ results in a sustained deflation to ensure that the return on money remains above one. To implement a sustained deflation, the government must contract its money supply, a policy financed through taxation. Since a sustained contraction of the money supply by a private entrepreneur is infeasible, in this situation there is only government money circulating in the economy.



(a) $0 < \gamma_t < (1/\beta)$ (b) $(m/\beta) < \gamma_t < (1/\beta)$

Figure 4: Dynamic system for government money, $m = \beta$

To implement a target value in the range $\beta < m < 1$, the government must tax private agents in the CM. To verify this claim, note that the government budget constraint can be written as

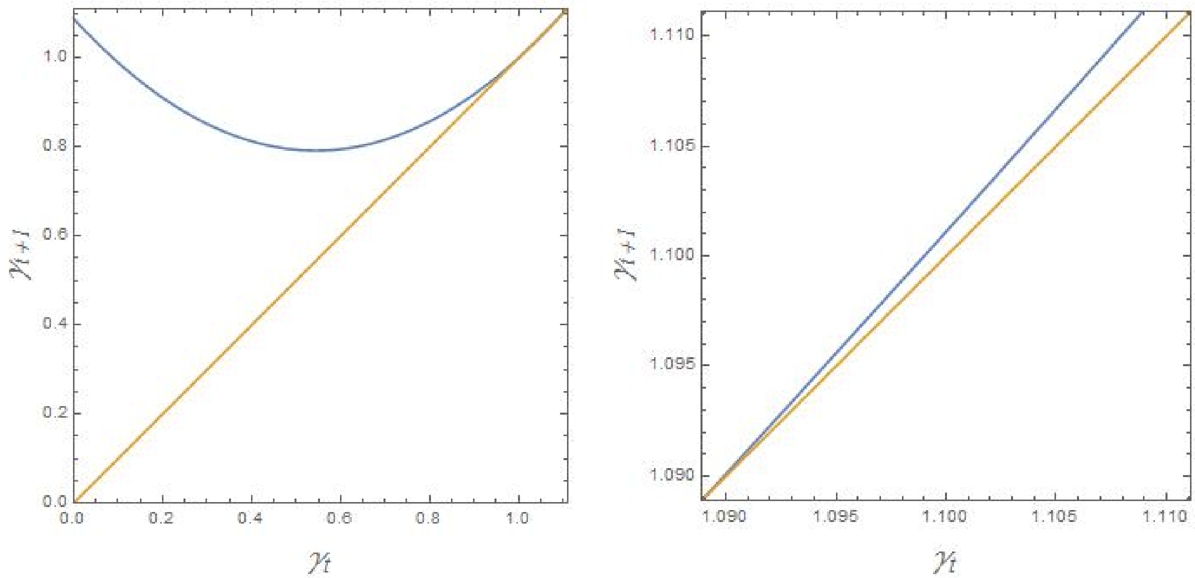
$$\tau_t = m(\gamma_t - 1)$$

in every period t. Because the unique steady state implies $\gamma_t = \beta^{-1}m$ for all $t \geq 0$, we must have

$$\tau_t = m\left(\beta^{-1}m - 1\right) > 0$$

at all dates $t \geq 0$. To implement its target value m, the government needs to persistently shrink the money supply by making purchases that exceed its sales in the CM, with the shortfall financed by taxes.

Finally, it is possible to implement an efficient allocation as the unique equilibrium outcome when we take the limit $m \to 1$. Thus, the joint goal of monetary stability and efficiency can be achieved by government policy. The only caveat is that the implementation of an efficient allocation requires government money to drive private money out of the economy. In a few pages we will see how government money will no longer be essential when private entrepreneurs have access to sufficiently productive capital.



(a) $0 < \gamma_t < (1/\beta)$

(b) $(m/\beta) < \gamma_t < (1/\beta)$

Figure 5: Dynamic system for government money, $\beta < m < 1$

6 Automata

In this section, we introduce private currencies issued by automata. This extension of the baseline model is motivated by the observation that digital currencies such as Bitcoin are associated with software protocols that fix, in an automatic fashion, the total amount of units of money that can be put into circulation.

Consider the benchmark economy described in Section 3 and add J automata, each programmed to issue a constant amount $H^j \in \mathbb{R}_+$ of tokens. Let $h_t^j \equiv \phi_t^j H^j$ denote the real value of the tokens issued by automaton $j \in \{1, ..., J\}$ and let $\mathbf{h}_t \equiv \left(h_t^1, ..., h_t^J\right) \in \mathbb{R}_+^J$ denote the vector of real values. If the units issued by automaton j are valued in equilibrium, then we must have

$$\frac{\phi_{t+1}^j}{\phi_t^j} = \gamma_{t+1} \tag{28}$$

at all dates $t \geq 0$. Here γ_{t+1} continues to represent the common real return across all currencies that are valued in equilibrium. Thus, condition (28) implies

$$h_t^j = h_{t-1}^j \gamma_t \tag{29}$$

for each j at all dates. The market-clearing condition in the asset market becomes

$$\sum_{i=1}^{J} h_t^j + \sum_{i=1}^{N} b_t^i = z\left(\gamma_{t+1}\right) \tag{30}$$

for all $t \geq 0$. Given these conditions, we can now provide a definition of equilibrium in the presence of automata.

Definition 5 *A perfect-foresight equilibrium is a sequence* $\{\mathbf{b}_t, \mathbf{h}_t, \gamma_t\}_{t=0}^{\infty}$ *satisfying (8), (29), (30),* $b_t^i \geq 0$*,* $h_t^j \geq 0$*, and* $0 \leq \gamma_t \leq \beta^{-1}$ *for all* $t \geq 0$*,* $i \in \{1, ..., N\}$*, and* $j \in \{1, ..., J\}$*.*

Our main result in this section is to show that the set of equilibrium allocations in the presence of automata is the same as in the economy with profit-maximizing entrepreneurs only.

Proposition 7 *The set of equilibrium allocations is the same as in the economy without automata. In particular, an equilibrium trajectory is characterized by a sequence* $\{\gamma_t\}_{t=0}^{\infty}$ *satisfying (10) and* $0 \leq \gamma_t \leq \beta^{-1}$ *at all dates* $t \geq 0$*.*

Proof. The free-entry condition implies $b_t^i = \gamma_t b_{t-1}^i$ for each $i \in \{1, ..., N\}$ at all dates. Then, we can derive the relation

$$\sum_{i=1}^{N} b_t^i = \gamma_t \sum_{i=1}^{N} b_{t-1}^i.$$

From (29), we find

$$\sum_{i=1}^{J} h_t^j = \gamma_t \sum_{i=1}^{J} h_{t-1}^j.$$

Then, the market-clearing condition implies

$$z\left(\gamma_{t+1}\right) = \sum_{i=1}^{J} h_t^j + \sum_{i=1}^{N} b_t^i = \gamma_t \left(\sum_{i=1}^{N} b_{t-1}^i + \sum_{i=1}^{J} h_{t-1}^j \right) = \gamma_t z\left(\gamma_t\right)$$

as initially claimed. Because the sequence of returns $\{\gamma_t\}_{t=0}^{\infty}$ determines the equilibrium allocation, we obtain the equivalence result as claimed. ■

The result that the introduction of automata does not change the fundamental properties of the private monetary arrangement is not surprising given that a profit-maximizing entrepreneur optimally chooses a constant nominal supply of his own brand in a perfectly competitive environment. The commitment implied by a machine programmed to issue a fixed amount of tokens does not alter the set of equilibrium allocations because market discipline limits the extent to which profit-maximizing issuers can resort to overissue as an individually optimal strategy.[11]

Because the market share of each entrepreneur is indeterminate, it is possible to have an equilibrium with the property that the tokens issued by automata drive out those issued by entrepreneurs. Given that entrepreneurs have no access to productive projects, the existence of equilibria with the property that some or all entrepreneurs remain idle has no welfare consequences in our baseline economy.

This observation motivates our next section. An interesting debate in monetary economics is whether productive firms should be able to issue their own money to fund capital expenditures and other operational expenses. To address this question, our next step is to consider an economy in which the entrepreneurs have access to a form of productive capital. As we will see, the introduction of productive capital will have deep consequences.

7 Productive Capital

So far, we have studied the properties of an economy where agents do not have access to any form of productive capital. Suppose now that each entrepreneur is endowed with a nontradable project that requires the CM good as input and that pays off at the beginning of the following date. Let $F(k)$ denote the payoff in terms of the CM good when $k \in \mathbb{R}_+$ is the amount invested. Suppose that the payoff function takes the form

$$F(k) = \begin{cases} (1+\rho)\,k \text{ if } 0 \leq k \leq \bar{\iota}, \\ (1+\rho)\,\bar{\iota} \text{ if } \bar{\iota} < k \leq 1, \end{cases}$$

with $1 + \rho \geq \beta^{-1}$ and $0 < \bar{\iota} < 1$.

Given the presence of productive capital, the entrepreneur's budget constraint is given by

$$x_t^i + \phi_t^i M_{t-1}^i + \sum\nolimits_{j \neq i} \phi_t^j M_t^j + k_t = \phi_t^i M_t^i + \sum\nolimits_{j \neq i} \phi_t^j M_{t-1}^j + F(k_{t-1})$$

in every period t. Because $1 + \rho \geq \beta^{-1}$, we obtain a corner solution $k_t = \bar{\iota}$.

Continue to assume that $\phi_{t+1}^i / \phi_t^i < \beta^{-1}$ for all i. Then, the entrepreneur's consumption

[11]The Bitcoin protocol issues bitcoins over time as a reward for the verification of transactions in the system. Since we are assuming that these costs are zero, the automaton does not issue new currency (as will be the case, in the long run, with Bitcoin).

is given by

$$x_t^i = \phi_t^i \left(M_t^i - M_{t-1}^i \right) + \rho \bar{\iota}$$

and his lifetime discounted utility is given by

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[\phi_\tau^i \left(M_\tau^i - M_{\tau-1}^i \right) + \rho \bar{\iota} \right].$$

That is, the currency-issuing business allows the entrepreneur to take advantage of the income stream associated with productive capital by providing an external source of funding.

The free-entry condition requires

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[\phi_\tau^i \left(M_\tau^i - M_{\tau-1}^i \right) + \rho \bar{\iota} \right] = 0$$

in each period $t \geq 0$. Given these changes in the environment, we can now define an equilibrium in the presence of productive capital.

Definition 6 *A perfect-foresight equilibrium is a sequence* $\{\mathbf{b}_t, \mathbf{k}_t, \gamma_t\}_{t=0}^{\infty}$ *satisfying (9),* $b_t^i \geq 0$*,* $0 \leq \gamma_t \leq \beta^{-1}$*,* $k_t^i = \bar{\iota}$*, and*

$$b_t^i - \gamma_t b_{t-1}^i + \rho \bar{\iota} = 0 \tag{31}$$

for all $t \geq 0$ *and* $i \in \{1, ..., N\}$*.*

Given free entry, the dynamic system becomes

$$z\left(\gamma_{t+1}\right) + \rho \bar{\iota} = \gamma_t z\left(\gamma_t\right). \tag{32}$$

Thus, the presence of productive capital fundamentally changes the properties of the dynamic system describing the evolution of the real return on money. In particular, autarky is no longer a steady state and there is no equilibrium with the property that the value of private currencies converges to zero.

Continue to assume the functional forms $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. Then, the dynamic system (32) reduces to

$$\frac{\sigma^{\frac{1+\alpha}{\eta+\alpha}} \left(\beta\gamma_{t+1}\right)^{\frac{1+\alpha}{\eta+\alpha}-1}}{\left[1-(1-\sigma)\beta\gamma_{t+1}\right]^{\frac{1+\alpha}{\eta+\alpha}}} + \rho \bar{\iota} = \frac{\beta^{\frac{1+\alpha}{\eta+\alpha}-1} \left(\sigma\gamma_t\right)^{\frac{1+\alpha}{\eta+\alpha}}}{\left[1-(1-\sigma)\beta\gamma_t\right]^{\frac{1+\alpha}{\eta+\alpha}}}. \tag{33}$$

The following proposition establishes the existence of a unique equilibrium allocation with the property that the real return on money is strictly greater than one (i.e., $\gamma_t = \gamma$ for all $t \geq 0$ with $1 < \gamma \leq \beta^{-1}$).

Proposition 8 *Suppose* $u(q) = (1-\eta)^{-1} q^{1-\eta}$ *and* $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, *with* $0 < \eta < 1$ *and* $\alpha \geq 0$. *Then, there exists a unique equilibrium allocation with the property* $\gamma_t = \gamma^s$ *for all* $t \geq 0$ *and* $1 < \gamma^s \leq \beta^{-1}$.

Proof. It can be shown that $d\gamma_{t+1}/d\gamma_t > 0$ for all $\gamma_t > 0$. When $\gamma_{t+1} = 0$, we have

$$\gamma_t = \frac{(\rho\bar{\iota})^{\frac{\eta+\alpha}{1+\alpha}}}{\sigma\beta^{\frac{1-\eta}{1+\alpha}} + (\rho\bar{\iota})^{\frac{\eta+\alpha}{1+\alpha}}(1-\sigma)\beta}.$$

Because $\gamma_t \in \left[0, \beta^{-1}\right]$ for all $t \geq 0$, a nonstationary solution would violate the boundary condition. Thus, the unique solution is stationary, $\gamma_t = \gamma^s$ for all $t \geq 0$, and must satisfy

$$\sigma^{\frac{1+\alpha}{\eta+\alpha}} (\beta\gamma^s)^{\frac{1+\alpha}{\eta+\alpha}-1} + \rho\bar{\iota}\left[1-(1-\sigma)\beta\gamma^s\right]^{\frac{1+\alpha}{\eta+\alpha}} = \beta^{\frac{1+\alpha}{\eta+\alpha}-1} (\sigma\gamma^s)^{\frac{1+\alpha}{\eta+\alpha}}$$

and

$$\frac{(\rho\bar{\iota})^{\frac{\eta+\alpha}{1+\alpha}}}{\sigma\beta^{\frac{1-\eta}{1+\alpha}} + (\rho\bar{\iota})^{\frac{\eta+\alpha}{1+\alpha}}(1-\sigma)\beta} \leq \gamma^s \leq \frac{1}{\beta}.$$

■

The presence of productive capital results in the global determinacy of equilibrium under free entry. Any belief implying a persistently declining value of private money is not consistent with the equilibrium conditions when entrepreneurs issue currency to fund productive projects. The existence of productive projects in the economy provides a fundamental value for the entrepreneur's currency-issuing business, which falsifies any privately held belief with the property that the exchange value of an entrepreneur's currency monotonically converges to zero. Figure 6 plots the dynamic system (33) with $\alpha = .5$, $\beta = .9$, $\eta = .5$, $\sigma = .9$, $\rho = .15$, and $\bar{\iota} = .5$.

Also, the presence of productive capital implies a strictly positive real return on money under free entry, reducing the opportunity cost of holding money for transaction purposes. In a perfectly competitive environment, free entry implies that the gains from intertemporal investment opportunities translate into a higher yield on privately issued currencies in the form of a sustained deflation. As a result, the ensuing equilibrium allocation Pareto dominates the allocation associated with price stability derived in the absence of productive capital. Moreover, the equilibrium real return on money approaches the efficient upper bound β^{-1} as the technological rate of return on capital increases. Therefore, the unique equilibrium allocation approaches the efficient allocation as the return on capital rises. In other words, a purely private monetary arrangement is consistent with Friedman's (1969) optimum quantity of money when capital is sufficiently productive.

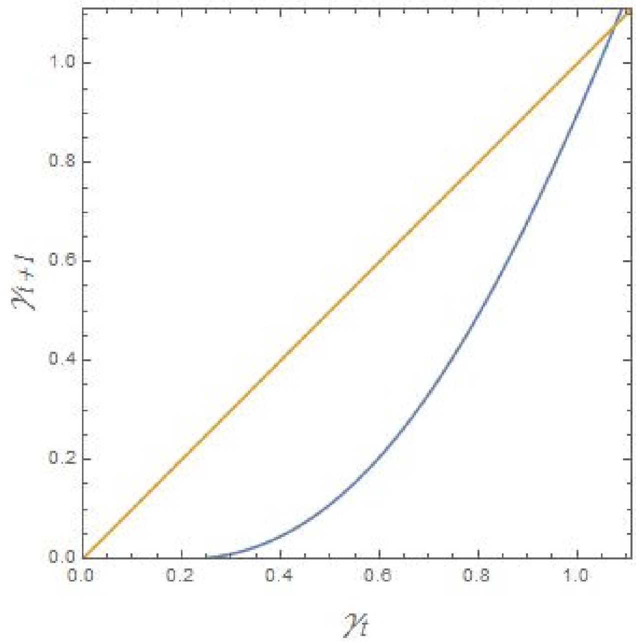


Figure 6: dynamic system (33)

To verify that a sustained deflation is consistent with individual optimization, note that the free-entry condition implies

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[\phi_{\tau}^{i} \left(M_{\tau}^{i} - M_{\tau-1}^{i}\right) + \rho\bar{\iota}\right] = 0$$

at all dates $t \geq 0$. Because consumption is nonnegative, we must have

$$\phi_{t}^{i} \left(M_{t}^{i} - M_{t-1}^{i}\right) + \rho\bar{\iota} = 0$$

in every period $t \geq 0$. If the currency issued by entrepreneur i is valued in equilibrium, we must have $M_{t}^{i} < M_{t-1}^{i}$. Thus, the nominal supply of each brand must be monotonically decreasing to be consistent with free entry. This means that, along the equilibrium trajectory, an entrepreneur's purchases exceed his sales in the CM, with the shortfall financed by the proceeds from investment in the productive technology. Thus, the presence of productive capital makes a sustained deflation feasible as an equilibrium outcome under free entry.

The implementation of the previously described allocation requires the enforcement of the plan consistent with profit maximization. As we have seen, an entrepreneur makes purchases that exceed his sales in the CM when his nominal supply is strictly decreasing. If an entrepreneur deviated from this plan by retaining the proceeds from capital investments, he would clearly be better off. Thus, it is necessary to punish entrepreneurs who opportunistically deviate from the plan consistent with individual optimization when the nominal supply is strictly decreasing. In the absence of productive capital, the nominal supply consistent with free entry is constant over time so that an opportunistic deviation is not an issue in the baseline model without capital. The consequences of the absence of an enforcement mechanism of the plan are an interesting avenue for future research, but go well beyond the scope of this paper.

The following example illustrates these important properties of an economy with productive capital and competing private currencies.

Example 7 *Suppose* $u(q) = 2\sqrt{q}$ *and* $w(q) = q$*. In addition, assume* $\sigma \to 1$*. In this case, the dynamic system is given by*

$$\gamma_{t+1} = \gamma_t^2 - \beta^{-1}\rho\bar{\iota} \equiv f(\gamma_t).$$

The unique fixed point in the range $\left[0, \beta^{-1}\right]$ *is*

$$\gamma^s \equiv \frac{1 + \sqrt{1 + 4\beta^{-1}\rho\bar{\iota}}}{2}$$

provided $\rho \leq \frac{1-\beta}{\bar{\iota}\beta}$*. Because* $f'(\gamma) > 0$ *for all* $\gamma > 0$ *and* $0 = f\left(\sqrt{\beta^{-1}\rho\bar{\iota}}\right)$*, it follows that* $\gamma_t = \gamma^s$ *for all* $t \geq 0$ *is the unique equilibrium trajectory. As we can see, the real return on money is strictly greater than one. If we take the limit*

$$\rho \to \frac{1-\beta}{\bar{\iota}\beta},$$

we find that the unique equilibrium allocation is socially efficient.

In our model, not only do the entrepreneurs have access to the technologies to produce nonfalsifiable tokens and keep track of trading histories but they also have the opportunity to invest in a productive project at each date. One can interpret an entrepreneur in our framework as the owner of an internet platform (i.e., a productive project) that issues money to be used for the purchase of goods through her platform. Thus, a monetary system with the previously described properties is closer to reality than one might initially imagine. And the desirable properties we have demonstrated serve as a useful guide to regulators, policy makers, and others interested in the future of digital currencies.

Indeed, the result that when entrepreneurs can issue currency to fund socially productive projects, a competitive monetary system is consistent with monetary stability and implements an allocation that is arbitrarily close to the efficient allocation is a key finding of this paper. This result vindicates Hayek's proposal for the denationalization of money and it links our research to the literature on the provision of liquidity by productive firms (Holmström and Tirole, 2011, and Dang, Gorton, Holmström, and Ordoñez, 2014).

The possibility of having a private monetary arrangement consistent with Friedman's optimum quantity of money has recently been studied by Monnet and Sanches (2015) and Andolfatto, Berentsen, and Waller (2016). Monnet and Sanches (2015) characterize a monetary arrangement in which privately issued debt claims circulate as a medium of exchange. In their analysis, the debt claims are associated with an explicit promise by the issuer to redeem them at some expected value. Andolfatto, Berentsen, and Waller (2016) study the properties of a monetary arrangement in which an institution with the monopoly rights on the economy's physical capital issues claims that circulate as a medium of exchange. However, these arrangements are fundamentally different from our analysis of currency competition.

8 Network Effects

Many discussions of currency competition highlight the importance of network effects. See, for example, Halaburda and Sarvary (2015). To evaluate these network effects, let us consider a version of the baseline model in which the economy consists of a countable infinity of identical locations indexed by $j \in \{..., -2, -1, 0, 1, 2, ...\}$. Each location contains a $[0, 1]$-continuum of buyers, a $[0, 1]$-continuum of sellers, and a countable infinity of entrepreneurs. All agents have the same preferences and technologies as previously described. For simplicity, we take the limit $\sigma \to 1$ so that each buyer is randomly matched with a seller with probability one and vice versa.

The main change from the baseline model is that sellers move randomly across locations. Suppose that a fraction $1 - \delta$ of sellers in each location j is randomly selected to move to location $j + 1$ at each date $t \geq 0$. Assume that the seller's relocation status is publicly revealed at the beginning of the decentralized market and that the actual relocation occurs after the decentralized market closes.

We start by showing the existence of a symmetric and stationary equilibrium with the property that the currency issued by an entrepreneur in location j circulates only in that location. In this equilibrium, a seller who finds out he is going to be relocated from location j to $j+1$ does not produce the DM good for the buyer in exchange for local currency because he believes that currency issued in location j will not be valued in location $j+1$. This belief can be self-fulfilling so that currency issued in location j circulates only in that location. In this case, the optimal portfolio choice implies the first-order condition

$$\delta \frac{u'(q(\mathbf{m},t))}{w'(q(\mathbf{m},t))} + 1 - \delta = \frac{1}{\beta \gamma_{t+1}^{i}}$$

for each currency i. Note that we have suppressed any superscript or subscript indicating the agent's location, given that we restrict attention to symmetric equilibria. Let $\hat{q}(\gamma)$ denote the DM output as an implicit function of the real return γ. Then, the demand for real balances in each location is given by

$$\hat{z}\left(\gamma_{t+1}\right) \equiv \frac{w\left(\hat{q}\left(\gamma_{t+1}\right)\right)}{\beta \gamma_{t+1}},$$

where γ_{t+1} denotes the common rate of return across all valued currencies in a given location. Because the market-clearing condition implies

$$\sum_{i=1}^{N} \phi_t^i M_t^i = \hat{z}\left(\gamma_{t+1}\right),$$

the equilibrium sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfies the law of motion

$$\hat{z}\left(\gamma_{t+1}\right) = \gamma_t \hat{z}\left(\gamma_t\right)$$

and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$.

Suppose $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. Then, the dynamic system describing the equilibrium evolution of γ_t is given by

$$\frac{\gamma_{t+1}^{\frac{1+\alpha}{\eta+\alpha}-1}}{\left[1-(1-\delta)\beta\gamma_{t+1}\right]^{\frac{1+\alpha}{\eta+\alpha}}} = \frac{\gamma_t^{\frac{1+\alpha}{\eta+\alpha}}}{\left[1-(1-\delta)\beta\gamma_t\right]^{\frac{1+\alpha}{\eta+\alpha}}} \tag{34}$$

and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$. The following proposition establishes the existence of a stationary equilibrium with the property that privately issued currencies circulate locally.

Proposition 9 *Suppose $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. There exists a stationary equilibrium with the property that the quantity traded in the DM is given by $\hat{q}(\delta) \in (0, q^*)$ satisfying*

$$\delta \frac{u'(\hat{q}(\delta))}{w'(\hat{q}(\delta))} + 1 - \delta = \beta^{-1}. \tag{35}$$

In addition, $\hat{q}(\delta)$ is strictly increasing in δ.

Proof. It is easy to show that the sequence $\gamma_t = 1$ for all $t \geq 0$ satisfies (34). Then, the solution to the optimal portfolio problem implies that the DM output must satisfy (35). Because the term $u'(q)/w'(q)$ is strictly decreasing in q, it follows that the solution $\hat{q}(\delta)$ to (35) must be strictly increasing in δ. ■

The previously described stationary allocation is associated with price stability across all locations, but production in the DM occurs only in a fraction $\delta \in (0, 1)$ of all bilateral meetings. Only a seller who is not going to be relocated is willing to produce the DM good in exchange for locally issued currency. A seller who finds out he is going to be relocated does not produce in the DM because the buyer can only offer him currencies that are not valued in other locations.

Now we construct an equilibrium in which the currency issued by an entrepreneur in location j circulates in other locations. To do so, we establish a currency-exchange system similar to that of Cavalcanti, Erosa, and Temzelides (1999). Suppose the entrepreneurs in all locations establish a currency-exchange system by which entrepreneur i in location j agrees to retire the currency issued by entrepreneur i in location $j-1$. In this case, the budget constraint for entrepreneur i in each location is given by

$$x_t^i + \phi_t^i M_{t-1}^i = \phi_t^i M_t^i.$$

In this equation, the term M_{t-1}^i refers to all currency of type i that is in circulation in a given location before the centralized market opens. In a symmetric equilibrium, the same amount of type-i currency that flowed from location j to $j+1$ in the previous period flowed into location j as relocated sellers moved across locations. As a result, M_{t-1}^i denotes the beginning-of-period type-i circulation in a given location. Given the common knowledge of a currency-exchange system, we build an equilibrium with the property that all sellers in a given location accept locally issued currency because they believe that these currencies will be valued in other locations. In this case, the optimal portfolio choice implies the first-order condition

$$\frac{u'(q(\mathbf{m}, t))}{w'(q(\mathbf{m}, t))} = \frac{1}{\beta \gamma_{t+1}^i}$$

for each currency i. Let $\bar{q}(\gamma)$ denote the DM output as an implicit function of the real return γ. Then, the demand for real balances in each location is given by

$$\bar{z}(\gamma_{t+1}) \equiv \frac{w(\bar{q}(\gamma_{t+1}))}{\beta \gamma_{t+1}}.$$

Because the market-clearing condition implies

$$\sum_{i=1}^{N} \phi_t^i M_t^i = \bar{z}\left(\gamma_{t+1}\right),$$

the equilibrium sequence $\{\gamma_t\}_{t=0}^{\infty}$ satisfies the law of motion

$$\bar{z}\left(\gamma_{t+1}\right) = \gamma_t \bar{z}\left(\gamma_t\right)$$

and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$.

Suppose $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. Then, the dynamic system describing the equilibrium evolution of γ_t is

$$\gamma_{t+1}^{\frac{1+\alpha}{\eta+\alpha}-1} = \gamma_t^{\frac{1+\alpha}{\eta+\alpha}} \tag{36}$$

and the boundary condition $0 \leq \gamma_t \leq \beta^{-1}$. The following proposition establishes the existence of a stationary equilibrium with the property that locally issued currencies circulate in several locations.

Proposition 10 *Suppose $u(q) = (1-\eta)^{-1} q^{1-\eta}$ and $w(q) = (1+\alpha)^{-1} q^{1+\alpha}$, with $0 < \eta < 1$ and $\alpha \geq 0$. There exists a stationary equilibrium with the property that the quantity traded in the DM is given by $\bar{q} \in (\hat{q}(\delta), q^*)$ satisfying*

$$\frac{u'(\bar{q})}{w'(\bar{q})} = \beta^{-1}. \tag{37}$$

Proof. It is straightforward to show that the sequence $\gamma_t = 1$ for all $t \geq 0$ satisfies (36). Then, the solution to the optimal portfolio problem implies that the DM output must satisfy (37). The quantities $\bar{q}$ and $\hat{q}(\delta)$ satisfy

$$\frac{u'(\bar{q})}{w'(\bar{q})} = \delta \frac{u'(\hat{q}(\delta))}{w'(\hat{q}(\delta))} + 1 - \delta.$$

Because $\delta \in (0,1)$, we must have $\bar{q} > \hat{q}(\delta)$ as claimed. ■

Because $\bar{q} > \hat{q}(\delta)$, the allocation associated with the currency-exchange system Pareto dominates the allocation associated with the local circulation of private currencies. Thus, we conclude that, in the presence of currency-exchange systems, the presence of network effects may be less relevant than argued by the previous literature.

9 Conclusion

In this paper, we have shown how a system of competing private currencies can work. Our evaluation of such a system is nuanced. While we offer glimpses of hope for it by showing the existence of stationary equilibria that deliver price stability, there are plenty of other less desirable equilibria. And even the best equilibrium does not deliver the socially optimum amount of money. Furthermore, at this stage of the research, we do not have any argument to forecast the empirical likelihood of each of these equilibria.

Although a system of competing currencies does not *necessarily* imply efficiency and stability, we have identified the characteristics of an efficient and stable private monetary arrangement under free entry, given that our framework is sufficiently flexible to consider relevant extensions of the baseline model. As we have seen, the introduction of productive capital fundamentally changes the properties of the model, implying the existence of a unique equilibrium allocation that can be arbitrarily close to the efficient allocation depending on the technological rate of return on capital. This result is illuminating as it identifies the characteristics of a socially beneficial private monetary system.

We have, nevertheless, just scratched the surface of the study of private currency competition. Many topics, including the analysis of the different degrees of moneyness of private currencies (including interest-bearing assets and redeemeable instruments), the role of positive transaction costs among different currencies, the entry and exit of entrepreneurs, the possibility of market power by currency issuers, or the consequences of the lack of enforceability of contracts are some of the avenues for future research that we hope to tackle shortly.

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