State-Dependent Intellectual Property Rights Policy*

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Abstract

What form of intellectual property rights (IPR) policy contributes to economic growth? Should a company with a large technological lead receive the same IPR protection as a company with a more limited advantage? Should technological followers be able to license the products of technological leaders? We develop a general equilibrium framework to investigate these questions. The economy consists of many industries and firms engaged in cumulative (step-by-step) innovation. IPR policy regulates whether followers in an industry can copy the technology of the leader and also how much they have to pay to license past innovations. With full patent protection, followers can catch up to the leader in their industry either by making the same innovation(s) themselves or by making some pre-specified payments to the technological leaders.

We prove the existence of a steady-state equilibrium and characterize some of its properties. We then quantitatively investigate the implications of different types of IPR policy on the equilibrium growth rate and welfare. The two major results of this exercise are as follows. First, the growth rate and welfare in the standard models used in the (growth) literature can be improved significantly by introducing a simple form of licensing. Second and more importantly, full patent protection is not optimal from the viewpoint of maximizing welfare; instead, welfare-maximizing (and growth-maximizing) policy involves state-dependent IPR protection, providing greater protection to technological leaders that are further ahead than those that are close to their followers. This form of the welfare-maximizing policy is a result of the "trickle-down" effect, which implies that providing greater protection to firms that are further ahead of their followers than a certain threshold increases the R&D incentives also for all technological leaders that are less advanced than this threshold.

Keywords: competition, economic growth, endogenous growth, industry structure, innovation, intellectual property rights, licensing, patents, research and development, trickle-down.

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1 Introduction

How should the intellectual property rights of a company be protected? Should a firm with a large technological lead receive the same intellectual property rights (IPR) protection as a company with a more limited technological lead? These questions are central to many discussions of patent and competition policy. A recent ruling of the European Commission, for example, has required Microsoft to share secret information about its products with other software companies (New York Times, December 22, 2004). There is a similar debate about whether Apple should make iPod's code available to competitors that are producing complementary products. Central to these debates is the substantial technological lead that these companies have built over their rivals, which was viewed by the European Commission both as a source of excessive monopoly power and as an impediment to further technological progress in the industry. A systematic analysis of these policy questions and a full investigation of the effects of intellectual property rights on growth and welfare require a framework incorporating *state-dependent* patent/IPR protection policy. By state-dependent IPR policy, we mean a policy that makes the extent of patent or intellectual property rights protection conditional on the technology gap between different firms in the industry. Existing work has investigated the optimal length and breadth of patents assuming an IPR policy that does not allow for licensing and is uniform. In this paper, we make a first attempt to develop a framework that is rich enough to investigate these issues and we use it to study the implications, and the optimal form, of various IPR policies.

Our basic framework builds on and extends the step-by-step innovation models of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001), where a number of (typically two) firms engage in price competition within an industry and undertake R&D in order to improve the quality of their product. The technology gap between the firms determines the extent of the monopoly power of the leader, and hence the price markups and profits. The purpose of R&D by the follower is to catch up and surpass the leader (as in standard Schumpeterian models of innovation, e.g., Reinganum, 1981, 1985, Aghion and Howitt, 1992, Grossman and Helpman, 1991), while the purpose of R&D by the leader is to escape the competition of the follower and increase its markup and profits. As in racing-type models in general (e.g., Harris and Vickers, 1985, 1987, Budd, Harris and Vickers, 1993), a large gap between the leader and the follower discourages R&D by both. Consequently, overall R&D and technological progress are greater when the technology gap between the leader and the follower is relatively small.[1] One may expect that full patent protection may be suboptimal in a world of step-by-step competition; by stochastically or deterministically allowing the follower to use the innovations of the technological leader, the likelihood of relatively small gap between leaders and followers, and thus the amount of R&D, may be raised.[2] Based on this intuition, one may further conjecture that state-dependent IPR policy, when feasible, should provide *less* protection to firms that are technologically more advanced relative to their competitors.

There are two problems with this intuition, however. First, it is derived from models with uniform IPR policy, where relaxation of patent protection always discourages R&D. The major contribution of our paper will be to show that state-dependent relaxation of patent protection can increase R&D. This force will lead to the *opposite* of the above conjecture and show that optimal IPR policy involves providing *more* protection to firms that are technologically more advanced. This is because of *trickle-down of incentives*; providing relatively low protection to firms with limited leads and greater protection to those that have greater leads not only improves the incentives of firms that are technologically advanced, but also encourages R&D by those that have limited leads because of the prospect of reaching levels of technology gaps associated with greater protection. Second, this conjecture is based on models that do not allow licensing of the leading-edge technology. Introducing licensing changes the trade-offs underlying the above intuition and the implied form of the optimal IPR policy.[3]

To investigate these issues systematically, we construct a general equilibrium model with step-by-step innovation, potential licensing of patents and state-dependent IPR policy. In our model economy, each firm can climb the technology ladder via three different methods: (i) by "catch-up R&D," that is, R&D investments applied to a variant of the technology of the leader; (ii) by "frontier R&D," that is, building on the patented innovations of the technological leader for a pre-specified *license fee*; and (iii) as a result of the expiration of the patent of the technological leader.

The presence of various different forms of technological progress in this model allows for a range of different policy regimes. The first is *full patent protection with no licensing*, which corresponds to the environment assumed in existing growth models (e.g., Aghion, Harris, Howitt and Vickers, 2001) and provides full (indefinite) patent protection to technological leaders, but does not allow any licensing agreements (it sets the license fees to infinity). The second is

[1] Aghion, Bloom, Blundell, Griffith and Howitt (2005) provide empirical evidence that there is greater R&D in British industries where there is a smaller technological gap between firms. See O'Donoghue, Scotchmer and Thisse (1998) for a discussion of how patent life may come to an end because of related innovations.

[2] This is conjectured, for example, in Aghion, Harris, Howitt and Vickers (2001, p. 481).

[3] See Scotchmer (2005) for the importance of incorporating these types of licensing agreements into models of innovation.

full patent protection with (compulsory) licensing. This regime allows technological followers to build on the leading-edge technology in return for a pre-specified license fee. Licensing is "compulsory" in this regime in the sense that the patent holder does not have the right to refuse to license its innovation to a follower that is willing to pay the pre-specified license fee. There is "full patent protection" in the sense that patents never expire and the license fee is equal to the gain in net present value accruing to the follower because of its use of the leading-edge technology.[4] The third regime is *uniform imperfect patent protection*, which deviates from the previous two benchmarks by allowing either expiration of patents and/or license fees that are less than the full benefit to the follower. The adjective uniform indicates that in this policy environment all industries are treated identically regardless of the technology gap between the leader and the follower. The final and most interesting policy regime is *state-dependent imperfect patent protection*, which deviates from full patent protection as a function of the technology gap between the leader and the follower in the industry (i.e., it allows technologically more advanced firms to receive a different amount of IPR protection). Each of these policy regimes captures a different conceptualization of IPR policy and is interesting in its own right (and naturally, the last regime is general enough to nest the other three).

We first prove the existence of a stationary (steady-state) equilibrium under any of these policy regimes and characterize a number of features of the equilibrium analytically. For example, we prove that with uniform IPR policy, R&D investments decline when the gap between the leader and the follower increases.

We then turn to a quantitative investigation of welfare-maximizing ("optimal") IPR policy. We provide a simple calibration of our baseline model and then derive the optimal IPR given this economy. This calibration exercise only requires the choice of two parameters and the functional form for the R&D production function. Despite its simplicity and parsimony, the model generates reasonable numbers for the allocation of the workforce between production and research and the magnitude of profits in GDP. Our quantitative investigation leads to two major results:

1. Allowing for (compulsory) licensing of patents increases the equilibrium growth and welfare

[4] Compulsory license fees may be based on the damage that the use of the technology causes to the technological leader (because of loss of profits) or on the gain to followers from the use of superior technology. In practice, licensing fees or patent infringement fees reflect both the benefits to the firm using the knowledge and the damage to the original inventor (see, e.g., Scotchmer, 2005). In our analysis, we allow license fees to be set at any level, thus incorporating both possibilities. The analysis can also be extended to allow for bilateral licensing arrangements between the leader and the follower in the industry, for example at some license fee that results from a bargain between them. We will discuss this possibility in subsection 3.3 and argue that compulsory licensing fees typically improve welfare and growth relative to bilateral agreements.

of the economy significantly. Intuitively, without such licensing, a large part of the R&D effort goes to duplication, and followers' R&D does not directly contribute to growth. Licensing implies that R&D by all firms—not just the leaders—contributes to growth and also increases the R&D incentives of followers. In our benchmark parameterization, allowing for licensing increases the steady-state equilibrium growth rate of the economy from 1.86% to 2.58% per annum and also has a significant effect on steady-state welfare.

2. More importantly, we show that welfare-maximizing IPR policy is state dependent and provides *greater protection to firms that are technologically more advanced* (relative to technological leaders that only have a small lead over their followers). In particular, because of the disincentive effect of relaxing IPR protection on R&D, uniform IPR policy (either by manipulating license fees or the duration of patents) has a minimal effect on growth and welfare. In contrast, state-dependent IPR policy can significantly increase innovation, growth and welfare. For example, in our baseline parameterization, optimal state-dependent IPR policy increases the growth rate to 2.96% relative to the growth rate of 2.63% under (optimal) uniform IPR policy.

The reason why optimal IPR policy provides greater protection to technological leaders that are further ahead than their rivals is the *trickle-down effect*. When a particular state for the technological leader (say being n^* steps ahead of the follower) is very profitable, this increases the incentives to perform R&D not only for leaders that are $n^* - 1$ steps ahead, but for *all* leaders with a lead of size $n \leq n^* - 1$. The trickle-down effect makes state-dependent IPR, with greater protection for firms that are technologically more advanced than their rivals, preferable to uniform IPR. Another implication of the trickle-down effect is also worth noting. As is well known, uniform relaxation of IPR protection *reduces* R&D incentives (because innovation is rewarded less). However, because of the trickle-down effect, state-dependent relaxation of IPR may *increase* (average) R&D investments—because increasing protection at technology gap n^* and reducing it at $n^* - k$ creates a big boost to the R&D of firms with technological lead of $n^* - k$ steps. We will show that for plausible parameter values the amount of R&D is greater under imperfect state-dependent IPR protection than under full IPR protection, and this will be the main reason why state-dependent IPR can have a significant positive effect on economic growth.

Our paper is a contribution both to the IPR protection and the endogenous growth literatures. Previous work in industrial organization and in growth theory emphasizes that ex-post monopoly rents and thus patents are central for generating the ex-ante investments in R&D

and technological progress, even though monopoly power also creates distortions (e.g., Arrow, 1962, Reinganum, 1981, Tirole, 1988, Romer, 1990, Grossman and Helpman, 1991, Aghion and Howitt, 1992, Green and Scotchmer, 1995, Scotchmer, 1999, Gallini and Scotchmer, 2002, O'Donoghue and Zweimuller, 2004).[5] Much of the literature discusses the trade-off between these two forces to determine the optimal length and breadth of patents. For example, Klemperer (1990) and Gilbert and Shapiro (1990) show that optimal patents should have a long duration in order to provide inducement to R&D, but a narrow breadth so as to limit monopoly distortions. A number of other papers, for example, Gallini (1992) and Gallini and Scotchmer (2002), reach opposite conclusions.

Another branch of the literature, including the seminal paper by Scotchmer (1999) and the recent interesting papers by Llobet, Hopenhayn and Mitchell (2006) and Hopenhayn and Mitchell (2001), adopts a mechanism design approach to the determination of the optimal patent and intellectual property rights protection system. For example, Scotchmer (1999) derives the patent renewal system as an optimal mechanism in an environment where the cost and value of different projects are unobserved and the main problem is to decide which projects should go ahead. Llobet, Hopenhayn and Mitchell (2006) consider optimal patent policy in the context of a model of sequential innovation with heterogeneous quality and private information. They show that allowing for a choice from a menu of patents will be optimal in this context. To the best of our knowledge, no other paper in the literature has considered state-dependent IPR policy or developed the general equilibrium framework for IPR policy analysis. As a first attempt, we only look at state-dependent patent length and license fees (though similar ideas can be applied to an investigation of the gains from making the breadth of patent awards state-dependent).

Our paper is most closely related to and extends the results of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001) on endogenous growth with step-by-step innovation.[6] Although our model builds on these papers, it also differs from them in a number of significant ways. First, we allow licensing agreements whereby followers can pay a pre-specified license fee for building on the leading-edge technology developed by other firms. We show that such licensing has significant effects on growth and welfare. Second, our economy incorporates a general IPR policy that can be state dependent. Third, in our economy there is a general equilibrium interaction between production and R&D, since they both compete

[5] Boldrin and Levine (2001, 2004) or Quah (2003) argue that patent systems are not necessary for innovation.

[6] Segal and Whinston (2005) analyze the impact of anti-trust policy on economic growth in a related model of step-by-step innovation.

for scarce labor.[7] Finally, we provide a number of analytical results for the general model (with or without IPR policy), while previous literature has focused on the special cases where innovations are either "drastic" (so that the leader never undertakes R&D) or very small, and has not provided existence or general characterization results for steady-state equilibria in this class of economies.

Lastly, our results are also related to the literature on tournaments and races, for example, Fudenberg, Gilbert, Stiglitz and Tirole (1983), Harris and Vickers (1985, 1987), Choi (1991), Budd, Harris and Vickers (1993), Taylor (1995), Fullerton and McAfee (1999), Baye and Hoppe (2003), and Moscarini and Squintani (2004). This literature considers the impact of endogenous or exogenous prizes on effort in tournaments, races or R&D contests. In terms of this literature, state-dependent IPR policy can be thought of as "state-dependent handicapping" of different players (where the state variable is the gap between the two players in a dynamic tournament). To the best of our knowledge, these types of schemes have not been considered in this literature.

The rest of the paper is organized as follows. Section 2 presents the basic environment. Section 3 proves the existence of a steady-state equilibrium and characterizes some of its key properties under both uniform and state-dependent IPR policy. In this section, we also briefly discuss how bargaining over license fees can be incorporated into our framework and why compulsory licensing fees would have a useful role even in the presence of bilateral bargaining between technology leaders and followers. Section 4 quantitatively evaluates the implications of various different types of IPR policy regimes on welfare and characterizes the welfare-maximizing state-dependent IPR policies. Section 5 concludes, while the Appendix contains the proofs of all the results stated in the text.

2 Model

We now describe the basic environment. The characterization of the equilibrium under the different policy regimes is presented in the next section.

2.1 Preferences and Technology

Consider the following continuous time economy with a unique final good. The economy is populated by a continuum of 1 individuals, each with 1 unit of labor endowment, which they

[7] This general equilibrium aspect is introduced to be able to close the model economy without unrealistic assumptions and makes our economy more comparable to other growth models (Aghion, Harris, Howit and Vickers, 2001, assume a perfectly elastic supply of labor). We show that the presence of general equilibrium interactions does not significantly complicate the analysis and it is still possible to characterize the steady-state equilibrium.

supply inelastically. Preferences at time t are given by

$$\mathbb{E}_t \int_t^\infty \exp\left(-\rho\left(s-t\right)\right) \log C\left(s\right) ds, \tag{1}$$

where $\mathbb{E}_t$ denotes expectations at time t, $\rho > 0$ is the discount rate and $C(t)$ is consumption at date t. The logarithmic preferences in (1) facilitate the analysis, since they imply a simple relationship between the interest rate, growth rate and the discount rate (see (2) below).

Let $Y(t)$ be the total production of the final good at time t. We assume that the economy is closed and the final good is used only for consumption (i.e., there is no investment), so that $C(t) = Y(t)$. The standard Euler equation from (1) then implies that

$$g(t) \equiv \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = r(t) - \rho, \tag{2}$$

where this equation defines $g(t)$ as the growth rate of consumption and thus output, and $r(t)$ is the interest rate at date t.

The final good Y is produced using a continuum 1 of intermediate goods according to the Cobb-Douglas production function

$$\ln Y(t) = \int_0^1 \ln y(j,t)\, dj, \tag{3}$$

where $y(j,t)$ is the output of jth intermediate at time t. Throughout, we take the price of the final good as the numeraire and denote the price of intermediate j at time t by $p(j,t)$. We also assume that there is free entry into the final good production sector. These assumptions, together with the Cobb-Douglas production function (3), imply that the final good sector has the following demand for intermediates

$$y(j,t) = \frac{Y(t)}{p(j,t)}, \quad \forall j \in [0,1]. \tag{4}$$

Intermediate $j \in [0,1]$ comes in two different *varieties*, each produced by one of two infinitely-lived firms. We assume that these two varieties are perfect substitutes and these firms compete a la Bertrand.[8] Firm $i = 1$ or 2 in industry j has the following technology

$$y(j,t) = q_i(j,t)\, l_i(j,t) \tag{5}$$

[8] A more general case would involve these two varieties being imperfect substitutes, for example, with the output of intermediate j produced as

$$y(j,t) = \left[\varphi y_1(j,t)^{\frac{\sigma-1}{\sigma}} + (1-\varphi)\, y_2(j,t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

with $\sigma > 1$. The model analyzed in the text corresponds to the limiting case where $\sigma \to \infty$. Our results can be easily extended to this more general case with any $\sigma > 1$, but at the cost of additional notation. We therefore prefer to focus on the case where the two varieties are perfect substitutes. It is nonetheless useful to bear this formulation with imperfect substitutes in mind, since it facilitates the interpretation of "distinct" innovations by the two firms (when the follower engages in "catch-up" R&D).

where $l_i(j,t)$ is the employment level of the firm and $q_i(j,t)$ is its level of technology at time t. Each consumer in the economy holds a balanced portfolio of the shares of all firms. Consequently, the objective function of each firm is to maximize expected profits.

The production function for intermediate goods, (5), implies that the marginal cost of producing intermediate j for firm i at time t is

$$MC_i(j,t) = \frac{w(t)}{q_i(j,t)} \tag{6}$$

where $w(t)$ is the wage rate in the economy at time t.

When this causes no confusion, we denote the *technological leader* in each industry by i and the follower by $-i$, so that we have:

$$q_i(j,t) \geq q_{-i}(j,t).$$

Bertrand competition between the two firms implies that all intermediates will be supplied by the leader at the "limit" price:[9]

$$p_i(j,t) = \frac{w(t)}{q_{-i}(j,t)}. \tag{7}$$

Equation (4) then implies the following demand for intermediates:

$$y(j,t) = \frac{q_{-i}(j,t)}{w(t)} Y(t). \tag{8}$$

2.2 Technology, R&D and IPR Policy

R&D by the leader or the follower stochastically leads to innovation. We assume that when the leader innovates, its technology improves by a factor $\lambda > 1$.

The follower, on the other hand, can undertake R&D to catch up with the frontier technology or to improve over the frontier technology.[10] The first possibility is *catch-up R&D* and can be thought of R&D to discover an *alternative* way of performing the same task as the current leading-edge technology. Because this innovation applies to the follower's variant of the product (recall footnote 8) and results from its own R&D efforts, we assume that it does

[9] If the leader were to charge a higher price, then the market would be captured by the follower earning positive profits. A lower price can always be increased while making sure that all final good producers still prefer the intermediate supplied by the leader i rather than that by the follower $-i$, even if the latter were supplied at marginal cost. Since the monopoly price with the unit elastic demand curve is infinite, the leader always gains by increasing its price, making the price given in (7) the unique equilibrium price.

[10] A third possibility is for the follower to climb the technology ladder step-by-step, meaning that, for example, when the current leader is at some technology rung $n_{ij}(t)$ and the follower itself is at $n_{-ij}(t) < n_{ij}(t) - 1$, it must first discover technology $n_{-ij}(t) + 1$, et cetera. We have investigated this type of environment with "slow catch-up" in a previous version of the paper. Since the general results are similar, we do not discuss this variation to save space.

not constitute infringement on the patent of the leader, and the follower does not have to make any license fee payments. Therefore, if the follower chooses the first possibility, it will have to retrace the steps of the technological leader (corresponding to its own variant of the product), but in return, it will not have to pay a patent license fee. For follower firm $-i$ in industry j at time t, we denote this type of R&D by

$$a_{-i}(j,t) = 0.$$

The alternative, *frontier R&D*, involves followers building on and improving the current leading-edge technology. If this type of R&D succeeds, the follower will have improved the leading-edge technology using the patented knowledge of the technological leader, and thus will have to pay a license fee to the leader. The license fees may result from bargaining between the leader and the follower or they may be compulsory license fees imposed by policy. In either case, one must first characterize the equilibrium for a given sequence of license fees, which will be the main part of our analysis. We specify the license fees and how they vary below, and consistent with our main focus, we refer to them as "policy," though we will also discuss how they can be determined via bargaining.[11] This strategy is denoted by

$$a_{-i}(j,t) = 1.$$

Throughout, we allow $a_{-i}(j,t) \in [0,1]$ for mathematical convenience, thus a should be interpreted as the probability of frontier R&D by the follower.

R&D by the leader, catch-up R&D by the follower, and frontier R&D by the follower may have different costs and success probabilities. We simplify the analysis by assuming that all three types of R&D have the same costs and the same probability of success. In particular, in all cases, we assume that innovations follow a controlled Poisson process, with the arrival rate determined by R&D investments. Each firm (in every industry) has access to the following R&D technology:

$$x_i(j,t) = F(h_i(j,t)), \tag{9}$$

where $x_i(j,t)$ is the flow rate of innovation at time t and $h_i(j,t)$ is the number of workers hired by firm i in industry j to work in the R&D process at t. This specification implies that within a time interval of Δt, the probability of innovation for this firm is $x_i(j,t)\Delta t + o(\Delta t)$.

We assume that F is twice continuously differentiable and satisfies $F'(\cdot) > 0$, $F''(\cdot) < 0$, $F'(0) < \infty$ and that there exists $\bar{h} \in (0,\infty)$ such that $F'(h) = 0$ for all $h \geq \bar{h}$. The

[11] It should already be noted that the follower will never license the technology of the leader for production purposes, since this would lead to Bertrand competition and zero ex post profits for both parties.

assumption that $F'(0) < \infty$ implies that there is no Inada condition when $h_i(j,t) = 0$. The last assumption, on the other hand, ensures that there is an upper bound on the flow rate of innovation (which is not essential but simplifies the proofs). Recalling that the wage rate for labor is $w(t)$, the cost for R&D is therefore $w(t)\,G(x_i(j,t))$ where

$$G(x_i(j,t)) \equiv F^{-1}(x_i(j,t)), \tag{10}$$

and the assumptions on F immediately imply that G is twice continuously differentiable and satisfies $G'(\cdot) > 0$, $G''(\cdot) > 0$, $G'(0) > 0$ and $\lim_{x\to\bar{x}} G'(x) = \infty$, where

$$\bar{x} \equiv F\left(\bar{h}\right) \tag{11}$$

is the maximal flow rate of innovation (with $\bar{h}$ defined above).

We next describe the evolution of technologies within each industry. Suppose that leader i in industry j at time t has a technology level of

$$q_i(j,t) = \lambda^{n_{ij}(t)}, \tag{12}$$

and that the follower $-i$'s technology at time t is

$$q_{-i}(j,t) = \lambda^{n_{-ij}(t)}, \tag{13}$$

where $n_{ij}(t) \geq n_{-ij}(t)$ and $n_{ij}(t)$, $n_{-ij}(t) \in \mathbb{Z}_+$ denote the technology rungs of the leader and the follower in industry j. We refer to $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ as the *technology gap* in industry j. If the leader undertakes an innovation within a time interval of Δt, then its technology increases to $q_i(j, t+\Delta t) = \lambda^{n_{ijt}+1}$ and the technology gap rises to $n_j(t+\Delta t) = n_j(t) + 1$ (the probability of two or more innovations within the interval Δt will be $o(\Delta t)$, where $o(\Delta t)$ represents terms that satisfy $\lim_{\Delta t\to 0} o(\Delta t)/\Delta t$).

When the follower is successful in catch-up R&D (i.e., $a_{-i}(j,t) = 0$) within the interval Δt, then its technology improves to

$$q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}},$$

and the technology gap variable becomes $n_{jt+\Delta t} = 0$. In contrast, if the follower is successful in frontier R&D and pays the license fee (i.e., $a_{-i}(j,t) = 1$), then it surpasses the leading-edge technology, so we have

$$q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}+1}$$

and the technology gap variable becomes $n_{jt+\Delta t} = 1$ (and from this point onwards, the labels i and $-i$ are swapped, since the previous follower now becomes the leader).

In addition to catching up with or surpassing the technology frontier with their own R&D, followers can also copy the technology frontier because IPR policy is such that some patents expire. In particular, we assume that patents expire at some policy-determined Poisson rate η, and after expiration, followers can costlessly copy the frontier technology, jumping to $q_{-i}(j, t + \Delta t) = \lambda^{n_{ijt}}$.[12]

This description makes it clear that there are two aspects to IPR policy: (i) the length of the patent (modeled as a Poisson rate of arrival of the termination of the patent); (ii) the license fees. We allow both of these to be state dependent, so they are represented by the following two functions:

$$\boldsymbol{\eta} : \mathbb{N} \to \mathbb{R}_+$$

and for all $t \geq 0$,

$$\hat{\boldsymbol{\zeta}}(t) : \mathbb{N} \to \mathbb{R}_+ \cup \{+\infty\}.$$

Here $\eta(n) \equiv \eta_n < \infty$ is the flow rate at which the patent protection is removed from a technological leader that is n steps ahead of the follower. When $\eta_n = 0$, this implies that there is full protection at technology gap n, in the sense that patent protection will never be removed. In contrast, $\eta_n \to \infty$ implies that patent protection is removed immediately once technology gap n is reached. Similarly, $\hat{\zeta}(n, t) \equiv \hat{\zeta}_n(t)$ denotes the patent fee that a follower has to pay in order to build upon the innovation of the technological leader, when the technology gap in the industry is n steps.[13] Our formulation imposes that $\boldsymbol{\eta} \equiv \{\eta_1, \eta_2, ...\}$ is time-invariant, while $\hat{\boldsymbol{\zeta}}(t) \equiv \left\{\hat{\zeta}_1(t), \hat{\zeta}_2(t), ...\right\}$ is a function of time. This is natural, since in a growing economy, license fees should not remain constant. Below, we will require that $\hat{\boldsymbol{\zeta}}$ grows at the same rate as aggregate output in the economy.

When $\hat{\zeta}_n(t) = 0$, there is no protection because followers can license the leading-edge technology at zero cost.[14] In contrast, when $\hat{\zeta}_n(t) = \infty$, licensing the leading-edge technology is prohibitively costly. Note however that $\hat{\zeta}_n < \infty$ does not necessarily imply that patent protection is imperfect. In particular, in what follows we interpret a situation in which the

[12] Alternative modeling assumptions on IPR policy, such as a fixed patent length of $T > 0$ from the time of innovation, are not tractable, since they lead to value functions that take the form of delayed differential equations.

[13] Throughout, we assume that $\boldsymbol{\zeta}$ is a policy choice and firms cannot contract around it. An alternative approach would be to allow firms to bargain over the level of license fees. In this case, it is plausible to presume that the legally-specified infringement penalties or license fees will affect the equilibrium in the bargaining game, so the effect of policies we investigate would still be present. We do not allow bargaining between firms over the license fees in order to simplify the analysis.

[14] Throughout, we interpret $\zeta_n(t) = 0$ as $\zeta_n(t) = \varepsilon$ with $\varepsilon \downarrow 0$, so that followers continue not to license the new technology without innovation (recall the comment in footnote 11).

license fee is equal to the net extra gain from surpassing the leader rather than being neck-and-neck (i.e., being at a technology gap of 0) as "full protection".[15] We also refer to a policy regime as *uniform* IPR protection if both $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\zeta}}(t)$ are constant functions of n, meaning that intellectual property law treats all firms and industries identically regardless of the technology gap between the leader and the follower (i.e., $\boldsymbol{\eta}^{uni} \equiv \{\eta, \eta, ...\}$ and $\hat{\boldsymbol{\zeta}}^{uni}(t) \equiv \left\{\hat{\zeta}(t), \hat{\zeta}(t), ...\right\}$). We also assume that there exists some $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$ and $\hat{\zeta}_n(t) = \hat{\zeta}_{\bar{n}}(t)$ for all $n \geq \bar{n}$.

Given this specification, we can now write the law of motion of the technology gap in industry j as follows:

$$n_j(t+\Delta t) = \begin{cases} n_j(t)+1 & \text{with probability} \quad x_i(j,t)\,\Delta t + o(\Delta t) \\ 0 & \text{with probability} \quad \left((1-a_{-i}(j,t))\,x_{-i}(j,t) + \eta_{n_j(t)}\right)\Delta t + o(\Delta t) \\ 1 & \text{with probability} \quad a_{-i}(j,t)\,x_{-i}(j,t)\,\Delta t + o(\Delta t) \\ n_j(t) & \text{with probability} \quad 1-\left(x_i(j,t) + x_{-i}(j,t) + \eta_{n_j(t)}\right)\Delta t - o(\Delta t)) \end{cases}. \tag{14}$$

Here $o(\Delta t)$ again represents second-order terms, in particular, the probabilities of more than one innovations within an interval of length Δt. The terms $x_i(j,t)$ and $x_{-i}(j,t)$ are the flow rates of innovation by the leader and the follower; $a_{-i}(j,t) \in [0,1]$ denotes whether the follower is trying to catch up with a leader or surpass it; and $\eta_{n_j(t)}$ is the flow rate at which the follower is allowed to copy the technology of a leader that is $n_j(t)$ steps ahead. Intuitively, the technology gap in industry j increases from $n_j(t)$ to $n_j(t)+1$ if the leader is successful. When $a_{-i}(j,t) = 1$, the technology gap in industry j becomes 1 if the follower is successful (flow rate $x_{-i}(j,t)$). Finally, the firms become "neck-and-neck" when the follower comes up with an alternative technology to that of the leader (flow rate $x_{-i}(j,t)$) without using the license ($a_{-i}(j,t) = 0$) or the patent expires at the flow rate η_{n_j}.

[15] In other words, we interpret "full protection" to correspond to a situation in which $\zeta_n(t) \geq V_1(t) - V_0(t)$, where V_1 refers to the net present value of a firm that is one step ahead of its rival and V_0 is the value of a firm that is neck-and-neck with its rival. Alternatively, full protection could be interpreted as corresponding to the case in which the follower pays a license fee equal to the loss of profits that it causes for the technology leader (see Scotchmer, 2005). In our model, this would correspond to $\zeta_n(t) = V_0(t) - V_{-1}(t)$, where V_{-1} is the net present value of a firm that is one step behind the technology leader. In all equilibria we compute below, we find that the second amount is significantly less than the first, thus our notion of full protection licensing fee is large enough to cover both possibilities. In any case, what value of ζ is designated as "full protection" does not have any bearing on our formal analysis, since we characterize the equilibrium for any $\boldsymbol{\zeta}$ and then find the welfare-maximizing policy sequence.

2.3 Profits

We next write the instantaneous "operating" profits for the leader (i.e., the profits exclusive of R&D expenditures and license fees). Profits of leader i in industry j at time t are

$$\begin{aligned}\Pi_i(j,t) &= [p_i(j,t) - MC_i(j,t)]\, y_i(j,t) \\ &= \left(\frac{w(t)}{q_{-i}(j,t)} - \frac{w(t)}{q_i(j,t)}\right)\frac{Y(t)}{p_i(j,t)} \\ &= \left(1-\lambda^{-n_j(t)}\right) Y(t) \qquad (15)\end{aligned}$$

where $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ is the technology gap in industry j at time t. The first line simply uses the definition of operating profits as price minus marginal cost times quantity sold. The second line uses the fact that the equilibrium limit price of firm i is $p_i(j,t) = w(t)/q_{-i}(j,t)$ as given by (7), and the final equality uses the definitions of $q_i(j,t)$ and $q_{-i}(j,t)$ from (12) and (13). The expression in (15) also implies that there will be zero profits in neck-and-neck industries, i.e., in those with $n_j(t) = 0$. Also clearly, followers always make zero profits, since they have no sales.

The Cobb-Douglas aggregate production function in (3) is responsible for the form of the profits (15), since it implies that profits only depend on the technology gap of the industry and aggregate output. This will simplify the analysis below by making the technology gap in each industry the only industry-specific payoff-relevant state variable.

The objective function of each firm is to maximize the net present discounted value of "net profits" (operating profits minus R&D expenditures and plus or minus patent fees). In doing this, each firm will take the sequence of interest rates, $[r(t)]_{t\geq 0}$, the sequence of aggregate output levels, $[Y(t)]_{t\geq 0}$, the sequence of wages, $[w(t)]_{t\geq 0}$, the R&D decisions of all other firms and policies as given.

2.4 Equilibrium

Let $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ denote the distribution of industries over different technology gaps, with $\sum_{n=0}^{\infty} \mu_n(t) = 1$. For example, $\mu_0(t)$ denotes the fraction of industries in which the firms are neck-and-neck at time t. Throughout, we focus on Markov Perfect Equilibria (MPE), where strategies are only functions of the payoff-relevant state variables.[16] This allows us to drop the dependence on industry j, thus we refer to R&D decisions by x_n for the technological

[16] MPE is a natural equilibrium concept in this context, since it does not allow for implicit collusive agreements between the follower and the leader. While such collusive agreements may be likely when there are only two firms in the industry, in most industries there are many more firms and also many potential entrants, making collusion more difficult. Throughout, we assume that there are only two firms to keep the model tractable.

leader that is n steps ahead and by a_{-n} and x_{-n} for a follower that is n steps behind. Let us denote the list of decisions by the leader and the follower with technology gap n at time t by $\xi_n(t) \equiv \langle x_n(t), p_i(j,t), y_i(j,t)\rangle$ and $\xi_{-n}(t) \equiv \langle a_{-n}(t), x_{-n}(t)\rangle$.[17] Throughout, $\boldsymbol{\xi}$ will indicate the whole sequence of decisions at every state, so that $\boldsymbol{\xi}(t) \equiv \{\xi_n(t)\}_{n=-\infty}^{\infty}$. We define an allocation as follows:

Definition 1 ***(Allocation)*** *Let $< \boldsymbol{\eta},\ [\hat{\boldsymbol{\zeta}}(t)]_{t\geq 0} >$ be the IPR policy sequences. Then an allocation is a sequence of decisions for a leader that is $n = 0, 1, 2, ...$ step ahead, $[\xi_n(t)]_{t\geq 0}$, a sequence of R&D decisions for a follower that is $n = 1, 2, ...$ step behind, $\left[\xi_{-n}(t)\right]_{t\geq 0}$, a sequence of wage rates $[w(t)]_{t\geq 0}$, and a sequence of industry distributions over technology gaps $[\boldsymbol{\mu}(t)]_{t\geq 0}$.*

For given IPR sequences $\boldsymbol{\eta}$ and $[\hat{\boldsymbol{\zeta}}(t)]_{t\geq 0}$, MPE strategies, which are only functions of the payoff-relevant state variables, can be represented as follows

$$\begin{aligned} \mathbf{x} &: \mathbb{Z} \times \mathbb{R}_+^2 \times [0,1]^\infty \to \mathbb{R}_+, \\ \boldsymbol{a} &: \mathbb{Z}_- \backslash \{0\} \times \mathbb{R}_+^2 \times [0,1]^\infty \to [0,1]. \end{aligned}$$

The first mapping represents the R&D decision of a firm (both when it is the follower and when it is the leader in an industry) as a function of the technology gap, $n \in \mathbb{Z}$, the aggregate level of output and the wage, $(Y, w) \in \mathbb{R}_+^2$, and R&D decision of the other firm in the industry, $\tilde{\mathbf{x}} \in [0,1]^\infty$. The second function represents the follower's decision of whether to direct its R&D to catching up with or surpassing the leading-edge technology (or more precisely, it represents the probability with which the follower will choose to undertake R&D to surpass the leading edge technology). Consequently, we have the following definition of equilibrium:

Definition 2 ***(Equilibrium)*** *Given an IPR policy sequence $< \boldsymbol{\eta},\ [\hat{\boldsymbol{\zeta}}(t)]_{t\geq 0} >$, a Markov Perfect Equilibrium is given by a sequence $[\boldsymbol{\xi}^*(t), w^*(t), Y^*(t)]_{t\geq 0}$ such that (i) $[p_i^*(j,t)]_{t\geq 0}$ and $[y_i^*(j,t)]_{t\geq 0}$ implied by $[\boldsymbol{\xi}^*(t)]_{t\geq 0}$ satisfy (7) and (8); (ii) R&D policies $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ are best responses to themselves, i.e., $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ maximizes the expected profits of firms taking aggregate output $[Y^*(t)]_{t\geq 0}$, wages $[w^*(t)]_{t\geq 0}$, government policy $< \boldsymbol{\eta},\ [\hat{\boldsymbol{\zeta}}(t)]_{t\geq 0} >$ and the R&D policies of other firms $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ as given; (iii) aggregate output $[Y^*(t)]_{t\geq 0}$ is given by (3); and (iv) the labor market clears at all times given the wage sequence $[w^*(t)]_{t\geq 0}$.*

[17] The price and output decisions, $p_i(j,t)$ and $y_i(j,t)$, depend not only on the technology gap, aggregate output and the wage rate, but also on the exact technology rung of the leader, $n_{ij}(t)$. With a slight abuse of notation, throughout we suppress this dependence, since their product $p_i(j,t)\,y_i(j,t)$ and the resulting profits for the firm, (15), are independent of $n_{ij}(t)$, and consequently, only the technology gap, $n_j(t)$, matters for profits, R&D, aggregate output and economic growth.

2.5 The Labor Market

Since only the technological leader produces, labor demand in industry j with technology gap $n_j(t) = n$ can be expressed as

$$l_n(t) = \frac{\lambda^{-n} Y(t)}{w(t)} \quad \text{for} \quad n \in \mathbb{Z}_+. \tag{16}$$

In addition, there is demand for labor coming for R&D from both followers and leaders in all industries. Using (9) and the definition of the G function, we can express industry demands for R&D labor as

$$h_n(t) = G(x_n(t)) + G(x_{-n}(t)) \quad \text{for} \quad n \in \mathbb{Z}_+ , \tag{17}$$

where $G(x_n(t))$ and $G(x_{-n}(t))$ refer to the demand of the leader and the follower in an industry with a technology gap of n. Note that in this expression, $x_{-n}(t)$ refers to the R&D effort of a follower that is n steps behind (conditional on its optimal choice of $a_{-n}(t) \in [0,1]$).

The labor market clearing condition can then be expressed as:

$$1 \geq \sum_{n=0}^{\infty} \mu_n(t) \left[\frac{1}{\omega(t) \lambda^n} + G(x_n(t)) + G(x_{-n}(t)) \right], \tag{18}$$

and $\omega(t) \geq 0$, with complementary slackness, where

$$\omega(t) \equiv \frac{w(t)}{Y(t)} \tag{19}$$

is the labor share at time t. The labor market clearing condition, (18), uses the fact that total supply is equal to 1, and demand cannot exceed this amount. If demand falls short of 1, then the wage rate, $w(t)$, and thus the labor share, $\omega(t)$, have to be equal to zero (though this will never be the case in equilibrium). The right-hand side of (18) consists of the demand for production (the terms with ω in the denominator), the demand for R&D workers from the neck-and-neck industries ($2G(x_0(t))$ when $n = 0$) and the demand for R&D workers coming from leaders and followers in other industries ($G(x_n(t)) + G(x_{-n}(t))$ when $n > 0$).

Defining the index of aggregate quality in this economy by the aggregate of the qualities of the leaders in the different industries, i.e.,

$$\ln Q(t) \equiv \int_0^1 \ln q_i(j,t)\, dj, \tag{20}$$

the equilibrium wage can be written as:[18]

$$w(t) = Q(t) \lambda^{-\sum_{n=0}^{\infty} n \mu_n(t)}. \tag{21}$$

[18] Note that $\ln Y(t) = \int_0^1 \ln q_i(j,t)\, l(j,t)\, dj = \int_0^1 \left[\ln q_i(j,t) + \ln \frac{Y(t)}{w(t)} \lambda^{-n_j} \right] dj$, where the second equality uses (16). Thus we have $\ln Y(t) = \int_0^1 [\ln q_i(j,t) + \ln Y(t) - \ln w(t) - n_j \ln \lambda]\, dj$. Rearranging and canceling terms, and writing $\exp \int n_j \ln \lambda dj = \lambda^{-\sum_{n=0}^{\infty} n \mu_n(t)}$, we obtain (21).

2.6 Steady State and the Value Functions

Let us now focus on steady-state (Markov Perfect) equilibria, where the distribution of industries $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ is stationary, $\omega(t)$ defined in (19) and g, the growth rate of the economy, are constant over time. We will establish the existence of such an equilibrium and characterize a number of its properties. If the economy is in steady state at time $t = 0$, then by definition, we have $Y^*(t) = Y_0 e^{g^* t}$ and $w^*(t) = w_0 e^{g^* t}$, where g^* is the steady-state growth rate. These two equations also imply that $\omega(t) = \omega^*$ for all $t \geq 0$. Throughout, we assume that the parameters are such that the steady-state growth rate g^* is positive but not large enough to violate the transversality conditions. This implies that net present values of each firm at all points in time will be finite. This enables us to write the maximization problem of a leader that is $n > 0$ steps ahead recursively.

First note that given an optimal policy $\hat{x}$ for a firm, the net present discounted value of a leader that is n steps ahead at time t can be written as:

$$V_n(t) = \mathbb{E}_t \int_t^{\infty} \exp(-r(s-t)) \left[\Pi(s) + Z(s) - w(s) G(\hat{x}(s))\right] ds$$

where $\Pi(s)$ is the operating profit at time $s \geq t$, $Z(s)$ is the patent license fees received (or paid) by a firm which is the leader and $w(s) G(\hat{x}(s))$ denotes the R&D expenditure at time $s \geq t$. All variables are stochastic and depend on the evolution of the technology gap within the industry.

Next taking as given the equilibrium R&D policy of other firms, $x_{-n}^*(t)$ and $a_{-n}^*(t)$, the equilibrium interest and wage rates, $r^*(t)$ and $w^*(t)$, and equilibrium profits $\{\Pi_n^*(t)\}_{n=1}^{\infty}$ (as a function of equilibrium aggregate output), this value can be written as (see the Appendix for the derivation of this equation):[19]

$$r^*(t) V_n(t) - \dot{V}_n(t) = \max_{x_n(t)} \left\{ \begin{array}{c} \left[\Pi_n^*(t) - w^*(t) G(x_n(t))\right] + x_n(t)\left[V_{n+1}(t) - V_n(t)\right] \\ + \left(\left(1 - a_{-n}^*(t)\right) x_{-n}^*(t) + \eta_n\right)\left[V_0(t) - V_n(t)\right] \\ + \left(a_{-n}^*(t) x_{-n}^*(t) + \eta_n\right)\left[V_{-1}(t) - V_n(t) + \hat{\zeta}_n\right] \end{array} \right\}, \quad (22)$$

where $\dot{V}_n(t)$ denotes the derivative of $V_n(t)$ with respect to time. The first term is current profits minus R&D costs, while the second term captures the fact that the firm will undertake an innovation at the flow rate $x_n(t)$ and increase its technology lead by one step. The remaining terms incorporate changes in value due to catch-up by the follower (flow rate $\left(1 - a_{-n}^*(t)\right) x_{-n}^*(t) + \eta_n$ in the second line) and due to the follower leapfrogging the leader

[19] Clearly, this value function could be written for any arbitrary sequence of R&D policies of other firms. We set the R&D policies of other firms to their equilibrium values, $x_{-n}^*(t)$ and $a_{-n}^*(t)$, to reduce notation in the main body of the paper.

(flow rate $a^*_{-n}(t)\, x^*_{-n}(t)$ in the third line). In this last case, the follower will make a payment of $\hat{\zeta}_n$ to the leader for the license.

In steady state, the net present value of a firm that is n steps ahead, $V_n(t)$, will also grow at a constant rate g^* for all $n \in \mathbb{Z}_+$. Let us then define the normalized values as

$$v_n(t) \equiv \frac{V_n(t)}{Y(t)} \tag{23}$$

for all $n \in \mathbb{Z}$, which will be independent of time in steady state, i.e., $v_n(t) = v_n$. Similarly, in what follows we assume that license fees are also scaled up by GDP, so that

$$\zeta_n \equiv \frac{\hat{\zeta}_n(t)}{Y(t)},$$

which will ensure the existence of a (stationary) steady-state equilibrium.

Using (23) and the fact that from (2), $r(t) = g(t) + \rho$, the recursive form of the steady-state value function (22) can be written as:

$$\rho v_n = \max_{x_n} \left\{ \begin{array}{c} \left(1 - \lambda^{-n}\right) - \omega^* G(x_n) + x_n [v_{n+1} - v_n] \\ + \left[\left(1 - a^*_{-n}\right) x^*_{-n} + \eta_n\right][v_0 - v_n] + a^*_{-n} x^*_{-n} [v_{-1} - v_n + \zeta_n] \end{array} \right\} \text{ for } n \in \mathbb{N}, \tag{24}$$

where x^*_{-n} is the equilibrium value of R&D by a follower that is n steps behind, and ω^* is the steady-state labor share (while x_n is now explicitly chosen to maximize v_n).

Similarly the value for neck-and-neck firms is

$$\rho v_0 = \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x^*_0 [v_{-1} - v_0] \right\}, \tag{25}$$

while the values for followers are given by

$$\rho v_{-n} = \max_{x_{-n}, a_{-n}} \left\{ \begin{array}{c} -\omega^* G(x_{-n}) + [(1 - a_{-n}) x_{-n} + \eta_n][v_0 - v_{-n}] \\ + a_{-n} x_{-n} [v_1 - v_{-n} - \zeta_n] + x^*_n [v_{-n-1} - v_{-n}] \end{array} \right\} \text{ for } n \in \mathbb{N}, \tag{26}$$

which takes into account that if the follower decides to build upon the leading-edge technology, when it innovates it will become the new leader but will have to pay the patent fee ζ_n.

For neck-and-neck firms and followers, there are no instantaneous profits, which is reflected in (25) and (26). In the former case this is because neck-and-neck firms sell at marginal cost, and in the latter case, this is because followers have no sales. These normalized value functions emphasize that, because of growth, the effective discount rate is $r(t) - g(t) = \rho$ rather than $r(t)$.

The maximization problems in (24)-(25) immediately imply that any steady-state equilibrium R&D policies, $\langle \boldsymbol{a}^*, \mathbf{x}^* \rangle$, must satisfy:

$$a^*_{-n} \left\{ \begin{array}{ll} = 1 & \text{if } v_1 - \zeta_n > v_0 \\ \in [0,1] & \text{if } v_1 - \zeta_n = v_0 \\ = 0 & \text{if } v_1 - \zeta_n < v_0 \end{array} \right. \tag{27}$$

and

$$x_n^* = \max\left\{G'^{-1}\left(\frac{[v_{n+1}-v_n]}{\omega^*}\right),0\right\} \tag{28}$$

$$x_{-n}^* = \max\left\{G'^{-1}\left(\frac{\left(1-a_{-n}^*\right)[v_0-v_{-n}]+a_{-n}^*[v_1-v_{-n}-\zeta_n]}{\omega^*}\right),0\right\} \tag{29}$$

$$x_0^* = \max\left\{G'^{-1}\left(\frac{[v_1-v_0]}{\omega^*}\right),0\right\}, \tag{30}$$

where the normalized value functions, the vs, are evaluated at the equilibrium, and $G'^{-1}(\cdot)$ is the inverse of the derivative of the G function. Since G is twice continuously differentiable and strictly concave, G'^{-1} is continuously differentiable and strictly increasing. These equations therefore imply that innovation rates, the x_n^*s, will increase whenever the incremental value of moving to the next step is greater and when the cost of R&D, as measured by the normalized wage rate, ω^*, is less. Note also that since $G'(0) > 0$, these R&D levels can be equal to zero, which is taken care of by the max operator.

The response of innovation rates, x_n^*, to the increments in values, $v_{n+1} - v_n$, is the key economic force in this model. For example, a policy that reduces the patent protection of leaders that are $n + 1$ steps ahead (by increasing η_{n+1} or reducing ζ_{n+1}) will make being $n + 1$ steps ahead less profitable, thus reduce $v_{n+1} - v_n$ and x_n^*. This corresponds to the standard *disincentive effect* of relaxing IPR policy. In contrast to existing models, however, here relaxing IPR policy can also create a *positive incentive effect*. This novel incentive effect has two components. First, as equation (29) shows, weaker patent protection in the form of lower license fees (lower ζ) may encourage further frontier R&D by the followers, directly contributing to aggregate growth. Second and perhaps somewhat more paradoxically, lower protection for technological leaders that are $n + 1$ steps ahead will tend to reduce v_{n+1}, thus increasing $v_{n+2} - v_{n+1}$ and x_{n+1}^*. We will see that this latter effect plays an important role in the form of optimal state-dependent IPR policy. In addition to the incentive effects, relaxing IPR protection may also create a beneficial *composition effect*; this is because, typically, $\{v_{n+1} - v_n\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that x_{n-1}^* is higher than x_n^* for $n \geq 1$ (see, e.g., Proposition 2). Weaker patent protection (in the form of shorter patent lengths) will shift more industries into the neck-and-neck state and potentially increase the equilibrium level of R&D in the economy. Finally, weaker patent protection also creates a beneficial "level effect" by influencing equilibrium markups and prices (as shown in equation (7) above) and by reallocating some of the workers engaged in "duplicative" R&D to production. This level effect will also feature in our welfare computations. The optimal level and structure of IPR policy in this economy will be determined by the interplay of these various forces.

Given the equilibrium R&D decisions $\langle \boldsymbol{a}^*,\ \mathbf{x}^* \rangle$, the steady-state distribution of industries across states $\boldsymbol{\mu}^*$ has to satisfy the following accounting identities:

$$\left(x_{n+1}^* + x_{-n-1}^* + \eta_{n+1}\right) \mu_{n+1}^* = x_n^* \mu_n^* \text{ for } n \in \mathbb{N}, \tag{31}$$

$$\left(x_1^* + x_{-1}^* + \eta_1\right) \mu_1^* = 2x_0^* \mu_0^* + \sum_{n=1}^{\infty} a_{-n}^* x_{-n}^* \mu_n^*, \tag{32}$$

$$2x_0^* \mu_0^* = \sum_{n=1}^{\infty} \left(\left(1 - a_{-n}^*\right) x_{-n}^* + \eta_n\right) \mu_n^*. \tag{33}$$

The first expression equates exit from state $n+1$ (which takes the form of the leader going one more step ahead or the follower catching up for surpassing the leader) to entry into the state (which takes the form of a leader from state n making one more innovation). The second equation, (32), performs the same accounting for state 1, taking into account that entry into this state comes from innovation by either of the two firms that are competing neck-and-neck and also from followers that perform frontier R&D. Finally, equation (33) equates exit from state 0 with entry into this state, which comes from innovation by a follower in any industry with $n \geq 1$.

The labor market clearing condition in steady state can then be written as

$$1 \geq \sum_{n=0}^{\infty} \mu_n^* \left[\frac{1}{\omega^* \lambda^n} + G\left(x_n^*\right) + G\left(x_{-n}^*\right) \right] \text{ and } \omega^* \geq 0, \tag{34}$$

with complementary slackness.

The next proposition characterizes the steady-state growth rate. As with all the other results in the paper, the proof of this proposition is provided in the Appendix.

Proposition 1 *Let the steady-state distribution of industries and R&D decisions be given by* $< \boldsymbol{\mu}^*,\ \boldsymbol{a}^*,\ \mathbf{x}^* >$*, then the steady-state growth rate is*

$$g^* = \ln \lambda \left[2\mu_0^* x_0^* + \sum_{n=1}^{\infty} \mu_n^* \left(x_n^* + a_{-n}^* x_{-n}^*\right) \right]. \tag{35}$$

This proposition clarifies that the steady-state growth rate of the economy is determined by three factors:

1. R&D decisions of industries at different levels of technology gap, $\mathbf{x}^* \equiv \{x_n^*\}_{n=-\infty}^{\infty}$.

2. The distribution of industries across different technology gaps, $\boldsymbol{\mu}^* \equiv \{\mu_n^*\}_{n=0}^{\infty}$.

3. Whether followers are undertaking R&D to catch up with the frontier or to surpass the frontier, $\boldsymbol{a}^* \equiv \{a_n^*\}_{n=-\infty}^{-1}$.

IPR policy affects these three margins in different directions as illustrated by the discussion above.

3 Existence and Characterization of Steady-State Equilibria

We now define a steady-state equilibrium in a more convenient form, which will be used to establish existence and derive some of the properties of the equilibrium.

Definition 3 ***(Steady-State Equilibrium)*** *Given an IPR policy $< \boldsymbol{\eta},\ \boldsymbol{\zeta} >$, a steady-state equilibrium is a tuple $< \boldsymbol{\mu}^*,\ \mathbf{v},\ \boldsymbol{a}^*,\ \mathbf{x}^*,\ \omega^*,\ g^* >$ such that the distribution of industries $\boldsymbol{\mu}^*$ satisfy (31), (32) and (33), the values $\mathbf{v} \equiv \{v_n\}_{n=-\infty}^{\infty}$ satisfy (24), (25) and (26), the R&D decisions $\boldsymbol{a}^*$ and $\mathbf{x}^*$ are given by (27), (28), (29) and (30), the steady-state labor share ω^* satisfies (34) and the steady-state growth rate g^* is given by (35).*

We next provide a characterization of the steady-state equilibrium, starting first with the case in which there is uniform IPR policy.

3.1 Uniform IPR Policy

Let us first focus on the case where IPR policy is uniform. This means $\eta_n = \eta < \infty$ and $\zeta_n = \zeta < \infty$ for all $n \in \mathbb{N}$ and we denote these by $\boldsymbol{\eta}^{uni}$ and $\boldsymbol{\zeta}^{uni}$. In this case, (26) implies that the problem is identical for all followers, so that $v_{-n} = v_{-1}$ for $n \in \mathbb{N}$. Consequently, (26) can be replaced with the following simpler equation:

$$\rho v_{-1} = \max_{x_{-1},a_{-1}} \left\{-\omega^* G\left(x_{-1}\right) + \left[\left(1-a_{-1}\right)x_{-1} + \eta\right]\left[v_0 - v_{-1}\right] + a_{-1}x_{-1}\left[v_1 - v_{-1} - \zeta\right]\right\}, \quad (36)$$

implying optimal R&D decisions for all followers of the form

$$x_{-1}^* = \max\left\{G'^{-1}\left(\frac{\max\left\langle\left[v_0 - v_{-1}\right], \left[v_1 - v_{-1} - \zeta\right]\right\rangle}{\omega^*}\right), 0\right\}. \quad (37)$$

Let us denote the sequence of value functions under uniform IPR as $\{v_n\}_{n=-1}^{\infty}$. We next establish the existence of a steady-state equilibrium under uniform IPR and characterize some of its most important properties. Establishing the existence of a steady-state equilibrium in this economy is made complicated by the fact that the equilibrium allocation cannot be represented as a solution to a maximization problem. Instead, as emphasized by Definition

3, each firm maximizes its value taking the R&D decisions of other firms as given; thus an equilibrium corresponds to a set of R&D decisions that are best responses to themselves and a labor share (wage rate) ω^* that clears the labor market. Nevertheless, there is sufficient structure in the model to guarantee the existence of a steady-state equilibrium and monotonic behavior of values and R&D decisions.

Proposition 2 *Consider a uniform IPR policy* $< \boldsymbol{\eta}^{uni}, \boldsymbol{\zeta}^{uni} >$ *and suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$. *Then a steady-state equilibrium* $< \boldsymbol{\mu}^*, \mathbf{v}, a_{-1}^*, \mathbf{x}^*, \omega^*, g^* >$ *exists. Moreover, in any steady-state equilibrium* $\omega^* < 1$. *In addition, if either* $\eta > 0$ *or* $x_{-1}^* > 0$, *then* $g^* > 0$. *For any steady-state R&D decisions* $< a_{-1}^*, \mathbf{x}^* >$, *the steady-state distribution of industries* $\boldsymbol{\mu}^*$ *is uniquely determined.*

In addition, we have the following results:

- $v_{-1} \leq v_0$ *and* $\{v_n\}_{n=0}^{\infty}$ *forms a bounded and strictly increasing sequence converging to some* $v_\infty \in (0, \infty)$.
- $x_0^* > x_1^*$, $x_0^* \geq x_{-1}^*$, *and* $x_{n+1}^* \leq x_n^*$ *for all* $n \in \mathbb{N}$ *with* $x_{n+1}^* < x_n^*$ *if* $x_n^* > 0$. *Moreover, provided that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ *and* $\zeta > 0$, $x_0^* > x_{-1}^*$.

Proof. See the Appendix. ∎

Remark 1 The condition that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ ensures that there will be positive R&D in equilibrium. If this condition does not hold, then there exists a trivial steady-state equilibrium in which $x_n^* = 0$ for all $n \in \mathbb{Z}_+$, i.e., an equilibrium in which there is no innovation and thus no growth (this follows from the fact that $x_0^* \geq x_n^*$ for all $n \neq 0$, see the Appendix for more details). Moreover, if $\eta > 0$, then this equilibrium would also involve $\mu_0^* = 1$, so that in every industry two firms with equal costs compete a la Bertrand and charge price equal to marginal cost, leading to zero aggregate profits and a labor share of output equal to 1. The assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$, on the other hand, is sufficient to rule out $\mu_0^* = 1$ and thus $\omega^* = 1$. If, in addition, the steady-state equilibrium involves some probability of catch-up or innovation by the followers, i.e., either $\eta > 0$ or $x_{-1}^* > 0$, then the growth rate is also strictly positive. A sufficient condition to ensure that $x_{-1}^* > 0$ when $\eta = 0$ is that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right) > 0$.[20]

[20] To see why this condition is sufficient suppose that $\eta = 0$ and also that $x_{-1}^* = 0$. Then (36) immediately implies $v_{-1} = 0$ and (24) implies $v_1 \geq \left(1-\lambda^{-1}\right)/\rho$. Moreover, from (37) and the fact that $\omega^* \leq 1$, we have $x_{-1}^* \geq G'^{-1}\left(v_1 - v_{-1} - \zeta\right) \geq G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right)$. Therefore, $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right) > 0$ contradicts the hypothesis that $x_{-1}^* = 0$, and implies $x_{-1}^* > 0$. The reason why $\eta > 0$ can, under some circumstances, contribute to positive growth is related to the composition effect discussed above.

In addition to the existence of a steady-state equilibrium with positive growth, Proposition 2 shows that the sequence of values $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to some v_{∞}, and more importantly that $\mathbf{x}^* \equiv \{x_n^*\}_{n=1}^{\infty}$ is a decreasing sequence, which implies that technological leaders that are further ahead undertake less R&D. Intuitively, the benefits of further R&D are decreasing in the technology gap, since greater values of the technology gap translate into smaller increases in the equilibrium markup (recall (15)). Moreover, the R&D level of neck and-and-neck firms, x_0^*, is greater than both the R&D level of technological leaders that are one step ahead and technological followers that are one step behind (i.e., $x_0^* > x_1^*$ and $x_0^* \geq x_{-1}^*$). This implies that with uniform policy neck-and-neck industries are "most R&D intensive," while industries with the largest technology gaps are "least R&D intensive". This is the basis of the conjecture mentioned in the Introduction that reducing protection given to technologically advanced leaders might be useful for increasing R&D by bringing them into the neck-and-neck state.

3.2 State-Dependent IPR Policy

We now extend the results from the previous section to the environment with state-dependent IPR policy, though results on monotonicity of values and R&D efforts no longer hold.[21]

Proposition 3 *Consider the state-dependent IPR policy* $< \boldsymbol{\eta}, \boldsymbol{\zeta} >$ *and suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta_1\right)\right) > 0$. *Then a steady-state equilibrium* $< \boldsymbol{\mu}^*, \mathbf{v}, a_{-1}^*, \mathbf{x}^*, \omega^*, g^* >$ *exists. Moreover, in any steady-state equilibrium* $\omega^* < 1$. *In addition, if either* $\eta_1 > 0$ *or* $x_{-1}^* > 0$, *then* $g^* > 0$.

Proof. See the Appendix. ■

Unfortunately, it is not possible to determine the optimal (welfare- or growth-maximizing) state-dependent IPR policy analytically. For this reason, in Section 4, we undertake a quantitative investigation of the form and structure of optimal state-dependent IPR policy using plausible parameter values.

3.3 Compulsory Versus Bargained License Fees

The analysis so far has characterized the steady-state equilibrium for a given sequence of license fees $\boldsymbol{\zeta}$. Our interpretation in the next section will be that this sequence of license fees is

[21] This is because IPR policies could be very sharply increasing at some technology gap, making a particular state very unattractive for the leader. For example, we could have $\eta_n = 0$ and $\eta_{n+1} \to \infty$, which would imply that $v_{n+1} - v_n$ is negative.

determined by policy—i.e., these fees correspond to compulsory licensing fees for intellectual property that has been patented. We will thus imagine a world in which once a company patents an innovation, the knowledge embedded in this innovation can be used by its competitors as long as they pay a pre-specified licensing fee. The analysis in the next section will show that such licensing fees will increase growth and welfare.

One may also wish to consider an alternative world in which license fees are determined by bilateral bargaining. To characterize the equilibrium in such a world, one must conduct exactly the same analysis as we have done in this section. In other words, one must first characterize the equilibrium for a given sequence of license fees, and then taking the license fees agreed by other firms as given, one ought to consider the bargaining problem between a leader and a follower. As already noted in footnote 11, regardless of whether license fees are set by policy or are bargained, no follower will pay a positive price for a license for production (since this will simply lead to zero ex post profits). The only issue then becomes whether the leader and the follower in an industry can agree on a license fee if such fees are not specified by policy.

The answer to this question depends on the exact bargaining protocol between the two firms and there are *two scenarios*, which have quite different implications. The first scenario corresponds to a situation in which the leader and the follower can write a contract specifying the licensing fee *before* the follower undertakes R&D. In this case, the gain to the follower from licensing would be $v_1 - v_0$ (conditional on success in the innovation), while the loss to the leader would be $v_0 - v_{-1}$. As long as the first quantity is greater than the second (which may not always be the case), some level of ζ would be agreed between the two firms. Once this level is determined, then the equilibrium characterization so far applies with this level of license fee. However, this license fee would be *uniform*, not state dependent, since the gain to the follower and the loss to the leader are independent of the technology gap in the industry. Since our analysis in the next section shows significant gains from state-dependent IPR policy (including licensing fees), some type of compulsory licensing fee policy would improve over this bargained solution.

The potential welfare improvements from compulsory licensing policy are greater in the second scenario, where bargaining over the license fee would be undertaken *after* the innovation of the follower. In this scenario, a bargained price may not even emerge. In particular, the follower would have committed itself to an innovation using the patented knowledge of the leader and would have to bargain after its innovation efforts are sunk. In this case, the value of the license to a follower in an industry with an n-step gap is $v_1 - v_{-n}$ instead of $v_1 - v_0$, because if it cannot obtain a license, the follower would still remain n steps behind the leader.

Clearly $v_1 - v_{-n} > v_1 - v_0$, reflecting a form of *holdup* of the follower by the leader. This holdup may imply that there would be no licensing fee that the two parties can agree on (for example, because $v_1 - v_{-n}$ will be typically greater than the ex post loss of the leader, $v_0 - v_{-1}$). This second scenario, which appears to us more likely than the scenario with full contracting on the licensing fee before R&D, significantly reduces the role for bilateral licensing arrangements and implies that the type of compulsory licensing policies considered in the next section may create substantial welfare and growth benefits.

4 Optimal IPR Policy: A Quantitative Investigation

In this section, we investigate the implications of various different types of IPR policies on R&D, growth and welfare using numerical computations of the steady-state equilibrium. Our purpose is not to provide a detailed calibration of the model economy but to highlight the broad quantitative characteristics of the model and its implications for optimal IPR policy under plausible parameter values. We focus on welfare-maximizing policy (growth-maximizing policies are discussed below). We will see that the structure of optimal IPR policy and the innovation gains from such policy are relatively invariant to the range of parameter values we consider.

4.1 Welfare

Our focus so far has been on steady-state equilibria (mainly because of the very challenging nature of transitional dynamics in this class of models). In our quantitative analysis, we continue to focus on steady states and thus look at steady-state welfare. In a steady-state equilibrium, welfare at time $t = 0$ can be written as

$$\begin{aligned} \text{Welfare}\,(0) &= \int_0^\infty e^{-\rho t} \ln\left(Y\,(0)\, e^{g^* t}\right) dt \\ &= \frac{\ln Y\,(0)}{\rho} + \frac{g^*}{\rho^2}, \end{aligned} \tag{38}$$

where the first-line uses the facts that all output is consumed, utility is logarithmic (recall (1)), output and consumption at date $t = 0$ are given by $Y\,(0)$, and in the steady-state equilibrium output grows at the rate g^*. The second line simply evaluates the integral. Next, note that

$$\begin{aligned}\ln Y(t) &= \int_0^1 \ln y(j,t)\,dj \\ &= \int_0^1 \ln\left(\frac{q_{-i}(j,t)\,Y(t)}{w(t)}\right)dj \\ &= \int_0^1 \ln q_{-i}(j,t)\,dj - \ln\omega(t) \\ &= \ln Q(t) - \ln\lambda\left(\sum_{n=0}^{\infty} n\mu_n(t)\right) - \ln\omega(t),\end{aligned} \tag{39}$$

where the first line simply uses the definition in (3), the second line substitutes for $y(j,t)$ from (8), the third line uses the definition of the labor share $\omega(t)$, and the final line uses the definition of $Q(t)$ from (20) together with the fact that in the steady state $q_i(j,t) = \lambda^n q_{-i}(j,t)$ in a fraction $\mu_n(t)$ of industries. The expression in (39) implies that output simply depends on the quality index, $Q(t)$, the distribution of technology gaps, $\boldsymbol{\mu}(t)$ (because this determines markups), and also on the labor share, $\omega(t)$. In steady-state equilibrium, the distribution of technology gaps and labor share are constant, while output and the quality index grow at the steady-state rate g^*. Therefore, for steady-state comparisons of welfare across economies with different policies, it is sufficient to compare two economies with the same level of $Q(0)$, but with different policies. We can then evaluate steady-state welfare with the distribution of industries given by their steady-state values in the two economies, and output and the quality index growing at the corresponding steady-state growth rates. Expression (39) also makes it clear that only the aggregate quality index $Q(0)$ needs to be taken to be the same in the different economies. Given $Q(0)$, the dispersion of industries in terms of the quality levels has no effect on output or welfare (though, clearly, the distribution of industries in terms of technology gaps between leaders and followers, $\boldsymbol{\mu}$, influences the level of markups and output, and thus welfare).

However, note one difficulty with welfare comparisons highlighted by equations (38) and (39); proportional changes in steady-state welfare due to policy changes will depend on the initial level of $Q(0)$, which is an arbitrary number. Therefore, proportional changes in welfare are not informative (though none of this affects ordinal rankings, thus welfare-maximizing policy is well defined and independent of the level of $Q(0)$). These two expressions also make it clear that changes in steady-state welfare will be the sum of two components: the first is the *growth effect,* given by g^*/ρ^2, whereas the second is due to changes in $\ln\lambda\left(\sum_{n=0}^{\infty} n\mu_n\right)/\rho - \ln\omega(0)$. Since changes in the labor share $\omega(0)$ are largely driven by the distribution of industries, we refer to this as the *distribution effect*. Policies will typically affect both of these quantities. In what follows, we give the welfare rankings of different policies and then report the relative magnitudes of the growth and the distribution effects. This will show that the growth effects will be one or two orders of magnitude greater than the distribution effects and dominate welfare comparisons. So if the reader wishes, he or she may think of the magnitudes of the changes in welfare as given by the proportional changes in growth rates.

4.2 Calibration

For our calibration exercise, we take the annual discount rate as 5%, i.e., $\rho_{year} = 0.05$. In all our computations, we work with the monthly equivalent of this discount rate in order to increase precision, but throughout the tables, we convert all numbers to their annual counterparts to facilitate interpretation.

The theoretical analysis considered a general production function for R&D given by (9). The empirical literature typically assumes a Cobb-Douglas production function. For example, Kortum (1993) considers a function of the form

$$\text{Innovation}\,(t) = B_0 \exp\,(\kappa t)\,(\text{R\&D inputs})^{\gamma}\,, \tag{40}$$

where B_0 is a constant and $\exp(\kappa t)$ is a trend term, which may depend on general technological trends, a drift in technological opportunities, or changes in general equilibrium prices (such as wages of researchers etc.). The advantage of this form is not only its simplicity, but also the fact that most empirical work estimates a single elasticity for the response of innovation rates to R&D inputs. Consequently, they essentially only give information about the parameter γ in terms of equation (40). A low value of γ implies that the R&D production function is more concave. For example, Kortum (1993) reports that estimates of γ vary between 0.1 and 0.6 (see also Pakes and Griliches, 1980, or Hall, Hausman and Griliches, 1988). For these reasons, throughout, we adopt a R&D production function similar to (40):

$$x = Bh^{\gamma} \tag{41}$$

where $B, \gamma > 0$. In terms of our previous notation, equation (41) implies that $G(x) = [x/B]^{\frac{1}{\gamma}}\, w$, where w is the wage rate in the economy (thus in terms of the above function, it is captured by the $\exp(\kappa t)$ term).[22] Equation (41) does not satisfy the boundary conditions we imposed so far

[22] More specifically, (41) can be alternatively written as

$$\text{Innovation}\,(t) = Bw\,(t)^{-\gamma}\,(\text{R\&D expenditure})^{\gamma}\,,$$

thus would be equivalent to (40) as long as the growth of $w(t)$ can be approximated by constant rate.

and can be easily modified to do so without affecting any of the results, since in all numerical exercises only a finite number of states are reached.[23] Following the estimates reported in Kortum (1993), we start with a benchmark value of $\gamma = 0.35$, and then report sensitivity checks for $\gamma = 0.1$ and $\gamma = 0.6$. The other parameter in (41), B, is chosen so as to ensure an annual growth rate of approximately 1.9%, i.e., $g^* \simeq 0.019$, in the benchmark economy which features indefinitely-enforced patents and no licensing. This growth rate together with $\rho_{year} = 0.05$ also pins down the annual interest rate as $r_{year} = 0.069$ from equation (2).

We choose the value of λ using a reasoning similar to Stokey (1995). Equation (35) implies that if the expected duration of time between any two consecutive innovations is about 3 years in an industry, then a growth rate of about 1.9% would require $\lambda = 1.05$.[24] This value is also consistent with the empirical findings of Bloom, Schankerman and Van Reenen (2005).[25] We take $\lambda = 1.05$ as the benchmark value, and check the robustness of the results to $\lambda = 1.01$ and $\lambda = 1.2$ (expected duration of 1 year and 12 years, respectively). Finally, without loss of generality, we normalize labor supply to 1. This completes the determination of all the parameters in the model except the IPR policy.

As noted above, we begin with the full patent protection regime without licensing, i.e., $\langle \boldsymbol{\eta} = \{0, 0, ...\}, \boldsymbol{\zeta} = \{\infty, \infty, ...\} \rangle$.[26] We then compare this to an economy with full patent protection and licensing, i.e., $\langle \boldsymbol{\eta} = \{0, 0, ...\}, \boldsymbol{\zeta} = \{\bar{\zeta}, \bar{\zeta}, ...\} \rangle$, where $\bar{\zeta} = v_1 - v_0$.[27] We move to a comparison of the optimal (welfare-maximizing) uniform IPR policy $\boldsymbol{\eta}^{uni}, \boldsymbol{\zeta}^{uni}$ to the optimal state-dependent IPR policy. Since it is computationally impossible to calculate the optimal

[23] For example, we could add a small linear term to the production function for R&D, (41), and also make it flat after some level $\bar{h}$. For example, the following generalization of (41),

$$x = \min \left\{ Bh^{\gamma} + \varepsilon h; B\bar{h}^{\gamma} + \varepsilon \bar{h} \right\}$$

for ε small and $\bar{h}$ large, makes no difference to our simulation results.

[24] In particular, in our benchmark parameterization with full protection without licensing, 24% of industries are in the neck-and-neck state. This implies that improvements in the technological capability of the economy is driven by the R&D efforts of the leaders in 76% of the industries and the R&D efforts of both the leaders and the followers in 24% of the industries. Therefore, the growth equation, (35), implies that $g \simeq \ln \lambda \times 1.24 \times x$, where x denotes the average frequency of innovation in a given industry. A major innovation on average every three years implies a value of $\lambda \simeq 1.05$.

[25] The production function for the intermediate good, (5), can be written as $\log(y(j,t)) = n(j,t)\log(\lambda) + \log(l(j,t))$, where $n(j,t)$ is the number of innovations to date in sector j and represents the "knowledge stock" of this industry. Bloom, Schankerman and Van Reenen (2005) proxy the knowledge stock in an industry by the stock of R&D in that industry and estimate the elasticity of sales with respect to the stock of R&D to be approximately 0.06. In terms of the exercise here, this implies that $\log(\lambda) = 0.06$, or that $\lambda \approx 1.06$.

[26] Here $\zeta = \infty$ stands for ζ sufficiently large so that there is no licensing. It does not need to be literally equal to infinity and in fact, in the theoretical analysis we presumed that it is equal to some finite number.

[27] For the interpretation of full patent protection as $\bar{\zeta} = v_1 - v_0$, recall the discussion in footnote 15. Note also that at a license fee of $\bar{\zeta}$, followers are indifferent between $a = 0$ and $a = 1$, and in computing the equilibrium in this case we always suppose that they choose $a = 1$. Thus alternatively one might wish to think that $\bar{\zeta} = v_1 - v_0 - \varepsilon$ for $\varepsilon \downarrow 0$.

value of each η_n and ζ_n, we limit our investigation to a particular form of state-dependent IPR policy, whereby the same η and ζ applies to all industries that have a technology gap of $n = 5$ or more. In other words, the IPR policy can be represented as:

IPR policy $\rightarrow$	*none*	(η_1, ζ_1)	(η_2, ζ_2)	(η_3, ζ_3)	(η_4, ζ_4)	(η_5, ζ_5)
Technology gap: $n \rightarrow$	0	1	2	3	4	5 6 7 8 9 10 11 . . ∞

We checked and verified that allowing for further flexibility (e.g., allowing η_5 and η_6 or ζ_5 and ζ_6 to differ) has little effect on our results.

The numerical methodology we pursue relies on *uniformization* and value function iteration. The details of the uniformization technique are described in the proof of Lemma 1 in the Appendix. On value function iteration, see Judd (1999). In particular, we first take the IPR policies $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ as given and make an initial guess for the equilibrium labor share ω^*. Then for a given ω^*, we generate a sequence of values $\{v_n\}_{n=-\infty}^{\infty}$, and we derive the optimal R&D policies, $\{x_n^*\}_{n=-\infty}^{\infty}$, $\{a_n^*\}_{n=-\infty}^{-1}$ and the steady-state distribution of industries, $\{\mu_n^*\}_{n=0}^{\infty}$. After convergence, we compute the growth rate g^* and welfare, and then check for market clearing in the labor market from equation (18). Depending on whether there is excess demand or supply of labor, ω^* is varied and the whole process is repeated until the entire steady-state equilibrium for a given IPR policy is computed. The process is then repeated for different IPR policies.

In the state-dependent IPR case, the optimal (welfare-maximizing) IPR policy sequences, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$, are computed one element at a time, until we find the welfare-maximizing value for that component, for example, η_1. We then move the next component, for example, η_2. Once the welfare-maximizing value of η_2 is determined, we go back to optimize over η_1 again, and this procedure is repeated recursively until convergence.

4.3 Full IPR Protection Without Licensing

We start with the benchmark with full protection and no licensing, which is the case that the existing literature has considered so far (e.g., Aghion, Harris, Howitt and Vickers, 2001). In terms of our model, this corresponds to $\eta_n = 0$ for all n and $\zeta_n = \infty$ for all n. Equation (27) implies that $a_{-n}^* = 0$ for all n. We choose the parameter B in terms of (41), so that the benchmark economy has an annual growth rate of 1.86%.

The value function for this benchmark case is shown in Figure 1 (with the solid line). The value function has decreasing differences for $n \geq 0$, which is consistent with the results in Proposition 2, and features a constant level for all followers (since there is no state dependence in the IPR policy). Figure 2 shows the level of R&D efforts for leaders and followers in this benchmark (with the solid lines). Again consistent with Proposition 2, this figure also shows that the R&D level of a leader declines as the technology gap increases and that the highest level of R&D is for firms that are neck-and-neck (i.e., at the technology gap of $n = 0$). Since there is no state-dependent IPR policy, all followers undertake the same level of R&D effort, which is also shown in the figure.

Figure 3 shows the distribution of industries according to technology gaps (again the solid line refers to the benchmark case). The mode of the distribution is at the technology gap of $n = 1$, but there is also a concentration of industries at technology gap $n = 0$, because $a^*_{-n} = 0$ implies that innovations by the followers take them to the "neck-and-neck" state.

The first column of Table 1 also reports the results for this benchmark simulation. As noted above, B is chosen such that the annual growth rate is equal to 0.0186, which is recorded at the bottom of Table 1 together with the initial consumption and welfare levels according to (3) and (38) respectively. The table also shows the R&D levels x^*_0 and x^*_{-1} (0.35 versus 0.22), the frequencies of industries with technology gaps of 0, 1 and 2. The steady-state value of ω is 0.95. Since labor is the only factor of production in the economy, ω^* should not be thought of as the labor share in GDP. Instead, $1 - \omega^*$ measures the share of pure monopoly profits in value added. In the benchmark parameterization, this corresponds to 5% of GDP, which is reasonable.[28] Finally, the table also shows that in this benchmark parameterization 3.2% of the workforce is working as researchers, which is also consistent with US data.[29] These results are encouraging for our simple calibration exercise, since with very few parameter choices, the model generates reasonable numbers, especially for the share of the workforce allocated to research.[30]

4.4 Full IPR Protection With Licensing

We next turn to full IPR protection with licensing. As specified above, we think of full IPR protection with licensing as corresponding to $\eta_n = 0$ for all n (so that patents never expire) and $\zeta_n = \bar{\zeta} \equiv v_1 - v_0$ for all n (so that the license fee for making use of a leading-edge

[28] Bureau of Economic Analysis (2004) reports that the ratio of before-tax profits to GDP in the US economy in 2001 was 7% and the after-tax ratio was 5%.

[29] According to National Science Foundation (2006), the ratio of scientists and engineers in the US workforce in 2001 is about 4%.

[30] Most endogenous growth models imply that a significantly greater fraction of the labor force should be employed in the research sector and one needs to introduce various additional factors to reduce the profitability of research or to make entry into research more difficult. In the current model, the step-by-step nature of innovation and competition plays this role and generates a plausible allocation of workers between research and production.

technology is equal to the net present discounted value gap between being a one step ahead leader and a neck-and-neck firm). Figures 1-3 show the corresponding value functions, R&D effort levels and distribution of industries for this case (with the dashed lines). Since there is no state-dependent policy, the general pattern is similar to that in the economy without licensing. There is no longer a spike in R&D effort at $n = 0$, however, since now firms always prefer to pay the license fee and jump ahead of the leading-edge technology. This makes the neck-and-neck state no longer special (in fact, as column 2 of Table 1 shows, in equilibrium there will be no industries in the neck-and-neck state). More importantly, the level of R&D by followers is considerably higher than in the benchmark case. In particular, x^*_{-1} is now 0.25 rather than 0.22. The resulting growth rate is 2.58% instead of 1.86%. Correspondingly, welfare increases by 5.76 points because of the growth effect (that is, g^*/ρ^2 increases by 5.76) and declines by 0.03 points because of the distribution effect (in particular, because of the change in the composition of markups). This case therefore illustrates the general pattern mentioned above, whereby the growth effect is one or two orders of magnitudes greater than the distribution effect and dominates the welfare implications of alternative policies.

It is important to note that, in this case, the boost to growth and welfare comes not from increased R&D effort, but from the fact that the R&D of the followers now also advances the technological frontier of the economy owing to licensing (recall equation (35)). In fact, column 2 of Table 1 shows that this considerably higher growth rate is achieved with a *lower* fraction of the workforce, only 2.6%, working in the research sector.

The contribution of licensing to growth and welfare, which is robust across different parameterizations of the model, is the first important implication of our analysis. Relative to existing models of step-by-step innovation, such as Aghion, Harris, Howitt and Vickers (2001), which do not allow for the possibility of licensing, here the R&D effort by followers can directly contribute to economic growth and this increases the equilibrium growth rate of the economy.

4.5 Optimal Uniform IPR Protection

We next turn to optimal IPR policy with licensing. That is, we impose that $\eta_n = \eta$ and $\zeta_n = \zeta$ for all n, and look for values of η and ζ that maximize the welfare in the economy. Column 3 of Table 1 shows that the welfare-maximizing values of η and ζ are both equal to 0 in the benchmark parameterization. This corresponds to zero license fees and indefinite duration of patents, so that followers can never copy the leading-edge technology without R&D, but they can always advance one step ahead of the leader when they are successful in their R&D efforts (without paying any license fees).

The resulting value function, R&D effort levels and industry distributions according to technology gaps are shown in Figures 4-6 (with the solid lines). The figures and column 3 of Table 1 show that the welfare-maximizing IPR policy discourages leaders (this can be seen from the fact that $v_1 - v_0$ declines significantly), but encourages R&D effort by the followers, since when successful they do not have to pay the license fee. The optimal uniform IPR increases the growth rate by only a small amount, however. While the growth rate of the economy with full IPR protection with licensing was 2.58%, it is now 2.63%. This increase in the growth rate also raises steady-state welfare. In particular, the growth effect increases welfare by 4 points, while in this case there is also a slight improvement in welfare because of the change in the distribution of markups (though this is again small, equivalent to 0.1 points, that is, 1/40th as important as the growth effect). Finally, optimal uniform IPR protection also lead to a modest rise in the share of the labor force working in research (from 2.6% to 2.7%).

4.6 Optimal State-Dependent IPR Without Licensing

We next turn to our second major question; whether state-dependent IPR makes a significant difference relative to the uniform IPR. To highlight the roles played by different components of IPR policy, we first investigate the nature of welfare-maximizing space-dependent IPR policy without licensing (so that the comparison is to the benchmark case in column 1). In particular, we set $\zeta_n = \infty$ for all n and look for the combination of $\{\eta_1, ..., \eta_5\}$ that maximizes the welfare. The results are shown in column 4 of Table 1.

Two features are worth noting. First, the growth rate increases noticeably relative to column 1; it is now 2.04% instead of 1.86%. Nevertheless, this increase is models at relative to the benefits of licensing. The increase in steady-state welfare is also correspondingly smaller. Therefore, state-dependent IPR policy with no licensing is not a substitute for licensing.

Second, we see an interesting pattern (which is in fact quite general in all of our quantitative investigations). The optimal state-dependent policy $\{\eta_1, ..., \eta_5\}$ provides *greater* protection to technological leaders that are further ahead. In particular, we find that the optimal policy involves $\eta_1 = 0.71$, $\eta_2 = 0.08$, and $\eta_3 = \eta_4 = \eta_5 = 0$. This corresponds to very little patent protection for firms that are one step ahead of the followers. In particular, since $\eta_1 = 0.71$ and $x^*_{-1} = 0.12$, in this equilibrium firms that are one step behind followers are more than *six times* as likely to catch up with the technological leader because of the expiration of the patent of the leader as they are likely to catch up because of their own successful R&D. Then, there is a steep increase in the protection provided to technological leaders that are two steps ahead, and η_2 is 1/12th of η_1. Perhaps even more remarkably, after a technology gap of three or more steps, optimal IPR involves full protection, and patents never expire.

This pattern of greater protection for technological leaders that are further ahead may go against a naïve intuition that state-dependent IPR policy should try to boost the growth rate of the economy by bringing more industries with large technology gaps (where leaders engage in little R&D) into neck-and-neck competition. This composition effect is present, but dominated by another, more powerful force, the *trickle-down* effect. The intuition for the trickle-down effect is as follows: by providing secure patent protection to firms that are three or more steps ahead of their rivals, optimal state-dependent IPR increases the R&D effort of leaders that are one and two steps ahead as well. This is because technological leaders that are only one or two steps ahead now face greater returns to R&D, which will not only increase their profits but also the security of their intellectual property. Mechanically, high levels of η_1 and η_2 reduce v_1 and v_2, while high IPR protection for more advanced firms increases v_n for $n \geq 3$, and this increases the R&D incentives of leaders at $n = 1$ or at $n = 2$. This pattern of increased R&D investments under state-dependent IPR contrasts with uniform IPR, which always reduces R&D by all firms. The possibility that imperfect state-dependent IPR protection can *increase* (rather than reduce) R&D incentives is a novel feature of our approach and will be illustrated further in the next subsection.

4.7 Optimal State-Dependent IPR With Licensing

Finally, we turn to the most general policy regime, which allows both state-dependent patent protection and licensing. In particular, we now choose combinations of $\{\eta_1, ..., \eta_5\}$ and $\{\zeta_1, ..., \zeta_5\}$ to maximize steady-state welfare. The results of this exercise are shown in column 5 of Table 1. The most natural comparison in this case is to the optimal uniform IPR policy with licensing in column 3, where uniform IPR policies η and ζ were chosen to maximize welfare. The value functions, R&D efforts and the industry distribution over different levels of technology gaps in this economy are shown in Figures 4-6 (with the dashed lines).

We see in column 5 that welfare-maximizing IPR policy involves $\eta_n = 0$ for all n, so that with compulsory licensing, optimal IPR involves infinite duration of patents (though this is not always the case, see Table 2). Nevertheless, IPR protection for technological leaders is not full. In particular, the welfare-maximizing policy involves $\zeta_1 = 0$, which implies that followers can build on the leading-edge technology that is one step ahead of their own knowledge without paying any license fees. From there on, ζ increases to $\zeta_2 = 0.98$, then to $\zeta_3 = 1.93$, and to $\zeta_4 = 1.97$. After five steps, the welfare-maximizing policy is equivalent to full patent protection, that is, $\zeta_5 = 1.98$ (note that $v_1 - v_0 = 1.98$). The resulting growth rate of the economy is

2.96%, which is significantly higher than the growth rate under uniform IPR policy, 2.63% in column 3. Steady-state welfare also increases by a corresponding amount relative to the case with optimal uniform IPR case. In particular, the growth effect on welfare is an increase of 1.32 points, while the distribution effect involves a slight deterioration in welfare, equivalent to 0.015 points. Overall, this benchmark case shows that state-dependent policies can increase growth and welfare significantly.

State-dependent policies again achieve this superior growth performance by exploiting the trickle-down effect, which we already saw in the case without licensing. In particular, ζ_n is an increasing sequence, so that technological leaders that are further ahead receive greater protection. As in the previous subsection, this pattern of IPR is used as a way of boosting the R&D effort of technological leaders that are one or two steps ahead of their rivals (see Figure 5). Since these leaders receive little protection and understand that they can increase both their profits and their IPR protection by undertaking further innovations, they have relatively strong innovation incentives and undertake high levels of R&D. Figure 5 makes it clear that state-dependent relaxation of IPR in this case increases total R&D in the economy relative to full protection. The dashed line in Figure 5 is almost everywhere above the solid line. Alternatively, Table 1 shows that the fraction of the labor force working in R&D increases to 3.9% from 2.6% under full IPR protection with licensing. This positive effect of relaxation of IPR on R&D incentives is a novel implication of our model, and is due to the trickle-down effect.

It is also worth noting that, under state-dependent IPR policy with licensing, the growth rate of the economy receives a further boost from the R&D effort of the followers, since, thanks to licensing, followers' R&D directly contributes to the advancing the technological frontier of the economy. Figure 5 shows that followers that are one step behind the frontier also have a higher R&D effort than even in the case with welfare-maximizing uniform IPR (which involved $\zeta_n = 0$ for all n). The reason for this pattern of R&D efforts is again the trickle-down effect, which increases the value of being a technological leader and thus the incentive of followers to undertake R&D. In contrast, the R&D level of followers that are more than one step behind is lower than in the economy with uniform IPR (though as the comparison of the fraction of the labor force working in research to other columns demonstrates, this is dominated by the increase in the R&D of the technological leaders and of followers that are one step behind).

Overall, the results show that state-dependent IPR policies can increase growth and steady-state welfare substantially, and that this is because of the trickle-down effect. The trickle-down effect is powerful, not only when we consider an economy without licensing, but also in the presence of licensing.

4.8 Robustness

Tables 2-5 show the robustness of the patterns documented in Figures 1-6 and in Table 1. In particular, each of these tables changes one of the two parameters λ and γ (increasing or reducing λ to 1.2 or 1.01, and increasing or reducing γ to 0.6 or 0.1) and shows the results corresponding to each one of the five different policy regimes and discussed so far. In each case, we also change the parameter B in equation (41) to ensure the growth rate of the benchmark economy with full IPR protection and without licensing is the same as in Table 1, that is, $g^* = 1.86\%$.

Notably, the qualitative, and even the quantitative, patterns in Table 1 are relatively robust. In all cases we see a significant increase in the growth rate and welfare when we allow licensing. The smallest increase is seen when $\gamma = 0.6$, presumably because with limited diminishing returns to R&D, incentives were already sufficiently strong without licensing. As a result, in this case, the growth rate increases only from 1.86% to 1.98%. In all other cases, allowing for licensing increases the growth rate to above 2.6%, which is a sizable increase relative to the baseline of 1.86%.

Moreover, in all cases, moving to state-dependent IPR policy increases the growth rate and welfare further, though the extent of the increase varies depending on parameters.

Perhaps, more noteworthy is the fact that in all cases, welfare-maximizing state-dependent IPR is shaped by the trickle-down effect. In all of the various parameterizations we have considered, there is little or no protection provided to technological leaders that are one step ahead, but IPR protection grows as the technology gap increases. This is the typical pattern implied by the trickle-down effect. In addition, in most, but not all, cases optimal IPR policy provides patents of infinite duration and only makes compulsory licensing fees state dependent. Table 2 and 5, which are for ($\lambda = 1.01,\ \gamma = 0.35$) and ($\lambda = 1.05,\ \gamma = 0.6$), provide instances where both the optimal length of patent enforcement and optimal licensing fees are used as part of the welfare-maximizing policy and are both state dependent.

Finally, we have also computed growth-maximizing policies. In all cases, these are very similar to the welfare-maximizing policies, which is not surprising in view of the fact that, as shown above, welfare comparisons are driven by the growth effects.

We therefore conclude that both the substantial benefits of licensing and the benefits of state-dependent policies are robust across different specifications.

4.9 Partial Equilibrium Calibration and Further Robustness

The calibration exercises reported in the previous subsections show that for a range of plausible parameters the trickle-down effect is powerful and induces a pattern of welfare-maximizing (and growth-maximizing) policy that provides greater IPR protection to firms that are technologically more advanced relative to their rivals than to those enjoying a more limited technology gap. This is a new and somewhat surprising finding. Despite the robustness exercises, the reader may wonder whether this result holds for a much broader range of parameter values. Given the computationally-intensive nature of the exercises reported so far, it is not possible to compute or report results for the entire range of parameters for λ, γ and ρ.

In this last subsection, we specialize the economy in three ways and report results for the entire set of parameter values. First, we fix ω, so that the general equilibrium feedback on the labor share is removed. Second, we take the function $G(\cdot)$ to be quadratic. Finally, as assumed by a number of papers in this literature (e.g., Aghion, Harris, Howitt and Vickers, 2001, or Aghion, Bloom, Blundell, Griffith and Howitt, 2005), we assume that the maximum technology gap between a leader and a follower is $n = 2$. Under these assumptions, there are only two possible values for η, η_1 and η_2, and two possible values for ζ, ζ_1 and ζ_2. State-dependent IPR policy here simply means $\eta_1 \neq \eta_2$ and/or $\zeta_1 \neq \zeta_2$. The pattern we have seen in the general equilibrium model, where technological leaders that are further ahead receive greater protection, in turn, corresponds to $\eta_1 > \eta_2$ and/or $\zeta_1 < \zeta_2$. Since there are only two parameters, ρ and λ, we can plot the distribution of optimal policies for a large range of values of these two parameters and see the robust patterns in the form of optimal IPR policy.

More specifically, for a policy vector $(\eta_1, \eta_2, \zeta_1, \zeta_2)$, the stationary equilibrium is characterized by the solution to the following set of recursive equations:

$$
\begin{aligned}
\rho v_2 &= \max_{x_2 \geq 0} \left\{ \begin{array}{c} \left(1-\lambda^{-2}\right) + \left[\left(1-a_{-2}^*\right) x_{-2}^* + \eta_2\right]\left[v_0 - v_2\right] \\ +a_{-2}^* x_{-2}^* \left[v_{-1} - v_2 + \zeta_2\right] \end{array} \right\}, \\
\rho v_1 &= \max_{x_1 \geq 0} \left\{ \begin{array}{c} \left(1-\lambda^{-1}\right) - x_1^2/2 + x_1\left[v_2 - v_1\right] \\ +\left[\left(1-a_{-1}^*\right) x_{-1}^* + \eta_1\right]\left[v_0 - v_1\right] + a_{-1}^* x_{-1}^* \left[v_{-1} - v_1 + \zeta_1\right] \end{array} \right\}, \\
\rho v_0 &= \max_{x_0 \geq 0} \left\{ -x_0^2/2 + x_0\left[v_1 - v_0\right] + x_0^*\left[v_{-1} - v_0\right] \right\}, \\
\rho v_{-1} &= \max_{x_{-1} \geq 0, a_{-1} \in [0,1]} \left\{ \begin{array}{c} -x_{-1}^2/2 + \left[\left(1-a_{-1}\right) x_{-1} + \eta_1\right]\left[v_0 - v_{-1}\right] \\ +a_{-1} x_{-1}\left[v_1 - v_{-1} - \zeta_1\right] + x_1^*\left[v_0 - v_{-1}\right] \end{array} \right\}, \\
&\text{and} \\
\rho v_{-2} &= \max_{x_{-2} \geq 0, a_{-2} \in [0,1]} \left\{ \begin{array}{c} -x_{-2}^2/2 + \left[\left(1-a_{-2}\right) x_{-2} + \eta_2\right]\left[v_0 - v_{-2}\right] \\ +a_{-2} x_{-2}\left[v_1 - v_{-2} - \zeta_2\right] \end{array} \right\}.
\end{aligned}
$$

Given the solution to these equations, we can determine the welfare-maximizing combination of policies as in our previous calibration exercise (using the same notion of steady-state welfare). For expositional convenience, we do this in two steps, depicted in Figures 7 and 8; first for η_1 and η_2 (setting $\zeta_1 = \zeta_2 = \infty$), and then for ζ_1 and ζ_2 (setting $\eta_1 = \eta_2 = 0$).

Figure 7 shows the pattern of welfare-maximizing policy for the range of parameters $\rho \in [0, 0.5]$ and $\lambda \in (1, 10]$. We can see that for all parameters $\eta_1 > \eta_2$. Thus there is always greater protection given to technological leaders that are two steps ahead than those that are only one step ahead. Figure 8 shows the pattern of optimal policies with only licensing fees for the range of parameters $\rho \in [0, 0.5]$ and $\lambda \in (1, 10]$. Once again, there is greater protection for technological leaders that are further ahead. In fact, in this case for all parameter values, the welfare-maximizing policy involves $\zeta_1 = 0$, meaning that there is no protection provided to technological leaders that are one step ahead. In contrast, ζ_2 is always strictly positive. This pattern again induces a greater R&D investment by technological leaders that are one step ahead of their rivals.

Finally, we have also computed welfare-maximizing policies when the entire vector $(\eta_1, \eta_2, \zeta_1, \zeta_2)$ is allowed to vary. In this case, the welfare-maximizing policy again always provides greater protection to firms that are further ahead. In addition, it typically makes greater use of license fees, but for a small range of parameters, both license fees and relaxation of patent protection are used simultaneously. To save space, we do not show these results, which are more difficult to depict in the figures.

Overall, these results illustrate that the patterns we found for a narrower range of parameters in the general equilibrium model hold more broadly in this partial equilibrium version of the model. In all cases, there is greater protection given to firms that are further ahead of their rivals, and in all cases, the reason for this is the trickle-down effect.

5 Conclusions

In this paper, we developed a general equilibrium framework to investigate the impact of the extent and form of intellectual property rights (IPR) policy on economic growth and welfare. The two major questions we focused on are whether licensing, which allows followers to build on the leading-edge technology in return of a license fee, has a major impact on the equilibrium growth rate and whether the same degree of patent protection should be given to companies that are further ahead of their competitors as those that are technologically close to their rivals.

In our model economy, firms engage in cumulative (step-by-step) innovation. Leaders can innovate in order to widen the technology gap between themselves and the followers, which enables them to charge higher markups. Followers innovate to catch up with or surpass the technological leaders in their industry. Followers can advance in three different ways. First, the patent of the technological leader may expire, allowing the follower in the industry to copy the leading-edge technology. Second, each follower can undertake "catch-up R&D" to improve its own variant of the product to catch up with the leader. Third, each follower can undertake "frontier R&D," building on and improving the leading-edge technology. In this latter case, when successful, a follower may have to pay a license fee to the technological leader.

In the model economy, IPR policy regulates the length of patents and whether licensing is possible and the cost of licensing. We characterized the form of the steady-state equilibrium and proved its existence under general IPR policies. We then used this framework to investigate the form of "optimal" (welfare-maximizing) IPR policy quantitatively.

The major findings of this quantitative exercise are as follows:

1. A move from an IPR policy without licensing to one that allows for licensing has a significant effect on the equilibrium growth rate and the welfare. For the benchmark parameterization of our model, licensing increases the growth rate from 1.86% to 2.58% per annum, which is a significant effect. There is a corresponding increase in welfare as well. These substantial increases are robust to a large range of variation in the parameters.

2. State-dependent IPR also leads to a significant improvement in the equilibrium growth rate and welfare. In our benchmark parameterization, welfare-maximizing IPR policy increases the growth rate of the economy from 2.58% under the best possible uniform IPR policy to 2.96% under state-dependent IPR policy. Perhaps more interesting than this substantial impact on both growth and welfare is the form of the optimal state-dependent IPR policy. Contrary to a naïve intuition, we find that the welfare-maximizing IPR policy provides greater protection to firms that are further ahead of their rivals than those that are technologically close to their competitors. Underlying this form of the optimal IPR policy is *the trickle-down effect*. The trickle-down effect implies that providing greater protection to sufficiently advanced technological leaders not only increases their R&D efforts but also raises the R&D efforts of all technological leaders that are less advanced than this level. This is because the reward to innovation now includes the greater protection that they will receive once they reach this higher level of technology. Our results suggest that the trickle-down effect is powerful both with and without licensing, and its form and magnitude are relatively insensitive to the exact

parameter values used in the quantitative investigation.

The analysis in this paper suggests that a move to a richer menu of IPR policies, in particular, a move towards optimal state-dependent policies with licensing, may significantly increase innovation, economic growth and welfare. The results also show that the form of optimal IPR policy may depend on the industry structure (and the technology of catch-up within the industry). It should be noted, however, that these conclusions are based on a quantitative evaluation of a rather simple model. Our objective has not been to obtain practical policy prescriptions and the exact effects of different policies implied by our model undoubtedly miss a host of important factors and ignore potential limitations on the form and complexity of IPR policies. Nevertheless, our results demonstrate a range of robust and new effects that should be part of the calculus of IPR and competition policy.

The next step in this line of research should be to investigate the robustness of these effects in different models of industry dynamics. It would also be useful to study whether the relationship between the form of optimal IPR policy and industry structure suggested by our analysis also applies when variation in industry structure has other sources (for example, differences in the extent of fixed costs causing differential gaps between technological leaders and followers across industries). The most important area for future work is a detailed empirical investigation of the form of optimal IPR policy, using both better estimates of the effects of IPR policy on innovation rates and also structural models that would enable the evaluation of the effects of different policies on equilibrium growth and welfare. We hope that the theoretical framework presented in this paper will be useful in developing models that can be estimated in future work.

Appendix: Proofs

Derivation of Equation (22)

Fix the equilibrium R&D policies of other firms, $x^*_{-n}(t)$ and $a^*_{-n}(t)$, the equilibrium interest and wage rates, $r^*(t)$ and $w^*(t)$, and equilibrium profits $\{\Pi^*_n(t)\}^\infty_{n=1}$. Then the value of the firm that is n steps ahead at time t can be written as:

$$V_n(t) = \max_{x_n(t)} \left\{ \left[\Pi^*_n(t) - w^*(t)\, G(x_n(t))\right] \Delta t + o(\Delta t) + \exp\left(-r^*(t+\Delta t)\Delta t\right) \left[\begin{array}{c} (x_n(t)\Delta t + o(\Delta t))\, V_{n+1}(t+\Delta t) \\ + \left(\eta_n \Delta t + \left(1 - a^*_{-n}(t)\right) x^*_{-n}(t)\Delta t + o(\Delta t)\right) V_0(t+\Delta t) \\ + \left(a^*_{-n}(t)\, x^*_{-n}(t)\Delta t + o(\Delta t)\right)\left(V_{-1}(t+\Delta t) + \hat{\zeta}_n\right) \\ + \left(1 - x_n(t)\Delta t - \eta_n \Delta t - x^*_{-n}(t)\Delta t - o(\Delta t)\right) V_n(t+\Delta t) \end{array} \right] \right\}. \tag{42}$$

The first part of this expression is the flow profits minus R&D expenditures during a time interval of length Δt. The second part is the continuation value after this interval has elapsed. $V_{n+1}(t)$ and $V_0(t)$ are defined as net present discounted values for a leader that is $n+1$ steps ahead and a firm in an industry that is neck-and-neck (i.e., $n = 0$). The second part of the expression uses the fact that in a short time interval Δt, the probability of innovation by the leader is $x_n(t)\Delta t + o(\Delta t)$, where $o(\Delta t)$ again denotes second-order terms. This explains the first line of the continuation value. For the remainder of the continuation value, note that the probability that the follower will catch up with the leader is $\left(1 - a^*_{-n}(t)\right) x^*_{-n}(t)\Delta t + o(\Delta t)$; in particular, if $a^*_{-n}(t) = 1$, this eventually will never happen, since the follower would be undertaking R&D not to catch up but to surpass the leader. This explains the third line, which applies when $a^*_{-n}(t) = 1$. There are two differences between the second and third lines; (i) in the third line, conditional on success by the follower, a leader moves to the position of a follower rather than a neck-and-neck firm (V_{-1} instead of V_0); (ii) it receives the state-dependent patent fee $\hat{\zeta}_n$. Finally, the last line applies when no R&D effort is successful and patents continue to be enforced, so that the technology gap remains at n steps. Now, subtract $V_n(t)$ from both sides, divide everything by Δt, and take the limit as $\Delta t \to 0$ to obtain (22). ■

Proof of Proposition 1

Equations (19) and (21) imply

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu^*_n(t)}}{\omega(t)}.$$

Since $\omega(t) = \omega^*$ and $\{\mu^*_n\}^\infty_{n=0}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Therefore,

$$g^* = \lim_{\Delta t \to 0} \frac{\ln Q(t+\Delta t) - \ln Q(t)}{\Delta t}.$$

Now note the following: during an interval of length Δt (i) in the fraction μ^*_n of the industries with technology gap $n \geq 1$ the leaders innovate at a rate $x^*_n \Delta t + o(\Delta t)$; (ii) in the same industries, the followers innovate at the rate $a^*_{-n} x^*_{-n} \Delta t + o(\Delta t)$; (iii) in the fraction μ^*_0 of the industries with technology gap of $n = 0$, both firms innovate, so that the total innovation rate is $2x^*_0 \Delta t + o(\Delta t)$); and (iv) each innovation increase productivity by a factor λ. Combining these observations, we have

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu^*_0 x^*_0 \Delta t + \sum_{n=1}^{\infty} \mu^*_n x^*_n \Delta t + o(\Delta t) + \sum_{n=1}^{\infty} a^*_{-n} x^*_{-n} \Delta t + o(\Delta t) \right].$$

Subtracting $\ln Q(t)$, dividing by Δt and taking the limit $\Delta t \to 0$ gives (35). ■

Proof of Proposition 2

We prove this proposition in four parts. (1) Existence of a steady-state equilibrium. (2) Properties of the sequence of value functions. (3) Properties of the sequence of R&D decisions. (4) Uniqueness of an invariant distribution given R&D policies.

Part 1: Existence of a Steady-State Equilibrium.

First, note that each x_n belongs to a compact interval $[0, \bar{x}]$, where $\bar{x}$ is the maximal flow rate of innovation defined in (11) above. Now fix a labor share $\tilde{\omega} \in [0,1]$ and a sequence $\langle \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$ of (Markovian) steady-state strategies for all other firms in the economy, and consider the dynamic optimization problem of a single firm. Our first result characterizes this problem and shows that given some $\mathbf{z} \equiv \langle \tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$, the value function of an individual firm is uniquely determined, while its optimal R&D choices are given by a convex-valued correspondence. In what follows, we denote sets and correspondences by uppercase letters and refer to their elements by lowercase letters, e.g., $a_{-1}(\mathbf{z}) \in A_{-1}[\mathbf{z}]$, $x_n(\mathbf{z}) \in X_n[\mathbf{z}]$.

Lemma 1 *Consider a uniform IPR policy $\langle \boldsymbol{\eta}^{uni}, \ \boldsymbol{\zeta}^{uni} \rangle$, and suppose that the labor share and the R&D policies of all other firms are given by $\mathbf{z} = \langle \tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$. Then the dynamic optimization problem of an individual firm leads to a unique value function $\mathbf{v}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+$ and optimal R&D policies $\hat{A}_{-1}[\mathbf{z}] \subset [0,1]$ and $\hat{\mathbf{X}}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \rightrightarrows [0, \bar{x}]$ are compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$ and upper hemi-continuous in $\mathbf{z}$ (where $\mathbf{v}[\mathbf{z}] \equiv \{v_n[\mathbf{z}]\}_{n=-1}^{\infty}$ and $\hat{\mathbf{X}}[\mathbf{z}] \equiv \left\{\hat{X}_n[\mathbf{z}]\right\}_{n=-1}^{\infty}$).*

Proof. Fix $\mathbf{z} = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}, \{\tilde{a}_n\}_{n=-\infty}^{-1} \right\rangle$, and consider the optimization problem of a representative firm, written recursively as:

$$\begin{aligned} \rho v_n = & \max_{x_n \in [0,\bar{x}]} \{ \left(1 - \lambda^{-n}\right) - \tilde{\omega} G(x_n) + x_n [v_{n+1} - v_n] \\ + & \quad \tilde{x}_{-1} (\tilde{a}_{-1} [v_{-1} - v_n + \zeta] + (1 - \tilde{a}_{-1}) [v_0 - v_n]) + \eta [v_0 - v_n] \} \text{ for } n \in \mathbb{N} \end{aligned}$$

$$\begin{aligned} \rho v_0 &= \max_{x_0 \in [0,\bar{x}]} \{ -\tilde{\omega} G(x_0) + x_0 [v_1 - v_0] + \tilde{x}_0 [v_{-1} - v_0] \} \\ \rho v_{-1} &= \max_{x_{-1} \in [0,\bar{x}], \ a_{-1} \in [0,1]} \{ -\tilde{\omega} G(x_0) + x_{-1} (a_{-1} [v_1 - v_{-1} - \zeta] + (1 - a_{-1}) [v_0 - v_{-1}]) \\ & +\eta [v_0 - v_{-1}] \}. \end{aligned}$$

We now transform this dynamic optimization problem into a form that can be represented as a contraction mapping using the method of "uniformization" (see, for example, Ross, 1996, Chapter 5). Let $\tilde{\boldsymbol{\xi}} = \left\langle \{\tilde{x}_n\}_{n=-1}^{\infty}, \{\tilde{a}_n\}_{n=-\infty}^{-1} \right\rangle$ and $p_{n,n'}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right)$ be the probability that the next state will be n' starting with state n when the firm in question chooses policies $\boldsymbol{\xi} \equiv \left\langle \{x_n\}_{n=-1}^{\infty}, \ \{a_n\}_{n=-\infty}^{-1} \right\rangle$ and the R&D policy of other firms is given by $\tilde{\boldsymbol{\xi}}$. Using the fact that, because of uniform IPR policy, $\langle x_{-n}, a_{-n} \rangle = \langle x_{-1}, a_{-1} \rangle$ for all $n \in \mathbb{N}$, these transition probabilities can be written as:

$p_{-1,0}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{(1-a_{-1})x_{-1}+\eta}{x_{-1}+\eta}$	$p_{-1,1}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{a_{-1}x_{-1}}{x_{-1}+\eta}$	
$p_{0,-1}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{\tilde{x}_0}{x_0+\tilde{x}_0}$	$p_{0,1}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{x_0}{x_0+\tilde{x}_0}$	
$p_{n,-1}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{a_{-1}\tilde{x}_{-1}}{x_n+\tilde{x}_{-1}+\eta}$	$p_{n,0}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{(1-a_{-1})\tilde{x}_{-1}+\eta}{x_n+\tilde{x}_{-1}+\eta}$	$p_{n,n+1}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) = \frac{x_n}{x_n+\tilde{x}_{-1}+\eta}$

.

Uniformization involves adding fictitious transitions from a state into itself, which do not change the value of the program, but allow us to represent the optimization problem as a contraction. For this purpose, define the transition rates ψ_n as

$$\psi_n\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\begin{cases} x_n+x_{-1}+\eta & \text{for } n\in\{1,2,...\} \\ x_{-1}+\eta & \text{for } n=-1 \\ 2x_n & \text{for } n=0 \end{cases}.$$

These transition rates are finite since $\psi_n\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)\le\psi\equiv 2\bar{x}+\eta<\infty$ for all n, where $\bar{x}$ is the maximal flow rate of innovation defined in (11) in the text (both $\bar{x}$ and η are finite by assumption).

Now following equation (5.8.3) in Ross (1996), we can use these transition rates and define the new transition probabilities (including the fictitious transitions from a state to itself) as:

$$\tilde{p}_{n,n'}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\begin{cases} \frac{\psi_n(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}})}{\psi}p_{n,n'}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right) & \text{if } n\neq n' \\ 1-\frac{\psi_n(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}})}{\psi} & \text{if } n=n' \end{cases}.$$

This yields equivalent transition probabilities

$\tilde{p}_{-1,-1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=1-\frac{x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{-1,0}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{(1-a_{-1})x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{-1,1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{a_{-1}x_{-1}}{2\bar{x}+\eta}$	
$\tilde{p}_{0,-1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,0}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=1-\frac{x_0+\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{x_0}{2\bar{x}+\eta}$	
$\tilde{p}_{n,0}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{(1-\tilde{a}_{-1})\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=1-\frac{x_n+\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n+1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{x_n}{2\bar{x}+\eta}$	$\tilde{p}_{n,-1}\left(\boldsymbol{\xi}\mid\tilde{\boldsymbol{\xi}}\right)=\frac{\tilde{a}_{-1}\tilde{x}_{-1}}{2\bar{x}+\eta}$

,

and also defines an effective discount factor β given by

$$\beta\equiv\frac{\psi}{\rho+\psi}=\frac{2\bar{x}+\eta}{\rho+2\bar{x}+\eta}.$$

Also let the per period return function (profit net of R&D expenditures) be

$$\hat{\Pi}_n(x_n)=\begin{cases} \frac{1-\lambda^{-n}-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{if } n\ge 1 \\ \frac{-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{otherwise} \end{cases}. \tag{43}$$

Using these transformations, the dynamic optimization problem can be written as:

$$\begin{aligned} v_n &= \max_{x_n,\ a_n}\left\{\hat{\Pi}_n(x_n)+\beta\sum_{n'}\tilde{p}_{n,n'}\left(\xi_n\middle|\,\tilde{\boldsymbol{\xi}}\right)\tilde{v}_{n'}\right\}, \text{ for all } n\in\mathbb{Z}, \\ &\equiv T\tilde{v}_n, \text{ for all } n\in\mathbb{Z}. \end{aligned} \tag{44}$$

where $\mathbf{v}\equiv\{v_n\}_{n=-1}^{\infty}$ and the second line defines the operator T, mapping from the space of functions $\mathbf{V}\equiv\{\mathbf{v}:\{-1\}\cup\mathbb{Z}_+\to\mathbb{R}_+\}$ into itself. T is clearly a contraction, thus, for given $\mathbf{z}=\langle\tilde{\omega},\tilde{a}_{-1},\{\tilde{x}_n\}_{n=-1}^{\infty}\rangle$, possesses a unique fixed point $\mathbf{v}^*\equiv\{v_n^*\}_{n=-1}^{\infty}$ (e.g., Stokey, Lucas and Prescott, 1989).

Moreover, $x_n\in[0,\bar{x}]$, $a_{-1}\in[0,1]$, and v_n for each $n=-1,0,1,...$ given by the right-hand side of (44) is continuous in a_n and x_n (a_n applying only for $n=-1$), so Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) implies that the set of maximizers $\left\langle\hat{A}_{-1},\left\{\hat{X}_n\right\}_{n=-1}^{\infty}\right\rangle$ exists, is nonempty and compact-valued for each $\mathbf{z}$ and is upper hemi-continuous in $\mathbf{z}=\langle\tilde{\omega},\tilde{a}_{-1},\{\tilde{x}_n\}_{n=-1}^{\infty}\rangle$. Moreover, concavity of v_n in a_n and x_n for each $n=-1,0,1,...$ implies that $\left\langle\hat{A}_{-1},\left\{\hat{X}_n\right\}_{n=-1}^{\infty}\right\rangle$ is also convex-valued for each $\mathbf{z}$, completing the proof. ■

Now let us start with an arbitrary $\mathbf{z}\equiv\langle\tilde{\omega},\tilde{a}_{-1},\tilde{\mathbf{x}}\rangle\in\mathbf{Z}\equiv[0,1]^2\times[0,\bar{x}]^{\infty}$. From Lemma 1, this $\mathbf{z}$ is mapped into optimal R&D decision sets $\hat{A}_{-1}[\mathbf{z}]$ and $\hat{\mathbf{X}}[\mathbf{z}]$, where $\hat{a}_{-1}\in\hat{A}_{-1}[\mathbf{z}]$ and $\hat{x}_n[\mathbf{z}]\in\hat{X}_n[\mathbf{z}]$.

From R&D policies $\langle \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$, we calculate $\boldsymbol{\mu}[\tilde{a}_{-1}, \tilde{\mathbf{x}}] \equiv \{\mu_n[\tilde{a}_{-1}, \tilde{\mathbf{x}}]\}_{n=0}^{\infty}$ using equations (31), (32) and (33). Then we can rewrite the labor market clearing condition (34) as

$$\begin{aligned} \omega &= \min\left\{\sum_{n=0}^{\infty} \mu_n \left[\frac{1}{\lambda^n} + G(\tilde{x}_n)\tilde{\omega} + G(\tilde{x}_{-n})\right]\tilde{\omega}; 1\right\}, \\ &\equiv \varphi(\tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}}) \end{aligned} \tag{45}$$

where due to uniform IPR, $\hat{x}_{-n} = \hat{x}_{-1}$ for all $n > 0$. Next, define the mapping (correspondence)

$$\Phi[\mathbf{z}] \equiv \left(\varphi(\mathbf{z}),\ \hat{A}_{-1}[\mathbf{z}],\ \hat{\mathbf{X}}[\mathbf{z}]\right)$$

, which maps $\mathbf{Z}$ into itself, that is,

$$\Phi\colon \mathbf{Z} \rightrightarrows \mathbf{Z}. \tag{46}$$

That Φ maps $\mathbf{Z}$ into itself follows since $\mathbf{z} \in \mathbf{Z}$ consists of $\tilde{a}_{-1} \in [0,1]$, $\tilde{\mathbf{x}} \in [0,\bar{x}]^{\infty}$ and $\tilde{\omega} \in [0,1]$, and the image of $\mathbf{z}$ under Φ consists of $\hat{a}_{-1} \in [0,1]$ and $\hat{\mathbf{x}} \in [0,\bar{x}]^{\infty}$, and moreover, (45) is clearly in $[0,1]$ (since the right-hand side is nonnegative and bounded above by 1). Finally, from Lemma 1, $\hat{A}_{-1}[\mathbf{z}]$ and $\hat{X}_n[\mathbf{z}]$ are compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$, and also upper hemi-continuous in $\mathbf{z}$, and φ is continuous. Using this construction, we can establish the existence of a steady-state equilibrium as follows.

We first show that the mapping $\Phi\colon \mathbf{Z} \rightrightarrows \mathbf{Z}$ constructed in (46) has a fixed point, and then establish that when $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ this fixed point corresponds to a steady state with $\omega^* < 1$. First, it has already been established that Φ maps $\mathbf{Z}$ into itself. We next show that $\mathbf{Z}$ is compact in the product topology and is a subset of a locally convex Hausdorff space. The first part follows from the fact that $\mathbf{Z}$ can be written as the Cartesian product of compact subsets, $\mathbf{Z} = [0,1] \times [0,1] \times \prod_{n=-1}^{\infty} [0,\bar{x}]$. Then by Tychonoff's Theorem (e.g., Aliprantis and Border, 1999, Theorem 2.57, p. 52; Kelley, 1955, p. 143), $\mathbf{Z}$ is compact in the product topology. Moreover, $\mathbf{Z}$ is clearly nonempty and also convex, since for any $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ and $\lambda \in [0,1]$, we have $\lambda\mathbf{z} + (1-\lambda)\mathbf{z}' \in \mathbf{Z}$. Finally, since $\mathbf{Z}$ is a product of intervals on the real line, it is a subset of a locally convex Hausdorff space (see Aliprantis and Border, 1999, Lemma 5.54, p. 192).

Next, φ is a continuous function from $\mathbf{Z}$ into $[0,1]$ and from Lemma 1, $\hat{A}_{-1}(\mathbf{z})$ and $\hat{X}_n(\mathbf{z})$ for $n \in \{-1\} \cup \mathbb{Z}_+$ are upper hemi-continuous in $\mathbf{z}$. Consequently, $\Phi \equiv \left\langle \varphi[\mathbf{z}],\ \hat{A}_{-1}[\mathbf{z}],\ \hat{\mathbf{X}}[\mathbf{z}] \right\rangle$ has closed graph in $\mathbf{z}$ in the product topology. Moreover, each one of $\varphi(\mathbf{z})$, $\hat{A}_{-1}(\mathbf{z})$ and $\hat{X}_n(\mathbf{z})$ for $n = -1, 0, ...$ is nonempty, compact and convex-valued. Therefore, the image of the mapping Φ is nonempty, compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$. The Kakutani-Fan-Glicksberg Fixed Point Theorem implies that if the function Φ maps a convex, compact and nonempty subset of a locally convex Hausdorff space into itself and has closed graph and is nonempty, compact and convex-valued $\mathbf{z}$, then it possesses a fixed point $\mathbf{z}^* \in \Phi(\mathbf{z}^*)$ (see Aliprantis and Border, 1999, Theorem 16.50 and Corollary 16.51, p. 549-550). This establishes the existence of a fixed point $\mathbf{z}^*$ of Φ.

To complete the proof, we need to show that the fixed point, $\mathbf{z}^*$, corresponds to a steady state equilibrium. First, since $\hat{a}_n\left(\omega^*, a^*_{-1}, \{x^*_n\}_{n=-1}^{\infty}\right) = a^*_{-1}$ and $\hat{x}_n\left(\omega^*, a^*_{-1}, \{x^*_n\}_{n=-1}^{\infty}\right) = x^*_n$ for $n \in \{-1\} \cup \mathbb{Z}_+$, we have that given a labor share of ω^*, $\left\langle a^*_{-1}, \{x^*_n\}_{n=-1}^{\infty} \right\rangle$ constitutes an R&D policy vector that is best response to itself, as required by steady-state equilibrium (Definition 3). Next, we need to prove that the implied labor share ω^* leads to labor market clearing. This follows from the fact that the fixed point involves $\omega^* < 1$, since in this case (45) will have an interior solution, ensuring labor market clearing. Suppose, to obtain a contradiction, that $\omega^* = 1$. Then, as noted in the text, we must have $\mu^*_0 = 1$. From (31), (32) and (33), this implies $x^*_n = 0$ for $n \in \{-1\} \cup \mathbb{Z}_+$. However, Lemma 2 implies that this is not possible when $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$. Consequently, (45) cannot be satisfied at $\omega^* = 1$, implying that $\omega^* < 1$. When $\omega^* < 1$, the labor market clearing condition (34) is satisfied at ω^* as an equality, so ω^* is an equilibrium given $\{x^*_n\}_{n=-1}^{\infty}$, and thus $\mathbf{z}^* = \left(\omega^*, a^*_{-1}, \{x^*_n\}_{n=-1}^{\infty}\right)$ is a steady-state equilibrium as desired.

Finally, if $\eta > 0$, then (33) implies that $\mu_0^* > 0$. Since $x_0^* > 0$ from Lemma 2, equation (35) implies $g^* > 0$. Alternatively, if $x_{-1}^* > 0$, then $g^* > 0$ follows from (35). This completes the proof of the existence of a steady-state equilibrium with positive growth.

Part 2: Properties of the Sequence of Value Functions.

Let $\langle a_{-1}, \{x_n\}_{n=-1}^{\infty} \rangle$ be the R&D decisions of the firm and $\{v_n\}_{n=-1}^{\infty}$ be the sequence of values, taking the decisions of other firms and the industry distributions, $\{x_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g, as given. By choosing $x_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$. Moreover, since flow profit satisfy $\pi_n \leq 1$ for all $n \geq -1$, $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$.

Proof of $v_1 > v_0$: Suppose, first, $v_1 \leq v_0$, then (30) implies $x_0^* = 0$, and by the symmetry of the problem in equilibrium (25) implies $v_0 = v_1 = 0$. As a result, from (29) we obtain $x_{-1}^* = 0$. Equation (24) implies that when $x_{-1}^* = 0$, $v_1 \geq \left(1 - \lambda^{-1}\right) / (\rho + \eta) > 0$, yielding a contradiction and proving that $v_1 > v_0$. □

Proof of $v_{-1} \leq v_0$: Suppose, to obtain a contradiction, that $v_{-1} > v_0$.

If $v_1 - \zeta \leq v_0$, (29) yields $x_{-1}^* = 0$. This implies $v_{-1} = \eta v_0 / (\rho + \eta)$, which contradicts $v_{-1} > v_0$ since $\eta / (\rho + \eta) < 1$. Thus we must have $v_1 - \zeta > v_0$, which implies that $a_{-1}^* = 1$. Imposing $a_{-1}^* = 1$, the value function of a neck-and-neck firm can be written as:

$$\begin{aligned} \rho v_0 &= \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x_0^* [v_{-1} - v_0] \right\}, \\ &\geq \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] \right\}, \\ &\geq \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_{-1} - \zeta] \right\}, \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_1 - v_{-1} - \zeta], \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_1 - v_{-1} - \zeta] + \eta [v_0 - v_{-1}], \\ &= \rho v_{-1}, \end{aligned} \tag{47}$$

which contradicts the hypothesis that $v_{-1} > v_0$ and establishes the claim. □

Proof of $v_n < v_{n+1}$: Suppose, to obtain a contradiction, that $v_n \geq v_{n+1}$. Now (28) implies $x_n^* = 0$, and (24) becomes

$$\rho v_n = \left(1 - \lambda^{-n}\right) + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0 - v_n\right] + \eta [v_0 - v_n] \tag{48}$$

Also from (24), the value for state $n+1$ satisfies

$$\rho v_{n+1} \geq \left(1 - \lambda^{-n-1}\right) + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0 - v_{n+1}\right] + \eta [v_0 - v_{n+1}]. \tag{49}$$

Combining the two previous expressions, we obtain

$$\begin{aligned} &\left(1 - \lambda^{-n}\right) + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0 - v_n\right] + \eta [v_0 - v_n] \\ \geq\ & 1 - \lambda^{-n-1} + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0 - v_{n+1}\right] + \eta [v_0 - v_{n+1}]. \end{aligned}$$

Since $\lambda^{-n-1} < \lambda^{-n}$, this implies $v_n < v_{n+1}$, contradicting the hypothesis that $v_n \geq v_{n+1}$, and establishing the desired result, $v_n < v_{n+1}$. Consequently, $\{v_n\}_{n=-1}^{\infty}$ is nondecreasing and $\{v_n\}_{n=0}^{\infty}$ is (strictly) increasing. Since a nondecreasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_∞, which must be strictly positive, since $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and has a nonnegative initial value. □

The above results combined complete the proof that values form an increasing sequence. ■

Part 3: **Properties of the Sequence of R&D Decisions.**

Proof of $x^*_{n+1} < x^*_n$: From equation (28),

$$\delta_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv \delta_n \tag{50}$$

would be sufficient to establish that $x^*_{n+1} < x^*_n$ whenever $x^*_n > 0$. We next show that this is the case.

Let us write:

$$\bar{\rho} v_n = \max_{x_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(x_n\right) + x^*_n \left[v_{n+1} - v_n\right] + x^*_{-1} \left[a^*_{-1}\left(v_{-1} + \zeta\right) + \left(1 - a^*_{-1}\right) v_0\right] + \eta v_0 \right\}, \tag{51}$$

where $\bar{\rho} \equiv \rho + x^*_{-1} + \eta$. Since x^*_{n+1}, x^*_n and x^*_{n-1} are maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (51) implies:

$$\begin{aligned}
\bar{\rho} v_{n+1} &= 1 - \lambda^{-n-1} - \omega^* G\left(x^*_{n+1}\right) + x^*_{n+1}\left[v_{n+2} - v_{n+1}\right] + x^*_{-1}\left[a^*_{-1}\left(v_{-1} + \zeta\right) + \left(1 - a^*_{-1}\right) v_0\right] + \eta v_0, \qquad (52)\\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(x^*_{n+1}\right) + x^*_{n+1}\left[v_{n+1} - v_n\right] + x^*_{-1}\left[a^*_{-1}\left(v_{-1} + \zeta\right) + \left(1 - a^*_{-1}\right) v_0\right] + \eta v_0, \\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(x^*_{n-1}\right) + x^*_{n-1}\left[v_{n+1} - v_n\right] + x^*_{-1}\left[a^*_{-1}\left(v_{-1} + \zeta\right) + \left(1 - a^*_{-1}\right) v_0\right] + \eta v_0, \\
\bar{\rho} v_{n-1} &= 1 - \lambda^{-n+1} - \omega^* G\left(x^*_{n-1}\right) + x^*_{n-1}\left[v_n - v_{n-1}\right] + x^*_{-1}\left[a^*_{-1}\left(v_{-1} + \zeta\right) + \left(1 - a^*_{-1}\right) v_0\right] + \eta v_0.
\end{aligned}$$

Now taking differences with $\bar{\rho} v_n$ and using the definitions of δ_ns, we obtain

$$\begin{aligned}
\bar{\rho} \delta_{n+1} &\leq \lambda^{-n}\left(1 - \lambda^{-1}\right) + x^*_{n+1}\left(\delta_{n+2} - \delta_{n+1}\right) \\
\bar{\rho} \delta_n &\geq \lambda^{-n+1}\left(1 - \lambda^{-1}\right) + x^*_{n-1}\left(\delta_{n+1} - \delta_n\right).
\end{aligned}$$

Therefore,

$$\left(\bar{\rho} + x^*_{n-1}\right)\left(\delta_{n+1} - \delta_n\right) \leq -k_n + x^*_{n+1}\left(\delta_{n+2} - \delta_{n+1}\right), \tag{53}$$

where

$$k_n \equiv (\lambda - 1)^2 \lambda^{-n-1} > 0.$$

Now to obtain a contradiction, suppose that $\delta_{n+1} - \delta_n \geq 0$. From (53), this implies $\delta_{n+2} - \delta_{n+1} > 0$ since k_n is strictly positive. Repeating this argument successively, we have that if $\delta_{n'+1} - \delta_{n'} \geq 0$, then $\delta_{n+1} - \delta_n > 0$ for all $n \geq n'$. However, we know from Part 2 of the proposition that $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $\delta_n \downarrow 0$, which contradicts the hypothesis that $\delta_{n+1} - \delta_n \geq 0$ for all $n \geq n' \geq 0$, and establishes that $x^*_{n+1} \leq x^*_n$. To see that the inequality is strict when $x^*_n > 0$, it suffices to note that we have already established (50), i.e., $\delta_{n+1} - \delta_n < 0$, thus if equation (28) has a positive solution, then we necessarily have $x^*_{n+1} < x^*_n$.

We next prove that $x^*_0 \geq x^*_{-1}$ and then show that under the additional condition $G'^{-1}\left(\left(1 - \lambda^{-1}\right) / (\rho + \eta)\right) > 0$, this inequality is strict.

Proof of $x^*_0 \geq x^*_{-1}$: Suppose first that $\zeta > v_1 - v_0$. Then (27) implies $a^*_{-1} = 0$, and (25) can be written as

$$\rho v_0 = -\omega^* G\left(x^*_0\right) + x^*_0\left[v_{-1} + v_1 - 2v_0\right]. \tag{54}$$

We have $v_0 \geq 0$ from Part 2 of the proposition. Suppose $v_0 > 0$. Then (54) implies $x^*_0 > 0$ and

$$\begin{aligned}
v_{-1} + v_1 - 2v_0 &> 0 \qquad (55)\\
v_1 - v_0 &> v_0 - v_{-1}.
\end{aligned}$$

This inequality combined with $a^*_{-1} = 0$, (30) and (37) yields $x^*_0 > x^*_{-1}$. Suppose next that $v_0 = 0$. Inequality (55) now holds as a weak inequality and implies that $x^*_0 \geq x^*_{-1}$. Moreover, since $G(\cdot)$ is strictly convex and x^*_0 is given by (30), (54) then implies $x^*_0 = 0$ and thus $x^*_{-1} = 0$.

We next show that when $\zeta \leq v_1 - v_0$, $x_0^* \geq x_{-1}^*$. In this case, $a_{-1}^* = 1$ is an optimal policy, so that

$$\begin{aligned} \rho v_0 &= -\omega^* G(x_0^*) + x_0^* [v_1 - v_0] + x_0^* [v_{-1} - v_0] \\ \rho v_{-1} &\geq -\omega^* G(x_0^*) + x_0^* [v_1 - v_{-1} - \zeta] + \eta [v_0 - v_{-1}]. \end{aligned}$$

Subtracting the second expression from the first, we obtain

$$\rho [v_0 - v_{-1}] \leq x_0^* [v_{-1} + \zeta - v_0] + (x_0^* + \eta)[v_{-1} - v_0],$$

and therefore

$$[v_0 - v_{-1}] \leq [v_{-1} + \zeta - v_0].$$

Part 2 of the proposition implies that $v_{-1} \leq v_0$, and therefore $v_{-1} + \zeta \geq v_0$. Next observe that with $a_{-1}^* = 1$, (30) and (37) imply that $x_0^* \geq x_{-1}^*$ if and only if $v_1 - v_0 \geq v_1 - v_{-1} - \zeta$, or equivalently if and only if $v_{-1} + \zeta \geq v_0$. Thus we have established that $x_0^* \geq x_{-1}^*$ both when $\zeta > v_1 - v_0$ and when $\zeta \leq v_1 - v_0$. □

We now have the following intermediate lemma.

Lemma 2 *Suppose that $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$, then $x_0^* > 0$ and $v_0 > 0$.*

Proof. Suppose, to obtain a contradiction, that $x_0^* = 0$. The first part of the proof then implies that $x_{-1}^* = 0$. Then (24) implies

$$\rho v_1 \geq 1 - \lambda + \eta [v_0 - v_1].$$

Equation (25) together with $x_0^* = 0$ gives $v_0 = 0$, and hence

$$v_1 - v_0 \geq \frac{1 - \lambda^{-1}}{\rho + \eta}.$$

Combined with this inequality, (30) implies

$$\begin{aligned} x_0^* &\geq \max\left\{G'^{-1}\left(\frac{1 - \lambda^{-1}}{\omega^*(\rho + \eta)}\right), 0\right\}, \\ &\geq \max\left\{G'^{-1}\left(\frac{1 - \lambda^{-1}}{\rho + \eta}\right), 0\right\}, \end{aligned}$$

where the second inequality follows from the fact that $\omega^* \leq 1$. The assumption that $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$ then implies $x_0^* > 0$, thus leading to a contradiction and establishing that $x_0^* > 0$. Strict convexity of $G(\cdot)$ together with $x_0^* > 0$ then implies $v_0 > 0$. ■

Proof of $x_0^* > x_{-1}^*$ when $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$ and $\zeta > 0$: Given Lemma 2, $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$ implies that $x_0^* > 0$. Then the first part of the proof implies that when $\zeta > v_1 - v_0$, $x_0^* > x_{-1}^*$. Next suppose that $0 < \zeta < v_1 - v_0$. Then the same argument as above implies that $x_0^* > x_{-1}^*$ if and only if $v_1 - v_0 > v_1 - v_{-1} - \zeta$, or equivalently if and only if $v_{-1} + \zeta > v_0$. Suppose this is not the case. Then from the first part of the proof, we have that $x_0^* = x_{-1}^* = 0$, and thus $v_{-1} = v_0 = 0$, which implies $v_{-1} + \zeta > v_0$ and thus $x_0^* > x_{-1}^*$. This yields a contradiction and completes the proof that $x_0^* > x_{-1}^*$ when $\zeta > 0$ and $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$. □

Proof of $x_0^* > x_1^*$: To prove that $x_0^* > x_1^*$, let us write the value functions v_2, v_1 and v_0 as in (52):

$$\begin{aligned} \bar{\rho} v_2 &= 1 - \lambda^{-2} - \omega^* G(x_2^*) + x_2^* [v_3 - v_2] + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0\right] + \eta v_0, \\ \bar{\rho} v_1 &\geq 1 - \lambda^{-1} - \omega^* G(x_2^*) + x_2^* [v_2 - v_1] + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0\right] + \eta v_0, \\ \bar{\rho} v_1 &\geq 1 - \lambda^{-1} - \omega^* G(x_0^*) + x_0^* [v_2 - v_1] + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0\right] + \eta v_0, \\ \bar{\rho} v_0 &= -\omega^* G(x_0) + x_0^* [v_1 - v_0] + \eta v_0 + x_{-1}^* v_0 + x_0^* [v_{-1} - v_0]. \end{aligned}$$

Now taking differences with $\bar{\rho}v_n$ and using the definitions of δ_ns as in (50), we obtain

$$\begin{aligned}
\bar{\rho}\delta_2 &\leq \lambda^{-1}\left(1-\lambda^{-1}\right)+x_2^*\left(\delta_3-\delta_2\right), &(56)\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right)+x_0^*\left(\delta_2-\delta_1\right)+x_{-1}^*\left[a_{-1}^*\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^*\right)v_0-v_0\right]-x_0^*\left[v_{-1}-v_0\right],\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right)+x_0^*\left(\delta_2-\delta_1\right)+x_{-1}^*a_{-1}^*\zeta+\left(x_{-1}^*a_{-1}^*-x_0^*\right)\left[v_{-1}-v_0\right],\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right)+x_0^*\left(\delta_2-\delta_1\right)+\left(x_{-1}^*a_{-1}^*-x_0^*\right)\left[v_{-1}-v_0\right].
\end{aligned}$$

Next recall from Part 2 that $v_{-1}-v_0\leq 0$. Moreover, the first part of the first part of the proof has established that $x_{-1}^*-x_0^*\leq 0$. Combining this with $a_{-1}^*\leq 1$ establishes that $\left[x_{-1}^*-x_0\right]\left[v_{-1}-v_0\right]\geq 0$, and the last inequality then implies

$$\bar{\rho}\delta_1\geq\left(1-\lambda^{-1}\right)+x_0^*\left(\delta_2-\delta_1\right).$$

Now combining this inequality with the first inequality of (56), we obtain

$$\left(\bar{\rho}+x_0^*\right)\left(\delta_2-\delta_1\right)\leq-\left(1-\lambda^{-1}\right)^2+x_2^*\left(\delta_3-\delta_2\right). \tag{57}$$

Part 2 has already established $\delta_2>\delta_3$, so that the right-hand side is strictly negative, therefore, we must have $\delta_2-\delta_1<0$, which implies that $x_0^*>x_1^*$ and completes the proof. □

The above results together complete the proof of Part 3. ■

Part 4: Uniqueness of the Invariant Distribution.

Lemma 3 *Consider a uniform IPR policy $\left\langle\boldsymbol{\eta}^{uni},\ \boldsymbol{\zeta}^{uni}\right\rangle$ and a corresponding steady-state equilibrium $\left\langle\boldsymbol{\mu}^*,\ \mathbf{v},\ a_{-1}^*,\ \mathbf{x}^*,\ \omega^*,\ g^*\right\rangle$. Then, there exists $n^*\in\mathbb{N}$ such that $x_n^*=0$ for all $n\geq n^*$.*

Proof. The first-order condition of the maximization of the value function (24) implies:

$$G'\left(x_n\right)\geq\frac{v_{n+1}-v_n}{\omega^*}\text{ and }x_n\geq 0,$$

with complementary slackness. $G'(0)$ is strictly positive by assumption. If $\left(v_{n+1}-v_n\right)/\omega^*<G'(0)$, then $x_n=0$. The second part of the proposition implies that $\{v_n\}_{n=-1}^\infty$ is a convergent and thus a Cauchy sequence, which implies that there exists $\exists n^*\in\mathbb{N}$ such that $v_{n+1}-v_n<\omega^*G'(0)$ for all $n\geq n^*$. ■

An immediate consequence of Lemma 3, combined with (31) is that $\mu_n=0$ for all $n\geq n^*$ (since there is no innovation in industries with technology gap greater than n^*). Thus the law of motion of an industry can be represented by a finite Markov chain. Moreover, because after an innovation by a follower, all industries jump to the neck-and-neck state (when $a_{-1}^*=0$) or to the technology gap of one (when $a_{-1}^*=1$), this Markov chain is irreducible (and aperiodic), thus converges to a unique steady-state distribution of industries. More formally, there exists n^* such that $x_{n^*}^*=0$ and $x_n^*=0$ for all $n>n^*$. Combined with the fact $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right)>0$ and that either $\eta>0$ or $x_{-1}^*>0$, this implies that the states $n>n^*$ are transient and can be ignored. Consequently, $\{\mu_n^*\}_{n=0}^\infty$ forms a finite and irreducible Markov chain over the states $n=0,1,...,n^*$. To see this, let $n^*=\min_{n\in\{0,...,n^{**}\}}\{n\in\mathbb{N}:v_{n+1}-v_n\leq\omega^*G'(0)\}$. Such an n^* exists, since the set $\{0,...,n^{**}\}$ is finite and nonempty because of the assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right)>0$. Then by construction $x_n^*>0$ for all $n<n^*$ and $x_{n^*}^*=0$ as desired. Now denoting the probability of being in state $\tilde{n}$ starting in state n after τ periods by $P^\tau\left(n,\tilde{n}\right)$, we have that $\lim_{\tau\to\infty}P^\tau\left(n,\tilde{n}\right)=0$ for all $\tilde{n}>n^*$ and for all n. Thus we can focus on the finite Markov chain over the states $n=0,1,...,n^*$, and $\{\mu_n^*\}_{n=0}^{n^*}$ is the limiting (invariant) distribution of this Markov chain. Given a_{-1}^* and $\{x_n^*\}_{n=-1}^{n^*}$, $\{\mu_n^*\}_{n=0}^{n^*}$ is uniquely defined. Moreover, the underlying Markov chain is irreducible (since $x_n^*>0$ for $n=0,1,...,n^*-1$, so that all states communicate with $n=0$ or $n=1$). Therefore, by Theorem 11.2 in Stokey, Lucas and Prescott (1989, p. 62) there exists a unique stationary distribution $\{\mu_n^*\}_{n=0}^\infty$. ■

Proof of Proposition 3

We prove this proposition using two crucial lemmas.

Lemma 4 *Consider the state-dependent IPR policy* $\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle$*, and suppose that* $\left\langle \boldsymbol{\mu}^*, \mathbf{v}, a^*_{-1}, \mathbf{x}^*, \omega^*, g^* \right\rangle$ *is a steady-state equilibrium. Then there exists a state* $n^* \in \mathbb{N}$ *such that* $\mu^*_n = 0$ *for all* $n \geq n^*$*.*

Proof. There are two cases to consider. First, suppose that $\{v_n\}_{n\in\mathbb{Z}_+}$ is strictly increasing. Then it follows from the proof of Lemma 3 that there exists a state $n^* \in \mathbb{N}$ such that $x^*_n = 0$ for all $n \geq n^*$, and as in the proof of Part 4 of Proposition 2, states $n \geq n^*$ are transient (i.e., $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n), so $\mu^*_n = 0$ for all $n \geq n^*$.

Second, in contrast to the first case, suppose that there exists some $n^{**} \in \mathbb{Z}_+$ such that $v_{n^{**}} \geq v_{n^{**}+1}$. Then, let $n^* = \min_{n\in\{0,\dots,n^{**}\}} \{n \in \mathbb{N}: v_{n+1} - v_n \leq \omega^* G'(0)\}$, which is again well defined. Then, optimal R&D decision (28) immediately implies that $x^*_n > 0$ for all states with $n < n^*$, and since $x^*_{n^*} = 0$, all states $n > n^*$ are transient and $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n, completing the proof. ■

Lemma 5 *Consider the state-dependent IPR policy* $\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle$ *and suppose that the labor share and the R&D policies of all other firms are given by* $\mathbf{z} = \langle \tilde{\omega}, \tilde{\boldsymbol{a}}, \tilde{\mathbf{x}} \rangle$*. Then the dynamic optimization problem of an individual firm leads to a unique value function* $\mathbf{v}[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$ *and optimal R&D policies* $\hat{\mathbf{A}}[\mathbf{z}] : \mathbb{Z}_- \setminus \{0\} \rightrightarrows [0,1]$ *and* $\hat{\mathbf{X}}[\mathbf{z}] : \mathbb{Z} \rightrightarrows [0, \bar{x}]$ *are compact and convex-valued for each* $\mathbf{z} \in \mathbf{Z}$ *and upper hemi-continuous in* $\mathbf{z}$ *(where* $\mathbf{v}[\mathbf{z}] \equiv \{v_n[\mathbf{z}]\}_{n=-1}^{\infty}$*,* $\hat{\mathbf{A}}[\mathbf{z}] \equiv \left\{\hat{A}_n[\mathbf{z}]\right\}_{n=-\infty}^{-1}$ *and* $\hat{\mathbf{X}}[\mathbf{z}] \equiv \left\{\hat{X}_n[\mathbf{z}]\right\}_{n=-1}^{\infty}$ *).*

Proof. The proof follows closely that of Lemma 1. In particular, again using uniformization, the maximization problem of an individual firm can be written as a contraction mapping similar to (44) there. The finiteness of the transition probabilities follows, since $\psi_n\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) \leq \psi \equiv 2\bar{x} + \max_n\{\eta_n\} < \infty$ (this is a consequence of the fact that $\bar{x}$ defined in (11) is finite and $\max_n\{\eta_n\}$ is finite, since each $\eta_n \in \mathbb{R}_+$ and by assumption, there exists $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$). This contraction mapping uniquely determines the value function $\mathbf{v}[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$.

Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) again implies that each of $\hat{A}_n(\mathbf{z})$ for $n \in \mathbb{Z}_- \setminus \{0\}$ and $\hat{X}_n(\mathbf{z})$ for $n \in \mathbb{Z}$ is upper hemi-continuous in $\mathbf{z} = \langle \tilde{\omega}, \tilde{\boldsymbol{a}}, \tilde{\mathbf{x}} \rangle$, and moreover, since v_n for $n \in \mathbb{Z}$ is concave in a_n and x_n, the maximizers of $\mathbf{v}[\mathbf{z}]$, $\hat{\mathbf{A}} \equiv \left\{\hat{A}_{-n}\right\}_{n=1}^{\infty}$ and $\hat{\mathbf{X}} \equiv \left\{\hat{X}_n\right\}_{n=-\infty}^{\infty}$, are nonempty, compact and convex-valued. ■

Now using the previous two lemmas, we can establish the existence of a steady-state equilibrium. This part of the proof follows that of Proposition 2 closely. Fix $\mathbf{z} = \left\langle \tilde{\omega}, \{\tilde{a}_n\}_{n=-\infty}^{-1}, \{\tilde{x}_n\}_{n=-\infty}^{\infty} \right\rangle$, and define $\mathbf{Z} \equiv [0,1] \times \prod_{n=-1}^{\infty} [0,1] \times \prod_{n=-\infty}^{\infty} [0, \bar{x}]$. Again by Tychonoff's Theorem, $\mathbf{Z}$ is compact in the product topology. Then consider the mapping Φ: $\mathbf{Z} \rightrightarrows \mathbf{Z}$ constructed as $\Phi \equiv \left(\varphi, \hat{\mathbf{A}}, \hat{\mathbf{X}}\right)$, where φ is given by (45) and $\hat{\mathbf{A}}$ and $\hat{\mathbf{X}}$ are defined in Lemma 5. Clearly Φ maps $\mathbf{Z}$ into itself. Moreover, as in the proof of Proposition 2, $\mathbf{Z}$ is nonempty, convex, and a subset of a locally convex Hausdorff space. The proof of Lemma 5 then implies that Φ has closed graph in the product topology and is nonempty, compact and convex-valued in $\mathbf{z}$. Consequently, the Kakutani-Fan-Glicksberg Fixed Point Theorem again applies and implies that Φ has a fixed point $\mathbf{z}^* \in \Phi(\mathbf{z}^*)$. The argument that the fixed point $\mathbf{z}^*$ corresponds to a steady-state equilibrium is identical to that in Proposition 2, and follows from the fact that within argument identical to that of Lemma 2, $G'^{-1}\left(\left(1 - \lambda^{-1}\right) / \left(\rho + \eta_1\right)\right) > 0$ implies that $x^*_0 > 0$. The result that $\omega^* < 1$ then follows immediately. Finally, as in the proof of Proposition 2, either $\eta_1 > 0$ or $x^*_{-1} > 0$ is sufficient for $g^* > 0$. ■

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Table 1. Benchmark Results

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.71	0
η_2	0	0	0	0.08	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	3.53	0	∞	0
ζ_2	∞	3.53	0	∞	0.98
ζ_3	∞	3.53	0	∞	1.93
ζ_4	∞	3.53	0	∞	1.97
ζ_5	∞	3.53	0	∞	1.98
$v_1 - v_0$	2.71	3.53	1.71	1.52	1.98
x_{-1}^*	0.22	0.25	0.27	0.12	0.32
x_0^*	0.35	0.41	0.27	0.25	0.30
μ_0^*	0.24	0	0	0.46	0
μ_1^*	0.33	0.48	0.51	0.19	0.44
μ_2^*	0.20	0.25	0.25	0.13	0.24
ω^*	0.95	0.93	0.94	0.96	0.94
Researcher ratio	0.032	0.026	0.027	0.028	0.039
$\ln C(0)$	33.78	35.47	35.57	34.20	36.16
g^*	0.0186	0.0258	0.0263	0.0204	0.0296
Welfare	683.0	719.8	722.0	692.1	735.1

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x_{-1}^*; the (annual) R&D rate of neck-and-neck competitors, x_0^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (38). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 2. $\lambda = 1.01$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.01	1.01	1.01	1.01	1.01
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	3.14	0.06
η_2	0	0	0	0.23	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	0.19	0	∞	0.04
ζ_2	∞	0.19	0	∞	0.10
ζ_3	∞	0.19	0	∞	0.10
ζ_4	∞	0.19	0	∞	0.10
ζ_5	∞	0.19	0	∞	0.10
$v_1 - v_0$	0.14	0.19	0.09	0.08	0.10
x_{-1}^*	1.08	1.27	1.29	0.64	1.51
x_0^*	1.67	1.95	1.29	1.25	1.38
μ_0^*	0.25	0	0	0.45	0.01
μ_1^*	0.33	0.50	0.50	0.20	0.45
μ_2^*	0.19	0.25	0.25	0.14	0.23
ω^*	0.99	0.99	0.99	0.99	0.99
Researcher ratio	0.008	0.007	0.007	0.007	0.010
$\ln C(0)$	10.28	11.88	11.90	10.66	12.66
g^*	0.0186	0.0257	0.0258	0.0203	0.0293
Welfare	213.1	247.8	248.3	221.4	265.0

Note: This table gives the results of the numerical computations with $\rho = 0.05$, $\lambda = 1.01$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x_{-1}^*; the (annual) R&D rate of neck-and-neck competitors, x_0^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (38). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 3. $\boldsymbol{\lambda = 1.2}$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.20	1.20	1.20	1.20	1.20
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.19	0
η_2	0	0	0	0.08	0
η_3	0	0	0	0.05	0
η_4	0	0	0	0.05	0
η_5	0	0	0	0.05	0
ζ_1	∞	25.07	0	∞	0
ζ_2	∞	25.07	0	∞	5.52
ζ_3	∞	25.07	0	∞	10.95
ζ_4	∞	25.07	0	∞	13.64
ζ_5	∞	25.07	0	∞	14.98
$v_1 - v_0$	20.29	25.07	13.86	9.62	14.98
x^*_{-1}	0.06	0.068	0.08	0.02	0.09
x^*_0	0.10	0.12	0.08	0.06	0.09
μ^*_0	0.22	0	0	0.56	0
μ^*_1	0.32	0.45	0.52	0.24	0.45
μ^*_2	0.20	0.26	0.26	0.10	0.24
ω^*	0.81	0.74	0.78	0.92	0.76
Researcher ratio	0.069	0.057	0.070	0.037	0.089
$\ln C(0)$	114.78	116.79	117.17	115.49	117.30
g^*	0.0186	0.0265	0.0282	0.0189	0.0306
Welfare	2303.0	2346.5	2354.7	2317.3	2358.3

Note: This table gives the results of the numerical computations with $\rho = 0.05$, $\lambda = 1.2$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x^*_{-1}; the (annual) R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (38). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 4. $\gamma = 0.1$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.1	0.1	0.1	0.1	0.1
η_1	0	0	0	3.18	0
η_2	0	0	0	0.04	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	3.12	0	∞	0
ζ_2	∞	3.12	0	∞	0.62
ζ_3	∞	3.12	0	∞	1.74
ζ_4	∞	3.12	0	∞	1.82
ζ_5	∞	3.12	0	∞	1.84
$v_1 - v_0$	2.21	3.12	1.64	0.49	1.88
x^*_{-1}	0.27	0.28	0.29	0.19	0.29
x^*_0	0.29	0.31	0.29	0.25	0.29
μ^*_0	0.31	0	0	0.77	0
μ^*_1	0.33	0.50	0.50	0.10	0.49
μ^*_2	0.17	0.25	0.25	0.06	0.25
ω^*	0.94	0.92	0.92	0.98	0.92
Researcher ratio	0.008	0.008	0.008	0.003	0.010
$\ln C(0)$	34.07	36.16	36.20	34.94	36.31
g^*	0.0186	0.0278	0.0280	0.0222	0.0286
Welfare	688.8	734.2	735.1	707.7	737.6

Note: This table gives the results of the numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.1$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x^*_{-1}; the (annual) R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (38). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 5. $\gamma = 0.6$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.6	0.6	0.6	0.6	0.6
η_1	0	0	0	0.61	0.01
η_2	0	0	0	0.18	0
η_3	0	0	0	0.07	0
η_4	0	0	0	0.03	0
η_5	0	0	0	0	0
ζ_1	∞	5.26	5.26	∞	0
ζ_2	∞	5.26	5.26	∞	0.62
ζ_3	∞	5.26	5.26	∞	1.52
ζ_4	∞	5.26	5.26	∞	1.98
ζ_5	∞	5.26	5.26	∞	2.35
$v_1 - v_0$	4.40	5.26	5.26	2.47	2.35
x^*_{-1}	0.13	0.18	0.18	0.03	0.31
x^*_0	0.64	0.85	0.85	0.27	0.25
μ^*_0	0.09	0	0	0.29	0.01
μ^*_1	0.26	0.42	0.42	0.12	0.29
μ^*_2	0.19	0.25	0.25	0.09	0.15
ω^*	0.94	0.94	0.94	0.94	0.93
Researcher ratio	0.073	0.044	0.044	0.084	0.097
$\ln C(0)$	33.19	33.88	33.88	33.91	35.48
g^*	0.0186	0.0198	0.0198	0.0229	0.0303
Welfare	671.3	685.6	685.6	687.3	721.6

Note: This table gives the results of the numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.6$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x^*_{-1}; the (annual) R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (38). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

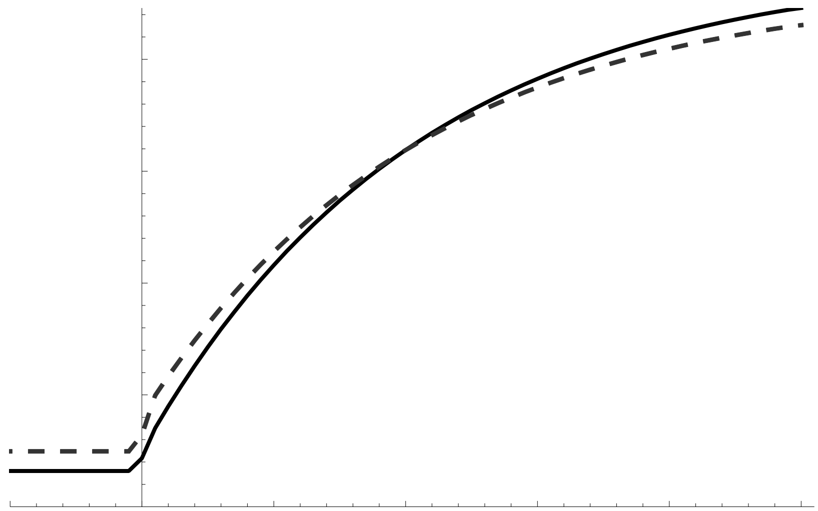

Figure 1. Value Functions.

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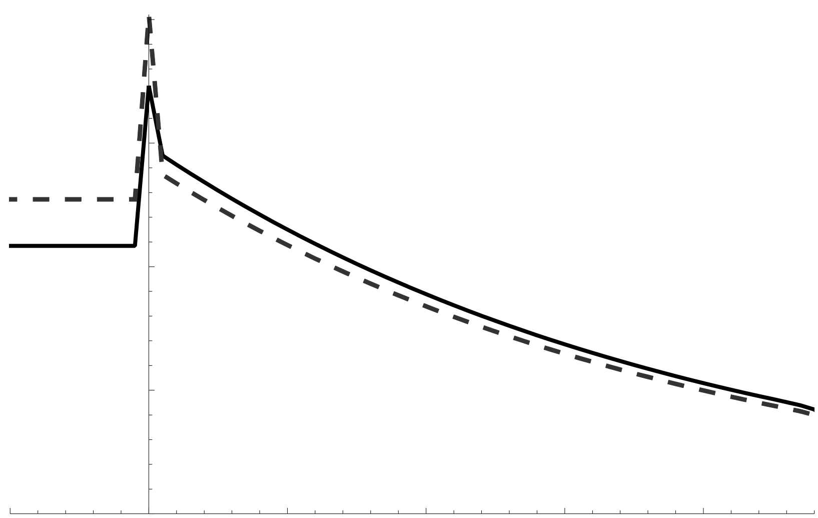


Figure 2. R&D Efforts.

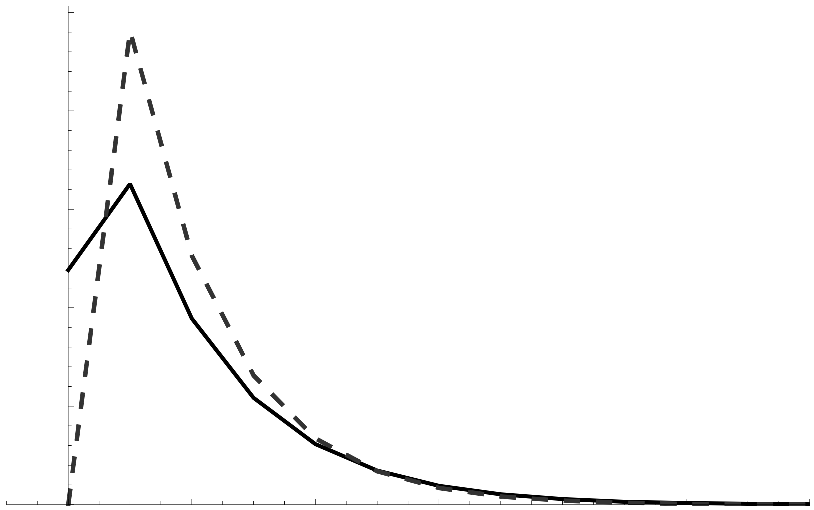


Figure 3. Industry Shares.

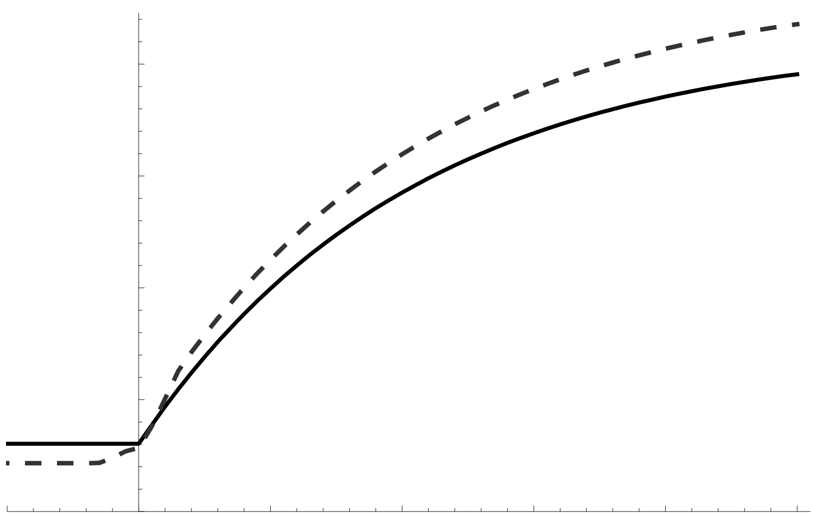


Figure 4. Value Functions.

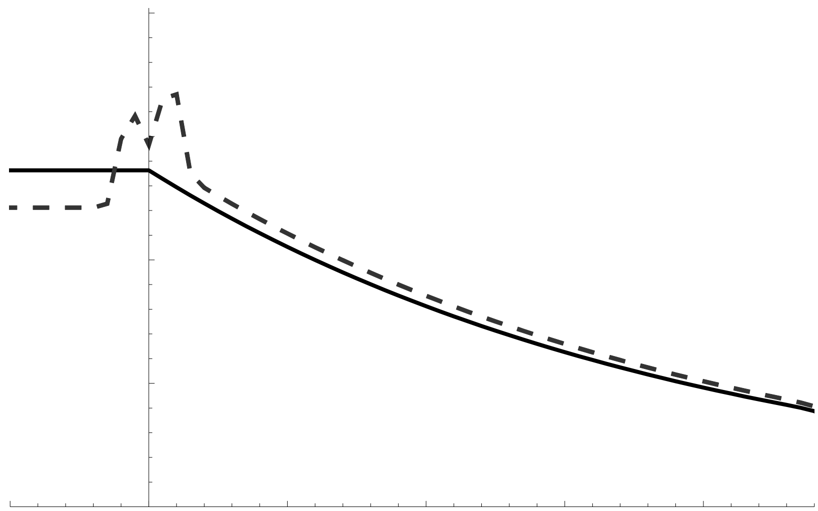


Figure 5. R&D Efforts.

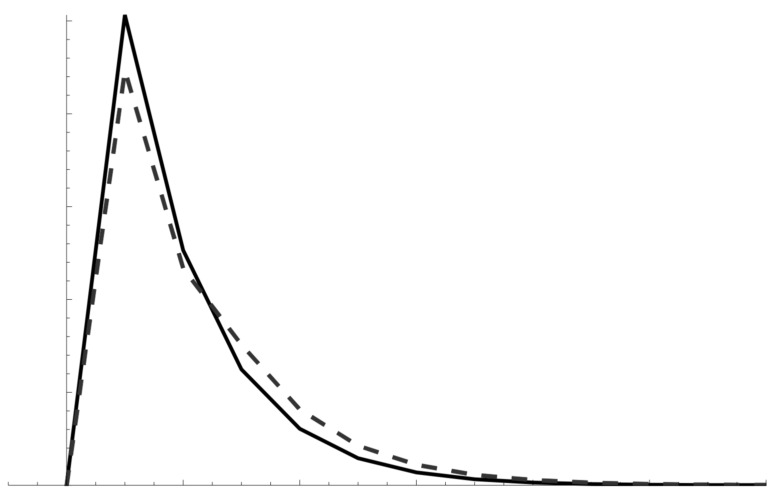
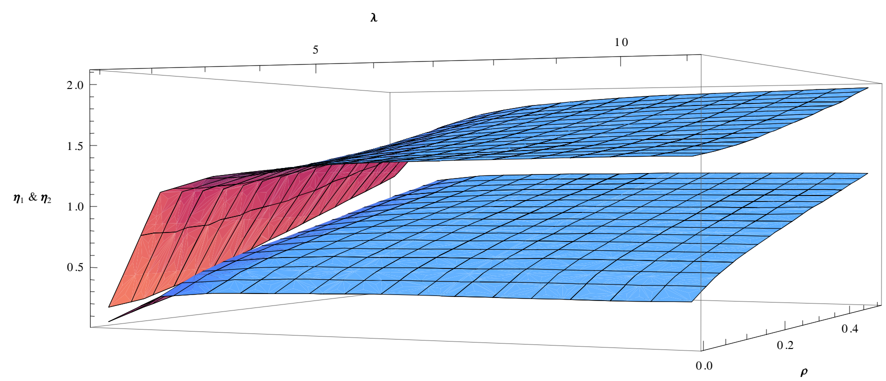
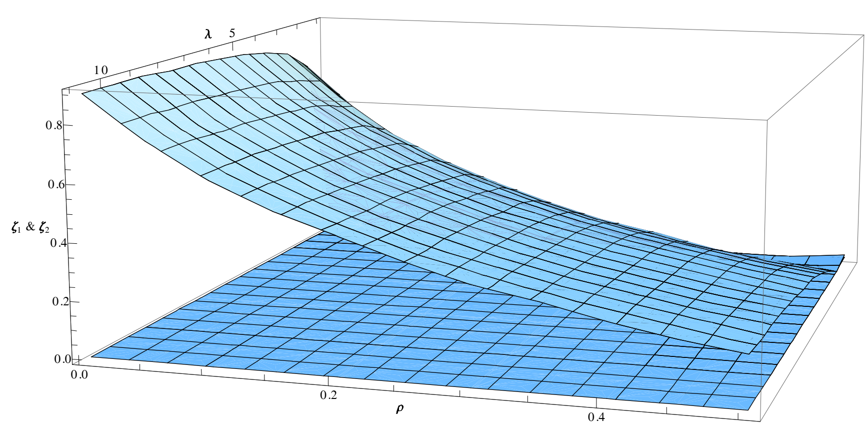


Figure 6. Industry Shares.



η_1−Upper Layer η_2−Lower Layer

Figure 7. Welfare Maximizing Flow Rates.



ζ_1−Lower Layer ζ_2−Upper Layer

Figure 8. Welfare Maximizing License Fees.