Price plans and the real effects of monetary policy *

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March 22, 2016

Preliminary draft

Abstract

We analyze a sticky price model where a firm chooses a price plan, namely a set of 2 prices. Changing the plan entails a menu cost, but either price in the plan can be charged at any point in time. The setup generates a persistent reference price level and short lived deviations from it, and a decreasing hazard function for price changes, consistent with some datasets. We analytically solve for the optimal policy and for the cumulative output response caused by a monetary shock. We show that the temporary price changes substantially increase the flexibility of the aggregate price level.

JEL Classification Numbers: E3, E5

Key Words: sticky prices, menu cost models, sales, temporary price changes, reference prices, price plans, price flexibility.

*First draft July 2014. We are grateful to seminar participants at Northwestern University, Banque de France and the 2015 Hydra conference, and to our discussants John Leahy and Oleksiy Kryvtsov for many useful comments. Part of the research for this paper was sponsored by the ERC advanced grant 324008.

1 Introduction

Transitory price changes, i.e. temporary deviations above or below a reference price level, appear in many datasets. Kehoe and Midrigan (2015) conclude that above 70% of price changes in the BLS data are temporary. Such price changes do not fit neatly in the simple theory of menu costs, but they make a large difference on the measurement of the frequency of price changes. This difference is apparent in the way different authors have approached the treatment of sales in the data: some authors, such as Bils and Klenow (2004), count sales as price changes since they view temporary price changes as a source of price flexibility. Others, such as Golosov and Lucas (2007) or Nakamura and Steinsson (2008), exclude sales from the counting of price changes believing they are not a useful instrument to respond to shocks. This paper proposes a simple theoretical model, that produces transitory as well as persistent price changes. The objective is to provide a theoretical benchmark to asses whether the temporary price changes matter for the propagation of monetary shocks.

We develop the analysis using a menu cost model with idiosyncratic shocks in the spirit of Eichenbaum, Jaimovich, and Rebelo (2011). The setup modifies the model of Golosov and Lucas (2007) by assuming that upon paying the menu cost the firm can choose 2 prices, instead of 1, say $P \equiv \{p_H, p_L\}$. We call the set P a price plan, which is a singleton in the standard model, and the firm is free to change prices as many times as it wishes within the plan. Changes of the plan instead require paying the fixed cost ψ. As customary, we assume that the firm's marginal cost are hit by idiosyncratic shocks which affect the firm's markup and create a motive for adjusting the price. We analytically solve for the optimal policy of a firm, and for the cumulative impulse response function of output to a once and for all monetary shock. The model generates a persistent "reference" price level and short lived deviations from it, as seen in many datasets. Moreover we show that the introduction of price plans generates a decreasing hazard function for price changes, a feature we find interesting because is connects with empirical evidence by e.g. Nakamura and Steinsson (2008) as well as by Campbell and Eden (2014). The ultimate objective is to compare the economy with 2-

price plan with an equivalent menu-cost economy featuring the same number of *plan* changes. We find that, for a small monetary shock, the introduction of the plan yields a cumulative output response that is 1/3 of the one produced by the menu cost economy. The smaller effect of the monetary shock is due to the extra flexibility, delivered by the price plans, to react to the monetary shock.

A few theoretical contributions analyze the relevance of temporary price changes using micro founded sticky price models. Guimaraes and Sheedy (2011) develop a model of sales, i.e. of temporary low prices, where the firms' profit maximizing behavior implies a randomization between a regular price and a low price. They assume that the timing for the adjustment of the regular price follows an exogenous rule a la Calvo. The real effects of monetary shocks in their model are essentially identical to the real effects produced by a Calvo model. Kehoe and Midrigan (2015) setup a model where the firm faces a regular menu cost for permanent price changes and a smaller menu cost for temporary price changes (which last for 1 month). The model implies that firms will not use the temporary price changes to respond to monetary policy shocks. Both of these models imply that temporary price changes are not a relevant measure of the firm's price flexibility to respond to aggregate shocks. Our results differ starkly from those because the implementation of a temporary price change is state dependent (as in Kehoe-Midrigan) and costless. Hence firms use the temporary price changes to respond to a monetary shock. We thus offer a theoretical model where the temporary price changes do matter for the transmission of monetary policy. Analyzing empirically whether firm use temporary price changes to respond to the aggregate macroeconomic conditions, as in the recent papers by Kryvtsov and Vincent (2014) and Anderson et al. (2015), is a useful way to select between these alternative models.

Our model is uniquely poised to analyze new evidence on high frequency (weekly or daily) price changes. It has been noticed empirically that high frequency observations tend to display a larger number of price changes, likely due to a time aggregation issue. Our model provides a tractable laboratory to analyze time-aggregation which is useful to the frequencies of price changes across dataset and to assess its consequences. For instance the daily Japanese data discussed in Sudo, Ueda, and Watanabe (2014) display a much larger number of price changes than the US weekly data of Eichenbaum, Jaimovich, and Rebelo (2011), in spite of the fact that the frequency of regular price changes is similar across the two dataset (the annual number of price changes the Japanese scanner daily data is just above three times the average number of price changes in the US weekly scanner data). We show that this is consistent with a model in which the US and Japan have identical real effects of monetary policy and the only difference pertains to the frequency with which the data are sampled.

We use our simple, but very tractable model, to compare the cumulative output effect of a once and for all monetary shock. In particular, we are interested in comparing models with price plans to models without them, i.e. to study what is change in the impulse response of output due to the extra price flexibility. When we conduct such a comparison we need to discuss what we kept constant across the two economies, i.e. the one with price plans and the one without them. The summary result is that we show that in this model, the extra flexibility of the price changes within a plan reduces the output effect to monetary policy to about a third of an otherwise similar menu cost economy, where we keep both economies with the same number of regular price changes. Below we explain this results in more detail.

The paper is organized as follows. Section 2 sets up the price setting problem and derives the firm optimal policy. Section 3 derives analytically the cumulative output response of the economy to an unexpected small monetary shock (increase of the money supply). The results in this section allow a straightforward comparison with the size of the real effects produced by the canonical menu cost model. The model with price plans produces two types of price changes: changes within the plan, i.e. oscillations between p_H and p_L, and changes of the plan. Thus when matching the model to the frequency of price changes in the data one has to take a stand on what constitutes a "price change": all types of price changes, including short lived temporary oscillations between p_H and p_L, or only the "regular" persistent (i.e. plan) changes. Section 4 formalizes the different definitions for the frequency of price changes and compares the real effects of a monetary shock across models with and without plans while keeping fixed alternative definitions of frequency. The existence of price plans allows to have lots of price changes that are reversed, since they alternative between the prices within the plan. These reversals have several implications. First, we produce an analytical characterization showing that the presence of price plans allows to have a large number of price changes and simultaneously being able to have many spells where prices stay at the model price for that interval, both features seen in scanner price data both in Eichenbaum, Jaimovich, and Rebelo (2011). Second, in Section 2.2, we show that these reversals gives a novel economically motivated reason for hazard rates of price changes to be decreasing in the duration of the price spells, a feature found in several data sets. Finally, in Section 3 we show how to characterize the output effect purely in terms of a notion of the frequency of price changes that is reminiscent of the notion of "regular" or "reference" price changes introduced by Eichenbaum, Jaimovich, and Rebelo (2011).[1] We find that if we compare two economies one with price plans and the other without them the cumulative impulse response function is 1/3 of the original. The latter economy is a standard menu cost model, where each price change is a change in plans, but the plans contain only one price, as opposed to the set $\{p_L, p_H\}$ of the baseline case. We obtain this results regardless of the rest of the parameters of the model, provided that the two economies have parameters so that they have the same average number of price plans per year. In other words, if we identify prices changes that occur when plans are changed as regular price changes, we obtain the 1/3 ratio of cumulative output impulse responses when the two economies have the same frequency of regular price changes. The economics of this difference is that on impact many firms switch from the low p_L to the high price wp_H as a reaction to permanent common positive cost shock, thus increasing the aggregate price level (i.e. with price changes within a plan), implying more flexible prices. We note that in steady state there are lots of firms that are ready to

[1] A related, complementary approach, is whether one can find empirical evidence on the cyclicality of sales, or in their response, either in frequency of depth, after a monetary shock.

make this change, which is the same reason why this model can produce many reversals. Controlling for the same number of regular price changes across the two economies makes them comparable so that the effect of the monetary shock in the aggregate price level that occur when price plans are changed is the same. Thus, the extra flexibility of the economy with plans is due exclusively to price changes within the plans. Section 5 concludes and discusses some extensions and directions for future work.

2 The firm's problem

Consider a firm whose (log) profit-maximizing price at time t, $p^*(t)$, follows the process

$$\mathrm{d}p^*(t) = \sigma \ \mathrm{d}W(t)$$

where $W(t)$ is a standard brownian motion with no drift and i.i.d. innovations with a variance per unit of time given by σ^2. We will call the profit-maximizing price $p^*(t)$ the *ideal* price. The firm that charges the (log) price $p(t)$ at time t has a loss, relative to what it will get charging the desired price, equal to $B\,(p(t) - p^*(t))^2$. The firm discounts losses at rate $r \geq 0$. A firm making decisions at time t observes the history of $\{p^*(s)\}$ from the initial time $s = 0$ up to the current time $s = t$.[2]

At any moment of time the firm has a price plan available. A price plan (or menu of prices) is given by two numbers $P \equiv \{p^L, p^H\}$ so that the firm can charge any (log) prices at t on this set, i.e. $p(t) \in P$ at t. At any time the firm can pay a cost ψ and change its price plan to any $P \in \mathbb{R}^2$. We let P_i, be the i^{th} price plan and let τ_i be the stopping time at which this i^{th} price plan was chosen, so this plan will be in effect between τ_i and τ_{i+1}. The stopping times and the price plans can depend on all the information available until the time that are chosen. The problem for the firm is then to choose the stopping times

[2]See Appendix B in Alvarez and Lippi (2014) for a detailed derivation of these expressions as a second order approximation to the general equilibrium problem in which firms face a CES demand for their goods.

$\tau_1 < \tau_2 < \tau_3 < \cdots$, the corresponding price plans $P_1, P_2, P_3, ...$, as well as the two prices $p(t) \in P_i$ at times $t \in [\tau_i, \tau_{i+1})$. The state of the firm's problem is given by the triplet: $\{p^*, p^L, p^H\}$, where p^* is the current desired price level, and where $\{p^L, p^H\}$ is the price plan currently available. Thus the firm's optimization problem is

$$V(p^*, p^L, p^H) =$$
$$\min_{\{\tau_i, P_i\}_{i=1}^{\infty}} \mathbb{E}\left[\sum_{i=1}^{\infty} \int_{\tau_{i-1}}^{\tau_i} e^{-r\,t} \min_{p(t)\in P_{i-1}} B\ (p(t) - p^*(t))^2\ dt + \sum_{i=1}^{\infty} e^{-r\tau_i}\psi \ \middle|\ p^* = p^*(0), P_0 = \{p^L, p^H\} \right]$$

where $\tau_0 = 0$ and $\{\tau_i, P_i\}_{i=1}^{\infty}$ are the (stopping) times and corresponding price plans. The key novel element compared to the standard menu-cost problem is the *min* operator which appears inside the square bracket: at each point in time the firm can freely choose to charge any of the prices specified by the Plan, for instance the plan P_0 lets the firm free to choose either p^L or p^H at any point in time.

Symmetry of the value function. We note that $V : \mathbb{R}^3 \to \mathbb{R}$ is symmetric in the following sense:

$$V(p^*, p^L, p^H) = V(p^* + \Delta, p^L + \Delta, p^H + \Delta) \ \text{ for all } \Delta, p^*, p^L, p^H, \tag{1}$$

a property that follows directly from the fact that the period return function is a function of the difference between the price charged $p(t)$ and the desired price $p^*(t)$.[3]

Considering the value function right after the decision of a change of plan, and in particular

[3] A second symmetry property is: $V(p^*, p^* - \tilde{g} - \Delta, p^* + \tilde{g}) = V(p^*, p^* - \tilde{g}, p^* + \tilde{g} + \Delta)$ for all $\Delta, p^*, \tilde{g}$ which follows from the symmetric objective function $(p(t) - p^*(t))^2$ and the lack of drift of the desired price.

right after the fixed cost ψ has been paid and right after the plan $\{p^L, p^H\}$ has been adopted:

$$
\begin{aligned}
& V(p^*, p^L, p^H) = \\
& \min_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-r\,t} \min_{p(t)\in\{p^L,p^H\}} B\,\left(p(t)-p^*(t)\right)^2\,dt + e^{-r\tau}\left[\psi + V^*\left(p^*(\tau)\right)\,\right] \,\middle|\, p^* = p^*(0)\right] \\
& \text{where } V^*\left(p^*\right) \equiv \min_{(\bar{p}^L,\bar{p}^H)\in\mathbb{R}^2} V\left(p^*, \bar{p}^L, \bar{p}^H\right)
\end{aligned}
$$

Using the property in equation (1) that V depends on differences only, then we can conclude that $V^*\left(p^*\right)$ is a constant which, abusing notation, we denote by $\bar{V}^*$. We can then measure the value function right after the fixed cost has been paid, but just before the price plan was chosen, to obtain

$$
\bar{V}^* = \min_{\tau, p^L, p^H} \mathbb{E}\left[\int_0^{\tau} e^{-r\,t} \min_{p(t)\in\{p^L,p^H\}} B\,\left(p(t)-p^*(t)\right)^2\,dt + e^{-r\tau}\left[\psi + \bar{V}^*\right] \,\middle|\, p^* = p^*(0)\right] \quad (2)
$$

where we used that, given the symmetry stated in equation (1), the value of p^* is immaterial for the level of the value function.

Centered and symmetric price plans. Let p_i^* be the desired price at the time at which the i^{th} price plan is decided, i.e. define $p_i^* \equiv p^*(\tau_i)$ where $\tau_i \leq t < \tau_{i+1}$. The optimal policy has the following property:

Lemma 1. *The optimal price plans are centered and symmetric, i.e. they satisfy*

$$P_i = \{p_i^* - \tilde{g}, p_i^* + \tilde{g}\}$$

This lemma follows because the objective function in the right hand side of equation (2) is symmetric, due to the symmetry of the objective function and due to the symmetry of the distribution of the standard Brownian motion. Hence, the optimal plan must be centered and symmetric.

Redefined value function. Using the property in equation (1) and Lemma 1, it follows that we can write the value function for a centered symmetric price plan, decided when the desired price was p_i^* and the current desired price is p^*, as

$$v(g\,;\,\tilde{g}) \;\equiv\; V(p^*\,,\,p_i^* - \tilde{g}\,,\,p_i^* + \tilde{g}) \qquad \text{with} \quad g(t) \equiv p^*(t) - p_i^* = p^*(t) - p^*(\tau_i)$$

where we define the new state variable g and the second equality uses the definition of p_i^*. In words, g measures the current desired price relative to the desired price at the time of the last change in plans. With this new definition the state g is reset to zero every time a new price plan is chosen. We refer to the state g as the *normalized desired price*. Figure 1 illustrates a sample path for the main variables of interest in this investigation: it displays the normalized desired price $g(t) = p*(t) - p^*(\tau_i)$, the (normalized) price charged by the firm at each point in time $p(t) - p^*(\tau_i) = \pm\tilde{g}$, and the barriers that determine the revision of the price plan $\pm\bar{g}$. The stationarity produced by the normalization of the prices (both desired and actual) simplifies the analysis substantially.

2.1 Firm's optimal policy

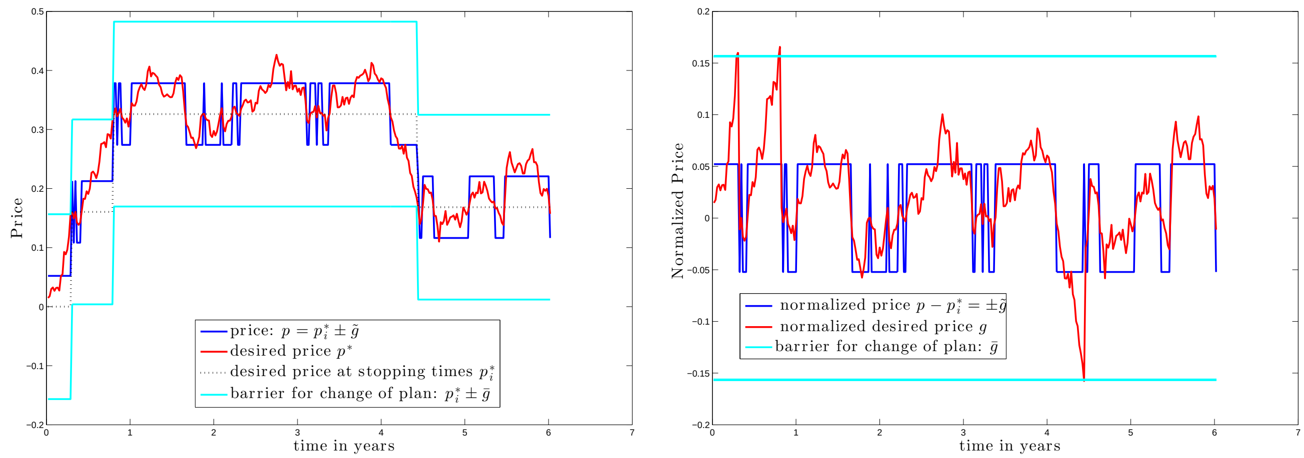
We look for an optimal policy that is described by two positive numbers $0 < \tilde{g} < \bar{g}$. The value of $\tilde{g}$ is the width of a centered price plan as defined in Lemma 1. The value of $\bar{g}$ is the smallest change in the desired price, since the last price plan was decided, which triggers a new price plan. The value of $\bar{g}$ defines the inaction set for which the centered price plan $\tilde{g}$ applies.

In the inaction region $g \in [-\bar{g}\ \bar{g}]$ the value function solves:

$$r\,v(g;\tilde{g}) = \min_{\hat{g} \in \{-\tilde{g}, \tilde{g}\}} B\ (g - \hat{g})^2 + \frac{1}{2}\sigma^2\, v''(g;\tilde{g})$$

Figure 1: Example of model generated sample paths

Sample path Normalized sample path



Note that: $\min_{\hat{g}\in\{-\tilde{g},\tilde{g}\}} (g-\hat{g})^2 = \min\{(g-\tilde{g})^2 \; , \; (g+\tilde{g})^2\} = g^2+\tilde{g}^2-2|g|\tilde{g}$, so we can write the value function in the inaction region $g \in [-\bar{g} \; \bar{g}]$ as:

$$r\, v(g;\tilde{g}) = B\, \left(g^2+\tilde{g}^2-2|g|\tilde{g}\right) + \frac{1}{2}\sigma^2\, v''(g;\tilde{g})$$

Since at any time price plans can be changed by paying a cost ψ the value function must satisfy for all g: $v(g;\tilde{g}) \leq \psi + \min_{\tilde{g}'} v(0;\tilde{g}')$. Note that $\min_{\tilde{g}'} v(0;\tilde{g}') = \bar{V}^*$ as defined in equation (2).

The value function $v(g;\tilde{g})$ and optimal thresholds $\tilde{g}, \bar{g}$ must satisfy the following conditions, where the primes denote the partial derivatives with respect to the first argument of the function:

$$
\begin{aligned}
\frac{\partial}{\partial \tilde{g}} v(0;\tilde{g}) &= 0\,, && (3)\\
v(\bar{g};\tilde{g}) &= \psi + v(0,\tilde{g})\,, && (4)\\
v'(\bar{g};\tilde{g}) &= 0\,, && (5)\\
r\,v(g,\tilde{g}) &= B\,\left(\,|g| - \tilde{g}\,\right)^2 + \frac{1}{2}\sigma^2\,v''(g;\tilde{g}) \;\text{ for all } -\bar{g} \le g \le \bar{g}, g \neq 0\,, && (6)\\
\lim_{g\uparrow 0} v(g;\tilde{g}) &= \lim_{g\downarrow 0} v(g;\tilde{g}) \;\;. && (7)
\end{aligned}
$$

These conditions are of two types. First in the inaction range the value function should satisfy the Hamilton-Jacobi-Bellman equation equation (6), except at interior points where the periods return is non-differentiable, where instead we have equation (7). Notice that the function $v(\cdot,\tilde{g})$ is symmetric around $g = 0$ and hence its first derivative should satisfy:

$$
v'(0;\tilde{g}) = 0\,, \tag{8}
$$

which is the familiar optimality conditions for the return point in an sS model. Equation (8) and the symmetry condition $v(g;\tilde{g}) = v(-g;\tilde{g})$ in $g \in [0,\bar{g}]$ can replace the boundary condition of equation (7). Second, we have the familiar value matching condition at the boundary of the inaction set for an sS model, namely equation (4), and the smooth pasting condition in equation (5). Lastly, the prices within the plan should be optimally decided, which requires equation (3) to hold.

Optimal reset prices $\tilde{g}$. We turn to the determination of $\tilde{g}$, given a value of the inaction threshold $\bar{g}$. Consider a firm that have just reset the price plan until this plan is changed, and that *takes as given* the value of $\bar{g}$. Let $\tau(\bar{g})$ the stopping time associated with threshold

$\bar{g}$. The relevant objective function for this problem is:

$$\min_{\tilde{g}} \mathbb{E}\left[\int_0^{\tau(\bar{g})} e^{-rt}\ \left(\tilde{g}^2 - 2\,\tilde{g}\,|g(t)| + g(t)^2\right)\ dt \;\middle|\; g(0)=0\right]$$

Note that this is a one dimensional quadratic minimization problem, with a convex objective function. The first order condition for this problem is:

$$\tilde{g} = \frac{\mathbb{E}\left[\int_0^{\tau(\bar{g})} e^{-rt}\,|g(t)|\,dt \;\middle|\; g(0)=0\right]}{\mathbb{E}\left[\int_0^{\tau(\bar{g})} e^{-rt}\,dt \mid g(0)=0\right]}$$

We compute the numerator and denominator separately: consider the following functions computing the corresponding expected discounted values as function of an arbitrary g:

$$a(g) = \mathbb{E}\left[\int_0^{\tau(\bar{g})} e^{-rt}\,|g(t)|\,dt \;\middle|\; g(0)=g\right] \quad , \quad d(g) = \mathbb{E}\left[\int_0^{\tau(\bar{g})} e^{-rt}\,dt \mid g(0)=g\right]$$

We are interested in evaluating them at $g=0$ to get: $\tilde{g} = a(0)/d(0)$. Analysis of these functions gives the following characterization of $\tilde{g}$.

Proposition 1. Let $\tilde{g}$ be the optimal decision rule for the prices within a plan given a barrier $\bar{g}$ for change of plans. The value of the prices is given by a function ρ of the variable $\phi \equiv r\,\bar{g}^2/\sigma^2$, satisfying:

$$\tilde{g} = \frac{a(0)}{d(0)} = \bar{g}\ \rho(\phi)$$

The function $\rho(\cdot)$ satisfies:

$$\rho(\phi) = \frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi}\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2\right)} \quad \text{so that} \quad \rho(0) = \frac{1}{3}\,,\ \rho'(\phi) < 0\,,\ \lim_{\phi\to\infty}\rho(\phi) = 0 \ \text{and} \ \lim_{\phi\to\infty}\rho(\phi)\,\sqrt{2\phi} = 1\,.$$

Thus the ratio $\tilde{g}/\bar{g}$ is equal to $1/3$ for small $\phi \equiv r\bar{g}^2/\sigma^2$, and an even smaller fraction for larger values. Indeed as ϕ becomes arbitrary large then the fraction converges to zero, with the value of $\tilde{g} \to \sigma/\sqrt{2r}$.

We now turn to the characterization of the optimal decision concerning the width of the inaction range, $\bar{g}$, taking as given $\tilde{g}$. This invokes solving explicitly the value function, which we show is indeed differentiable at $g = 0$, in spite of the fact that the objective function is not. The following proposition, together with Proposition 1, shows that indeed the optimal policy is given by two thresholds $\tilde{g}, \bar{g}$ and gives a complete characterization of the value of these thresholds.

PROPOSITION 2. The optimal policy of the problem is given by two thresholds $\bar{g}$ and $\tilde{g}$. The value of $\bar{g}$ is the unique solution of the equation:

$$\eta^2 \, r \, \frac{\psi}{B} = \kappa \left(\eta \bar{g} \right) \quad \text{with} \quad \eta = \sqrt{2 \, r/\sigma^2} \, , \tag{9}$$

where the function κ is given by:

$$\kappa \left(x \right) \equiv \left[1 - 2 \, \rho \left(x^2/2 \right) \right] \left[x^2 - 2 \, x \, \frac{\left(e^x + e^{-x} - 2 \right)}{\left(e^x - e^{-x} \right)} \right]$$

where the function $\rho(\cdot)$ is given in Proposition 1. The function κ is strictly increasing, with $\kappa(0) = 0$, $\lim_{x \to \infty} \kappa(x) = \infty$, for small values we have: $\kappa(x) = x^4/36 + o(x^4)$, and for large values $\kappa(x)/x^2 \to 1$ as $x \to \infty$. The value of $\tilde{g}$ is obtained by $\tilde{g} = \bar{g} \, \rho \, (\eta^2 \, \bar{g}^2/2)$.

We can use Proposition 2 to give a simple approximation to the solution $\bar{g}$, which is very accurate for small values of $(r^2/\sigma^4) \, \psi/B$. Note that this is accurate if either the fixed cost ψ is small, or if r is small. In this case we can disregard the term of order higher than x^4 and write:

$$\tilde{g} = \frac{1}{3} \, \bar{g} \quad \text{and} \quad \bar{g} = \left(18 \, \frac{\psi}{B} \, \sigma^2 \right)^{1/4} \tag{10}$$

Note also that this approximation for $\bar{g}$ does not invoke r. Indeed differentiating equation (9) with respect to r one can show that $\partial \bar{g}/\partial r = 0$ when evaluated at $r = 0$.It is interesting to compare the expression for $\bar{g}$ in equation (10) with the one that obtains in the standard menu cost model, which we refer to as the Golosov-Lucas model, or simple GL model for short. In the GL model $\tilde{g} = 0$ since each price plan has only one price. The expression for $\bar{g}$ in such a model is identical expect that the factor 18 it has a factor 12, or in other words, it will lead to the same value of $\bar{g}$ if it had a fixed cost three times higher. This is intuitive: if the firm had the same fixed cost but it has access to price plans the firm choses to have a wider band, by a factor of $3^{1/4}$ or approximately 32% wider. Note that otherwise it is the same quartic root expression than in Barro (1972) or Dixit (1991). Indeed the quartic root is not obvious at all in this context, since the period objective function is not quadratic -as in these two papers -it is given by $(\tilde{g} - |g|)^2$, which includes the absolute value.[4]

While small values of $r^2\psi/B$ are the ones that typically apply in the standard menu cost model, it is also of interest to analyze the approximation for large values of $r^2\psi/B$ which is useful in models which feature poisson adjustments at a rate λ a la Calvo: for these models the proper discount rate is $r + \lambda$ and hence large values of ϕ are interesting. As the value of $\phi = r^2\psi/B$ becomes large, then the approximate solution becomes $\bar{g} = \sqrt{(\psi/B)/r}$, a limit identical to the one worked out in Alvarez, Le Bihan, and Lippi (2014) for the analogous case. Hence, for large values of $r^2\psi/B$ the presence of price plans with two values makes no difference on the threshold $\bar{g}$.

2.2 Hazard Rates

An interesting feature of the presence of price plans is that it produces a downward sloping function for the hazard rate of price changes, a feature that it is commonly found in the majority of the empirical studies. In this section we explain how to compute the hazard rates in this model and we explain why they are decreasing.

We compute the instantaneous hazard rate of price changes in two versions of our model. The first version, which is the one developed so far, has price plans that change when the absolute value of the normalized desired price $|g|$ reaches a critical value, the threshold that

[4]For instance, Dixit (1991) finds that when the period objective function is purely $|g|$ the approximation for the optimal rule has a cubic root.

we denote by $\bar{g}$. We refer to this version as the menu cost version, and we denote the hazard rate for a price with duration $t > 0$ as $h_{MC}(t)$. In Appendix C we also consider a version of the model where price plans are changed at (exogenous) exponentially distributed times, in which case we denote the hazard rate for price changes by $h_{exp}(t)$. In both cases we provide an analytical solution to the hazard rate of price changes. These analytical expressions depends on only one parameter, namely N_p: the expected number of price plans changes per unit of time. In the benchmark price plan model, the expected number of price plan changes per unit of time has a simple expression $N_p = \bar{g}^2/\sigma^2$, an expression whose derivation and interpretation we return in Section 4. In the version where price plans are changes at exponentially distributed times, N_p is simply the expected number of price plan changes per unit of time. Both hazard rates are downward slopping, very much so for low durations, behaving approximately as $1/2t$ for low t, and they asymptote to different constants. The asymptote for h_{MC} is a multiple of the number of price plan changes per year, namely $\pi^2/2\, N_p \approx 5\, N_p$. While the asymptote for in the exponential case is simply N_p. Appendix B provides more information and the exact definition of the hazard rates, and on their analytical characterization. We summarize that analysis in the following proposition:

Proposition 3. The hazard rate h_{MC} for the baseline model with price plans is:

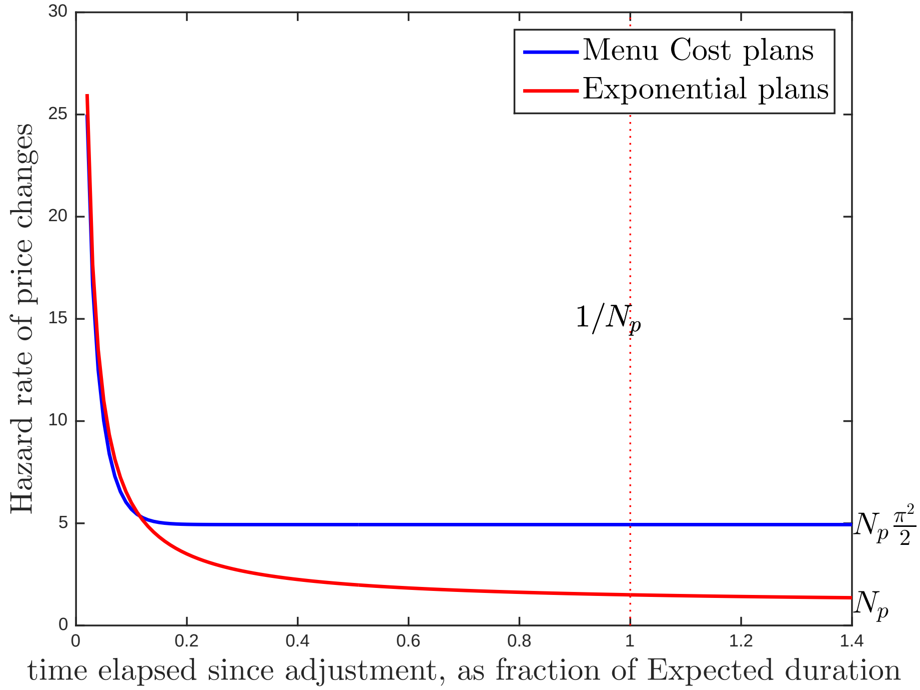
$$h_{MC}(t) = \sum_{m=1,3,5,\ldots}^{\infty} m^2\, N_p\, \frac{\pi^2}{2}\, \theta(t, m; N_p) \quad \text{where} \tag{11}$$

$$\theta(t, m; N_p) \equiv \frac{e^{-t\, m^2\, N_p\, \frac{\pi^2}{2}}}{\sum_{m'=1,3,5,\ldots}^{\infty} e^{-t\,(m')^2\, N_p\, \frac{\pi^2}{2}}}$$

where for each $t > 0$, the $\theta(t, \cdot; N_p)$ are non-negative and add up to one over $m = 1, 3, 5, \ldots$. The hazard h_{MC} has the following properties:

$$h'_{MC}(t) < 0 \text{ for } t > 0\,,\ \lim_{t \to 0} h_{MC}(t) = \infty\,,\ \lim_{t \to 0} h_{MC}(t)\, t = \frac{1}{2}\,,\ \text{ and } \lim_{t \to \infty} h_{MC}(t) = \frac{\pi^2}{2} N_p\,.$$

Figure 2: The hazard rate of price changes in two models



For the case with exponentially distributed price plans times we have:

$$h_{exp}(t) = N_p + \frac{1}{2\,t} \text{ for all } t > 0\,. \tag{12}$$

Figure 2 plots the two hazard rates. As explain in the proposition the hazard rate depend on one parameter, the expected number of plan changes, and hence $1/N_p$ is the expected time between price changes. In the figure we normalize N_p to one, so that duration, i.e. time, in the horizontal axes can be interpreted relative to average duration of a plan. As it can be seen they are very similar for short durations, say for durations below 10% of the expected duration of a price plan, and very similar to the function $1/2t$. They differ in the level of asymptotic hazard rate, which is almost reached if prices last the expected value of a price plan, and much sooner for the model based on menu costs.

We finish with an intuitive discussion of why in the model with price plans hazard rates are decreasing, while in the model without them they aren't. For instance, in the standard Calvo model of price setting without plans, hazard rates are constant by assumption. Likewise, hazard rates are increasing in the canonical menu cost model, such as in Golosov and Lucas (2007), since right after a price change the firm charges the profit maximizing price in the menu cost, so that the probability to observe a new price change right after an adjustment is near zero. Instead, in the case of price plans with with two prices, the firm is indifferent between charging $p_i^* \pm \tilde{g}$ right after a price change. Given that the upper threshold is preferred when $g > 0$ and the lower threshold is preferred when $g < 0$, the fact that $g = 0$ right after a price change makes it very likely that its sign with reverse many times, which triggers lots of price changes. We can also understand why $h(t) \approx 1/(2t)$ for small duration t. The reason is that a Brownian motion has, for a small enough time interval, approximately the same probability of an increase than a decrease, so if $g(t) > 0$, but $g(t)$ is small, then with probability roughly $1/2$ it returns to zero, and thus the hazard rate is $1/(2t)$.

3 Cumulative output response to a monetary shock

In this section we analyze the aggregate output's cumulative impulse response to an unexpected once and for all monetary shock. We consider an economy whose cross section distribution of prices is invariant at the time the monetary shock occurs.

Price changes. The distribution of price changes takes four values. We denote a price change at time t as $\Delta p(t)$. Price changes happens either within a plan, when g crosses zero, or when there is a change of plans.

A price change within a plan is given by $\Delta p(t) = \pm 2\,\tilde{g}$ since the price changes from the top (bottom) to the bottom (top) of the barriers. This price change happens at times $\tau_i < t < \tau_{i+1}$ whenever $g(t)$ hits zero. In particular if g crosses zero from above, then $\Delta p(t) = -2\,\tilde{g}$ and it if crosses from below $\Delta p(t) = +2\,\tilde{g}$.

A price change at the time of a plan-change occurs is $|g(\tau_i)| = \bar{g}$. Immediately after the plan-change g is reset to zero, and the firm is indifferent to charge a new price at the bottom or the top of the new barrier. We follow the rule of resetting the price (under the new plan) at the the same barrier, so that $|\Delta p(\tau_i)| = \bar{g}$. This is because if $g(\tau_i) = \bar{g}$, then at times t just before τ_i the firm charges the price at the top barrier, then it will charge a price for t right after τ_i at the top barrier, which gives the price change:[5]

$$\Delta p(\tau_i) = (p^*(\tau_i) + \tilde{g}) - (p^*(\tau_{i-1}) + \tilde{g}) = \bar{g}$$

Price gaps. As customary in the literature we find it useful to define the *price gap* as the difference between the price charged and the desired price. Denoting price gaps by $\hat{p}(t)$ the next line gives its definition and notes that price gaps are related to the normalized desired price g as follows:

$$\begin{aligned}\hat{p}(t) &\equiv p(t) - p^*(t) = [p^*(\tau_i) + \tilde{g} \text{ sgn}\,(g(t))] - [p^*(\tau_i) + g(t)] \\ &= \tilde{g} \text{ sgn}\,(g(t)) - g(t)\,, \quad \text{for } \tau_i \le t < \tau_{i+1}\end{aligned}$$

where sgn(x) is defined as

$$\text{sgn}(x) = \begin{cases} +1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Prices and Output's IRF to a monetary shock. We are interested in measuring the effect of an unexpected once and for all monetary shock of size $\delta > 0$ on output. In particular we consider the impulse response of output to such a shock, and focus on the area below such as IRF as a summary measure. To be precise, we start the economy at steady state, with zero inflation, and consider the effect of such shock. We let $\mathcal{P}(t, \delta)$ be the response of

[5] Alternatively, we could have followed the rule of resetting the price at the opposite barrier (which will make the price change smaller). In this case we will get: $\Delta p(\tau_i) = (p^*(\tau_i) - \tilde{g}) - (p^*(\tau_{i-1}) + \tilde{g}) = \bar{g} - 2\,\tilde{g}$. A similar argument shows that $\Delta p(\tau_i) = -\bar{g}$ or $\Delta p(\tau_i) = -\bar{g} + 2\,\tilde{g}$ if the price changes was triggered when $g(\tau_i) = -\bar{g}$.

the aggregate price level to a monetary shock of size δ after $t \geq 0$ units of time. Using the notation in Alvarez and Lippi (2014) we can denote the IRF of prices $t \geq 0$ periods after the shock by:

$$\mathcal{P}(t, \delta) = \Theta(\delta) + \int_0^t \theta(s, \delta) ds \tag{13}$$

The notation in equation (13) uses that the aggregate price level, just before the shock, is normalized to zero. So that $\Theta(\delta)$ is the instantaneous jump in the price level at the time of the monetary shock and $\theta(\delta, t)$ is the contribution to the price level at time t.

We consider models where the effect of output is proportional to the difference between the monetary shock and the price level, i.e. denoting by $Y(t, \delta)$ the impulse response of aggregate output t units of time after the shock of size δ as: $Y(t, \delta) = (1/\epsilon)[\delta - \mathcal{P}(t, \delta)]$, where ϵ is an elasticity that translates the increase in real balances (or real wages) into increases in output. Below we consider the summary measure $\mathcal{M}(\delta)$, the cumulative impulse response function:

$$\mathcal{M}(\delta) = \int_0^\infty Y(t, \delta)\, dt \equiv \int_0^\infty \frac{1}{\epsilon} \left[\delta - \mathcal{P}(t, \delta)\right] \, dt \tag{14}$$

Our approach to characterize equation (14) is to compute the corresponding measure for each firm, and then aggregate over firms using the steady state distribution.

The firm's expected contribution to cumulative output IRF. As announced, in what follows we compute $\hat{m}(g)$, the contribution to output of a firm that starts with a normalized price g until the first plan change. By contribution we refer to the expected discounted value of the (minus) the price gap defined above. A firm with a high price gap charges a high price and thus it contributes negatively to output. We stop the computation once a price plan is changed, because at the beginning of a price plan prices are as likely to go up as to go down and so their expected contribution to output is zero. We will use this function to compute the cumulative impulse response of output to a once and for all monetary shock by integrating the contribution of each firm $\hat{m}(g)$ over the measure of firms at the time of the shock $f(g)$. We start by defining $\hat{m}(g)$:

$$\hat{m}(g) = -\mathbb{E}\left[\int_0^{\tau} \hat{p}(t)\, dt \mid g(0) = g\right] \tag{15}$$

where τ is the stopping time indicating the next change of price plans. Note that by definition we have

$$\hat{m}(g) = m(g) - \tilde{g}\, D(g)$$

where

$$m(g) = \mathbb{E}\left[\int_0^{\tau(\bar{g})} g(t)\, dt \mid g(0) = g\right]$$

$$D(g) = s(g) - s(-g)$$

$$s(g) = \mathbb{E}\left[\int_0^{\tau(\bar{g})} 1_{g(t)\geq 0}\, dt \mid g(0) = g\right]$$

The solution for $m(g)$ is found in closed form by noting that $m(g)$ obeys the following differential equation $0 = -g + \frac{\sigma^2}{2} m''(g)$ with boundary conditions $m(0) = 0$ and $m(\bar{g}) = 0$, which gives $m(g) = -g\left(\frac{g^2 - \bar{g}^2}{3\sigma^2}\right)$.[6] The next lemma solves for $D(g)$.

Lemma 2. Given thresholds $\tilde{g}, \bar{g}$ the function $D(g)$ is:

$$D(g) \equiv s(g) - s(-g) = \begin{cases} \frac{g}{\sigma^2}\left[\bar{g} + g\right] & \text{if } g \in [-\bar{g}, 0] \\ \frac{g}{\sigma^2}\left[\bar{g} - g\right] & \text{if } g \in [0, \bar{g}] \end{cases}$$

Intuitively the function $s(g)$ measures the expected time that the normalized desired price will be positive $g(t) > 0$ until the next change of price plan, conditional on the current value

[6] We derived a more general version of this solution in Alvarez, Le Bihan, and Lippi (2014) to analyze the model by Golosov and Lucas and compare it to other models.

of g. Thus $D(g)$ measures the differential between the time spent with positive $g(t)$ in excess of the time spent with negative $g(t)$ is negative, conditional on the current g.

Output cumulative IRF's to a monetary shock. We use the function $\hat{m}$ derived above, as well as the invariant distribution of normalized desired prices $f(g)$, to define cumulative to compute the cumulative impulse of aggregate output for a once and form all shock to the money supply of size δ. The invariant density function is easily shown to be a triangular tent, namely $f(g) = (\bar{g} - |g|)/\bar{g}^2$ for $g \in (-\bar{g}, \bar{g})$.[7] We assume that the economy is in a steady state that corresponds to zero inflation when the unexpected permanent shock occurs. The shock increases permanently money, nominal wages, and aggregate nominal demand by δ log points. The cumulative response of output, relative to the steady state level, is given by:

$$\mathcal{M}(\delta) = \int_{-\bar{g}}^{\bar{g}} \hat{m}\,(g+\delta)\; f\,(g)\; dg \tag{16}$$

Note that the shock δ displaces all the normalized desired prices, increasing them by δ. We use results from Alvarez and Lippi (2014) to justify why the decision rules of the firms ($\tilde{g}$ and $\bar{g}$) are kept constant and equal to their steady state values after a small monetary shock occurs. We have:

$$\begin{aligned}\mathcal{M}(\delta) = \mathcal{M}'(0)\delta + o(\delta) &= \delta \int_{-\bar{g}}^{\bar{g}} \hat{m}'\,(g)\; f\,(g)\; dg \\ &= \delta \left[\int_{-\bar{g}}^{\bar{g}} m'\,(g)\; f\,(g)\; dg - \tilde{g} \int_{-\bar{g}}^{\bar{g}} D'\,(g)\; f\,(g)\; dg \right] + o(\delta)\end{aligned} \tag{17}$$

since by definition $\mathcal{M}(0) = 0$. We summarize the solution of this integration in the next lemma, where we use $N_p = \sigma^2/\bar{g}$ to denote the expected number of plan changes per period (see Proposition 6 below for a proof):

LEMMA 3. The cumulative output effect after a small monetary shock δ is $\mathcal{M}(\delta) =$

[7] This follows since the invariant density solves the Kolmogorov forward equation: $f''(g) = 0$ which immediately implies the linearity, with the boundary conditions $f(\bar{g}) = 0$.

$\delta\ \mathcal{M}'(0) + o(\delta)$ where

$$\mathcal{M}'(0) = \frac{\bar{g}}{18\ \sigma^2} = \frac{1}{18 N_p} \tag{18}$$

The lemma shows that the cumulated output effect is a decreasing function of the number of price plans per period, N_p. This is intuitive: the monetary shock is permanent so it requires a permanent adjustment of the prices changed by the firms, which can only occur when the price plan of the firm is changed.

One of the objectives of our analysis is to compare the output effect in equation (18) with the corresponding effect in a standard menu cost model a la Golosov and Lucas (labelled GL henceforth) with a threshold for price adjustment equal to $\bar{g}$. Notice that the expected number of price changes per period in GL is $N_{GL} = \sigma^2/\bar{g}^2$. The cumulative output effect is $\mathcal{M}_{GL}(\delta) = \int_{-\bar{g}}^{\bar{g}} m\,(g+\delta)\ f\,(g)\ dg$, which is given by $\mathcal{M}_{GL}(\delta) = \delta\ \mathcal{M}'_{GL}(0) + o(\delta)$ where

$$\mathcal{M}'_{GL}(0) = \int_{-\bar{g}}^{\bar{g}} m'\,(g)\ f\,(g)\ dg = \frac{\bar{g}^2}{6\,\sigma^2} = \frac{1}{6\,N_{GL}}$$

which is the real cumulative output effect in the Golosov and Lucas model, i.e. in a menu cost model without the price plans.[8]

The next proposition compares the real cumulative effects in the two models:

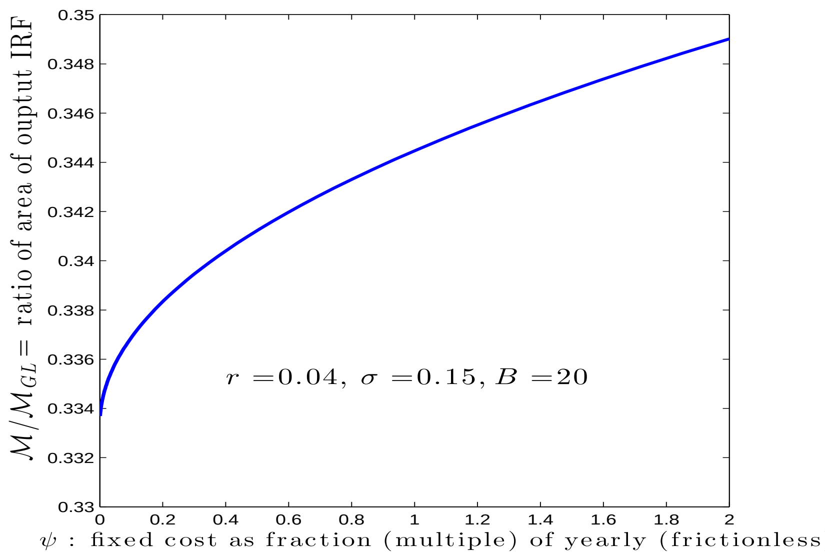
PROPOSITION 4. *Let N_p be the per-period mean number of Plan changes and N_{GL} be the mean number of price changes in the canonical menu cost model without plans. The ratio of the cumulative output responses in the two models is:*

$$\lim_{\delta\downarrow 0,\, \frac{\psi}{B}\downarrow 0,} \frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \lim_{\delta\downarrow 0, r\downarrow 0} \frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \frac{1}{3}\ \frac{N_{GL}}{N_p},$$

$$\lim_{\delta\downarrow 0} \frac{\partial}{\partial\,\frac{\psi}{B}} \frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} \geq 0 \quad \text{and} \quad \lim_{\delta\downarrow 0,\, \frac{\psi}{B}\uparrow\infty} \frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = 1\,.$$

[8]For more details see our computation in Alvarez, Le Bihan, and Lippi (2014) where the difference in the sign of δ and g is due to the difference in the interpretation, in particular in Alvarez, Le Bihan and Lippi $-g$ was the price gap, as opposed to the normalized desired price.

Figure 3: Accuracy of approximation: Ratio of areas under output's IRF



Note: Each ratio is computed for a value of the fixed cost ψ for the economy with price plans. The GL economy (the economy without price plans) is chosen with a fixed cost so that the total number of price changes is equal to the number of price plans changes in the economy with price plans.

The proposition shows that, for small fixed cost ψ/B or small discount factor r, price plans give up to 3 times more flexibility -and hence three times less effect on output, than the menu cost model *with the same number of regular price changes* i.e. assuming $N_{GL} = N_p$ (in Section 4 we discuss other comparisons with different measures of price changes). As the fixed cost increases, prices can spend more time far away from the desired value, and hence it is less likely that they cross the threshold for which they move from one value of the price plan to another. In this sense, a high fixed cost implies more effect of monetary policy in the model without price plans and also more similar effects in the model with price plans.

While the results of proposition Proposition 4 hold for small values of ψ/B or of r, very large values will be implausible. For instance Figure 3 plots the ratio of the areas under the IRF for two economies defined as in Proposition 4 for different values of the fixed cost ψ in the economy with price plans. The fixed cost ψ is measured as a ratio of the annual profits for a firm in the frictionless case, for instance the value of 1 indicates that the menu cost equals the profits in one year. The interpretation of B, the benefit of adjusting, is the curvature of the profit function, which in the simple case can be taken to be $B = (1/2)\eta(\eta - 1)$, where η is the elasticity of demand. So a markup of about 15%, or an elasticity of about 7, implies a value of $B \approx 20$. We have chosen a high but reasonable interest rate $r = 0.04$, since as $r \to 0$ the ratio converges to 1/3. As can be seen from Figure 3, even for unreasonably large values of the fixed cost ψ, such as two years of frictionless profits, the ratio of the area under the output IRFs is smaller than 0.35, as opposed to the limit value of 1/3 obtained in Proposition 4. Thus we conclude that, for this model we can safely use the approximation that the ratio $\mathcal{M}/\mathcal{M}_{GL} = 1/3$ for the comparison in Proposition 4.

Impact Effect. It is interesting to compare the impact effect in the model with price plans with the impact effect in the Golosov-Lucas model. As shown in Alvarez and Lippi (2014), in the standard menu cost model, the impact effect on prices, which we denote by $\Theta(\delta)$, is of second order of δ. This is because, for a shock $\delta < \bar{g}$, the mass of firms that change price on impact is given by $\delta^2/(2\,\bar{g}^2)$. The zero impact effect is a pervasive feature of time-dependent models almost by definition, since in those setting the firm decision to adjust prices does not depend on the state (i.e. on the value of the price). This feature is likewise widespread in state-dependent models where the density of the invariant distribution of the firm's desired price is zero at the boundaries of the inaction region.[9] It is intuitive that a zero density at the boundaries of the inaction region implies that a small shift of the support of the distribution, say of size δ, triggers a very small mass of adjustments in canonical menu cost models, since this mass is roughly given by the product between the (near zero) density at the boundary and δ. While this basic observation remain true in the model with plans concerning the mass of firms that adjust their plan, it is not true concerning within-plan price changes. This is because a shift of the invariant distribution pushes many firms to change the sign of the normalized desired price, thus triggering a price change. Since the invariant distribution of

[9] This result can be seen analytically from equations (19) and (20) in Caballero and Engel (2007).

desired prices has a high mass at zero, this generates a large impact effect on the aggregate price level in the model with plans, which we formally analyze next.[10]

We consider now the impact effect in the model with price plans. On impact there are two types of price changes, those that come with a change of price plan, and those within the existing price plans. Those that come from a change in price plans are given by an expression identical to the one in Golosov and Lucas, and those are of second order. We let $\tilde{\Theta}(\delta)$ denote the impact effect on prices due to price changes within the existing price plan. The effect is given by the mass of firms whose negative desired price $g < 0$ becomes positive following the shock times the size of their price change $2\tilde{g}$. Using the form of $f(g)$ we obtain

$$\tilde{\Theta}(\delta) = \tilde{g}\left[1 - \frac{\delta^2}{2\,\bar{g}^2} - \frac{(\bar{g}-\delta)^2}{\bar{g}^2}\right]$$

Note that one can understand this simple expression by computing the fraction of firms with normalized desired price g that the shock shifts from a negative to a positive desired price. For a small δ this fraction is $f(0)\,\delta$. The effect on price of this is $2\,\tilde{g}\,f(0)\,\delta$. Thus, we have:

$$\tilde{\Theta}(\delta) = \tilde{\Theta}(0) + \tilde{\Theta}'(0)\,\delta + o(\delta) = 2\,\tilde{g}\,f(0)\,\delta + o(\delta) = 2\,\frac{\tilde{g}}{\bar{g}}\,\delta + o(\delta)\,. \tag{19}$$

We highlight that the first equality in equation (19) will hold for other cases, i.e. even if f is different (see for instance the extension that assumes costly price changes within the plan, developed in Appendix D). The next proposition summarizes these results:

[10] The non-zero impact effect of the model with plans is a robust feature of models with plans. It also appears in the version of the model where the times of the plan changes have an exogenous constant hazard (see Appendix C). Interestingly, it also appears if the within-plan change of prices is subject to a small but positive menu cost, see Appendix D.

Proposition 5. Replacing the optimal value of $\tilde{g}$ when $r \to 0$ we obtain:

$$\lim_{r\to 0}\lim_{\delta\to 0}\frac{\tilde{\Theta}(\delta)}{\delta} = \lim_{\frac{\psi}{B}\to 0}\lim_{\delta\to 0}\frac{\tilde{\Theta}(\delta)}{\delta} = \frac{2}{3}$$
$$\frac{\partial}{\partial\frac{\psi}{B}}\lim_{\delta\to 0}\frac{\Theta(\delta)}{\delta} \le 0 \text{ and } \lim_{\frac{\psi}{B}\to\infty}\lim_{\delta\to 0}\frac{\tilde{\Theta}(\delta)}{\delta} = 0$$

Thus, for either small discount rate r or small fixed cost ψ/B, the response of the aggregate price level on impact is 2/3 of the monetary expansion, very close to the flexible price, and hence the impact effect on output is $Y(0) = (1/\epsilon)(1/3)\delta$. As in the case of the ratio of the areas under the IRF, as the fixed cost becomes arbitrarily large, the impact effect diminishes.

4 Mapping the frequency of price changes to data

The previous section compared the real effects of a monetary shock between an economy with price plans and a traditional menu cost economy. The frequency of Plan changes N_p played a key role in the analysis. But it is not straightforward to measure N_p in the data because the model with price plans produces two types of price changes: changes within the plan, i.e. oscillations between p_H and p_L and changes of the plan. Thus when matching this model to the frequency of price changes that is seen in the data one has to take a stand on how to count the number price changes: all types of price changes, including short lived temporary oscillations between p_H and p_L, or only the "regular" persistent (i.e. plans) changes? Below we formalize this idea and compare the real effects in the model with plans with those of a menu cost model, using various measures of price changes.

To do this it is necessary to setup a discrete time / discrete state representation of the model, for three reasons. First, in the continuous time version the expected number of price changes within a price plan diverges to $+\infty$ (see equation (21) below for a proof). Second, the source of the empirical study in Eichenbaum, Jaimovich, and Rebelo (2011) comes from a grocery chain where price changes are decided (and recorded) weekly. Third, and more importantly, this allows us to evaluate the effect of introducing price plans into an otherwise stylized version of the Golosov-Lucas model by keeping the *total* number of price changes fixed. From this analysis we will conclude that the effect of introducing price plans keeping the *total* number of price changes fixed is to double the real cumulative effects of a monetary shock.

Frequency of price changes in a discrete time version of the model. The discrete state / discrete time representation has time periods of length Δ and the normalized desired price following

$$g(t+\Delta) - g(t) = \begin{cases} +\sqrt{\Delta}\,\sigma & \text{with probability } 1/2 \\ -\sqrt{\Delta}\,\sigma & \text{with probability } 1/2 \end{cases} \tag{20}$$

We assume that g reaches $\pm\bar{g}$ after an integer number of periods (or steps), we define this value as $\bar{n} = \bar{g}/\left[\sqrt{\Delta}\sigma\right]$, an integer greater or equal than 2. The requirement that it is greater or equal than 2 allows us to have price changes within a price plan. Let $g(t)$ be the discrete time process following equation (20) holds for $-\bar{g} < g < \bar{g}$. Let $\tau(\bar{g})$ the stopping time denoting the first time at which $|g(t)|$ reaches $\bar{g}$.

We define N to be the total number of price changes per unit of time, N_p the number of price plans per unit of time, and N_w the number of price changes per unit of time without a price plan change, we have $N = N_w + N_p$. First we notice that N_w depends only on N_p and Δ, and no other parameters. This is because the computation for N_w requires knowing the number of steps $\bar{n}$ that are necessary to get from $g = 0$ to $|g| = \bar{g}$ (at which time the plan is terminated), which is given by $\bar{n} = \sqrt{N_p/\Delta}$ (see Proposition 6). Thus the value of N_w, which in principle depends on $\bar{g}, \sigma, \Delta$, can be fully characterized in terms of 2 parameters only: N_p and Δ.

The next proposition characterizes the expected number of plans per unit of time, N_p:

Proposition 6. Let $\Delta > 0$ be the length of the discrete-time period. Assume that

$\bar{g}/(\sigma\sqrt{\Delta})$ is an integer larger than 2. Given the threshold $\bar{g}$, the number of plan changes per unit of time is $N_p = \frac{\sigma^2}{\bar{g}^2}$.

It is immediate to realize that N_p is independent of Δ and hence its value coincides with the number of adjustments of the continuous time model, i.e. the limit for $\Delta \to 0$. We now establish two inequalities bounding N_w as a function of the length of the time period Δ, and of the number of price plans per unit of time N_p. The inequality follows directly from Doob's uncrossing inequality applied to our set-up.

Proposition 7. Let $\Delta > 0$ be the length of the time period, and $\bar{g}$ be the width of the inaction band. The expected number of price changes within a plan N_w (per unit of time) has the following bounds

$$\frac{1}{\sqrt{\frac{\Delta}{N_p} + \frac{\Delta}{2}\left[\frac{1+\sqrt{\Delta N_p}}{1-\sqrt{\Delta N_p}}\right]}} \leq N_w \leq 2\sqrt{\frac{N_p}{\Delta}} - \frac{N_p}{2} \tag{21}$$

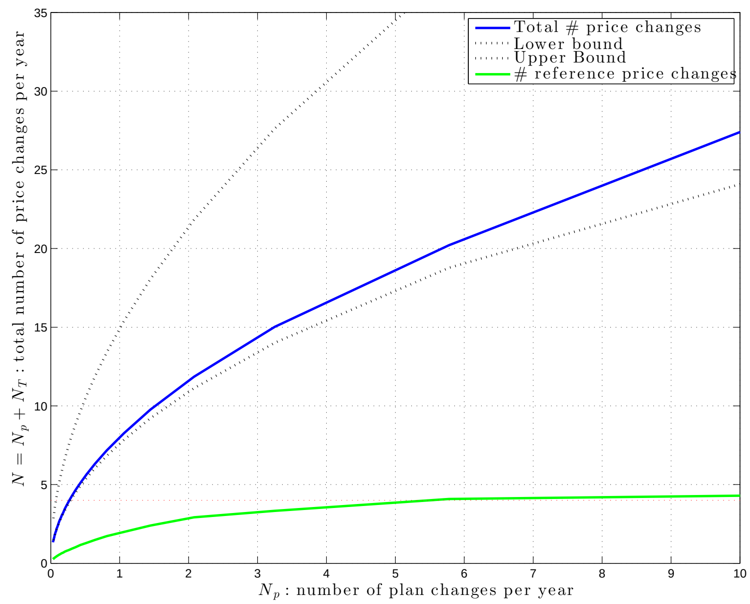
Note that both the lower and upper bound for N_w are increasing in N_p and decreasing in Δ. As $\Delta \to 0$, then $N_w \to \infty$, and indeed N_w behaves as $\Delta^{-1/2}$ for small values of Δ.[11]

Next we use the lower bound in equation (21) to derive an approximation for N as a function of Δ and N_p which is accurate for small Δ. To this end we first define the function $N = \mathcal{N}(\Delta, N_p)$ which gives the total number of price changes as function of Δ and N_p. This implicitly defines the function $N_p = \mathcal{N}_p(\Delta, N)$. We have the following approximation:

$$N \approx \sqrt{\mathcal{N}_p/\Delta} \quad \text{for small } \Delta \qquad \text{or formally} \quad \lim_{\Delta \to 0} \frac{\mathcal{N}_p(\Delta, N)}{\Delta N^2} = 1 \tag{22}$$

[11] Note also that both bounds for N_w are increasing in N_p, at least for small N_p. This is because as N_p is large, the price gap is reset to values closer to zero more often, which is the time when price changes without a price plan tend to happen. Finally note that fixing $\Delta > 0$, and letting $N_p \to 0$ then $N_w \to 0$, and hence $N = N_p + N_w \to 0$. Summarizing, letting $N \equiv N_w + N_p = \mathcal{N}(\Delta, N_p)$ we have $\mathcal{N}(0, N_p) = \infty$ for $N_p > 0$ and $\mathcal{N}(\Delta, 0) = 0$ for $\Delta > 0$, with the upper bound and lower bound of $\mathcal{N}(\Delta, N_p)$ being increasing in N_p and decreasing in Δ.

Figure 4: Total number of price changes N as function of N_p, for $\Delta = 1/52$



Note: The upper and lower bound are denoted by dashed lines. The simulation uses a weekly frequency $\Delta = 1/52$; different values of N_p on the horizontal axis correspond to different values of σ.

Figure 4 illustrates the upper and lower bounds using $\Delta = 1/52$, so that the time period is a week, and plot the total number of price changes N as a function of the number of price plan changes. Indeed we plot the exact number of price changes, via simulation, as well as the upper and lower bound described above. Note that, as explained above, the total number of price changes in this discrete version of the model is only a function of N_p and Δ.

The previous results provide novel insights on the measurement of flexibility for an actual economy featuring both temporary and permanent price changes. It is common practice to measure the flexibility of an economy by the measured frequency of the price changes, at least since Blinder (1994). Indeed this statistic is central in Klenow and Malin (2010) who also carefully distinguish between permanent and short-lived price changes. Our theory is useful to interpret those findings: the result in Lemma 3 establishes that it is the number of permanent changes N_p (i.e. Plan changes) that concur to determine the output effect of a monetary shock, not the overall number of price changes. Yet another implication of our theory is novel and interesting to interpret the data: equation (22) shows that, for a given N_p, the total number of price changes depends on the length of the decision period Δ. This result provides an interpretation for the findings of Sudo, Ueda, and Watanabe (2014) who analyze the frequency of price changes using a unique daily dataset for selected Japanese restaurants finding that (i) the frequency of posted prices is about three times as flexible as the frequency of US scanner data, and that (ii) regular prices are almost as flexible as those in the U.S.[12] Our theory suggests that the much higher frequency of posted prices reflects the different time aggregation frequency, and in particular that they should scale approximately with the square root of the time period. Figure 10 of Sudo, Ueda, and Watanabe (2014) compares the daily, weekly, monthly, quarterly and annual frequency of price changes computed using their daily scanner prices. They find that the probabilities of price changes are quite sensitive to the time interval, but that they vary less than the ratio of the square root of the time interval. For instance from Figure 10 one can see that the ratio between daily and monthly probability of price changes in their data set is about 3 times at the start of their sample and even smaller at the end, while $\sqrt{30} \approx 5.5$. This observation per se is not informative about the real effects of a monetary shock: what matters for aggregate flexibility is the number of regular price changes, which appears similar between these economies.

Fraction of time spent at the reference price. In this section we analyze the prevalence of reference prices. Reference prices are defined as the modal price during an interval of time, say during $[0, T]$ a concept introduced by Eichenbaum, Jaimovich, and Rebelo (2011). The idea is to highlight that while there are many price changes during a time interval, prices spend a large fraction of time at the modal value during this interval, i.e. prices are often at the "reference price". The comparison between the large frequency of price changes and the

[12] Sudo, Ueda, and Watanabe (2014) computes daily probabilities of price changes of about 13%, depending of the type of good. See their Table 1 where they estimate daily probabilities which they express by linearly extrapolating them to monthly probabilities of of about 415%, which corresponds to about 50 price changes per year. In page 245 of Eichenbaum, Jaimovich, and Rebelo (2011) they report a probability of a price changes corrected by the presence of measurement error, of 0.29 per week, or about 15 price changes per year. Thus the ratio of 50/15 gives 3.3 more prices changes per year when measured at daily as opposed to weekly frequency.

prevalence of reference prices captures the idea that prices return to the previous values.

To analyze this effect, we compute a statistic fixing a time interval of length T and a fraction $\alpha \in [0, 1]$. The statistic $F(T, \alpha)$ is the fraction of sample periods of length T in which the firm price spends at least αT time at the modal price. We will show that it is possible to have $F(T, \alpha) \approx 1$ even for α close to one, and at the same time that we have an arbitrarily large number of price changes, N. Next we formally define reference prices and the statistic F. We fix an interval $[0, T]$ and index each price-path in the interval by ω, so the prices for this path are denoted by $p(\omega, t)$ for each $t \in [0, T]$. We let μ_T be the measure of these sample paths. We will fix a path ω and define three concepts. First we define the set of prices observed in a interval of length T for a given price path ω:

$$\mathbb{P}(\omega) \equiv \{y : y = p(\omega, t) \text{ for some } t \in [0, T]\}$$

Second we define the modal price in an interval of length T for a given price path ω, or the reference price:

$$p^{ref}(\omega) \equiv \text{ mode of } \mathbb{P}(\omega)$$

Third, we define the duration of the reference price as the time spent at the modal price in $[0, T]$ for a given sample path ω:

$$d^{ref}(\omega) \equiv \int_0^T 1_{\{p(\omega,t)=p^{ref}(\omega)\}} \, dt$$

Finally, the statistic $F(T, \alpha)$ measures the mass of price paths of length T for which the duration of the reference price is higher than αT:

$$F(T, \alpha) = \mu_T \left(\omega : d^{ref}(\omega) \geq \alpha \, T\right)$$

Recall that our model assumes that the (log of the) ideal price follows a random walk. We will next consider both the continuous time as well as the discrete-time version, with a time period of length $\Delta > 0$ and symmetric steps of size $\pm\sigma\sqrt{\Delta}$, each with probability one half. In the continuous time case the log of the ideal price follows a standard BM with variance per period of time σ^2, and we denote it as the $\Delta = 0$ case. We have:

Proposition 8. Fix $\sigma^2 > 0$ and let $\Delta \geq 0$ be the length of the time period, with $\Delta = 0$ denoting the case of a Brownian motion. Consider an interval length $T > 0$, a fraction $1/2 \leq \alpha < 1$ and a number $0 < \epsilon \leq 1$. Then there exists a threshold value $G > 0$ such that for all $\bar{g} \geq \sigma\, G$ then $F(T, \alpha) \geq 1 - \epsilon$. The threshold G depends on ϵ, α and T but it is independent of σ.

In words, the proposition states that for any fraction $\alpha \in (0, 1)$, it is possible to choose a value of $\bar{g}$ large enough so that the price will equal the reference price at least a fraction α of the time "on average". Recall that in our continuous time model with $\bar{g}$ large, but finite, the expected number of price changes per unit of time diverges towards ∞. Thus, our model can simultaneously have prices spending a large fraction of time at the reference price, as well as lots of price changes. The extreme results of infinitely many price changes holds in the continuous time limit, which is a tractable but unrealistic idealization when dealing with price spells that have very small duration. For this reason we complemented these two analytical results with the discrete time approximations and with numerical simulations for a weekly version of the model.

Quantitative Evaluation. Table 1 summarizes the main implications of our model concerning the patterns of price setting behavior. We use a weekly time period (i.e. $\Delta = 1/52$) and a range of N_p. The statistics in the table depend only on 2 parameters: Δ, which is fixed at the weekly frequency, and N_p, for which we consider 4 values reported in the first line of the table (one can use equation (10) to map these values to different primitives, e.g. values of the fixed cost ψ). The table also reports some statistics for the "reference price", defined as the modal price in a quarter (following EJR11). It appears that the frequency of plan changes and reference price changes do not coincide but that they are quite close and strongly

Table 1: Summary statistics on price setting behavior (weekly model)

# Price changes per year ($N = N_p + N_w$)	20	15	10	5.1
# Plan changes per year (N_p)	5.8	3.3	1.4	0.4
# Reference Price changes per year*	4.1	3.3	2.4	1.2
Prob. of Price changes (per week)	0.35	0.27	0.18	0.10
Prob. of Reference price change (per quarter)	1.0	0.8	0.6	0.29
Fraction of time spent at reference price	0.50	0.63	0.79	0.95
Fraction of time spent below the reference price	0.25	0.17	0.10	0.02

* The Reference price is defined as the modal price in a quarter. All results refer to a discrete time model where the time period is the week, i.e. $\Delta = 1/52$.

correlated. The entries in the second column are the ones that closely replicate the frequency of total weekly price changes reported by Eichenbaum, Jaimovich, and Rebelo (2011), with a total number of about 15 price changes per year.[13] In our model (which differs from theirs because, among others, we focus on unit root shocks), this implies about 3 Reference price changes per year, a value that is above the one of Eichenbaum, Jaimovich, and Rebelo (2011). We use the calibration to compute a measure of the prevalence of reference prices, i.e. what fraction of the time the prices spend at the reference price, as well as below it. The second to last row of the table shows that the stickier the plan (i.e. the smaller N_p), the larger the fraction of time that prices spend at their modal value, consistently with the theoretical result in Proposition 8. The remaining time is equally split in visits to other prices that are both "above" and well as "below" the reference price. The calibration in the second column shows that the prices spend about 63% of the time at their modal value, a statistic that is almost identical to the one by Eichenbaum, Jaimovich, and Rebelo (2011) and not far from the BLS statistics of Kehoe and Midrigan (2015) on the prevalence of reference prices which indicate that prices spend about 75% of the time at the modal price (see their Table 4).[14]

[13]They estimate that weekly price changes have a duration of 0.18 quarters of a year, and correcting for measurement error, a duration of 0.27 quarters.

[14]We compare our model to the statistics in Eichenbaum, Jaimovich, and Rebelo (2011) because their data is weekly. Instead the BLS data used by Kehoe and Midrigan (2015) is monthly, which requires further time aggregation. Also the value of the period length T to compute the fraction of time at the modal price is different, it is four times longer in statistics computed by Kehoe and Midrigan (2015) relative to those computed by Eichenbaum, Jaimovich, and Rebelo (2011).

The cumulated real output effects in models with and without plans. In Table 2 we compare the models with and without price plans. Two comparisons are done: the first one imposes the two models to have the same number of plan changes, the second one imposes that the two models have the same number of total price changes.

As a benchmark, and for consistency with e.g. Eichenbaum, Jaimovich, and Rebelo (2011), we use a weekly model with a total number of about 15 price changes per year. Note that, according to Table 1 this corresponds to a frequency of plan changes, N_p, around 3. In Table 2 we refer to the number of price changes for the model without price plans as N_{GL}, and use N_p for the number of plan changes for the model with price plans. The second row of Table 2 compares the area under the impulse response of a monetary shock for an economy with price plans and one without price plans, where both economies have the same number of total price changes $N = 15$. It appears that the monetary shock has larger real effects in the economy with price plans. This has a simple explanation: when the two models are rigged to produce the same *total* number of price changes, the model with plans has only a small fraction of adjustments that allow the firm to respond permanently to the aggregate shock. Instead, every adjustment in the menu cost model is a permanent one. Because of this, the real effects are larger in the model with plans *that are calibrated to the same number of total price changes.*

An analytical approximation allows us to highlight how the total number of price changes N and the length-of-time period Δ (which we assumed to be weekly) affect the result in the second row of the table. The ratio of the area under the output's IRF is $\frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \frac{N_{GL}}{3\,N_p}$, which can be written in terms of the time period length Δ and the total number of price changes, assumed the same in both models: $N_{GL} = N$ for the Golosov-Lucas and $N = \mathcal{N}(\Delta, N_p)$ in the model with plans. Using the approximation in equation (22) we have:

$$\lim_{\delta \to 0} \frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \frac{N_{GL}}{3\,N_p} = \frac{1}{3}\frac{N}{\mathcal{N}_p\,(\Delta, N)} \approx \frac{1}{3}\frac{1}{N\,\Delta} \tag{23}$$

Indeed as $\Delta \to 0$ the ratio of the effects diverges towards infinity. To see this notice that using

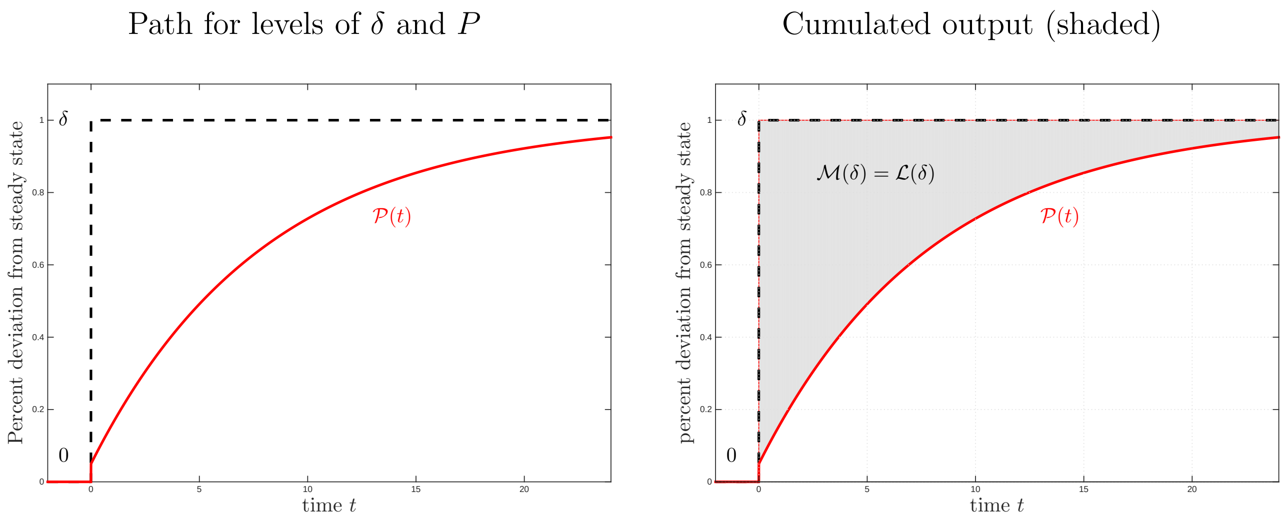
Table 2: Comparison of Effects of a Monetary Shock

	With Price Plans	Without Price Plans	Ratio $\frac{\text{With Price Plans}}{\text{Without Price Plans}}$
Theoretical Expressions	$\mathcal{M}(\delta) \approx \frac{\delta}{6}\frac{1}{3\,N_p}$	$\mathcal{M}_{GL}(\delta) \approx \frac{\delta}{6}\frac{1}{N_{GL}}$	$\frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \frac{N_{GL}}{3\,N_p}$
Both models have $N = 15$ (based on $\Delta = 1/52$)	$\mathcal{M}(\delta) \approx \frac{\delta}{6}\frac{1}{9}$ ($N_p = 3$)	$\mathcal{M}_{GL}(\delta) \approx \frac{\delta}{6}\frac{1}{15}$ ($N_{GL} = 15$)	$\frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} \approx 1.7$
Both models have $N_p = 3$ (based on $\Delta = 1/52$)	$\mathcal{M}(\delta) \approx \frac{\delta}{6}\frac{1}{9}$ ($N \approx 15$)	$\mathcal{M}_{GL}(\delta) \approx \frac{\delta}{6}\frac{1}{3}$ ($N_{GL} = 3$)	$\frac{\mathcal{M}(\delta)}{\mathcal{M}_{GL}(\delta)} = \frac{1}{3}$

the lower bound of the number of total price changes we have $N = N_p + \left(\sqrt{\frac{\Delta}{N_p}} + \frac{\Delta}{2}\left[\frac{1+\sqrt{\Delta N_p}}{1-\sqrt{\Delta N_p}}\right]\right)^{-1}$, which can be regarded as N_p as a function of N and Δ. Note that, keeping fixed $N > 0$, as $\Delta \to 0$ then $N_p \to 0$. Thus, to match a given number of price changes with a small time period Δ, the number of plan-changes has to be very small, and thus the effect of a monetary shock becomes arbitrarily large.

Finally, the third row of Table 2 compares the real effects under the assumption that $N_p = N_{GL}$, i.e. that the number of *plan* changes is the same as the number of price changes in the menu cost model.Under this assumption the two setups have the same "low-frequency" ability to respond to the monetary shock. We see this case as a good approximation to the practice of "ignoring" temporary price fluctuations (e.g. by focusing on reference prices) when calibrating menu cost models to the data. The heuristic behind this practice is that temporary price changes are not counted because they are not seen as a regular form of price adjustment (e.g. Golosov and Lucas (2007) and Nakamura and Steinsson (2008)). In terms of the model, this means calibrating the menu cost model and the model with plans to the same frequency of plan changes, N_p. It is apparent though that the model with plans will feature more price changes per period, due to the presence of the temporary changes. The table shows that those high frequency price changes provide a significant amount of flexibility to

Figure 5: Impulse response functions to permanent shocks



respond to the shock in the short run, so that the cumulative output effect under this scenario is 1/3 of the one in the menu cost model, as was shown in Proposition 4.

4.1 The cyclicality of sales over the business cycle

We conclude by discussing the model's implications for the cyclicality of sales in an economy featuring both aggregate productivity and monetary shocks. We first consider the propagation of a permanent aggregate productivity shock in the model. For an economy with sticky prices we derive a result that relates the cumulated response of output after a monetary shock to the response after a productivity shock. We use this result to discuss the model's implications for the cyclicality of sales.

DEFINE IDEAL PRICE AND ILLUSTRATE EQUIVALENT NATURE OF PRODUCTIVITY AND MONETARY SHOCKS

We introduce some notation that helps making the point in a simple and transparent way. Let δ_m denote the monetary shock analyzed above and δ_z denote a permanent innovation to aggregate productivity. Let Y_t^f denote aggregate (log) output (in deviation from the steady state level) in the economy without price setting frictions (e.g. where $\psi = 0$) and Y_t^s the (log) output (in deviation from the steady state level) in the economy with the menu cost friction (the superscripts $\{f, s\}$ refer to flexible and sticky, respectively). We let $Y_t^j(\delta_m, \delta_z)$ denote output as a function of monetary as productivity shocks, which appear respectively as the first and second argument, where $j = \{f, s\}$. We note that $Y_t^f(\delta_m, 0) = 0$ and that $Y_t^f(0, \delta_z) = \delta_z$ for all t.[15]

Proposition 9. Consider permanent monetary and productivity shocks of size δ. The following relation holds: $Y_t^s(\delta, 0) - Y_t^f(\delta, 0) = Y_t^f(0, \delta) - Y_t^s(0, \delta)$, or

$$Y_t^s(\delta, 0) = \delta - Y_t^s(0, \delta) \tag{24}$$

which implies the following equivalence:

$$\mathcal{L}(\delta) \equiv \int_0^\infty \Big(\delta - Y_t^s(0, \delta)\Big) dt \;=\; \mathcal{M}(\delta) \tag{25}$$

COMMENTS: TO BE ADDED Whether the frequency of temporary variation on prices (say sales) are pro or counter cyclical depends on the nature of shocks. For positive demand shocks (i.g. increases in M), prices go up, and fraction of increases goes up. For positive supply shocks (i.e. increases in TFP) prices go down, and fraction of decreases goes up. In both cases, plans gives more flexibility, but one type of shocks has deflationary pressures, the other inflationary ones.

EMPIRICAL EVIDENCE ON CYCLICALITY OF SALES: Analyzing empirically whether firm use temporary price changes to respond to the aggregate macroeconomic conditions, as in the recent papers by Kryvtsov and Vincent (2014) and Anderson et al. (2015), is a useful

[15] See the online appendix B of Alvarez and Lippi (2014) for a formal derivation of those results.

way to select between these alternative models. Sudo, Ueda, and Watanabe (2014) argue that the frequency of temporary sales is influenced by macro business cycle.

5 Concluding remarks and extensions

Three substantive extensions of the baseline menu cost model are discussed in appendices. These extensions allow us to better relate our contribution to the existing literature and verify the robustness of the results presented in the main body of the paper.

Appendix C analyzes the consequences of introducing price plans $P \equiv \{p_H, p_L\}$ in a setting where the adjustment times for the plan are exogenous and follow the standard Calvo model. It is shown that this simple extension generates a hump-shaped impulse response of output, as well as decreasing hazard rates.

Appendix D discusses a modified model which assumes that price changes within the plan are not free, although they are cheaper than changes of the plan. The main finding of this extension is that the model retains the feature that the impact effect of a small monetary shock on the price level can be substantial. This shows that our baseline result is robust i.e. that it does not depend on the assumption of a zero cost of price changes within the plan.[16]

Finally, Appendix E changes the baseline model by considering innovations to the firm's desired price that are iid (temporary) rather than permanent. Temporary shocks are useful to relate to the model by Eichenbaum, Jaimovich, and Rebelo (2011).

References

Alvarez, Fernando, Katarína Borovičková, and Robert Shimer. 2015. "Decomposing Duration Dependence in a Stopping Time Model." Mimeo, University of Chicago.

Alvarez, Fernando, Herve Le Bihan, and Francesco Lippi. 2014. "Small and Large Price Changes and the Propagation of Monetary Shocks." NBER Working Papers 20155, National Bureau of Economic Research, Inc.

[16]This result, formally given in Proposition 13 extends the result of Proposition 5. The economics is that even in the presence of a small adjustment cost within the plan there is a non-negligible mass of firms in the neighborhood of the price-adjustment threshold (within a plan).

Alvarez, Fernando, Robert Shimer, and Fabrice Tourre. 2015. "Seniority rules, rigid wages and unemployment duration." Mimeo, University of Chicago.

Alvarez, Fernando E. and Francesco Lippi. 2014. "Price setting with menu costs for multi product firms." *Econometrica* 82 (1):89–135.

Anderson, Eric, Benjamin A. Malin, Emi Nakamura, Duncan Simester, and Jn Steinsson. 2015. "Informational Rigidities and the Stickiness of Temporary Sales." Research Department Staff Report 513, Federal Reserve Bank of Minneapolis.

Barro, Robert J. 1972. "A Theory of Monopolistic Price Adjustment." *Review of Economic Studies* 39 (1):17–26.

Bils, Mark and Peter J. Klenow. 2004. "Some Evidence on the Importance of Sticky Prices." *Journal of Political Economy* 112 (5):947–985.

Blinder, Alan S. 1994. *On Sticky Prices: Academic Theories Meet the Real World*, chap. 4. The University of Chicago Press, 117–154.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 2007. "Price stickiness in Ss models: New interpretations of old results." *Journal of Monetary Economics* 54 (Supplement):100–121.

Calvo, Guillermo A. 1983. "Staggered prices in a utility-maximizing framework." *Journal of Monetary Economics* 12 (3):383–398.

Campbell, Jeffrey R. and Benjamin Eden. 2014. "Rigid prices: Evidence from US scanner data." *International Economic Review* 55 (2):423–442.

Dixit, Avinash. 1991. "Analytical Approximations in Models of Hysteresis." *Review of Economic Studies* 58 (1):141–51.

Eichenbaum, Martin, Nir Jaimovich, and Sergio Rebelo. 2011. "Reference Prices, Costs, and Nominal Rigidities." *American Economic Review* 101 (1):234–62.

Golosov, Mikhail and Robert E. Jr. Lucas. 2007. "Menu Costs and Phillips Curves." *Journal of Political Economy* 115:171–199.

Guimaraes, Bernardo and Kevin D. Sheedy. 2011. "Sales and Monetary Policy." *American Economic Review* 101 (2):844–76.

Kehoe, Patrick and Virgiliu Midrigan. 2015. "Prices are sticky after all." *Journal of Monetary Economics* forthcoming (0):–.

Klenow, J. Peter and Benjamin Malin. 2010. "Microeconomic Evidence on Price-Setting." *Handbook of monetary economics* 3:231–284.

Kolkiewicz, A.W. 2002. "Pricing and hedging more general double-barrier options." *Journal of Computational Finance* 5 (3):1–26.

Kryvtsov, Oleksiy and Nicolas Vincent. 2014. "On the Importance of Sales for Aggregate Price Flexibility." Working Papers 14-45, Bank of Canada.

Nakamura, Emi and Jon Steinsson. 2008. "Five Facts about Prices: A Reevaluation of Menu Cost Models." *The Quarterly Journal of Economics* 123 (4):1415–1464.

Sudo, Nao, Kozo Ueda, and Kota Watanabe. 2014. "Micro Price Dynamics during Japan's Lost Decades." *Asian Economic Policy Review* 9 (1):44–64.

APPENDICES – FOR ONLINE PUBLICATION ONLY

A Proofs

Proof. (of Proposition 1). The functions $a(g)$ and $d(g)$ are the solution to the following o.d.e.'s and boundary conditions:

$$
\begin{aligned}
r\,a(g) &= |g(t)| + \frac{\sigma^2}{2} a''(g) \text{ for all } g \in [-\bar{g}, \bar{g}]\,,\ g \neq 0 \\
a(-\bar{g}) &= a(\bar{g}) = 0 \text{ and } a'(0) = 0 \\
r\,d(g) &= 1 + \frac{\sigma^2}{2} d'(g) \text{ for all } g \in [-\bar{g}, \bar{g}] \\
d(-\bar{g}) &= d(\bar{g}) = 0\,.
\end{aligned}
$$

First we develop the expressions for a. The function a must be symmetric around $g = 0$ so that $a(g) = a(-g)$ for all $g \in [0, \bar{g}]$, thus:

$$
a(g) = \begin{cases} +g/r + A_1\, e^{\sqrt{2r/\sigma^2}\, g} + A_2\, e^{-\sqrt{2r/\sigma^2}\, g} & \text{if } g \in [0, \bar{g}] \\ -g/r + A_2\, e^{\sqrt{2r/\sigma^2}\, g} + A_1\, e^{-\sqrt{2r/\sigma^2}\, g} & \text{if } g \in [-\bar{g}, 0] \end{cases}
$$

We have the following three boundary conditions:

$$
\begin{aligned}
1 &= r\,(A_2 - A_1)\ \sqrt{2r/\sigma^2} \\
0 &= \bar{g} + r\,A_1\, e^{\sqrt{2r/\sigma^2}\,\bar{g}} + r\,A_2\, e^{-\sqrt{2r/\sigma^2}\,\bar{g}}
\end{aligned}
$$

Hence

$$
-\bar{g} - \frac{e^{-\sqrt{2r/\sigma^2}\,\bar{g}}}{\sqrt{2r/\sigma^2}} = r\,A_1\, \left(e^{\sqrt{2r/\sigma^2}\,\bar{g}} + e^{-\sqrt{2r/\sigma^2}\,\bar{g}} \right)
$$

$$
r\,(A_1 + A_2) = \frac{1}{\sqrt{2r/\sigma^2}} + 2rA_1 = \frac{1}{\sqrt{2r/\sigma^2}} - 2\frac{\bar{g} + \frac{e^{-\sqrt{2r/\sigma^2}\,\bar{g}}}{\sqrt{2r/\sigma^2}}}{e^{\sqrt{2r/\sigma^2}\,\bar{g}} + e^{-\sqrt{2r/\sigma^2}\,\bar{g}}}
$$

since we are interested in:

$$
r\,a(0) = r\ (A_1 + A_2) = \bar{g} \left[\frac{1}{\sqrt{2r/\sigma^2}\,\bar{g}} - 2\frac{1 + \frac{e^{-\sqrt{2r/\sigma^2}\,\bar{g}}}{\sqrt{2r/\sigma^2}\bar{g}}}{e^{\sqrt{2r/\sigma^2}\,\bar{g}} + e^{-\sqrt{2r/\sigma^2}\,\bar{g}}} \right]
$$

For d we have, as shown in Alvarez, Le Bihan, and Lippi (2014) that:

$$r\,d(0) = \frac{e^{\sqrt{2r/\sigma^2}\bar{g}} + e^{-\sqrt{2r/\sigma^2}\bar{g}} - 2}{e^{\sqrt{2r/\sigma^2}\bar{g}} + e^{-\sqrt{2r/\sigma^2}\bar{g}}}$$

We can write:

$$r\,a(0) = \bar{g}\, \frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi}\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}\right)}$$
$$r\,d(0) = \frac{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2}{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}}$$

Thus

$$\tilde{g} = \frac{r\,a(0)}{r\,d(0)} = \bar{g}\ \rho(\phi) = \bar{g}\, \frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi}\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2\right)}$$

The properties of ρ follows directly from this expression. The limit as $\phi \to 0$ follows by letting $x = \sqrt{2\phi}$ into the expression for $\tilde{g}/\bar{g}$ and expanding the exponentials, canceling to obtain:

$$\begin{aligned}
\frac{\tilde{g}}{\bar{g}} &= \frac{e^{x} - e^{-x} - 2x}{x\left[e^{x} + e^{-x} - 2\right]} = \frac{2(x + x^3/3! + x^5/5! + \cdots) - 2x}{x\left[2(1 + x^2/2 + x^4/4! + \cdots) - 2\right]} \\
&= \frac{2(x^3/3! + x^5/5! + \cdots)}{x\left[2(x^2/2 + x^4/4! + \cdots)\right]} = \frac{x^3(2/3!) + x^5(2/5!) + x^7(2/7!) + \cdots}{x^3(2/2!) + x^5(2/4!) + x^7(2/6!) + \cdots} \\
&= \frac{(2/3!) + x^2(2/5!) + x^4(2/7!) + \cdots}{(2/2!) + x^2(2/4!) + x^4(2/6!) + \cdots}
\end{aligned}$$

Taking the limit $x \to 0$ we obtain $\tilde{g}/\bar{g} = (2/3!)/(2/2!) = 1/3$. That ρ is decreasing it follows by inspection of the previous expression since each of the coefficients of x is smaller in the numerator. That the limit as $\phi \to 0$ of $\rho(\phi)\sqrt{2\phi} \to 1$ follows immediately since $\phi > 0$. This also implies that $\rho \to 0$ as $\phi \to \infty$. $\square$

Proof. (of Proposition 2). We first establish an intermediate result in the following lemma.

LEMMA 4. Let $\tilde{g} \geq 0$ be an arbitrary width for the reset prices. We derive an equation solving for the optimal inaction threshold $\bar{g}$, where we write the width of the price threshold as $\tilde{g} = \gamma\,\bar{g}$ for a constant $0 \leq \gamma$. The optimal inaction threshold $\bar{g}$ must solve:

$$\eta^2 r \frac{\psi}{B} = \varphi\left(x\,;\,\gamma\right) \quad \text{with } x \equiv \eta\,\bar{g}\,,\ \text{ and } \varphi\left(x\,;\,\gamma\right) \equiv \left(1 - 2\,\gamma\right)\left(x^2 - 2\,x\,\frac{\left[e^{x} + e^{-x} - 2\right]}{\left[e^{x} - e^{-x}\right]}\right) \tag{26}$$

The function $\varphi(x;\gamma)$ is: i) strictly increasing in $x \geq 0$ for each $0 \leq \gamma < 1/2$, ii) strictly decreasing in γ for each $x > 0$, and iii) for $0 \leq \gamma < 1/2$, then $\lim_{x\to\infty} \varphi(x;\gamma)/x^2 = 1$, and $\lim_{x\to 0} \varphi(x;\gamma)/\left(x^4/12\right) = 1$.

Using Lemma 4 we obtain the function κ by simply replacing ρ into φ and using that

$x^2/2 \equiv (\eta \bar{g})^2/2 = r\,\bar{g}^2/\sigma^2 \equiv \phi$, where ϕ is defined in Proposition 1. Since $\varphi(x,\gamma)$ is increasing in x and decreasing in γ, and since ρ is decreasing in γ, then κ is strictly increasing in x. Since $\rho'(0)$ is finite, then we can just substitute the limit value of $\rho(0) = 1/3$ and obtain $\kappa(x) = (1-2/3)x^4/12 + o(x^4) = x^4/36 + o(x^4)$. For large x we established that $\rho(x) \to 0$ and hence $\lim_{x\to\infty} \kappa(x)/x^2 = \lim_{x\to\infty} \varphi(x)/x^2 = 1$. $\square$

Proof. (of Lemma 4). The solution of the value function in inaction is given by the sum of a particular solution and the solution to the homogenous function. The particular solution $v^p(g)$ is:

$$v^p(g) = \frac{B}{r}\left[g^2 + \tilde{g}^2 - 2|g|\tilde{g} + \frac{\sigma^2}{r}\right]$$

and the homogenous solutions $v^f(g)$, for $g > 0$ is:

$$v^f(g) = A_1\, e^{\eta_1\, g} + A_2\, e^{\eta_2\, g} \quad \text{where } \eta_i = \pm\sqrt{2\,r/\sigma^2} \tag{27}$$

Thus the solution of v is:

$$v(g;\tilde{g}) = \begin{cases} (B/r)g^2 - (B\,2\,\tilde{g}/r)\,g + (B/r)\left[\tilde{g}^2 + \frac{\sigma^2}{r}\right] & \\ +A_1\, e^{\sqrt{2\,r/\sigma^2}\,g} + A_2\, e^{-\sqrt{2\,r/\sigma^2}\,g} & \text{for all } g \in (0,\bar{g}] \\ (B/r)g^2 + (B\,2\,\tilde{g}/r)\,g + (B/r)\left[\tilde{g}^2 + \frac{\sigma^2}{r}\right] & \\ +A_2\, e^{\sqrt{2\,r/\sigma^2}\,g} + A_1\, e^{-\sqrt{2\,r/\sigma^2}\,g} & \text{for all } g \in [-\bar{g},0) \end{cases} \tag{28}$$

Value matching and smooth pasting are:

$$A_1 + A_2 + \psi = (B/r)\bar{g}^2 - (B\,2\,\tilde{g}/r)\,\bar{g} + A_1\, e^{\sqrt{2\,r/\sigma^2}\,\bar{g}} + A_2\, e^{-\sqrt{2\,r/\sigma^2}\,\bar{g}} \tag{29}$$

$$0 = 2(B/r)\bar{g} - (B\,2\,\tilde{g}/r) + \sqrt{2\,r/\sigma^2}\left[A_1\, e^{\sqrt{2\,r/\sigma^2}\,\bar{g}} - A_2\, e^{-\sqrt{2\,r/\sigma^2}\,\bar{g}}\right] \tag{30}$$

If v is differentiable at $g = 0$, then equation (8) implies:

$$\sqrt{2\,(r/\sigma^2)}\,[A_1 - A_2] = B\,2\,(\tilde{g}/r) \tag{31}$$

For $r > 0$ we can rewrite this system as:

$$a_1 + a_2 + r\frac{\psi}{B} = (\bar{g} - 2\,\tilde{g})\,\bar{g} + a_1\, e^{\eta\,\bar{g}} + a_2\, e^{-\eta\,\bar{g}} \tag{32}$$

$$0 = 2\,(\bar{g} - \tilde{g}) + \eta\left[a_1\, e^{\eta\,\bar{g}} - a_2\, e^{-\eta\,\bar{g}}\right] \tag{33}$$

$$a_1 - a_2 = 2\,\tilde{g}/\eta \tag{34}$$

Solving for a_1 and replacing it we get:

$$\begin{aligned} r\frac{\psi}{B} &= (\bar{g} - 2\,\tilde{g})\,\bar{g} + a_2\left(e^{\eta\,\bar{g}} + e^{-\eta\,\bar{g}} - 2\right) + 2\,\tilde{g}\left(\frac{e^{\eta\,\bar{g}} - 1}{\eta}\right) \\ 0 &= 2\,(\bar{g} - \tilde{g}) + \eta\,a_2\left[e^{\eta\,\bar{g}} - e^{-\eta\,\bar{g}}\right] + 2\,\tilde{g}\,e^{\eta\,\bar{g}} \text{ or} \\ a_2 &= -2\,\frac{\bar{g} + \tilde{g}\,(e^{\eta\,\bar{g}} - 1)}{\eta\,\left[e^{\eta\,\bar{g}} - e^{-\eta\,\bar{g}}\right]} \end{aligned}$$

Solving for $\bar{g}$ we get

$$r\frac{\psi}{B} = (\bar{g} - 2\tilde{g})\,\bar{g} + 2\tilde{g}\left(\frac{e^{\eta\bar{g}} - 1}{\eta}\right) - \frac{[e^{\eta\bar{g}} + e^{-\eta\bar{g}} - 2]}{(e^{\eta\bar{g}} - e^{-\eta\bar{g}})}\,2\,\left[\frac{\bar{g}}{\eta} + \tilde{g}\left(\frac{e^{\eta\bar{g}} - 1}{\eta}\right)\right] \tag{35}$$

Thus we can define

$$\begin{aligned} \eta^2 r\frac{\psi}{B} &= \varphi\,(x;\gamma) \quad \text{with } x \equiv \eta\,\bar{g}\,,\ \tilde{g} \equiv \gamma\,\bar{g} \text{ and } \phi(\cdot) \text{ defined as} \\ \varphi\,(x;\gamma) &\equiv x\,(x - 2\,\gamma\,x) + 2\,\gamma\,x\,(e^x - 1) - 2\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]}\,(x + \gamma\,x\,(e^x - 1)) \end{aligned}$$

or equivalently

$$\varphi\,(x;\gamma) \equiv x\,(x - 2\,\gamma\,x) + 2\,\gamma\,x\,(e^x - 1) - 2\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]}\,(x + \gamma\,x\,(e^x - 1)) \tag{36}$$

We can rewrite this expression as follows. Collecting the first two terms we have:

$$x\,(x - 2\,\gamma\,x) = x^2\,(1 - 2\,\gamma)$$

and that collecting the remaining terms we have:

$$\begin{aligned} & 2\,\gamma\,x\,(e^x - 1)\left[1 - \frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right] - 2\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right]\,x \\ & 2\,\gamma\,x\,(e^x - 1)\left[\frac{e^x - e^{-x} - e^x - e^{-x} + 2}{e^x - e^{-x}}\right] - 2\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right]\,x \\ & 2\,\gamma\,x\,(e^x - 1)\left[\frac{-2e^{-x} + 2}{e^x - e^{-x}}\right] - 2\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right]\,x \\ & = 2\,\gamma\,x\,2\,\frac{(e^x - 1)(1 - e^{-x})}{e^x - e^{-x}} - 2\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right]\,x \\ & = 2\,\gamma\,x\,2\,\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right] - 2\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right]\,x \\ & = (2\,\gamma\ - 1)\ x\,2\,\left[\frac{e^x + e^{-x} - 2}{e^x - e^{-x}}\right] \end{aligned}$$

Thus we can write:

$$\varphi(x;\gamma) \equiv (1-2\,\gamma)\left(x^2 - 2\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]}x\right)$$

To see that φ is increasing in x note that:

$$\frac{\partial \varphi(x;\gamma)}{\partial x} \equiv (1-2\,\gamma)\left(2\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]}\right)\left(x\frac{[e^x + e^{-x}]}{[e^x - e^{-x}]} - 1\right) \geq 0$$

since

$$\begin{aligned} 2\,\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]} &= 2\,\frac{(e^x - 1)\,(1 - e^{-x})}{[e^x - e^{-x}]} \geq 0 \quad \text{and} \\ x\,\frac{[e^x + e^{-x}]}{[e^x - e^{-x}]} &= \frac{x + x^3/2! + x^5/4! + x^7/6! + \cdots}{x + x^3/3! + x^5/5! + x^7/7! + \cdots} \geq 1 \end{aligned}$$

To see that $\lim_{x\to\infty} \varphi(x;\gamma) = \infty$ note that:

$$\lim_{x\to\infty} \frac{\varphi(x;\gamma)}{x} = \lim_{x\to\infty} x - 2\lim_{x\to\infty} \frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]} = \infty - 2 = \infty\,.$$

and thus:

$$\lim_{x\to\infty} \frac{\varphi(x;\gamma)}{x^2} = 1$$

Finally to obtain the expansion for small x we write:

$$\begin{aligned} \frac{\varphi(x;\gamma)}{(1-2\gamma)} &= x^2 - 2\frac{[e^x + e^{-x} - 2]}{[e^x - e^{-x}]}x = x^2\left(1 - 2\,\frac{1}{x}\,\left[\frac{x^2/2 + x^4/4! + \cdots}{x + x^3/3! + x^5/5! + \cdots}\right]\right) \\ &= x^2\left(1 - \frac{x^2 + x^4(2/4!) + x^6(2/6!) + \cdots}{x^2 + x^4/3! + x^6/5! + \cdots}\right) \\ &= x^2\left(\frac{x^4/3! + x^6/5! + \cdots - x^4(2/4!) - x^6(2/6!) + \cdots}{x^2 + x^4/3! + x^6/5! + \cdots}\right) \\ &= x^2\left(\frac{x^4(1/3! - 2/4!) + x^6(1/5! - 2/6!) + \cdots}{x^2 + x^4/3! + x^6/5! + \cdots}\right) \\ &= x^2\left(\frac{x^2(1/3! - 2/4!) + x^4(1/5! - 2/6!) + \cdots}{1 + x^2/3! + x^4/5! + \cdots}\right) \\ &= x^4\frac{1}{12} + o(x^4) \end{aligned}$$

□

Proof. (of Lemma 2)By symmetry we have:

$$\mathbb{E}\left[\int_0^{\tau(\bar{g})} 1_{g(t)<0}\, dt \mid g(0) = g\right] = s(-g)$$

So

$$D(g) = s(g) - s(-g) = \mathbb{E}\left[\int_0^{\tau(\bar{g})} 1_{g(t)\geq 0}\, dt \mid g(0) = g\right] - \mathbb{E}\left[\int_0^{\tau(\bar{g})} 1_{g(t)<0}\, dt \mid g(0) = g\right]$$

For s we have:

$$\begin{aligned}
0 &= 1 + \frac{\sigma^2}{2} s''(g) \quad \text{for } g \in [0, \bar{g}] \\
0 &= \frac{\sigma^2}{2} s''(g) \quad \text{for } g \in [-\bar{g}, 0) \\
s(\bar{g}) &= s(-\bar{g}) = 0 \\
\lim_{g\uparrow 0} s(g) &= \lim_{g\downarrow 0} s(g) \\
\lim_{g\uparrow 0} s'(g) &= \lim_{g\downarrow 0} s'(g)
\end{aligned}$$

Note that the ode is different for negative and positive values of g since in the negative values $1_{g(t)\geq 0} = 0$. The boundary conditions may require some explanation. That $s(\bar{g}) = s(-\bar{g}) = 0$ follows because when $|g(t)| = \bar{g}$ the barrier is reached, so $t = \tau(\bar{g})$. The continuity of s and its derivative at $g = 0$ follow because the brownian regularizes the discontinuous function $1_{g\geq 0}$. Thus the function is linear for negative values and quadratic for positive values with coefficients a_0, a_1, q_0, q_1 to be determined by the four boundary conditions

$$s(g) = \begin{cases} a_0 + a_1 g & \text{if } g \in [-\bar{g}, 0] \\ q_0 + q_1 g - \frac{1}{\sigma^2} g^2 & \text{if } g \in [\bar{g}, 0] \end{cases}$$

The conditions are :

$$\begin{aligned}
0 &= a_0 - a_1 \bar{g} \\
0 &= q_0 + q_1 \bar{g} - \frac{1}{\sigma^2}\left(\bar{g}^2\right) \\
a_0 &= q_0 \\
a_1 &= q_1
\end{aligned}$$

the solution is thus:

$$s(g) = \begin{cases} \frac{\bar{g}^2}{2\sigma^2} + \frac{\bar{g}\, g}{2\sigma^2} & \text{if } g \in [-\bar{g}, 0] \\ \frac{\bar{g}^2}{2\sigma^2} + \frac{\bar{g}\, g}{2\sigma^2} - \frac{1}{\sigma^2} g^2 & \text{if } g \in [0, \bar{g}] \end{cases}$$

This gives the desired result. □

Proof. (of Proposition 4) We have

$$\begin{aligned}
\mathcal{M}'(0) &= \frac{\bar{g}^2}{6\,\sigma^2} - 2\,\tilde{g}\int_0^{\bar{g}} D'(g)\,f(g)\,dg \\
&= \frac{\bar{g}^2}{6\,\sigma^2} - \frac{2\,\tilde{g}}{\sigma^2\bar{g}}\int_0^{\bar{g}} [\bar{g}-2g]\left[1-\frac{g}{\bar{g}}\right]dg \\
&= \frac{\bar{g}^2}{6\,\sigma^2} - \frac{2\,\tilde{g}}{\sigma^2\bar{g}}\frac{\bar{g}^2}{6} = \frac{\bar{g}^2}{6\,\sigma^2}\left[1-\frac{2\,\tilde{g}}{\bar{g}}\right]
\end{aligned}$$

where we use that $f(g)$ is symmetric and $D(g)$ is negative symmetric around $g=0$. Based on this we can compare the cumulative response the cumulative impulse response into the fraction of the corresponding menu cost economy:

$$\lim_{\delta\downarrow 0}\frac{\mathcal{M}_{GL}(\delta)}{\mathcal{M}(\delta)} = \frac{\mathcal{M}'_{GL}(0)}{\mathcal{M}'(0)} = \frac{1}{1-\frac{2\,\tilde{g}}{\bar{g}}}$$

The previous expression takes a given the location of the threshold $\tilde{g}$. Next we evaluate this expression for the optimal value the the threshold. Using Proposition 1 considering the limit of small discount factor or fixed cost , and given that $\rho = \tilde{g}/\bar{g} \to 1/3$ as $r \downarrow 0$ or $\psi/B \to 0$. Using Proposition 2 we know that $\bar{g}$ is increasing in ψ/B, and using Proposition 1 we have that ρ is decreasing in $\bar{g}$, which establish the inequality as well at the limit as $\psi/B \to \infty$. This gives the desired result. $\square$

Proof. (of Proposition 6) Let $T(g)$ be the expected time until the next change of price plan, i.e. until $|g_n|$ reaches $\bar{g}$. We can index the state by $i = 0, \pm 1, ..., \pm\bar{n}$. We have the discrete time version of the Kolmogorov backward equation:

$$T_i \;=\; \Delta \;+\; \frac{1}{2}\,[T_{i+1}+T_{i-1}] \;\; \text{for all } i = 0, \pm 1, \pm 2, ..., \pm(\bar{n}-1)\;.$$

and at the boundaries we have $T_{\bar{n}} = T_{-\bar{n}} = 0$. We use a guess a verify strategy, guessing a solution of the form:

$$T_i = a_0 + a_2\,i^2 \;\; \text{for all } i = 0, \pm 1, \pm 2, ..., \pm\bar{n}\;.$$

for some constants a_0, a_2. Inserting this into the KBE we obtain

$$a_0 + a_2\,i^2 \;=\; \Delta \;+\; \frac{1}{2}\left[a_0 + a_2\,(i+1)^2 + a_0 + a_2\,(i-1)^2\right] \;\; \text{for all } i = 0, \pm 1, \pm 2, ..., \pm(\bar{n}-1)\;.$$

so that $a_2 = -\Delta$. Using this value, into the equation for the boundary condition, we get:

$$a_0 - \Delta\,(\bar{n})^2 \;=0\,,\;\Longrightarrow\; a_0 = \Delta\,(\bar{n})^2\;.$$

and since $\bar{n}\,\sqrt{\Delta}\,\sigma = \bar{g}$ and $T_0 = a_0$ we have:

$$T_0 = a_0 = \Delta\,(\bar{n})^2 = \Delta\left(\frac{\bar{g}}{\sqrt{\Delta}\sigma}\right)^2 = \left(\frac{\bar{g}}{\sigma}\right)^2 = \frac{1}{N_p}$$

□

Proof. (of Proposition 7) We now derive formally the expression that give the inequalities described in equation (21). The proof focus on the case $\bar{n} \geq 2$ (see the discussion following equation (20)), which is equivalent to $N_p\Delta \leq 1/4$. We first obtain an upper bound on the number of price changes within a price plans. We first state the 2 parts of the inequality in two lemmas, and then prove each of them.

Lemma 5. Let $\Delta > 0$ be the length of the time period, and $\bar{g}$ be the width of the inaction band. Let n_w be the expected number of price changes during a price plan. We have:

$$n_w \leq \frac{2}{\sqrt{\Delta}} \frac{1}{\sqrt{N_p}} - \frac{1}{2} \tag{37}$$

Hence the total number of price changes per unit of time within price plans, denoted by N_w, and equal to $n_w N_p$, satisfies:

$$N_w \leq 2\sqrt{\frac{N_p}{\Delta}} - \frac{N_p}{2}$$

where $N_p = \sigma^2/\bar{g}^2$ and $\bar{g}/(\sigma\sqrt{\Delta}) = 1/\sqrt{N_p\Delta}$ is an integer larger than 2.

It is straightforward to obtain an upper bound on expected number of all price changes $N = N_p + N_w$. We obtain:

$$N = N_w + N_p \leq 2\sqrt{\frac{N_p}{\Delta}} + \frac{N_p}{2}$$

Next we obtain a lower bound on the number of price changes within a plan.

Lemma 6. The expected number of price changes per unit of time within a plan N_w has the following lower bound:

$$N_w \geq \frac{1}{\sqrt{\frac{\Delta}{N_p}} + \frac{\Delta}{2}\left[\frac{1+\sqrt{\Delta N_p}}{1-\sqrt{\Delta N_p}}\right]},$$

where $N_p = \sigma^2/\bar{g}^2$ and $\bar{g}/(\sigma\sqrt{\Delta}) = 1/\sqrt{N_p\Delta}$ is an integer larger than 2.

Proof. (of Lemma 5) We first start with a lemma that relates the expected number of up-crossings within a plan and the expected number of plans.

Lemma 7. In the discrete-time discrete-state model we have: $n_w = 2\,E\left[U(\tau)\right] - \frac{1}{2}$.

Proof of Lemma 7. We relate the price changes within a price plan to the number of up-crossing, $U(\tau)$, and number of down-crossings, $D(\tau)$, between $g = 0$ and $g = \sqrt{\Delta}\,\sigma$. We assume that the optimal policy within a plan that has just started at $g(0) = 0$ has a price $\tilde{g} > 0$ if $g \geq \sqrt{\Delta}\,\sigma$ and price $-\tilde{g} < 0$ if $g \leq 0$. We focus on upcrossing where g goes from $g(t) = 0$ to $g(t + \Delta) = \sqrt{\Delta}\,\sigma$, so there is a price increase. For a down crossing, $g(t)$ goes from $g(t) = \sqrt{\Delta}\,\sigma$ to $g(t + \Delta) = 0$, so there is price decrease. We will denote by $U(\tau)$ the number of up-crossings, and $D(\tau)$ the number of down-crossings at the time where the price plan ends. Notice that in any path from $g(0) = 0$ to $g(\tau) = +\bar{g}$ there are $U(\tau) = D(\tau) + 1$

up-crossings, while in any path where $g(\tau) = -\bar{g}$ there are $U(\tau) = D(\tau)$ up-crossings. Since the number of price changes is the sum of up-crossings plus down-crossings, and since the price plan is as likely to end with $g(\tau) = \bar{g}$ as well as with $g(\tau) = -\bar{g}$, thus

$$\Pr\{U(\tau) - D(\tau) = 1\} = \Pr\{U(\tau) - D(\tau) = 0\} = \frac{1}{2}\,.$$

and hence: $n_w = 2\,E\left[U(\tau)\right] - \frac{1}{2}$. This finishes the proof of the lemma.

We now return to the proof of Lemma 5 and use Doob's inequality for the expected number of up-crossings obtaining:

$$(b-a)E\left[U(\tau)\right] \leq \sup_{t=0,\Delta,2\Delta,\ldots} \left(a + E\left[\,|g(t)|\,\right]\right)$$

so that using the values $a = 0$, $b = \sqrt{\Delta}\,\sigma$ and that $E\left[\,|g(t)|\,\right] \leq \bar{g}$ we have

$$E\left[U(\tau)\right] \leq \frac{\bar{g}}{\sqrt{\Delta}\,\sigma}$$

Hence:

$$n_w = 2\,E\left[U(\tau)\right] - \frac{1}{2} \leq 2\,\frac{\bar{g}}{\sqrt{\Delta}\,\sigma} - \frac{1}{2} = \frac{2}{\sqrt{\Delta}}\,\frac{1}{\sqrt{N_p}} - \frac{1}{2}\,. \;\square$$

Proof. (of Lemma 6) The proof proceeds in several steps. First we define a stopping time that counts consecutive price changes, the first an increase of size $2\tilde{g}$ and the second a decrease of $2\tilde{g}$, starting from a normalized desired price $g = 0$ and ending in the same value $g = 0$. Call this event a cycle. Because of the Markovian nature of g and because it starts and ends at the same value then consecutive cycles are independent so that the expected number of cycles is, by the fundamental law of renewal theory, the inverse of the expected duration of such a cycle. We know that by construction each cycle has 2 price changes of the same absolute value, $2\tilde{g}$. Second we decompose this into two events, whose expected values we compute separately. Third we use the fundamental theorem of renewal theory to compute the expected number of price changes per unit of time which do not involve a change in price plan. We use the following normalization for price changes within a plan:

$$p(t) = \begin{cases} p^*(t) + \tilde{g} & \text{if } g(t) > 0 \\ p^*(t) - \tilde{g} & \text{if } g(t) \leq 0 \end{cases} \tag{38}$$

Note that the normalization consists on charging $p(t) = p^*(t) - \tilde{g}$ when $g(t) = 0$. The normalization affects the definition below, but not the final result.

1. Define the stopping times τ^u and τ^d as:

$$\tau^u = \min\left\{t : p(t) - p(t-\Delta) = +2\,\tilde{g}\,,\; g(t) = \sqrt{\Delta}\sigma\,,\; g(0) = 0\,,\; t = \Delta, 2\Delta, \ldots\right\} \tag{39}$$

$$\tau^d = \min\left\{t : p(t) - p(t-\Delta) = -2\,\tilde{g}\,, g(t) = 0\,,\; g(0) = \sqrt{\Delta}\sigma\,,\; t = \Delta, 2\Delta, \ldots\right\} \tag{40}$$

In words τ^u is the time elapsed until the first price increase starting from a state where $g = 0$, i.e. at the beginning of a price plan. Instead τ^d is the time elapsed until the first price decrease starting from the state where $g = \sigma\sqrt{\Delta}$, i.e. after a price increase has just occurred. Note that at τ^d the state is the same as in the beginning of a price plan. The expected value of $\tau^u + \tau^d$ gives the expected value of a cycle of at least one price increase followed by a price decrease, within a price plan. In this cycle the initial state is equal to the final one, namely $g = 0$. Notice that in each cycle there are at least two price changes, one (or more) increases and one (or more) decreases. There could be more than two price changes because in each τ^u there could be price decreases and during each τ^d there can be price increases caused by changes of the plan.

2. We compute the expected value of τ^u and τ^d separately.

   (a) We discuss how to compute $E[\tau^u]$. For this quantity we use the operator T^u, for which $T^u(0) = E[\tau^u]$. The operator T^u is the expected first time for which g goes from 0 to $\sqrt{\Delta}\,\sigma$, which coincides with a price increase, conditional on $g(0) = 0$. Note that there may be none or several plan changes before this event occurs. The function T_u solves:

   $$T^u(i) = \Delta + \frac{1}{2}\left[T^u(i-1) + T^u(i+1)\right] \quad \text{for } i = -1, -2, ..., -\bar{n}+1$$

   which is a version of the backward Kolmogorov equation, and the boundary conditions: $T^u(-\bar{n}) = T^u(0)$, because when the price plan ends it is restarted at $g = 0$, or index $i = 0$, and $T^u(0) = \Delta + (1/2)T^u(-1)$, because at $g = \sqrt{\Delta}\sigma$, which is index $i = 1$ there is a price increase, and we stop counting time. We show that $T^u(i) = a + b\,i + c\,i^2$. First, the Kolmogorov Backward equation implies that $c = -\Delta$. We use this into the two boundary conditions. The boundary condition $T^u(-\bar{n}) = T^u(0)$ gives $a = a + b\bar{n} - \Delta(\bar{n})^2 = 0$ or $b = -\Delta(\bar{n})$. The boundary condition at $i = 0$ gives $a = \Delta + (1/2)[a - b - \Delta]$, or $a + b = \Delta$. These equations imply that $T^u(0) = a = \Delta - b = \Delta(1 + \bar{n})$.

   (b) Now we discuss how to compute $E[\tau^d]$. For this quantity we use the operator T^d, for which $T^d(1) = E[\tau^d]$. The operator T^d is the expected time for which g goes from $\sqrt{\Delta}\,\sigma$ to 0, which coincides with a price decrease, conditional on $g(0) = \sqrt{\Delta}\,\sigma$. Note that there may be none or several price plan changes before this event occurs, as well as none, one, or more price increases. The function T^d solves:

   $$T^d(i) = \Delta + \frac{1}{2}\left[T^d(i-1) + T^d(i+1)\right] \quad \text{for } i = 1, 2, ..., \bar{n}-1$$

   which is a version of the backward Kolmogorov equation, and the boundary conditions. At the top we have $T^d(\bar{n}) = T^u(0) + T^d(1)$, since at this point there is a price plan change which returns the process to $g = 0$ and thus there must be an increase in prices within a plan before we can have a decrease. The other boundary condition is $T^d(1) = \Delta + (1/2)T^d(2)$ which uses the fact that a price decrease within a price plan must occur when $g = \sqrt{\Delta}\sigma$ which correspond to the

$i = 1$ index. In this event we stop counting time. We try a solution of the type $T^d(i) = \alpha + \beta\, i + \gamma\, i^2$. Using the Kolmogorov Backward equation we obtain that $\gamma = -\Delta$. Using the boundary condition at the top, as well as the solution for $T^u(0)$, we obtain:

$$\alpha + \beta\bar{n} - \Delta\bar{n}^2 = \Delta(1 + \bar{n}) + \alpha + \beta - \Delta$$

This implies that $\beta = \Delta(\bar{n}^2 + 1)/(\bar{n} - 1)$. The other boundary gives:

$$\alpha + \beta - \Delta = \Delta + (1/2)\,[\alpha + \beta 2 - \Delta 4]$$

or $\alpha = (1/2)\alpha$ which implies $\alpha = 0$. Hence we have

$$T^d(1) = \beta - \Delta = \Delta(\bar{n}^2 + 1 - \bar{n} + 1)/(\bar{n} - 1) = \Delta\bar{n} + \Delta\frac{2}{\bar{n} - 1}$$

3. Now we use the previous result to obtain the desired expression for N_w. First note that

$$T^u(0) + T^d(1) = E[\tau^u] + E[\tau^d] = 2\Delta\bar{n} + \Delta\frac{2 + \bar{n} - 1}{\bar{n} - 1} = 2\Delta\bar{n} + \Delta\frac{1 + \bar{n}}{\bar{n} - 1}$$

Because the cycles start and end at $g = 0$ and consecutive cycles are independent, we can use the Fundamental theorem of renewal theory. Hence the expected number of cycles per unit of time is $1/(E[\tau^u] + E[\tau^d])$. Also recall that in each cycle there are at least two price changes, hence the expected number of price changes N_w per unit of time is at least two times the (reciprocal of) expected duration of the cycle, i.e.:

$$N_w \geq \frac{2}{2\Delta\bar{n} + \Delta\frac{1+\bar{n}}{\bar{n}-1}} = \frac{1}{\Delta\bar{n} + \Delta\frac{1+\bar{n}}{2(\bar{n}-1)}} \,.$$

Using $\sqrt{\Delta}\,\sigma\,\bar{n} = \bar{g}$ and $\bar{n} = \sqrt{1/(\Delta N_p)}$ we can write

$$N_w \geq \left(\sqrt{\frac{\Delta}{N_p}} + \frac{\Delta}{2}\left[\frac{1 + \sqrt{1/(\Delta N_p)}}{\sqrt{1/(\Delta N_p)} - 1}\right]\right)^{-1} \,.$$

$\square$

Proof. (of Proposition 8) The proof proceed by first defining a subset of the path at which the price spent at least αT of the time at the reference price. This will give a lower bound for F. The advantage is that this lower bound is easier to compute. Then we will show the proposition for the lower bound.

We first consider the most delicate case, i.e the continuous time case with $\Delta = 0$. We note that, without loss of generality given the symmetry in the model, we will consider that at the beginning $[0, T]$ the normalized desired price $g(0)$ is positive. Using the invariant distribution for the normalized desired prices, and conditioning that $g = g(0) > 0$ we have that it has density $f(g) = 2/\bar{g} - 2g/\bar{g}^2$. Fixing $g = g(0) > 0$ we can consider the path of price that will follow during $[0, T]$. If $0 < g(\omega, t) < \bar{g}$ for all $0 < t < \alpha T$ then the price will remain at $p(\omega, t) = p^*_-(\omega) + \tilde{g}$ where $p^*_-(\omega)$ is the ideal price at the start of the current price plan corresponding to this price path. Thus if $\alpha > 1/2$ the reference price in this path is $p^*_-(\omega) + \tilde{g}$. If otherwise for $g(\omega, t) = 0$ or $g(\omega, t) = \bar{g}$ at some $0 < t < \alpha T$, then the reference price may be a different number. If g reaches the upper bound there will be a new price plan. If g reaches zero then there will be price change within the plan. While in either of these two events, it is possible that, depending of what happens subsequently, that $p^*_-(\omega)$ still be the reference price, but we will ignore them obtaining a lower bound on F. We will denote our lower bound as $\tilde{F}(T, \alpha)$ which is given by:

$$F(T, \alpha) \geq \tilde{F}(T, \alpha) \equiv \int_0^{\bar{g}} \Pr\left\{0 < B(t) < \frac{\bar{g}}{\sigma} \text{ for all } t \in [0, \alpha T] \,|\, B(0) = \frac{g}{\sigma}\right\} f(g)\, dg$$

where B is a standard Brownian motion (BM). We will compute lower bounds for the probability of a BM not hitting a barrier as follows. First for short we will denote this probability as:

$$Q\left(\alpha T, \frac{\bar{g}}{\sigma} \,\Big|\, \frac{g}{\sigma}\right) \equiv \Pr\left\{0 < B(t) < \frac{\bar{g}}{\sigma} \text{ for all } t \in [0, \alpha T] \,|\, B(0) = \frac{g}{\sigma}\right\}$$

so we can write:

$$\tilde{F}(T, \alpha) = \int_0^{\bar{g}} Q\left(\alpha T, \frac{\bar{g}}{\sigma} \,\Big|\, \frac{g}{\sigma}\right) f(g)\, dg$$

We will break the interval $[0, \bar{g}]$ into three parts. Let $n \geq 2$ be an integer and let $a = \frac{1}{n}\frac{\bar{g}}{\sigma}$ so that:

$$Q\left(\alpha T, \frac{\bar{g}}{\sigma} \,\Big|\, \frac{g}{\sigma}\right) \geq \begin{cases} 0 & \text{if } \frac{g}{\sigma} \in [0\,,\, a) \\ Q\left(\alpha T, 2\,a \,\Big|\, a\right) & \text{if } \frac{g}{\sigma} \in [a\,,\, a(n-1)] \\ 0 & \text{if } \frac{g}{\sigma} \in (a(n-1)\,,\, a\,n] \end{cases}$$

where the inequality for the middle range it follows from immediately by the assumption that $\bar{g}/\sigma = na$ for $n \geq 2$. The density for the first hitting time of either of two barriers for a BM, from which we can obtain Q as follows:

$$Q\left(\alpha T, n\,a \,\Big|\, a\right) = \frac{2\pi}{a^2 n^2} \sum_{j=0}^{\infty} (2j+1)\,(-1)^j \cos\left[\pi(2j+1)\frac{(n-2)}{2n}\right] \int_{\alpha T}^{\infty} \exp\left(-\frac{(2j+1)^2\,\pi^2\,t}{2n^2 a^2}\right) dt$$

$$= \sum_{j=0}^{\infty} (-1)^j \cos\left[\pi(2j+1)\frac{(n-2)}{2n}\right] \frac{4}{(2j+1)\,\pi} \exp\left(-\frac{(2j+1)^2\,\pi^2\,\alpha T}{2n^2 a^2}\right)$$

and for $n = 2$ we get:

$$Q\left(\alpha T, 2\,a \,\Big|\, a\right) = \frac{2\pi}{a^2 4} \sum_{j=0}^{\infty} (2j+1)\,(-1)^j \int_{\alpha T}^{\infty} \exp\left(-\frac{(2j+1)^2\,\pi^2\,t}{8a^2}\right) dt$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{4}{(2j+1)\,\pi} \exp\left(-\left(\frac{(2j+1)\,\pi}{2}\right)^2 \frac{\alpha T}{2a^2}\right)$$

Clearly $Q\left(\alpha T, 2\,a \,\middle|\, a\right)$ is increasing in a, since for larger a the BM starts further away from the two barriers. Using the Gregory-Leibniz's formula for π, for any $\alpha T > 0$ we have:

$$\lim_{a\to\infty} Q\left(\alpha T, 2\,a \,\middle|\, a\right) = \sum_{j=0}^{\infty} (-1)^j \frac{4}{(2j+1)\,\pi} = 1$$

Thus, for any $\delta > 0$ we can find an A_δ such that $Q\left(\alpha T, 2\,a \,\middle|\, a\right) \geq 1 - \delta$ for $a > A_\delta$.

We also have:

$$\int_0^a f(g)dg + \int_{(n-1)a}^{na} f(g)dg = \frac{2}{n}$$

Thus

$$\tilde{F}(T, \alpha) \geq \left[1 - \frac{2}{n}\right] Q\left(\alpha T, 2\,a \,\middle|\, a\right) \geq \left[1 - \frac{2}{n}\right](1-\delta) \geq 1 - \epsilon$$

Hence setting n and a large enough we show the desired result. In particular set n and δ to satisfy

$$\frac{n}{n-2} < \frac{1}{1-\epsilon} \text{ and } 0 < \delta < 1 - \frac{n}{n-2}(1-\epsilon)$$

and let G to be higher than $G \geq nA_\delta$.

Now we briefly comment on the differences with the discrete time model with period length $\Delta > 0$. The summary is that steps of the proof are identical. There are only two minor differences. One is that the invariant distribution of g takes a discrete number of values, but it is also triangular, so the calculations that involve f are virtually identical. The second difference is that probability $Q\left(\alpha T, 2\,a \,\middle|\, a\right)$ does not take the expression we used above. Nevertheless, it is easy to see that for any $\Delta > 0$ one can in fact chose a large enough so that $Q\left(\alpha T, 2\,a \,\middle|\, a\right) = 0$. For instance, if $a/\left(\sigma\sqrt{\Delta}\right) > T\alpha/\Delta$ or equivalently if $a \geq T\,\alpha\,\sigma/\sqrt{\Delta}$ then $Q\left(\alpha T, 2\,a \,\middle|\, a\right) = 0$ since it will take at least $a/\left(\sigma\sqrt{\Delta}\right)$ consecutive ups (or downs) to hit either of the barriers. Thus in the discrete time case with length Δ we can take $n > 2/\epsilon$ (using the expression derived from the continuous case) and $G/\sigma \geq n\,T\,\alpha/\sqrt{\Delta} \geq 2\,T\,\alpha/\left(\sqrt{\Delta}\,\epsilon\right)$.

□

B Derivation of Hazard Rates

We compute the instantaneous hazard rate of price changes in two versions of our model. The first version has price plans that change when the (absolute value of the) normalized desired price $|g|$ reaches a critical value, a threshold that we denote by $\bar{g}$. We refer to this version as the menu cost version. We also consider another version where price plans are changed at exponentially distributed plans. In both cases we provide an analytical solution to the hazard rate of price changes. These analytical expressions depends on only one parameter, namely N_p: the expected number of price plans changes per unit of time. Both hazard rates are downward slopping, very much so for low durations (behaving approximately at $1/2t$), and they asymptote to different constants.

Hazard rate when price plans changes subject to menu cost. To describe the hazard rate in this case we discuss the mathematical objects we use to define them and compute them. These results comes from the analysis in Alvarez, Shimer, and Tourre (2015), which themselves borrow some results from Kolkiewicz (2002). In our model a price change occurs when either a new price plan is in place or when within the same price plan prices are changed. In either case, at the instant right before price change takes place, the value of the desired normalized price satisfies $g = 0$. Thus, we compute the hazard rate for the following objects. We take $g(0) = \epsilon$, with $0 < \epsilon < \bar{g}$ and consider the following three stopping times:

$$\tilde{\tau}(\epsilon) = \inf_t \{\sigma W(t) \leq 0 \,|\, \sigma W(0) = \epsilon\} \tag{41}$$

$$\bar{\tau}(\epsilon) = \inf_t \{\sigma W(t) \geq \bar{g} \,|\, \sigma W(0) = \epsilon\} \tag{42}$$

$$\tau(\epsilon) = \min \{\bar{\tau}(\epsilon)\, ,\, \tilde{\tau}(\epsilon)\} \tag{43}$$

where W is a standard Brownian notion, so that we can use the desired normalized price until a price plan as $g(t) = \sigma W(t)$. The stopping time $\tilde{\tau}$ gives the first time that the desired normalized price g reaches back to 0, and hence the price changes, in logs, by $2\,\tilde{g}$. Instead $\bar{\tau}$ gives the firms that the desired normalized price g reaches the upper barrier $\bar{g}$, and hence there is a new price plan, which new price. Thus, a price changes occurs, the first time that either event takes place, which is denoted by the stopping time τ. Note that in all cases we started with a normalized desired price equal to ϵ. Since right after price change $g = 0$, we will compute the limit of these stopping times as $\epsilon \to 0$. We require $g(0) = \epsilon$ to be small but strictly positive, because if we set $g(0)$ exactly equal to zero, then the distribution of τ is degenerate, i.e. $\tilde{\tau} = 0$ with probability one.[17] A convenient expressions for the distribution of $\bar{\tau}(\epsilon)$ and $\tilde{\tau}(\epsilon)$ can be found in Kolkiewicz (2002) expressions (15) and (16). In Alvarez, Shimer, and Tourre (2015), we derive the hazard rates, and compute the limit as $\epsilon \to 0$. Letting $h(t)$ the hazard rate of price changes, and adapting the expression in Alvarez, Shimer, and Tourre (2015) we obtain equation (11) in Proposition 3.

Hazard rate when price plans have exponentially distributed durations Again a price change occurs when either a new price plan is in place or within the same price plan. In this version we simply assume that price plans are changed at durations that are exponentially distributed, and independent of the normalized desired price g. This exponential distribution is assumed to have expected duration denoted by $1/N_p$, so N_p is the expected number of price plans per unit of time. Price changes within a plan are given by the stopping time as $\hat{\tau}$, define in equation (42). The price changes that occur within a price plan are described by the same (limit of) the stopping time $\tilde{\tau}$ define above. Thus the stopping time for price changes is given

[17] Give the symmetry of the problem we could have defined $\epsilon < 0$ and concentrate on the fist time that it comes back to zero, or it reaches $-\bar{g}$. Clearly we obtain the same stopping times.

by:

$$\tau(\epsilon) = \min\{\hat{\tau}\,,\, \tilde{\tau}(\epsilon)\} \tag{44}$$

where W is $g(t)$ are defined as above. Since $\hat{\tau}$ and $\tilde{\tau}(\epsilon)$ are independent, then the hazard rate is simple the sum of the two hazard rates. The hazard rate corresponding to $\hat{\tau}$ is simply N_p. The hazard rate corresponding to $\tilde{\tau}$ can be computed as the hazard rate corresponding to the first time that a BM (with zero and volatility σ) and that starts at $\epsilon > 0$ and reaches 0. This stopping time is distributed according to the stable Levy law with density and CDF equal to[18]:

$$f(t;\epsilon) = \frac{\epsilon}{\sigma\sqrt{2\,\pi\,t^3}}\, e^{-\frac{\epsilon^2}{2\,t\,\sigma^2}}$$
$$F(t;\epsilon) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\frac{\epsilon^2}{2\,t\,\sigma^2}}} e^{-z^2}\, dz\,.$$

Defining the hazard rate in terms of f and F, and taking ϵ to zero we obtain:

$$\tilde{h}(t) \equiv \lim_{\epsilon\to 0}\frac{f(t;\epsilon)}{1-F(t;\epsilon)} = \lim_{\epsilon\to 0}\frac{\frac{\epsilon}{\sigma\sqrt{2\,\pi\,t^3}}\, e^{-\frac{\epsilon^2}{2\,t\,\sigma^2}}}{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\frac{\epsilon^2}{2\,t\,\sigma^2}}} e^{-z^2}\, dz} = \frac{\frac{1}{\sigma\sqrt{2\,\pi\,t^3}}}{\frac{2}{\sqrt{\pi}}\sqrt{\frac{1}{2\,t\,\sigma^2}}} = \frac{1}{2\,t}\,,$$

where we use L'Hopital rule to evaluate the limit. Thus we have equation (12) in Proposition 3.

We briefly comment on the nature the limit hazard rates displayed in equation (11) and equation (12). We note that in continuous time both cases h_{MC} and h_{Exp} are not hazard rates that corresponds to a proper survivor function. The survivor function that correspond to $\epsilon = 0$ has $S(0) = 1$ and $S(t) = 0$ for all $t > 0$. The hazard rates in equation (11) and equation (12) are the limits of the approximation as $\epsilon \to 0$, so they should be regarded as approximations that are accurate for very small ϵ, or alternatively, as the hazard rates conditional on surviving a very small duration.[19]

C Plans with exponentially distributed duration

In this section we consider an alternative model to the menu cost model. Specifically, we assume that the duration of the price plans is exogenous and has a constant hazard rate λ, so that the duration of a plan is exponentially distributed. This version model corresponds to the well known Calvo (1983) pricing, if the price plan is a singleton. Thus this section can also be viewed as introducing price plans, or menu of prices, into the Calvo price setting. The reason for exploring this case is the pervasive use of the Calvo pricing in the sticky price literature. First we discuss the optimal value for $\tilde{g}$. Then we characterize output's cumulative

[18]See Alvarez, Borovičková, and Shimer (2015) for a derivation for the case of a BM with drift.

[19]The derivation in Alvarez, Borovičková, and Shimer (2015) takes the second limit, i.e. the hazard rates conditional on a strictly positive duration.

IRF to a monetary shock.

Optimal threshold $\tilde{g}$. The determination of the optimal threshold $\tilde{g}$ follows exactly the same logic as in the case where the firm must pay a fixed cost, and thus price plans has duration given by the first time a top or bottom thresholds $\bar{g}$ or $-\bar{g}$ is hit. Instead in this case the stopping time is given by an exponentially distributed random variable, independent of g. Using the same first order condition as in Section 2.1.

PROPOSITION 10. The optimal threshold for the exponentially distributed price plan is:

$$\tilde{g} = \frac{\sigma}{\sqrt{2\,(r+\lambda)}} \tag{45}$$

The result in equation (45) is intuitive: the threshold is increasing in σ since for higher values of it the deviations will be larger to each side. It is decreasing in $r+\lambda$ because this decreases the duration of the price plan, hence it is more likely that gaps will be smaller. Note also that it is the same as the limit obtained in Proposition 2 as $\bar{g} \to \infty$.

The firm's contribution to the IRF. The logic of the firm's contribution to the cumulative output response after a shock is the same as in the benchmark case discussed in the main text, so that the price gap is $\hat{p}(t) \equiv p(t) - p^*(t) = \tilde{g}\,\mathrm{sgn}\,(g(t)) - g(t)$ for $\tau_i \leq t < \tau_{i+1}$ as in Section 3. The difference concerns the stopping time that determines the change of plan, so the definition of $\hat{m}$ is the same as in equation (15). In this set-up the we have that $s(g)$ is:

$$s(g) = \mathbb{E}\left[\int_0^\tau 1_{g(t)\geq 0}\, dt \mid g(0) = g\right] = \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[1_{g(t)\geq 0} \middle|\, g(0) = g\right]\, dt \tag{46}$$

where we use that $e^{-\lambda t}$ is the probability that the price plan survived at time t, and $\mathbb{E}\left[1_{g(t)\geq 0} \middle|\, g(0) = g\right]$ is the fraction of paths that at t have positive $g(t)$, conditional on $g(0) = g$. The following lemma gives an expression for s:

LEMMA 8. The derivative of $D(g) = s(g) - s(-g)$, where s is equation (46) is given by:

$$D'(g) = \frac{2}{\sigma}\,\frac{e^{-\frac{g\sqrt{2\,\lambda}}{\sigma}}}{\sqrt{2\,\lambda}} \tag{47}$$

The invariant distribution of the normalized desired prices is described by the density $f(g)$ which is a Laplace distribution, i.e.:

$$f\,(g) = \frac{\sqrt{2\,\lambda}\,/\sigma}{2}\, e^{-\sqrt{2\,\lambda}/\sigma\,|g|} \quad \text{for all } g\,.\text{[20]} \tag{48}$$

[20] This is easily seen by noticing that the invariant density solves the Kolmogorov forward equation: $\lambda\, f\,(g) = \frac{\sigma^2}{2}\, f''\,(g)$ and also that $\int_0^\infty f(g)dg = 1/2$.

Notice that the definition of the cumulative real output effect in equation (16) is, again except for the specification of τ, the same. Likewise, equation (17) also holds. Simple computations than leads to

Lemma 9. With exponentially distributed revisions of plan the cumulative output effect after a small monetary shock δ is $\mathcal{M}(\delta) = \delta\ \mathcal{M}'(0) + o(\delta)$ where

$$\mathcal{M}'(0) = \frac{1}{2\ \lambda} = \frac{1}{2N_p} \tag{49}$$

For comparison with the well known Calvo pricing with $N_C = \lambda$ price adjustments per period we define $\mathcal{M}_C(\delta) = \int_{-\bar{g}}^{\bar{g}} m\,(g+\delta)\ f\,(g)\ dg$ as the cumulative impulse response where $f(g)$ is the same exponential density defined above.[21] Simple analysis along the lines followed above reveals that the cumulative real effect of a small monetary shock in the Calvo model are given by:

$$\mathcal{M}_C(\delta) \ = \ \delta\ \frac{1}{\lambda} + o(\delta) \ \approx \ \delta\ \frac{1}{N_C}$$

Proposition 11. Assume plans are adjusted at the exogenous constant rate λ. Let $N_p = \lambda$ be the mean number of plan changes per period. Let N_C denote the mean number of price changes per period in a Calvo model without plans. The ratio of the cumulative output responses in the two models is:

$$\lim_{\delta \downarrow 0, r \downarrow 0} \frac{\mathcal{M}(\delta)}{\mathcal{M}_C(\delta)} = \frac{N_C}{2\ N_p} \tag{50}$$

The proposition shows that, as was observed for the menu cost model, the introduction of the plans introduces a flexibility that reduces the real effects of monetary shocks *assuming the number of plan changes is the same across models*, i.e. $N_p = N_C$.

Table 3: Synopsis of theoretical effect of price-plans across models: $\mathcal{M}'(0)$

"Menu cost model"		"Calvo model"	
Without Price Plans	With Price Plans	Without Price Plans	With Price Plans
$\frac{1}{6\ N}$	$\frac{1}{18\ N_p}$	$\frac{1}{N}$	$\frac{1}{2\ N_p}$

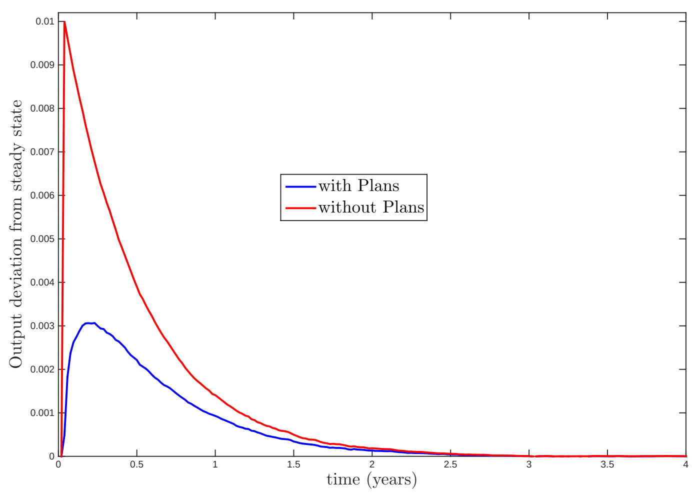
Note: N denotes the total number of price changes, N_p denotes the total number of plan changes.

[21] As noted above, the price gap in the Calvo model is $\hat{p} = -g$. Since the density f is symmetric around zero this is also the density of price gaps.

Table 3 provides a summary of the effects of introducing price plans in the various models where the notation there uses N_p the number of plans and N for the total number of price changes in the model without plans. The cumulative output response in a model with exponentially distributed plan's adjustments is 1/2 of the effect in the corresponding Calvo model, as it appears comparing the expressions in the third and fourth panels of the table with $N = N_p$. This result is to be compared with the one in Proposition 4 where, for small r the ratio was 1/3.[22]

Figure 6 reports the impulse response (numerically computed) to a 1 percent monetary shock ($\delta = 0.01$). The exercise assumes that the number of plan adjustment per period equals the number of price adjustments in the Calvo model (without plans): $N_p = N = \lambda$. The figure confirms that area under the IRF in Calvo is twice the area in the model with plans. Interestingly the figure shows that the introduction of plans gives rise to a hump-shaped profile of the output response: output does not respond on impact because the model with plans has a large impact effect of the monetary shock. This is very different from the model without plans and it is due to the fact that in the model with plans there is a mass point of firms responding on impact to the shock, as was also shown for the menu cost model in Section 3, whereas in the model with plans the max of firms responding on impact is typically negligible.

Figure 6: Output impulse response in model with exponential adjustments



Note: Response to a 1 percent $\delta = 0.01$ monetary shock. Parameters are $\sigma = 0.20, \lambda = 2, r = 0$. The number of plan adjustment per period equal the number of price adjustments in Calvo: $N_p = N = \lambda$.

Impact effect It is immediate to see that, as was the case for the menu cost model, the introduction of the plans leads to a non-negligible mass of adjustments on impact when

[22] The table also shows that for models without price plans, the area under the output's IRF in the menu cost model is 1/6 of the area in a Calvo model, a result first proved by Alvarez, Le Bihan, and Lippi (2014). For models with price plans, the table shows that ratio of the cumulated real effects is even smaller: the real effects of the menu cost model with plans is 1/9 of the real effect of a Calvo model with plans.

the shock occurs. This happens because the monetary shock δ shifts the distribution of the normalized desired prices $f(g)$ given in equation (48) and a mass of agents $\int_0^\delta f(g)\,dg$ switches from negative to positive values of g, therefore switching from the low to the high price within the price plan, i.e. each firm increases its price by $2\tilde{g}$. The next proposition summarizes this result

Proposition 12. The impact effect of a monetary shock δ on the aggregate price level is:

$$\lim_{r \to 0} \tilde{\Theta}(\delta) = \lim_{r \to 0} \tilde{g} \int_0^\delta f(g)\,dg = \delta \lim_{r \to 0} \sqrt{\frac{\lambda}{\lambda + r}} = \delta \quad .$$

The proof follows immediately by using the density in equation (48) and the expression for $\tilde{g}$ in equation (45). This result shows that the impact effect that results from the firm adjustments on impact yields an immediate jump of the price level of the same size of the monetary shock, so that output does not change at all on impact, as seen in the impulse response of Figure 6.

C.1 Proofs for the model with "Calvo" plans

Proof. (of Proposition 10)

$$\begin{aligned}
\tilde{g} &= \frac{\mathbb{E}\left[\int_0^\tau e^{-rt}\,|g(t)|\,dt \;\middle|\; g(0) = 0\right]}{\mathbb{E}\left[\int_0^\tau e^{-rt}\,dt \middle|\; g(0) = 0\right]} \\
&= \frac{\int_0^\infty \lambda\, e^{-(r+\lambda)t}\,\mathbb{E}\left[\,|g(t)|\,dt \;\middle|\; g(0) = 0\right]\,dt}{\int_0^\infty \lambda\, e^{-(r+\lambda)t}\,dt} = \frac{\int_0^\infty e^{-(r+\lambda)t}\,\sigma\sqrt{t\,2/\pi}\,dt}{\int_0^\infty e^{-(r+\lambda)t}\,dt} \\
&= \int_0^\infty (r+\lambda)\,e^{-(r+\lambda)t}\,\sigma\sqrt{2\,t/\pi}\,dt = \int_0^\infty (r+\lambda)\,e^{-(r+\lambda)t}\,\sigma\sqrt{2\,t\,/\pi}\,dt \\
&= \frac{\sigma}{\sqrt{2\,(r+\lambda)}}
\end{aligned}$$

where we use that $g(t)$ is, conditional on $g(0) = 0$, normally distributed with mean 0 and variance $\sigma^2\,t$, and hence $\mathbb{E}\left[\,|g(t)|\,dt \;\middle|\; g(0) = 0\right] = \sigma\sqrt{2\,t/\pi}$. The last line follows by performing the integration. $\square$

Proof. (of Lemma 8) Since $g(t)$ in normally distributed with mean g and variance $\sigma^2\,t$, so denoting by Φ the CDF of a standard normal, we can write

$$s(g) = \int_0^\infty e^{-\lambda\,t}\left[1 - \Phi\left(\frac{-g}{\sigma\sqrt{t}}\right)\right]dt\,, \text{ and } s(-g) = \int_0^\infty e^{-\lambda\,t}\left[1 - \Phi\left(\frac{g}{\sigma\sqrt{t}}\right)\right]dt\,.$$

Thus we have:

$$s'(g) = \int_0^\infty \frac{e^{-\lambda t}}{\sqrt{\sigma^2 t}} \phi\left(\frac{-g}{\sigma\sqrt{t}}\right) dt\,, \text{ and } s'(-g) = -\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{\sigma^2 t}} \phi\left(\frac{g}{\sigma\sqrt{t}}\right) dt$$

$$D'(g) = s'(g) + s'(-g) = 2\int_0^\infty \frac{e^{-\lambda t} e^{-\frac{1}{2}\left(\frac{g}{\sigma\sqrt{t}}\right)^2}}{\sqrt{2\pi\sigma^2 t}} dt$$

Note we can write:

$$e^{-\lambda t - \frac{1}{2}\left(\frac{g}{\sigma\sqrt{t}}\right)^2} = e^{-\frac{g^2+2\sigma^2\lambda t^2}{2\sigma^2 t}} = e^{-\frac{g\sqrt{2\lambda}}{\sigma}} e^{-\frac{g^2+2\sigma^2\lambda t^2-2g\sigma\sqrt{2\lambda}t}{2\sigma^2 t}} = e^{-\frac{g\sqrt{2\lambda}}{\sigma}} e^{-\frac{(g-\sigma\sqrt{2\lambda}t)^2}{2\sigma^2 t}}$$
$$= e^{-\frac{g\sqrt{2\lambda}}{\sigma}} e^{-\frac{(1-\sqrt{2\lambda}t(\sigma/g))^2}{2(\sigma/g)^2 t}}$$

Hence we can write:

$$D'(g) = 2\, e^{-\frac{g\sqrt{2\lambda}}{\sigma}} \int_0^\infty \frac{e^{-\frac{(1-\sqrt{2\lambda}t(\sigma/g))^2}{2(\sigma/g)^2 t}}}{\sqrt{2\pi\sigma^2 t}} dt$$
$$= 2\,\frac{e^{-\frac{g\sqrt{2\lambda}}{\sigma}}}{g} \int_0^\infty \frac{e^{-\frac{(1-\sqrt{2\lambda}t(\sigma/g))^2}{2(\sigma/g)^2 t}}}{\sqrt{2\pi(\sigma/g)^2 t}} dt = 2\,\frac{e^{-\frac{g\sqrt{2\lambda}}{\sigma}}}{g} \int_0^\infty \frac{e^{-\frac{(1-\sqrt{2\lambda}t(\sigma/g))^2}{2(\sigma/g)^2 t}}}{\sqrt{2\pi(\sigma/g)^2 t^3}}\, t\, dt$$

Using that the last term is the expected value of an inverse gaussian so that

$$\int_0^\infty \frac{e^{-\frac{(1-\sqrt{2\lambda}t(\sigma/g))^2}{2(\sigma/g)^2 t}}}{\sqrt{2\pi(\sigma/g)^2 t^3}}\, t\, dt = \frac{g/\sigma}{\sqrt{2\lambda}}$$

we have:

$$D'(g) = 2\,\frac{e^{-\frac{g\sqrt{2\lambda}}{\sigma}}}{g}\frac{(g/\sigma)}{\sqrt{2\lambda}} = \frac{2}{\sigma}\frac{e^{-\frac{g\sqrt{2\lambda}}{\sigma}}}{\sqrt{2\lambda}}$$

□

Proof. (of Proposition 11) Differentiating the definition of $\mathcal{M}$ we have:

$$\mathcal{M}'(0) = \frac{1}{\lambda} - 2\,\tilde{g}\int_0^\infty D'(g)\; f(g)\; dg = \frac{1}{\lambda} - 2\,\tilde{g}\int_0^\infty \frac{2}{\sigma}\frac{e^{-\frac{g\sqrt{2\lambda}}{\sigma}}}{\sqrt{2\lambda}}\frac{\sqrt{2\lambda}/\sigma}{2} e^{-\sqrt{2\lambda}/\sigma\, g}\, dg$$
$$= \frac{1}{\lambda} - 2\,\frac{\tilde{g}}{\sigma^2}\int_0^\infty e^{-\frac{g\sqrt{2\lambda}}{\sigma}} e^{-\sqrt{2\lambda}/\sigma\, g}\, dg = \frac{1}{\lambda} - 2\,\frac{\tilde{g}}{\sigma^2}\int_0^\infty e^{-\frac{g\,2\sqrt{2\lambda}}{\sigma}}\, dg$$
$$= \frac{1}{\lambda} - 2\,\frac{\tilde{g}}{\sigma^2}\frac{1}{\frac{2\sqrt{2\lambda}}{\sigma}} = \frac{1}{\lambda} - 2\,\frac{\tilde{g}}{\sigma^2}\frac{\sigma}{2\sqrt{2\lambda}}$$
$$= \frac{1}{\lambda} - \frac{\tilde{g}}{\sigma\sqrt{2\lambda}}$$

Replacing the optimal value of $\tilde{g}$:

$$\mathcal{M}'(0) = \frac{1}{\lambda} - \frac{\sigma}{\sigma\,\sqrt{2\,(r+\lambda)}\,\sqrt{2\,\lambda}}$$

Considering the case when $r \downarrow 0$:

$$\mathcal{M}'(0) = \frac{1}{\lambda} - \frac{1}{2\lambda}$$

Using that $\mathcal{M}_C(\delta) = \delta\left(1 - e^{-\lambda\,t}\right)$,

$$\lim_{\delta\downarrow 0, r\downarrow 0} \frac{\mathcal{M}_C(\delta)}{\mathcal{M}(\delta)} = \frac{1/\lambda}{\frac{1}{\lambda} - \frac{1}{2\lambda}} = 2$$

□

D Costly adjustments within plan

This appendix generalizes the model of the paper by assuming that prices changes within the plan, i.e. changes back and forth between the low and the high price within the plan, are also costly. In particular we assume the firm must pay a menu cost ν to change the price within the plan, and a larger menu cost ψ to change the plan.

Let $\nu > 0$ be the cost for a price change within the plan $-\tilde{g} \rightleftarrows +\tilde{g}$. Our baseline model assumes that $\nu = 0$. This modified problem gives rise to 2 value functions $v_h(\cdot), v_l(\cdot)$; symmetric : $\quad v_h(g) = v_l(-g)$, since the value of a given normalized price g depends on the price currently charged, i.e. $\pm\tilde{g}$.

In such a setting the optimal policy is given by 3 thresholds: $-\underline{g} \le 0 \le \tilde{g} < \bar{g}$, such that the profit maximizing firm sets the price $\tilde{g}$ as long as $g \in (-\underline{g}, \bar{g})$ and $-\tilde{g}$ for $g \in (-\bar{g}, \underline{g})$. We have that $\tilde{g}, \bar{g}$ and value functions $v_h(\cdot), v_l(\cdot)$ solve for all g:

$$r\,v_h(g) \le B\,\left(g - \tilde{g}\right)^2 + \frac{\sigma^2}{2}\,v_h''(g)$$
$$r\,v_l(g) \le B\,\left(g + \tilde{g}\right)^2 + \frac{\sigma^2}{2}\,v_l''(g)$$

with equality if inaction is optimal, and

$$v_h(g) \le \nu + v_l(g) \;\text{ and }\; v_l(g) \le \nu + v_h(g;\tilde{g})$$
$$v_h(g) \le \psi + v_h(0) \;\text{ and }\; v_l(g) \le \psi + v_l(0;\tilde{g})$$

if either changing from high to low price (or vice-versa) or if changing the plan is optimal. Thus at least one of this inequality must hold as equality at each g.

Solving the problems requires solving a system of 5 equations in 5 unknowns: $\underline{g}, \tilde{g}, \bar{g}$ and the 2 parameters of the second order ODE for the bellman equation. The five equations are given by:

$$\text{2 value matching at } \underline{g} \text{ and } \bar{g}: \quad v_h(-\underline{g}) = \nu + v_l(-\underline{g}) \quad , \quad v_h(\bar{g}) = \psi + v_h(0)$$

$$\text{smooth pasting at } \underline{g}: \quad v_h'(-\underline{g}) = v_l'(-\underline{g}) = -v_h'(\underline{g}) \quad \text{by symmetry}$$

$$\text{smooth pasting at } \bar{g}: \quad v_h'(\bar{g}) = 0 \quad \text{and the optimal return } \ v_h'(\tilde{g}) = 0 \ .$$

This system can be solved numerically to deliver the three optimal thresholds $-\underline{g} \leq 0 < \tilde{g} < \bar{g}$. The classic menu cost problem with one price is obtained when $\nu = \psi$ so that $\tilde{g} = 0, \bar{g} > 0$ and $\underline{g} = \bar{g}$. The price plan model discussed in the paper has $\psi > 0$ and $\nu = 0$ so that $\tilde{g} > 0, \bar{g} > 0$ and $\underline{g} = 0$.

Next, for given thresholds, we compute the density of the price gaps $f(g)$ as well as the density of high prices $\tilde{p}$, which we denote by $f_f(g)$, and the density of low prices $-\tilde{p}$, which we denote by $f_l(g)$. The density $f(g)$ is the usu The Kolmogorov forward equation $0 = f''(g)\sigma^2/2$, which implies a linear density function, and the boundary conditions $f(\bar{g}) = f(-\bar{g}) = 0$ (due to the fact that these are exit points and no mass can be accumulated here) imply that the density $f(g)$ is

$$f(g) = \begin{cases} \frac{\bar{g}+g}{\bar{g}^2} & \text{for } g \in [-\bar{g}, 0] \\ \frac{\bar{g}-g}{\bar{g}^2} & \text{for } g \in [0, \bar{g}] \end{cases} \tag{51}$$

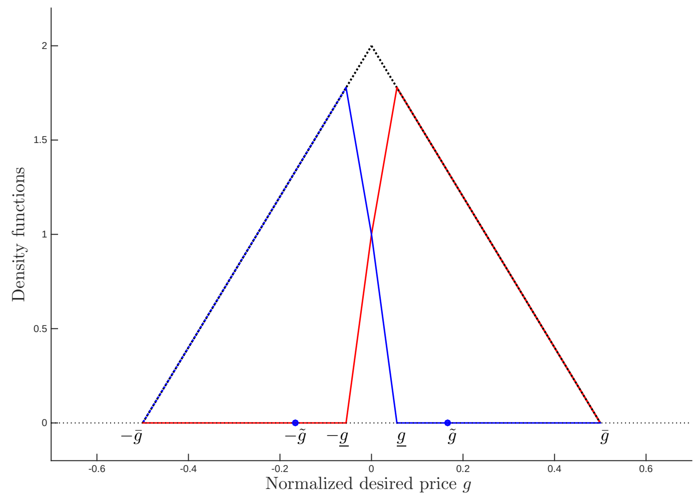
Notice that the densities $f_h(g), f_l(g)$ follow the same Kolmogorov equation, hence they are linear, but they have different boundaries. In particular we have that the density $f_h(g)$ is continuous in $(-\bar{g}, \bar{g})$, the density is zero between $[-\bar{g}, -\underline{g})$, it is upward sloping between $-(\underline{g}, 0)$, it is upward sloping between $(0, \underline{g})$, and it coincides with $f(g)$ between $(\underline{g}, \bar{g})$. Using linearity and the boundary conditions $f_h(-\underline{g}) = 0$, $f_h(0) = f(0)/2$ and $f_h(\underline{g}) = f(\underline{g})$ yields the following density for high prices

$$f_h(g) = \begin{cases} 0 & \text{for } g \in [-\bar{g}, -\underline{g}) \\ \frac{1}{2\bar{g}} + \frac{1}{2\bar{g}\underline{g}}\, g & \text{for } g \in [-\underline{g}, 0) \\ \frac{1}{2\bar{g}} + \left(\frac{1}{2\bar{g}\underline{g}} - \frac{1}{\bar{g}^2}\right) g & \text{for } g \in [0, \underline{g}) \\ \frac{1}{\bar{g}} - \frac{1}{\bar{g}^2} g & \text{for } g \in [\underline{g}, \bar{g}] \end{cases} \tag{52}$$

The density for low prices $f_l(g)$ is the symmetric counterpart of $f_h(g)$, in particular we have that $f_l(-g) = f_h(g)$. Figure 7 plots the two densities as an illustration of for the case in which $\nu > 0$ so the price gaps in the interval $(-\underline{g}, \underline{g})$ are associated with both high and low prices.

Next we use the solution in equation (52) to discuss the impact effect of a monetary shock. The next proposition shows that the impact effect is still approximate the same than the impact effect in a model where $\nu = 0$, i.e. first order with respect to the shock δ, as the fixed cost ν is small enough. More formally, the proposition states that the impact effect is continuous in ν, so that for small values of ν, which are necessary to get many temporary price changes as in the data, the impact effect is close to the impact of a model where $\nu = 0$:

Figure 7: Density function for high and low prices when $\nu > 0$



Proposition 13. Continuity of the impact effect on $\underline{g}$. Fix a $0 < \delta < \bar{g}$ and $\epsilon > 0$. Then there exist a $\underline{G}(\epsilon, \delta)$ such that for all $0 < \underline{g} < \underline{G}(\epsilon, \delta)$ the impact effect $|\tilde{\Theta}(\delta; \underline{g}) - \tilde{\Theta}(\delta; 0)| < \epsilon$.

Note that the optimal threshold $\underline{g} \to 0$ as the fixed cost $\nu \to 0$, so that the impact effect can be made arbitrarily close to the impact discussed in Proposition 5 in the main text.

E Price plans with iid shocks (TO BE FINISHED)

The models above assumed that the dynamics of the desired price followed a random walk, i.e. that innovations were permanent. While this extreme assumption is useful for analytical reasons, some previous quantitative analyses have focused on models where the distribution of the desired prices is stationary.

Assume the desired price x is drawn from a normal distribution with cdf $F(x)$ with zero mean (a normalization). The value function for a firm with price p before the price is drawn is

$$v(p) = \int_x \min \left\{ B(p-x)^2 + \ \beta v(p) \ , \ \psi + \min_{\hat{p}} \left(B(\hat{p}-x)^2 + \beta v(\hat{p}) \right) \right\} \ dF(x) \tag{53}$$

with the inaction region $\underline{x}, \bar{x}$ is a pair of barriers (for each p) defined by

$$B(p - \underline{x})^2 + \beta v(p) = \psi + \min_{\hat{p}} B(\hat{p} - \underline{x})^2 + \beta v(\hat{p})$$

and

$$B(p - \bar{x})^2 + \beta v(p) = \psi + \min_{\hat{p}} B(\hat{p} - \bar{x})^2 + \beta v(\hat{p})$$

such that for every p the inaction range is $\underline{x}(p) < x < \bar{x}(p)$ it is optimal to keep the price unchanged.

Upon adjustment, i.e. when for a given p we have $x \notin (\underline{x}(p), \bar{x}(p))$ then we have the optimal return point $\hat{p}(x)$ that satisfies

$$2B(\hat{p} - x) + \beta v'(\hat{p}) = 0 \qquad \text{or} \qquad \hat{p} = x - \frac{\beta v'(\hat{p})}{2B} \tag{54}$$

Notice that the optimal return point depends only on x (i.e. conditional on adjusting it is independent of the state p).

Inaction region. Define $\bar{p}$ as the value of $p > 0$ above which the value function is flat, i.e. it has a zero derivative, formally $v'(p) = 0$ for all $|x| > \bar{p}$. Notice that under this assumption if $p < \bar{p}$ and $x \geq \bar{p}$ then the new price is $\hat{p}(x) = x$ since $v'(x) = 0$.

Formally $\bar{p}$ solves

$$v(\bar{p}) = \int_x \min \left\{ B(\bar{p} - x)^2 + \ \beta v(\bar{p}) \ , \ \psi + \min_{\hat{p}} \left(B(\hat{p} - x)^2 + \beta v(\hat{p}) \right) \right\} \ dF(x)$$

If the state happens to be at $p > \bar{p}$ (as after a large shock) the price will be adjusted if the new price is sufficiently away from it. Otherwise it is left unchanged. The inaction region at $\bar{p}$ is $\underline{g}, \bar{g}$, where these values solve the following:

for $\bar{p} < x < \bar{p} + \bar{g}$ we have inaction, where $B\bar{g}^2 + \beta v(\bar{p}) = \psi + \beta v(\bar{p} + \bar{g}) \implies \bar{g} = \sqrt{\psi / B}$

for $\bar{p} - \underline{g} < x < \bar{p}$ we have inaction, where $B\underline{g}^2 + \beta v(\bar{p}) = \psi + \min_{\hat{p}} \left(B(\hat{p} - \bar{p} + \underline{g})^2 + \beta v(\hat{p}) \right)$

E.1 Introducing price plans

As before assume the desired price x is drawn from a normal distribution with cdf $F(x)$ with zero mean (a normalization). The firm now chooses a plan, namely a pair of prices $\mathcal{P} \equiv \{p_L, p_H\}$. Changing the plan involves a menu cost ψ, but price changes within the plan (from one price to the other) are free. The value function for a firm with price plan $\mathcal{P}$ before the price is drawn is

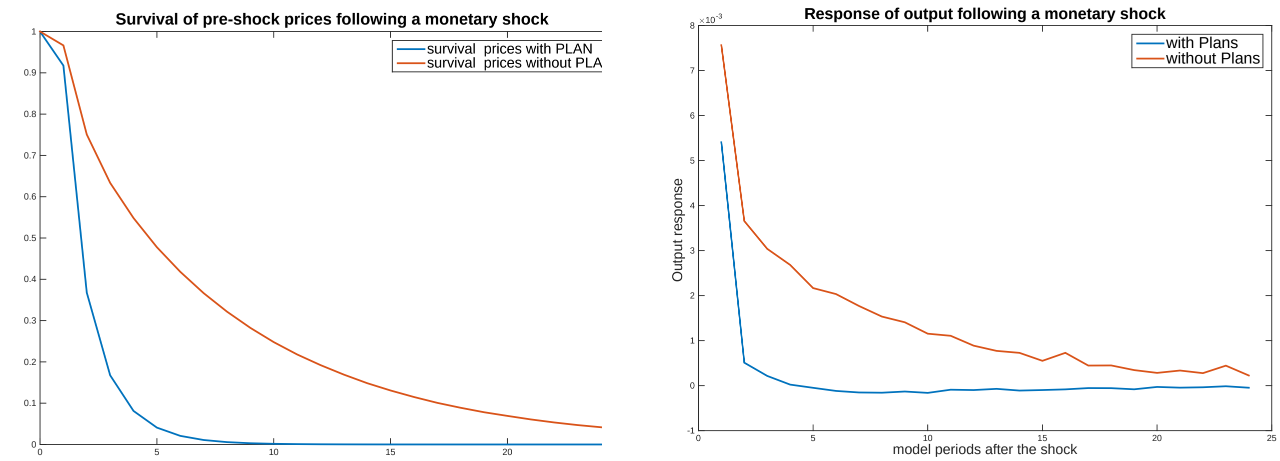
$$v(p_L, p_H) = \int_x \min \left\{ \min_{p_j \in \mathcal{P}} \left[B(p_j - x)^2 + \ \beta v(p_L, p_H) \right] , \ \psi + \min_{\{p'_L, p'_H\}} \left(B(\hat{p} - x)^2 + \beta v(p'_L, p'_H) \right) \right\} \ dF(x) \tag{55}$$

A new plan involves choosing 2 prices. We can think of one price as the optimal response to this period shock (which is expected to last only one period). The other price will be used to hedge against future expected shocks. Thus if one of the two prices is positive, the other one will be negative (remember that the distribution of desired prices is symmetric around zero).

The following symmetry properties are useful to speed up the computation:

$$v(p_L, p_H) = v(p_H, p_L) \qquad \text{symmetry around the 45\% line}$$

Figure 8: Comparison of IRF in the economy with iid shocks



Note: The simulation compares two economies, with and without price plans. Both economies are calibrated to have the same number of plan changes per quarter (for the economy without plans every price change is a plan change).

$$v(p_L, p_H) = v(-p_L, -p_H) \qquad \text{symmetry around the -45\% line}$$