Comparing Risks by Acceptance and Rejection*

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Abstract

Stochastic dominance is a partial order on risky assets ("gambles") that is based on the uniform preference—of all decision-makers of an appropriate class—for one gamble over another. We modify this requirement, first, by taking into account the status quo (given by the current wealth) and the possibility of rejecting gambles, and second, by comparing rejections that are substantive (that is, uniform over wealth levels or over utilities). This yields two new stochastic orders: "wealth-uniform dominance" and "utility-uniform dominance." Unlike stochastic dominance, these two orders are *complete*: any two gambles can be compared. Moreover, they are equivalent to the orders induced by, respectively, the Aumann–Serrano (2008) index of riskiness and the Foster–Hart (2009a) measure of riskiness.

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1 Introduction

A *risky asset* (or *gamble*) yields uncertain returns according to a given probability distribution; these returns may be positive (gains) or negative (losses).[1] How can gambles be compared to one another? Which is "less risky"? While different decision-makers regard gambles differently—each according to his own risk posture—we would like to capture in these comparisons the gambles' inherent riskiness. That is, we want to find *objective* ways to compare gambles: independently of the specific decision-maker, and depending only on the gambles themselves (i.e., on their outcomes and probabilities). Such objective comparisons do exist, for instance, for the "return" of gambles (compare their expectations) and their "spread" (compare their variances). Moreover, these comparisons yield *complete* orders (i.e., any two gambles can be compared). The aim of this paper is to do the same for riskiness—that is, to provide *complete and objective orders of riskiness.*

Let g and h denote two gambles. There are situations where it is clear that g is less risky than h; this is certainly so when g is obtained from h by increasing some gain, by decreasing some loss, or by replacing a lottery with its expectation. Combining these kinds of transformations yields the well-known *second-degree stochastic dominance* order (see Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970, 1971, Machina and Rothschild 2008). As it turns out, there is an equivalent approach that leads to the same comparison: g (second-degree) stochastically dominates h if and only if *all risk-averse* decision-makers prefer[2] g to h. This is a natural approach since it is risk, after all, to which risk-averse decision-makers are averse, and this aversion can be used to compare the riskiness of the two gambles. However, it seldom happens that all decision-makers agree which one of the two gambles is preferred to the other. Indeed, in general

[1]The values a gamble takes should be understood as *net* changes to the current wealth, and not the final wealth.

[2]To streamline the text, we say "greater than" rather than "greater than or equal to": all comparisons should thus be understood in the *weak* sense. For example, "prefers" means "prefers or is indifferent to," and "rejected less" means "rejected less often or as often." Also, all the orders we consider are *weak orders* (from which the strict part and the indifference part are easily deduced as usual).

some will prefer the first to the second, others the second to the first—and then stochastic dominance is silent. Formally, this means that the stochastic dominance order between gambles is a *partial* (rather than complete) *order*—in fact, very partial and far from complete.

Suppose now that decision-makers may choose whether to accept or reject a gamble.[3] But then, if both g and h are rejected, does it really matter in this case that, say, g is preferred to h? This suggests a way to weaken the requirement of stochastic dominance, by asking only that g be accepted more, and thus rejected less, than h. We thus define a new order on gambles, which we call "acceptance dominance": *g acceptance dominates h* if every time that g is rejected (by a risk-averse decision-maker) then so is h. The higher aversion to h than to g is now expressed in the fact that h is rejected more than g.

Clearly, the acceptance dominance requirement is a weakening of the stochastic dominance requirement: if g is preferred to h and g is rejected, then surely h is rejected (since the status quo is preferred to g, which in turn is preferred to h). Therefore the acceptance dominance order extends the stochastic dominance order. That it is an actual extension can be seen, for example, by taking a gamble g with positive expectation and doubling its outcomes (i.e., multiplying all gains and losses by 2); the resulting gamble $h = 2g$ turns out always to be acceptance dominated by g, even though it can never be stochastically dominated by g (see Section 3.2 and Remark (2) in Section A.1).

While acceptance dominance allows one to compare more pairs of gambles than does stochastic dominance, it is still a very partial order: for general gambles g and h, there are instances where g is rejected and h is accepted, and other instances where h is rejected and g is accepted. The reason is that the requirement that "*in every instance* that g is rejected h must also be rejected" is a strong requirement, and thus hard to satisfy for general gambles g and h. After all, g and h may take very different values, with different ranges, and

[3]A gamble is *rejected* whenever staying put at the current wealth (the "status quo") is preferable to taking the gamble, and *accepted* otherwise. See Section 5 (c) for further discussion.

it may be too much to expect that for *all* utility functions and *all* wealth levels, the rejection of one gamble g will always imply the rejection of another gamble h.

All this leads to the following idea: make the comparison only when the rejection is "substantive," in the sense that it holds over a significant range of decision problems. That is, only a stronger, "uniform" rejection—not just a single instance, but rather a whole range of rejections—should qualify as evidence of the riskiness of gambles; thus, the gamble that is uniformly rejected less is less risky and dominates the other. Since an acceptance or rejection decision is characterized by a utility function and a current wealth level, there are two simple ways to impose uniformity: one in which the rejection is *wealth-uniform* (i.e., rejection by one utility function at *all* wealth levels), and the other in which it is *utility-uniform* (i.e., rejection at one wealth level by *all* utilities).[4]

We thus obtain two new orders on gambles, which we call "wealth-uniform dominance" and "utility-uniform dominance." Formally, g *wealth-uniformly dominates* h if any risk-averse utility function that rejects g at all wealth levels also rejects h at all wealth levels; and g *utility-uniformly dominates* h if any wealth level at which all risk-averse utility functions reject g is also a wealth level at which they all reject h. That is, in the former case g is wealth-uniformly rejected less than h, and in the latter g is utility-uniformly rejected less than h. Clearly, these two new orders extend acceptance dominance, and thus a fortiori stochastic dominance: if every time that g is rejected h is also rejected, then any uniform rejection of g implies the same uniform rejection of h.

What may come as a surprise is that each one of these two uniform dominance orders is in fact a *complete order*: any two gambles can be compared. That is, for any g and h, either g wealth-uniformly dominates h, or h wealth-uniformly dominates g; also, either g utility-uniformly dominates h, or h utility-uniformly dominates g (but in general the two orders may yield dif-

[4]To ensure the soundness of these uniform requirements some standard regularity conditions will be imposed on the class of utility functions that are considered; see Section 2.2 and the discussion in Section 5 (d) and (e).

ferent comparisons). Thus, when one considers *strong* aversion to gambles (i.e., aversion that is uniform with respect to wealth or utility), there is no longer any ambiguity: of any two gambles, one is *always* rejected more.

Recently, two numerical measures of riskiness were introduced: the "economic" index R^{AS} developed by Aumann and Serrano (2008), and the "operational" measure R^{FH} of Foster and Hart (2009a). We will show a noteworthy connection between our orders and these measures of riskiness: wealth-uniform dominance is equivalent to R^{AS}, and utility-dominance to R^{FH}. That is, g wealth-uniformly dominates h if and only if the AS-riskiness index $R^{\mathrm{AS}}(g)$ of g is less than or equal to the AS-riskiness index $R^{\mathrm{AS}}(h)$ of h, i.e., $R^{\mathrm{AS}}(g) \leq R^{\mathrm{AS}}(h)$; and g utility-uniformly dominates h if and only if the FH-riskiness measure $R^{\mathrm{FH}}(g)$ of g is less than or equal to the FH-riskiness measure $R^{\mathrm{FH}}(h)$ of h, i.e., $R^{\mathrm{FH}}(g) \leq R^{\mathrm{FH}}(h)$.

One can draw a parallel between our approach to riskiness and standard decision and consumer theory. There are two models for rational choice: one is based on comparing alternatives, which yields a "preference order"; the other is based on assigning a number to each alternative, which yields a "utility function." The connection between the two is that a utility function represents a preference order if and only if the preferred outcome has a higher utility. Similarly, in the context of riskiness, the present paper provides the "order" approach, while the papers of Aumann and Serrano (2008) and Foster and Hart (2009a) provide the "numerical index" approach; the results stated in the previous paragraph yield the connections (see also the discussion in Section 5 (a)).

Another interesting observation is that the two ways of getting uniform dominance further emphasize the "duality" between the AS-index and the FH-measure (pointed out in Foster and Hart 2009a, Section VI.A, (iii)): the AS-index looks for the critical utility regardless of wealth, whereas the FH-measure looks for the critical wealth regardless of utility.

To summarize: taking into consideration the *status quo* given by the current wealth level and the possibility to *reject* gambles has enabled us to go beyond the standard stochastic dominance and compare gambles in terms of their inherent and objective riskiness. Our approach may be summed up in the following three basic principles:

1. A gamble g is *less risky* than a gamble h whenever risk-averse decision-makers are *less averse* to g than to h.

2. *Aversion* to a gamble is conveyed by its *rejection*.

3. *Rejection* of different gambles should be compared whenever it is substantive, i.e., *uniform* over a range of decisions.

Putting these together yields:

A gamble g is less risky than a gamble h whenever
g is uniformly rejected less than h by risk-averse decision-makers.

This principle yields two orders—*wealth-uniform dominance* and *utility-uniform dominance*—which are complete orders, and moreover equivalent to the Aumann and Serrano (2008) index of riskiness and the Foster and Hart (2009a) measure of riskiness, respectively.

The contribution of this paper is twofold: first, in showing how natural and simple modifications allow one to complete the stochastic dominance order, and thus compare any two gambles; and second, in providing a new, ordinal approach to riskiness. Moreover, all this is carried out in one unified and standard framework, which helps provide additional understandings and insights into these concepts and their connections.

The paper is organized as follows. Section 2 includes the standard setup and preliminaries. The various orders on gambles—from stochastic dominance, through acceptance dominance, to our two uniform dominance orders—are presented in Section 3, together with the main results. Section 4 is devoted to further results on wealth-uniform dominance and the Aumann and Serrano (2008) index of riskiness. We conclude in Section 5 with a discussion of various pertinent issues and possible extensions. The proofs, together with some additional results, are relegated to the Appendix.

2 Preliminaries

A *gamble* g is a real-valued random variable[5] with positive expectation and some negative values (i.e.,[6] $\mathbf{E}[g] > 0$ and $\mathbf{P}[g < 0] > 0$); for simplicity, assume that g takes finitely many values.[7] Let $\mathcal{G}$ denote the collection of all such gambles. For each gamble g in $\mathcal{G}$, denote by $M_g := \max g$ the maximal gain of g, and by $L_g := \max(-g) = -\min g$ its maximal loss; $\max|g| = \max\{M_g, L_g\}$ is its overall bound. One should view a gamble as the *net* returns of a risky asset; that is, the values of g represent the possible changes in wealth that would occur if g is accepted (the positive values of g are gains, and the negative ones, losses).

A (von Neumann and Morgenstern) *risk-averse utility function* u is a strictly increasing and concave function[8] $u : \mathbb{R}_+ \to \mathbb{R}$. "Risk aversion" is represented by the concavity assumption: the utility of a sure outcome of a is always at least as large as the expected utility of a random variable with expectation a; i.e., $u(\mathbf{E}[X]) \geq \mathbf{E}[u(X)]$ for any random variable X. Let $\mathcal{U}$ denote the collection of all such utility functions u.

2.1 Accepting and Rejecting Gambles

A decision-maker is characterized by a utility function $u \in \mathcal{U}$ and a *wealth* level $w > 0$. The decision-maker *accepts* a gamble $g \in \mathcal{G}$ if $\mathbf{E}[u(w+g)] > u(w)$, and *rejects* g if $\mathbf{E}[u(w+g)] \leq u(w)$; i.e., a gamble is accepted if the expected utility from accepting is higher than from staying put, and is rejected otherwise.[9]

Remark. Since the utility $u(x)$ is not defined for $x \leq 0$, acceptance of g is

[5]The probability space on which this random variable is defined is irrelevant; only the distribution of the gamble matters. We chose to work with random variables g rather than their distributions G for convenience, as $\mathbf{E}[g]$ appears simpler than $\mathbf{E}_G[\cdot]$.

[6]$\mathbf{E}$ and $\mathbf{P}$ denote expectation and probability, respectively.

[7]See Section 5 (g).

[8]$\mathbb{R} = (-\infty, \infty)$ is the set of real numbers, and $\mathbb{R}_+ = (0, \infty)$ the set of positive numbers.

[9]The decision in the case of indifference (i.e., when $\mathbf{E}[u(w+g)] = u(w)$) does not matter. We could have acceptance instead of rejection, or even leave this undefined; while some of the inequalities in the proofs may change from strict to weak and vice versa, none of the final results are affected.

considered only at wealth levels w such that[10] $w + g > 0$, or $w > L_g$. Thus statements such as "g is accepted by u at *all* w" should be understood to refer to *all* $w > L_g$.

2.2 Regular Utilities

We will use two standard assumptions on utility functions propounded by Arrow (1965, Lecture 2; 1971, page 96); these assumptions amount to certain monotonicity relations between decisions and wealth levels. The first is that *acceptance increases*[11] *with wealth*: if u accepts a gamble g at wealth level w then u accepts g also at any higher wealth level $w' > w$. The second is that *acceptance decreases with relative wealth*: scaling up both the gamble and the wealth by the same factor that is greater than 1 decreases acceptance (and thus scaling down by a factor that is less than 1 increases acceptance); that is, if u rejects g at wealth level w then u also rejects[12] λg at wealth level λw for every $\lambda > 1$ (equivalently, if u accepts g at w then u accepts λg at λw for every $0 < \lambda < 1$).

For another way to state these conditions, assume[13] that the utility functions $u \in \mathcal{U}$ are twice continuously differentiable (i.e., of class C^2) and $u'(x) > 0$ for every $x > 0$. The Arrow–Pratt coefficient of *absolute risk aversion* ("ARA") of u at x is $\rho_u(x) := -u''(x)/u'(x)$, and the coefficient of *relative risk aversion* ("RRA") is $\tilde{\rho}_u(x) := -xu''(x)/u'(x) = \rho_u(x)/(1/x)$ (see Arrow 1965, 1971, and Pratt 1964). The first condition of "acceptance increasing with wealth" corresponds to *Decreasing Absolute Risk Aversion* ("DARA"): ρ_u is a decreasing function of wealth, i.e., $\rho_u(x') \leq \rho_u(x)$ for all

[10] $w + g > 0$ means that $w + x > 0$ for all values x of g. It is convenient to put $u(x) := -\infty$ for $x \leq 0$ (which makes u concave over all $\mathbb{R}$), and then for every $w \leq L_g$ we have $\mathbf{E}[u(w+g)] = -\infty \leq u(w)$, and so g is indeed rejected at such w.

[11] Recall that "increasing" and "decreasing" should always be understood in the weak sense (i.e, they mean "nondecreasing" and "nonincreasing," respectively); when needed, we will use "strictly" explicitly.

[12] λg is the gamble where all outcomes of g have been multiplied by the factor λ (and the probabilities are unchanged).

[13] The differentiability assumptions do not matter, as our concepts and constructs are continuous with respect to pointwise convergence of the utility functions (and the gambles are bounded).

$x' > x$ (see Pratt 1964, Yaari 1969, Dybvig and Lippman 1983). Similarly, the second condition of "acceptance decreasing with relative wealth" corresponds to *Increasing Relative Risk Aversion* ("IRRA"): $\tilde{\rho}_u$ is an increasing function of wealth, i.e., $\tilde{\rho}_u(x') \geq \tilde{\rho}_u(x)$ for all $x' > x$. Let $\mathcal{U}_{\mathrm{DA}} := \{u \in \mathcal{U} : \rho_u(x)$ decreases in $x\}$ and $\mathcal{U}_{\mathrm{IR}} := \{u \in \mathcal{U} : \tilde{\rho}_u(x)$ increases in $x\}$ denote these two collections of utilities.

Two special families of utilities (that belong to both $\mathcal{U}_{\mathrm{DA}}$ and $\mathcal{U}_{\mathrm{IR}}$) are the *Constant Absolute Risk Aversion* ("CARA") utilities $\bar{v}_\alpha$ for $\alpha > 0$, where $\bar{v}_\alpha(x) := -\exp(-\alpha x)$ (and thus $\rho_{\bar{v}_\alpha}(x) = \alpha$ for all x), and the *Constant Relative Risk Aversion* ("CRRA") utilities $\tilde{v}_\gamma$ for $\gamma \geq 0$, where $\tilde{v}_\gamma(x) := x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and $\tilde{v}_1(x) = \log(x)$ (and thus $\tilde{\rho}_{\tilde{v}_\gamma}(x) = \gamma$ for all $x > 0$); let $\mathcal{U}_{\mathrm{CA}} := \{\bar{v}_\alpha : \alpha > 0\}$ and $\mathcal{U}_{\mathrm{CR}} := \{\tilde{v}_\gamma : \gamma \geq 0\}$.

A final requirement imposed on a utility function u will be that no gamble should always be accepted by u: for every $g \in \mathcal{G}$ there is $w > L_g$ such that u rejects g at w. Intuitively, when this condition is not satisfied it means that u is willing to accept too many risks, and so is not sufficiently risk-averse (and thus hardly qualifies to attest to the riskiness of gambles).[14] Let $\mathcal{U}_{\mathrm{sr}}$ denote this collection of utility functions ("sr" stands for "some rejection").

Altogether, we will denote by $\mathcal{U}^* := \mathcal{U}_{\mathrm{DA}} \cap \mathcal{U}_{\mathrm{IR}} \cap \mathcal{U}_{\mathrm{sr}}$ the resulting class of utilities; as we will see in Section A.4 in the Appendix, $u \in \mathcal{U}^*$ if and only if ρ_u decreases, $\tilde{\rho}_u$ increases, and $\lim_{x \to 0^+} u(x) = -\infty$ (the last condition is equivalent to $\inf_{x>0} \tilde{\rho}_u(x) = \lim_{x \to 0^+} \tilde{\rho}_u(x) \geq 1$). In particular, $\mathcal{U}^*$ contains all CRRA utilities $\tilde{v}_\gamma$ with RRA coefficient $\gamma \geq 1$, utilities that appear consistent with observed behavior;[15] it also contains utilities that are CARA from some wealth on.[16]

We emphasize that we chose to work throughout with one collection of utilities, $\mathcal{U}^*$, for convenience only; see also the discussion in Section 5 (d) and (e).

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[14]For example, $u(x) = \sqrt{x}$ does *not* belong to $\mathcal{U}_{\mathrm{sr}}$ (for instance, it always accepts the half-half gamble on \$4 and $-$\$1), whereas both $\log(x)$ and $-1/x$ do belong to $\mathcal{U}_{\mathrm{sr}}$; see the next paragraph.

[15]E.g., see Meyer and Meyer (2005), Palacios-Huerta and Serrano (2006).

[16]Take for instance $\hat{v}_\alpha(x) := (\log(\alpha x) - 1)/e$ for $x \leq 1/\alpha$ and $\hat{v}_\alpha(x) := -\exp(-\alpha x)$ for $x \geq 1/\alpha$; then $\rho_{\hat{v}_\alpha}(x) = 1/x$ for $x \leq 1/\alpha$ and $\rho_{\hat{v}_\alpha}(x) = \alpha$ for $x \geq 1/\alpha$ and so $\hat{v}_\alpha \in \mathcal{U}^*$ for each $\alpha > 0$.

2.3 Numerical Measures of Riskiness

How can one *quantify* the intrinsic riskiness of gambles—that is, assign to each gamble a real number that measures its riskiness? Again, we want to do this in an objective way, independent of any specific decision-maker. Just as the "return" of the gamble (its expectation) and the "spread" of the gamble (its standard deviation) depend only on the gamble itself (i.e., its distribution: outcomes and probabilities) and are thus objective measures, so the riskiness of the gamble should be.

Two such recent approaches are the "economic" *index of riskiness* developed by Aumann and Serrano (2008),[17] and the "operational" *measure of riskiness* of Foster and Hart (2009a).[18] Although based on quite different considerations, they turn out to be similar in many ways, and to share several useful properties (besides being objective measures), such as monotonicity with respect to (first- and second-degree) stochastic dominance; see Aumann and Serrano (2008), Foster and Hart (2009a; Section VI.A compares the two approaches), and Foster and Hart (2009b).

Formally, for every gamble $g \in \mathcal{G}$:

- $R^{\mathrm{AS}}(g)$, the *Aumann–Serrano index of riskiness* of g, is given by

$$\mathbf{E}\left[\exp\left(-\frac{1}{R^{\mathrm{AS}}(g)}g\right)\right] = 1.$$

That is, consider the equation $\mathbf{E}\left[\exp\left(-\alpha g\right)\right] = 1$; it has a unique positive solution $\alpha = \alpha^*(g) > 0$, i.e.,

$$\mathbf{E}\left[\exp\left(-\alpha^*(g)\, g\right)\right] = 1, \tag{1}$$

and then

$$R^{\mathrm{AS}}(g) := \frac{1}{\alpha^*(g)} \tag{2}$$

[17] This index was used in the technical report of Palacios-Huerta, Serrano, and Volij (2004); see the footnote on page 810 of Aumann and Serrano (2008).

[18] For a discussion of some of the earlier work, see Section VIII in Aumann and Serrano (2008) and Section VI.D in Foster and Hart (2009a). The reason one is called an "index" and the other a "measure" is explained in Foster and Hart (2009, Section VI.A); see also Section 5 (a).

(see Aumann and Serrano 2008, Theorems A and B).

This means that among the CARA utilities $\bar{v}_\alpha \in \mathcal{U}_{\mathrm{CA}}$, the one with coefficient $\alpha = \alpha^*(g)$ is always indifferent between accepting and rejecting g, whereas all those with a higher absolute risk aversion coefficient $\alpha > \alpha^*(g)$ always reject g, and all those with a lower one $\alpha < \alpha^*(g)$ always accept g; here "always" stands for "at all wealth levels w." We can informally say that $1/R^{\mathrm{AS}}(g)$ is the *critical risk aversion* level for g.

- $R^{\mathrm{FH}}(g)$, the *Foster–Hart measure of riskiness* of g, is given by

$$\mathbf{E}\left[\log\left(1 + \frac{1}{R^{\mathrm{FH}}(g)} g\right)\right] = 0.$$

That is, consider the equation $\mathbf{E}\left[\log(r+g)\right] = \log(r)$; it has a unique positive solution $r = R^{\mathrm{FH}}(g) > L_g$, i.e.,

$$\mathbf{E}\left[\log\left(R^{\mathrm{FH}}(g) + g\right)\right] = \log\left(R^{\mathrm{FH}}(g)\right) \tag{3}$$

(see Foster and Hart 2009a, Theorem 1).

That is, $R^{\mathrm{FH}}(g)$ is the wealth level where the CRRA utility $\tilde{v}_1 \in \mathcal{U}_{\mathrm{CR}}$ with RRA coefficient 1 (i.e., $\tilde{v}_1(x) = \log(x)$) is indifferent between accepting and rejecting the gamble g; at any higher wealth level $w > R^{\mathrm{FH}}(g)$ the utility $\tilde{v}_1$ accepts g, and at any lower wealth level $w < R^{\mathrm{FH}}(g)$ it rejects g. Informally, $R^{\mathrm{FH}}(g)$ is the *critical wealth* level for g: to avoid decreasing wealth and bankruptcy, g is rejected at any wealth level w below the measure of riskiness $R^{\mathrm{FH}}(g)$ of g; see Foster and Hart (2009a).

3 Comparing Gambles

This section presents the various orders on gambles, starting from the known stochastic dominance and ending with our two new uniform dominance orders in Sections 3.4 and 3.5, which also include the statements of the main results.

Let g and h be gambles in $\mathcal{G}$; the objective is to find out when g "dominates" h, in the sense that g is less risky than h, and thus risk-averse decision-makers are less averse to g than to h.

3.1 Stochastic Dominance

The first approach is based on "desirability": if every risk-averse decision-maker prefers g to h, this is a clear indication that g is less risky than h. This yields the classical comparison known as "second-degree stochastic dominance"[19] (see Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970, 1971, Machina and Rothschild 2008): *g stochastically dominates h*, which we denote[20] $g \geqslant_{\mathrm{S}} h$ ("S" stands for "stochastic"), if the expected utility that g yields is always at least as large as that of h, i.e.,

$$\mathbf{E}\left[u(w+g)\right] \geq \mathbf{E}\left[u(w+h)\right] \tag{4}$$

for every $u \in \mathcal{U}$ and every[21] w. Thus, given the choice between g and h, every risk-averse decision-maker prefers g to h.

As is well known (see for instance the above references), g second-degree stochastically dominates h if and only if there are h' and h'' that are defined on the same probability space as g, such that the following holds: $g \geq h'$ (i.e., in each state the realization of g is no less than the realization of h'); h'' is obtained from h' by a sequence of mean-preserving spreads (which replace an outcome x of h' with a lottery whose expectation equals x); and h'' has the same distribution as h. That is, g can only have higher gains, lower losses, or fewer lotteries, than h (more precisely, h'').

Stochastic dominance yields a clear and uncontroversial order on gambles;

[19] "Second degree" refers to risk-averse utility functions (i.e., in $\mathcal{U}$: strictly increasing, and concave), whereas "first degree" refers to utility functions that are just strictly increasing (and not necessarily concave). The second-degree order is thus an extension of the first-degree order: if g first-degree stochastically dominates h, then g second-degree stochastically dominates h (nevertheless, many authors restrict "second-degree" only to the additional comparisons that go beyond the first-degree order, and then deal only with pairs of gambles with identical expectations).

Since risk and risk aversion are of the essence in this paper, the second-degree order is the relevant one, and so "stochastic dominance" will always be of the second degree.

[20] Recall that we deal throughout with the *weak* versions of the orders.

[21] At this point the reader may ask why is the wealth w used at all, as (4) is equivalent to $\mathbf{E}\left[u(g)\right] \geq \mathbf{E}\left[u(h)\right]$ for all strictly increasing and concave u (take $\tilde{u}(x) := u(w+x)$). Although this is indeed irrelevant for stochastic dominance (and also for acceptance dominance; see below), it will become significant for our uniform dominance orders.

however, it is a very partial order,[22] with most pairs of gambles g and h being incomparable: neither one stochastically dominates the other.

3.2 Acceptance Dominance

To weaken the requirement in (4), consider a decision-maker who rejects both g and h; does it really matter in this case that he prefers g to h? This suggests an alternative comparison criterion: g is accepted more than h, and so is rejected less than h, by all risk-averse decision-makers. We thus say that[23] g *acceptance dominates* h, denoted $g \geqslant_{\mathrm{A}} h$ ("A" stands for "Acceptance"), if the following holds:

$$\begin{array}{c}\text{if } g \text{ is rejected by } u \text{ at } w \\ \text{then } h \text{ is rejected by } u \text{ at } w,\end{array} \tag{5}$$

for every $u \in \mathcal{U}$ and every $w > 0$. Formally:

$$\mathbf{E}\left[u(w+g)\right] \leq u(w) \quad \text{implies} \quad \mathbf{E}\left[u(w+h)\right] \leq u(w). \tag{6}$$

It is immediate to see that $\geqslant_{\mathrm{A}}$ is a partial order (i.e., reflexive and transitive). Moreover, (4) implies (6): if $\mathbf{E}\left[u(w+g)\right] \geq \mathbf{E}\left[u(w+h)\right]$, then $\mathbf{E}\left[u(w+g)\right] \leq u(w)$ implies $\mathbf{E}\left[u(w+h)\right] \leq u(w)$. Thus the acceptance dominance order extends the stochastic dominance order: $g \geqslant_{\mathrm{S}} h$ implies $g \geqslant_{\mathrm{A}} h$.

To see that acceptance dominance in fact goes beyond stochastic dominance, take any gamble $g \in \mathcal{G}$ and put $h := 2g$ (i.e., double all the outcomes; any factor larger than 1 would work just as well). Although g cannot stochastically dominate h (for instance, because $\mathbf{E}\left[g\right] < \mathbf{E}\left[h\right] = 2\mathbf{E}\left[g\right]$), it turns out that g acceptance dominates h, i.e., $g \geqslant_{\mathrm{A}} h$. Indeed, for every concave

[22] An *order* (sometimes called "preorder" or "partial order") $\geqslant_*$ is a binary relation that is reflexive (i.e., $g \geqslant_* g$ for any g) and transitive (i.e., $g \geqslant_* h$ and $h \geqslant_* k$ imply $g \geqslant_* k$, for any g, h, k). It is a *complete* (sometimes called "linear") order if every pair g, h can be compared (i.e., either $g \geqslant_* h$ or $h \geqslant_* g$ holds for any g, h).

[23] A more apt, if cumbersome, name would be "acceptance *stochastic* dominance" (the term "stochastic" refers to the fact that only the distributions of the gambles—i.e., values and probabilities—matter). Since all the orders in this paper are "stochastic," we will drop this word for simplicity (except in the case of the original stochastic dominance of Section 3.1).

function u we have $2u(w+x) \geq u(w+2x) + u(w)$, and so $2\mathbf{E}[u(w+g)] \geq \mathbf{E}[u(w+h)] + u(w)$. Therefore (6) is satisfied: if $u(w) \geq \mathbf{E}[u(w+g)]$—i.e., g is rejected—then necessarily $u(w) \geq \mathbf{E}[u(w+h)]$—i.e., h is also rejected.

Although acceptance dominance allows us to compare more gambles than stochastic dominance, it is still only a partial order, and for general gambles g and h neither one will acceptance dominate the other. For example, let $g \in \mathcal{G}$ be the gamble where one gains 20 or loses 10 with equal probabilities of $1/2, 1/2$, and $h \in \mathcal{G}$ the gamble where one gains 50 with probability $2/3$ and loses 20 with probability $1/3$; then $u_1(x) = \log(x)$ at $w_1 = 21$ accepts g and rejects h, whereas $u_2(x) = -1/x$ at $w_2 = 39$ rejects g and accepts h.

In Section A.1 in the Appendix we will provide a precise characterization of acceptance dominance: it turns out to amount to stochastic dominance between "dilutions" of the given gambles, where "diluting" a gamble means taking it with a probability that may be less than one.

3.3 Uniform Rejection

As we have argued in the Introduction, the reason that acceptance dominance does not allow one to compare most risks is that requiring (5) for each and every instance, i.e., for every u and w, is too strong a condition. The values of g and h may be very different, and then it would be hard to deduce from the fact that a certain utility function u rejects g at a certain wealth level w that exactly the same occurs for h.

This suggests that one seek stronger evidence of "aversion" to g before requiring that the same hold for h. Thus, rather than a single instance of rejection, one should consider a whole range of rejections: "uniform rejection." Since an acceptance/rejection decision is characterized by a utility function u and a wealth level w, there are two natural ways of doing so: uniformly over the wealth levels, and uniformly over the utility functions. These two "uniform dominance" orders will be the subject of the next two sections. For simplicity we will from now on restrict ourselves to regular utility functions $u \in \mathcal{U}^*$, which guarantees that the uniform conditions do not become vacuous (see however Section 5 (d) and (e)).

3.4 Wealth-Uniform Dominance

We start with "wealth-uniformity": a gamble g is *wealth-uniformly rejected* by u if g is rejected by u at *all* wealth levels w.

Our first uniform order is thus defined as follows: a gamble $g \in \mathcal{G}$ *wealth-uniformly dominates* a gamble $h \in \mathcal{G}$, denoted $g \geqslant_{\mathrm{WU}} h$ ("WU" stands for "Wealth-Uniform"), whenever:

$$\begin{array}{c}\text{if } g \text{ is rejected by } u \text{ at } \textit{all } w > 0 \\ \text{then } h \text{ is rejected by } u \text{ at } \textit{all } w > 0,\end{array} \qquad \textbf{[WU]}$$

for every utility $u \in \mathcal{U}^*$. That is, if g is wealth-uniformly rejected, then so is h.

Formally:

$$\begin{array}{rl} & \Big(\mathbf{E}\left[u(w+g)\right] \leq u(w) \text{ for all } w > 0\Big) \\ \text{implies} & \Big(\mathbf{E}\left[u(w+h)\right] \leq u(w) \text{ for all } w > 0\Big) \end{array}$$

for every $u \in \mathcal{U}^*$ (compare (6)). This captures the idea that a less risky gamble is rejected less often; however, only when the rejection of g occurs at *all* wealth levels—a strong premise—do we require that the same hold for h.

It is immediate to see that wealth-uniform dominance is a partial order (reflexive and transitive), and that it extends acceptance dominance: $g \geqslant_{\mathrm{A}} h$ implies $g \geqslant_{\mathrm{WU}} h$.

Our main result here is:

Theorem 1 *Wealth-uniform dominance $\geqslant_{\mathrm{WU}}$ is a* complete *order on $\mathcal{G}$ that extends stochastic dominance and acceptance dominance. Moreover, for any two gambles g and h in $\mathcal{G}$,*

$$g \geqslant_{\mathrm{WU}} h \quad \textit{if and only if} \quad R^{\mathrm{AS}}(g) \leq R^{\mathrm{AS}}(h), \tag{7}$$

where R^{AS} denotes the Aumann–Serrano index of riskiness.

Thus, while it may appear from its definition that wealth-uniform dominance, just like acceptance dominance, is only a partial order (i.e., not every pair of gambles may be compared), it turns out to be complete: for any $g, h \in \mathcal{G}$, either $g \geqslant_{\mathrm{WU}} h$ or $h \geqslant_{\mathrm{WU}} g$. Moreover, (7) says that $\geqslant_{\mathrm{WU}}$ is equivalent to the order induced by the Aumann–Serrano index R^{AS}; in other words, R^{AS} *represents* the wealth-uniform dominance order.[24] The uniqueness of R^{AS} up to monotonic transformations is now an immediate consequence:

Corollary 2 (i) *A real-valued function Q on $\mathcal{G}$ represents the $\geqslant_{\mathrm{WU}}$ order if and only if Q is ordinally equivalent to R^{AS} (i.e., there exists a strictly increasing function ϕ such that $Q(g) = \phi\left(R^{\mathrm{AS}}(g)\right)$ for all $g \in \mathcal{G}$).*

(ii) *A real-valued function Q on $\mathcal{G}$ that is positively homogeneous of degree one*[25] *represents the $\geqslant_{\mathrm{WU}}$ order if and only if Q is a positive multiple of R^{AS} (i.e., there exists a constant $c > 0$ such that $Q(g) = cR^{\mathrm{AS}}(g)$ for all $g \in \mathcal{G}$).*

Corollary 2 should be compared to the main results of Aumann and Serrano (2008, Theorems D and A). We have started from a simple "riskiness" order on gambles—wealth-uniform dominance—and then showed that this order is uniquely (up to monotonic or linear transformations) represented by the Aumann–Serrano index. This parallels the standard route of decision theory and consumer theory, which starts with an order on outcomes (a "preference" order) and then represents it by a numerical index (a "utility function"). In our approach the WU-order yields the AS-index directly, without any postulates (whereas Aumann and Serrano need continuity and strict monotonicity with respect to first-order stochastic dominance; see Theorem D and (6.1) in their paper); Section 4.2 below discusses this further.

The proof of Theorem 1 is relegated to Section A.3 in the Appendix; the proof of Corollary 2 is completely standard and thus omitted. For additional results see Section 4.

[24] Just like a consumer's utility function represents his preference order on commodity bundles. Note that in our case the order $\geqslant_{\mathrm{WU}}$ and the riskiness function R^{AS} go in opposite directions: WU-dominance corresponds to lower riskiness.

[25] I.e., $Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$ and $g \in \mathcal{G}$.

3.5 Utility-Uniform Dominance

We come now to "utility-uniformity": a gamble g is *utility-uniformly rejected* at wealth level w if g is rejected by *all* utility functions $u \in \mathcal{U}^*$ at w.

Our second uniform dominance order is thus defined as follows: a gamble $g \in \mathcal{G}$ *utility-uniformly dominates* a gamble $h \in \mathcal{G}$, denoted $g \geqslant_{\mathrm{UU}} h$ ("UU" stands for "Utility-Uniform"), whenever:

$$\begin{array}{c} \text{if } g \text{ is rejected by } \textit{all } u \in \mathcal{U}^* \text{ at } w \\ \text{then } h \text{ is rejected by } \textit{all } u \in \mathcal{U}^* \text{ at } w, \end{array} \qquad [\mathbf{UU}]$$

for every wealth level $w > 0$. That is, if g is utility-uniformly rejected, then so is h. Formally:

$$\begin{array}{rl} & \Big(\mathbf{E}\left[u(w+g)\right] \leq u(w) \text{ for all } u \in \mathcal{U}^*\Big) \\ \text{implies} & \Big(\mathbf{E}\left[u(w+h)\right] \leq u(w) \text{ for all } u \in \mathcal{U}^*\Big) \end{array}$$

for every $w > 0$ (compare (6)). This is another way to capture the idea that a less risky gamble is rejected less often; but now, only when the rejection of g at a certain wealth level is by *all* decision-makers do we require the same for h. The fact that this is a strong premise makes the requirement reasonable.

It is immediate to see that utility-uniform dominance is also a partial order (reflexive and transitive), and that it extends acceptance dominance: $g \geqslant_{\mathrm{UU}} h$ implies $g \geqslant_{\mathrm{A}} h$.

Our main result here is:

Theorem 3 *Utility-uniform dominance $\geqslant_{\mathrm{UU}}$ is a* complete *order on $\mathcal{G}$ that extends stochastic dominance and acceptance dominance. Moreover, for any two gambles g and h in $\mathcal{G}$,*

$$g \geqslant_{\mathrm{UU}} h \quad \textit{if and only if} \quad R^{\mathrm{FH}}(g) \leq R^{\mathrm{FH}}(h), \tag{8}$$

where R^{FH} denotes the Foster–Hart measure of riskiness.

Thus every two gambles $g, h \in \mathcal{G}$ can be compared by utility-uniform dominance: either $g \geqslant_{\mathrm{UU}} h$ or $h \geqslant_{\mathrm{UU}} g$. Moreover, (8) says that the Foster–Hart measure R^{FH} represents this order. We have:

Corollary 4 (i) *A real-valued function Q on $\mathcal{G}$ represents the $\geqslant_{\mathrm{UU}}$ order if and only if Q is ordinally equivalent to R^{FH} (i.e., there exists a strictly increasing function ϕ such that $Q(g) = \phi\left(R^{\mathrm{FH}}(g)\right)$ for all gambles $g \in \mathcal{G}$).*

(ii) *A real-valued function Q on $\mathcal{G}$ that is positively homogeneous of degree one represents the $\geqslant_{\mathrm{UU}}$ order if and only if Q is a positive multiple of R^{FH} (i.e., there exists $c > 0$ such that $Q(g) = cR^{\mathrm{FH}}(g)$ for all $g \in \mathcal{G}$).*

Thus utility-uniform dominance determines R^{FH} uniquely up to monotonic transformations, and together with homogeneity up to a multiplication by a constant (recall however that R^{FH} has a clear "operational" interpretation that pins it down completely; see Foster and Hart 2009a). Corollary 4 thus captures only the "ordinal" aspects of the Foster–Hart measure (see also Section 5 (a)).

Theorem 3 is proved in Section A.4 in the Appendix.

4 Variations on Wealth-Uniformity

In this section we provide an alternative approach to the wealth-uniform order $\geqslant_{\mathrm{WU}}$, and connect our work to that of Aumann and Serrano (2008).

4.1 Wealth-Bounded Dominance

Acceptance dominance requires that if g is rejected by u at w then h is also rejected by u at w; wealth-uniform dominance gets the same conclusion (that h is rejected by u at w), but from a much stronger premise (that g is rejected by u at all w'). But perhaps one should take some middle ground between the rejection of g at a *single* wealth level in (5) and its rejection at *all* wealth levels in [WU]: namely, rejection at a certain *range* of wealths.

Indeed, the premise in (5) that g is rejected by u at w tells us very little about the values of u outside the interval $[w + \min g, w + \max g] = [w - L_g, w + M_g]$. But $w + h$ may well have outcomes that are far away from this interval, which explains why it is hard, except in special cases, to deduce that every u that rejects g at w also rejects h at w (this is why acceptance dominance is only a partial order). It also suggests that, in order to deduce that h is rejected at w, one may want to strengthen the premise on g and require that g be rejected not just at w itself, but also at all wealth levels in a certain interval around w—an interval that is determined by (the ranges of outcomes of) g and h.

We thus introduce another order: a gamble $g \in \mathcal{G}$ *wealth-boundedly dominates* a gamble $h \in \mathcal{G}$, denoted $g \geqslant_{\mathrm{WB}} h$ ("WB" stands for "Wealth-Bounded"), whenever there exists a *bound* $b < \infty$ (that depends only on g and h) such that

$$\begin{array}{c}\text{if } g \text{ is rejected by } u \text{ at all } w' \text{ with } |w' - w| \leq b \\ \text{then } h \text{ is rejected by } u \text{ at } w,\end{array} \qquad \textbf{[WB]}$$

for every utility $u \in \mathcal{U}^*$ and wealth $w > 0$.

We emphasize that the bound b depends only on the two gambles g and h, but applies to all utility functions $u \in \mathcal{U}^*$ and all wealth levels $w > 0$. The special case of $b = 0$ corresponds to (5), and so [WB] enables us to compare more gambles than acceptance dominance.[26] Moreover, since we are dealing with utility functions $u \in \mathcal{U}^* \subset \mathcal{U}_{\mathrm{DA}}$ for which rejection decreases with wealth, rejection at all w' in the interval $[w - b, w + b]$ is equivalent to rejection at its higher end $w + b$, and so [WB] can be restated as "if g is rejected at $w + b$ then h is rejected at w," or (replace w with $w + b$),

$$\text{if } g \text{ is rejected at } w \text{ then } h \text{ is rejected at } w - b$$

(equivalently, "if h is accepted at w then g is accepted at $w + b$"). Thus, although we no longer require, as in acceptance dominance, that the rejection of g at w implies the rejection of h at the exact same wealth level w, we do require that it implies the rejection of h at a wealth level that is lower than w by at most b; equivalently, acceptance of h at w implies acceptance of g not necessarily at the same w, but at a wealth level that is higher by at most

[26] One could also say that $b = \infty$ corresponds to [WU].

b. Moreover, as we will see in Section A.3 in the Appendix (see the Remark immediately following the Proof of Theorem 1 and Proposition 5), one can always take $b = L_g + M_h \leq \max|g| + \max|h|$, and so b is of the same order of magnitude as the outcomes of the two gambles.[27]

It is immediate that $\geqslant_{\mathrm{WB}}$ is a partial order, lying between $\geqslant_{\mathrm{A}}$ and $\geqslant_{\mathrm{WU}}$: if $g \geqslant_{\mathrm{A}} h$ then $g \geqslant_{\mathrm{WB}} h$ (take $b = 0$), and if $g \geqslant_{\mathrm{WB}} h$ then $g \geqslant_{\mathrm{WU}} h$ (when g is rejected at all w we can apply [WB] to each w separately). Thus $\geqslant_{\mathrm{WB}}$ may compare fewer pairs of gambles than the order $\geqslant_{\mathrm{WU}}$; however, it turns out that these two orders are equivalent (and thus, recalling Theorem 1, complete).

Proposition 5 *Wealth-uniform dominance and wealth-bounded dominance are equivalent: $g \geqslant_{\mathrm{WU}} h$ if and only if $g \geqslant_{\mathrm{WB}} h$, for every g, h in $\mathcal{G}$.*

4.2 The Duality Axiom and the Duality Order

The approach of Aumann and Serrano (2008) is based on their *duality axiom*. Though on the face of it this postulate seems very reasonable, on closer inspection it turns out to be relatively complex, and its rationale not entirely straightforward. In particular, it involves *two* decision-makers *and* the index itself (besides the two gambles that are compared).

Formally, Aumann and Serrano (2008) proceed as follows. They call a utility function[28] $u \in \mathcal{U}$ *uniformly no less risk-averse* than a utility function $v \in \mathcal{U}$, written $u \trianglerighteq v$, if whenever the utility u accepts a gamble g at some wealth w, the utility v accepts that gamble g at any wealth w'; and u is called *uniformly more risk-averse than* v, written $u \triangleright v$, if $u \trianglerighteq v$ and $v \ntrianglerighteq u$. The *duality* requirement on a real-valued function Q defined on $\mathcal{G}$ is that, for any

[27] Adding the requirement $b \leq L_g + M_h$, or $b \leq \max|g| + \max|h|$, to [WB] would make the resulting order appear no longer transitive (though it *is* transitive by Proposition 5 and Theorem 1). A note of caution: if g is rejected at w, and $w' = w - b$ with $b = L_g + M_h$, then all the outcomes of $w' + h$ are less than or equal to all the outcomes of $w + g$ (i.e., $\max(w' + h) = w' + M_h \leq w - L_g \leq \min(w + g)$), and so in particular $\mathbf{E}[u(w' + h)] \leq \mathbf{E}[u(w + g)] \leq u(w)$; however, this does *not* immediately imply the rejection of h at w', which amounts to the stronger inequality $\mathbf{E}[u(w' + h)] \leq u(w')$.

[28] In this section we work with general utilities in $\mathcal{U}$ (see Remark (2) following the Proof of Proposition 6 in Section A.5 in the Appendix).

$$
\begin{tikzcd}
u \text{ ACCEPTS } h \arrow[r, dashed] \arrow[d, "u \rhd v"'] \arrow[dr, Rightarrow] & u \text{ ACCEPTS } g \arrow[d, "u \rhd v"] \\
v \text{ ACCEPTS } h \arrow[r, dashed] & v \text{ ACCEPTS } g
\end{tikzcd}
$$

Figure 1: The Duality Axiom

two gambles $g, h \in \mathcal{G}$, any two utilities $u, v \in \mathcal{U}$, and any wealth w:

$$
\begin{array}{ll}
\text{if} & \left\{ \begin{array}{l} u \rhd v, \\ Q(h) > Q(g), \text{ and} \\ u \text{ accepts } h \text{ at } w, \end{array} \right. \\
\text{then} & v \text{ accepts } g \text{ at } w.
\end{array} \tag{9}
$$

What duality says is that if a more risk-averse utility u accepts h, then a less risk-averse utility v should accept a less risky gamble g, where "less risky" is taken according to the yet-to-be-determined index Q. Thus one deduces something about the pair (v, g) (namely, that v accepts g) from an assumption on the pair (u, h) (namely, that u accepts h); see Figure 1. This requires replacing one utility with another, and at the same time also one gamble with another (represented by the diagonal arrow in Figure 1). Now the premise that $u \rhd v$ allows one to replace the utilities while keeping the gamble fixed (these are the vertical arrows): from (u, h) to (v, h), or, alternatively, from (u, g) to (v, g) (indeed, if u accepts h then v accepts h, and the same holds for g). What is missing is a reason to replace the gambles while keeping the utility fixed (i.e., the horizontal arrows): from (u, h) to (u, g), or from (v, h) to (v, g).

This discussion suggests dealing directly with these "horizontal" implications and dispensing with the rest—which is precisely what our wealth-uniform dominance does.

To see this formally, we define another order on gambles: $g \geqslant_{\mathrm{D}} h$ ("D" stands for "Duality") whenever

$$\begin{array}{c}\text{if } g \text{ is rejected by } v \text{ at } w \\ \text{then } h \text{ is rejected by } u \text{ at } w,\end{array} \tag{10}$$

for any two utilities $u, v \in \mathcal{U}$ with $u \rhd v$ and any wealth[29] w; equivalently, if h is accepted by u at w then g is accepted by v at w. Using the $\geqslant_{\mathrm{D}}$ order, the duality axiom (9) can be restated as: for any gambles $g, h \in \mathcal{G}$,

$$Q(g) < Q(h) \quad \text{implies} \quad g \geqslant_{\mathrm{D}} h. \tag{11}$$

How does this relate to our approach? We have:

Proposition 6 *For any two gambles $g, h \in \mathcal{G}$,*

$$g \geqslant_{\mathrm{WU}} h \quad \textit{if and only if} \quad g \geqslant_{\mathrm{D}} h.$$

Thus, the two orders $\geqslant_{\mathrm{WU}}$ and $\geqslant_{\mathrm{D}}$ turn out to be identical. That should not be surprising, since, after all, the wealth-uniform dominance $\geqslant_{\mathrm{WU}}$ essentially provides the missing horizontal implications in Figure 1 (see Section A.5 in the Appendix for a precise proof). In view of Theorem 1, it follows that

Corollary 7 *A real-valued function Q on $\mathcal{G}$ represents the $\geqslant_{\mathrm{D}}$ order if and only if Q is ordinally equivalent to R^{AS}.*

Corollary 7 does not yet yield the result of Aumann and Serrano (2008), since the duality axiom is weaker than the requirement that Q represents

[29]Note the similarity with (5); however, while one agent v is assumed to reject g, it is another agent u who rejects h. The fact that $u \rhd v$, i.e., u is *(wealth-)uniformly* more risk-averse than v, implies that the rejection becomes a wealth-uniform rejection; see Proposition 6.

the $\geqslant_{\mathrm{D}}$ order (i.e., $Q(g) \leq Q(h)$ *if and only if* $g \geqslant_{\mathrm{D}} h$; compare this with (11))—which explains the need of Aumann and Serrano (2008) to appeal to additional axioms: either homogeneity (in their Theorem A), or continuity together with monotonicity with respect to first-order stochastic dominance (in their Theorem D); see also Section X.N of their paper for counterexamples without these additional conditions. In Section A.5 in the Appendix we provide simple alternative proofs for these two results of Aumann and Serrano (2008), based on our Proposition 6 (whose proof is also simple).

In summary, there are now two alternative approaches to the Aumann–Serrano index of riskiness.[30] The first is based on the duality postulate: Theorem D in Aumann and Serrano (2008) (see also our Proposition 19 in Section A.5), or Corollary 7 above. The second is based on wealth-uniform, or wealth-bounded, dominance: Corollary 2 or Proposition 5. We believe that the second one, as introduced in the present paper, captures the Aumann–Serrano riskiness in a starker and more basic form; after all, at the basis of the duality axiom lies wealth-uniform dominance, and so using this order directly simplifies and streamlines the whole approach.[31]

5 Discussion

This section discusses a number of issues and presents some possible extensions.

(a) Ordinal approach to riskiness. The approach to riskiness in this paper is *ordinal*, in the sense that we compare gambles (and the end results are complete orders), whereas the numerical measures of Aumann and Serrano

[30] Another one is the axiomatization in Foster and Hart (2009b).

[31] For wealth-uniform dominance we require the decision-makers to have monotonic decisions (i.e., $u \in \mathcal{U}_{\mathrm{DA}}$; see Section 5 (d) and (e), and the Remark at the end of Section A.3 in the Appendix). We regard this natural regularity condition as a small price to pay to get rid of the *two* decision-makers and the more complex rationale of the duality postulate. To further clarify this, note that the "horizontal" implications in Figure 1 hold for utilities in $\mathcal{U}_{\mathrm{DA}}$, but not for general utilities in $\mathcal{U}$. To overcome this, Aumann and Serrano (2008; see the Proof of Theorem A) take *two* utilities u, v with $u \rhd v$, which implies the existence of CARA utilities "between" u and v—and for CARA utilities the horizontal implications hold.

(2008) and Foster and Hart (2009a) may be viewed as *cardinal*, as they associate a numerical value to each gamble;[32] Theorems 1 and 3, respectively, provide the connections.

Now the Foster–Hart measure of riskiness has a clear (and "operational") interpretation: the critical wealth levels below which accepting gambles may lead in the long run to decreasing wealth and bankruptcy; therefore, applying a monotonic transformation to it may not make much sense. In contrast, the derivation and interpretation of the Aumann–Serrano index of riskiness—which have to do with the critical risk aversion coefficient—are less conclusive in pinning down this index within the class of all its monotonic, or linear, transformations (cf. Theorems D and A in Aumann and Serrano 2008; see also Section IV.C there). In a sense, the Aumann–Serrano index seems to be more of an ordinal concept, whereas the Foster–Hart measure is more cardinal.

Finally, note that the Foster and Hart (2009a) approach applies to general setups that go beyond utility and expected utility.

(b) Duality between wealth and utility. As noted already in Foster and Hart (2009a, Section VI.A), the constructions of the Aumann–Serrano index and the Foster–Hart measure exhibit an interesting duality between wealth and utility. The approach of the current paper further underscores this duality: wealth-uniformity yields the critical utility (and the AS-index), and utility-uniformity yields the critical wealth (and the FH-measure).

(c) Status quo. A basic ingredient that enabled us to go beyond the classical stochastic dominance is the status quo, i.e., the current wealth level (indeed, stochastic dominance looks only at the final outcomes). Allowing the decision-makers to reject gambles—rather than just choose which one they prefer—yields additional comparisons between gambles (even before going to the uniform dominance orders). For example, the fact that *all* risk-averse decision-makers reject λg more often than g, for any gamble $g \in \mathcal{G}$ and any factor $\lambda > 1$ (as acceptance dominance shows; see Remark (2) in Section A.1),

[32] In parallel to decision theory: preference orders (ordinal) vs. utility functions (cardinal).

is a "universal" property that lies below the radar of stochastic dominance (because the status quo and rejection are not seen there).

The idea of status quo together with acceptance and rejection is of course not new (see, e.g., Yaari 1969), and it is already embodied in the Aumann and Serrano (2008) and Foster and Hart (2009a) approaches. More generally, the relevance and significance of the status quo has been pointed out in many setups, theoretical and behavioral (e.g., Kahneman and Tversky 1979, Rabin 2000, and many others).

(d) Regular utilities. We have chosen to use one class of utilities for both wealth-uniform and utility-uniform dominance; this is more elegant and makes the comparisons clearer. However, as we will see in the Appendix (see the remarks at the end of Sections A.3 and A.4), for wealth-uniform dominance we could replace $\mathcal{U}^*$ by, say, $\mathcal{U}_{\mathrm{DA}}$, and for utility-uniform dominance, by $\mathcal{U}_{\mathrm{IR}} \cap \mathcal{U}_{\mathrm{sr}'}$. This is yet another "duality": of Arrow's two conditions (recall Section 2.2), the one that acceptance increases with wealth is used for $\geqslant_{\mathrm{WU}}$ and R^{AS}, whereas the other one that acceptance decreases with relative wealth is used for $\geqslant_{\mathrm{UU}}$ and R^{FH}.

(e) General utilities. Without some regularity assumptions on the acceptance/rejection decisions, the uniformity requirements used in our orders become vacuous (and so one does not go beyond acceptance dominance). Indeed, a utility function whose risk-aversion coefficient oscillates up and down[33] will have decisions that oscillate between rejection and acceptance as the wealth changes—see Proposition 11 in Appendix A.2—and will thus contradict the behavior of utilities with monotonic decisions (for an explicit example, see Section 5.3 in Hart 2009); but how reasonable are decision-makers that, say, accept a gamble g at wealth \$1000 and at wealth \$1002, but reject it at wealth \$1001?

(f) Comparing the two uniform orders. An interesting issue is to understand the similarities and the differences between the two uniform dominance

[33] For some of our results it suffices to require monotonic decisions from some wealth on.

orders. As they both extend stochastic dominance and acceptance dominance, they agree on these comparisons. Also, they become more and more similar as the riskiness of the gambles increases; this follows from Proposition 4 in Section VI.A of Foster and Hart (2009a). Beyond that, are there other interesting cases where the two orders agree?

(g) The class of gambles. To avoid inessential technical issues, we have kept throughout the assumption that each gamble takes only finitely many values. It should not be difficult to relax this and replace it with, say, boundedness (at least from below).

Our gambles $g \in \mathcal{G}$ have positive expectation and take some negative values; this is the interesting case. Indeed, a random variable f with $\mathbf{E}[f] \leq 0$ is rejected by every risk-averse decision-maker, and a (nontrivial) random variable $k \geq 0$ is accepted by every decision-maker, and so any gamble g in $\mathcal{G}$ acceptance dominates f and is acceptance dominated by k; i.e., $k \geqslant_{\mathrm{A}} g \geqslant_{\mathrm{A}} f$ for all[34] $g \in \mathcal{G}$.

(h) Beyond expected utility. This paper (as well as Aumann–Serrano 2008 and Foster–Hart 2009a) deals with pure risk in the standard von Neumann and Morgenstern expected utility setup. It would be interesting to go beyond that and consider more general setups—such as subjective probability, uncertainty, and various non-expected-utility models—and try to capture universal notions of "more risky" and/or "more uncertain."

(i) Characterizations of dominance. First- and second-degree stochastic dominance have equivalent characterizations (in terms of lower values and mean-preserving spreads). Restricting the utilities to some of the classes in this paper ($\mathcal{U}_{\mathrm{DA}}$, or $\mathcal{U}_{\mathrm{IR}}$, or $\mathcal{U}^*$) affects the stochastic order, and it would be of interest to obtain appropriate characterizations (cf. Whitmore 1970 and the survey of Levy 1992); the same applies to acceptance dominance.

[34] One may thus define the riskiness of these f and k to be ∞ and 0, respectively.

A Appendix

The Appendix contains the proofs and some additional results.[35]

A.1 Acceptance Dominance

As indicated in Section 3.2, acceptance dominance is connected to stochastic dominance through the concept of "dilution."

Let $g \in \mathcal{G}$ and $0 < \alpha \leq 1$; the *α-dilution* of the gamble g, denoted $\alpha * g$, is the gamble where g obtains with probability α and the outcome 0 with probability $1 - \alpha$; thus, if g takes the values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$ (where $p_1 + p_2 + ... + p_n = 1$), then $\alpha * g$ takes the values $x_1, x_2, ..., x_n, 0$ with respective probabilities $\alpha p_1, \alpha p_2, ..., \alpha p_n, 1-\alpha$. Since $\mathbf{E}\left[u(w + \alpha * g)\right] = \alpha \mathbf{E}\left[u(w + g)\right] + (1 - \alpha)u(w)$, or

$$\mathbf{E}\left[u(w + \alpha * g)\right] - u(w) = \alpha \Big(\mathbf{E}\left[u(w + g)\right] - u(w)\Big), \tag{12}$$

it follows that g is accepted by u at w if and only if $\alpha * g$ is accepted by u at w: dilution does not affect acceptance and rejection (but, as we will see below, it may well affect stochastic dominance).

The result is:

Proposition 8 *Let $g, h \in \mathcal{G}$. Then $g \geqslant_{\mathrm{A}} h$ if and only if there exist $0 < \alpha, \beta \leq 1$ such that $\alpha * g \geqslant_{\mathrm{S}} \beta * h$.*

Proof. One direction is immediate: $\alpha * g \geqslant_{\mathrm{S}} \beta * h$ implies $\alpha * g \geqslant_{\mathrm{A}} \beta * h$, which is equivalent to $g \geqslant_{\mathrm{A}} h$ by (12).

Conversely, assume that $g \geqslant_{\mathrm{A}} h$. We will show that there exists $\lambda > 0$ such that

$$\mathbf{E}\left[u(w + g)\right] - u(w) \geq \lambda \Big(\mathbf{E}\left[u(w + h)\right] - u(w)\Big) \tag{13}$$

for all $u \in \mathcal{U}$ and $w > 0$. When $0 < \lambda \leq 1$ we get from (13)

$$\mathbf{E}\left[u(w + g)\right] \geq \lambda \mathbf{E}\left[u(w + h)\right] + (1 - \lambda)u(w) = \mathbf{E}\left[u(w + \lambda * h)\right],$$

[35]Some of the arguments below are standard; insofar as they were short enough, we have preferred to provide self-contained proofs rather than refer the reader to various other sources.

i.e., $g \geqslant_{\mathrm{S}} \lambda * h$, and when $\lambda \geq 1$ we get from (13)

$$\mathbf{E}\left[u\left(w+\lambda^{-1} * g\right)\right]=\lambda^{-1}\mathbf{E}\left[u(w+g)\right]+\left(1-\lambda^{-1}\right)u(w) \geq \mathbf{E}\left[u(w+h)\right],$$

i.e., $\lambda^{-1} * g \geqslant_{\mathrm{S}} h$.

The proof of (13) will use a standard separation argument (which essentially amounts to the fact that an affine inequality $A_1(x) \leq 0$ is a consequence of another affine inequality $A_2(x) \leq 0$ if and only if there is $\lambda > 0$ such that $A_2(x) \geq \lambda A_1(x)$ for all x).

First, note that (13) trivially holds at every $0 < w \leq L_h$ (since the right-hand side equals $-\infty$), and that $g \geqslant_{\mathrm{A}} h$ implies $L_h \geq L_g$ (since otherwise $u(x) = x$ would reject g and accept h at any w with $L_h < w < L_g$). Let

$$Y := \left\{\Big(\mathbf{E}\left[u(w+g)\right]-u(w), \mathbf{E}\left[u(w+h)\right]-u(w)\Big) : u \in \mathcal{U} \text{ and } w > L_h\right\}.$$

The set $Y \subset \mathbb{R}^2$ is nonempty and convex, since $\mathcal{U}$ is nonempty and convex (given $u_1, u_2 \in \mathcal{U}$, $w_1, w_2 > L_h$, and $0 < \theta < 1$, take $w_0 := \min\{w_1, w_2\} > L_h$ and $u_0 \in \mathcal{U}$ given by $u_0(x) := \theta u_1(x + w_1 - w_0) + (1-\theta)u_2(x + w_2 - w_0)$ for every $x > 0$).

Now $g \geqslant_{\mathrm{A}} h$ says that Y is disjoint from the convex set $Z := \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \leq 0 < z_2\}$, and so Y and Z can be separated: there exists $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\mu \neq (0,0)$, such that

$$\inf_{y \in Y} \mu \cdot y \geq \sup_{z \in Z} \mu \cdot z. \tag{14}$$

We have $\mu_1 \geq 0$, since otherwise the right-hand side equals ∞ (take $(z_1, 1) \in Z$ with $z_1 \to -\infty$); similarly, $\mu_2 \leq 0$ (take $(0, z_2) \in Z$ with $z_2 \to \infty$). Therefore $\sup_{z \in Z} \mu \cdot z \leq 0$; finally, taking $(0, z_2) \in Z$ with $z_2 \to 0^+$ shows that $\sup_{z \in Z} \mu \cdot z = 0$. Hence (14) becomes

$$\mu \cdot y \geq 0 \tag{15}$$

for every $y \in Y$. We cannot have $\mu_1 = 0$, since then $\mu_2 < 0$ (recall that $\mu \neq (0,0)$), but $u_1(x) = x$ and $w_1 > L_h$ yield $(y_1, y_2) \in Y$ with $y_2 > 0$, contradicting (15). Similarly, we cannot have $\mu_2 = 0$, since then $\mu_1 > 0$, but $u_2 = \bar{v}_\alpha \in \mathcal{U}$ with $\alpha > \alpha^*(g)$ and $w_2 > L_h$ yield $(y_1, y_2) \in Y$ with $y_1 < 0$, contradicting (15). Therefore $\mu_1 > 0 > \mu_2$, and (15) yields (13) for all $w > L_h$, with $\lambda = -\mu_2/\mu_1 > 0$. $\square$

Remarks. (1) In Proposition 8 one can always take at least one of α and β to be equal to 1 (because $\alpha * g \geqslant_{\mathrm{S}} \beta * h$ is equivalent to $(\gamma\alpha) * g \geqslant_{\mathrm{S}} (\gamma\beta) * h$ for any $\gamma > 0$ with $\gamma\alpha, \gamma\beta \leq 1$; see also (13)).

(2) For every $g \in \mathcal{G}$ and every $\lambda > 1$ we have $g \geqslant_{\mathrm{A}} \lambda g$. Indeed, $g \geqslant_{\mathrm{S}} (\lambda^{-1} * (\lambda g))$, since for every function u that is concave, $u(w+x) \geq \lambda^{-1}u(w + \lambda x) + (1 - \lambda^{-1})u(w)$, and so $\mathbf{E}\left[u(w+g)\right] \geq \mathbf{E}\left[u(w + \lambda^{-1} * (\lambda g))\right]$. Thus rescaling all outcomes by a factor λ larger than 1 can only increase rejection. However, g *cannot* stochastically dominate λg (for instance, since $\mathbf{E}\left[w+g\right] < \mathbf{E}\left[w + \lambda g\right]$ and $u(x) = x$ belongs to $\mathcal{U}$).

(3) The result of Proposition 8 connecting acceptance dominance and stochastic dominance is quite general, and holds in many setups. For instance, we may replace $\mathcal{U}$ with other collections of utilities, such as the strictly increasing utilities (which yield the first-degree stochastic dominance), or any of the other classes in this paper: $\mathcal{U}_{\mathrm{DA}}$, $\mathcal{U}_{\mathrm{IR}}$, and $\mathcal{U}^*$. What matters is that the collection of utilities should be a convex set, and, for any $g \in \mathcal{G}$, it should contain some utilities u_1, u_2 such that $\mathbf{E}\left[u_1(w_1 + g)\right] > u(w_1)$ and $\mathbf{E}\left[u_2(w_2 + g)\right] < u(w_2)$ for some (large enough) w_1, w_2. Also, one may replace the gambles with random variables that take values in general linear spaces (where separation theorems apply; for instance, $\mathbb{R}^d$).

A.2 Acceptance, Rejection, and Risk Aversion

From now on we assume that $u \in \mathcal{U}$ is twice continuously differentiable, i.e., of class C^2, and $u'(x) > 0$ for every $x > 0$. Recall that the Arrow–Pratt risk aversion coefficient $\rho \equiv \rho_u$ of u is given by $\rho(x) = -u''(x)/u'(x)$ for every x. The basic result we will use is that a higher risk aversion coefficient yields a concave transformation of the utility function, and thus more rejection (cf. Arrow 1965, 1971 and Pratt 1964—see in particular Theorem 1 there).[36] Specifically:

Proposition 9 *Let $u_1, u_2 \in \mathcal{U}$ be two utility functions with absolute risk aversion coefficients ρ_1 and ρ_2, respectively, and let $I \subset (0, \infty)$ be an interval*[37] *where $\rho_1(x) \geq \rho_2(x)$ for every $x \in I$. Then for every $w > 0$ and $g \in \mathcal{G}$ such that $w + g \subset I$, if u_2 rejects g at w then u_1 rejects g at w.*

Proof. Let ψ be such that $u_1 = \psi \circ u_2$; then ψ is strictly increasing (since u_1 and u_2 are such), and concave (since for every $x \in I$ we have $\psi'(u_2(x)) = u_1'(x)/u_2'(x)$, hence $(\log \psi'(u_2(x)))' = (\log u_1'(x))' - (\log u_2'(x))' = -\rho_1(x) + \rho_2(x) \leq 0$, and so $\psi'' \leq 0$).

Therefore $\mathbf{E}\left[u_2(w+g)\right] \leq u_2(w)$ implies

$$\mathbf{E}\left[u_1(w+g)\right] = \mathbf{E}\left[\psi\left(u_2(w+g)\right)\right] \leq \psi\left(\mathbf{E}\left[u_2(w+g)\right]\right) \leq \psi(u_2(w)) = u_1(w)$$

(the concavity of ψ was used in the first inequality, and the monotonicity of ψ and the assumption on u_2 in the second). □

Recall that, for $\alpha > 0$, the CARA utility $\bar{v}_\alpha \in \mathcal{U}_{\mathrm{CA}}$ satisfies $\rho_{\bar{v}_\alpha}(w) = \alpha$ for every w. The definition (1) of $\alpha^* \equiv \alpha^*(g)$ implies that for every w we have $\mathbf{E}\left[\bar{v}_{\alpha^*}(w+g)\right] = \bar{v}_{\alpha^*}(w)$. Therefore

Lemma 10 *Let $g \in \mathcal{G}$ and $\bar{v}_\beta \in \mathcal{U}_{\mathrm{CA}}$. If $\beta < \alpha^*(g)$ then $\bar{v}_\beta$ accepts g at all $w > L_g$, and if $\beta \geq \alpha^*(g)$ then $\bar{v}_\beta$ rejects g at all w.*

Proof. Use Proposition 9, or direct computation. □

Proposition 11 *Let $u \in \mathcal{U}$, $g \in \mathcal{G}$, and $w > L_g$, and put $I := [w - L_g, w + M_g] \equiv [w + \min g, w + \max g]$.*

(i) *If $\rho_u(w') \geq \alpha^*(g)$ for every $w' \in I$ then u rejects g at w.*

[36] One may easily prove that the converse—which is not needed in the current paper—also holds: a utility function that *always* rejects more than another utility function must be a concave transformation of it, and thus have a higher risk aversion coefficient.

[37] The interval I can be open or closed at either end, and its upper end can be ∞. As usual, $w + g \subset I$ means that $w + x \in I$ for every value x of g.

(ii) *If* $\rho_u(w') < \alpha^*(g)$ *for every* $w' \in I$ *then* u *accepts* g *at* w.

(iii) *If* u *rejects* g *at* w *then there exists* $w' \in I$ *such that* $\rho_u(w') \geq \alpha^*(g)$.

(iv) *If* u *accepts* g *at* w *then there exists* $w' \in I$ *such that* $\rho_u(w') < \alpha^*(g)$.

Proof. We will only prove (i) and (ii), since (iv) is equivalent to (i) and (iii) is equivalent to (ii). Put $\alpha^* \equiv \alpha^*(g)$.

(i) Since $\bar{v}_{\alpha^*}$ rejects g at w (by Lemma 10) and $\rho_{\bar{v}_{\alpha^*}}(w') = \alpha^*$ for every w', Proposition 9 with $u_1 = u$ and $u_2 = \bar{v}_{\alpha^*}$ implies that u rejects g at w.

(ii) Let $\beta := \max_{w' \in I} \rho_u(w')$; then $0 < \beta < \alpha^*$ (use the continuity of ρ_u and $w - L_g > 0$). Since $\bar{v}_\beta$ accepts g at w (by Lemma 10), applying Proposition 9 with $u_1 = \bar{v}_\beta$ and $u_2 = u$ implies that u accepts g at w. □

Proposition 11 is essentially (4.3.2) of Aumann and Serrano (2008); we have proved it here directly for completeness (their arguments are slightly more elaborate).

A.3 Wealth-Uniform Dominance

Proof of Theorem 1 and Proposition 5. We will prove these two results together.

First, we claim that

$$g \geqslant_{\mathrm{WU}} h \quad \text{implies} \quad \alpha^*(g) \geq \alpha^*(h).$$

Indeed, if $\alpha^*(g) < \alpha^*(h)$ then put $\beta := \alpha^*(g)$ and let $u = \hat{v}_\beta \in \mathcal{U}^*$ be given by[38] $\rho_u(x) := \max\{1/x, \beta\}$. Since $\rho_u(x) \geq \beta = \alpha^*(g)$ for every $x > 0$, Proposition 11 (i) implies that u rejects g at all w; since $\rho_u(x) = \beta < \alpha^*(h)$ for every $x \geq 1/\beta$, Proposition 11 (ii) implies that u accepts h at all w such that $w - L_h \geq 1/\beta$, which contradicts $g \geqslant_{\mathrm{WU}} h$.

Second, we claim that

$$\alpha^*(g) \geq \alpha^*(h) \quad \text{implies} \quad g \geqslant_{\mathrm{WB}} h \text{ with } b = L_g + M_h. \tag{16}$$

[38] See footnote 16.

Indeed, if $\alpha^*(g) \geq \alpha^*(h)$ and g is rejected by u at $w + b$, then there exists $w' \in [w + b - L_g, w + b + M_g]$ such that $\rho_u(w') \geq \alpha^*(g) \geq \alpha^*(h)$ (the first inequality by Proposition 11 (iii), the second by assumption), and so $\rho_u(w') \geq \alpha^*(h)$ for every $w' \leq w + b - L_g$ (since ρ_u is decreasing); for $b = L_g + M_h$ we have $w + b - L_g = w + M_h$, which implies that h is rejected at w (by Proposition 11 (i))—and we have proved (16).

Since $g \geqslant_{\mathrm{WB}} h$ immediately implies $g \geqslant_{\mathrm{WU}} h$ (see Section 4.1), altogether we have obtained

$$g \geqslant_{\mathrm{WU}} h \quad \text{if and only if} \quad g \geqslant_{\mathrm{WB}} h \quad \text{if and only if} \quad \alpha^*(g) \geq \alpha^*(h); \tag{17}$$

recalling (2) completes the proof. □

Remark. The proof above (specifically, (16)) shows that one can take $b \equiv b(g, h) = L_g + M_h \leq \max|g| + \max|h|$ as the bound in [WB].

We can now provide some further insights into wealth-uniform dominance and thus, a fortiori, the Aumann–Serrano index. For each utility function $u \in \mathcal{U}^*$, classify the gambles $g \in \mathcal{G}$ into three *rejection classes* relative to u:

(R1) g is always accepted by u;[39]

(R2) g is sometimes rejected and sometimes accepted by u; and

(R3) g is always rejected by u.

This is a rough classification, since in case (R2) we ignore exactly when g is rejected and when it is accepted (when $u \in \mathcal{U}_{\mathrm{DA}}$ there is a *critical wealth* level, denote it $\bar{w}_u(g)$, where the transition from rejection to acceptance occurs). Since the amount of rejection increases from (R1) to (R2) to (R3), we will say that (R3) is the "highest rejection class," and (R1) the "lowest."

As usual, derive from the weak order $\geqslant_{\mathrm{WU}}$ its *strict* part $>_{\mathrm{WU}}$ and its *indifference* part $\sim_{\mathrm{WU}}$: that is, $g >_{\mathrm{WU}} h$ if and only if $g \geqslant_{\mathrm{WU}} h$ holds but $h \geqslant_{\mathrm{WU}} g$ does not, and $g \sim_{\mathrm{WU}} h$ if and only if both $g \geqslant_{\mathrm{WU}} h$ and $h \geqslant_{\mathrm{WU}} g$ hold. We have:

[39] Recall (Section 2.1) that "g is always accepted" means that g is accepted at all wealth levels $w > L_g$.

Proposition 12 *Let $g, h \in \mathcal{G}$. Then:*

(i) $g \geqslant_{WU} h$ *if and only if for every $u \in \mathcal{U}^*$ the rejection class of g is no higher than the rejection class of h.*

(ii) $g \sim_{\mathrm{WU}} h$ *if and only if for every $u \in \mathcal{U}^*$ the rejection class of g is the same as the rejection class of h. Moreover, when they are both of class* (R2) *the critical wealth levels $\bar{w}_u(g)$ and $\bar{w}_u(h)$ satisfy* $|\bar{w}_u(g) - \bar{w}_u(h)| \leq \max|g| + \max|h|$.

(iii) $g >_{\mathrm{WU}} h$ *if and only if there exists $u \in \mathcal{U}^*$ for which g is of rejection class* (R3) *and h is of rejection class* (R1)*; i.e., g is always rejected and h is always accepted.*

The proof is omitted, as it all straightforwardly follows from the previous results. Note that taking the negation of (iii) and interchanging g and h yields the following statement (compare [WU]): $g \geqslant_{\mathrm{WU}} h$ if and only if for every $u \in \mathcal{U}^*$, if g is rejected by u at all w then h is rejected by u at *some*[40] w.

Remark. The proofs above show that for the wealth-uniform results one can replace $\mathcal{U}^*$ with any collection of utilities $\mathcal{U}^{*\mathrm{W}} \subset \mathcal{U}$ that satisfies: (i) $\mathcal{U}^{*\mathrm{W}} \subset \mathcal{U}_{\mathrm{DA}}$; and (ii) for every $\beta > 0$ there is $u \in \mathcal{U}^{*\mathrm{W}}$ such that $\inf_{x>0} \rho_u(x) = \lim_{x\to\infty} \rho_u(x) = \beta$. In particular, one may take $\mathcal{U}^{*\mathrm{W}} = \mathcal{U}_{\mathrm{DA}}$.

A.4 Utility-Uniform Dominance

We start by characterizing $\mathcal{U}_{\mathrm{sr}}$.

Proposition 13 *Let $u \in \mathcal{U}$ and put*[41] $u(0^+) := \lim_{x\to 0^+} u(x)$. *Then:*

(i) $u(0^+) = -\infty$ *if and only if for every $g \in \mathcal{G}$ there is $\delta > 0$ such that g is rejected by u at all $w \in (L_g, L_g + \delta)$.*

(ii) $u(0^+) > -\infty$ *if and only if there is $g \in \mathcal{G}$ and $\delta > 0$ such that g is accepted by u at all $w \in (L_g, L_g + \delta)$.*

[40] One may take this statement as the definition of wealth-uniform dominance.

[41] The limit exists (and is either finite or $-\infty$) since u is an increasing function.

Proof. If $u(0^+) = -\infty$ then, for $p := \mathbf{P}[g = -L_g] > 0$, we have

$$\mathbf{E}[u(w+g)] - u(w) \leq pu(w - L_g) + (1-p)u(w + M_g) - u(w) \to -\infty$$

as w decreases to L_g. Therefore $\mathbf{E}[u(w+g)] - u(w) < 0$ for all w close enough to L_g, and g is rejected there.

If $u(0^+) > -\infty$ then let $0 < \varepsilon < 1/2$ be small enough so that $(1-\varepsilon)u(2) + \varepsilon u(0^+) > u(1)$ (recall that u is increasing). Then the gamble $g \in \mathcal{G}$ that takes the values 1 and -1 with probabilities $1-\varepsilon$ and ε, respectively, is accepted by u at all $w > 1 = L_g$ that are close enough to 1.

These two implications, together with the fact that the clauses in (i) and (ii) on rejection and acceptance, respectively, are clearly contradictory, yield the converse implications in both (i) and (ii). $\square$

Corollary 14 *Let $u \in \mathcal{U}_{\mathrm{DA}}$, particularly $u \in \mathcal{U}^*$; then $u \in \mathcal{U}_{\mathrm{sr}}$ if and only if $u(0^+) = -\infty$.*

Proof. In case (i) of Proposition 13 we get some $d \equiv d_g > 0$, possibly $d = \infty$, such that u rejects g at all $w \leq L_g + d$ and accepts g at all $w > L_g + d$; in case (ii), u accepts g at all $w > L_g$ (and so $d_g = 0$)—therefore g is sometimes rejected if and only if case (i) occurs. $\square$

Lemma 15 *Let $u \in \mathcal{U}$. If there is $c > 0$ such that $\sup_{0<x\leq c} \tilde{\rho}_u(x) < 1$ then $u(0^+) > -\infty$.*

Proof. Let $\gamma := \sup_{0<x\leq c} \tilde{\rho}_u(x)$; then $0 \leq \gamma < 1$. For every $x \in (0, c]$ we have: $(\log u'(x))' = -\rho_u(x) = -\tilde{\rho}_u(x)/x \geq -\gamma/x$, and so $\log u'(c) - \log u'(x) \geq \int_x^c (-\gamma/y)\,\mathrm{d}y = -\gamma \log(c/x)$. Therefore $u'(x) \leq ax^{-\gamma}$ (with $a := u'(c)c^\gamma > 0$), from which we get $u(c) - u(x) \leq \int_x^c ay^{-\gamma}\,\mathrm{d}y = a\left(c^{1-\gamma} - x^{1-\gamma}\right)/(1-\gamma)$, and so $u(x) \geq B := u(c) - ac^{1-\gamma}/(1-\gamma)$ and $u(0^+) \geq B$. $\square$

Corollary 16 *If $u \in \mathcal{U}^*$ then $\tilde{\rho}_u(x) \geq 1$ for all $x > 0$.*

Proof. If $\tilde{\rho}_u(c) < 1$ for some $c > 0$, then $\tilde{\rho}_u(x) \leq \gamma := \tilde{\rho}_u(c) < 1$ (since $u \in \mathcal{U}_{\mathrm{IR}}$ and so $\tilde{\rho}_u$ is increasing); Lemma 15 implies that $u(0^+)$ is finite, and so $u \notin \mathcal{U}_{\mathrm{sr}}$ by Corollary 14. $\square$

Lemma 17 *Let $u \in \mathcal{U}^*$ and $g \in \mathcal{G}$. Then u rejects g at all $w \leq R^{\mathrm{FH}}(g)$.*

Proof. By the definition of the Foster–Hart measure, $\tilde{v}_1(x) = \log(x)$ rejects g at all $w \leq R^{\mathrm{FH}}(g)$. The result follows from Proposition 9 with $u_1 = u$ and $u_2 = \tilde{v}_1$ (and so $\tilde{\rho}_2(x) = 1$ for every $x > 0$), and $I = (0, \infty)$. □

We can now prove Theorem 3.

Proof of Theorem 3. Lemma 17, together with the fact that at each $w > R^{\mathrm{FH}}(g)$ the CRRA utility $\tilde{v}_1 = \log$, which belongs to $\mathcal{U}^*$, does not reject g, implies that "g is rejected by all $u \in \mathcal{U}^*$ at w" *if and only if* "$w \leq R^{\mathrm{FH}}(g)$." So the condition in [UU] translates to "if $w \leq R^{\mathrm{FH}}(g)$ then $w \leq R^{\mathrm{FH}}(h)$," which is equivalent to "$R^{\mathrm{FH}}(g) \leq R^{\mathrm{FH}}(h)$." □

Remarks. (1) The proofs above show that for the utility-uniform results one can replace $\mathcal{U}^*$ with any collection of utilities $\mathcal{U}^{*\mathrm{U}} \subset \mathcal{U}$ that satisfies: (i) $\tilde{v}_1 \equiv \log \in \mathcal{U}^{*\mathrm{U}}$; and (ii) $\inf_{x>0} \tilde{\rho}_u(x) \geq 1$ for every $u \in \mathcal{U}^{*\mathrm{U}}$. In particular, one may take $\mathcal{U}^{*\mathrm{U}} = \mathcal{U}_{\mathrm{IR}} \cap \mathcal{U}_{\mathrm{sr}'}$, where $\mathcal{U}_{\mathrm{sr}'}$ requires rejection at arbitrarily small wealth levels (i.e., $u \in \mathcal{U}_{\mathrm{sr}'}$ if and only if for every $g \in \mathcal{G}$ and $\delta > 0$ there is $w \in (L_g, L_g + \delta)$ such that u rejects g at w).

(2) If we drop the condition that each utility function u will sometimes reject any gamble—i.e., $u \in \mathcal{U}_{\mathrm{sr}}$—then the only wealth levels w where all utilities reject a gamble g are $w \leq L_g$, and so the resulting order, denote it $\geqslant_{\mathrm{UU0}}$, satisfies $g \geqslant_{\mathrm{UU0}} h$ if and only if $L_g \leq L_h$.

A.5 The Duality Axiom and the Duality Order

Here we provide proofs for the statements of Section 5. We now work with general utilities in $\mathcal{U}$ (but see Remark (2) after the Proof of Proposition 6).

We start with the characterization of the "uniformly more risk-averse" relation $\trianglerighteq$ between utility functions.[42]

Lemma 18 *Let $u, v \in \mathcal{U}$. Then $u \trianglerighteq v$ if and only if* $\inf_w \rho_u(w) \geq \sup_w \rho_v(w)$.

[42] This is (4.1.2) in Aumann and Serrano (2008); we prove it here for completeness (and the proof is short).

Proof. If $u \trianglerighteq v$ but $\inf_w \rho_u(w) < \sup_w \rho_v(w)$, then there exist w_1, w_2 such that $\rho_u(w_1) < \rho_v(w_2)$; the continuity of ρ (recall that the utilities are C^2) implies that there are $\varepsilon > 0$ small enough and $\beta > 0$ such that $\rho_u(w_1') < \beta < \rho_v(w_2')$ for every $w_1' \in [w_1 - \varepsilon, w_1 + \varepsilon]$ and $w_2' \in [w_2 - \varepsilon, w_2 + \varepsilon]$. Let g take the values ε and $-\varepsilon$ with probabilities $\exp(\beta\varepsilon)/(1+\exp(\beta\varepsilon))$ and $1/(1+\exp(\beta\varepsilon))$, respectively; then $\alpha^*(g) = \beta$ (since $\mathbf{E}[\exp(-\beta g)] = 1$), and Proposition 11 (i) and (ii) implies that g is accepted by u at w_1 and rejected by v at w_2, contradicting $u \trianglerighteq v$. Conversely, if $\inf_w \rho_u(w) \geq \sup_w \rho_v(w)$ and u accepts g at some w_1, then Proposition 11 (iv) implies that there is w_2 such that $\rho_u(w_2) < \alpha^*(g)$, and so $\sup_w \rho_v(w) \leq \inf_w \rho_u(w) \leq \rho_u(w_2) < \alpha^*(g)$, which, by Proposition 11 (ii), implies that g is accepted by v at any w—that is, $u \trianglerighteq v$. □

Proof of Proposition 6. In view of (17), we need to show that $g \geqslant_{\mathrm{D}} h$ if and only if $\alpha^*(g) \geq \alpha^*(h)$.

Assume that $\alpha^*(g) \geq \alpha^*(h)$, and let $u \trianglerighteq v$ be such that g is rejected by v at w. Then $\inf \rho_u \geq \sup \rho_v \geq \alpha^*(g) \geq \alpha^*(h)$ (the first inequality by Lemma 18, the second by Proposition 11 (iii), and the last one by assumption); therefore h is rejected by u at any w by Proposition 11 (i).

Conversely, assume that $\alpha^*(g) < \alpha^*(h)$. Take β such that $\alpha^*(g) < \beta < \alpha^*(h)$ and consider the two CARA utilities $u = \bar{v}_\beta$ and $v = \bar{v}_{\alpha^*(g)}$. Then $u \vartriangleright v$ (by Lemma 18), g is rejected by v at all w, but h is accepted by u at all $w > L_h$ (by Lemma 10), and so $g \geqslant_{\mathrm{D}} h$ does *not* hold. □

Remarks. (1) The duality order $\geqslant_{\mathrm{D}}$, and thus the duality axiom (which is equivalent to (11)), would not be affected if we were to replace in the definitions (10) and (9) the strong relation $u \vartriangleright v$ with its weaker form $u \trianglerighteq v$ (indeed: the Proof of Proposition 6 uses only $u \trianglerighteq v$).

(2) The duality order $\geqslant_{\mathrm{D}}$, and thus the duality axiom, would not be affected if we were to require in the definitions (10) and (9) the utilities u, v to be monotonic, i.e., $u, v \in \mathcal{U}_{\mathrm{DA}}$ and/or $u, v \in \mathcal{U}_{\mathrm{IR}}$ (indeed: the Proof of Proposition 6, on the one hand, did not use any of these monotonicity properties, and on the other hand, the specific utilities that appeared there were CARA utilities that belong to both $\mathcal{U}_{\mathrm{DA}}$ and $\mathcal{U}_{\mathrm{IR}}$). However, we *cannot* add the requirement that $u, v \in \mathcal{U}_{\mathrm{sr}}$, since then the relation $u \trianglerighteq v$ becomes vacuous (by Lemma 18, because $\sup_w \rho_u(w) = \infty$ for every $u \in \mathcal{U}_{\mathrm{sr}}$).

Based on Proposition 6 we can provide simple proofs for the main results, Theorems A and D, of Aumann and Serrano (2008).

Proposition 19 (i) *A continuous and first-degree monotonic real-valued function Q on $\mathcal{G}$ satisfies the duality axiom if and only if Q is ordinally equivalent to R^{AS}.*

(ii) *A positively homogeneous of degree one real-valued function Q on $\mathcal{G}$ satisfies the duality axiom if and only if Q is a positive multiple of R^{AS}.*

Proof. Theorem 1 and Proposition 6 show that the duality axiom (see (11)) is equivalent to

$$Q(g) < Q(h) \quad \text{implies} \quad R^{\mathrm{AS}}(g) \leq R^{\mathrm{AS}}(h), \tag{18}$$

or

$$R^{\mathrm{AS}}(g) < R^{\mathrm{AS}}(h) \quad \text{implies} \quad Q(g) \leq Q(h) \tag{19}$$

(take the negation of (18) and interchange g and h). Thus R^{AS} satisfies the duality axiom; since it is clearly homogeneous, continuous, and first-order monotonic (cf. Section V in Aumann and Serrano 2008), it remains to prove its uniqueness in (i) and (ii). This will readily follow once we show that in both cases we have

$$Q(g) < Q(h) \quad \text{if and only if} \quad R^{\mathrm{AS}}(g) < R^{\mathrm{AS}}(h). \tag{20}$$

To prove this in case (i), assume that $Q(g) < Q(h)$; then $Q(g) < Q(h+\varepsilon)$ for small enough $\varepsilon > 0$ (by continuity of Q), implying that $R^{\mathrm{AS}}(g) \leq R^{\mathrm{AS}}(h+\varepsilon)$ (by (18)), and so $R^{\mathrm{AS}}(g) < R^{\mathrm{AS}}(h)$ (by monotonicity of R^{AS} we have $R^{\mathrm{AS}}(h+\varepsilon) < R^{\mathrm{AS}}(h)$). Conversely, assume that $R^{\mathrm{AS}}(g) < R^{\mathrm{AS}}(h)$; then $R^{\mathrm{AS}}(g) < R^{\mathrm{AS}}(h+\varepsilon)$ for small enough $\varepsilon > 0$ (by continuity of R^{AS}), implying that $Q(g) \leq Q(h+\varepsilon)$ (by (19)), and so $Q(g) < Q(h)$ (by monotonicity of Q). This completes the proof of (20) in case (i).

The proof in case (ii) is similar. Assume that $Q(g) < Q(h)$; using the homogeneity of Q we get $Q(g) < (1-\varepsilon)Q(h) = Q((1-\varepsilon)h)$ for small enough $\varepsilon > 0$, implying that $R^{\text{AS}}(g) \le R^{\text{AS}}((1-\varepsilon)h) = (1-\varepsilon)R^{\text{AS}}(h)$ (by (18) and the homogeneity of R^{AS}), and so $R^{\text{AS}}(g) < R^{\text{AS}}(h)$. Conversely, assume that $R^{\text{AS}}(g) < R^{\text{AS}}(h)$; then $R^{\text{AS}}(g) < (1-\varepsilon)R^{\text{AS}}(h) = R^{\text{AS}}((1-\varepsilon)g)$ for small enough $\varepsilon > 0$, implying that $Q(h) \le Q((1-\varepsilon)g) = (1-\varepsilon)Q(g)$ (by (19) and the homogeneity of Q), and so $Q(g) < Q(h)$. This completes the proof of (20) in case (ii). □

The role of the additional assumptions in Aumann and Serrano (2008)—either homogeneity, or continuity together with monotonicity—becomes clear now: they are needed to go from (18) to (20).

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