Discontinuity in Labor Laws, Firm Size Distribution and Reallocation *

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November 14, 2012

Abstract

In France, firms with 50 employees or more face substantially more regulation than firms with less than 50. As a result, the size distribution of firms is visibly distorted: there are many firms with exactly 49 employees. We introduce a simple dynamic model that emphasizes the sunk cost aspect of the regulation, and we estimate the model using SMM by fitting these salient features of the size distribution. The key finding is that the legislation acts like a sunk cost equivalent to more than 1 year of the average employee's salary. Removing the regulation improves labor allocation across firms, leading to a productivity gain of around 0.3%, holding the number of firms fixed. However, if firm entry is elastic, the steady-state gains are significantly smaller.

Keywords: firm size distribution, regulation, threshold effect, reallocation.

JEL: O1; O40; E23

Preliminary and Incomplete - Please do not circulate

*We thank Nick Bloom and Jeff Campbell for useful discussions. The views expressed here are those of the authors and do not necessarily represent those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

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1 Introduction

In many countries, small firms face lighter regulation than large firms. Regulation, broadly defined, takes many forms, from hygiene and safety rules, to mandatory elections of employee representatives, to larger payroll taxes. The rationale for exempting small firms from some regulations is that the compliance cost is too high relative to their sales. A necessary consequence, however, is that regulations are phased in as the firm grows, generating an implicit marginal tax. Because regulations are typically phased in at a few finite points, they are sometimes referred to as "threshold effects": for instance, in the case of France, a first important set of regulations applies to firms with more than 10 employees, and a second important set of regulations applies to firms with more than 50 employees. As a result, the firm size distribution is distorted, with few firms with exactly 10 (or 50) employees and a large number of firms with 9 (or 49) employees. Figure 1 plots the firm size distribution in our French data, illustrating this well-known pattern.

These distortions have generated a large attention in public policy circles; for instance numerous commissions drew attention to this issue (see for instance Cahuc and Kramarz (2004)). In spite of this interest, there are few studies that use an explicit model to nderstand evaluate these policies. On the positive side, a structural model is needed to understand the exact sources of distortion. Are regulations equivalent to higher fixed costs, higher proportional taxes on labor, or to a sunk cost? It is not obvious how the regulations should be modeled, given their scope and complexity (The appendix gives the list of regulations). The puzzle that quickly emerges is, why are there *any* firms at all with exactly 50 employees given the higher fixed costs? Our intuition is that many of these regulations might be better approximated as a sunk cost (i.e. a one-time investment). This in turn can generate a distribution with some firms that have exactly 50 employees. The reason is that, with sunk costs, firms are reluctant to have more than 50 employees the first time that they reach that limit, but they do not care about the limit in subsequent periods. the normative side, what are the potential benefits of removing, or smoothing, the regulation thresholds? In this paper, we introduce and estimate a simple structural model that takes into account the phase-in of the regulation. We model the regulation through two parameters: (a) a sunk cost of compliance, which captures both the cost of learning the regulation the cost of any one-time investment that it requires; (b) a per-period higher payroll tax. Hence, a firm has to decide when to pay the sunk cost; once it has paid the sunk cost, it can decide in any given period whether to be above or below the threshold.

Our model can be solved using standard stochastic dynamic optimization techniques (Dixit and Pindyck (1994), Stokey (2008)), and we obtain the cross-sectional distribution in closed form. This is useful for intuition, and it is useful when we turn to the estimation because simulating accurately the highly skewed cross-sectional distribution of firms is very challenging. We fit the model to our French data and we use our model estimates to infer the cost of the regulation. Holding the number of firms constant, we find a productivity loss of 0.3% due to misallocation of labor across firms. However when we allow the number of firms to adjust, we find a much smaller effect, around 0.04%. This suggests that these regulations may not have large aggregative effects.

The rest of the paper is organized as follows. We first discuss the related literature. Section 2 presents the data and some reduced-form evidence that motivates our analysis. Section 3 discusses the model. Section 4 covers our estimation method (SMM) and presents the empirical results. Section 5 uses these estimates to conduct some policy experiments. Section 6 concludes.

Related Literature Our paper is related to a recent growing literature which studies the effect of misallocation on aggregate productivity and welfare. Building on Hopenhayn and Rogerson (1993), Restuccia and Rogerson (2008), and Guner et al. (2008) suggest that misallocation is an important determinant of total factor productivity. Hsieh and Klenow (2009) and Bartelsman et al. (2009) present empirical evidence consistent with higher misallocation in poorer countries with lower TFP.

In the study by Restuccia and Rogerson (2008), distortions arise due to implicit taxes, which are difficult to measure or infer. The regulations that we discuss are a prime example of these distortions, and they very clearly affect the firm distribution, consistent with these studies. While our aim is more modest than the macro studies, since we focus on one particular distortion, we believe that our focus allows a more credible identification of the effect of government regulation on firms outcomes. In particular, we evaluate whether it is feasible to match the distortion in the firm size distribution, which is the *prima facie* evidence of the regulation.

There are several existing studies documenting the size distortion (Cahuc and Kramarz (2004),Ceci-Renaud and Chevalier (2011)) but we are not aware of any structural modeling that tries to apprehend the costs of the distortion. While finishing this paper, we became aware of a very recent working paper Garicano et al. (2012) that shares some of our goals and approach. The key differences between our papers are that we focus on the sunk cost element of the regulation and aim to fit the distribution around the threshold, whereas they focus on the labor tax element and aim at the entire firm distribution. We use similar data but different estimation methods and different targets. The results are quite different, in part because of the different models and in part because they estimate larger costs and higher measurement error than we do. Overall, our results are complementary.

2 Motivating Evidence

We first describe briefly the institutional background, then we present our data sources, and finally we show some simple reduced-form evidence of the threshold effects.

2.1 Institutional Background

This section draws heavily from Ceci-Renaud and Chevalier (2011). Labor laws in France as well as various accounting and legal rules apply for firms with more than 10, 11, 20, or 50 employees.

These regulations are not all based on the same definition of "employee". Labor laws, which are likely the most important, are based on the full-time equivalent workforce, computed as an average over the last 12 months. The full-time equivalent workforce takes into account part-time workers, as well a temporary workers, but not trainees or subsidized employment (*contrats aidés*). On the other hand, several accounting rules are based on sales as well as employment.

The main additional regulations as the firm reaches 50 employees are: - possibly mandatory designation of an employee representative; - a committee for hygiene, safety and work conditions must be formed and trained; - a *comité d'entreprise* must be formed, that must meet at least every two months; this committee, that must have some office space and receives a subsidy equal to 0.2% of the total payroll, has both social objectives (e.g., organizing cultural or sports activities for employees) and economic role (mostly on a consultatory basis); - higher payroll tax subsidizing training which goes from 0.9% to 1.5% (*formation professionelle*); - in case of firing of more than 9 workers, a special legal process must be followed (*plan social*). This legal process implies potentially a large cost and high uncertainty for the firm.

We emphasize that these are just a subset of the regulations which apply. This is enough to give a glimpse of why one may expect them to be important, and also the difficulty of modeling these rules in a simple model: while some are simply monetary rules (e.g., higher taxes), many add an element of uncertainty, and many incorporate some organizational work.

2.2 Data

We use a panel data of firms assembled by the French National Statistical Institute (INSEE), that covers the 1994-2000 period. This panel, known as BRN (Bénéfices Réels Normaux), contains standard accounting information on total compensation costs, value added, current operating surplus, gross productive assets, etc. The BRN data are exhaustive of all private companies with a sales turnover of more than 3.5 million

Francs (around 500,000 Euros) and liable to corporate taxes under the standard regime.

2.3 Preliminary data analysis

Figure 1 plots the distribution of employment for the entire period (1994-2000) and is truncated at 100 employees. Figure 2 zooms on on the distribution between 40 and 60 employees. There are clearly large discontinuities around the thresholds of 10 and 50 employees. On the other hand, the threshold for 20 employees appears less significant. Many surveys reveal “rounding” of employment, but this figure shows the opposite pattern.

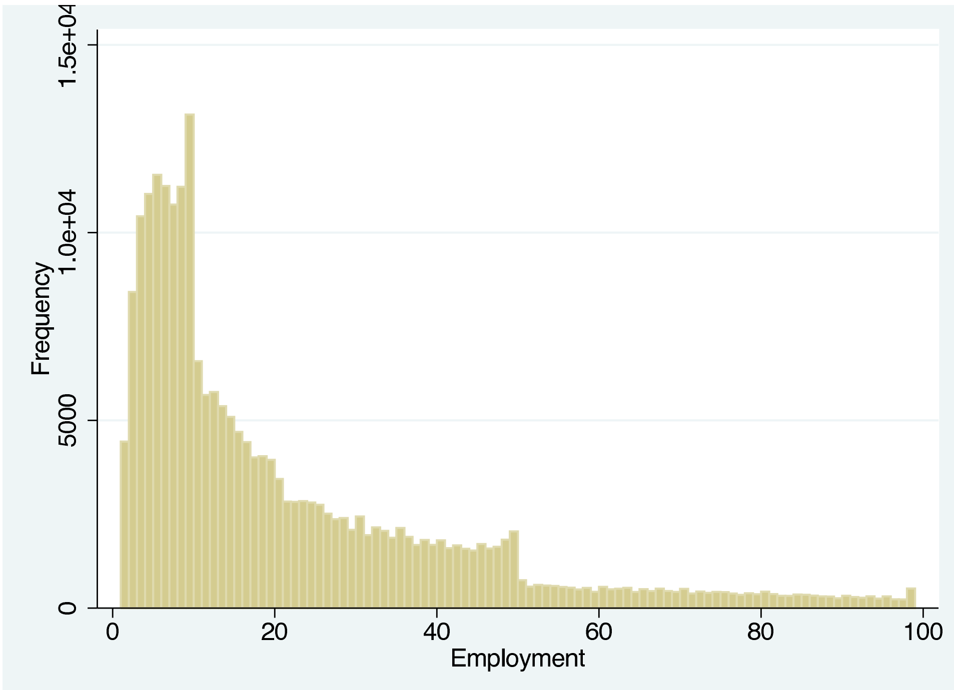

Figure 1: histogram

Table 1 displays the following descriptive statistics: bins of the distribution of employment over the range $20-70$ normalized by the fraction of firms between 20 and 70. and the probability of employment in a given bin at time t conditional on having employment in the same bin at time $t-1$. This table reveals that firms in the bin 45-49 are significantly more likely to remain in that bin next year (59% compared to 50% for the bin 35-39 and 38% for the bin 55-59). This suggests that the presence of the threshold leads to inaction and hence slows down the growth of employment.

A useful way to summarize the break in this distribution is to run a regression of the log frequency on log size, with or without a structural break at size 50. Figure 2.3 plots the regression lines. The $R^2 = 0.975$.

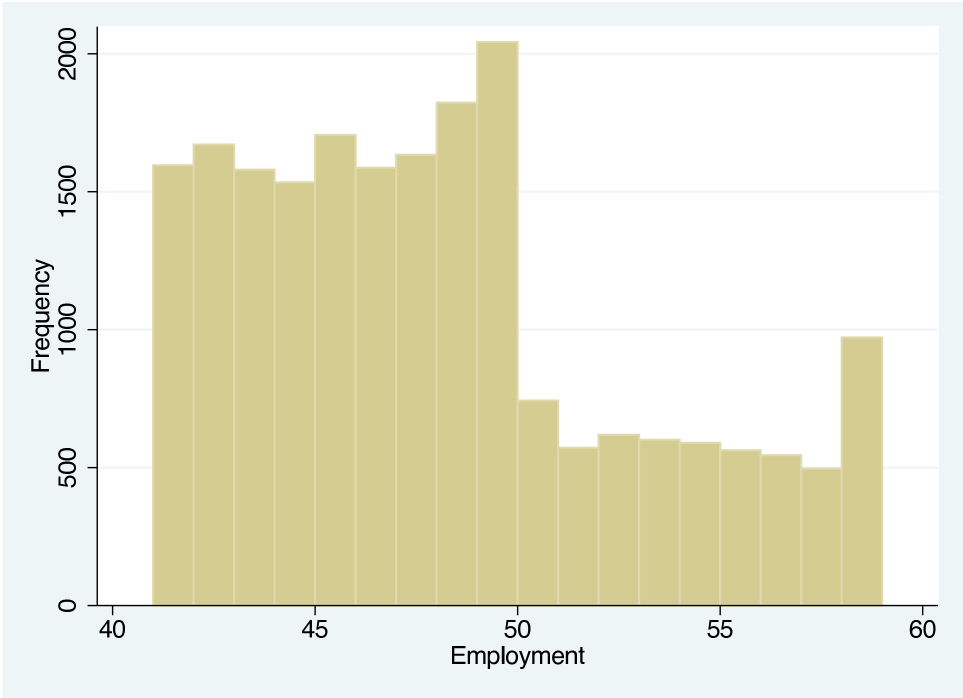

Figure 2: histogram zoom

Table 1: Descriptive Statistics

	Distribution of Employment %
[21 − 40)	66.48
[40 − 45)	10.02
[45 − 50)	10.91
[50 − 55)	3.97
[55 − 60)	3.20
[60 − 70)	5.39

	$Pr(n_t \in bin \mid n_{t-1} \in bin)$
35 - 39	50.06
45 - 49	59.13
55 - 59	38.12

The presence of a structural break is clearly visible from this figure.

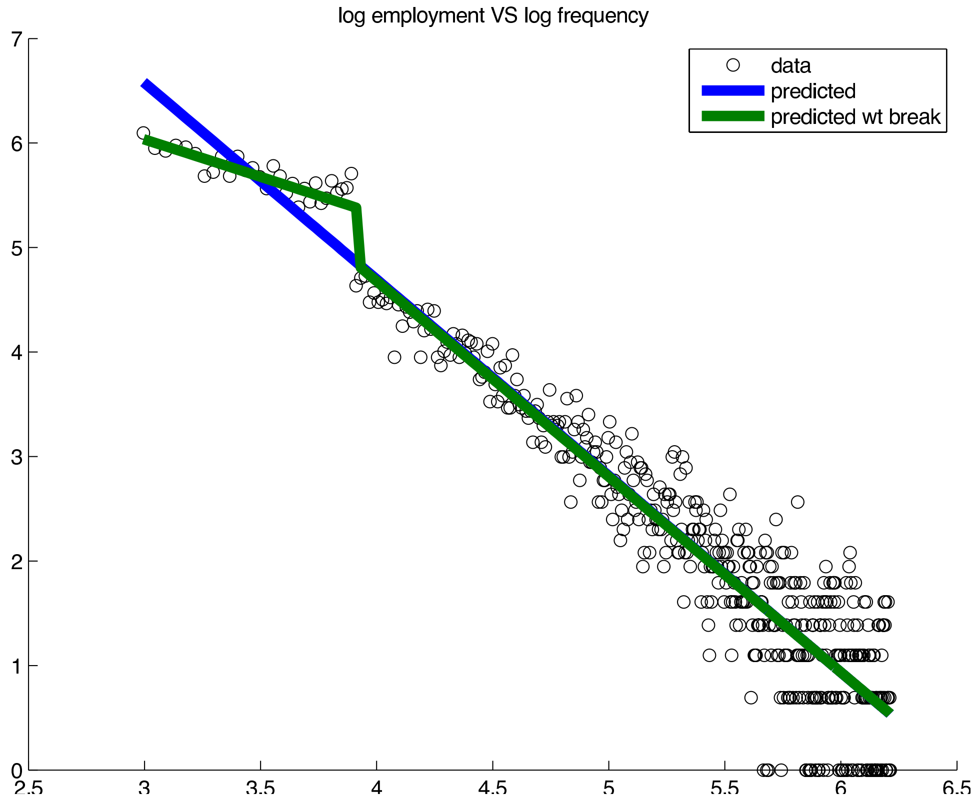

Figure 3: Regression Log Employment on Log Frequency

Next, we estimate a probit characterizing the probability of not adjusting employment. Explanatory variables are a set of dummies variables indicating whether or not last period employment was $45, ..., 55$, the growth rate of production, last period employment, and a set of time dummies capturing aggregate shocks. This estimation uses firms with an average level of employment between 20 and 100 over the period 1994-2000. Table 2 reports the coefficients. The probability of inaction increases for firm with a number of employees between 45 and 49. The largest increase is observed for firms of size 49.

Finally, Figure 2.3 plots labor productivity by size. While there is substantial noise in this figure, it is noteworthy that the peak of labor productivity is obtained for 49 employees. Our model will replicate this feature: because firms are reluctant to go over the threshold, they hire less labor than they would, generating large output per worker.

Variable	Coefficient	(Std. Err.)
Production Growth Rate	-0.211	(0.021)
Log of Previous Period Employment	-0.287	(0.013)
Size 45	0.101	(0.039)
Size 46	0.093	(0.041)
Size 47	0.152	(0.040)
Size 48	0.315	(0.036)
Size 49	0.620	(0.032)
Size 50	-0.017	(0.062)
Size 51	-0.074	(0.074)
Size 52	-0.141	(0.073)
Size 53	-0.067	(0.071)
Size 54	0.008	(0.070)
Size 55	-0.134	(0.076)

Table 2: Estimation results : Probit

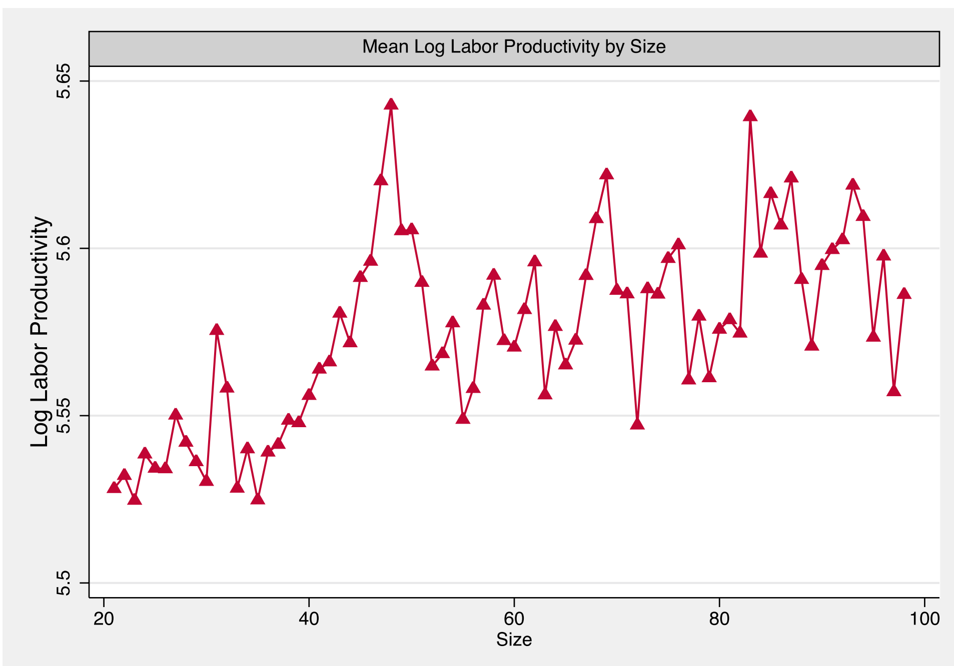

Figure 4: Mean Labor Productivity by Size

3 Model

In this section, we introduce and solve a simple dynamic model of production and employment, based on Lucas (1978). For simplicity we assume that there is only one superscript. Firms face a regulation which requires them to pay a sunk cost the first time that their employment exceeds the superscript $\underline{n}$, and firms face higher marginal and fixed costs each period if they currently have more than $\underline{n}$ employees. We start with a partial-equilibrium model, which is the basis of our estimation strategy. Section 5 embeds our model of the firm in a general equilibrium framework to perform some policy experiments.

3.1 Model assumptions

Time is continuous and there is no aggregate uncertainty. There is a continuum of firms, which are ex-ante homogeneous but differ in their realization of idiosyncratic shocks. Each firm operates a decreasing-return to scale, labor-only production:

$$y = e^z n^\alpha,$$

where $\alpha \in (0, 1)$ and e^z is the exogenous productivity (e denotes the exponential function). For simplicity we assume that exit is exogenous and occurs at rate λ. Note that we abstract from fixed cost in this problem; given that we assume exogenous exit, this is without loss of generality. Fixed costs do not affect the employment decision, and we do not use profits data in our estimation.

We assume that log productivity z follows a Brownian motion,

$$dz = \mu dt + \sigma dW_t.$$

This specification is attractive not only because of its tractability, but because it is consistent with two robust features of the data: (i) firm-level shocks are highly persistent, if not permanent; (ii) the firm size distribution follows a Pareto distribution. The geometric brownian motion dynamics generate a stationary distribution that is Pareto.

We also assume that all firms enter with the same productivity z_0. This simplification has little impact on our results since we do not focus our estimation on very small firms (which is where the entrants start).

Employment n can be costlessly adjusted, and the wage is w. For simplicity, we assume that n is a continuous choice (i.e., we do not impose indivisibility). If n is greater than $\underline{n}$, a proportional tax τ applies to the wage rate and a fixed cost c_f has to be paid. We assume that the proportional tax applies to all employment, including that below $\underline{n}$, but this is without loss of generality, since we allow the fixed cost c_f to be negative (i.e., the tax could apply only to employment in excess of $\underline{n}$). The first-time a firm crosses the superscript $\underline{n}$, it has to pay a sunk cost F. This cost captures the investment necessary to comply with the regulation, including the physical cost of buying an equipment, but also the informational costs such as learning about the regulation and perhaps consulting with a lawyer. These costs may also reflect managerial time that is lost.

The presence of the sunk cost makes this a dynamic optimization problem. Let $s \in \{0, 1\}$ denote whether a firm has already paid the sunk cost in the past. The state of the firm is summarized by (z, s).

3.2 Static subproblem

We first study the static problem, to determine the firm profit function which will enter the dynamic optimization.[1] To find the optimal labor demand and profit of the firm, we first solve the firm's problem conditional on operating below the threshold, then we find the solution conditional on operating above the threshold, and finally we find the overall solution by combining these results.

The current-period profit function for a firm which operates below the threshold is:

$$\pi^b(z) = \max_{0 \le n < \underline{n}} \left\{ e^z n^\alpha - wn \right\}. \tag{1}$$

The superscript b stands for "below the threshold". Optimal employment is given by:

$$n^b(z) = \begin{cases} \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} e^{\frac{z}{1-\alpha}} & , \text{ if } z < \underline{z} \\ \underline{n}^- & , \text{ if } z \ge \underline{z}. \end{cases}$$

where $\underline{z} = \log\left(\underline{n}^{1-\alpha} \frac{w}{\alpha}\right)$ and $\underline{n}^-$ indicates a value just below $\underline{n}$. Profits are given by the formula

$$\begin{aligned} \pi^b(z) &= e^{\frac{z}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha), \text{ if } z < \underline{z} \\ &= e^z \underline{n}^\alpha - w\underline{n}, \text{ if } z \ge \underline{z}. \end{aligned}$$

The current-period profit function for a firm that decides to operate above the threshold, and hence to face

[1] This section thus does not depend on assumption that z is a Brownian motion.

the regulation, is:

$$\pi^a(z) = \max_{n \geq \underline{n}} \left\{ e^z n^\alpha - w(1+\tau)n - c_f \right\}. \tag{2}$$

where the superscript a stands for "above the threshold". The firm operates above the threshold if z is greater than a cutoff value $\overline{z}$, defined as the solution to

$$e^{\frac{z}{1-\alpha}} \left(\frac{\alpha}{w(1+\tau)} \right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f = e^z \underline{n}^\alpha - w\underline{n}.$$

It is easy to see that $\overline{z} > \underline{z}$, provided that there is a cost of operating above the threshold: $\tau\overline{n} + c_f > 0$. We will maintain this realistic assumption throughout the paper.

Summarizing, optimal employment if the firm decides to operate above the threshold is

$$n^a(z) \quad = \quad \begin{cases} \underline{n} & \text{if } z < \overline{z}, \\ \left(\frac{\alpha}{w(1+\tau)} \right)^{\frac{1}{1-\alpha}} e^{\frac{z}{1-\alpha}} & , \text{ if } z \geq \overline{z}, \end{cases} \tag{3}$$

This leads to profits

$$\begin{aligned} \pi^a(z) \quad &= \quad e^{\frac{z}{1-\alpha}} \left(\frac{\alpha}{w(1+\tau)} \right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f, \text{ if } z \geq \overline{z}, \\ \pi^a(z) \quad &= \quad e^z \underline{n}^\alpha - w(1+\tau)\underline{n} - c_f, \text{ if } z < \overline{z}. \end{aligned}$$

Combining our results, we can now write the firm profit, as a function of the current productivity and state $s \in \{0, 1\}$. Recall that $s = 0$ means that the firm has not paid the sunk cost and hence is forced to operate below the threshold, whereas a firm with $s = 1$ can *choose* to operate either below or above the threshold. Mathematically,

$$\pi(z, 0) = \pi^b(z),$$

$$\pi(z, 1) = \max \left\{ \pi^a(z), \pi^b(z) \right\}.$$

We can obtain a formula for $\pi(z, 1)$ by noting the following: (i) if $z < \underline{z}$, $\pi^b(z) > \pi^a(z)$, since the firm pays lower wages and fixed costs; (ii) for $z > \underline{z}$, the firm will decide to operate above the threshold; (iii) if

$z \in (\underline{z},\overline{z})$, it is optimal to remain just below the threshold. Hence,

$$\begin{aligned}
\pi(z,1) &= e^{\frac{z}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) \text{ for } z<\underline{z}, \\
&= e^{z}\underline{n}^{\alpha} - w\underline{n} \text{ for } \underline{z} \le z \le \overline{z}, \\
&= e^{\frac{z}{1-\alpha}}\left(\frac{\alpha}{w(1+\tau)}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) - c_f \text{ for } z>\overline{z}.
\end{aligned}$$

For completeness, we also state the

$$\begin{aligned}
\pi(z,0) &= e^{\frac{z}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) \text{ for } z<\underline{z}, \\
&= e^{z}\underline{n}^{\alpha} - w\underline{n} \text{ for } z \ge \underline{z}.
\end{aligned}$$

and the employment demand:

$$\begin{aligned}
n(z,0) &= \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} e^{\frac{z}{1-\alpha}} \text{ for } z<\underline{z}, \\
&= \underline{n}^{-} \text{ for } z>\underline{z}.
\end{aligned}$$

$$\begin{aligned}
n(z,1) &= \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} e^{\frac{z}{1-\alpha}} \text{ for } z<\underline{z}, \\
&= \underline{n} \text{ for } \underline{z} \le z \le \overline{z}, \\
&= \left(\frac{\alpha}{w(1+\tau)}\right)^{\frac{1}{1-\alpha}} e^{\frac{z}{1-\alpha}} \text{ for } z \ge \overline{z}.
\end{aligned}$$

Figure 5 plots profit as a function of employment for a high, medium and low value of productivity. The left column is the case of a proportional wage tax while the right column represents a fixed cost. This figure illustrates that firms with low productivity decide to operate below the threshold, since it is where their profits are highest. The high productivity firms operate above the threshold. The medium productivity may operate exactly at (just below) the threshold.

3.3 Dynamic optimization

Given the process for z, and the probability of exit λ, the firm's value maximization problem can be written formally as choosing a stopping time T to cross the threshold. Formally, for a firm that has prodctivity z

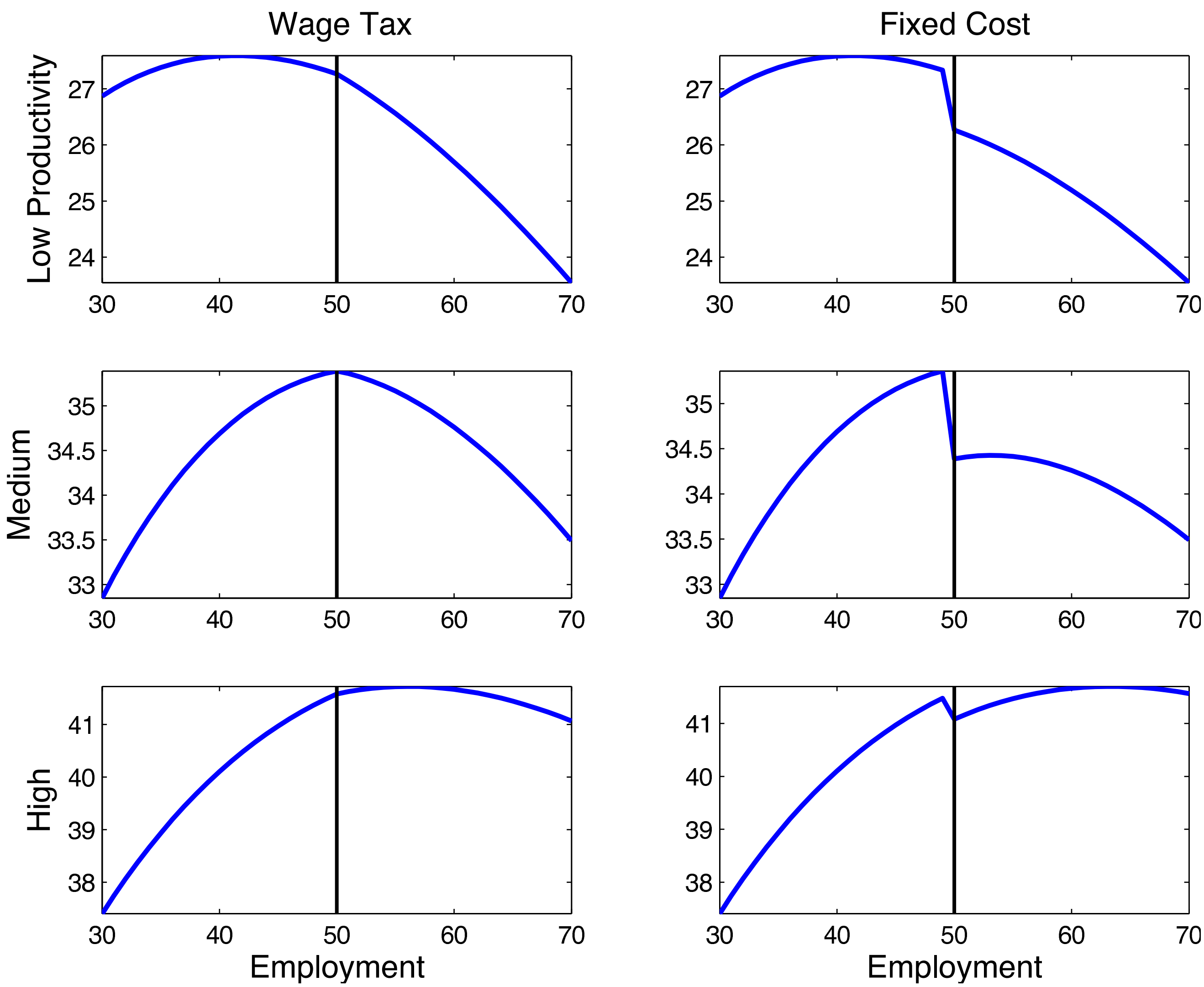

Figure 5: Profit and employment for different values of productivity

today:

$$V(z,0)=\sup_{T\geq 0} E\left[\int_0^T e^{-(r+\lambda)t}\pi(z_t,0)dt+\left(\int_T^\infty e^{-(r+\lambda)t}\pi(z_t,1)dt-Fe^{-(r+\lambda)T}\right)\right]. \qquad (4)$$

Intuitively, the firm will make the switch if its productivity becomes large enough; denote by z^* the cutoff that triggers the firm to pay the sunk cost. A standard option value argument implies that z^* will be greater than $\overline{z}$. (Note that in writing this expression, we normalized the exit value to zero; since exit is exogenous, this is without loss of generality.)

This section presents the solution of the model using directly some results in Stokey (2008) for a general option exercise problem.[2] First, we rewrite the problem explicitely as choosing a cutoff z^* :

$$V(z,0)=\sup_{z^*\geq z} E_z\left[\int_0^{T(z^*)} e^{-(r+\lambda)t}\pi(z_t,0)dt+e^{-(r+\lambda)T(z^*)}\left(V(z^*,1)-F\right)\right], \qquad (5)$$

with

$$V(z^*,1)\equiv E_{z^*}\left[\int_0^\infty e^{-(r+\lambda)t}\pi(z_t,1)dt\right],$$

and with R_1 and R_2 the roots of the quadratic $\frac{\sigma^2}{2}R^2+\mu R-(\lambda+r)=0$, i.e. with $J=\sqrt{\mu^2+2(r+\lambda)\sigma^2}$, we have $R_1=\frac{-\mu-J}{\sigma^2}<0$, and $R_2=\frac{-\mu+J}{\sigma^2}>0$.

The next proposition derives the optimal policy. In the language of Stokey (2008), R_1 discounts the time the process z will spend between $\overline{z}$ and z^*.

Proposition. *The solution to the Problem in Equation (5) is z^{star}, the unique value satisfying:*

$$-R_1\int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\left[\pi^a(z)-\pi^b(z)\right]dz=(r+\lambda)F. \qquad (6)$$

Proof. see appendix. □

For given structural parameters $\{\alpha,\overline{n},\mu,\sigma,\tau,c_f,F,r,\lambda\}$, this equation allows us to find z^* numerically easily. We conclude this subsection by noting some intuitive comparative statics: higher uncertainty, higher sunk costs, or higher fixed costs, all make it optimal to wait longer before crossing the threshold. This is the standard real option effect.

Corollary. *z^* is increasing in σ^2,F,τ_w,τ_f and $\underline{n}$*

Proof. Differentiation of Equation (6) gives the results. □

[2] An alternative solution method, using the more intuitive Hamilton-Jacobi-Bellman equations and smooth pasting conditions, is presented in the appendix.

3.4 Stationary Distribution

Given our interest in the size distribution, we derive the joint cross-sectional distribution over (z, s) in closed form. Denote the probability density function as $f(z, s)$. Recall that firms enter with $z = z_0$, and z then evolves according to a Brownian motion with parameters (μ, σ). Firms switch from $s = 0$ to $s = 1$ as soon as z reaches z^*, and exit upon the realization of a Poisson process with parameter λ. We can write the Kolmogorov Forward equation, which reflects the conservation of the total number of firms, net of exit:

$$-\mu \frac{\partial f(z,0)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 f(z,0)}{\partial z^2} = \lambda f(z,0), \tag{7}$$

which holds for all $z < z_0$ and all $z \in (z_0, z^*)$. (See Dixit and Pindyck (1994), appendix of chapter 3, for a heuristic derivation, and chapter 8 for an application similar to our case.) The equation needs not hold for $z = z_0$, since there is entry of new firms.

The same equation applies to firms which have made the switch:

$$-\mu \frac{\partial f(z,1)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 f(z,1)}{\partial z^2} = \lambda f(z,1), \tag{8}$$

which holds for all $z \in (-\infty, z^*)$ and for all $z \in (z^*, +\infty)$.

Last, we need to state the boundary conditions. The first one is simply the requirement that f is a density, i.e.

$$\int_{-\infty}^{+\infty} f(s,1)ds + \int_{-\infty}^{+\infty} f(s,0)ds = 1.$$

To derive the other boundary conditions, the easiest approach is to approximate the Brownian motion with a discrete random walk, as in Dixit and Pindyck (1994). This yields the conditions

$$f(z^*, 0) = 0,$$

and $f(., 0)$ must be continuous at z_0, while $f(., 1)$ must be continuous at z^* :

$$\lim_{s \to z_0^-} f(s,0) = \lim_{s \to z_0^+} f(s,0),$$

$$\lim_{s \to z_-^*} f(s,1) = \lim_{s \to z_+^*} f(s,1).$$

Finally, a balance condition holds for $z = z^*$, reflecting that the number of firms which reach z^* and have

$s=0$ is equal to the number of firms which enter at $s=1$ with $z=z^*$, and is equal to the number of firms with $s=1$ which exit in any time period: this leads to

$$-\frac{\sigma^2}{2}f'(z^*,0)=\lambda\int_{-\infty}^{\infty}f(s,1)ds.$$

Given these equations, solving for the cross-sectional distribution involves some simple algebra, which is relegated to the appendix. The result is:

$$\begin{aligned} f(z,0) &= \frac{\beta_1\beta_2}{\beta_1-\beta_2}\left(e^{\beta_2(z-z_0)}-e^{\beta_1(\hat{z}-z_0)}e^{\beta_2(z-\hat{z})}\right),\ \text{for } z<z_0, \\ &= \frac{\beta_1\beta_2}{\beta_1-\beta_2}\left(e^{\beta_1(z-z_0)}-e^{\beta_1(\hat{z}-z_0)}e^{\beta_2(z-\hat{z})}\right),\ \text{for } z^*>z>z_0, \end{aligned}$$

and

$$\begin{aligned} f(z,1) &= \frac{\beta_1\beta_2}{\beta_1-\beta_2}e^{\beta_1(\hat{z}-z_0)}e^{\beta_2(z-\hat{z})},\ \text{for } z<z^*, \\ &= \frac{\beta_1\beta_2}{\beta_1-\beta_2}e^{\beta_1(z-z_0)},\ \text{for } z>z^*. \end{aligned}$$

This expression implies that z has an exponential distribution in the upper tail. Since log employment and log sales are both proportional to z, this implies that employment and sales follow Pareto distributions. The exponent of the employment distribution is $\beta_1(1-\alpha)-1$ (i.e. the p.d.f. of n is proportional to n^{ξ} with $\xi=\beta_1(1-\alpha)-1$). Note that this implies some restrictions on β_1 to ensure that employment be finite. This in turn restricts the parameters μ,λ,σ^2.[3]

Figure 6 presents a numerical example of the implied distribution of employment. For now focus on the left column. The top row shows the distribution without distortion - it is is Pareto. The bottom row depicts the distribution with a per-period fixed cost. (The results are similar with a wage tax.) There is a substantial "hole" in the distribution with no firms whatsoever between 50 and 67 employees. This figure presents an empirical challenge, because Last, the middle row shows the impact of a sunk cost on the firm size distribution. The sunk cost model does not suffer from the same deficiency as the fixed cost model: there are no holes in the distribution, and in particular some firms have exactly 50 employees. These are firms that crossed the threshold in the past and that were subsequently hit by negative productivity shocks.

To establish the economic relevance of these regulations, we now turn to the data and propose a simple structural estimation of our model.

[3] However, our estimated parameters satisfy these restrictions, so we do not need to impose them in practice.

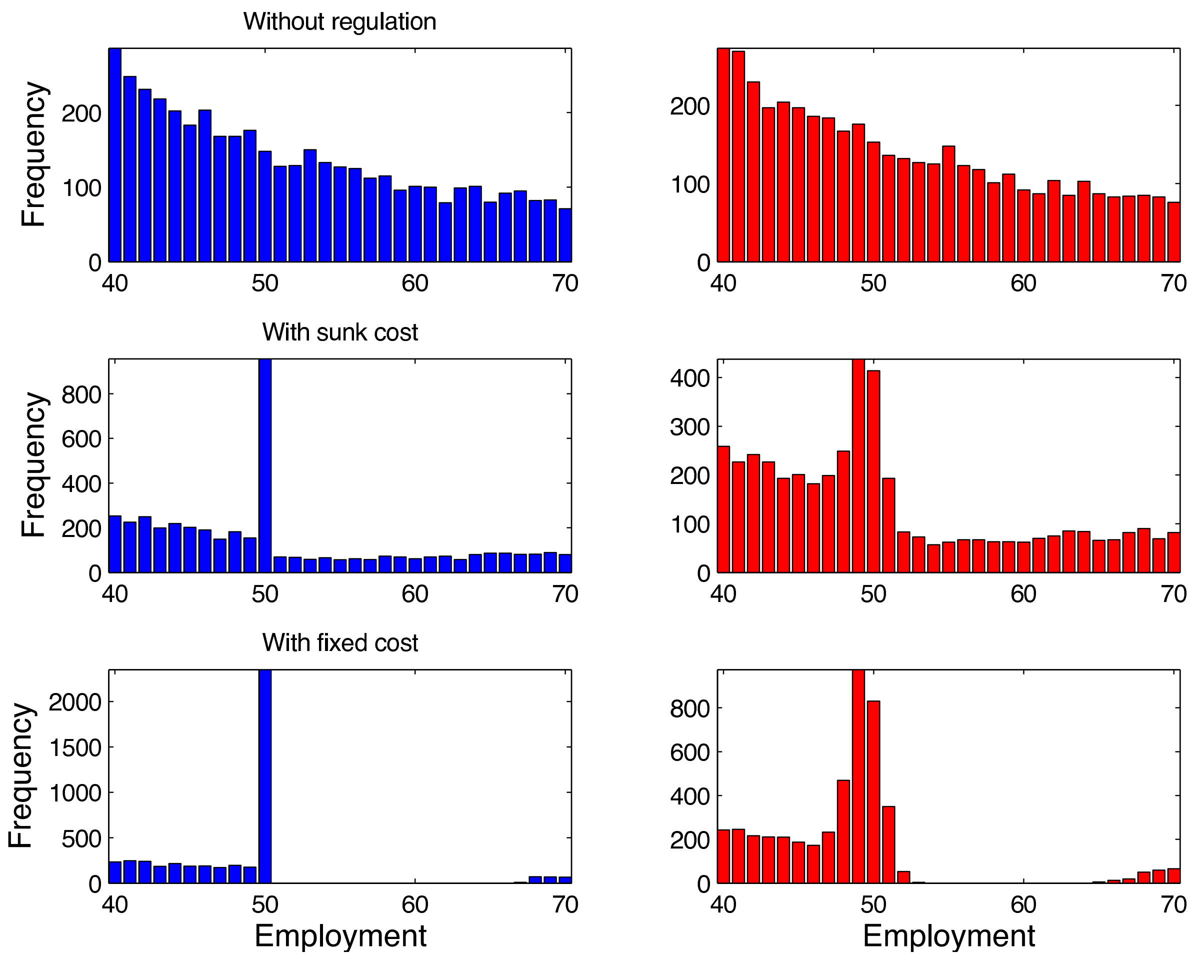

Figure 6: Theoretical distribution of employment around the threshold

Parameters	Definition	
r	interest rate	fixed
α	curvature profit function	fixed
z_0	Entry TFP level	fixed
w	wages	normalized
λ	death probability	estimated
μ	drift	estimated
σ	std dev shocks	estimated
τ	proportional tax on wages	estimated
F	sunk cost	estimated
σ_{mrn}	measurement error	estimated

Table 3: Economic Parameters

4 Estimation

This section proposes a simple estimation of our model using indirect inference. We take advantage of our closed form solutions which make calculating model moments computationally easy. Given that the data is likely to have some noise, we incorporate classical measurement error in (log) employment. The full set of structural parameters is the vector $\theta = (r, w, \alpha, z_0, \lambda, \mu, \sigma, \tau, F, \sigma_{MRE})$. We partition this vector into two vectors, i.e. $\theta = (\theta_p, \theta_e)$ where $\theta_p = (r, w, \alpha, z_0)$ includes parameters that are set a priori, and $\theta_e = (\lambda, \mu, \sigma, \tau, F, \sigma_{MRE})$ is the vector of estimated parameters.

Like calibration, indirect inference works by selecting a set of statistics of interest, which the model is asked to reproduce. These statistics are called sample auxiliary parameters $\hat{\Psi}$. For an arbitrary value of θ_e, we use the structural model to generate S statistically independent simulated data set and compute simulated auxiliary parameters $\Psi^s(\theta_e)$. The parameter estimate $\hat{\theta}_e$ is then derived by searching over the parameter space to find the parameter vector which minimizes the criterion function:

$$\hat{\theta}_e = \arg\min_{\theta_e \in \Theta_e} \left(\hat{\Psi} - \frac{1}{S}\Psi^s(\theta_e)\right)' W \left(\hat{\Psi} - \frac{1}{S}\Psi^s(\theta_e)\right)$$

where W is a weighting matrix and Θ_e the estimated parameters space. We choose the identity matrix as using the optimal weighting matrix is fraught with a well-known small sample bias problem (cf. Altonji and Segal (1996)). This procedure will generate a consistent estimate of θ_e. The minimization is performed using numerical techniques. We assume the economy is at its steady state. To simulate the model, we draw from the stationary distribution derived in the previous section. The variance of the estimator is estimated using 1000 bootstrap repetitions.

Fraction of firms in a bin normalized by the fraction of firms between 40-60	
40-45	0.3565
45-50	0.3883
50-55	0.1413
55-60	0.1140
Median net employment growth	0
Variance of logged employment variations	0.0485
Slope of the power law for firms>100	1.1417
$\frac{\text{Employment Share - Firms larger than 200}}{\text{Employment Share - Firms larger than 100}}$	0.8326

Table 4: Auxiliary Parameters

4.1 Predefined Parameters

The variable employment in the dataset is measured as the arithmetic average of the number of employees at the end of each quarter. And heavy rounding can be expected. We explicity introduce measurement error into the simulated moments to mimic the bias these impute into the actual data moments by multiplying employment by, respectively, mrn_{it} that is i.i.d over firm and time and follow a log-normal distribution with mean $-\frac{1}{2}\sigma^2_{\text{mrn}}$ and standard deviation σ_{mrn}. The real interest rate r is set to 5 percent. The wage rate is normalized to 1.

4.2 Auxiliary Parameters and Identification

Table 4.2 lists the auxiliary parameters. We match bins of the distribution of employment around the thresholds, trend in aggregate employment variation and the employment share of firms larger than 200 employees divided by the employment share of firms larger than 100 employees and the slope of the power law.

Identification of the models's parameters is achieved by a combination of functional form and distributional assumptions. Heuristically, the median net employment growth is informative about the drift μ. The bins between 40 and 60 are informative regarding the frictions parameters τ and F and the variance of measurement error σ_{mrn}. The variance of logged employment variations is informative about the variance of TFP shocks σ and the variance of measurement error σ_{mrn}. The employment share of large firms and the slope of the power law are informative regarding the variance of TFP shocks σ, the drift μ and the exit rate λ.

	$F = 0$		$\tau_w = 0$	
λ	0.0347	(0.0065)	0.0478	(0.0076)
μ	-0.0004	(0.0066)	0.0018	(0.0039)
σ	0.0717	(0.0110)	0.0765	(0.0074)
σ_{mre}	0.0370	(0.0140)	0.0111	(0.0046)
τ	0.0027	(0.0006)	0	
F	0		1.0840	(0.1212)

Table 5: Estimation Results

4.3 Estimation Results

Table 5 reports the structural parameters estimates with standard deviation in parenthesis. The first column estimates the model without sunk costs: $F = 0$ and the second column estimates the model without the increase in the proportional tax on wages: $\tau = 0$.

The data are consistent with a legislation that acts like a sunk cost of about one year of a worker wages or a small proportional tax on wages of 0.25%. The estimates for the drift and the variance are not very sensitive to the specifications of the distortions. Shocks to TFP are estimated to be 7% per year and the drift is not significantly different from zero.

The variance of measurement is much larger for the model with a proportional tax than a model with a sunk cost. The main reason is that the model with a proportional tax implies that firms do not want to operate on the right side of the threshold. Even with a small estimated tax, the model predicts that when it is optimal to cross the threshold, firms operate at a size of at least 56 employees. Without measurement errors, there would be no firms with an employment level between 50 and 56. The model with a sunk cost does not suffer from this feature. Some firms operate naturally to the right of the threshold: firms that have crossed the threshold in the past and that were subsequently hit by negative productivity shocks. As a result, the amount of measurement error is much lower in the sunk model with an estimated standard deviation of 1% compared to the model with a proportional tax that implies a measurement error of 3%.

The exit rate is estimated to be 3.47% which is relatively low but it is because our model do not allow for endogenous exit. Hence the mode underpredicts the exit rate of small firms and is better description of the exit behavior of larger firms.

Table 4.3 reports the model fit. The model is able to fit the moments well.

We examine the capacity of the model to account for the firm size distribution and the employment share of firms of different size. The numbers reported are normalized by the fraction of firm of more than 100 employees and the share of employment of firms of more than 100 employees. Although not directly targeted in the estimation, the model does a reasonable job in accounting for these moments.

	Data	$F=0$	$\tau_w=0$
Slope Power Law	1.1417	1.1338	1.1481
$\frac{\text{Employment Share Firms} > 200}{\text{Employment Share Firms} > 100}$	0.8326	0.8263	0.8278
Med $\Delta \log n \vert n > 20$	0	0	0.0035
$V(\Delta \log n \vert n > 20)$	0.0485	0.0470	0.0501
$\frac{\#[40,45)}{\#[40,60)}$	0.3565	0.3445	0.3436
$\frac{\#[45,50)}{\#[40,60)}$	0.3883	0.3858	0.3851
$\frac{\#[50,55)}{\#[40,60)}$	0.1413	0.1489	0.1489
$\frac{\#[55,60)}{\#[40,60)}$	0.1140	0.1208	0.1224

Table 6: Auxiliary Parameters

	Data	$F=0$	$\tau_w=0$
Fraction of firms of more than 200 employees	0.4711	0.4132	0.4139
Fraction of firms of more than 500 employees	0.1615	0.1284	0.1290
Fraction of firms of more than 1000 employees	0.0674	0.0531	0.0534
Fraction of firms of more than 1000 employees	0.0059	0.0068	0.0069
Share of Employment - firms of more than 200 employees	0.8333	0.8263	0.8278
Share of Employment - firms of more than 500 employees	0.6191	0.6421	0.6449
Share of Employment - firms of more than 1000 employees	0.4722	0.5306	0.5339
Share of Employment - firms of more than 5000 employees	0.2113	0.3407	0.3443

Table 7: Firm Size Distribution and Employment Share - Normalized by firms of more than 100 employees

5 Policy Experiments

In the previous section, we estimated the regulatory cost as perceived by firms. In this section, we use our estimates to infer the effect of the regulation on macroeconomic aggregates.

From the point of view of a social planner, the regulation misallocates labor across firms and hence reduces total factor productivity. To demonstrate this, we consider three experiments, which differ in the set of equilibrium feedback that they allow. The first subsection discusses the conceptual framework for our experiments, and the second subsection presents and discusses the results.

5.1 Three Experiments

For the purpose of estimation, we do not need to take a stand on the determination of the number of firms or of market prices: given the observed number of firms and factor prices, we use cross-sectional information to identify our parameters. However, for our policy experiments, it can matter whether the number of firms, or prices adjust in response to a change in the regulation. Our three experiments differ in the assumptions they make about equilibrium price feedback.

Our first experiment abstracts from all feedbacks and considers the effect of removing entirely the regulation, holding the wage, the interest rate and the number of firms fixed. Concretely, we solve the firm problem with the sunk cost, and without the sunk cost, and obtain the policy functions $n(z; w, \theta)$, $y(z; w, \theta)$ and $z^*(w, \theta)$ where for clarity we now index all policy functions by the wage w as well as θ, the vector of parameters (which includes the sunk cost). We then calculate aggregate employment and output using the cross-sectional distribution $f(z; \theta)$:

$$N(w, \theta) = \int_{-\infty}^{\infty} n(z; w, \theta) f(z; \theta) dz,$$

$$Y(w, \theta) = \int_{-\infty}^{\infty} y(z; w, \theta) f(z; \theta) dz.$$

We present the percentage change in N and Y. This experiment implicitly assumes a perfectly elastic supply of labor and capital, and an inelastic supply of firms.

Our second experiment asks, how much of an increase in output can we obtain, holding total employment constant? That is, suppose we try to solve the allocation problem:

$$\max_{\{n(z)\}_{z=-\infty}^{\infty}} \int_{-\infty}^{\infty} e^{z} n(z)^{\alpha} f(z;\theta) dz$$
$$s.t. \quad : \quad \int_{-\infty}^{\infty} n(z) f(z;\theta) dz \leq N,$$

where $n(z)$ is the amount of labor allocated to firms with productivity z. This program gives an optimal allocation of labor and determines the aggregate production function (expressing maximum possible aggregate output given aggregate labor). This calculation shows how much more output can be obtained if labor is reallocated optimally, starting from the distorted distribution.

Our third experiment adds endogenous entry and labor supply to the model by embedding our firm dynamics in a general equilibrium framework as in Hopenhayn and Rogerson (1993).

We first describe the model. There is a representative agent with utility function

$$\int_{0}^{\infty} e^{-\rho t} u(C_t, 1 - N_t) dt.$$

This agent supplies work to the market, at the wage w_t, and buys or sells assets at the interest rate r_t. In equilibrium, the only assets are the firms. There is no aggregate variation, since a law of large numbers apply, and macroeconomic aggregates are constant. As a result, the interest rate is $r_t = r = \rho$.

For a given wage, we can solve the value function $V(z, s; w)$, policy functions $n(z, s; w)$ and $z^*(w)$, and stationary distribution $f(z, s; w)$ as above. We have added the wage as an explicit argument to these functions to emphasize the dependence. Since all firms enter with a productivity z_0, the free entry condition reads,

$$k = V(z_0, 0; w). \tag{9}$$

Assume that the flow of firms entering is E per unit of time. The stationary distribution of firms is then $Mf(z, s; w)$. With exogenous exit at rate λ, the flow of entrants per unit of time E must equal λM in a stationary equilibrium.

Total output is then given by

$$Y(w;\theta) = M \int_{-\infty}^{\infty} e^{z} n(z; w, \theta)^{\alpha} f(z;\theta) dz, \tag{10}$$

and total labor is given by

$$N(w;\theta) = M\int_{-\infty}^{\infty} n(z;w,\theta)f(z;\theta)dz. \tag{11}$$

The labor market clearing condition is

$$\frac{u_2(C,1-N)}{u_1(C,1-N)} = w, \tag{12}$$

We choose the utility function

$$U(C_t,N_t) = \log(C_t) - B\frac{N_t^{1+\phi}}{1+\phi}$$

Labor supply is given by the first order condition

$$B.C_tN_t^{\phi} = w_t, \tag{13}$$

and the goods market constraint is

$$C_t + E_tk = Y_t. \tag{14}$$

A stationary equilibrium is then given by $\{Y,C,E,M,w,N\}$ such that $E = \lambda M$ and the equations (9)-(14) are satisfied.

To understand this model, note that the free entry condition pins down the equilibrium wage. Given this wage, the number of firms adjusts so that labor demand equals labor supply; that is, there is a perfectly elastic supply of firms at a given cost.

We first consider the case of perfectly inelastic labor supply ($B = +\infty$ and $N = \overline{N}$), and then we consider the case of an elastic labor supply ($B < +\infty$). We calibrate the entry cost k to generate a wage equal to one. We also set B such that total employment is 0.25.

Finally, we extend this model by relaxing the assumption that entry is perfectly elastic at cost k. To generate an upward-sloping supply of entrants to the economy, we suppose that there is a pool N of potential entrants, which differ in their entry cost. The entry cost is distributed according to the cumulative distribution function H. In a given period, only potential entrants with an entry cost below $V(z_0,0;w)$ will enter. Denote k^* the threshold value for k. The flow of entrants E will equal $NH(k^*)$ and the free entry condition is $V(z_0,0;w) = k^*$.

We close by mentioning three issues that affect all experiments. First, we need to take a stand on whether the regulation cost is a real resource cost or is a transfer. In reality it is likely that both components are present, and the reality is in the middle. Hence we will present the results for the two possible assumptions.

		Y	N	w	M	C
Partial Eq	Tranfer	1.31	1.34	na	na	na
	Cost	1.32	1.34	na	na	na
Planner	Tranfer	0.42	na	na	na	na
	Cost	0.43	na	na	na	na
General Eq	Transfer	-0.06	-0.05	0.01	-1.32	0.06
	Cost	-0.06	-0.05	0.01	-1.33	0.06

Table 8: Sunk Cost - Policy Experiments

Second, the calculations above focus on steady-state effects and abstract from transitional dynamics. We believe this is appropriate to examine the long-run productive effects of the regulation. But of course this makes the welfare comparison less clear-cut as we discuss below. Last, our calculations have little to say on the desirability of the regulations themselves.

5.2 Results

Tables 8 presents the results of our three experiments, for the case where the regulation is a transfer and for the case where it is a real resource cost for the sunk model.

The first experiment shows that removing the regulation leads to a significant increase in output and employment as many medium-sized firms go over the threshold and hence increase labor demand. Total labor productivity falls as many firms that were previously constrained in their employment are now able to increase it.

Our second experiment shows what happens if we force total employment to remain constant. This is equivalent to taking the results of the first experiment and increasing the wage to make employment return to its initial value. In this case, the output gain is more modest. Very large firms and very small firms contract because of the higher wage. But intermediate firms grow as they now go over the threshold. The productivity gains from the reform are significant. We note that this result goes some way towards addressing the observation that France has relatively less medium-sized firms than comparable countries (Bartelsman et al. (2003, 2004)).

Our third experiment yields quite different results. Allowing the number of firms to adjust reduces dramatically the steady-state output gains. On the one hand, removing the regulation still increases labor demand by making firms constrained close to the threshold grow. On the other hand, removing the regulation increases the value of entry since firms are more likely to be at their efficient size in the future). Given this higher firm value at entry, the wage increases to keep the free entry condition satisfied. This higher wage in turn reduces labor demand. The total effect on average labor demand per firm depends on parameters.

For our parameter values, the first effect dominates and labor demand per firm goes up, which requires the number of firms to go down. In a sense this is intuitive: without regulations, we can have larger firms. We hence need fewer firms.

The economy adjusts by economizing on entry costs. Overall, output actually falls slightly, but the reduced entry costs imply that consumption rises.

All these results hold regardless of whether the regulation is a deadweight cost or a transfer. Hence, the misallocation effect appears to be more important than the direct cost of the regulation.

6 Conclusion

Our paper studies a particular regulation which has a clear effect on the firm size distortion. We find plausible estimates of the costs of the regulation. Their aggregate effects are significant, but are limited if the number of firms can adjust. We think this type of "case study" is complementary to broader macro approaches. There are several interesting extensions. First, incorporating labor adjustment costs or search frictions would be useful to take into account the costly control over the size of the labor force. Second, we could add capital to the model. The regulation may also encourage capital-labor substitution close to the threshold. Finally, we have abstracted from the existence of other thresholds (at 10 and 20 employees), and incorporating them might be important.

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A Appendix (Not for Publication)

A.1 Proof of Proposition 1

First, note that the function $V(.,1)$ is twice continuously differentiable (see Stokey (2008) Chapter 5.6 for a proof). Using the previously computed $\pi(z,1)$ gives:

$$\begin{aligned} V(z^*,1) &= \frac{1}{J}\left[\int_{z^*}^{\infty} e^{R_2(z^*-z)}\pi(z,1)dz + \int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,1)dz\right], \\ &= \frac{1}{J}\left[\int_{z^*}^{\infty} e^{R_2(z^*-z)}\pi^a(z)dz + \int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\pi^a(z)dz\right. \\ &\quad \left. + \int_{\underline{z}}^{\overline{z}} e^{R_1(z^*-z)}\pi^b(z)dz + \int_{-\infty}^{\underline{z}} e^{R_1(z^*-z)}\pi^b(z)dz\right]. \end{aligned}$$

Define, for all $x \leq z^*$,

$$\begin{aligned} H(x,z^*) &\equiv E_x\left[\int_0^{T(z^*)} e^{-(r+\lambda)t}\pi(z_t,0)dt + e^{-(r+\lambda)T(z^*)}\left(V(z^*,1)-F\right)\right], \\ &= \frac{1}{J}\left[\int_x^{z^*} e^{R_2(x-z)}\pi(z,0)dz + \int_{-\infty}^{x} e^{R_1(x-z)}\pi(z,0)dz - e^{R_2(x-z^*)}\int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,0)dz\right] \\ &\quad + e^{R_2(x-z^*)}\left(V(z^*,1)-F\right). \end{aligned}$$

Then, $V(x,0) = \sup_{z^*\geq x} H(x,z^*)$. Note that $H(x,z^*)$ is twice continuously differentiable. The FOC for a maximum at $z^* \geq \overline{z}$ is

$$\begin{aligned} 0 &\leq f_{z^*}(x,z^*) \\ &= \frac{1}{J}\left[e^{R_2(x-z^*)}\pi(z^*,0) + R_2 e^{R_2(x-z^*)}\int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,0)dz\right] \\ &\quad + \frac{1}{J}\left[-e^{R_2(x-z^*)}\pi(z^*,0) - R_1 e^{R_2(x-z^*)}\int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,0)dz\right] \\ &\quad - R_2 e^{R_2(x-z^*)}\left(V(z^*,1)-F\right) + e^{R_2(x-z^*)}V_{z^*}(z^*,1) \\ &= e^{R_2(x-z^*)}\left[\frac{R_2-R_1}{J}\int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,0)dz - R_2\left(V(z^*,1)-F\right) + V_{z^*}(z^*,1)\right], \end{aligned}$$

with equality if $z^* > \overline{z}$. Hence,

$$\begin{aligned}
& V(z^*,1) \\
= \; & \frac{1}{J}\left[\int_{z^*}^{\infty} e^{R_2(z^*-z)}\pi^a(z)dz + \int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\pi^a(z)dz + \int_{\underline{z}}^{\overline{z}} e^{R_1(z^*-z)}\pi^b(z)dz + \int_{-\infty}^{\underline{z}} e^{R_1(z^*-z)}\pi^b(z)dz\right] \\
& V_z(z^*,1) \\
= \; & \frac{R_2}{J}\int_{z^*}^{\infty} e^{R_2(z^*-z)}\pi(z,1)dz + \frac{R_1}{J}\int_{-\infty}^{z^*} e^{R_1(z^*-z)}\pi(z,1)dz \\
= \; & \frac{R_2}{J}\int_{z^*}^{\infty} e^{R_2(z^*-z)}\pi^a(z)dz \\
& + \frac{R_1}{J}\left[\int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\pi^a(z)dz + \int_{\underline{z}}^{\overline{z}} e^{R_1(z^*-z)}\pi^b(z)dz + \int_{-\infty}^{\underline{z}} e^{R_1(z^*-z)}\pi^b(z)dz\right].
\end{aligned}$$

Plugging ing the FOC gives

$$(R_1 - R_2)\int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\left[\pi^a(z) - \pi^b(z)\right]dz + R_2 JF = 0,$$

which simplifies to

$$R_1\int_{\overline{z}}^{z^*} e^{R_1(z^*-z)}\left[\pi^a(z) - \pi^b(z)\right]dz + (r+\lambda)F = 0.$$

It is easy to see that there exists a unique value of z^* that satisfies the preceding equality. Moreover, one can compute these integrals easily given our formulas for $\pi^a(z)$ and $\pi^b(z)$.

A.2 Alternative Derivation of Optimal Policy using Dynamic Programming

We start by writing the Hamilton-Jacobi-Bellman equation satisfied by V :

$$(r+\lambda)V(z,1) = \pi(z,1) + \mu V_z(z,1) + \frac{\sigma^2}{2}V_{zz}(z,1), \tag{15}$$

for any z, and

$$(r+\lambda)V(z,0) = \pi(z,0) + \mu V_z(z,0) + \frac{\sigma^2}{2}V_{zz}(z,0), \tag{16}$$

for $z < z^*$. Note that $\pi(z,0)$ and $\pi(z,1)$ are only C^1 (continuous differentiable): the second derivative is discontinuous at $z = \underline{z}$ for $\pi(.,0)$ and $\pi(.,1)$, and at $z = \overline{z}$ for $\pi(.,1)$.

The boundary conditions given by value matching:

$$V(z^*,1) = V(z^*,0) - F, \tag{17}$$

and by the smooth pasting condition:

$$V_z(z^*, 1) = V_z(z^*, 0). \tag{18}$$

The general solution of the associated homogeneous ODE (i.e., without the term π) is $A_1 e^{R_2 z} + A_2 e^{R_1 z}$, where R_1 and R_2 are the roots of the quadratic

$$\frac{\sigma^2}{2} X^2 + \mu X - (r + \lambda) = 0, \tag{19}$$

i.e.$R_2 = \frac{-\mu + \sqrt{\mu^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} > 0$ and $R_1 = \frac{-\mu - \sqrt{\mu^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} < 0$.

The specific forms of $\pi(z, 0)$ and $\pi(z, 1)$ make it possible to find particular solutions. Starting with the first equation, we guess that

$$\begin{aligned} \widetilde{V}(z, 0) &= b_0 e^{\frac{z}{1-\alpha}}, \text{ for } z < \underline{z}, \\ &= b_1 e^z + b_2, \text{ for } z > \underline{z}, \end{aligned}$$

is a solution of 16, for constants b_0, b_1, b_2 to be determined.

$\widetilde{V}$ satisfies the ODE for $z <\underline{z}$, provided that b_0 solves:

$$(r + \lambda) b_0 = \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1 - \alpha) + \mu \frac{b_0}{1 - \alpha} + \frac{\sigma^2}{2} \frac{b_0}{(1 - \alpha)^2},$$

or

$$b_0 = \frac{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1 - \alpha)}{r + \lambda - \frac{\mu}{1-\alpha} - \frac{\sigma^2}{2(1-\alpha)^2}}.$$

For $z >\underline{z}$, we require that

$$(r + \lambda) (b_1 e^z + b_2) = e^z \underline{n}^\alpha - w \underline{n} + \mu b_1 e^z + \frac{\sigma^2}{2} b_1 e^z,$$

i.e.

$$b_2 = -\frac{w \underline{n}}{r + \lambda},$$

$$b_1 = \frac{\underline{n}^\alpha}{r + \lambda - \mu - \frac{\sigma^2}{2}}.$$

The general solution of the first equation is thus

$$\begin{aligned}
V(z,0) &= \widetilde{V}(z,0) + A_1 e^{R_2 z} + A_2 e^{R_1 z} \\
&= \frac{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{r+\lambda-\frac{\mu}{1-\alpha}-\frac{\sigma^2}{2(1-\alpha)^2}} e^{\frac{z}{1-\alpha}} + A_1 e^{R_2 z} + A_2 e^{R_1 z}, \text{ for } z < \underline{z} \\
&= \frac{\underline{n}^{\alpha}}{r+\lambda-\mu-\frac{\sigma^2}{2}} e^{z} - \frac{w\underline{n}}{r+\lambda} + A_1 e^{R_2 z} + A_2 e^{R_1 z}, \text{ for } z \geq \underline{z}.
\end{aligned}$$

Turning to the second equation, we again look for one solution, which we guess as

$$\begin{aligned}
\widetilde{V}(z,1) &= e^{\frac{z}{1-\alpha}} b_3, \text{ for } z < \underline{z}, \\
&= e^{z} b_4 + b_5, \text{ for } \overline{z} > z > \underline{z}, \\
&= e^{\frac{z}{1-\alpha}} b_6 + b_7, \text{ for } z > \overline{z}.
\end{aligned}$$

The scalars b_3, b_4, b_5, b_6, b_7 must satisfy:

$$\begin{aligned}
b_3 &= \frac{\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{r+\lambda-\frac{\mu}{1-\alpha}-\frac{\sigma^2}{2(1-\alpha)^2}} = b_0, \\
b_4 &= \frac{\underline{n}^{\alpha}}{r+\lambda-\mu-\frac{\sigma^2}{2}} = b_1, \\
b_5 &= -\frac{w\underline{n}}{r+\lambda} = b_2, \\
b_6 &= \frac{\left(\frac{\alpha}{w(1+\tau)}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{r+\lambda-\frac{\mu}{1-\alpha}-\frac{\sigma^2}{2(1-\alpha)^2}} = \frac{b_0}{(1+\tau)^{\frac{\alpha}{1-\alpha}}}, \\
b_7 &= -\frac{c_f}{r+\lambda},
\end{aligned}$$

and the general solution is

$$V(z,1) = \widetilde{V}(z,1) + A_3 e^{R_2 z} + A_4 e^{R_1 z}$$

Finally we need to determine A_1, A_2, A_3, A_4 and z^*. A standard argument implies that $A_3 = 0$ (the investment option values goes to 0 if $z \to \infty$). Moreover, $A_4 = 0$ since as $z \to -\infty$ the firm value remains finite. Last, $A_2 = 0$ for the same reason. The two scalars A_1 and z^* are thus determined by the following system of two equations in two unknowns:

$$\widetilde{V}(z^*,1) = \widetilde{V}(z^*,0) + A_1 e^{R_2 z^*} - F,$$

$$\widetilde{V}_z(z^*,1) = \widetilde{V}_z(z^*,0) + A_1 R_2 e^{R_2 z^*}.$$

Given the formulas for $\widetilde{V}$ and that $z^* > \overline{z} > \underline{z}$, this can be rewritten as:

$$e^{\frac{z^*}{1-\alpha}} b_6 + b_7 = \frac{\underline{n}^\alpha}{r+\lambda-\mu-\frac{\sigma^2}{2}} e^{z^*} - \frac{w\underline{n}}{r+\lambda} + A_1 e^{R_2 z^*} - F$$

$$e^{\frac{z^*}{1-\alpha}} \frac{b_6}{1-\alpha} = \frac{\underline{n}^\alpha}{r+\lambda-\mu-\frac{\sigma^2}{2}} e^{z^*} + A_1 R_2 e^{R_2 z^*}.$$

This characterizes entirely the solution. It is easy to verify that this yields the same equation the results obtained in the main text using the results of Stokey (2008).

A.3 Derivation of the Stationary Cross-Sectional Distribution

To solve for f, first note that the general solution of the ODE 7 is

$$f(z,0) = D_0 e^{\beta_1 z} + D_1 e^{\beta_2 z},$$

where $\beta_1 < 0 < \beta_2$ are the two real roots of the characteristic equation:

$$\lambda = -\mu X + \frac{\sigma^2}{2} X^2.$$

This equation must be solved separately on each interval. Given that f is a density, the exponential terms which do not go to 0 must disappear. This yields the following simpler form:

$$\begin{aligned} f(z,0) &= C_1 e^{\beta_2 z}, \text{ for } z < z, \\ &= C_2 e^{\beta_1 z} + C_3 e^{\beta_2 z}, \text{ for } z^* > z > z, \end{aligned}$$

and

$$\begin{aligned} f(z,1) &= C_4 e^{\beta_2 z}, \text{ for } z < z^*, \\ &= C_5 e^{\beta_1 z}, \text{ for } z > z^*. \end{aligned}$$

The boundary conditions can then be expressed as a system of five linear equations in five unknowns. First, f is a p.d.f., i.e. its integral is one:

$$\frac{C_1}{\beta_2}e^{\beta_2 z} + \frac{C_2}{\beta_1}\left(e^{\beta_1 z^*} - e^{\beta_1 z}\right) + \frac{C_3}{\beta_2}\left(e^{\beta_2 z^*} - e^{\beta_2 z}\right) + \frac{C_4}{\beta_2}e^{\beta_2 z^*} - \frac{C_5}{\beta_1}e^{\beta_1 z^*} = 1.$$

Second, $f(.,0)$ is continuous at z :

$$C_1 e^{\beta_2 z} = C_2 e^{\beta_1 z} + C_3 e^{\beta_2 z},$$

Third, $f(.,0)$ is continuous at z^* :

$$C_2 e^{\beta_1 z^*} + C_3 e^{\beta_2 z^*} = 0,$$

Fourth, $f(.,1)$ is continuous at z^* :

$$C_5 e^{\beta_1 z^*} = C_4 e^{\beta_2 z^*}.$$

And finally the boundary condition at z^* :

$$-\frac{\sigma^2}{2}\left(C_2\beta_1 e^{\beta_1 z^*} + C_3\beta_2 e^{\beta_2 z^*}\right) = \lambda\left(\frac{C_4}{\beta_2}e^{\beta_2 z^*} - \frac{C_5}{\beta_1}e^{\beta_1 z^*}\right).$$

This is easily solved either analytically or using a linear solver, yielding the results in the main text.