Ratings-Based Price Discrimination*

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PRELIMINARY AND INCOMPLETE

Abstract

A long-lived consumer interacts with an infinite sequence of short-lived firms in a stationary Gaussian setting. Firms use the consumer's *rating*—an aggregate measure of her purchase history—to learn about the consumer's type, and thus set prices. The focus is on linear Markov equilibria when ratings discount past purchases exponentially. We find that equilibrium prices are lower in expectation than in a static benchmark due to the strategic effect of privacy concerns. The precision of the information conveyed in equilibrium by a rating is non-monotone in its persistence level. Firms may prefer more persistent ratings than under public histories, whereas high-value consumers may prefer more or less persistent ratings to uninformative ratings. Total surplus is instead maximized by uninformative ratings. Finally, hidden ratings that are not observed by the consumer reduce the sensitivity of demand and increase the firms' profits. Our analysis thus sheds light on the role that *transparency* and *persistence* of consumer data can have on market outcomes.

Keywords: Ratings; Price Discrimination; Signaling; Ratchet Effect; Brownian Motion; Persistence; Transparency.

JEL codes: C73, D82, D83.

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1 Introduction

Value creation in online markets depends critically on identifying consumers' preferences on the basis of behavioral and contextual data. The data is stored by intermediaries who classify consumers in "intent groups" or "interest segments" that enable targeting of products and advertising messages. Thus, the aggregation of large amounts of consumer-generated data into a succinct *rating* is a key technological driver of content personalization.

The precision of the consumer-level information available to marketers and retailers can, however, introduce privacy concerns. These concerns are not necessarily related to leakages or illicit uses of consumers' private information: the distributional implications of consumer categorization—as with all forms of market segmentation—are ambiguous. Furthermore, when the use of individual information impacts the distribution of surplus, the mechanisms by which it is collected and transmitted in turn determine consumers' willingness to reveal information about their preferences.

In this paper, we investigate the allocation and welfare consequences of rating consumers based on their purchase histories and using this information for price discrimination purposes. We seek to provide an equilibrium model of privacy concerns. We focus on the effect of a rating's *persistence* and *transparency* on the level and terms of trade. We then turn to optimal ratings (within a specific class) and address the following welfare questions: is a long memory detrimental to consumers? How quickly should consumers' actions be forgotten by the market? Should consumers be aware of the "bucket" firms have placed them in?[1] We address these questions both from consumers' and firms' perspectives. The former perspective can provide insights into guidelines for consumer protection and other types of regulation. The latter perspective has perhaps greater predictive power, under the assumptions that firms trade information without frictions and that consumers do not participate in the market for their own data.

We cast our analysis in a canonical ratchet-effect setting—where any information collected about a consumer is used to set future prices—with some key modifications. First, we consider a linear-quadratic-Gaussian model with stationary learning in continuous time. Second, we introduce noise in the observation of the consumer's actions, i.e., the quantity purchased is recorded with error. Third, an intermediary aggregates consumers' past purchase signals into a one-dimensional rating, and then reveals this rating to short-lived firms that use it to set prices. Our model can then be seen as one of very sophisticated third-degree price discrimination. In this setting, the consumer does not control the flow of information

[1] Data brokers make few attempts at improving transparency. One exception is the Oracle/Bluekai Registry http://www.bluekai.com/registry/ that reveals to consumers which interest groups they belong to.

directly. She can, however, distort the level of her purchases away from the myopic optimum, so to manipulate her own rating, and hence, affect the prices she will face in the future.

In our model, we explicitly ignore any "horizontal" aspect of information revelation. Such aspects would be directly beneficial to the consumer, by facilitating the matching of content to her tastes. Instead, we isolate a "vertical" aspect of information revelation that leads to price discrimination. Furthermore, we choose a formulation where price discrimination is harmful to consumers in a one-period interaction. This allows us to focus on the equilibrium effect of information revelation in a dynamic model, which can benefit the consumer by affecting the dynamics of prices over time.

Main results We derive the following results. For any exponential rating—where the weight on past signals decays exponentially with time—there exists a unique equilibrium. Because the incentives to manipulate prices depend on the consumer's true type, the sensitivity α of the consumer's purchases to θ determines the signal-to-noise ratio of the quantity signal. In equilibrium, high types have a stronger incentive to manipulate beliefs downward, resulting in a lower sensitivity of the consumer's actions to the underlying type, which reduces the information contained in a rating.

A single rating (i.e., a unique persistence level) yields the same equilibrium outcome as revealing the full history of signals. The fully revealing rating does not, however minimize the amount of information revealed by the consumer, nor does it maximize the amount of information conveyed to the firms. Indeed, a more persistent rating than the fully revealing one motivates the consumer to reveal more information. This rating transmits more information to the firms, as the loss in precision from aggregating the signals is second order. Conversely, a less persistent rating leads to less information transmitted both due to the direct technological effect and to the consumer's equilibrium behavior.

We then turn to the welfare implications of the rating's persistence. We show that the firm's (ex ante) expected profits can be written in terms of the mean and variance of the equilibrium price. The mean of the equilibrium price is proportional to the mean type and U-shaped in the persistence of the rating. We interpret the prior mean of the type as an observable characteristic of a market segment. Thus, for very valuable market segments, firms would prefer to operate under no information, so to eliminate the ratchet effect and encourage the consumer to purchase. Conversely, the variance of the equilibrium price is inverse-U shaped in the persistence parameter. It is related to, but not monotone in the precision of the firms' posterior beliefs. Thus, for less valuable segments, firms benefit from informative ratings. In these cases, the firm-optimal rating is more persistent than the maximally informative rating: firms are willing to trade-off the precision of the information they receive in equilibrium for greater sensitivity of the consumer's demand to her true type, which increases the ex ante variance of the price.

Consumer surplus is (almost) the mirror image of producer surplus: if the mean of the consumer's type is sufficiently low, the consumer-optimal rating is uninformative. Conversely, if the mean of the consumer's type is sufficiently high, consumers prefer an informative rating to not being tracked, because the presence of an informative rating leads firms to lower prices. Consumer surplus, however, displays an additional effect, whereby a purchase strategy that is responsive to the true type (i.e., buys more when the type is high) is per se beneficial. This third effect can lead the consumer-optimal rating to be more or less persistent than the unique "public" rating that yields the same equilibrium outcome as observing the full history of signals. Overall, consumer surplus can be maximized by a more or less persistent rating, depending on the discount rate and the persistence of the underlying type.

Finally, we turn to the case of ratings that are hidden to the consumer. With hidden ratings, both the firms and the consumer can signal their private information—the firms signal the rating and the consumer signals her type. Because prices are observed without noise, in a pure-strategy equilibrium, the contemporaneous price perfectly reveals the current rating. Therefore, the current price carries a signal of future prices that affects the consumer's incentives to manipulate the rating. The strength of the consumer's incentive to manipulate the rating is related to the properties of the consumer's continuation value, which is a decreasing and convex function of the current price.[2]

As a result, the advantage of reducing prices is greater when prices are low and the consumer buys more units. In particular, a signal of high future prices reduces the value of manipulation and limits the incentives for the consumer to scale back her purchases. Conversely, a low price signals a good opportunity to manipulate beliefs downward, and induces a smaller expansion of the consumer's purchases. The signaling component is in addition to the dynamic incentives that affect the consumer's choice. Therefore, relative to the case of a publicly observed rating, the consumer's demand is less sensitive to the current price when ratings are hidden. Opacity therefore allows firms to maintain higher prices, and (numerical simulations show that) increases their profits.

Related Literature This paper is most closely related to the literature on behavior-based price discrimination, e.g., Taylor (2004) and Acquisti and Varian (2005). The main results in this literature and their implications for consumer privacy are discussed at length in the surveys by Fudenberg and Villas-Boas (2006, 2015) and by Acquisti, Taylor, and Wagman (2016). In recent work, Cummings, Ligett, Pai, and Roth (2016) and Shen and Miguel

[2]This property is the analog of the convexity of the indirect utility function in a static model.

Villas-Boas (2017) study two-period models in which advertisement messages are targeted on the basis of the information about consumers' purchase activity. These papers highlight trade-offs similar to ours, where the value of targeted advertising (which could be negative) impacts the equilibrium price of the first-period good and the amount of information revealed by the consumer.

Our paper leverages the construction of a linear Gaussian *rating* pioneered in Hörner and Lambert (2017), which allows us to examine the role of persistence and transparency of the firms' information on economic outcomes in a tractable way. Relative to Hörner and Lambert (2017), we maintain the assumptions of short-lived firms and additive signals. In other words, the agent does not control the precision of the information directly. However, while we consider only a restrictive class of ratings, performing equilibrium analysis is more complex than in the career concerns model of Hörner and Lambert (2017) because we consider a privately informed agent. This introduces new considerations regarding the equilibrium precision of signals. Moreover, the value of signaling is not linear in our model, which means the agent's optimal action depends on the level of the firms' beliefs. Therefore, in our paper, it matters whether the agent knows his own rating.

The reduction and the distribution of surplus when firms learn about the consumer's type through an informative rating is due to the ratchet effect (Freixas, Guesnerie, and Tirole, 1985; Laffont and Tirole, 1988). The ratchet effect is also the key economic force underlying the analysis of privacy in a model with multiple principals. See, e.g., Calzolari and Pavan (2006) for the case of two principals and Dworczak (2017) for the case of a single transaction followed by an aftermarket.[3]. Finally, our welfare analysis addresses the question of optimal memory and information design in markets (Kovbasyuk and Spagnolo, 2016).

Finally, in related contributions, Heinsalu (2017) analyzes a dynamic game with noisy signaling and a Gaussian structure, abstracting from information design, while Di Pei (2016) analyzes performance ratings in a model with privately informed agents and Poisson learning.

2 Model

Players, timing and preferences. A long-lived consumer faces a continuum of short-run firms in a repeated game in continuous time over an infinite horizon. Firms are indexed by time $t \in [0, \infty)$, and each produces a good at zero production costs. At each time t, the stage game involves firm t as the single supplier of the good at that instant. The timing is

[3]The ratchet effect appears, with a different interpretation or motivation, in relational contracts with and without private information (Halac, 2012; Fong and Li, 2016) and in dynamic games with symmetric uncertainty (Cisternas, 2017b).

as follows: (i) firm t posts a unit price P_t for its good, and then (ii) the consumer chooses to purchase a quantity Q_t given the observed price. Once the transaction occurs, time evolves and the same interaction repeats with a different firm.

The consumer discounts the future at rate $r > 0$. Consuming $Q_t = q$ units of the good at price $P_t = p$ delivers a flow utility

$$u(\theta, q, p) := (\theta - p)q - \frac{q^2}{2} \tag{1}$$

where $\theta_t = \theta$ is the consumer's *type* at t. We assume that throughout the analysis that the consumer's type process is mean-reverting,

$$d\theta_t = -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ_t^\theta, \tag{2}$$

and stationary.[4] Thus, the consumer's type is Gaussian, and stationarity implies that

$$\mathbb{E}[\theta_t] = \mu \quad \text{and} \quad \text{Cov}[\theta_t, \theta_s] = \frac{\sigma_\theta^2}{2\kappa} e^{-\kappa|t-s|}. \tag{3}$$

Because production costs are normalized to zero, firms maximize their expected revenue from their interaction with the consumer. Namely, if the consumer's demand for the good is given by $p \mapsto Q(p)$ (and so, the realized purchase at time t is $Q_t = Q(P_t)$), firm t posts a price that maximizes

$$p\mathbb{E}[Q(p)|Y_t], \tag{4}$$

where Y_t is a random variable observed by firm t and that is specified shortly, $t \geq 0$.

Information. We adopt the following information structure. The consumer's type process $(\theta_t)_{t\geq 0}$ is her private information. The consumer's purchased quantity is recorded with an exogenous error by an (unmodeled) intermediary that observes a signal process

$$d\xi_t = Q_t dt + \sigma_\xi dZ_t^\xi, \ t > 0,$$

where $(Z^\xi)_{t\geq 0}$ is a Brownian motion independent of $(Z^\theta)_{t\geq 0}$.

The intermediary aggregates the past signals of the consumer's action into a one-dimensional variable Y_t, which is the only source of information available to firm t. Building on Hörner and Lambert (2017), we restrict attention to *exponential* ratings. Specifically, a consumer

[4] Mathematically, stationarity boils down to $\theta_0 \sim \mathcal{N}(\mu, \sigma_\theta^2/2\kappa)$ and independent of $(Z_t^\theta)_{t\geq 0}$.

rating process $(Y_t)_{t\geq 0}$ is any Ito process

$$dY_t = -\phi Y_t dt + d\xi_t,\ t > 0, \tag{5}$$

where $\phi > 0$. In particular, the linearity of the previous process leads to an exponential discounting of recorded purchases of the form

$$Y_t = Y_0 e^{-\phi t} + \int_0^t e^{-\phi(t-s)} d\xi_s. \tag{6}$$

where Y_0 is the initial value of the rating. We refer to Y_t as the consumer's time-t *rating* and to $\phi > 0$ as the *persistence* level of the rating—the main policy variable under study. We denote a consumer rating process of persistence $\phi > 0$ by $(Y_t^\phi)_{t\geq 0}$ whenever convenient.

In the baseline specification we assume that the consumer observes the entire history $Y^t := (Y_s : 0 \leq s \leq t)$. We discuss the cases in which (i) firm t also observes the entire history Y^t (equivalently, ξ^t) in Section 4, and (ii) firm t observes Y_t only, but the rating is hidden to the consumer in Section 6, $t \geq 0$. Finally, the random vector (θ_0, Y_0) is assumed to be normally distributed such that the joint process $(\theta_t, Y_t)_{t\geq 0}$ is a stationary Gaussian process in equilibrium.

Strategies and Equilibrium Concept At any time $t \geq 0$, the consumer observes the current posted price p, the value of the rating Y_t, and her own type θ_t. A feasible strategy for the consumer is any process $(Q_t)_{t\geq 0}$ taking values in $\mathbb{R}$ that is measurable with respect to (p, θ_t, Y_t) and that satisfies standard integrability conditions. Instead, since the firms only observe the current value of the rating, firm t must choose a price P_t that is measurable with respect to Y_t, $t \geq 0$. Given a pair (P_t, Q_t), the consumer's continuation value at time t is given by

$$\mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\theta_s, Q_s, P_s) ds \right].$$

where $\mathbb{E}_t[\cdot]$ denotes the consumer's conditional expectation operator.

A *linear Markov* strategy for the consumer is a feasible strategy Q_t that is an affine function of the current value of (p, θ_t, Y_t), with coefficients that are constant, i.e., independent of time and of her private history. Similarly for firm t, replacing (p, θ_t, Y_t) by Y_t. Abusing notation, we denote linear Markov strategies by $Q(p, \theta_t, Y_t)$ and $P(Y_t)$ for the consumer and firm, respectively, where $Q : \mathbb{R}^3 \to \mathbb{R}$ and $P : \mathbb{R} \to \mathbb{R}$ are linear. Observe that the weight that $Q(p, \theta_t, Y_t)$ attaches to the current price is simply the *sensitivity of demand*.

Definition 1 (Linear Markov Equilibrium)**.** *A pair (Q, P) of linear Markov strategies is a*

Nash equilibrium if

(i) *At history Y_t, $p = P(Y_t)$ maximizes $p\mathbb{E}[Q(p, \theta_t, Y_t)|Y_t]$*

(ii) *At history $(P(Y_t), \theta_t, Y_t)$, the process $(Q(P(Y_s), \theta_t, Y_s))_{s\geq t}$ maximizes the consumer's continuation utility given $P_s = P(Y_s)$, $s \geq t$.*

In this equilibrium concept, optimality of the consumer's strategy is verified only on the path of play, i.e., when firms set prices according to $P_t = P(Y_t)$ for all $t \geq 0$ (part (ii)). As a result, the outcome of the game is supported by prices based on the belief that the consumer's demand responds to contemporaneous off-path price variation with a sensitivity that coincides with the weight attached to current posted prices on the path of play. The justification is twofold. First, a deviation by a zero measure set of firms (in particular, by a single firm) does not affect the consumer's payoff when time is continuous. Second, when time is discrete and noise has full support the consumer's (linear) best response given a sequence of prices does not differentiate between a current off-path price versus an on-path one. In fact, since firms do not observe past prices, the continuation game remains unchanged after the deviation.[5]

A natural question that therefore arises is how to determine the equilibrium sensitivity of demand if such an off-equilibrium analysis is bypassed. In this line, the advantage of continuous-time lies partly on enabling the determination of a value for demand sensitivity independently from all the equilibrium conditions. Moreover, such value is in fact the limit of (endogenously determined) demand sensitivities along a sequence of discrete time games indexed by period length, under appropriately scaled noise. We elaborate more on this topic in Section 3.2.

Discussion of modeling choices First, instead of studying the reputation dynamics of a long-lived firm that practices behavior-based price discrimination, we focus on a consumer who is in the market for different products over time. In other words, the consumer interacts with a different (imperfectly competitive) industry in each period, and only returns to the same industry after a long time. In this sense, our simplifying assumptions are that the consumer is never in the market for the same product twice (i.e., firms are short lived), and that each firm is a local monopolist. Finally, we observe that the assumption of short-lived firms is useful to examine the role of ratings but that, under public (noisy) purchase histories, our Markovian equilibrium remains such in a model with a single firm.

Second, we restrict attention to one-dimensional, continuous ratings. The rating Y_t, as opposed to a coarse categorization of the consumer's tastes, is an approximation of the

[5]This is formally illustrated in Appendix B where a discretized version of the model is examined.

products offered by big-data brokers. Brokers collect data about consumers' behavior from a wide variety of sources. Data brokers' clients (e.g., retail firms) can access information about several attributes for each individual prospect. However, even a multidimensional map of a consumer's preferences does not often contain detailed time-level information. Thus, the rating does not correspond to a full history of consumers' actions. On the other hand, information on each attribute is collected in detailed categories (e.g., point estimates of annual income vs. coarse income brackets), which we approximate with a continuous rating.

Third, an exponential rating as in (6) is the most tractable aggregator of past signals that retains some attractive properties: it has a natural counterpart in Bayesian updating when all signals are public (because optimal filtering results in a specific ϕ that, as we shall see, depends on equilibrium behavior); it is analogous to stochastic forgetting—perhaps a more realistic representation of the lifespan of Internet cookies than a deterministic duration.[6]

3 Equilibrium Analysis

3.1 Firms' Beliefs

When the consumer conditions her demand on her type, firm t must form a belief regarding θ_t in order to optimally choose a price. But if the time-s realized purchase is linear in (θ_s, Y_s), $s \leq t$, the pair (θ_t, Y_t) is Gaussian, which results in firm t's posterior belief $\theta_t|Y_t$ being normally distributed. Let $M_t := \mathbb{E}[\theta_t|Y_t]$ and $\Sigma_t := \mathbb{E}[(\theta_t - M_t)^2|Y_t]$ denote the corresponding mean and variance.[7]

Importantly, because M_t will be an affine function of Y_t, we can ultimately pay attention to strategies that are linear in (θ_t, M_t). Specifically, will be interested in linear Markov equilibria in which, along the path of play, (i) the consumer purchases admit the representation

$$Q_t = \alpha\theta_t + \beta M_t + \delta\mu, \tag{7}$$

and (ii) the induced process $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian.[8] We refer to this type of equilibrium simply as *stationary linear Markov equilibrium.*

The next result characterizes stationary Gaussian pairs $(\theta_t, Y_t)_{t\geq 0}$ under (7) with the consistency requirement that $M_t = \mathbb{E}[\theta_t|Y_t]$:

[6] For example, different Internet browsers retain information for varying amounts of time, and different types of cookies have different automatic expiration dates.

[7] For notational simplicity, we omit the dependence of the expectation operator—and hence, of (M_t, Σ_t)—on the equilibrium strategy.

[8] That the constant term in the consumer's equilibrium purchasing process is proportional to μ is just a convenient normalization. It follows from the type and the beliefs being centered around μ.

Proposition 1. *Suppose that $(Y_t, \theta_t)_{t\geq 0}$ is stationary Gaussian under (7). Then,*

(i) there is $\lambda \in \mathbb{R}$ such that $M_t = \mu + \lambda[Y_t - \bar{Y}]$, with $\bar{Y} = \mu(\alpha+\beta+\delta)/\phi$, and solving

$$\lambda = \frac{\alpha\sigma_\theta^2(\phi-\beta\lambda)}{\alpha^2\sigma_\theta^2 + \sigma_\xi^2\kappa(\phi-\beta\lambda+\kappa)}; \tag{8}$$

(ii) $(Y_0, \theta_0) \sim \mathcal{N}([\bar{Y}, \mu]^\top, \Lambda)$ is independent of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$ where

$$\Lambda = \begin{bmatrix} \frac{\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi-\beta\lambda+\kappa)}{2\kappa(\phi-\beta\lambda)(\phi-\beta\lambda+\kappa)} & \frac{\alpha\sigma_\theta^2}{2\kappa(\phi-\beta\lambda+\kappa)} \\ \frac{\alpha\sigma_\theta^2}{2\kappa(\phi-\beta\lambda+\kappa)} & \frac{\sigma_\theta^2}{2\kappa} \end{bmatrix}, \quad \textit{and}$$

(iii) $\phi - \beta\lambda > 0$.

Conversely, if $Q_t = \alpha\theta_t + \beta(\mu+\lambda[Y_t-\bar{Y}]) + \delta\mu$, with λ and $\bar{Y}$ as in (i), and (ii) and (iii) hold, $(Y_t,\theta_t)_{t\geq 0}$ is stationary Gaussian and $M_t = \mu + \lambda[Y_t - \bar{Y}]$, $t \geq 0$.

Part (i) in the Proposition states that, in a stationary linear Markov equilibrium, the posterior mean process is also stationary Gaussian, with a constant sensitivity to changes in the rating λ that satisfies (8). Part (ii) instead states that when $(Y_t,\theta_t)_{t\geq 0}$ is stationary Gaussian—and thus, a non-degenerate long-run distribution for the joint process exists—(Y_0, θ_0) must correspond to an independent draw from the long-run distribution $\mathcal{N}([\bar{Y}, \mu]^\top, \Lambda)$; it then follows that $(Y_t, \theta_t) \sim \mathcal{N}([\bar{Y}, \mu]^\top, \Lambda)$ for all $t \geq 0$. Finally, part (iii) states that the long-run distribution is non-degenerate when Λ, as a covariance matrix, is positive definite, which in turn happens when $\phi - \beta\lambda > 0$. This is intuitive: the equilibrium rating will admit a long-run distribution whenever it is mean-reverting, i.e., when $\phi-\beta\lambda > 0$, as the original persistence ϕ is modified due to the contribution coming from βM_t in the purchase process (7). In what follows, we assume that firms form beliefs using (α, β, δ) such that $(Y_t,\theta_t)_{t\geq 0}$ is stationary Gaussian from their perspectives.

To conclude, two observations. First, because the noise in the rating prevents deviations from equilibrium behavior to be observed by firms, $M_t = \mu + \lambda[Y_t - \bar{Y}]) + \delta\mu$ holds on and off the path of play (i.e., it holds *path-by-path* of $(Y_t)_{t\geq 0}$). Thus, the consumer can control future firms' beliefs by affecting the evolution of the rating. Specifically, given any feasible purchasing strategy $(Q_t)_{t\geq 0}$, the law of motion of $(M_t)_{t\geq 0}$ is given by

$$dM_t = \left[-\phi\left(M_t - \mu + \lambda\bar{Y}\right) + \lambda Q_t\right]dt + \lambda\sigma_\xi dZ_t^\xi. \tag{9}$$

Second, by the projection theorem for Gaussian random variables

$$\lambda = \frac{\mathrm{Cov}[\theta_t, Y_t]}{\mathrm{Var}[Y_t]} = \frac{\Lambda_{12}}{\Lambda_{11}} \text{ and } \Sigma = \mathrm{Var}[\theta_t] - \frac{\mathrm{Cov}[\theta_t, Y_t]^2}{\mathrm{Var}[Y_t]} = \frac{\sigma_\theta^2}{2\kappa}\left[1 - \lambda\Lambda_{12}\right]$$

where Λ_{ij} is the (i, j) entry of Λ. In particular, the *gain function*

$$G(\phi, \alpha, \beta) \;\; := \;\; \lambda\Lambda_{12} = \frac{\alpha\lambda(\phi, \alpha, \beta)}{\phi + \kappa - \beta\lambda(\phi, \alpha, \beta)} \in [0, 1], \tag{10}$$

measures how much firm learn about the consumer's type, where the dependence of λ on (ϕ, α, β) is made explicit. This gain function will play a critical role in sections 4 and 5.

3.2 Consumer's Problem

The last step required to state the consumer's problem consists of specifying the price process that she will face. Because consumers observe their own ratings, they are able to predict the equilibrium price. However, specifying the consumer's response to an (off-path) price $p \neq P_t$ (i.e., the sensitivity of demand) is critical to compute the monopoly price P_t.

We solve for an equilibrium supported by a demand sensitivity of -1. Heuristically, this can be understood in two steps. First, since we are interested in Markov strategies, the weight on p does not depend on whether we are at an on- or off-path history—as stated, firms do not observe past prices, so the continuation game and the other equilibrium coefficients are unaffected by the deviation. The second step comes from dynamic programming (e.g., computing the first-order condition from equation (12)): the consumer's best-response problem attaches a weight equal to -1 to the candidate equilibrium price. The rigorous justification nevertheless comes from showing that -1 is the only possible value for the limit of demand sensitivities along a sequence of discrete time games indexed by period length, as the latter shrinks to zero—refer to Remark 1 for more details. The monopoly price in such an equilibrium is characterized in the next lemma.

Lemma 1 (Monopoly Price)**.** *Along the path of play of a linear Markov equilibrium with unit demand sensitivity and realized purchases (7), firms choose prices according to*

$$P_t = (\alpha + \beta)M_t + \delta\mu, \; t \geq 0. \tag{11}$$

The key behind this result is to recognize that the slope of the consumer's demand is the same as in a static model. Intuitively, this can be seen from (12) below, noting that (a) the price P_t enters the flow payoff only, and (b) that Q_t affects the continuation value linearly.

Thus, the signaling motive affects the intercept but not the slope of the consumer's demand. The consumer then solves the following problem:

$$\max_{(Q_t)_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-rt}\left[(\theta_t - P_t)Q_t - \frac{Q_t^2}{2}\right]dt\right]$$

subject to

$$\begin{aligned}
d\theta_t &= -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ_t^\theta \\
dM_t &= (-\phi[M_t - \mu + \lambda\bar{Y}] + \lambda Q_t)dt + \lambda\sigma_\xi dZ_t^\xi \\
P_t &= (\alpha+\beta)M_t + \delta\mu.
\end{aligned}$$

In a stationary linear Markov equilibrium, the consumer's best reply along the path of play is exactly (7), i.e., $Q_t = \alpha\theta_t + \beta M_t + \delta\mu$. Such an equilibrium can be characterized via dynamic programming.

Remark 1 (On Markov strategies and the sensitivity of demand)**.** Using the traditional discretization for $(\theta_t, Y_t)_{t\geq 0}$, Appendix B examines a sequence of discrete-time versions of the model indexed by their period length. Along this sequence, it is shown that (i) optimal linear best-responses along the path of play are also optimal after observing off-path prices, and (ii) the weight that the linear policy attaches to the current price converges to -1 as the period length goes to zero. Thus, the Markov assumption places no restriction in discrete time, and unit demand sensitivity is a limiting property of the consumer's best-response problem (due to the impact of its purchase on its continuation value becoming asymptotically linear).

3.3 Characterization of Linear Markov Equilibria

Let $V(\theta, M)$ denote the value of the consumer's problem when the state's current value is $(\theta, M) \in \mathbb{R}^2$. If (α, β, δ) is such that the policy delivered by the HJB equation

$$\begin{aligned}
rV(\theta, M) = \sup_{q\in\mathbb{R}} &\left\{ \big(\theta - \underbrace{[(\alpha+\beta)M + \delta\mu]}_{=P_t}\big)q - \frac{q^2}{2} - \kappa(\theta-\mu)V_\theta \right. \\
&\left. + (\lambda q - \phi[M - \mu + \lambda\bar{Y}])\frac{\partial V}{\partial M}(\theta, M) + \frac{\lambda^2\sigma_\xi^2}{2}\frac{\partial^2 V}{\partial M^2} + \frac{\sigma_\theta^2}{2}\frac{\partial^2 V}{\partial \theta^2} \right\} \quad (12)
\end{aligned}$$

(subject to standard transversality conditions) coincides with (7), then the coefficients (α, β, δ) fully determine a linear Markov equilibrium.

The combination of (i) quadratic flow payoffs and (ii) Gaussian types and shocks make the learning and signaling analysis tractable. In particular, the consumer's best-response problem is a linear-quadratic optimization problem, so we look for a quadratic value function

$$V(\theta, M) = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M$$

solving (12), and thus for a linear best response—imposing then the condition that the firms correctly anticipate the consumer's behavior, it is easy to find equations for the equilibrium coefficients (α, β, δ). To fully characterize stationary linear Markov equilibria, however, the resulting system must be coupled with the equation for λ displayed in (8), so as to pin down the equilibrium sensitivity of beliefs. The rest of the coefficients of the value function can in turn be determined in an iterative fashion.

To state the main result of this section, define the auxiliary function

$$f(\phi, \alpha) := -\frac{\alpha^2(r+2\phi)}{2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)} \tag{13}$$

which is well defined over $(\phi, \alpha) \in \mathbb{R}_+ \times [0, 1]$ and satisfies $-\alpha/2 < f(\phi, \alpha) \leq 0$ on the same region. In addition, abusing notation, let

$$\lambda(\phi, \alpha, \beta) \;\; := \;\; \frac{\sigma_\theta^2\alpha(\alpha+\beta) + \kappa\sigma_\xi^2(\kappa+\phi) - \sqrt{[\sigma_\theta^2\alpha(\alpha+\beta) + \kappa\sigma_\xi^2(\kappa+\phi)]^2 - 4\kappa(\sigma_\theta\sigma_\xi)^2\alpha\beta\phi}}{2\beta\kappa\sigma_\xi^2} \tag{14}$$

denote the the unique positive root of (8) when $\alpha > 0$ and $\beta < 0$.

Proposition 2 (Existence and uniqueness)**.** *There exists a unique stationary linear Markov equilibrium with coefficients (α, β, δ). In this equilibrium, $0 < \alpha < 1$ is characterized as the unique solution to the equation*

$$(r+\kappa+\phi)(x-1) - x\lambda(\phi, x, f(\phi, x))f(\phi, x) = 0, \;\; x \in [0, 1], \tag{15}$$

and $\beta = f(\phi, \alpha) \in (-\alpha/2, 0)$. Finally,

$$\delta \;\; = \;\; \frac{\kappa(\alpha-1) + [\alpha + 2f(\phi, \alpha)][\phi - (\alpha + f(\phi, \alpha))\lambda(\phi, \alpha, f(\phi, \alpha))]}{2(r+\phi) + (\alpha + f(\phi, \alpha))\lambda(\phi, \alpha, f(\phi, \alpha))}. \tag{16}$$

Because we are interested in the role of persistence on market outcomes, in what follows, we make explicit the dependence of (α, β, δ) on $\phi > 0$ whenever required.

3.4 Equilibrium Properties

Before discussing some properties of the equilibrium found, it is instructive to examine a *static benchmark*, i.e., a one-shot analog of the dynamic setting under analysis.

A consumer with preferences as in (1) interacts with a firm only once. The firm's prior about the consumer's type has mean $\mu \in \mathbb{R}$ and variance $\text{Var}[\theta] > 0$. Finally, suppose that, before interacting with the consumer, a public signal Y about the consumer's type is realized. The signal is general (i.e., it is not necessarily a rating, or Gaussian), and let $Y = \emptyset$ if the signal is uninformative.

Abusing notation, let $M := \mathbb{E}[\theta|Y]$. Given a posted price p, maximization of the consumer's flow payoff (1) yields a demand of unit slope $Q(p) = \theta - p$. The outcome of the static Nash equilibrium then entails a purchase and price given by $Q = \theta - M/2$ and $P = M/2$. Thus, from an ex ante perspective,

$$\mathbb{E}\left[P\right] = \frac{\mu}{2}. \tag{17}$$

We now turn to properties of the equilibrium purchases and prices in the dynamic game and compare them to the static benchmark. Let P_t^ϕ, M_t^ϕ denote the equilibrium price and beliefs as a function of the persistence parameter $\phi > 0$. In particular, observe that

$$\mathbb{E}[P_t^\phi] = [\alpha(\phi) + \beta(\phi) + \delta(\phi)]\mu \tag{18}$$

is the equilibrium price from an ex ante perspective.

Proposition 3 (Equilibrium Properties)**.**

(i) The coefficient $\alpha(\cdot)$ *satisfies*

$$1/2 < \frac{r+\kappa+\phi}{r+\kappa+2\phi} < \alpha(\phi) < 1, \quad \textit{for all } \phi > 0.$$

(ii) The coefficient $\alpha(\cdot)$ *is decreasing as* $\phi \to 0$ *and increasing as* $\phi \to \infty$*; and if* $r \geq \kappa$*, then* $\alpha(\cdot)$ *is quasiconvex.*[9]

(iii) Limit coefficients: $\lim_{\phi\to 0} \alpha(\phi) = \lim_{\phi\to\infty} \alpha(\phi) = 1$*;* $\lim_{\phi\to 0} \beta(\phi) = \lim_{\phi\to\infty} \beta(\phi) = -1/2$*; and* $\lim_{\phi\to 0} \delta(\phi) = \lim_{\phi\to\infty} \delta(\phi) = 0$*.*

(iv) Expected price: $\mathbb{E}[P_t^\phi] \in (\mu/3, \mu/2)$ *for all* $\phi > 0$ *if* $\mu \neq 0$*, and* $\mathbb{E}[P_t^\phi] \equiv 0$ *if* $\mu = 0$*. Moreover,* $\mathbb{E}[(P_t^\phi - \mu/2)^2] \to 0$ *as* $\phi \to 0$ *and* ∞*, for all* $t \geq 0$*.*

[9] Numerical simulations suggest that this property holds for all $(r, \kappa) \in \mathbb{R}_+^2$.

(v) Exogenous noise: for all $\phi > 0$, the coefficient $\alpha(\phi)$ and the expected price $\mathbb{E}[P_t^{\phi}]$ are increasing in σ_ξ/σ_θ.

In our model, the consumer manipulates the firms' beliefs to lead them into thinking she is a lower type. Part (i) shows that the advantage of reducing the firms' beliefs M (and thus the price) is higher for higher types, who buy more units. They will thus reduce their purchases more, resulting in a value of $\alpha < 1$, i.e., below the static benchmark. Part (ii) shows this effect is strongest for an intermediate persistence level ϕ.

Part (iii) studies what happens when ratings become uninformative. With exponential ratings, this can happen in two ways: the rating can be fully persistent $\phi = 0$, in which case it is not sensitive to new information; or it can have no memory, in which case new information is forgotten instantaneously. In other words, the intermediary need not add noise to the purchase signals (which we do not allow) to generate an uninformative rating, as extreme persistence levels render Y_t useless to the firms. In both cases, the equilibrium strategies converges to the static benchmark.

Part (iv) shows that, for all ratings $\phi > 0$, the ex ante expectation of the price is below the static benchmark. Thus, in dynamic contexts, a strategic buyer can induce firms to charge low prices on average relative to a one-shot interaction. As the rating becomes uninformative, however, the expected prices converge to their static level.

Finally, part (v) shows that the persistence level of the rating and the exogenous noise in the purchase signals have dramatically different effects on the equilibrium outcome. As the noise in the signal σ_ξ increases, the consumer's incentive to manipulate beliefs decreases, since firms' beliefs are less sensitive to the rating. Consequently, the coefficient α becomes higher, i.e., closer to the static benchmark. Because consumer's incentives to "hide" are reduced, the equilibrium price consequently rises in expectation. This contrasts with the effect of persistence, as both α and the expected price level attain their minimum at some interior values of ϕ.

The left panel of Figure 1 (below) illustrates the coefficient $\alpha(\phi)$, while the right panel shows the firms' posterior variance $\Sigma(\phi)$. In both cases, the dashed line corresponds to values of ϕ for which α is increasing.

4 Information Revelation

This section examines the interplay between the persistence of a rating and the corresponding degree of (equilibrium) learning by the firms she faces. Intuitively, since there is a tension

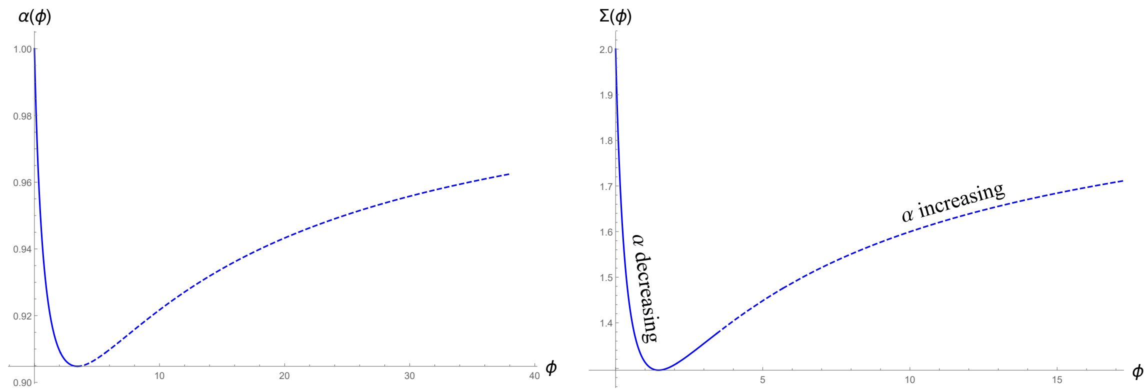

Figure 1: $(r, \sigma_\theta, \sigma_\xi, \kappa) = (3, 1, 2, 1)$.

between optimally manipulating prices and signaling willingness to pay, it is useful to evaluate how much information the consumer strategically transmits via the rating.

Fix $\alpha > 0$ and $\beta < 0$. We define the "public" persistence benchmark as

$$\nu(\alpha, \beta) := \kappa + \frac{\gamma(\alpha)\alpha(\alpha+\beta)}{\sigma_\xi^2}. \tag{19}$$

The function $\gamma(\alpha)$ is the unique positive root of the quadratic $x \mapsto \alpha^2 x^2/\sigma_\xi^2 + 2\kappa x - \sigma_\theta^2 = 0$. It corresponds to the steady state variance of beliefs for an observer who has access to the entire history of signals (ξ_t), when the underlying quantity process places weight α on θ_t.

The following result establishes that, when $\nu(\alpha, \beta) > 0$, the belief process the outside observer is, in fact, an exponential rating or discount rate $\phi = \nu(\alpha, \beta)$; and conversely, if consumer's behavior is such that $\nu(\alpha, \beta) > 0$, those beliefs can be induced by an exponential rating with the persistence level $\nu(\alpha, \beta)$.[10]

Proposition 4 (Learning under Public Histories). *Consider (7) with $\alpha > 0$ and $\beta < 0$, and suppose that $\nu(\alpha, \beta) > 0$. If firms observe the full history of signals and their beliefs are stationary, their posterior mean admits the representation*

$$M_t \quad = \quad \frac{\mu}{\nu(\alpha, \beta)} \left(\kappa - \frac{\alpha\gamma(\alpha)\delta}{\sigma_\xi^2} \right) + \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} Y_t^{\nu(\alpha,\beta)} \tag{20}$$

where $(Y_t^{\nu(\alpha,\beta)})_{t\geq 0}$ is a stationary Gaussian rating of persistence $\phi = \nu(\alpha, \beta)$.

Conversely, if firms only observe a stationary Gaussian rating of persistence $\phi = \nu(\alpha, \beta) >$

[10] Hörner and Lambert (2017) establish an analogous result. Our setup differs from theirs in that (a) the informativeness of the signal process depends on the consumer's strategy (α), and (b) the consumer's actions—an input into the rating—depend on the level of the firms' beliefs M_t.

0, *M_t in (i) in Proposition 1 satisfies*

$$\lambda(\nu(\alpha,\beta),\alpha,\beta)=\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} \quad \text{and} \quad \mu-\lambda(\nu(\alpha,\beta),\alpha,\beta)\bar{Y}=\frac{\mu}{\nu(\alpha,\beta)}\left(\kappa-\frac{\alpha\gamma(\alpha)\delta}{\sigma_\xi^2}\right).$$

At this stage, we wish to characterize the ratings ϕ that induce the same *equilibrium* outcome (i.e., allowing the consumer's behavior to respond) as the observation of the full history of purchase signals. We formalize this notion as follows.

Definition 2 (Concealing Information)**.** *We say that a rating with persistence $\phi>0$ does not conceal information about the consumer's* equilibrium *behavior if and only if it satisfies*

$$\phi=\nu(\alpha(\phi),f(\phi,\alpha(\phi))), \tag{21}$$

where $\alpha(\phi)$ is the solution to (15) and $\beta(\phi)=f(\phi,\alpha(\phi))$ is given in (13).

Condition (21) can be intuitively understood as follows. If each firm t had observed the entire history ξ^t, $t\geq 0$, Bayes rule would imply that past observations are discounted at a rate given by (19). Therefore, if the equilibrium coefficients for a rating with persistence ϕ generate a discount factor $\nu=\phi$, then the aggregation of signals into a rating does not conceal any further information about the consumer's history.[11] In other words, any fixed point of the map $\phi\mapsto\nu(\alpha(\phi),\beta(\phi))$ attains an equilibrium outcome of the game with observable purchase signals.

A key property of our model is that $\alpha(\cdot)$ is decreasing at any fixed point $\phi=\nu$. This result is instrumental to the welfare analysis in Section 5.

Proposition 5 (Revealing Ratings)**.** *Suppose that a rating with persistence $\phi>0$ does not conceal information about the consumer's equilibrium behavior. Then, $\alpha(\cdot)$ is strictly decreasing at ϕ.*

To understand the result, consider a persistence level $\phi>0$ that induces the same beliefs as observing the full history of signals. Then, a marginal reduction of ϕ has two effects. First, because beliefs are an affine function of the rating, beliefs also acquire more persistence. As a result, any change in the rating resulting from a change in demand now has a more prolonged effect on prices, which makes the consumer more wary of signaling her type. Second, because now the rating attaches excessively large importance to past behavior, it covaries less with the consumer's current type. Bayesian updating thus punishes this extra

[11] In fact, Lemma 5 in the Appendix establishes that $\phi=\nu(\alpha,\beta)$ defined in (19) is the unique minimizer of the firms' posterior variance Σ given (α,β) fixed.

persistence by reducing the sensitivity of the belief—and hence, of prices—to changes in the rating, which makes the consumer less concerned about purchasing more when her type is higher.[12] The second effect then dominates, reflecting that, due to the inherent linearity of Gaussian learning, the strength of the impulse-response of beliefs is essentially determined by the sensitivity with which these react to news.[13]

We establish the existence and uniqueness of a fixed point in the next Proposition.

Proposition 6 (Uniqueness of Revealing Rating)**.** *There exists a unique* $\phi^* \in \mathbb{R}_+$ *solving* $\phi = \nu(\alpha(\phi), f(\phi, \alpha(\phi)))$*. Such* ϕ^* *satisfies* $\kappa < \phi < \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2}$*.*

Thus, information concealment is a generic property of ratings in the linear Markov equilibrium under study. Behind the existence of a ϕ satisfying (21) there is the simple idea: since $(\theta_t, \xi_t)_{t\geq 0}$ is Gaussian under a linear strategy, observing the full time series of $(\xi_t)_{t\geq 0}$ leads to a belief process that is also an affine function of a rating determined by Bayesian updating. Therefore, in a linear Markov equilibrium of this kind, the consumer does not alter her behavior when firms are instead supplied with the corresponding rating as a summary statistic of the purchase history; but this means that the firms learn everything that is available given the coefficient α in the consumer's strategy and the noise in $(\xi_t)_{t\geq 0}$.

The fact that there is a unique ϕ^* satisfying (21) indirectly establishes the uniqueness of Markovian equilibrium for the case of public signals. Some intuition can be obtained by considering the consumer's best reply to the firms' conjecture about her strategy when all signals are public: if the firms expect low sensitivity of quantities to the underlying types, they also view the signals as uninformative, but the consumer then has no reason to hide. The opposite holds if the firms assign a large weight to the purchase signals. Thus, the firms' conjecture and the sensitivity of the consumer's actual behavior are strategic substitutes.

The degree of persistence ϕ^* that conceals no information about the consumer's behavior does not, however, maximize learning in equilibrium. Intuitively, this is because changing the persistence of the rating leads to a change in the consumer's behavior. In particular, as ϕ affects α, a different persistence level $\phi \neq \phi^*$ may induce the consumer to reveal more information. As a result, the precision of the firms' beliefs may increase even if the new rating conceals some of the information contained in the purchase signals. In the next result, we establish a property of the gain function $G(\phi, \alpha, \beta)$ defined in (10).

Proposition 7 (Maximizing Learning)**.** *At* $(\phi, \alpha, \beta) = (\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$, $G_\phi = G_\beta = 0$ *and* $G_\alpha > 0$*. Since* $\alpha(\cdot)$ *is strictly decreasing at* ϕ^**, there exists a rating* $\phi < \phi^*$ *that generates more information.*

[12] As shown in the proof of Lemma 5, $\lambda_\phi(\phi, \alpha, \beta) = \lambda(\phi, \alpha, \beta)/[\phi + \kappa] > 0$ at $\phi = \nu(\alpha, \beta)$.

[13] The trade-off between persistence and sensitivity also arises in signal-jamming models. See, for instance, Cisternas (2017a) in the context of career concerns.

By definition of ϕ^*, $G_\phi(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*))) = 0$, so changing the persistence of the rating only has a second-order effect on learning, everything else equal. Increasing α, however, has a first-order positive effect on learning, as the rating now covaries more with the consumer's type. Interestingly, increasing β marginally at $(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$ has no first-order effect on the amount of information transmitted. In fact, since β is corresponds to the coefficient on the firm's belief in the consumer's purchase process, a small change in it does not affect the informativeness of (ξ_t) in the case when the whole history of such signals is revealed. But since at ϕ^* the rating perfectly accounts for the contribution of past beliefs to (ξ_t), a small change in β has negligible effects on learning.[14]

From now on we refer to $G^*(\phi) := G(\phi, \alpha(\phi), f(\phi, \alpha(\phi))) \in [0, 1]$ as the equilibrium gain factor given a rating $\phi > 0$. Figure 2 (parametrically) plots the latter as a function of the difference $\phi - \nu(\phi)$. It shows that the gain factior attains it unique maximum to the left of the vertical axes. The vertical axes corresponds to the fixed point of the $\nu(\cdot)$ map, i.e., to the fully revealing rating ϕ^* (in this example, this is attained by $\phi \approx 1.5$).

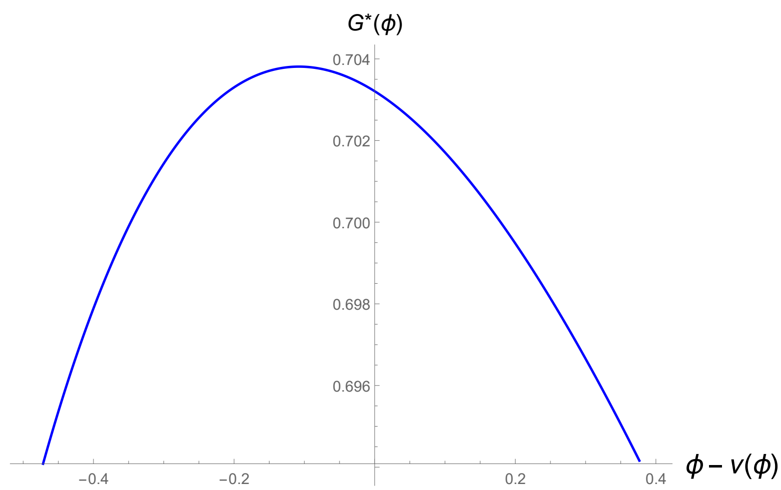

Figure 2: $(r, \sigma_\theta, \sigma_\xi, \kappa) = (3, 1, 2, 1)$.

5 Welfare Effects of Persistence

5.1 Static Benchmark

We revisit the static benchmark and derive its welfare properties. Recall that the equilibrium price and quantity in the static benchmark are given by $P = M/2$ and $Q = \theta - M/2$,

[14]This need not be the case at other degrees of persistence different from ϕ^*. When ϕ conceals information about the consumer's behavior, increasing β can lead to a first-order change in G due to the information conveyed by regressing θ_t on (unobserved to the firm) past beliefs $(M_s)_{s\leq t}$ embedded in the rating.

respectively, where $M = \mathbb{E}[\theta|Y]$ and Y is the public signal available.

As a result, ex ante consumer surplus and profits are given by

$$CS_Y^{\text{static}} = \frac{1}{2}\mathbb{E}\left[(\theta - P)^2\right] \quad \text{and} \quad \Pi_Y^{\text{static}} = \mathbb{E}[PQ] = \mathbb{E}[P^2].$$

However, using that $\mathbb{E}[P] = \mu/2$ and $\text{Cov}[\theta, M] = \text{Var}[M]$ (consequence of the law of iterated expectations), these expressions can be conveniently rewritten as

$$CS_Y^{\text{static}} = \frac{1}{2}\text{Var}[\theta] + \underbrace{\mathbb{E}[P]\left(\mu - \frac{3}{2}\mathbb{E}[P]\right)}_{=\mu^2/8} - \frac{3}{8}\text{Var}[M], \tag{22}$$

$$\Pi_Y^{\text{static}} = \frac{\mu^2}{4} + \frac{\text{Var}[M]}{4}. \tag{23}$$

In a static model, finer information structures, induce greater variability in the firm's posterior mean, which unambiguously hurts the consumer and benefits the firms.[15] From the firm's perspective, better information allows it to better tailor its price to the demand θ, thus improving profits from an ex ante perspective. From the consumer's perspective, more precise information results in a higher degree of correlation between her type and the price. In other words, the consumer is now more likely to face a higher price whenever her demand is high, which reduces her surplus on average.

5.2 Dynamic Setting

We now analyze the effect of the rating's persistence on firm profits and consumer surplus separately, beginning with the former. Omitting the dependence of (α, β, δ) on ϕ for notational convenience, firm's t ex ante profits are given by

$$\Pi(\phi) := \mathbb{E}\Big[\underbrace{[(\alpha+\beta)M_t + \delta\mu])}_{=P_t}\underbrace{[\alpha\theta + \beta M_t + \delta\mu]}_{=Q_t}\Big] = \mathbb{E}[(P_t^\phi)^2] = \mathbb{E}[P_t^\phi]^2 + \text{Var}[P_t^\phi],$$

where the second equality follows $\mathbb{E}[Q_t|Y_t] = P_t$. By stationarity, both $\Pi(\phi)$ and $CS(\phi)$ (below) are independent of time.

The expression for $\Pi(\phi)$ highlights two drivers of firm profits in our environment, i.e., the ex ante mean and variance of the equilibrium price. Recall that expected prices are given by

$$\mathbb{E}[P_t^\phi] = [\alpha(\phi) + \beta(\phi) + \delta(\phi)]\mu,$$

[15]For $\mathcal{G}_1 \subseteq \mathcal{G}_2$ two sigma algebras, $\text{Var}[\mathbb{E}[\theta|\mathcal{G}_i]] = \mathbb{E}[(\mathbb{E}[\theta|\mathcal{G}_i])^2] - \mu^2$, $i = 1, 2$. But since $\mathbb{E}[\theta|\mathcal{G}_1] = \mathbb{E}[\mathbb{E}[\theta|\mathcal{G}_2]|\mathcal{G}_1]$, Jensen's inequality yields $\mathbb{E}[(\mathbb{E}[\theta|\mathcal{G}_1])^2] \leq \mathbb{E}[\mathbb{E}[(\mathbb{E}[\theta|\mathcal{G}_2])^2|\mathcal{G}_1]] = \mathbb{E}[(\mathbb{E}[\theta|\mathcal{G}_2])^2]$.

where $\alpha(\phi) + \beta(\phi) + \delta(\phi) \in (1/3, 1/2)$ for all $\phi > 0$ from Proposition 3. In other words, all three coefficients in the consumer's strategy determine the behavior of the average price.

Proposition 8 (Average Equilibrium Price)**.** *Suppose that* $\kappa \geq \max\{r, \sigma_\theta/\sqrt{3}\sigma_\xi\}$*. Then,* $\alpha'(\phi^*) + \beta'(\phi^*) + \delta'(\phi^*) > 0$*. Thus,* $\mathbb{E}[P_t^\phi] > \mathbb{E}[P_t^{\phi^*}]$ *in a neighborhood to the right of* ϕ^**.*

Under the conditions of Proposition 8, concealing some information by making the rating ϕ less persistent than the fully revealing benchmark ϕ^* can actually increase the expected price. This is slightly surprising in light of the fact that $\alpha'(\phi^*) < 0$, and hence, making the rating less persistent conceals information *and* reduces the amount of information revealed by the consumer. But this result is about the average price, not the extent of learning, and the constant term $\delta(\phi)$ increases sufficiently to offset the first effect.

Now we turn to the role of learning. By the projection theorem for Gaussian random variables,

$$\mathrm{Var}[P_t^\phi] = \mathrm{Var}[M_t](\alpha(\phi) + \beta(\phi))^2 = \mathrm{Var}[\theta_t]G^*(\phi)(\alpha(\phi) + \beta(\phi))^2.$$

Thus, the ex ante variance of the firms' posterior beliefs is proportional to the variance of the fundamental, scaled by the equilibrium gain factor G^* that measures the extent of the firms' learning. However, the variance of the equilibrium price also depends by the sensitivity of the consumer's actions to θ and M. Therefore, the shape of the equilibrium coefficients determine the relationship between the amount of learning and the variance of the equilibrium price.

Proposition 9 (Variance of the Equilibrium Price)**.**

(i) $\alpha'(\phi^*) + \beta'(\phi^*) < 0$*. Thus,* $\mathrm{Var}[P_t^\phi] > \mathrm{Var}[P_t^{\phi^*}]$ *in a neighborhood to the left of* ϕ^**.*

(ii) If $r \geq \kappa$*,* $\alpha(\phi) + \beta(\phi)$ *is decreasing over* $(0, \phi^*]$*. Thus,* $\mathrm{Var}[P_t^\phi] > \mathrm{Var}[P_t^{\phi^\dagger}]$ *for some* $\phi < \phi^\dagger := \arg\max_{\phi \in [0,\phi^*]} G^*(\phi)$*.*

This result shows that the variance of the equilibrium price increases locally if ratings become more persistent than the fully revealing one. Furthermore, if $r \geq \kappa$ (a condition that does not appear necessary from numerical simulations), firms would prefer a rating even more persistent than the learning-maximizing one, as this induces a more sensitive equilibrium purchase process (i.e., a higher $\alpha + \beta$).

We now turn to the consumer's side. In particular, let

$$CS(\phi) := \mathbb{E}\Big[\int_t^\infty e^{-r(t-s)}\Big\{(\theta - \underbrace{[(\alpha+\beta)M_s + \delta\mu]}_{=P_s})\underbrace{(\alpha\theta + \beta M_s + \delta\mu)}_{=Q_s} - \underbrace{\frac{(\alpha\theta + \beta M_s + \delta\mu)^2}{2}}_{=Q_s^2/2}\Big\}ds\Big]$$

denote the (normalized) ex ante discounted consumer surplus from a time-t perspective.

Proposition 10 (Consumer Surplus)**.** *Ex ante consumer surplus is given by*

$$CS(\phi) \;=\; \alpha(\phi)\left(1-\frac{\alpha(\phi)}{2}\right)\frac{\sigma_\theta^2}{2\kappa}+\mathbb{E}[P_t^\phi]\left(\mu-\frac{3}{2}\mathbb{E}[P_t^\phi]\right)+A(\phi)\frac{\sigma_\theta^2}{2\kappa}G^*(\phi).$$

where

$$A(\phi):=\frac{\alpha(\phi)^2}{2}+\beta(\phi)-\frac{3}{2}(\alpha(\phi)+\beta(\phi))^2<0,\quad \textit{for all } \phi>0.$$

Because we know (Proposition 3) that $\mathbb{E}[P_t^\phi] > \mu/3$, the second term in $CS(\phi)$ is negative and decreasing in the expected price. In other words, the ratchet-type forces identified here help the consumer through a lower price. Opposing this dynamic benefit are two forces. First, by shading down her demand ($\alpha < 1$), the consumer moves away from her static optimum. This reduces consumer surplus, as reflected in the first term, which is increasing in α since we know from Proposition 3 that $\alpha(\phi) \in (1/2, 1)$. And second, the consumer transmits information about her willingness to pay to future firms. This makes the price positively vary with the consumer's type and reduces her surplus proportionally to the firms' information gain $G^*(\phi)$.

5.3 Optimal Persistence

The results in the previous subsection have implications for the consumer- and firm-optimal persistence level. Because both consumer and producer surplus can be written in the form $a + b\mu^2$, we now concentrate on the extreme cases of $\mu = 0$ and $\mu \to \infty$. Each case helps us isolate one dimension of the conflicting interests of firms and consumers. Furthermore, these two cases correspond to market segments where, based on publicly observable variables, the consumer's average willingness to pay is very low (high). In particular, the case $\mu = 0$ identifies the role information transmission, and the case $\mu \to \infty$ focuses our attention on the average price level.

Proposition 11 (Optimal Persistence)**.**

(i) If $\mu = 0$, the consumer's optimal rating is ϕ^{CS} is either zero or infinite, and the firm's optimal rating ϕ^{PS} is interior.

(ii) As $\mu \to \infty$, the consumer's optimal rating is ϕ^{CS} is interior, and the firm's optimal rating ϕ^{PS} is either 0 or infinite.

(iii) For any $\mu \geq 0$, total surplus is maximized by $\phi = 0$ or $\phi \to \infty$.

To summarize, under some conditions, firms do not want intermediaries to reveal the entire history of signals. Every numerical simulation suggests that the firm's optimal rating in the case $\mu = 0$ is, in fact, $\phi^{PS} < \phi^{\dagger}$.[16] That is, for markets with low average willingness to pay, the firms' ideal rating is more persistent than the one maximizing learning. For markets with high average willingness to pay, firms would prefer to commit to not observing any information. Because firms always suffer a dynamic loss due to consumers strategically manipulating prices—i.e., $\alpha + \beta + \delta < 1/2$—they would prefer uninformative ratings in this case, as equilibrium prices rise to the static benchmark on average.

Consumers prefer anonymity to tracked purchases when their average willingness to pay is low. When $\mu = 0$, prices are identically equal to zero in expectation in both the static and the dynamic cases; the benefit of lowering prices from an ex ante perspective is thus absent. The consumer then prefers either $\phi = 0$ or infinite, as no learning takes place in either case ($G^*(\phi) \to 0$ as $\phi \to 0, \infty$). But the equilibrium price is constant as a result, which induces the consumer to choose her static optimum. However, consumers prefer an informative rating to anonymous purchases when their average willingness to pay is high. The limit case $\mu \to \infty$ highlights the main tension present in market segments with high willingness to pay: more than information transmission, it is the desire of consumers (firms) to pay low (charge high) prices.

Whether consumers benefit from low or high persistence depends on parameter values. Numerical simulations suggest the consumer's ideal persistence level (for the case $\mu \to \infty$) can be higher or lower than the learning-maximizing $\phi^{\dagger}$ and the fully revealing ϕ^*. Furthermore, as is intuitive, the optimal persistence level appears to be increasing in the discount rate r. Figure 3 illustrates the expected price level (normalized by μ) as a function of the firms' posterior variance: in the left panel, the discount rate is low ($1/2 = r < \kappa$), and the price attains its minimum for a rating that is more persistent than the variance-minimizer; conversely, in the right panel ($\kappa < r = 3$), the price is minimized by a less persistent rating than the one that maximizes the firms' learning. In both cases, the parameter ϕ increases in the direction of the arrow.

The intuition for the comparative statics with respect to r is based on the the value of manipulating the firms' beliefs. Very patient consumers value having a long-term impact on the price. Thus, a very persistent rating increases their benefit of reducing the quantity they purchase *today* and leads firms to lower the current price. These consumers may then prefer ratings with high persistence, i.e., ratings that can potentially "trap them" for long in a specific category, in the hope of manipulating the category they end up in. Conversely,

[16] And every numerical simulation shows that $\phi^{\dagger} := \arg\max_{\phi \in [0,\phi^*]} G^*(\phi)$ is the global maximizer of G^*.

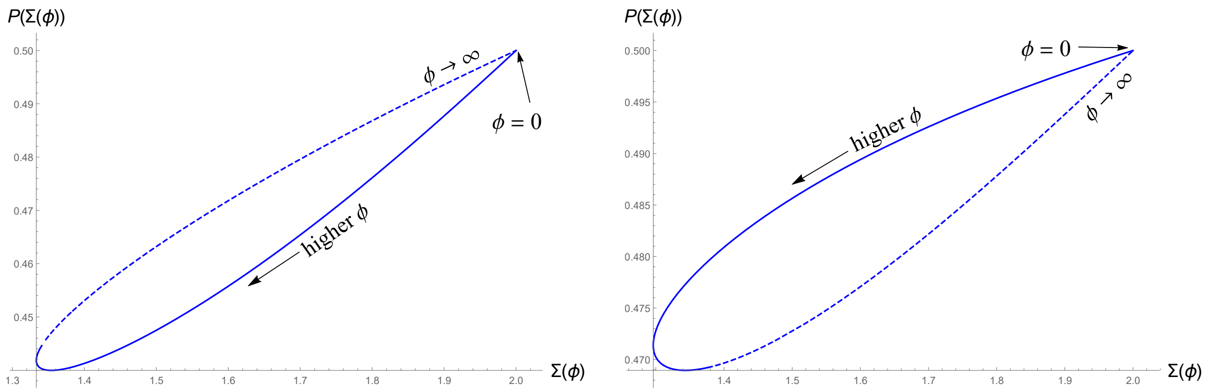

Figure 3: Expected Price Level, $(\sigma_\theta, \sigma_\xi, \kappa) = (1, 2, 1)$; $r = 1/2$ (left); $r = 3$ (right)

impatient consumers care about having an immediate effect on the price; consequently, a higher rating that is very sensitive to new information, but forgets it quickly, maximizes their incentives to manipulate—and minimizes the price.

6 Hidden Ratings

In this section we study the case in which the rating Y_t observed by firm t is hidden to the consumer. When this occurs, firms' beliefs are private, and hence, observing a price today can provide the consumer with information about future prices.

In this context, we say that a strategy for the consumer is *linear Markov* if it corresponds to a linear function of (θ_t, p), where p is the contemporaneous price. Because the information set for each of the firms remains unchanged, the notion of linear strategy for the firms is as in the previous section. Thus, the objects of interest are

$$\begin{aligned} Q(\theta, p) &= q_0 + q_1\theta + q_2 p \ \text{ and} \\ P(Y) &= p_0 + p_1 Y. \end{aligned}$$

Under a pricing strategy of this form, the consumer learns her categorization in real time along the path of play. In discrete time, with full-support noise, deviations in the posted price cannot be detected, as the noise in ξ could have taken the rating to take any value; i.e., all prices are on path. In continuous time, however, the price process that results from a linear Markov pricing strategy will have continuous paths; thus deviations in posted prices can be detected.

Because this issue is a consequence of continuous time only, we assume that the consumer simply ignores the deviation (and thus responds to both the current deviation and future prices with the same strategy $Q(\theta, p)$). Consequently, we see the program solved in this section as the limiting case of a sequence of discrete-time games in which the period length shrinks to zero under appropriately scaled noise. As in the baseline model, the focus is on stationary linear Markov equilibria:

Definition 3 (Equilibrium with Hidden Rating)**.** *A linear profile (Q, P) is a stationary linear Markov equilibrium if*

(i) $P(Y_t) = p_0 + p_1 Y_t$ *maximizes* $p\mathbb{E}_t[q_0 + q_1\theta_t + q_2 p | Y_t]$,

(ii) $Q_t = Q(\theta_t, P_t)$, $t \geq 0$, *is optimal for the consumer when firms price according to* $P(Y_t) = p_0 + p_1 Y_t$, $t \geq 0$, *and*

(iii) $(Y_t, \theta_t)_{t\geq 0}$ *is stationary Gaussian.*

Clearly, any such an equilibrium must entail $q_2 \neq 0$. In this case, it is easy to see that firm t will choose a price of the form $P(Y) = -[q_0 + q_1 M_t(Y_t)]/2q_2$. We therefore look for an equilibrium in which $M_t = m_0 + m_1 Y_t$, and thus

$$p_0 = -\frac{q_0 + q_1 m_0}{2q_2} \text{ and } p_1 = -\frac{q_1 m_1}{2q_2}. \tag{24}$$

With this in hand, the on-path purchases process can be written as

$$Q_t = q_0 + q_1\theta_t + q_2 \underbrace{\left[-\frac{q_0 + q_1 M_t}{2q_2}\right]}_{P_t=} = \underbrace{\frac{q_0}{2}}_{\delta:=} + \underbrace{q_1}_{\alpha:=}\theta_t + \underbrace{-\frac{q_1}{2}}_{\beta:=} M_t.$$

Observe that this process is exactly as in (7), with the additional restriction that $\beta = -\alpha/2$. As a result, the characterization of a stationary outcome presented in Section 3 also applies to this case by simply replacing β with $-\alpha/2$ in Proposition 1.

Recall from Proposition 2 that the equilibrium under study reduced to a single equation (15) in the variable α. Moreover, we had $\beta = f(\phi, \alpha) \in (-\alpha/2, 0)$ for all $\alpha \in [0, 1]$. While in this case the price process differs from the public ratings case, the fact that $\beta = -\alpha/2$ leads, interestingly, to an almost identical characterization of equilibrium behavior.

Proposition 12 (Equilibrium Characterization)**.** *There exists a unique stationary linear Markov equilibrium. In this equilibrium,* $q_1 \in (0, 1)$, *and is characterized as the unique solution to*

$$g^{hidden}(\alpha) := (r + \kappa + \phi)(\alpha - 1) + \alpha\lambda(\phi, \alpha, -\alpha/2)\frac{\alpha}{2} = 0,\ \alpha \in \mathbb{R}_+. \tag{25}$$

In addition, $m_1 = \lambda(\phi, q_1, -q_1/2)$*, where* $(\phi, \alpha, \beta) \in \mathbb{R}^2_+ \times \mathbb{R}_- \mapsto \lambda(\phi, \alpha, \beta)$ *is defined in (14), and*

$$q_2 = -\frac{2(r + 2\phi)}{2(r + 2\phi) + q_1\lambda(\phi, q_1, -q_1/2)}.$$ [17]

The previous Proposition states that there is always a unique stationary linear Markov equilibrium, and this one has positive demand sensitivity to changes in the consumer's type. Moreover, such sensitivity is determined by the same equation for α in the public case replacing $f(\phi, \alpha)$ by $-\alpha/2$.

We conclude the analysis by comparing the equilibrium outcomes in the hidden- vs. observable-rating case, as a function of the observable variables (θ_t, P_t) for the consumer, and Y_t for the firms. To unify notation, let

$$\begin{aligned} Q_t &= q_0^{\mathsf{obs}} + q_1^{\mathsf{obs}}\theta_t + q_2^{\mathsf{obs}}P_t \\ P_t &= p_0^{\mathsf{obs}} + p_1^{\mathsf{obs}}Y_t \end{aligned} \tag{26}$$

denote the realized demand and prices along the path of play of a linear Markov equilibrium when the rating is observed by the consumer (in particular, $q_1^{\mathsf{obs}} = \alpha$) and replace '$\mathsf{obs}$' by '$\mathsf{hidden}$' when the rating is hidden. Observe that all these coefficients are a function of ϕ:

Proposition 13 (Role of Transparency)**.** *In the linear Markov equilibria, for all* $\phi > 0$*:*

(i) Signaling: $q_1^{obs}(\phi) > q_1^{hidden}(\phi)$.

(ii) Sensitivity of demand: $-1 < q_2^{obs}(\phi) < q_2^{hidden}(\phi) < 0$.

(iii) Price volatility: $0 < p_1^{obs}(\phi) < p_1^{hidden}(\phi)$.

With hidden ratings, both the firms and the consumer can signal their private information. In particular, the contemporaneous price reveals the current rating. Thus, the price carries a signal of future prices. In particular, a high price signals the rating is high, which reduces the value of manipulation and limits the extent to which the consumer scales back. This is due to a convexity property of the consumer's value, as a function of the current price. This property is the analog of the convexity of the indirect utility function in a static model. Thus, the advantage of reducing prices is greater when prices are low and the consumer buys more units. Conversely, a low price signals a good opportunity to manipulate beliefs downward, and induces a smaller expansion of the consumer's purchases. Therefore, demand is less sensitive to price when ratings are hidden.

[17] Refer to the proof for the expression for q_0.

7 Conclusion

In this paper, we investigate the allocation and welfare consequences of rating consumers based on their purchase histories and using the information so-gained to price discriminate. Strategic consumers react to the possibility of firms ratcheting up prices by taking actions that reduce the amount of information revealed to the market via the rating. More specifically, we have focused on the effects of a rating's persistence and transparency on the level and terms of trade in a setting where a consumer purchases a good from a different monopolist in every period.

In the future, it would be useful to relax several of our simplifying assumptions. For instance, we could contrast the leading case of exponential ratings, i.e., a stochastic memory, with the case of a moving window, i.e., a deterministic memory (Hörner and Lambert, 2017). Deriving the fully optimal rating in our setting appears challenging but not entirely out of reach. Perhaps the most interesting direction would involve formalizing a market demand for consumer ratings, and so endogenizing the intermediary's equilibrium persistence and transparency policy.

Appendix A: Proofs

Proofs for Section 3

Proof of Proposition 1. Suppose that $(Y_t, \theta_t)_{t\geq 0}$ is stationary Gaussian. Then, by the projection theorem for Gaussian random variables and stationarity,

$$M_t = \underbrace{\mathbb{E}[\theta_t]}_{=\mu} + \underbrace{\frac{\mathrm{Cov}[\theta_t, Y_t]}{\mathrm{Var}[Y_t]}}_{\lambda:=}[Y_t - \underbrace{\mathbb{E}[Y_t]}_{\bar{Y}:=}] =: \rho + \lambda Y_t,\ t \geq 0,$$

λ and $\bar{Y}$ to be determined. In this case, we can write (7) as

$$Q_t = \delta\mu + \alpha\theta_t + \beta M_t = \underbrace{\delta\mu + \beta\rho}_{\hat{\delta}:=} + \alpha\theta_t + \underbrace{\beta\lambda}_{\hat{\beta}:=} Y_t, \tag{A.1}$$

and thus $(\theta_t, Y_t)_{t\geq 0}$ has linear dynamics given by

$$\begin{aligned} dY_t &= \underbrace{[-(\phi - \hat{\beta})Y_t + \hat{\delta} + \alpha\theta_t]dt + \sigma_\xi dZ_t^\xi}_{=-\phi Y_t dt + d\xi_t \text{ under (A.1)}} \qquad \text{(A.2)} \\ d\theta_t &= -\kappa(\theta - \mu)dt + \sigma_\theta dZ_t^\theta,\ t > 0. \end{aligned}$$

Defining the matrices

$$X := \begin{bmatrix} Y \\ \theta \end{bmatrix},\ A_0 := \begin{bmatrix} \hat{\delta} \\ \kappa\mu \end{bmatrix},\ A_1 := \begin{bmatrix} \phi - \hat{\beta} & -\alpha \\ 0 & \kappa \end{bmatrix},\ B := \begin{bmatrix} \sigma_\xi & 0 \\ 0 & \sigma_\theta \end{bmatrix} \text{ and } Z := \begin{bmatrix} Z_t^\xi \\ Z_t^\theta \end{bmatrix}$$

we can thus write $dX_t = [A_0 - A_1 X_t]dt + BdZ_t$, $t > 0$, which has as unique solution

$$X_t = e^{-A_1 t}X_0 + \int_0^t e^{-A_1(t-s)}A_0 dt + \int_0^t e^{-A_1(t-s)}BdZ_s. \tag{A.3}$$

Moreover, because $(X_t)_{t\geq 0}$ has a linear dynamic, its stationary solution is obtained by letting X_0 being normally distributed and independent of $(Z_t)_{t\geq 0}$, with a mean $\vec{\mu} \in \mathbb{R}^2$ and covariance matrix Λ satisfying the equations

$$\begin{aligned} \mathbb{E}[X_t] = \hat{\mu} &\iff e^{-A_1 t}\hat{\mu} + [A_1^{-1} - e^{-A_1 t}A_1^{-1}]A_0 = \hat{\mu} \text{ and} \\ \mathrm{Var}[X_t] = \Lambda &\iff e^{-A_t t}\Lambda e^{-A_1^T t} + e^{-A_1 t}\mathrm{Var}\left[\int_0^t e^{A_1 s}BdZ_s\right]e^{-A_1^T t} = \Lambda, \end{aligned}$$

where Var[$\cdot$] denotes the covariance matrix operator. In fact, these equations follow directly from (A.3) using the independence restriction and imposing stationarity.

Observe that the first condition leads to $\vec{\mu} = A_1^{-1} A_0$ provided A_1 is invertible, which happens when $\phi - \beta\lambda \neq 0$—we assume this in what follows. Regarding the second condition, differentiating it and using that $\text{Var}\left[\int_0^t e^{A_1 s} B dZ_s\right] = \int_0^t e^{A_1 s} B^2 e^{A_1^T s} ds$ yields

$$-A_1 \Lambda - \Lambda A_1^T + B^2 = 0.$$

Using that $\vec{\mu} = (\mathbb{E}[Y_t], \mathbb{E}[\theta_t])^T$ and that $\Lambda_{11} = \text{Var}[Y_t]$, $\Lambda_{12} = \Lambda_{21} = \text{Cov}[\theta_t, Y_t]$ and $\Lambda_{22} = \text{Var}[\theta_t]$, it is then easy to verify that the previous system has as solution

$$\mathbb{E}[\theta_t] = \mu, \qquad \mathbb{E}[Y_t] = \bar{Y} = \frac{\hat{\delta} + \alpha\mu}{\phi - \hat{\beta}}, \qquad \text{Var}[\theta_t] = \frac{\sigma_\theta^2}{2\kappa}, \tag{A.4}$$

$$\text{Var}[Y_t] = \frac{1}{2(\phi - \hat{\beta})}\left[\sigma_\xi^2 + \frac{\alpha^2 \sigma_\theta^2}{\kappa(\phi - \hat{\beta} + \kappa)}\right] \text{ and } \text{Cov}[\theta_t, Y_t] = \frac{\alpha \sigma_\theta^2}{2\kappa(\phi - \hat{\beta} + \kappa)}. \tag{A.5}$$

The last step required to confirm that the previous expressions indeed correspond to the first two moments of stationary Gaussian process is to verify that Λ is positive definite. Since $\sigma_\theta^2 / 2\kappa > 0$, this boils down to

$$\det(\Lambda) > 0 \Leftrightarrow \frac{\sigma_\xi^2 \kappa(\phi - \hat{\beta} + \kappa)^2 + \alpha^2 \sigma_\theta^2 \kappa}{(\phi - \hat{\beta} + \kappa)^2 (\phi - \hat{\beta})} > 0 \Leftrightarrow \phi - \underbrace{\hat{\beta}}_{=\beta\lambda} > 0.$$

This proves (ii) and (iii).

To finish the proof, we need to determine $\lambda > 0$ and $\rho := \mu - \lambda \bar{Y} \in \mathbb{R}$ that are consistent with Bayes' rule. Using (A.4)–(A.5), and after some simplification,

$$\lambda \; := \; \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]} = \frac{\alpha \sigma_\theta^2 (\phi - \beta\lambda)}{\alpha^2 \sigma_\theta^2 + \kappa \sigma_\xi^2 (\phi + \kappa - \beta\lambda)} \text{ and} \tag{A.6}$$

$$\rho \; = \; \mu - \lambda \bar{Y} = \mu - \frac{\overbrace{\delta\mu + \beta\rho}^{\hat{\delta}} + \alpha\mu}{(\phi - \beta\lambda)} \lambda$$

$$\Rightarrow \rho \; = \; \mu \frac{[\phi - \beta\lambda] - [\delta + \alpha]\lambda}{\phi} \tag{A.7}$$

In particular,

$$\bar{Y} = \frac{\delta\mu + \beta\rho + \alpha\mu}{\phi - \hat{\beta}} = \frac{\mu}{\phi - \beta\lambda}\left[\alpha + \beta + \delta - \frac{\beta\lambda[\beta + \alpha + \delta]}{\phi}\right] = \frac{\mu[\alpha + \beta + \delta]}{\phi},$$

thus proving (i). The converse part of the Proposition is true by the previous constructive argument. This concludes the proof. $\square$

Proof of Lemma 1. Consider a linear Markov strategy $Q(p, \theta, Y)$ for the consumer with weight equal to -1 on the contemporaneous price. Fix $t > 0$. Because all firms $s < t$ assumed that (7) is followed along the path of play, $M_t = \rho + \lambda Y_t$ from the perspective of firm t. Thus, we can write

$$Q_t = Q(p, \theta_t, M_t) = q_0 + \alpha\theta_t + q_2 M_t - p$$

for some coefficients q_0, α and q_2. Importantly, the weight on the price does not change under this linear transformation.

Firm t therefore solves

$$\max_p \; p\mathbb{E}[q_0 + \alpha\theta_t + q_2 M_t - p] \Leftrightarrow P(M_t) = \underbrace{\frac{q_0}{2}}_{p_0:=} + \underbrace{\frac{\alpha + q_2}{2}}_{p_1:=} M_t. \tag{A.8}$$

Along the path of play of a Markov equilibrium, therefore, firms expect

$$\begin{aligned} Q_t &= q_0 + \alpha\theta_t + q_2 M_t - P_t \\ &= \frac{q_0}{2} + \alpha\theta_t + \frac{q_2 - \alpha}{2} M_t. \end{aligned}$$

But this shows that if the firms expect a purchase process $Q_t = \delta\mu + \alpha\theta_t + \beta M_t$ to be realized, they will price according to $P_t = \delta\mu + (\alpha + \beta)M$. Once the coefficients (α, β, δ) are determined, simple algebra shows that prices are supported by the belief that the consumer follows the Markov strategy

$$Q(p, \theta_t, Y_t) = 2\delta\mu + \rho[\alpha + 2\beta] + \alpha\theta_t + \lambda(\phi, \alpha, \beta)[\alpha + 2\beta]Y_t - p,$$

where $\lambda(\phi, \alpha, \beta)$ is given in (14). This concludes the proof. $\square$.

Proof of Proposition 2. Guess a quadratic solution $V = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M$ to the HJB equation (12), i.e.,

$$\begin{aligned} rV(\theta, M) = \sup_{q \in \mathbb{R}} \; &\Big\{(\theta - [(\alpha + \beta)M + \delta\mu])q - q^2/2 - \kappa(\theta - \mu)V_\theta \\ &+ (\lambda q - \phi[M - \rho])\frac{\partial V}{\partial M}(\theta, M) + \frac{\lambda^2\sigma_\xi^2}{2}\frac{\partial^2 V}{\partial M^2} + \frac{\sigma_\theta^2}{2}\frac{\partial^2 V}{\partial \theta^2}\Big\} \end{aligned}$$

where we have used that $\mu - \lambda\bar{Y} = \rho$ in the drift of $(M_t)_{t\geq 0}$, and ρ is defined in (A.7).

The first-order condition reads

$$\begin{aligned} q &= \theta - [\delta\mu + (\alpha+\beta)M] + \lambda[v_2 + 2v_3 M + v_5\theta] \\ &= -\delta\mu + \lambda v_2 + [1+\lambda v_5]\theta + [2\lambda v_3 - (\alpha+\beta)]M \end{aligned} \tag{A.9}$$

which leads to the following system matching coefficient conditions:

$$\begin{aligned} \delta\mu &= -\delta\mu + \lambda v_2 \\ \alpha &= 1 + \lambda v_5 \\ \beta &= 2\lambda v_3 - (\alpha+\beta). \end{aligned} \tag{A.10}$$

By the envelope theorem, moreover,

$$\begin{aligned} (r+\phi)[v_2 + 2v_3 M + v_5\theta] &= -(\alpha+\beta)[\delta\mu + \alpha\theta + \beta M] - \kappa(\theta-\mu)v_5 \\ &\quad +[\lambda(\delta\mu + \alpha\theta + \beta M) - \phi(M-\rho)]2v_3 \end{aligned} \tag{A.11}$$

which yields the following system

$$\begin{aligned} (r+\phi)v_2 &= -(\alpha+\beta)\delta\mu + \kappa\mu v_5 + [\lambda\delta\mu + \phi\rho]2v_3 \\ (r+2\phi)2v_3 &= -(\alpha+\beta)\beta + 2v_3\lambda\beta \\ (r+\kappa+\phi)v_5 &= -(\alpha+\beta)\alpha + 2v_3\lambda\alpha \end{aligned} \tag{A.12}$$

and using that v_2, v_3 and v_5 can be written as a function of α, β and $\delta\mu$, this system becomes

$$\begin{aligned} (r+\phi)\frac{2\delta\mu}{\lambda} &= -(\alpha+\beta)\delta\mu + \kappa\mu\frac{\alpha-1}{\lambda} + [\lambda\delta\mu + \phi\rho]\frac{\alpha+2\beta}{\lambda} \\ (r+2\phi)\frac{\alpha+2\beta}{\lambda} &= \underbrace{-(\alpha+\beta)\beta + \beta(\alpha+2\beta)}_{=(\beta)^2} \\ (r+\kappa+\phi)\frac{\alpha-1}{\lambda} &= \underbrace{-(\alpha+\beta)\alpha + \alpha(\alpha+2\beta)}_{=\alpha\beta}. \end{aligned} \tag{A.13}$$

which assumes that $\lambda \neq 0$. In fact, as we show below, $\alpha \neq 0$ in any stationary linear Markov equilibrium; but this coupled with $\phi - \beta\lambda > 0$ in any such an equilibrium yields $\lambda \neq 0$.

We now establish a key lemma:

Lemma 2. *Any stationary linear Markov equilibrium must satisfy* $\alpha \in (0,1)$.

Proof. Consider a stationary linear Markov equilibrium (α, β, δ) with a sensitivity of beliefs λ satisfying (8). Straightforward integration shows that the consumer's value function is quadratic, and thus the system of equations (A.13) holds. Observe that $\alpha > 0$. In fact, $\alpha = 0$ cannot hold in equilibrium: $M_t = \mu$ in this case due to $\lambda = 0$, which implies that it is optimal for the consumer to behave myopically by choosing $Q_t = \theta_t - \mu$, a contradiction. If instead $\alpha < 0$, the last equation in (A.13) yields

$$\phi - \beta\lambda = (r + \kappa)\left(\frac{1}{\alpha} - 1\right) + \frac{\phi}{\alpha} < 0,$$

which is a contradiction with the equilibrium being stationary ((iii) in Proposition 1).

The case $\alpha = 1$ can be easily ruled out: the last equation in the system (A.13) then yields that $\beta = 0$ (as $\lambda > 0$), but the second equation then implies that $\alpha = 0$, a contradiction.

Suppose now that $\alpha > 1$. We solve for λ and β from the last two equations of system (A.13). We then obtain the following expression:

$$\phi - \beta\lambda =: L = \frac{\phi - \alpha(\kappa + r) + \kappa + r}{\alpha}.$$

Solving this expression for $\phi(\alpha, L)$ and substituting into the equation for λ (A.6), we obtain the following expression:

$$\frac{\alpha L \sigma_\theta^2}{\kappa(\kappa + L)\sigma_\xi^2 + \alpha^2\sigma_\theta^2} + \frac{(\alpha - 1)(\kappa + L + r)(3\alpha(\kappa + L + r) - 3\kappa + L + r)}{\alpha(2\alpha(\kappa + L + r) - 2\kappa - r)} = 0.$$

It is immediate to see, however, that $\alpha > 1$ and $L > 0$ imply the left-hand side of this expression is strictly positive. This means any solution to the system (A.13) with $\alpha > 1$ violates the condition for stationarity $\phi - \beta\lambda > 0$. This concludes the proof of the lemma. □

We continue with the proof of the proposition. Since $\alpha \in (0, 1)$ and $\lambda > 0$ in any stationary linear Markov equilibrium, the last equation in (A.13) implies that $\beta \neq 0$. Multiplying the second equation by $\alpha \neq 0$ and the third by $\beta \neq 0$ in the same system then yields

$$\begin{aligned} & (r + 2\phi)\alpha(\alpha + 2\beta) = (r + \kappa + \phi)\beta(\alpha - 1) \\ \Rightarrow \quad & \beta = f(\phi, \alpha) := \frac{-(\alpha)^2(r + 2\phi)}{2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)} \in (-\frac{\alpha}{2}, 0) \text{ when } \alpha \in (0, 1). \end{aligned}$$

We conclude that

$$\lambda = \lambda(\phi,\alpha,f(\phi,\alpha)) := \frac{\left[\ell^2(\phi,\alpha,f(\phi,\alpha)) - 4\kappa\sigma_\xi^2\sigma_\theta^2 f(\phi,\alpha)\alpha\phi\right]^{1/2} - \ell(\phi,\alpha,f(\phi,\alpha))}{-2\kappa\sigma_\xi^2 f(\phi,\alpha)} > 0, \quad \text{(A.14)}$$

where

$$\ell(\phi,\alpha,\beta) = \sigma_\theta^2\alpha[\alpha+\beta] + \kappa\sigma_\xi^2(\phi+\kappa).$$

This is because $\lambda(\phi,\alpha,\beta)$, defined as the ratio in (A.14) replacing $f(\phi,\alpha)$ by β, is the unique positive root of (8) when $\alpha > 0$ and $\beta < 0$. Observe also that $\alpha^2 + \alpha f(\phi,\alpha) = \alpha[\alpha + f(\phi,\alpha)] \geq \alpha^2/2 > 0$ when $\alpha \in [0,1]$, and so $\ell(\alpha) > 0$ over the same range.

Letting $g(\alpha) := (r+\kappa+\phi)(\alpha-1) - \lambda(\phi,\alpha,f(\phi,\alpha))\alpha f(\phi,\alpha)$ for $\alpha \in [0,1]$, we are then left with α satisfying the equation

$$g(\alpha) = 0. \quad \text{(A.15)}$$

Lemma 3. *There is a unique $\alpha \in (0,1)$ satisfying the previous equation.*

Proof: Fix $\phi > 0$. Observe that

- As $\alpha \to 1$: $f(\phi,\alpha) \to -1/2$ and $\lim_{\alpha\to 1}\lambda(\phi,\alpha,f(\phi,\alpha)) > 0$ for all $\phi > 0$. Hence, $\lim_{\alpha\to 1} g(\alpha) > 0$.
- As $\alpha \to 0$: $f(\phi,\alpha) \to 0$ and $f(\phi,\alpha)\lambda(\phi,\alpha,f(\phi,\alpha)) \to 0$ for all $\phi > 0$. Hence, $\lim_{\alpha\to 0} g(\alpha) < 0$.

The existence of $\alpha \in (0,1)$ then follows from continuity of $g(\cdot)$.

To show uniqueness, we prove that $\alpha \mapsto -\lambda(\phi,\alpha,f(\phi,\alpha))\alpha f(\alpha)$ is strictly increasing in $[0,1]$. To this end, notice first that since $-\lambda(\phi,\alpha,f(\phi,\alpha))f(\phi,\alpha) > 0$ in $[0,1]$, it suffices to show that $\alpha \mapsto H(\phi,\alpha) := -\lambda(\phi,\alpha,f(\phi,\alpha))f(\phi,\alpha)$ is strictly increasing. Towards a contradiction, suppose that there is $\hat{\alpha} \in (0,1)$ s.t. $H_\alpha(\phi,\hat{\alpha}) = 0$, where H_α denotes the partial derivative of H with respect to α. But this occurs if and only if

$$\ell_\alpha(\phi,\hat{\alpha})\underbrace{\left[\ell(\phi,\hat{\alpha}) - \left(\ell^2(\phi,\hat{\alpha}) - 4\kappa\sigma_\xi^2\sigma_\theta^2 f(\phi,\hat{\alpha})(\phi,\hat{\alpha})\phi\right)^{1/2}\right]}_{<0,\ \text{as}\ f<0} = 2\kappa(\sigma_\theta\sigma_\xi)^2[f_\alpha(\phi,\hat{\alpha})\hat{\alpha} + f(\phi,\hat{\alpha})]\phi.$$

Moreover, straightforward algebra shows that

$$f_\alpha(\phi,\alpha)\alpha = \underbrace{f(\phi,\alpha)}_{<0} - \underbrace{\frac{\alpha^2(r+2\phi)(r+\kappa+\phi)}{[2\alpha(r+2\phi) - (r+\kappa+\phi)(\alpha-1)]^2}}_{>0} < 0 \ \text{ for } \ \alpha \in [0,1]$$

Thus, $\ell_\alpha(\phi,\hat{\alpha}) = \sigma_\theta^2[2\alpha + f_\alpha(\phi,\hat{\alpha})\hat{\alpha} + f(\phi,\hat{\alpha})] > 0$, otherwise the left-hand side of the previous condition is positive, while the right-hand side is negative.

Rearranging terms, squaring both sides, and dividing by $4\kappa(\sigma_\theta\sigma_\xi)^2\phi$ in the first-order condition yields

$$\begin{aligned} 0 \;=\; & \underbrace{\ell_\alpha(\phi,\hat{\alpha})\{\ell(\phi,\hat{\alpha})[-f_\alpha(\phi,\hat{\alpha})\hat{\alpha} - f(\phi,\hat{\alpha})] + \ell_\alpha(\phi,\hat{\alpha})f(\phi,\hat{\alpha})\hat{\alpha}\}}_{A:=} \\ & + \underbrace{\kappa(\sigma_\theta\sigma_\xi)^2[-f_\alpha(\phi,\hat{\alpha})\hat{\alpha} - f(\phi,\hat{\alpha})]^2\phi}_{>0} \end{aligned} \tag{A.16}$$

so we must have that $A < 0$. Using that $\ell(\phi,\alpha) = \sigma_\theta^2\alpha[\alpha + f(\phi,\alpha)] + \kappa\sigma_\xi^2(\phi+\kappa)$ and that $-f_\alpha(\phi,\hat{\alpha})\hat{\alpha} - f(\phi,\hat{\alpha}) > 0$, we conclude that

$$\begin{aligned} [\hat{\alpha}^2 + \hat{\alpha}f(\phi,\hat{\alpha})][-f_\alpha(\phi,\hat{\alpha})\hat{\alpha} - f(\phi,\hat{\alpha})] + [2\hat{\alpha} + \hat{\alpha}f_\alpha(\phi,\hat{\alpha}) + f(\phi,\hat{\alpha})]\hat{\alpha}f(\phi,\hat{\alpha}) &< 0 \\ \Leftrightarrow \hat{\alpha}^2[-\hat{\alpha}f_\alpha(\phi,\hat{\alpha}) + f(\phi,\hat{\alpha})] &< 0 \end{aligned}$$

But $-\hat{\alpha}f_\alpha(\phi,\hat{\alpha}) + f(\phi,\hat{\alpha}) = \alpha^2(r+2\phi)(r+\kappa+\phi)/[2\alpha(r+2\phi) - (r+\kappa+\phi)(\alpha-1)]^2 > 0$, we reach to a contradiction. Since $H_\alpha(\phi,\alpha) > 0$ must hold for some $\alpha \in [0,1]$, the continuity of H_α implies that $\alpha \mapsto H(\phi,\alpha)$ is strictly increasing, which concludes the proof. $\square$

Now we turn to δ. Recall from the first equation in (A.13) that

$$(r+\phi)\frac{2\delta\mu}{\lambda} = -(\alpha+\beta)\delta\mu + \kappa\mu\frac{\alpha-1}{\lambda} + [\lambda\delta\mu + \phi\rho]\frac{\alpha+2\beta}{\lambda},$$

where ρ defined in (A.7) is itself a function of δ:

$$\rho = \mu\frac{[\phi - \beta\lambda] - [\delta+\alpha]\lambda}{\phi}.$$

Plugging this expression in the previous equation, straightforward algebra shows that

$$\underbrace{\left[\frac{2(r+\phi)}{\lambda} + \alpha + \beta\right]}_{>0}\delta\mu \;=\; \mu\left[\frac{\kappa(\alpha-1)}{\lambda} + \frac{\alpha+2\beta}{\lambda}[\phi - (\alpha+\beta)\lambda]\right].$$

If $\mu = 0$ this equation is trivially satisfied (and $v_2 = 0$, leaving the rest of the system unaffected). In this case, the price and purchase process along the path of play have no intercept. If $\mu \neq 0$, we have that

$$\Rightarrow \delta = \left.\frac{\kappa(\alpha-1)+[\alpha+2\beta][\phi-(\alpha+\beta)\lambda]}{2(r+\phi)+(\alpha+\beta)\lambda}\right|_{(\lambda,\beta)=(\lambda(\phi,\alpha,f(\phi,\alpha)),f(\phi,\alpha))} \tag{A.17}$$

and so δ admits a solution for all range of parameters. Thus, we can always write the intercept of the purchase process as $\delta\mu$.

To conclude the proof of the proposition:

1. From the three matching coefficient conditions (A.10), v_2, v_3 and v_5 are determined using δ, α and β:

$$v_2 = \frac{2\delta\mu}{\lambda},\ v_3 = \frac{\alpha+2\beta}{2\lambda} > 0, \quad \text{and} \quad v_5 = \frac{\alpha-1}{\lambda} < 0.$$

As for v_1 and v_4 (corresponding to θ and θ^2 in the value function) these can be obtained via the envelope theorem in the HJB equation. Namely:

$$\begin{aligned}(r+\kappa)[v_1+2v_4\theta+v_5M] &= (\delta\mu+\alpha\theta+\beta M)\,[1+v_5\lambda]-v_5\phi\,[M-\rho] \\ &\quad -2v_4\kappa(\theta-\mu)\end{aligned}$$

leads to the system

$$\begin{aligned}(r+\kappa+\phi)v_5 &= \beta\underbrace{[1+v_5\lambda]}_{=\alpha \text{ from (A.10)}} = \alpha\beta \text{ which we already had, and} \\ 2(r+\kappa)v_4 &= \alpha[1+\lambda v_5]-2v_4\kappa \Rightarrow v_4 = \frac{\alpha^2}{2(r+2\kappa)} \\ (r+\kappa)v_1 &= \delta\mu\alpha+v_5\phi\rho \Rightarrow v_1 = \frac{\delta\mu\alpha}{r+\kappa}+\frac{\phi\rho\alpha\beta}{(r+\kappa+\phi)(r+\kappa)}\end{aligned} \tag{A.18}$$

The coefficient v_0 can be found by equating the constant terms in the HJB equation (and there is no constraint on it).

2. Finally, the equilibrium law of motion of the firms' beliefs is given by

$$dM_t = (\lambda[\delta\mu+\alpha\theta_t+\beta M_t]-\phi(M_t-\rho))dt+\lambda\sigma_\xi dZ_t^\xi \tag{A.19}$$

and so the belief is mean reverting (with rate $\phi-\lambda\beta>0$) around the trend

$$\frac{1}{\phi-\lambda\beta}\left(\lambda\delta\mu+\alpha\theta_t-\phi(M_t-\rho)\right).$$

We conclude that standard transversality conditions hold as $(\theta_t)_{t\geq 0}$ is also mean reverting and the flow payoff quadratic. This concludes the proof. $\square$.

Proof of Proposition 3. (iii) Limit value of the coefficients. Recall that $\alpha(\phi) \in (0,1)$ is uniquely defined as

$$\begin{aligned} g(\alpha(\phi)) \;&:=\; (r+\kappa+\phi)(\alpha(\phi)-1) \\ &\quad + \frac{\alpha(\phi)}{2\kappa\sigma_\xi^2}\underbrace{[\sqrt{[\ell(\phi,\alpha(\phi))]^2 - 4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi} - \ell(\phi,\alpha(\phi))]}_{L(\phi):=} = 0 \end{aligned}$$

where $\ell(\phi,\alpha) := \alpha^2\sigma_\theta^2 + \kappa\sigma_\xi^2(\phi+\kappa) + \sigma_\theta^2\alpha f(\phi,\alpha)$. Since $|f(\phi,\alpha)| < 1/2$ for all $\alpha \in [0,1]$ and $\alpha(\phi) \in (0,1)$ for all $\phi > 0$, we have that $0 \leq -4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi \to 0$ as $\phi \to 0$. In addition, $\ell(\phi,\alpha) > \kappa^2\sigma_\xi^2$. Thus,

$$\begin{aligned} 0 \leq L(\phi) \;&=\; \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi}{\sqrt{[\ell(\phi,\alpha(\phi))]^2 - 4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi} + \ell(\phi,\alpha(\phi))} \\ &<\; \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi}{2\kappa^2\sigma_\xi^2} \to 0 \;\text{ as }\; \phi \to 0. \end{aligned} \tag{A.20}$$

We conclude that $\lim\limits_{\phi\to 0} \alpha(\phi)$ exists and takes value 1.

As for the limit, notice that since $\ell(\phi,\alpha(\phi)) \geq \kappa\sigma_\xi^2\phi$ and $\alpha f(\cdot) < 0$

$$0 \leq L(\phi) \;=\; \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)}{\sqrt{\left[\frac{\ell(\phi,\alpha(\phi))}{\phi}\right]^2 - \frac{4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)}{\phi}} + \frac{\ell(\phi,\alpha(\phi))}{\phi}} \leq \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)}{4\sigma_\xi^2\kappa}$$

But since $\alpha(\cdot)$ and $f(\phi,\alpha)$ are bounded, $L(\phi)$ is bounded, and using $g(\alpha(\phi)) = 0$,

$$\underbrace{\frac{\alpha(\phi)L(\phi)}{2\kappa\sigma_\xi^2}}_{\text{bounded}}\underbrace{\frac{1}{(r+\kappa+\phi)}}_{\to 0 \text{ as } t\to\infty} = 1 - \alpha(\phi) \to 0 \text{ as } \;\phi \to \infty.$$

Regarding the limit value of $\beta(\phi)$, this one follows directly from

$$\beta(\phi) \;=\; f(\phi,\alpha(\phi)) = \frac{-\alpha(\phi)^2(r+2\phi)}{2(r+2\phi)\alpha(\phi) - (r+\kappa+\phi)(\alpha(\phi)-1)}$$

As for $\delta(\phi)$, recall that

$$\delta(\phi) \quad = \quad \frac{\kappa(\alpha(\phi)-1)+[\alpha(\phi)+2\beta(\phi)][\phi-(\alpha(\phi)+\beta(\phi))\lambda(\phi)]}{2(r+\phi)+(\alpha(\phi)+\beta(\phi))\lambda(\phi)}.$$

However, Lemma 4 below shows that $\lambda \to 0$ as $\phi \to 0$. Using that $\alpha(\phi) \to 1$ and $\alpha(\phi) + 2\beta(\phi) \to 0$ as $\phi \to 0$, and that $\alpha(\phi)+\beta(\phi) > 0$, we conclude that $\delta(\phi) \to 0$ as $\phi \to 0$. The same lemma shows that $\lambda \to \sigma_\theta^2/\kappa\sigma_\xi^2$ as $\phi \to \infty$. Thus,

$$\delta(\phi) \quad = \quad \frac{\overbrace{\frac{\kappa(\alpha(\phi)-1)}{\phi}}^{\to 0}+\overbrace{[\alpha(\phi)+2\beta(\phi)]}^{\to 0}\overbrace{[1-\frac{(\alpha(\phi)+\beta(\phi))\lambda}{\phi}]}^{\to 1}}{\underbrace{\frac{2(r+\phi)+(\alpha(\phi)+\beta(\phi))\lambda}{\phi}}_{\to 2}} \to 0 \ \text{ as } \phi \to \infty.$$

(i) <u>Bounds on α.</u> To simplify notation, denote $\lambda(\phi, \alpha(\phi), f(\phi, \alpha(\phi)))$ by λ

$$\begin{aligned} \lambda \quad &= \quad \frac{2\sigma_\theta^2\alpha(\phi)\phi}{\sqrt{\ell^2(\phi,\alpha(\phi))-4\kappa(\sigma_\theta\sigma_\xi)^2\alpha(\phi)f(\phi,\alpha(\phi))\phi}+\ell(\phi,\alpha(\phi))} \\ &< \quad \frac{\sigma_\theta^2\alpha(\phi)\phi}{\ell(\phi,\alpha(\phi))} \qquad\qquad (A.21) \\ &< \quad \frac{2\phi}{\alpha(\phi)} \end{aligned}$$

where in the last inequality we used that $\ell(\phi,\alpha(\phi)) > \sigma_\theta^2\alpha(\phi)[\alpha(\phi)+f(\phi,\alpha(\phi))] > \sigma_\theta^2\alpha(\phi)^2/2$ (which follows from $f(\phi,\alpha(\phi)) > -\alpha/2$).

On the other hand, using the definition of α and that $-f(\phi,\alpha) > 0$

$$\begin{aligned} 0 \quad &= \quad (r+\kappa+\phi)(\alpha(\phi)-1)+\underbrace{\lambda(\phi)}_{\leq 2\phi/\alpha(\phi)}\ \alpha(\phi)\underbrace{[-f(\phi,\alpha(\phi))]}_{\leq \alpha(\phi)/2} \\ &< \quad (r+\kappa+\phi)(\alpha(\phi)-1)+\phi\alpha(\phi) \\ \Rightarrow \alpha(\phi) \quad &> \quad \frac{r+\kappa+\phi}{r+\kappa+2\phi} > \frac{1}{2}, \quad \text{for all} \ \ \phi > 0. \end{aligned}$$

(ii) 1. <u>Limiting properties of $\alpha'(\cdot)$</u>: To be completed.

2. <u>Quasiconvexity of α</u>. Consider the system of three equations (A.13) that characterizes an equilibrium. To prove the property, it is now more useful to solve the last two equations

for λ and β. We obtain

$$\begin{aligned}
\lambda(\phi,\alpha) &= -\frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)}, \\
\beta(\phi,\alpha) &= -\frac{\alpha^2(r+2\phi)}{\kappa+\alpha(-\kappa+r+3\phi)+r+\phi}.
\end{aligned}$$

Substituting into equation (A.6) defining λ, we obtain

$$\lambda(\phi,\alpha) = \frac{\alpha\sigma_\theta^2(\phi-\beta(\alpha,\phi)\lambda(\phi,\alpha))}{\kappa\sigma_\xi^2(\phi+\kappa-\beta(\alpha,\phi)\lambda(\phi,\alpha))+\alpha^2\sigma_\theta^2},$$

and thus the equilibrium locus becomes

$$\begin{aligned}
g(\phi,\alpha) &:= \frac{\alpha\sigma_\theta^2(\kappa-\alpha(\kappa+r)+r+\phi)}{\alpha^3\sigma_\theta^2-\alpha\kappa r\sigma_\xi^2+\kappa\sigma_\xi^2(\kappa+r+\phi)} \\
&\quad +\frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)} = 0.
\end{aligned}$$

Now let $s := \sigma_\xi^2/\sigma_\theta^2$. The partial derivative $\partial g/\partial\alpha$ can be written as

$$\begin{aligned}
\frac{\partial g(\alpha,\phi)}{\partial\alpha} &= -\frac{(\kappa+r+\phi)\left(4\alpha(\kappa-\phi)+\alpha^2(-\kappa+r+3\phi)-3(\kappa+r+\phi)\right)}{\alpha^4(r+2\phi)} \\
&\quad +\frac{\alpha^3(3\alpha(\kappa+r)-2(\kappa+r+\phi))+\kappa s\left((\kappa+r+\phi)^2-\alpha^2 r(\kappa+r)\right)}{\left(\alpha^3+\kappa s(\kappa-\alpha r+r+\phi)\right)^2}.
\end{aligned}$$

The numerator in the first term is quadratic in α. Because $r \geq \kappa$, the coefficient on α^2 is negative. It is then easily verified that the entire expression is positive when evaluated at $\alpha \in \{1/2, 1\}$. Therefore, the first term is positive. Moreover, both the denominator and the numerator of the second term are increasing in s. Thus, if the numerator is negative, the whole expression can be bounded from below by setting $s = 0$. In this case we obtain

$$\left.\frac{\partial g(\alpha,\phi)}{\partial\alpha}\right|_{s=0} = \alpha^{-3}\left(\alpha^2\left(\kappa(\kappa+r)-2r\phi-3\phi^2\right)-2\alpha(2\kappa+r)(\kappa+r+\phi)+3(\kappa+r+\phi)^2\right).$$

The term in parentheses on the right-hand side is quadratic in α. One can then verify that this expression is positive when evaluated at $\alpha = 1/2$, $\alpha = 1$, and at the unique critical point $\alpha = -(2\kappa+r)(\kappa+r+\phi)/(-\kappa^2-\kappa r+2r\phi+3\phi^2)$. Therefore, $g(\phi,\alpha)$ is increasing in α (for $\alpha \in [1/2, 1]$).

Next, consider the second partial derivative

$$\frac{\partial^2 g(\alpha,\phi)}{(\partial\phi)^2} = -\frac{2(\alpha-1)^2(2\kappa+r)^2}{(r+2\phi)^3} - \frac{2\alpha^5\kappa s\,(\alpha^2+\kappa^2 s)}{(\alpha^3+\kappa s(\kappa-\alpha r+r+\phi))^3}.$$

By inspection, the first term is nonpositive and the second term is strictly negative. Therefore, the second partial with respect to ϕ is strictly negative. Combined with the fact that g is increasing in α, the Implicit Function Theorem implies that the solution $\alpha(\phi)$ to the equation $g(\alpha,\phi)=0$ is increasing in ϕ at every critical point.

(iv) 1. Average price between $\mu/3$ and $\mu/2$. Omitting the dependence on ϕ, observe that $\mathbb{E}[P_t] = \delta\mu + (\alpha+\beta)\mathbb{E}[M_t] = [\delta+\alpha+\beta\mu]$. Now, adding the second and third equation in the system (A.13) that (δ,α,β) satisfies (proof of Proposition 2) yields

$$(\alpha+2\beta)(\alpha+\beta)\lambda = (r+2\phi)(\alpha+2\beta) + (r+\kappa+\phi)(\alpha-1) + (\alpha+\beta)^2\lambda.$$

Thus,

$$\begin{aligned}
\delta &= \frac{\kappa(\alpha-1)+[\alpha+2\beta][\phi-(\alpha+\beta)\lambda]}{2(r+\phi)+(\alpha+\beta)\lambda}\\
&= \frac{\kappa(\alpha-1)+(\alpha+2\beta)\phi-(r+2\phi)(\alpha+2\beta)-(r+\kappa+\phi)(\alpha-1)-(\alpha+\beta)^2\lambda}{2(r+\phi)+(\alpha+\beta)\lambda}\\
&= \frac{-(r+\phi)[2(\alpha+\beta)-1]-(\alpha+\beta)^2\lambda}{2(r+\phi)+(\alpha+\beta)\lambda}
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathbb{E}[P_t] &= [\delta+\alpha+\beta]\mu\\
&= \mu\frac{-(r+\phi)[2(\alpha+\beta)-1]-(\alpha+\beta)^2\lambda+2(r+\phi)(\alpha+\beta)+(\alpha+\beta)^2\lambda}{2(r+\phi)+(\alpha+\beta)\lambda}\\
&= \mu\frac{r+\phi}{2(r+\phi)+(\alpha+\beta)\lambda} < \frac{\mu}{2} \;\text{ if }\; \mu\neq 0,
\end{aligned}$$

where the last inequality comes from $(\alpha+\beta)\lambda>0$ for all $\phi>0$.

On the other hand, from (A.21), and omitting the dependence of α and β on ϕ,

$$(\alpha+\beta)\lambda < (\alpha+\beta)\frac{\sigma_\theta^2\alpha\phi}{\ell(\phi,\alpha)} < (\alpha+\beta)\frac{\sigma_\theta^2\alpha(\phi)\phi}{\sigma_\theta^2\alpha[\alpha+\beta]} = \phi$$

where in the second inequality we used that $\ell(\phi,\alpha) > \sigma_\theta^2\alpha[\alpha + f(\phi,\alpha)] = \sigma_\theta^2\alpha[\alpha+\beta]$. Thus,

$$\frac{\mathbb{E}[P_t^\phi]}{\mu} = \frac{r+\phi}{2(r+\phi)+(\alpha+\beta)\lambda} > \frac{r+\phi}{2(r+\phi)+\phi} > 1/3$$

if $\mu \neq 0$, where the last inequality follows from $r>0$.

2. Convergence of prices. To show the convergence of prices, we start with a preliminary Lemma, where we use the notation $\lambda(\phi)$ for $\lambda(\phi,\alpha(\phi), f(\phi,\alpha(\phi)))$:

Lemma 4. $\lim\limits_{\phi\to 0}\lambda(\phi) = 0$, $\lim\limits_{\phi\to\infty}\lambda(\phi) = \sigma_\theta^2/\kappa\sigma_\xi^2$ *and* $\lim\limits_{\phi\to 0}\lambda(\phi)/\phi = 2\sigma_\theta^2/[\sigma_\theta^2 + 2\sigma_\xi^2\kappa^2]$.

Proof. We first show that $\lim\limits_{\phi\to 0}\lambda(\phi) = 0$. To this end, recall that

$$\lambda(\phi) = -\frac{\sqrt{[\ell(\phi,\alpha(\phi))]^2 - 4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi} - \ell(\phi,\alpha(\phi))}{2\kappa\sigma_\xi^2 f(\phi,\alpha(\phi))}$$

Denote the numerator of $-\lambda(\phi)$ by $L(\phi)$, and recall from (i.1) in the proof of the proposition that $L(\phi)\to 0$ as $\phi\to 0$. Moreover, since $\alpha(\cdot)$ is strictly bounded away from zero and converges to 1 as $\phi\to 0$, we have that $\phi\mapsto f(\phi,\alpha(\phi))$ is also strictly bounded away from zero and converges to -1/2 as $\phi\to 0$. It follows that $\lambda(\phi)\to 0$ when $\phi\to 0$.

On the other hand,

$$\begin{aligned}\lambda(\phi) = -\frac{L(\phi)}{2\kappa\sigma_\xi^2 f(\phi,\alpha(\phi))} &= \frac{4\kappa(\sigma_\xi\sigma_\theta)^2\alpha(\phi)}{2\kappa\sigma_\xi^2\left(\sqrt{\left[\frac{\ell(\phi,\alpha(\phi))}{\phi}\right]^2 - \frac{4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)}{\phi}} + \frac{\ell(\phi,\alpha(\phi))}{\phi}\right)} \\ &\to \frac{4\kappa(\sigma_\xi\sigma_\theta)^2}{2\kappa\sigma_\xi^2[\kappa\sigma_\xi^2+\kappa\sigma_\xi^2]} = \frac{\sigma_\theta^2}{\kappa\sigma_\xi^2} \text{ as } \phi\to\infty,\end{aligned}$$

and thus the second limit holds.

Finally, observe that since $\ell(\phi,\alpha) := \alpha^2\sigma_\theta^2 + \kappa\sigma_\xi^2(\phi+\kappa) + \sigma_\theta^2\alpha f(\phi,\alpha)$ and $\alpha(\phi) + f(\phi,\alpha(\phi))\to 1/2$ as $\phi\to 0$, we have that $\ell(\phi,\alpha(\phi))\to\sigma_\theta^2/2+\sigma_\xi^2\kappa^2$ as $\phi\to 0$. Also,

$$\frac{\lambda(\phi)}{\phi} = \frac{4\kappa(\sigma_\xi\sigma_\theta)^2\alpha(\phi)}{2\kappa\sigma_\xi^2\left(\sqrt{[\ell(\phi,\alpha(\phi))]^2 - 4\kappa(\sigma_\xi\sigma_\theta)^2 f(\phi,\alpha(\phi))\alpha(\phi)\phi} + \ell(\phi,\alpha(\phi))\right)},$$

from where it is straightforward to conclude that $\lim\limits_{\phi\to 0}\lambda(\phi)/\phi = 2\sigma_\theta^2/[\sigma_\theta^2+2\sigma_\xi^2\kappa^2]$. This ends the proof of the lemma. ∎

Using the lemma, we first show that $\lim_{\phi\to\infty} \mathrm{Var}[\lambda(\phi)Y_t] = \lim_{\phi\to 0} \mathrm{Var}[\lambda(\phi)Y_t] = 0$, where $\lambda(\phi)$ stands for $\lambda(\phi, \alpha(\phi), f(\phi, \alpha(\phi)))$. In fact, from the proof of Proposition 1,

$$\mathrm{Var}[Y_t] = \frac{1}{2(\phi - \beta(\phi)\lambda(\phi))}\left[\sigma_\xi^2 + \frac{\alpha(\phi)\sigma_\theta^2}{\kappa[\phi - \beta(\phi)\lambda(\phi) + \kappa]}\right].$$

Since, $(\beta(\phi), \lambda(\phi)) \to (-1/2, \sigma_\theta^2/\kappa\sigma_\xi^2)$ as $\phi \to \infty$, it follows that $\lim_{\phi\to\infty} \mathrm{Var}[Y_t] = 0$ and so $\lim_{\phi\to\infty} \mathrm{Var}[\lambda(\phi)Y_t] = 0$ as $\phi \to \infty$. As for the other limit,

$$\begin{aligned} \mathrm{Var}[\lambda(\phi)Y_t] &= \frac{\lambda^2(\phi)}{2(\phi - \beta(\phi)\lambda(\phi))}\left[\sigma_\xi^2 + \frac{\alpha(\phi)\sigma_\theta^2}{\kappa[\phi - \beta(\phi)\lambda(\phi) + \kappa]}\right] \\ &= \underbrace{\frac{1}{2(\frac{\phi}{\lambda(\phi)} - \beta(\phi))}}_{\to\ \text{constant}} \underbrace{\lambda(\phi)}_{\to 0} \underbrace{\left[\sigma_\xi^2 + \frac{\alpha(\phi)\sigma_\theta^2}{\kappa[\phi - \beta(\phi)\lambda(\phi) + \kappa]}\right]}_{\to \sigma_\xi^2 + \sigma_\theta^2/\kappa^2} \to 0 \text{ as } \phi \to 0. \end{aligned}$$

Using this, and recalling that $M_t = \rho(\phi) + \lambda(\phi)Y_t$, $P_t = \delta\mu + (\alpha+\beta)M_t$, and $(\alpha(\phi), \beta(\phi)) \to (1, -1/2)$ as $\phi \to 0, \infty$, we conclude that

$$\lim_{\phi\to 0} \mathrm{Var}[P_t] = \lim_{\phi\to\infty} \mathrm{Var}[P_t] = \lim_{\phi\to\infty} \mathrm{Var}[M_t] = \lim_{\phi\to 0} \mathrm{Var}[M_t] = 0.$$

Finally, by the projection formula for Gaussian random variables, $\mathbb{E}[M_t] = \mu$, and so $\mathbb{E}[(M_t - \mu)^2] = \mathrm{Var}[M_t] \to 0$ as $\phi \to 0, \infty$. In addition,

$$\mathbb{E}[P_t^*] = \delta\mu + (\alpha+\beta)\mathbb{E}[M_t] = \delta + (\alpha+\beta)\mu.$$

But since $\delta(\phi) \to 0$ as $\phi \to 0$ and ∞, $\mathbb{E}[P_t] \to \mu/2$. Using that

$$(\mathbb{E}[(P_t - \mu/2)^2])^{1/2} \le (\underbrace{\mathbb{E}[(P_t - \mathbb{E}[P_t])^2]}_{=\mathrm{Var}[P_t]})^{1/2} + [(\underbrace{\delta(\phi) + (\alpha(\phi) + \beta(\phi))\mu}_{=\mathbb{E}[P_t]} - \mu/2)^2]^{1/2} \to 0$$

as $\phi \to 0$ and ∞, we conclude.

(v) To be completed.

$\square$

Proofs for Section 4

Proof of Proposition 4. When realized purchases $(Q_t)_{t\geq 0}$ follow (7), recorded purchases obey

$$d\xi_t = (\delta\mu + \alpha\theta_t + \beta M_t)dt + \sigma_\xi dZ_t^\xi,$$

where the process $(M_t)_{t\geq 0}$ satisfies the filtering equation

$$dM_t = -\kappa(M_t - \mu)dt + \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2}[d\xi_t - (\delta\mu + [\alpha+\beta]M_t dt)] \tag{A.22}$$

and $\gamma(\alpha)$ is the unique positive solution to $x \mapsto -2\kappa x + \sigma_\theta^2 - (\alpha x/\sigma_\xi)^2 = 0$ (Liptser and Shiryaev, 1977). As a function of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$, therefore,

$$dM_t = \left(-\left[\kappa + \frac{\alpha^2\gamma(\alpha)}{\sigma_\xi^2}\right] M_t + \kappa\mu + \frac{\alpha^2\gamma(\alpha)}{\sigma_\xi^2}\theta_t\right) dt + \frac{\alpha\gamma(\alpha)}{\sigma_\xi} dZ_t^\xi.$$

Now, let

$$\lambda = \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} \quad \text{and} \quad \rho = \frac{1}{\nu(\alpha,\beta)}\left(\kappa\mu - \frac{\alpha\gamma(\alpha)\delta\mu}{\sigma_\xi^2}\right)$$

where $\nu(\alpha,\beta)$ is defined by (19), i.e.,

$$\nu(\alpha,\beta) := \kappa + \frac{\alpha\gamma(\alpha)[\alpha+\beta]}{\sigma_\xi^2}.$$

In particular, observe that $dM_t = [-(\kappa+\lambda\alpha)M_t + \kappa\mu + \lambda\alpha\theta_t]dt + \lambda\sigma_\xi dZ_t^\xi$.

With this in hand, consider $(Y_t)_{t\geq 0}$ evolving according to

$$dY_t = [-\nu(\alpha,\beta)Y_t + \delta\mu + \beta\rho + \alpha\theta_t + \beta\lambda Y_t]dt + \sigma_\xi dZ_t^\xi.$$

From the proof of Proposition 1, if (Y_0, θ_0) are independent of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$ and

$$\mathbb{E}[Y_0] = \frac{\delta\mu + \beta\rho + \alpha\mu}{\nu(\alpha,\beta) - \beta\lambda},\ \text{Var}[Y_0] = \frac{1}{2(\nu(\alpha,\beta)-\beta\lambda)}\left[\sigma_\xi^2 + \frac{\alpha^2\sigma_\theta^2}{\kappa(\nu(\alpha,\beta)-\beta\lambda+\kappa)}\right] \text{ and}$$
$$\text{Cov}[\theta_0, Y_0] = \frac{\alpha\sigma_\theta^2}{2\kappa(\nu(\alpha,\beta)-\beta\lambda+\kappa)},$$

the pair $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian (*cf.* (A.1)–(A.5)), as $\phi - \beta\lambda = \nu(\alpha,\beta) - \beta\alpha\gamma(\alpha)/\sigma_\xi^2 =$

$\kappa+\alpha^2\gamma(\alpha)/\sigma_\xi^2>0$. Denote by $(Y^{\nu(\alpha,\beta)})_{t\geq 0}$ the ratings process satisfying these conditions, as

$$Y_t = e^{-\nu(\alpha,\beta)t}Y_0 + \int_0^t e^{-\nu(\alpha,\beta)(t-s)}d\xi_s \; t\geq 0,$$

as a function of the public history, with $\nu(\alpha,\beta)>0$ by assumption.

Defining $X_t=\rho+\lambda Y_t^{\nu(\alpha,\beta)}$, it is easy to verify that

$$\begin{aligned} dX_t &= [\lambda(\delta\mu+\alpha\theta_t+\beta X_t)-\nu(\alpha,\beta)[X_t-\rho]]+\lambda\sigma_\xi dZ_t^\xi \\ &= [-(\kappa+\lambda\alpha)X_t+\kappa\mu+\lambda\alpha\theta_t]dt+\lambda\sigma_\xi dZ_t^\xi \end{aligned}$$

where in the last equality we used that $\nu(\alpha,\beta)=\kappa+\lambda(\alpha+\beta)$ and that $\lambda\delta\mu+\nu(\alpha,\beta)=\mu\kappa$. We conclude that M_t-X_t satisfies $d[M_t-X_t]=-(\kappa+\lambda\alpha)[M_t-X_t]dt$, and therefore that $M_t-X_t=[M_0-X_0]e^{-(\kappa+\lambda\alpha)t}$ for all $t\geq 0$.

Notice, however, that since $(X_t)_{t\geq 0}$ is stationary, stationarity of $(M_t)_{t\geq 0}$ implies that $M_0-X_0\equiv 0$ a.s. To see this, notice first that M_0-X_0 cannot be random: otherwise $\text{Var}[M_t]=\text{constant }\forall t\geq 0$ becomes

$$\underbrace{\text{Var}[X_t]}_{\text{independent of } t} + e^{-2[\kappa+\lambda\alpha]t}\text{Var}[M_0-X_0]+2e^{-[\kappa+\lambda\alpha]t}\underbrace{\text{Cov}[X_t,M_0-X_0]}_{\text{independent of } t}=\text{constant},$$

which cannot hold for all $t\geq 0$. Thus, $M_0-X_0=C\in\mathbb{R}$. However, it is easy to see that $\mathbb{E}[M_t]=\text{constant}$ implies that $C=0$. Consequently, if beliefs are stationary,

$$M_t=X_t=\rho+\lambda Y_t^{\nu(\alpha,\beta)}=\left[\frac{1}{\nu(\alpha,\beta)}\left(\kappa\mu-\frac{\alpha\gamma(\alpha)\delta\mu}{\sigma_\xi^2}\right)\right]+\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2}Y_t^{\nu(\alpha,\beta)}, \;\text{ for all }\; t\geq 0.$$

To prove the converse, notice that when $\alpha>0$, (iii) in Proposition 1 implies that λ solving (8) is strictly positive when the rating of persistence $\phi=\nu(\alpha,\beta)>0$ is stationary. Also, if, in addition, $\beta<0$, (8) admits a unique strictly positive root. We now show that under such a rating,

$$\lambda=\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} \quad\text{and}\quad \rho:=\mu-\lambda\bar{Y}=\frac{1}{\nu}\left(\kappa\mu-\frac{\alpha\gamma(\alpha)\delta\mu}{\sigma_\xi^2}\right).$$

To show the first equality we show that $\alpha\gamma(\alpha)/\sigma_\xi^2>0$ solves (8). In what follows, we omit the dependence of ν on (α,β) and of γ on α. To this end, rewrite (8) at $\phi=\nu$ as

$$-\kappa\sigma_\xi^2\beta\lambda^2+\lambda[\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\nu+\kappa)+\alpha\sigma_\theta^2\beta]-\alpha\sigma_\theta^2\nu=0.$$

However,

$$
\begin{aligned}
\lambda\kappa\sigma_\xi^2(\nu+\kappa) &= \lambda\kappa\sigma_\xi^2\left(2\kappa+\frac{\alpha\gamma}{\sigma_\xi^2}\left[\alpha+\beta\right]\right) \\
\alpha\sigma_\theta^2\nu &= \alpha\sigma_\theta^2\left(\kappa+\frac{\alpha\gamma}{\sigma_\xi^2}\left[\alpha+\beta\right]\right)
\end{aligned}
$$

So,

$$
\begin{aligned}
0 &= \lambda\alpha^2\sigma_\theta^2+2\kappa^2\sigma_\xi^2\lambda+2\kappa\lambda\alpha^2\gamma+\kappa\lambda\alpha\gamma\beta-\kappa\lambda^2\sigma_\xi^2\beta-\alpha\sigma_\theta^2\kappa \\
&\quad -\frac{\alpha^3\gamma\sigma_\theta^2}{\sigma_\xi^2}-\frac{\alpha^2\gamma\sigma_\theta^2\beta}{\sigma_\xi^2}+\alpha\sigma_\theta^2\beta\lambda \\
&= \lambda\alpha^2\sigma_\theta^2+2\kappa^2\sigma_\xi^2\lambda+\kappa\lambda\alpha^2\gamma-\alpha\sigma_\theta^2\kappa-\frac{\alpha^3\gamma\sigma_\theta^2}{\sigma_\xi^2} \\
&\quad +\beta\left[\kappa\lambda\alpha\gamma-\kappa\lambda^2\sigma_\xi^2-\frac{\alpha^2\gamma\sigma_\theta^2}{\sigma_\xi^2}+\alpha\sigma_\theta^2\lambda\right]
\end{aligned}
$$

Setting $\lambda=\alpha\gamma/\sigma_\xi^2$, the first and last term of the first line in the second equality cancel out, and the last bracket vanishes. Thus, we are left with

$$
0=2\kappa\alpha\underbrace{\left[2\kappa\gamma+\frac{\alpha^2\gamma^2}{\sigma_\xi^2}-\sigma_\theta^2\right]}_{\equiv 0},
$$

which is true by definition of γ.

Finally, to show the second equality, we can multiplying both sides of (A.7) (proof of Proposition 1) by $\phi=\nu$ to obtain

$$
\nu\rho=\mu\{(\nu-\beta\lambda)-[\delta+\alpha]\lambda\}
$$

But since $\lambda=\alpha\gamma/\sigma_\xi^2$,

$$
\nu=\kappa+\frac{\alpha\gamma}{\sigma_\xi^2}\left(\alpha+\beta\right)=\kappa+\lambda\left(\alpha+\beta\right)\Rightarrow\nu-\beta\lambda=\kappa+\lambda\alpha
$$

yielding $\nu\rho=\mu\{\kappa+\lambda\alpha-[\delta+\alpha]\lambda\}=\kappa\mu-\delta\mu\alpha\gamma/\sigma_\xi^2$ as desired. This concludes the proof.□

Before proceeding with the rest of the results, we state the following lemma, which is at the heart of Definition 2. As a byproduct, we obtain a useful result about the partial derivative of $\lambda(\phi,\alpha,\beta)$ with respect to its first argument at $\phi=\nu(\alpha,\beta)$, and which is used subsequently in the analysis. Let $\lambda_\phi(\nu(\alpha,\beta),\alpha,\beta)$ denote such derivative.

Lemma 5. *$\phi\mapsto G(\phi,\alpha,\beta)$ has a unique minimizer located at $\phi=\nu(\alpha,\beta)$ defined in (19). Moreover, $\lambda_\phi(\nu(\alpha,\beta),\alpha,\beta)=\lambda(\nu(\alpha,\beta),\alpha,\beta)/(\nu+\kappa)$.*

Proof. For notational simplicity, we omit any dependence on variables unless it is strictly necessary. We first verify that ν is an extreme point of $\phi\mapsto G(\phi,\alpha,\beta)$, and verify the desired equality in the process. To this end, let λ denote the unique positive solution to

$$\lambda=\frac{\alpha\sigma_\theta^2(\phi-\beta\lambda)}{\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2[\phi+\kappa-\beta\lambda]},$$

and recall that

$$G=\frac{\alpha\lambda}{\phi+\kappa-\beta\lambda}.$$

Thus, $G_\phi=0$ if and only if $\lambda_\phi(\phi+\kappa)=\lambda$. We now check that the desired equality is satisfied at $(\nu(\alpha,\beta),\alpha,\beta)$.

From the proof of Proposition 4, $\lambda=\alpha\gamma/\sigma_\xi^2$ at $\phi=\nu$, and hence, the claim reduces to showing that $\lambda_\phi(\nu(\alpha,\beta),\alpha,\beta)=\alpha\gamma(\alpha)/\sigma_\xi^2(\nu+\kappa)$. However, it is easy to check that

$$\lambda_\phi=\frac{\alpha\sigma_\theta^2[1-\beta\lambda_\phi][\alpha^2\sigma_\theta^2+\kappa^2\sigma_\xi^2]}{[\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi+\kappa-\beta\lambda)]^2}.$$

Also, $\nu+\kappa-\beta\lambda(\nu(\alpha,\beta),\alpha,\beta)=2\kappa+\alpha^2\gamma/\sigma_\xi^2=\sigma_\theta^2/\gamma$, where the last equality comes from the definition of γ. Thus,

$$\begin{aligned}\left.[\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi+\kappa-\beta\lambda)]^2\right|_{\phi=\nu}=\frac{\sigma_\theta^4[\alpha^2\gamma+\kappa\sigma_\xi^2]^2}{\gamma^2}\quad&=\quad\frac{\sigma_\theta^4[\alpha^2\overbrace{(\alpha^2\gamma^2+2\kappa\gamma\sigma_\xi^2)}^{=\sigma_\theta^2\sigma_\xi^2,\text{ by def. of }\gamma}+\kappa^2\sigma_\xi^4]}{\gamma^2}\\&=\quad\frac{\sigma_\theta^4\sigma_\xi^2[\alpha^2\sigma_\theta^2+\kappa^2\sigma_\xi^2]}{\gamma^2}.\end{aligned}$$

We conclude that at $(\nu(\alpha,\beta),\alpha,\beta)$,

$$\lambda_\phi=\frac{\gamma^2\alpha}{\sigma_\theta^2\sigma_\xi^2}[1-\beta\lambda_\phi]\Rightarrow\lambda_\phi\underbrace{[\sigma_\theta^2\sigma_\xi^2+\gamma^2\alpha\beta]}_{=2\kappa\gamma\sigma_\xi^2+\gamma^2\alpha^2+\gamma^2\alpha\beta}=\gamma^2\alpha\Rightarrow\lambda_\phi=\underbrace{\frac{\gamma\alpha}{\sigma_\xi^2}}_{\lambda(\nu)}\underbrace{\frac{1}{2\kappa+\frac{\alpha\gamma(\alpha+\beta)}{\sigma_\xi^2}}}_{1/(\nu+\kappa)},$$

which shows that ν is an extreme point of $\phi\mapsto G(\phi,\alpha,\beta)$.

On the other hand, it is easy to verify that at an extreme point ϕ,

$$G_{\phi\phi} = -\frac{\alpha\sigma_\theta^2}{2\kappa}\frac{\lambda_{\phi\phi}(\phi+\kappa)}{[\phi+\kappa-\beta\lambda]^2},$$

so the sign of $G_{\phi\phi}$ is determined by $\lambda_{\phi\phi}$ at that point. We now show that $\lambda_{\phi\phi}(\phi) < 0$ for all $\phi > 0$, and hence that any extreme point of $\phi \mapsto G(\phi, \alpha, \beta)$ must be a strict local minimum. But this is enough to guarantee that $\phi \mapsto G(\phi, \alpha, \beta)$ has a unique extreme point, and hence a global minimum that corresponds to ν.

Recall that $\lambda(\phi, \alpha, \beta) = [\sqrt{\ell^2(\phi, \alpha, \beta) - 4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi} - \ell(\phi, \alpha, \beta)]/[-2\kappa\sigma_\xi^2\beta]$ where $\ell(\phi, \alpha, \beta) := \alpha^2\sigma_\theta^2 + \alpha\sigma_\theta^2\beta + \kappa\sigma_\xi^2(\phi+\kappa)$. Thus

$$\begin{aligned}
\lambda_\phi &= \underbrace{\frac{1}{[-2\kappa\sigma_\xi^2\beta]}}_{=:A>0}\left[\frac{\kappa\sigma_\xi^2\ell(\phi,\alpha,\beta) - 2\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha}{\sqrt{\ell^2(\phi,\alpha,\beta) - 4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi}} - \kappa\sigma_\xi^2\right] \\
\Rightarrow \lambda_{\phi\phi} &= B(\phi)\underbrace{\left\{(\kappa\sigma_\xi^2)^2(\ell^2(\phi,\alpha,\beta) - 4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi) - (\kappa\sigma_\xi^2\ell(\phi,\alpha,\beta) - 2\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha)^2\right\}}_{L(\phi):=}
\end{aligned}$$

where $B(\phi) := A/[\ell^2(\phi, \alpha, \beta) - 4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi]^{3/2} > 0$. However,

$$\begin{aligned}
L(\phi) &= -4\kappa^3\sigma_\xi^6\sigma_\theta^2\beta\alpha\phi + 4\kappa^2\sigma_\xi^4\sigma_\theta^2\beta\alpha\ell(\phi,\alpha,\beta) - 4\kappa^2(\sigma_\xi\sigma_\theta)^4\beta^2\alpha^2 \\
&= \underbrace{-4\kappa^2\sigma_\xi^4\sigma_\theta^2\alpha\beta}_{>0, \text{ as } \beta<0}\underbrace{[\kappa\sigma_\xi^2\phi - \ell(\phi,\alpha,\beta) + \sigma_\theta^2\beta\alpha]}_{=-\alpha^2\sigma_\theta^2-\kappa^2\sigma_\xi^2 \text{ by def. of } \ell(\phi,\alpha,\beta)} < 0
\end{aligned}$$

concluding the proof. $\square$

Proof of Proposition 5. Recall that $\alpha(\phi)$ is defined as the unique $\alpha \in (0, 1)$ solving

$$(r+\kappa+\phi)(\alpha(\phi)-1) + \alpha(\phi)H(\phi, \alpha(\phi)) = 0.$$

where

$$H(\phi, \alpha) := -\lambda(\phi, \alpha, f(\phi, \alpha))f(\phi, \alpha) = \frac{\sqrt{\ell^2(\phi,\alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha f(\phi,\alpha)\phi} - \ell(\phi,\alpha)}{2\kappa\sigma_\xi^2}$$

and $\ell(\phi, \alpha) = \sigma_\theta^2\alpha[\alpha + f(\phi, \alpha)] + \kappa\sigma_\xi^2[\phi + \kappa]$. Also, recall from the proof of Lemma 3 in the proof of proposition 2 that $\alpha \mapsto H(\phi, \alpha)$ is strictly increasing over $[0, 1]$.

Thus, denoting the partial derivatives of H with respect to variable $x \in \{\phi, \alpha\}$ as H_x,

$$\alpha'(\phi)\left[r+\kappa+\phi+H(\phi,\alpha(\phi))+H_\alpha(\phi,\alpha(\phi))\right]=1-\alpha(\phi)-\alpha(\phi)H_\phi(\phi,\alpha(\phi))$$

Consequently, because $H > 0$, we conclude that the sign of the derivative of α is determined by the sign of the right-hand side of the previous expression. We now show that the latter side is negative at any point ϕ s.t. $\phi = \kappa + \frac{\alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi)+\beta(\phi)]}{\sigma_\xi^2}$.

To simplify notation, let $\Delta := \sqrt{\ell^2(\phi,\alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha f(\phi,\alpha)\phi}$ and denote the partial derivative of ℓ and f wrt to ϕ as ℓ_ϕ and f_ϕ. Omitting the dependence on $(\phi, \alpha(\phi))$,

$$H_\phi = \frac{1}{2\kappa\sigma_\xi^2}\left[\frac{\ell\ell_\phi - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha[\phi f_\phi + f]}{\Delta} - \ell_\phi\right]. \tag{A.23}$$

Using that $\ell_\phi = \sigma_\theta^2\alpha f_\phi + \kappa\sigma_\xi^2$ we can write

$$H_\phi \quad = \quad \frac{\kappa\sigma_\xi^2[\ell-\Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha f}{2\kappa\sigma_\xi^2\Delta} + \frac{\sigma_\theta^2\alpha f_\phi[\ell-\Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha\phi f_\phi}{2\kappa\sigma_\xi^2\Delta} \tag{A.24}$$

But observe that the first term satisfies

$$\frac{\kappa\sigma_\xi^2[\ell-\Delta] - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha f}{2\kappa\sigma_\xi^2\Delta} = -f\frac{\partial\lambda}{\partial\phi}(\phi,\alpha,f),$$

i.e. it is the sensitivity of the belief to a change in the persistence of the rating holding $(\alpha, \beta) = (\alpha(\phi), f(\phi, \alpha(\phi)))$ fixed. From the proof of proposition 5, we know that

$$\frac{\partial\lambda}{\partial\phi}(\nu,\alpha,\beta) = \frac{\lambda(\nu,\alpha,\beta)}{\nu+\kappa}$$

at $\nu = \kappa + \alpha\gamma(\alpha+\beta)/\sigma_\xi^2$; we conclude therefore the previous equality at the point of interest.

On the other hand, the second term can be written as

$$\frac{\sigma_\theta^2 f_\phi}{\Delta}\left[\alpha\frac{\ell-\Delta}{2\kappa\sigma_\xi^2} - \phi\alpha\right] = \frac{\sigma_\theta^2 f_\phi}{\Delta}\left[(r+\kappa+\phi)(\alpha-1)-\phi\alpha\right]$$

where we used that $\alpha H = \alpha(\Delta - \ell)/2\kappa\sigma_\xi^2$. We deduce that

$$1-\alpha-\alpha H_\phi \quad = \quad \underbrace{1-\alpha+\frac{\lambda\alpha\beta}{\phi+\kappa}}_{A:=} - \underbrace{\frac{\sigma_\theta^2\alpha f_\phi}{\Delta}\left[(r+\kappa+\phi)(\alpha-1)-\phi\alpha\right]}_{B:=} \tag{A.25}$$

Straightforward differentiation shows that

$$f_\phi = \frac{\partial}{\partial\phi}\left(\frac{-\alpha^2(r+2\phi)}{2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)}\right) = \frac{\alpha^2(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]^2} < 0,$$

so $B > 0$. As for the other term, we use that $(r+\kappa+\phi)(\alpha-1)-\lambda\alpha\beta = 0$ to conclude

$$A = \frac{(\phi+\kappa)(1-\alpha)+\lambda\alpha\beta}{\phi+\kappa} = \frac{r(\alpha-1)}{\phi+\kappa} < 0.$$

This concludes the proof. □

To prove Proposition 6 we need a preliminary lemma regarding the behavior of $\alpha+\beta$ at any ϕ where information is not concealed. Recall that this condition was given by (21):

$$\phi = \nu(\phi) = \kappa + \frac{\alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi)+\beta(\phi)]}{\sigma_\xi^2}.$$

Lemma 6. *$\alpha(\phi)+\beta(\phi)$ is strictly decreasing at any $\phi > 0$ satisfying (21). Moreover, if $r > \kappa$, $\alpha(\phi)+\beta(\phi)$ is decreasing when $\alpha(\phi)$ is decreasing.*

Proof. Omitting the dependence of α and β on ϕ, write

$$\alpha+\beta = \alpha\left[1-\frac{\alpha(r+2\phi)}{2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)}\right] =: \alpha h(\phi,\alpha), \tag{A.26}$$

and observe that

$$\frac{d(\alpha+\beta)}{d\phi} = \alpha'[h+\alpha h_\alpha]+\alpha h_\phi,$$

where

$$h_\phi(\alpha,\phi) = \frac{\alpha(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]^2} < 0.$$

Consequently, it suffices to show that $h+\alpha h_\alpha > 0$ at any ϕ where $\alpha(\cdot)$ is decreasing.

To prove the first part of the Proposition, we show the stronger result that $h+\alpha h_\alpha > 0$ over $[\kappa,\infty)$, the set where any point satisfying (21) must lie in, as α, γ and $\alpha+\beta$ are all positive. To this end, notice that

$$\begin{aligned} h+\alpha h_\alpha > 0 \quad \Leftrightarrow \quad & [2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)][(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)] \\ & -\alpha(r+2\phi)[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)] \\ & +\alpha^2(r+2\phi)[2(r+2\phi)-(r+\kappa+\phi)] > 0 \\ \Leftrightarrow \quad & 2(r+2\phi)^2\alpha^2+(r+\kappa+\phi)^2(\alpha-1)^2-\alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) > 0. \end{aligned}$$

If $\phi \geq \kappa$, however,

$$\begin{aligned}
& 2(r+2\phi)^2\alpha^2 - \alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) + \underbrace{(r+\kappa+\phi)^2(\alpha-1)^2}_{>0} \\
\geq \;& \alpha(r+2\phi)[2\underbrace{(r+2\phi)}_{>r+\kappa+\phi}\alpha - 3\alpha(r+\kappa+\phi) + 2(r+\kappa+\phi)] \\
\geq \;& \alpha(r+2\phi)(r+\kappa+\phi)[2-\alpha] > 0.
\end{aligned}$$

which concludes this part of the proof.[18]

To prove the second statement, notice that since $\phi > 0$ and $\alpha < 1$,

$$\begin{aligned}
& 2(r+2\phi)^2\alpha^2 - \alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) + \underbrace{(r+\kappa+\phi)^2(\alpha-1)^2}_{>0} \\
\geq \;& \alpha(r+2\phi)[\alpha(\phi - r - 3\kappa) + 2(r+\kappa+\phi)] \\
\geq \;& \alpha(r+2\phi)[-r-3\kappa+2(r+\kappa)] = \alpha(r+2\phi)[r-\kappa].
\end{aligned}$$

which is non-negative when $r \geq \kappa$. This concludes the proof. □

Proof of Proposition 6. To show existence, let $L(\phi) := \phi - \nu(\phi)$. Also, recall that $\gamma(\alpha)$ is defined as the unique strictly positive root of $0 = \sigma_\theta^2 - 2\kappa\gamma - (\alpha\gamma/\sigma_\xi)^2$, and so

$$\gamma(\alpha) = \frac{\sigma_\xi^2}{\alpha^2}\left[\sqrt{\kappa^2 + \alpha^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa\right].$$

Since $\alpha \in (1/2, 1)$, γ is bounded and we have $L(0) < 0$ and $L(\phi) > 0$ for ϕ large. We conclude the existence of ϕ s.t. $L(\phi) = 0$ by continuity of $L(\cdot)$.

To establish the bounds, observe first that since $\alpha > 0$ and $\alpha + \beta > \alpha/2$, $\phi > \kappa$. On the other hand, since $\beta < 0$ and $\alpha < 1$,

$$\nu(\phi) < \kappa + \frac{\alpha(\phi)^2\gamma(\alpha(\phi))}{\sigma_\xi^2} = \sqrt{\kappa^2 + \alpha(\phi)^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} < \sqrt{\kappa^2 + \frac{\sigma_\theta^2}{\sigma_\xi^2}}.$$

Finally, we address uniqueness. To this end, we show now that at any point $\phi > 0$ where $L(\phi) = 0$, $\nu'(\phi) < 0$, which in turn implies uniqueness (as two of such points would imply the existence of an intermediate point where $L(\cdot)$ vanishes satisfying $\nu' > 0$, a contradiction).

[18] Observe that when $\alpha \in [1/2, 2/3]$ the desired inequality trivially holds, as all the terms involved are non negative.

Notice that

$$\nu'(\phi) = \frac{d}{d\phi}\left(\frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2}\right)(\alpha(\phi)+\beta(\phi)) + \left(\frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2}\right)\frac{d(\alpha(\phi)+\beta(\phi))}{d\phi}.$$

Moreover, from Lemma 6, $\alpha(\phi)+\beta(\phi)$ is strictly decreasing at a $\phi > 0$ s.t. $L(\phi) = 0$. Since $\alpha + \beta > 0$ and $\alpha\gamma(\alpha(\phi)) > 0$ is suffices to show that

$$\frac{d}{d\phi}\left(\frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2}\right) < 0$$

at any such point. But,

$$\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} = \frac{1}{\alpha}\left[\sqrt{\kappa^2 + \alpha^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa\right] = \frac{\sigma_\theta^2}{\sigma_\xi^2}\frac{1}{\sqrt{\frac{\kappa^2}{\alpha} + \frac{\sigma_\theta^2}{\sigma_\xi^2}} + \frac{\kappa}{\alpha}}$$

which is a strictly increasing function of α. Since $\alpha(\cdot)$ is strictly decreasing at any point satisfying (21), the proof is completed. □

Proof of Proposition 7. Denote partial derivatives with subindices. We already know from Lemma 5 that $G_\phi(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*))) = 0$. On the other hand, straightforward differentiation shows that $G_\beta = 0$ if and only if $\alpha\lambda_\beta(\phi+\kappa) + \alpha\lambda^2 = 0$, where

$$\lambda_\beta = \frac{\kappa\sigma_\xi^2\lambda^2 - \lambda\sigma_\theta^2\alpha}{-2\kappa\sigma_\xi^2\beta\lambda + \sigma_\theta^2\alpha[\alpha+\beta] + \kappa\sigma_\xi^2(\phi+\kappa)}.$$

Since $\alpha\lambda > 0$, we aim to show that

$$[\kappa\sigma_\xi^2\lambda - \sigma_\theta^2\alpha][\phi+\kappa] + \lambda[-2\kappa\sigma_\xi^2\beta\lambda + \sigma_\theta^2\alpha[\alpha+\beta] + \kappa\sigma_\xi^2(\phi+\kappa)] = 0$$

at ϕ^*. However, recall that $\sigma_\theta^2 = 2\kappa\gamma - (\alpha\gamma/\sigma_\xi)^2$, and that, at ϕ^*, $\lambda = \alpha\gamma/\sigma_\xi^2$. Thus,

$$\alpha\sigma_\theta^2 = 2\kappa\gamma\alpha + \alpha\frac{\alpha^2\gamma^2}{\sigma_\xi^2} = 2\kappa\sigma_\xi^2\lambda + \alpha\sigma_\xi^2\lambda^2.$$

Plugging this in the first bracket and factoring by $\lambda > 0$ reduces the desired equality to

$$\begin{aligned} -\kappa\sigma_\xi^2(\phi+\kappa) - \alpha\sigma_\xi^2\lambda(\phi+\kappa) - 2\kappa\sigma_\xi^2\beta\lambda + \sigma_\theta^2\alpha[\alpha+\beta] + \kappa\sigma_\xi^2(\phi+\kappa) &= 0 \\ \Leftrightarrow -\alpha\sigma_\xi^2\lambda(\phi+\kappa) - 2\kappa\sigma_\xi^2\beta\lambda + \sigma_\theta^2\alpha[\alpha+\beta] &= 0, \end{aligned}$$

where the last equivalence results from the first and last terms canceling out. However, using that $\phi+\kappa = 2\kappa+\alpha\gamma(\alpha+\beta)/\sigma_\xi^2 = 2\kappa+\lambda(\alpha+\beta)$ and $\alpha\sigma_\theta^2 = 2\kappa\sigma_\xi^2\lambda+\alpha\sigma_\xi^2\lambda^2$ at ϕ^*,

$$\begin{aligned} & -\alpha\sigma_\xi^2\lambda(\phi+\kappa)-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta] \\ = \; & -2\kappa\sigma_\xi^2\alpha\lambda-\alpha\sigma_\xi^2\lambda^2(\alpha+\beta)-2\kappa\sigma_\xi^2\beta\lambda+2\kappa\sigma_\xi^2\lambda(\alpha+\beta)+\alpha\sigma_\xi^2\lambda^2(\alpha+\beta)=0. \end{aligned}$$

On the other hand, it is easy to verify that $G_\alpha > 0$ if and only if $\alpha\lambda_\alpha(\phi+\kappa)+[\lambda(\phi+\kappa)-\beta\lambda^2] > 0$, where

$$\lambda_\alpha = \frac{\sigma_\theta^2[\phi-2\lambda\alpha-\lambda\beta]}{-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta]+\kappa\sigma_\xi^2(\phi+\kappa)}$$

As a result, we are interested in the sign of

$$\alpha\sigma_\theta^2(\phi+\kappa)[\phi-2\lambda\alpha-\lambda\beta]+\lambda[\phi+\kappa-\beta\lambda][-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta]+\kappa\sigma_\xi^2(\phi+\kappa)]$$

at ϕ^*. But again using that $\phi+\kappa = 2\kappa+\alpha\gamma(\alpha+\beta)/\sigma_\xi^2 = 2\kappa+\lambda(\alpha+\beta)$ and $\alpha\sigma_\theta^2 = 2\kappa\sigma_\xi^2\lambda+\alpha\sigma_\xi^2\lambda^2$ at that point, we have that

$$\begin{aligned} \alpha\sigma_\theta^2(\phi+\kappa)[\phi-2\lambda\alpha-\lambda\beta] &= \lambda\sigma_\xi^2[2\kappa+\alpha\lambda](\phi+\kappa)[\kappa-\lambda\alpha] \\ \phi+\kappa-\beta\lambda &= 2\kappa+\alpha\lambda, \end{aligned} \tag{A.27}$$

so the expression of interest can be written as

$$\lambda[2\kappa+\alpha\lambda]\{[\kappa\sigma_\xi^2-\sigma_\xi^2\alpha\lambda](\phi+\kappa)-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta]+\kappa\sigma_\xi^2(\phi+\kappa)\}.$$

Since $\lambda[2\kappa+\alpha\lambda] > 0$, it suffices to show that

$$[\kappa\sigma_\xi^2-\sigma_\xi^2\alpha\lambda](\phi+\kappa)-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta]+\kappa\sigma_\xi^2(\phi+\kappa) > 0$$

However, from the first part of this proof, $-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta] = \alpha\sigma_\xi^2\lambda(\phi+\kappa)$. Thus,

$$[\kappa\sigma_\xi^2-\sigma_\xi^2\alpha\lambda](\phi+\kappa)\underbrace{-2\kappa\sigma_\xi^2\beta\lambda+\sigma_\theta^2\alpha[\alpha+\beta]}_{=\alpha\sigma_\xi^2\lambda(\phi+\kappa)}+\kappa\sigma_\xi^2(\phi+\kappa) = 2\kappa\sigma_\xi^2(\phi+\kappa) > 0.$$

This concludes the proof. $\square$

Proofs for Section 5

Proof of Proposition 8. Recall from the proof of Proposition 3 that

$$\alpha(\phi)+\beta(\phi)+\delta(\phi)=\frac{r+\phi}{2(r+\phi)+[\alpha(\phi)+\beta(\phi)]\lambda(\phi,\alpha(\phi),\beta(\phi))},$$

where $\beta(\phi)=f(\phi,\alpha(\phi))$. Also, $\mathbb{E}[P_t^\phi]=[\alpha(\phi)+\beta(\phi)+\delta(\phi)]\mu$.

Thus, omitting the dependence on variables,

$$[\alpha+\beta+\delta]'>0\Leftrightarrow(\alpha+\beta)\lambda-(r+\phi)[\lambda(\alpha+\beta)'+(\alpha+\beta)\mathrm{D}_\phi\lambda]>0.$$

where $\mathrm{D}_\phi\lambda$ denotes the total derivative $\lambda(\phi,\alpha(\phi),f(\phi,\alpha(\phi)))$.

However, $(\alpha+\beta)'=\alpha'[1+f_\alpha]+f_\phi$ and

$$\mathrm{D}_\phi\lambda=\lambda_\phi+\alpha'[\lambda_\alpha+\lambda_\beta f_\alpha]+\lambda_\beta f_\phi.$$

Thus, we want to show that at ϕ^*,

$$(\alpha+\beta)\underbrace{[\lambda-(r+\phi)\lambda_\phi]}_{(i)}-(r+\phi)f_\phi\underbrace{[\lambda+(\alpha+\beta)\lambda_\beta]}_{(ii)}$$
$$-(r+\phi)\alpha'\underbrace{[(1+f_\alpha)\lambda+(\lambda_\alpha+\lambda_\beta f_\alpha)(\alpha+\beta)]}_{(iii)}>0.$$

To this end, notice that since $\alpha+\beta$ and

$$-(r+\phi)f_\phi=-(r+\phi)\frac{\alpha^2(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]^2}$$

are strictly positive, and α is strictly decreasing at ϕ^* (so $-(r+\phi^*)\alpha'(\phi^*)>0$), it suffices to show that (i)–(iii) are non-negative under the condition stated in the Proposition.

(i): From the proof of Lemma 5, $\lambda_\phi=\lambda/(\phi+\kappa)$ at $(\phi^*,\alpha(\phi^*),f(\phi^*,\alpha(\phi^*)))$. Thus, at this point,

$$\lambda-(r+\phi^*)\lambda_\phi=\frac{\lambda}{r+\phi^*}[\kappa-r],$$

which is non-negative when $\kappa\geq r$.

(ii): Using that

$$G(\phi,\alpha,\beta)=\frac{\alpha\lambda(\phi,\alpha,\beta)}{\phi+\kappa-\beta\lambda(\phi,\alpha,\beta)}\Rightarrow\lambda(\phi,\alpha,\beta)=\frac{(\phi+\kappa)G(\phi,\alpha,\beta)}{\alpha+\beta G(\phi,\alpha,\beta)}$$

we have that

$$\lambda_\beta = (\phi+\kappa)\frac{G_\beta[\alpha+\beta G]-G[G+\beta G_\beta]}{(\alpha+\beta G)^2}.$$

However, we know from Proposition 7 that $G_\beta = 0$ at $(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$. We conclude that at $(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$,

$$\lambda_\beta = -\frac{(\phi+\kappa)G^2}{(\alpha+\beta G)^2} = -\frac{\lambda^2}{\phi^*+\kappa} < 0.$$

As a result, using that $\phi = \kappa + \lambda[\alpha+\beta]$ at $(\phi, \alpha, \beta) = (\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$,

$$\begin{aligned}\lambda + (\alpha+\beta)\lambda_\beta &= \frac{1}{\phi^*+\kappa}[\lambda(\phi^*+\kappa) - \lambda^2(\alpha+\beta)] \\ &= \frac{2\kappa\lambda(\phi^*, \alpha(\phi^*), \beta(\phi^*))}{\phi^*+\kappa} > 0.\end{aligned}$$

(iii): From the proof of Lemma 6, $1 + f_\alpha(\phi, \alpha(\phi)) > 0$ at $\phi = \phi^*$.[19] On the other hand, since $f_\alpha < 0$ (proof of Lemma 3), $\lambda_\beta f_\alpha > 0$ at ϕ^*. Thus, it suffices to show that $\lambda_\alpha > 0$. However, from the proof of Proposition 7,

$$\lambda_\alpha > 0 \;\text{ at }\; \phi^* \Leftrightarrow \phi^* - 2\lambda\alpha - \lambda\beta > 0 \;\text{ at }\; \phi^*.$$

Since $\phi^* = \kappa + (\alpha(\phi) + \beta(\phi))\lambda(\phi, \alpha(\phi), \beta(\phi))|_{\phi=\phi^*}$,

$$\begin{aligned}\phi^* - [2\lambda\alpha + \lambda\beta]|_{\phi=\phi^*} = \kappa - \lambda\alpha|_{\phi=\phi^*} &= \kappa - \frac{[\alpha]^2\gamma(\alpha)}{\sigma_\xi^2}\Big|_{\phi=\phi^*} \\ &= 2\kappa - \sqrt{\kappa^2 + \alpha\frac{\sigma_\theta^2}{\sigma_\xi^2}}\Big|_{\phi=\phi^*} \\ &\underbrace{>}_{\alpha<1} 2\kappa - \sqrt{\kappa^2 + \frac{\sigma_\theta^2}{\sigma_\xi^2}}\Big|_{\phi=\phi^*}\end{aligned}$$

and the latter is non-negative when $\kappa > \sigma_\theta/\sqrt{3}\sigma_\xi$. This concludes the proof. $\square$

Proof of Proposition 9. Part (i) follows directly from: 1) $G\phi = G_\beta = 0$ and $G_\alpha > 0$ at $(\phi^*, \alpha(\phi^*), f(\phi^*, \alpha(\phi^*)))$ (Proposition 7); 2) $\alpha'(\phi^*) < 0$ (Proposition 5); and 3) $\alpha'(\phi^*) + \beta'(\phi^*) < 0$ (Lemma 6). Part (ii) instead follows from α being strictly decreasing between zero and $\arg\min \alpha > \phi^*$ (quasiconvexity result in Proposition 3 coupled with Proposition

[19] In fact, in that section $(\alpha+\beta)' = (\alpha)'[h + \alpha h_\alpha] + \alpha h_\phi$, where $h(\phi, \alpha) = \alpha(\alpha-1)(r+2\kappa)/[2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)]$. But since $f_\phi(\phi, \alpha) = \alpha h_\phi(\phi, \alpha)$ and $(\alpha+\beta)' = (\alpha)'[1+f_\alpha] + f_\phi$, it follows that $1 + f_\alpha = h + \alpha h_\alpha$, and the right-hand side is strictly positive for $\phi \geq \kappa$.

5), and from $\alpha+\beta$ decreasing if α is decreasing (Lemma 6). $\square$

Proof of Proposition 10. Recall that $\mathrm{Var}[\theta_t|Y_t] = \sigma_\theta^2/2\kappa - \mathrm{Cov}[\theta_t, Y_t]^2/\mathrm{Var}(Y_t)$, $\lambda := \mathrm{Cov}[\theta_t, Y_t]/\mathrm{Var}(Y_t)$ and that $M_t = \mu + \lambda(Y_t - \bar{Y})$. Thus, $\mathbb{E}[\theta^2] = \sigma_\theta^2/2\kappa + \mu^2$, and

$$\mathbb{E}[M_t^2] = \mathbb{E}[\theta_t M_t] = \underbrace{\frac{\mathrm{Cov}[\theta_t, Y_t]^2}{\mathrm{Var}(Y_t)}}_{=\mathrm{Var}[M_t]} + \mu^2 \quad = \quad \frac{\sigma_\theta^2}{2\kappa} G^*(\phi) + \mu^2.$$

By stationarity, $CS(\phi) = \mathbb{E}[Q_t(\theta_t - P_t - Q_t/2)]$ where $P_t = \delta\mu(\phi) + (\alpha(\phi) + \beta(\phi))M_t$ and $Q_t = \delta(\phi)\mu + \alpha(\phi)\theta_t + \beta(\phi)M_t$. Thus, omitting the dependence on ϕ,

$$\begin{aligned} CS(\phi) = \quad & \delta\mu \left[\mathbb{E}[\theta_t]\left(1 - \frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t] - \frac{3\delta\mu}{2}\right] \\ + \quad & \alpha \left[\mathbb{E}[\theta_t^2]\left(1 - \frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t\theta_t] - \frac{3\delta\mu}{2}\mathbb{E}[\theta_t]\right] \\ + \quad & \beta \left[\mathbb{E}[\theta_t M_t]\left(1 - \frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t^2] - \frac{3\delta\mu}{2}\mathbb{E}[M_t]\right] \end{aligned} \tag{A.28}$$

and so, using the expressions for the first two moments of (θ_t, M_t),

$$\begin{aligned} CS(\phi) = \quad & \frac{\sigma_\theta^2}{2\kappa} G^*(\phi) \underbrace{\left[-\alpha\left(\alpha + \frac{3\beta}{2}\right) + \beta\left(1 - \frac{\alpha}{2}\right) - \beta\left(\alpha + \frac{3\beta}{2}\right)\right]}_{=-3(\alpha+\beta)^2/2 + (\alpha)^2/2 + \beta =: A_G(\phi)} + \frac{\sigma_\theta^2}{2\kappa}\left[\alpha(\phi)\left(1 - \frac{\alpha(\phi)}{2}\right)\right] \\ & + \mu^2 \underbrace{\left[-\alpha\left(\alpha + \frac{3\beta}{2}\right) + \beta\left(1 - \frac{\alpha}{2}\right) - \beta\left(\alpha + \frac{3\beta}{2}\right) + \alpha\left(1 - \frac{\alpha}{2}\right)\right]}_{=-3(\alpha+\beta)^2/2 + \alpha + \beta} \\ & \underbrace{+ \delta\mu^2 \left[\left(1 - \frac{3(\alpha+\beta)}{2} - \frac{3\alpha}{2} - \frac{3\beta}{2}\right) - \frac{3\delta}{2}\right]}_{=\mu^2\delta[(1-3(\alpha+\beta)) - 3\delta/2]}. \end{aligned}$$

Collecting terms in the last two lines yields

$$\begin{aligned}
& \mu^2\left[-3(\alpha+\beta)^2/2+\alpha+\beta+\delta-3\delta(\alpha+\beta)-\frac{3\delta^2}{2}\right] \\
= \ & \mu^2\left[\alpha+\beta+\delta-\frac{3}{2}\underbrace{\left\{(\alpha+\beta)^2+2\delta(\alpha+\beta)+\delta^2\right\}}_{(\alpha+\beta+\delta)^2}\right] \\
= \ & \mu^2(\alpha+\beta+\delta)\left[1-\frac{3}{2}(\alpha+\beta+\delta)\right], \\
= \ & \mathbb{E}[P_t^\phi]\left(\mu-\frac{3}{2}\mathbb{E}[P_t^\phi]\right).
\end{aligned}$$

Finally, notice that we can rewrite

$$A(\phi) := -3(\alpha+\beta)^2/2+\alpha^2/2+\beta = \underbrace{-\alpha[\alpha+2\beta]}_{<0}+\underbrace{\beta[1-\alpha]}_{<0}-3(\beta)^2/2<0.$$

On the other hand, since $-1/2<\beta<0$ and $\alpha>0$, and $0<\alpha+\beta<1$,

$$A(\phi)=\frac{\alpha(\phi)^2}{2}+\beta(\phi)-\frac{3}{2}(\alpha(\phi)+\beta(\phi))^2>0-\frac{1}{2}-\frac{3}{2}=-2.$$

Obs. When $\kappa>r$, it can be guaranteed that $A(\phi)<-3/8$. To see this, straightforward algebra shows that if $A(\phi)<-3/8$ and only if $12(\alpha+\beta)^2-8\beta\geq 3+4\alpha^2$, where the dependence on ϕ is omitted. Using that $\beta=f(\phi,\alpha)$, the previous condition translates to

$$\begin{aligned}
& 12[\alpha^2(r+2\phi)-(r+\kappa+\phi)\alpha(\alpha-1)]^2+8\alpha^2(r+2\phi)[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)] \\
> \ & (3+4\alpha^2)[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]^2.
\end{aligned}$$

which can be further transformed to

$$\begin{aligned}
(\alpha-1)\big\{&-4\alpha^2(\alpha-3)(r+2\phi)^2-8\alpha^3(r+2\phi)(r+\kappa+\phi)+8(r+\kappa+\phi)^2\alpha^2(\alpha-1) \\
&-8\alpha^2(r+2\phi)(r+\kappa+\phi)+12(r+\kappa+\phi)(r+2\phi)\alpha-3(r+\kappa+\phi)^2(\alpha-1)\big\}>0
\end{aligned}$$

Letting $\phi\to 0$, the expression inside $\{\cdot\}$ converges to $8r^2-16r(r+\kappa)+12(r+\kappa)r=4r[r-\kappa]$, which is negative is $\kappa>r$. Since $\alpha<1$, we conclude that the desired inequality holds for ϕ small enough. This concludes the proof.

□

Proof of Proposition 11. To show (i), notice that $A(\phi)<0$ and $\alpha(1-\alpha/2)<1/2$. Thus,

$CS(\phi)$ is bounded from above by its static counterpart with $Y=\emptyset$ when $\mu=0$. However, recall that $\alpha(\phi)\to 1$, $\beta(\phi)\to -1/2$. In addition, from Lemma 4, $\lim_{\phi\to 0}\lambda(\phi,\alpha(\phi),f(\phi,\alpha(\phi)))=0$ and $\lim_{\phi\to\infty}\lambda(\phi,\alpha(\phi),f(\phi,\alpha(\phi)))=\sigma_\theta^2/\kappa\sigma_\xi^2$. Thus,

$$G^*(\phi) \;=\; \frac{\alpha\lambda}{\phi+\kappa-\beta\lambda}\to 0 \text{ as } \phi\to 0,\infty. \tag{A.29}$$

Thus, the upper bound can be achieved asymptotically as $\phi\to 0$ and ∞ in this case. In the firm's case, the interior maximum follows from $\mathrm{Var}[P_t^\phi]=\sigma_\theta^2[\alpha+\beta]^2G^*(\phi)/2\kappa>0$ and $\mathrm{Var}[P_t^\phi]\to 0$ as $\phi\to 0,\infty$.

Regarding (ii), the consumer will prefer an interior optimum in this case due to

$$\mathbb{E}[P_t^\phi]\left(\mu-\frac{3}{2}\mathbb{E}[P_t^\phi]\right)\in\left(\frac{\mu^2}{8},\frac{\mu^2}{6}\right) \text{ and } \mathbb{E}[P_t^\phi]\left(\mu-\frac{3}{2}\mathbb{E}[P_t^\phi]\right)\to\frac{\mu^2}{8} \text{ as } \phi\to 0,\infty.$$

The firm instead prefers a corner solution due to $\mathbb{E}[P_t^\phi]\in(\mu/3,\mu/2)$ and $\mathbb{E}[P_t^\phi]\to\mu/2$ as $\phi\to 0,\infty$.

Finally, to show (iii), observe that ex ante total surplus is given by

$$CS(\phi)+\Pi(\phi)=\frac{\sigma_\theta^2}{2\kappa}\underbrace{\alpha(\phi)\left(1-\frac{\alpha(\phi)}{2}\right)}_{<1/2,\ \text{as}\ \alpha\in(1/2,1)}+\underbrace{\mathbb{E}[P_t^\phi]\left(\mu-\frac{\mathbb{E}[P_t^\phi]}{2}\right)}_{<\mu^2/8\ \text{as}\ \mathbb{E}[P_t^\phi]<\mu/2}+\underbrace{\left[\frac{\alpha^2}{2}+\beta-\frac{(\alpha+\beta)^2}{2}\right]}_{=\beta[1-\alpha]-\beta^2/2<0}\frac{\sigma_\theta^2}{2\kappa}G^*(\phi)$$

from where ex ante total surplus achieves its upper bound $\sigma_\theta^2/4\kappa+\mu^2/8$ asymptotically as $\phi\to 0,\infty$. ☐

Proofs for Section 6

Proof of Proposition 12. Recall from the proof of Proposition 1 that, starting from

$$Q_t=\delta\mu+\alpha\theta_t+\beta M_t \text{ and } M_t=\rho+\lambda Y_t$$

λ and ρ satisfy (A.6)–(A.7) given by

$$\begin{aligned}\lambda &= \frac{\alpha\sigma_\theta^2(\phi-\beta\lambda)}{\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi+\kappa-\beta\lambda)}\\ \rho &= \mu-\frac{\hat\delta+\alpha\mu}{\phi-\hat\beta}\lambda=\mu-\frac{\delta\mu+\beta\rho+\alpha\mu}{\phi-\beta\lambda}\lambda,\end{aligned}$$

respectively. In this hidden case, we replace $(\delta\mu, \alpha, \beta)$ by $(q_0/2, q_1, -q_1/2)$; let (m_0, m_1) denote the corresponding values for (ρ, λ) after this change. In particular, observe that, as in the baseline model, for the outcome to be stationary it must be that $\phi - \beta\lambda = \phi + q_1 m_1/2 > 0$. Also, $q_1 \neq 0$ (otherwise, the price is constant, which leads to a demand with unit weight on the type). But this in turn implies that m_1 (the analog of λ in this case) is strictly positive.

We can then write

$$\begin{aligned} dP_t = -\frac{q_1 m_1}{2q_2} dY_t &= -\frac{q_1 m_1}{2q_2}[(Q_t - \phi Y_t)dt + \sigma_\xi dZ_t^\xi] \\ &= \left[-\frac{q_1 m_1}{2q_2} Q_t - \phi\left(P_t + \frac{q_0 + q_1 m_0}{2q_2}\right)\right] dt - \frac{q_1 m_1}{2q_2}\sigma_\xi dZ_t^\xi, \end{aligned} \tag{A.30}$$

and the consumer's problem is to maximize her utility as in Section 3 with prices given by (24), subject to (2) and the previous law of motion of prices.

We guess a value function $V = v_0 + v_1\theta + v_2 P + v_3 P^2 + v_4\theta^2 + v_5\theta P$, which gives the first-order condition

$$q = \theta - P - \frac{q_1 m_1}{2q_2}\underbrace{[v_2 + 2v_3 P + v_5\theta]}_{V_P} = -\frac{q_1 m_1}{2q_2} v_2 + \left[1 - \frac{q_1 m_1}{2q_2} v_5\right]\theta + \left[-1 - \frac{q_1 m_1}{q_2} v_3\right] P.$$

As a result, we obtain the matching coefficients conditions

$$q_0 = -\frac{q_1 m_1}{2q_2} v_2,\ q_1 = 1 - \frac{q_1 m_1}{2q_2} v_5 \ \text{ and } q_2 = -1 - \frac{q_1 m_1}{q_2} v_3 \tag{A.31}$$

Moreover, by the envelope theorem

$$(r+\phi)[v_2 + 2v_3 P + v_5\theta] = q\left[-1 - v_3\frac{q_1 m_0(q_1)}{q_2}\right] - 2v_3\phi\left[P + \frac{q_0 + q_1 m_0}{2q_2}\right] - \kappa v_5(\theta - \mu)$$

which leads to the system

$$\begin{aligned} (r+\phi)v_2 &= q_0\left[-1 - v_3\frac{q_1 m_1}{q_2}\right] - 2v_3\phi\frac{q_0 + q_1 m_0}{2q_2} + \kappa\mu v_5 \\ 2(r+\phi)v_3 &= q_2\left[-1 - v_3\frac{q_1 m_1}{q_2}\right] - 2v_3\phi \\ (r+\phi)v_5 &= q_1\left[-1 - v_3\frac{q_1 m_1}{q_2}\right] - \kappa v_5. \end{aligned} \tag{A.32}$$

Using that v_2, v_3, v_5 can be written as a function of q_0, q_1, q_2, and dividing by q_2 in each equation, we obtain the following system

$$
\begin{aligned}
-(r+\phi)\frac{2q_0}{q_1m_1} &= q_0 + 2\phi\frac{q_2+1}{q_1m_1}\frac{q_0+q_1m_0}{2q_2} + \kappa\mu\frac{2(1-q_1)}{q_1m_1} \\
-2(r+2\phi)\frac{q_2+1}{q_1m_1} &= q_2 \\
(r+\phi+\kappa)\frac{2(1-q_1)}{q_1m_1} &= q_1. \qquad \text{(A.33)}
\end{aligned}
$$

Observe that the last equation is independent of the other two, while the second one is linear in q_1. Thus, we can solve for q_1 and q_2 sequentially. Finally, since $m_0 = \frac{1}{\phi}\left[\mu\left(\phi - \frac{q_1m_1}{2}\right) - \frac{q_0m_1}{2}\right]$, the equation for q_0 turns out to be linear too. We proceed by finding q_1 first.

It is immediate that $q_1 \in (0,1)$: if instead $q_1 < 0$, the last equation in (A.33) reads

$$
\underbrace{\phi + \frac{q_1}{2}m_1}_{=\phi-\beta\lambda} = (r+\kappa)\left(\frac{1}{q_1}-1\right) + \frac{q_1}{\phi} < 0,
$$

which contradicts stationarity. But if $q_1 \geq 1$, the same equation implies that $q_1 = 0$, or that $q_1 \neq 0$ but $m_1 < 0$, both contradictions; in particular, the latter follows from $m_1 > 0$ in a stationary linear Markov equilibrium with $q_1 > 0$ (due to $\phi - \beta\lambda = \phi + q_1m_1/2 > 0$.

Since $\alpha = q_1 > 0$ and $\beta = -q_1/2 < 0$, the unique possible value for m_1 is given by $\lambda(\phi, q_1, -q_1/2)$ as defined in (14), as this is the unique positive root of (8). The last equation in (A.33) then reads $g^{\mathsf{hidden}}(q_1) := (r+\kappa+\phi)(q_1-1) - q_1\lambda(\phi, q_1, -q_1/2)[-q_1/2] = 0$. Moreover, since

$$
\begin{aligned}
m_1 &= \frac{-\ell(q_1) + \sqrt{\ell(q_1)^2 + 8q_1^2\kappa\sigma_\xi^2\sigma_\theta^2\phi}}{q_1\kappa\sigma_\xi^2}, \quad \text{where} \quad \ell(q_1) = \frac{q_1^2}{2}\sigma_\theta^2 + \kappa\sigma_\xi^2(\phi+\kappa), \\
\Rightarrow m_1q_1^2 &= -\frac{q_1^3\sigma_\theta^2}{2\kappa\sigma_\xi^2} - q_1(\phi+\kappa) + q_1\left[\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2q_1^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2}. \qquad \text{(A.34)}
\end{aligned}
$$

Thus, we can rewrite $g^{\mathsf{hidden}}(q_1) = 0$ as

$$
\underbrace{q_1\left\{\left[\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2q_1^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2}\right\}}_{H(q_1):=} + q_1(2r+\phi+\kappa) - 2(r+\phi+\kappa) = 0.
$$

It is clear that $H(\cdot) > 0$. To show that $H'(q_1) > 0$ when $q_1 \in (0,1)$ observe that

$$
\begin{aligned}
H' > 0 \;\; &\Leftrightarrow \;\; 2\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)\frac{q_1\sigma_\theta^2}{\kappa\sigma_\xi^2} + \frac{4q_1\sigma_\theta^2\phi}{\kappa\sigma_\xi^2} > \frac{2q_1\sigma_\theta^2}{\kappa\sigma_\xi^2}\left[\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2q_1^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2} \\
&\Leftrightarrow \;\; \left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right) + 2\phi > \left[\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2q_1^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2} \\
&\Leftrightarrow \;\; \left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + 4\phi\left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right) + 4\phi^2 > \left(\frac{q_1^2\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2q_1^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2} \\
&\Leftrightarrow \;\; 4\phi(\phi + \kappa) + 4\phi^2 > 0
\end{aligned}
$$

which is true. Thus, H is strictly increasing. Also, $H(0) < 0$ and setting $q_1 = 1$ yields

$$
H(1) = \left[\left(\frac{\sigma_\theta^2}{2\kappa\sigma_\xi^2} + \phi + \kappa\right)^2 + \frac{2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{\sigma_\theta^2}{2\kappa\sigma_\xi^2} - \phi - \kappa > 0.
$$

Thus, there is a unique $q_1^* \in (0,1)$ such that $H(q_1^*) = 0$. In addition, $H(q_1) \to q_1\kappa + q_1(2r + \kappa) - 2(r + \kappa)$ as $\phi \to 0$, so $q_1^* = 1$. Similarly, as $\phi \to \infty$, $f_\phi(q_1) \approx q_1\phi + \phi(q_1 - 2)$, so $q_1^* = 1$ also holds.

Returning to q_2, it is easy to see from the second equation in (A.33)

$$
q_2 = -\frac{2(r + 2\phi)}{q_1 m_1 + 2(r + 2\phi)} \in (-1, 0).
$$

Finally, recall that q_0 satisfies the equation

$$
-(r + \phi)\frac{2q_0}{q_1 m_1} = q_0 + \phi\frac{q_2 + 1}{q_2 q_1 m_1}[q_0 + q_1 m_0] + \kappa\mu\frac{2(1 - q_1)}{q_1 m_1}
$$

and that

$$
m_0 = \frac{1}{\phi}\left[\mu\left(\phi - \frac{q_1 m_1}{2}\right) - \frac{q_0 m_1}{2}\right].
$$

Also, from the equation for q_2, $(q_2 + 1)/q_2 q_1 m_1 = -1/2(r + 2\phi)$. Thus, the coefficient that accompanies q_0 is given by

$$
\frac{1}{q_1 m_1}\left[-2(r + \phi) - q_1 m_1 + \frac{\phi}{2(r + 2\phi)}q_1 m_1 - \frac{\phi}{2(r + 2\phi)}q_1^2 m_1^2\right].
$$

But observe that $\phi q_1 m_1 / 2(r + 2\phi) \in (0, q_1 m_1/4)$, and so the second term dominates the third. We conclude that the previous expression is strictly negative, which implies that the equation for q_0 admits a solution under no restriction over the set of parameters. To conclude:

1. The rest of the unknowns are determined as follows. First, v_2, v_3 and v_5 are determined from the matching coefficient conditions (A.31) using q_0, q_1 and q_2 (all these equations admit a solution). v_1 and v_4 can be obtained via the envelope theorem. Namely:

$$\begin{aligned}(r+\kappa)[v_1+2v_4\theta+v_5P] &= (q_0+q_1\theta+q_2P)\left[1-v_5\frac{q_1m_1}{2q_2}\right]-v_5\phi\left[P+\frac{q_0+q_1m_0}{2q_2}\right]\\ &\quad -2v_4\kappa(\theta-\mu)\end{aligned}$$

yields two additional equations

$$\begin{aligned}2(r+\kappa)v_4 &= \underbrace{q_1\left[1-v_5\frac{q_1m_1}{2q_2}\right]}_{=q_1 \text{ from (A.31)}}-2v_4\kappa \Rightarrow v_4=\frac{q_1^2}{2(r+2\kappa)}\\ (r+\kappa)v_1 &= q_0q_1-v_5\phi\frac{q_0+q_1m_0}{2q_2}\Rightarrow v_1=\frac{q_0q_1}{r+\kappa}-\frac{v_5\phi(q_0+q_1m_1)}{2q_2(r+\kappa)}\end{aligned}$$

The coefficient v_0 in turn corresponds to

$$v_0=\frac{1}{r}\left[-q_0^2+v_2\left(q_0\frac{q_1m_1}{2q_2}-\phi\frac{q_0+q_1m_0}{2q_2}\right)+v_1\kappa\mu+\sigma_\theta^2v_3+\left(\frac{q_1m_1\sigma_\xi}{2q_2}\right)^2v_4\right]$$

which is determined comes from equating the constant terms in the HJB equation.

2. The law of motion of equilibrium prices is given by

$$\begin{aligned}dP_t &= \left[-\frac{q_1m_1}{2q_2}\underbrace{(q_0+q_1\theta_t+q_2P_t^*)}_{q_t=}-\phi\left(P_t+\frac{q_0+q_1m_0}{2q_2}\right)\right]dt-\frac{q_1m_1}{2q_2}\sigma_\xi dZ_t^\xi\\ &= \left[-\left(\frac{q_1m_1}{2}+\phi\right)P_t-\frac{q_1^2m_1}{2q_2}\theta_t-\left(\frac{q_0q_1m_1}{2q_2}+\phi\frac{q_0+q_1m_0}{2q_2}\right)\right]dt-\frac{q_1m_1}{2q_2}\sigma_\xi dZ_t^\xi.\end{aligned}$$

Since $q_1m_1/2+\phi>0$ prices are mean reverting with sensitivity to new information given by $-q_1m_1/2q_2>0$. Because the equilibrium dynamics (θ, P^*) are (coupled) mean reverting, usual transversality conditions hold hold.

□

Proof of Proposition Recall that $q_1^{\text{obs}} = \alpha$ and q_1^{hidden} are defined solutions to

$$\begin{aligned}
&-2(r+\kappa+\phi) + \alpha\,(2r+\kappa+\phi) \\
&+\alpha\left\{\left[\left(\frac{\sigma_\theta^2\alpha[\alpha+f(\alpha)]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 - \frac{4\sigma_\theta^2\alpha f(\alpha)\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{\sigma_\theta^2\alpha[\alpha+f(\alpha)]}{\kappa\sigma_\xi^2}\right\} = 0, \quad \text{and} \\
&-2(r+\phi+\kappa) + \alpha(2r+\phi+\kappa) \\
&+\alpha\left\{\left[\left(\frac{\alpha^2\sigma_\theta^2}{2\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 + \frac{2\alpha^2\sigma_\theta^2\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{\alpha^2\sigma_\theta^2}{2\kappa\sigma_\xi^2}\right\} = 0,
\end{aligned}$$

respectively (in the observable case, $g(\cdot)$ can be rearranged to appear as above). Also, recall that $f(\alpha) \in (-\alpha/2, 0)$, where we have omitted the dependence on ϕ. Thus, define

$$y \mapsto h_\alpha(y) := \left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 - \frac{4\sigma_\theta^2\alpha y\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}$$

and notice that the first equation can be written as $g^{\text{obs}}(\alpha) := -2(r+\kappa+\phi)+\alpha\,(2r+\kappa+\phi)+\alpha h_\alpha(f(\alpha)) = 0$, whereas the second can be written as $g^{\text{hidden}}(\alpha) := -2(r+\kappa+\phi)+\alpha\,(2r+\kappa+\phi)+\alpha h_\alpha(-\alpha/2) = 0$. We now show that $h_\alpha(\cdot)$ is strictly decreasing over $(-\alpha/2, 0)$.

To this end, observe that $h'_\alpha(y) < 0$ if and only if

$$\begin{aligned}
&\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)\frac{\sigma_\theta^2\alpha}{\kappa\sigma_\xi^2} - \frac{2\sigma_\theta^2\alpha\phi}{\kappa\sigma_\xi^2} < \frac{\sigma_\theta^2\alpha}{\kappa\sigma_\xi^2}\left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 - \frac{4\sigma_\theta^2\alpha y}{\kappa\sigma_\xi^2}\right]^{1/2} \\
\Leftrightarrow \quad &\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right) - 2\phi < \left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 - \frac{4\sigma_\theta^2\alpha y}{\kappa\sigma_\xi^2}\right]^{1/2}.
\end{aligned}$$

If the left-hand side is negative, the result follows immediately.

Suppose to the contrary that $\left(\sigma_\theta^2\alpha[\alpha+y]/\kappa\sigma_\xi^2+\phi+\kappa\right) - 2\phi > 0$. Squaring both sides of the inequality under study yields

$$-4\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)\phi + 4\phi^2 < -\frac{4\sigma_\theta^2\alpha y\phi}{\kappa\sigma_\xi^2} \Leftrightarrow 0 < \frac{\sigma_\theta^2\alpha^2}{\kappa\sigma_\xi^2}+\kappa, \tag{A.35}$$

which is true. We conclude that $g^{\text{pub}}(\alpha) < g^{\text{hidden}}(\alpha)$ for all $\alpha \in [0,1]$. But since $g^{\text{obs}}(\alpha)$ and $g^{\text{hidden}}(\alpha)$ are increasing (proofs of Propositions 2 and 12), it follows that $q_1^{\text{obs}}(\phi) > q_1^{\text{hidden}}(\phi)$.

As for (ii) and (iii), recall that in the observable case $Q_t = \delta\mu + \alpha\theta_t + \beta M_t$ and $P_t =$

$\delta\mu + (\alpha+\beta)M_t$, and thus

$$q_1^{\mathsf{obs}}(\phi) = \alpha,\ q_2^{\mathsf{obs}}(\phi) = \frac{\beta}{\alpha+\beta} \ \text{ and } \ p_1^{\mathsf{obs}}(\phi) = (\alpha+\beta)\lambda(\phi,\alpha,\beta).$$

(ii): From the second equation for the system that defines (δ,α,β) in the observable case,

$$(r+2\phi)\frac{\alpha+2\beta}{m_1} = \beta^2 \Rightarrow q_2^{\mathsf{obs}}(\phi) = \frac{\beta}{\beta+\alpha} = \frac{-2(r+2\phi)}{2(r+2\phi)-2\lambda\beta}.$$

On the other hand, from the private case,

$$q_2 = \frac{-2(r+2\phi)}{2(r+2\phi)+q_1m_1}.$$

Thus, we must compare $-2\lambda\beta$ with q_1m_1. However, from the third equation for the system that defines (δ,α,β),

$$(r+\kappa+\phi)\frac{\alpha-1}{\lambda} = \alpha\beta \Rightarrow (r+\kappa+\phi)\frac{2(1-\alpha)}{\alpha} = -2\lambda\beta.$$

Also, from the third equation in the system (A.33) for (q_0,q_1,q_2) in the private case,

$$(r+\kappa+\phi)\frac{2(1-q_1)}{q_1m_1} = q_1 \Rightarrow (r+\kappa+\phi)\frac{2(1-q_1)}{q_1} = q_1m_1$$

But since $1 > q_1^{\mathsf{obs}} > q_1^{\mathsf{hidden}} > 0$ we conclude that $0 < -2\lambda\beta < q_1m_1$, from where $-1 < q_2^{\mathsf{obs}} < q_2^{\mathsf{hidden}} < 0$.

(iii) Finally, from the second equation for (δ,α,β) in (A.13)

$$p_1^{\mathsf{obs}}(\phi) = (\alpha+\beta)\lambda = \frac{[\lambda\beta]^2}{r+2\phi} - \beta\lambda \ = \ \frac{4[\lambda\beta]^2 - 4\beta\lambda}{4(r+2\phi)}$$
$$\frac{[-2\lambda\beta + (r+2\phi)]^2 - (r+2\phi)^2}{4(r+2\phi)}.$$

However, using the expression for q_2 in the private case,

$$p_1^{\mathsf{hidden}} := -\frac{q_1m_1}{2q_2} = \frac{[q_1m_1]^2 + 2(r+2\phi)q_1m_1}{4(r+2\phi)} = \frac{[q_1m_1 + (r+2\phi)]^2 - (r+2\phi)^2}{4(r+2\phi)}.$$

Since $0 < -2m_1\beta < q_1m_1$, we deduce that $0 < p_1^{\mathsf{obs}} < p_1^{\mathsf{hidden}}$, concluding the proof. $\square$

Appendix B: Discretized Public Ratings Model

This appendix introduces a sequence of discrete-time counterparts of the continuous-time setting examined in which each game is indexed by the length of the interaction between the consumer and any myopic firm. The purpose is twofold. First, to illustrate that the Markov "restriction" (i.e., the study of history-independent coefficients) has no bite in discrete time. Second, that a sensitivity of demand equal to -1 is the limiting value of the sensitivities of demand along the sequence of games studied as the period length shrinks to zero.

A consumer interacts with a sequence of short-run firms in a stochastic game of period length $\Delta > 0$. Specifically, at each $t \in \{0, \Delta, 2\Delta, 3\Delta, ...\}$ the consumer shops for a product that is supplied by a single firm (*firm* t). The timing of events over $[t, t+\Delta)$ is as in Section 2: first, firm t posts a price; second, having observed this price, the consumer chooses how much to buy; third, the purchase is recorded with noise, and subsequently incorporated in the rating. The same sequence of events then repeats at $[t+\Delta, t+2\Delta)$, but now with a firm different from firm $s \in \{0, \Delta, ..., t\}$.

The discretized model consists of the dynamics

$$
\begin{aligned}
\theta_{t+\Delta} &= \theta_t - \kappa\Delta(\theta_t - \mu) + \sqrt{\Delta}\epsilon^{\theta}_{t+\Delta} \\
Y_{t+\Delta} &= Y_t - \phi\Delta Y_t + Q_t\Delta + \sqrt{\Delta}\epsilon^{\xi}_{t+\Delta}
\end{aligned}
$$

where $\epsilon^{\theta}_t \sim \mathcal{N}(0, \sqrt{\Delta}\sigma^2_{\theta})$ and $\epsilon^{\xi}_t \sim \mathcal{N}(0, \sqrt{\Delta}\sigma^2_{\xi})$ are independent across time, and the sequences independent from one another. Finally, the consumer's utility over period $[t, t+\Delta)$ given $(\theta_t, P_t, Q_t) = (\theta, p, q)$ is given by

$$
u^{\Delta}(\theta, p, q) = \left((\theta - p)q - \frac{q^2}{2}\right)\Delta.
$$

As in the main body of the paper, if the firms conjecture a consumer strategy $Q(p, \theta, M) = q_0 + q_1\theta + q_2 M - q_3 p$, $q_3 > 0$, they will set a price according to $P(M) = \frac{q_0 + (q_1+q_2)M}{2q_3}$, which leads to realized purchases along the path of play simply following

$$
Q_t = \frac{q_0}{2} + q_1\theta_t + \frac{q_2 - q_1}{2}M_t.
$$

Letting $\delta := q_0/2$, $\alpha := q_1$ and $\beta := (q_2 - q_1)/2$, we can then write

$$
P_t = \frac{\delta + (\alpha+\beta)M_t}{q_3} \;\text{ and }\; Q_t = \delta + \alpha\theta_t + \beta M_t,\; t \in \{0, \Delta, 2\Delta, ...\}. \;(*)
$$

For simplicity, we restrict the analysis to the natural case $\alpha > 0$, $\beta < 0$ and $\alpha + \beta > 0$, as these are the properties of the equilibrium studied in the baseline model.

We now proceed in three steps. First, we find an expression for the weight that the consumer's best-response attaches to the current price when firms conjecture a consumer linear strategy and set prices as in (*). Call this weight $\hat{q}_3$. Second, we show that at any history where firm t sets a price different than the one prescribed by (*), the consumer optimally responds with the same linear strategy from the previous step; thus $\hat{q}_3$ is effectively the sensitivity of demand. Third, we show that $\hat{q}_3$ goes to 1 as $\Delta \searrow 0$. Importantly, these steps hold under any linear conjecture by the firms (in particular, for $q_3 \neq \hat{q}_3$), and thus, $\hat{q}_3 = 1$ is a property of the consumer's best response along the sequence of games.

Step 1. Since from each firm's perspective Y_t was driven by past purchases that followed (*), $M_t := \mathbb{E}[\theta_t|Y_t] = \rho + \lambda Y_t$ for some $\rho \in \mathbb{R}$ and $\lambda > 0$, where Y_t carries information up to the time $t - \Delta$ purchase. In this case,

$$\begin{aligned} & M_{t+\Delta} - M_t = \lambda[Y_{t+\Delta} - Y_t] = \lambda[-\phi\Delta(M_t - \rho)/\lambda + Q_t\Delta + \sqrt{\Delta}\epsilon^{\xi}_{t+\Delta}] \\ \Rightarrow\quad & M_{t+\Delta} = M_t - \phi\Delta(M_t - \rho) + \lambda Q_t \Delta + \lambda\sqrt{\Delta}\epsilon^{\xi}_{t+\Delta} \end{aligned} \tag{A.36}$$

Let V denote the consumer's value function when facing prices given by prices as just stated. Then, the following Bellman equation holds

$$\begin{aligned} V(\theta, M) &= \max_{q\in\mathbb{R}} \left\{ \left[\left(\theta - \frac{\delta + (\alpha+\beta)M}{q_3} \right) q - q^2/2 \right] \Delta + e^{-r\Delta}\mathbb{E}[V(\theta', M')|(M,\theta)] \right\} \\ s.t. & \\ \theta' &= \theta - \kappa\Delta(\theta - \mu) + \sqrt{\Delta}\epsilon^{\theta} \\ M' &= M - \phi\Delta(M - \rho) + \lambda q \Delta + \lambda\sqrt{\Delta}\epsilon^{\xi}. \end{aligned} \tag{A.37}$$

We look for a quadratic value, i.e., $V(\theta, M) = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M$. Leting $X := (\theta, M)$, we have that $V(X') = V(X) + DV(X)(X' - X) + \frac{1}{2}(X' - X)^{\top} D^2 V(X' - X)$, and straightforward algebra shows that the Bellman equation further reduces to

$$\begin{aligned} V(\theta, M) = \max_{q\in\mathbb{R}} \Bigg\{ & \left[\left(\theta - \frac{\delta + (\alpha+\beta)M}{q_3} \right) q - q^2/2 \right] \Delta + e^{-r\Delta} V(\theta, M) \\ & + e^{-r\Delta}\Delta(-\kappa[\theta - \mu]V_{\theta} + [-\phi(M - \rho) + \lambda q]V_M + \frac{1}{2}V_{\theta\theta}[\Delta\kappa^2(\theta - \mu)^2 + \sigma_{\theta}^2]) \\ & + e^{-r\Delta}\Delta V_{\theta M}[-\kappa(\theta - \mu)(-\phi\Delta(M - \rho) + q\lambda\Delta)] \\ & + e^{-r\Delta}\Delta\frac{1}{2}V_{MM}[\phi^2\Delta(M - \rho)^2 + \lambda^2 q^2 \Delta + \lambda^2\sigma_{\xi}^2 - 2\phi\lambda\Delta(M - \rho)q]) \Bigg\} \end{aligned}$$

The first-order condition then reads

$$[1-e^{-r\Delta}\lambda^2\Delta V_{MM}]q = \theta - \underbrace{\frac{\delta+(\alpha+\beta)M}{q_3}}_{p=} + e^{-r\Delta}(\lambda V_M + \Delta V_{\theta M}[-\kappa(\theta-\mu)\lambda] - V_{MM}\phi\Delta(M-\rho)\lambda)$$

Thus, the contemporaneous price has weight $-\hat{q}_3$ in the consumer's linear best-response, where

$$\hat{q}_3 = \frac{1}{1-e^{-r\Delta}\lambda^2\Delta V_{MM}} = \frac{1}{1-2e^{-r\Delta}\lambda^2\Delta v_3}.$$

In step 3, we show that, fixing q_3 (which enters as a parameter in the consumer's best-response problem and thus affects v_3), $\Delta v_3 \searrow 0$. Thus, for $\Delta > 0$ small, (ii) the right-hand side of the Bellman equation is a concave problem, and (ii) any linear best-response exhibits $\hat{q}_3 \approx 1$.

Step 2. Consider now a history at which firm t posts a price $p \neq [\delta + (\alpha+\beta)M_t]/q_3$. In this case, it is easy to see that the consumer's problem is of the form

$$\begin{aligned} \max_{q\in\mathbb{R}} \quad & \left\{\left[(\theta-p)\,q - q^2/2\right]\Delta + e^{-r\Delta}\mathbb{E}[V(\theta',M')|(M,\theta)]\right\} \\ s.t. \quad & \theta' = \theta - \kappa\Delta(\theta-\mu) + \sqrt{\Delta}\epsilon^{\theta} \\ & M' = M - \phi\Delta(M-\rho) + \lambda q\Delta + \lambda\sqrt{\Delta}\epsilon^{\xi}. \end{aligned} \tag{A.38}$$

In fact, since the deviation is not observed by subsequent firms, the consumer's continuation value following the deviation is precisely given by V found by solving the Bellman equation of the previous step. As a result, the consumer's optimal strategy is determined by the same first-order condition.

Step 3. Straightforward algebraic manipulation shows that v_3 is determined by setting to zero the coefficient on M^2 in the Bellman equation. Specifically, the condition is

$$4q_3^2\Delta\lambda^2 v_3^2 + 2q_3v_3[q_3[(1-\Delta\phi)^2 - e^{\Delta r}] - 2\Delta(1-\Delta\phi)\lambda(\alpha+\beta)] + e^{\Delta r}\Delta(\alpha+\beta)^2 = 0.$$

Letting $\Gamma := q_3[(1-\Delta\phi)^2 - e^{\Delta r}] - 2\Delta(1-\Delta\phi)\lambda(\alpha+\beta)$, the two solutions are given by

$$v_3^{\pm} = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\Delta^2\lambda^2 e^{\Delta r}(\alpha+\beta)^2}}{2q_3\Delta\lambda^2}.$$

We now show that $\Delta v_3^{\pm} \searrow 0$ as $\Delta \searrow 0$ (but as we show below, v_3^- is the root associated with the equilibrium examined in the paper).

To this end, observe that λ also depends on Δ. A calculation presented at the end of this appendix in fact shows that this value is defined as the positive root of the equation

$$F(\Delta,\lambda):=\lambda-\frac{\sigma_\theta^2\alpha(1-\kappa\Delta)[2(\phi-\beta\lambda)-(\phi-\beta\lambda)^2\Delta]}{\sigma_\xi^2[2\kappa-\kappa^2\Delta][(\phi-\beta\lambda)(1-\kappa\Delta)+\kappa]+\sigma_\theta^2\alpha^2[2-\kappa\Delta-(\phi-\beta\lambda)(1-\kappa\Delta)\Delta]}=0.$$

It is easy to verify that at $\Delta=0$ the previous equation reduces to the quadratic function that determines the sensitivity of beliefs in the continuous-time game analyzed. Call such sensitivity λ_0, and thus $F(0,\lambda_0)=0$. Moreover, since $\beta<0$ and $\lambda_0>0$

$$\begin{aligned}\frac{\partial F}{\partial\lambda}(0,\lambda_0) &= \frac{(\sigma_\theta^2\alpha^2+\sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2+\beta\sigma_\theta^2\alpha[\sigma_\theta^2\alpha^2+\kappa^2\sigma_\xi^2]}{(\sigma_\theta^2\alpha^2+\sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2}\\ &> \frac{\sigma_\theta^4\alpha^3[\alpha+\beta]+\sigma_\xi^2\sigma_\theta^2\alpha\kappa^2[2\alpha+\beta]}{(\sigma_\theta^2\alpha^2+\sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2}>0 \qquad \text{(A.39)}\end{aligned}$$

where the last inequality follows from $\alpha+\beta>0$. By the implicit function theorem, therefore, the exists $\varepsilon>0$ and a unique continuously differentiable function $\lambda(\Delta)$ such that $\lambda(0)=\lambda_0$, $F(\Delta,\lambda(\Delta))=0$, and $\lambda(\Delta)>0$, for all $\Delta\in[0,\varepsilon]$.

Since $\lambda(\cdot)$ is bounded in that set, we conclude that

$$\Delta v_3^\pm=\frac{-\Gamma\pm\sqrt{\Gamma^2-4\Delta^2\lambda^2(\Delta)e^{\Delta r}(\alpha+\beta)^2}}{2q_3\lambda^2(\Delta)}\to 0$$

as $\Delta\searrow 0$, due to $\Gamma:=q_3[(1-\Delta\phi)^2-e^{\Delta r}]-2\Delta(1-\Delta\phi)\lambda(\alpha+\beta)$ also vanishing in the limit. This concludes step 3.

Before showing that $F(\Delta,\lambda)=0$ defines the sensitivity of beliefs consistent with Bayesian updating, two observations.

1. It is easy to see that when $q_3=1$, then, as $\Delta\searrow 0$,

$$v_3^\pm\to\frac{2\lambda_0(\alpha+\beta)+(r+2\phi)\pm\sqrt{[2\lambda_0(\alpha+\beta)+(r+2\phi)]^2-4\lambda^2(\alpha+\beta)^2}}{4\lambda_0^2}$$

the right-hand side being the two roots for the equation that v_3 must satisfy in the continuous-time program.[20] However, an equilibrium condition of the continuous time model is $2\lambda v_3=\alpha+2\beta$, and so either

$$2\lambda v_3^+=\alpha+2\beta \;\text{ or }\; 2\lambda v_3^-=\alpha+2\beta$$

[20]This follows from inserting the first-order condition (A.9) as a function of the v's in (A.11), and then solving the quadratic equation for v_3 that results from equating the coefficient on M to zero.

must hold. However, the previous conditions reduce to

$$r + 2\phi \pm \sqrt{(r+2\phi)^2 + 4\lambda(\alpha+\beta)(r+2\phi)} = 2\beta\lambda.$$

Since $\beta < 0$ in the equilibrium found, only v_3^- converges to the value of v_3 in the equilibrium studied.

2. Rational expectations implies that $\hat{q}_3 = q_3$. Straightforward algebra shows that, using v_3^-, this condition reduces to

$$\begin{aligned} 2(q_3-1) &= -\Gamma(q_3) - \sqrt{\Gamma^2(q_3) - 4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2} \\ &= \frac{4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2}{-\Gamma(q_3) + \sqrt{\Gamma^2(q_3) - 4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2}}. \end{aligned} \tag{A.40}$$

where the dependence of Γ on q_3 is being made explicit. For sufficiently small Δ, however, $(1-\Delta\phi)^2 - e^{\Delta r} < 0$ and so $-\Gamma(q_3) > 0$ for all $q_3 \geq 1$. The linearity of both $2(q_3-1)$ and $\Gamma(q_3)$ in q_3 then yields the existence of q_3^* such that the previous equality holds.

Equation for λ. For notational simplicity, we show this for $\mu = \rho = \delta = 0$ as the means and intercepts do not affect the sensitivity of beliefs.

Define the matrices

$$X := \begin{bmatrix} \theta \\ Y \end{bmatrix};\ A_\Delta := \begin{bmatrix} 1-\kappa\Delta & 0 \\ \alpha\Delta & 1-(\phi-\beta\lambda)\Delta \end{bmatrix};\ B := \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_\xi \end{bmatrix}\ \vec{\epsilon} := \begin{bmatrix} \epsilon^\theta \\ \epsilon^\xi \end{bmatrix}$$

where the shocks are orthogonal Gaussian white noise processes, and notice that

$$X_{(j+1)\Delta} = A_\Delta X_{j\Delta} + \sqrt{\Delta} B\vec{\epsilon}_{(j+1)\Delta},\ j \in \mathbb{N}.$$

The solution to this difference equation is given by

$$X_{(j+1)\Delta} = A_\Delta^{j+1} X_0 + \sqrt{\Delta} A_\Delta^{j+1} \sum_{i=0}^{j} A_\Delta^{-(j+1-i)} B\vec{\epsilon}_{(j+1-i)\Delta}.$$

To obtain a stationary Gaussian process, therefore, we impose first that X_0 is Gaussian and independent of $(\vec{\epsilon}_{j\Delta})_{j\in\mathbb{N}}$. Moreover, stationary requires that $\vec{\mu} := \mathbb{E}[X_0] = 0$, so as to obtain $\mathbb{E}[X_{j\Delta}] = 0$ for all $j \in \mathbb{N}$. In addition, letting Λ_Δ denote the candidate covariance matrix of

$(X_{j\Delta})_{j\in\mathbb{N}}$, we must have that

$$\Lambda_\Delta = A_\Delta^{j+1}\Lambda_\Delta(A_\Delta^{j+1})^\top + \Delta A_\Delta^{j+1}\left[\sum_{i=0}^{j} A_\Delta^{-(j+1-i)}B^2(A_\Delta^{-(j+1-i)})^\top\right](A_\Delta^{j+1})^\top,\ \forall j\in\mathbb{N}.$$

Moreover, taking consecutive differences leads to

$$\begin{aligned} 0 \;=\; & A_\Delta^j \Big\{ A_\Delta\Lambda_\Delta A_\Delta^\top - \Lambda_\Delta \\ & + \underbrace{\Delta A_\Delta\left[\sum_{i=0}^{j} A_\Delta^{-(j+1-i)}B^2(A_\Delta^{-(j+1-i)})^\top\right]A_\Delta^\top - \Delta\left[\sum_{i=0}^{j-1} A_\Delta^{-(j-i)}B^2(A_\Delta^{j-i})^\top\right]}_{=\Delta B^2}\Big\}(A_\Delta^j)^\top \end{aligned} \tag{A.41}$$

and thus Λ_Δ is defined by the equation

$$A_\Delta\Lambda_\Delta A_\Delta^\top - \Lambda + \Delta B^2 = 0.$$

Straightforward algebra leads to the following equations for the unknowns $\Lambda_{11} = \theta_{\mathrm{j}\Delta}$, $\Lambda 12 = \Lambda_{21} = \mathrm{Cov}[\theta_{j\Delta}, Y_{j\Delta}]$, and $\Lambda_{22} = \mathrm{Var}[Y_{j\Delta}]$, $j\in\mathbb{N}$:

$$\begin{aligned} \Lambda_{11}(1-\kappa\Delta)^2 - \Lambda_{11} + \Delta\sigma_\theta^2 &= 0 \\ \Lambda_{11}\alpha\Delta(1-\kappa\Delta) + \Lambda_{12}(1-(\phi-\beta\lambda)\Delta)(1-\kappa\Delta) - \Lambda_{12} &= 0 \\ \Lambda_{11}(\alpha\Delta)^2 + 2\Lambda_{12}(1-(\phi-\beta\lambda)\Delta)\alpha\Delta + \Lambda_{22}(1-(\phi-\beta\lambda)\Delta)^2 - \Lambda_{22} + \Delta\sigma_\xi^2 &= 0 \end{aligned} \tag{A.42}$$

This system has as a solution

$$\begin{aligned} \Lambda_{11}(\Delta) &= \frac{\sigma_\theta^2}{2\kappa-\kappa^2\Delta} \\ \Lambda_{12}(\Delta) &= \frac{\alpha\sigma_\theta^2(1-\kappa\Delta)}{[2\kappa-\kappa^2\Delta][\phi-\beta\lambda+\kappa-(\phi-\beta\lambda)\kappa\Delta]} \\ \Lambda_{22}(\Delta) &= \frac{1}{2(\phi-\beta\lambda)-(\phi-\beta\lambda)^2\Delta}\left[\sigma_\xi^2 + \frac{\sigma_\theta^2\alpha^2\Delta}{2\kappa-\kappa^2\Delta} + \frac{2\alpha[1-(\phi-\beta\lambda)\Delta]\sigma_\theta^2\alpha(1-\kappa\Delta)}{[2\kappa-\kappa^2\Delta][\phi-\beta\lambda+\kappa-(\phi-\beta\lambda)\kappa\Delta]}\right]. \end{aligned}$$

(in particular, observe that we recover the continuous-time expressions for all the Λ's by letting $\Delta\to 0$ and replacing λ by λ_0.) To conclude, by the projection theorem for Gaussian random variables,

$$M_t = \mu_1 + \frac{\mathrm{Cov}[\theta_t, Y_t]}{\mathrm{Var}[Y_t]}[Y_t-\mu_2] \underbrace{=}_{\mu_1=\mu_2=0} \frac{\mathrm{Cov}[\theta_t, Y_t]}{\mathrm{Var}[Y_t]}Y_t$$

which leads to λ_Δ satisfying the equation

$$\lambda = \frac{\Lambda_{12}(\Delta, \lambda)}{\Lambda_{22}(\Delta, \lambda)}.$$

After straightforward algebra, the equation reduces to

$$\lambda = \frac{\sigma_\theta^2 \alpha (1-\kappa\Delta)[2(\phi-\beta\lambda)-(\phi-\beta\lambda)^2\Delta]}{\sigma_\xi^2[2\kappa-\kappa^2\Delta][(\phi-\beta\lambda)(1-\kappa\Delta)+\kappa]+\sigma_\theta^2\alpha^2[2-\kappa\Delta-(\phi-\beta\lambda)(1-\kappa\Delta)\Delta]}.$$

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