MACRO IMPLICATIONS OF HOUSEHOLD FINANCE

Preliminary and Incomplete *

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Abstract

Our paper examines the impact of heterogeneous trading opportunities on the distribution of asset shares and wealth in an equilibrium model. We distinguish between "passive" traders who hold fixed portfolios of equity and bonds, and "active" traders who adjust their portfolios to changes in the investment opportunity set. In the presence of non-participants, the fraction of active traders is critical for asset prices, because only these traders respond to variation in state prices and hence help to clear the market, not the fraction of agents that participate in asset markets. We develop a new method for computing equilibria in this class of economies. This method relies on an optimal consumption sharing rule and an aggregation result for state prices that allows us to solve for equilibrium prices and allocations without having to search for market-clearing prices in each asset market. In a calibrated version of our model, we show that the heterogeneity in trading opportunities allows for a closer match of the wealth and asset share distribution as well as the moments of asset prices.

Keywords: Asset Pricing, Risk Sharing, Limited Participation (JEL code G12)

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1 Introduction

The correlation of household consumption and income in the data presents a challenge for models with unlimited trading opportunities. This observation started the work on incomplete market models which imposes exogenous restrictions on trading opportunities. More recently, more evidence has emerged about the positive correlation of household wealth and household participation in asset markets and the share of stocks in their portfolio. Even among those households who participate in asset markets, there are substantial differences in their investment strategies and realized portfolio returns that are not easily explained by preference heterogeneity or differences in non-tradable risk exposure.[1] Standard incomplete market models cannot address this dimension of household heterogeneity.

Our paper introduces heterogeneity in trading opportunities in an otherwise standard endowment economy with a large number of agents who are subject to both aggregate and idiosyncratic shocks, and who have constant relative risk aversion (CRRA) preferences with coefficient α. We distinguish between four different trading technologies; each household has access to only one of these: (i) *complete* traders who trade a complete menu of assets, (ii) *z-complete* traders who trade claims whose payoffs are contingent on aggregate shocks (e.g. bonds of different maturities, equity etc.) but not idiosyncratic shocks, (iii) *diversified* investors who trade a fixed portfolio of bonds and stocks, and (iv) *non-participants* who only have access to a savings account. All of these households face solvency constraints as well. The last two trade fixed portfolios of riskless and risky assets, but the last two do not.

We distinguish between "passive" traders –non-participants and diversified investors– who trade a fixed portfolio of safe and risky assets and "active" traders -z-complete and complete traders– who adjust their portfolio to changes in the investment opportunity set. At the micro level, this distinction allows us to match the heterogeneity in portfolio composition and returns that was documented in the data, but it also affects aggregate outcomes. Passive traders cannot respond to differences in state prices by reallocating consumption across different aggregate states of the world tomorrow. Instead, passive traders only respond to changes in average state prices that show up in the risk-free rate or the expected return on the market. Thus, they bear none of the residual aggregate risk created by non-participants themselves and hence they shift this residual aggregate risk to the active traders. Their presence concentrates aggregate risk among the market participants, because they consume "too much" in low consumption growth states and "too little" in high aggregate consumption growth states. Hence, the active traders bear the residual aggregate risk in the economy. They are "active" traders in that they actively manage their portfolio to take

[1] Campbell (2006) refers to the body of literature that documents this heterogeneity as "household finance". See Campbell (2006)'s AFA presidential address for a comprehensive discussion of these and other issues related to household finance.

advantage of variation in the aggregate price of risk. As we show, active traders respond to the variation in state prices by concentrating their consumption in "cheap" aggregate states (states with low state prices for aggregate consumption). Correspondingly, state prices will be higher in recessions to induce active traders to consume less, and lower in expansions. The non-participants and diversified traders are being forced to take the other side of these trades, consuming more in "expensive" aggregate states.

When this segment of active traders is small enough, this mechanism contributes large and counter-cyclical volatility to state prices across different aggregate states of the world tomorrow, while the larger segment of passive traders keeps today's expectation of state prices tomorrow constant. The first effect delivers large, counter-cyclical risk premia while the second effect keeps the risk-free rate stable. Furthermore, the presence of active traders keeps passive traders from accumulating wealth; active traders easily accumulate more wealth than passive traders. In the quantitative section, we calibrate the size of each trader segment to match the asset prices in the data, and we show that the interaction between these traders brings heterogeneous agent models closer to matching the wealth and the asset share distribution, without resorting to other sources of household heterogeneity.

Incomplete market economies with a large number of agents who trade in multiple assets are hard to analyze, even more so when different agents can trade different menus of assets. Our paper develops a new method for computing equilibria in a class incomplete market economies with heterogeneous trading opportunities, and we apply this method to solve a calibrated version of our model. Instead of directly imposing the trading restrictions on the recursive representation of the household's consumption and portfolio choice problem, we impose measurability restrictions on the household's time zero trading problem. These restrictions govern how net wealth is allowed to vary across different states of the world, similar to the measurability constraints in Aiyagari, Marcet, Sargent, and Seppala (2002) and Lustig, Sleet, and Yeltekin (2006). We use the multipliers on these constraints to derive a consumption sharing rule for households and an analytical expression for state prices. Importantly, the household's consumption sharing rule does not depend on the trading technology, only the dynamics of the multipliers do. State prices only depend on a weighted average of these multipliers –the $-1/\alpha$-th moment. We refer to this simply as the aggregate multiplier. It summarizes the aggregate shadow cost of the binding measurability and solvency constraints. This extends the aggregation result by Lustig (2005), who considers a complete markets environment.

To implement our algorithm, we use a recursive net savings function as an accounting device. This function allows us to determine the individual's multiplier updating rule as a function of the updating rule for the aggregate multiplier and the restrictions implied by our asset structure. These two updating rules – the aggregate multiplier updating rule and the individual's multiplier updating rule – completely determine the equilibrium of our economy. Different trading technologies only change the individual and aggregate multiplier updating rules, but they do not change our aggregation result. In the computational section, we construct the net savings function and the individual multiplier rule, taking as given some initial aggregate multiplier updating rule. Next, we solve for a new aggregate multiplier updating rule by simulating a process for the aggregate multiplier given the conjectured rule. Finally, we iterate on the aggregate multiplier updating rule until convergence is achieved.

Quantitatively, our approach has several major advantages. First, our aggregation result implies that we only need to forecast a single moment of the multiplier distribution, regardless of the number and nature of the different trading technologies. Also, our aggregation result allows us to directly compute the pricing kernel as a function of this moment. There is no need to search for the vector of prices that clears the various asset markets, as in the standard methods (Lucas (1994) and Krusell and Smith (1997)). Searching for market-clearing prices is hard because, in general, we do not know the mapping from the wealth distribution to prices. In addition, the updating rule for the multipliers involves solving a simple system of equations.

In the quantitative section of the paper, we show that the interaction between a small segment of active traders and a larger segment of passive traders is key to understanding asset prices and the wealth distribution. Due to this interaction, equilibrium state prices are highly volatile but their conditional expectation –and hence the risk-free rate– is not. Passive traders consume too much in low growth states (recessions) and too little in high growth states (expansions). Since there is no predictability in aggregate consumption growth, changes in the risk-free rate do nothing to clear the market in each aggregate state tomorrow. Instead, changes in the average state price and hence the risk-free rate change the average consumption growth path of non-participants, a large fraction of the population, by the *same amount* in all aggregate states tomorrow, thus creating even more aggregate risk in the economy. Instead, the equilibrium state prices are highly volatile *across* aggregate states to induce the small segment of active traders to adjust their consumption growth in different aggregate states of the world by enough. The active traders consume less in low growth states when state prices are high and more in high growth states when state prices are low. The share of total wealth owned by the active traders declines in low aggregate consumption growth states, because these take highly leveraged equity positions. As a result, the conditional volatility of state prices increases after each recession: a larger adjustment in state prices is needed to clear the goods markets.

The distinction between non-participants and diversified traders is critical. As long as all households can trade a claim to the market, regardless of the composition of the different trading groups, the risk premia are the same as in the representative agent economy, i.e. small and constant. This being the case, everyone bears the same amount of aggregate risk in equilibrium, the ability to reallocate consumption across different aggregate states of the world is redundant and the distinction between active and passive traders is moot, because the aggregate multiplier adjustment to state prices is constant, i.e. there is no spread between the prices in different states. Risk premia are identical to those in the representative agent version of our economy. However, if we exclude some households from actively trading shares in total financial wealth or the market, this irrelevance result, first derived by Krueger and Lustig (2006), disappears and the distinction between active and passive traders starts to matter.[2] In a calibrated version of this economy, we show that non-participants would be willing to pay 15 % of consumption every year to gain complete access to financial markets –to become a z-complete trader. Exclusion from financial markets is costly only because of the induced volatility of state prices and the passive trader's inability to exploit this spread in state prices.

In the quantitative section, we show that the z-complete traders accumulate five times more wealth on average than non-participants. Passive traders have an even stronger precautionary motive to save, but they earn much lower returns than the active traders, and hence fail to accumulate assets. As a result, our model comes closer to matching the actual distribution of wealth among US households than standard models, especially in the tails. The model also replicates the strong correlation between wealth and the portfolio share of risky assets –the main stylized fact of household portfolios–, even if households are ex ante identical. In addition, the failure of passive traders to accumulate wealth after a history of good shocks contributes to a break-down of the standard self-insurance mechanism. As a result, the idiosyncratic component of consumption growth volatility is highest for non-participants and then decreases monotonically in the degree of sophistication of traders. The "aggregate component" of household consumption growth volatility follows the reverse pattern, as we discussed, with active traders choosing high market beta investment strategies. As a result, the correlation of consumption growth with stock returns increases in wealth, consistent with Mankiw and Zeldes (1991) and Brav, Constantinides, and Geczy (2002a), who find lower risk aversion estimates off the Euler equation for stock returns for wealthier households. To an econometrician observing data generated by our model, heterogeneity in trading technologies may appear as heterogeneity in preferences. Using consumption and return data generated by our model, we estimate the elasticity of intertemporal substitution, and we find higher estimates for active traders than for passive traders. We find EIS estimates in the $[.92, 1.1]$ range using the bond returns and in the $[.48, .78]$ range for stock returns. This is consistent with the findings of Vissing-Jorgensen (2002), who, using CEX consumption data, reports estimates in the range $[.3, .4]$ for stock returns and $[.8, 1]$ for bond returns.

Related Literature With respect to our methodological goal, this paper is closely related to Krusell and Smith (1997) and (1998). KS developed a computational method that solves for and

[2] One of the key assumptions for this result is that aggregate shocks are i.i.d. and that the idiosyncratic shocks are independent of the aggregate shocks.

estimates approximate pricing functions, using the mean of the wealth distribution as the state variable. Their approach, which requires one to solve for and estimate a separate pricing function for each of the assets that are traded, has been used extensively and is in many respects the standard approach in the literature. The ability of the KS method to closely approximate prices using only the mean of the wealth distribution relies on approximate aggregation. Building on work by Krueger and Lustig (2006), we show that there is exact aggregation in our model, as long as all there are no non-participants. In this case, there is an equilibrium with an invariant wealth distribution. Unfortunately, this model's asset pricing implications will be at odds with the data.

In contrast to KS, we make use of an analytic asset pricing result that expresses state prices as a function of the growth rates of aggregate consumption and a single moment of the multiplier distribution. The algorithm consists of a search for the optimal forecasting function for this single moment of the multiplier distribution rather than a search for a menu of pricing functions. Moreover, as we show in our example, our approach works even when the aggregation result does not hold, in the case with non-participants. This model's asset pricing predictions line up better with the data. However, unlike KS, we do not include capital in our model. This is a substantial simplification, though we believe that our method can readily be extended to include capital.

Standard incomplete market models cannot match the dispersion of the wealth distribution in the data. In the literature, preference heterogeneity (Krusell and Smith (1997)) or concern for status Roussanov (2007) have been explored to generate more dispersion. Our paper focus exclusively on heterogeneity in trading technologies; we show that this mechanism alone generates the same skewness and kurtosis as in the data. However, the middle class in our model still accumulates too much wealth relative to the rich. There is a growing literature on the asset pricing effects of limited stock market participation, starting with Saito (1996) and Basak and Cuoco (1998). Our paper is the first -we believe- to document the importance of distinguishing between active and passive traders for understanding asset prices and the wealth distribution. Other papers have focussed mostly on heterogeneity in *preferences* (e.g. see Krusell and Smith (1998) for heterogeneity in the rate of time preference and Vissing-Jorgensen (2002), Guvenen (2003) and Gomez and Michaelides (2007) for heterogeneity in the willingness of households to substitute intertemporally) and the heterogeneity in *participation decisions* (e.g. see Guvenen (2003) and Vissing-Jorgensen (2002)), rather than trading opportunities. There has been substantial progress on the empirical front in carefully documenting the heterogeneity of household investment decisions. In a comprehensive dataset of Swedish households, Calvet, Campbell, and Sodini (2006) find that sophisticated investors realize higher Sharpe ratios, but at the cost of incurring more volatility. Indeed, the z-complete and complete traders in our model realize much higher returns, but they adopt a sophisticated trading strategy that exploits the time variation in the risk premium to do so. Campbell (2006) argues that some households voluntarily limit the set of assets they decide to trade for fear of making mistakes, at the cost of forgoing higher returns. To capture this, we introduce "diversified investors", who simply trade a claim to the market.

There is an active debate about the effects of limited participation on asset prices. Guvenen (2003) argues that limited participation goes a long way towards explaining the equity premium in a model with a bond- and a stockholder. In this model, stockholders are more willing to substitute intertemporally. Vissing-Jorgensen (2002) documents that this is a feature of the data. Guvenen's model abstracts from idiosyncratic risk. In more recent work, Gomez and Michaelides (2007) also consider a model with bond-and stockholders, but they add idiosyncratic risk. Their model produces a large risk premium, which they attribute to imperfect risk sharing among stockholders, not to the exclusion of households from equity markets. In our benchmark model, we show analytically that market segmentation only affects the risk-free rate, but not risk premia, as long as there is no predictability in aggregate consumption growth and all traders can trade the market –a claim to all diversifiable income. We do not model the participation decision, but we show that the costs of non-participation are too large in a model with volatile state prices to be simply explained by standard cost arguments. Instead, one might have to appeal to information and cognitive ability differences.[3] However, we argue that this is not implausible, given that complex trading strategies need to be implemented to fully realize the welfare gains of participation.

This paper is organized as follows. section 2 describes the environment, the preferences and trading technologies for all households. section 3 characterizes the equilibrium allocations and prices using cumulative multipliers that record all the binding measurability and solvency constraints. section 4 describes a recursive version of this problem that we can actually solve. This section also describes conditions under which market segmentation does not affect the risk premium. Finally, in section 5 we study a calibrated version of our economy.

2 Model

In this section we describe the environment, and we describe the household problem for each of different asset trading technologies. We also define an equilibrium for this economy.

2.1 Environment

This is an endowment economy with a unit measure of households who are subject to both aggregate and idiosyncratic income shocks. Households are ex ante identical, except for the access to trading technologies. Ex post, the households differ in terms of their idiosyncratic income shock realizations. Some of the households will be able to trade a complete set of securities, but others will trade a more limited set of securities. All of the households face the same stochastic process

[3] In the data, education is a strong predictor of equity ownership (see Table I in Campbell (2006)).

for idiosyncratic income shocks, and all households start with the same present value of tradeable wealth.

In the model time is discrete, infinite, and indexed by $t = 0, 1, 2, ...$ The first period, $t = 0$, is a planning period in which financial contracting takes place. We use $z_t \in Z$ to denote the aggregate shock in period t and $\eta_t \in N$ to denote the idiosyncratic shock in period t. z^t denotes the history of aggregate shocks, and, similarly, η^t, denotes the history of idiosyncratic shocks for a household. The idiosyncratic events η are i.i.d. across households. We use $\pi(z^t, \eta^t)$ to denote the unconditional probability of state (z^t, η^t) being realized. The events are first-order Markov, and we assume that

$$\pi(z^{t+1}, \eta^{t+1}|z^t, \eta^t) = \pi(z_{t+1}|z_t)\pi(\eta_{t+1}|z_{t+1}, \eta_t).$$

Since we can appeal to a law of large number, $\pi(z^t, \eta^t)/\pi(z^t)$ also denotes the fraction of agents in state z^t that have drawn a history η^t. We use $\pi(\eta^t|z^t)$ to denote that fraction. We introduce some additional notation: $z^{t+1} \succ z^t$ or $y^{t+1} \succ y^t$ means that the left hand side node is a successor node to the right hand side node. We denote by $\{z^\tau \succ z^t\}$ the set of successor aggregate histories for z^t including those many periods in the future; ditto for $\{\eta^\tau \succ \eta^t\}$. When we use $\succeq$, we include the current nodes z^t or η^t in the summation.

There is a single final good in each period, and the amount of it is given by $Y(z^t)$, which evolves according to

$$Y(z^t) = \exp\{z_t\}Y(z^{t-1}), \tag{2.1}$$

with $Y(z^1) = \exp\{z_1\}$. This endowment good comes in two forms. The first form is diversifiable capital income, which is not subject to the idiosyncratic shock, and is given by $(1 - \gamma)Y(z^t)$. The other form is non-diversifiable income which is subject to idiosyncratic risk and is given by $\gamma Y(z^t)\eta_t$; hence γ is the share of income that is non-diversifiable.

Households trade assets in securities markets and they trade the final good in spot markets that re-open in every period. A fraction μ_1 of households can trade claims that are contingent on both their aggregate and their idiosyncratic state (z^t, η^t), a fraction μ_2 can trade claims contingent on the aggregate state z^t, a fraction μ_3 can only trade claims to a share of diversifiable income, and a faction μ_4 can only trade non-contingent contracts to deliver units of the final good in the next time the spot market reopens.

We refer to the first set of households as the complete traders since they are able to trade a complete set of Arrow securities. We refer to the second set as the z-complete traders since they can only offset aggregate risk but not idiosyncratic risk through their asset trading. We refer to the third set of households as the diversified investors since they are trading a claim to total financial wealth or equivalently a claim to all tradeable income. We will refer to the fourth set of households as non-participants, since they only have a savings account.

Since the return on the tradeable income claim is measurable with respect to the asset trading structures of the complete and z-complete traders, we assume w.l.o.g. that the households in the first two partitions can also trade the claim to diversifiable income. Since this is not the case for the non-participants in the fourth partition, we assume that they cannot trade or hold this claim.

$\varpi(z^t)$ denotes the price of a claim to diversifiable income in aggregate state z^t. In each node, total diversifiable income is given by $(1-\gamma)Y(z^t)$. We use $q\left[(z^{t+1}, \eta^{t+1}), (z^t, \eta^t)\right]$ to denote the price of a unit claim to the final good in state (z^{t+1}, η^{t+1}) acquired in state (z^t, η^t). The absence of arbitrage implies that there exist aggregate state prices $q(z_{t+1}, z^t)$ such that

$$q\left[\left(z^{t+1}, \eta^{t+1}\right), \left(z^t, \eta^t\right)\right] = \pi(\eta^{t+1}|z^{t+1}, \eta^t)q(z_{t+1}, z^t),$$

where $q(z_{t+1}, z^t)$ denotes the price of a unit of the final good in aggregate state z^{t+1} given that we are in aggregate history z^t. From these, we can back out the present-value state prices recursively as follows:

$$\pi(z^t, \eta^t)P(z^t, \eta^t) = q(z_t, z^{t-1})q(z_{t-1}, z^{t-2}) \cdots q(z_1, z^0)q(z_0).$$

We use $\widetilde{P}(z^t, \eta^t)$ to denote the Arrow-Debreu prices $P(z^t)\pi(z^t, \eta^t)$. Let $m(z^{t+1}|z^t) = P(z^{t+1})/P(z^t)$ denote the stochastic discount factor that prices any random payoffs. We assume there is always a non-zero measure of z-complete or complete traders to guarantee the uniqueness of the stochastic discount factor.

The diversified traders effectively hold a *fixed portfolio of equity and bonds.* Following Abel (1999), we define equity as a leveraged claim to consumption. Let ϕ denote the leverage parameter, let $b_t(z^t)$ denote the supply of one-period risk-free bonds, and let R_t^f denote the risk-free rate. We can decompose the aggregate payout that flows from the tradeable income claim $(1-\gamma)Y(z^t)$ into a dividend component $d_t(z^t)$ from *equity* and a bond component $R_t^f(z^{t-1})b(z^{t-1}) - b(z^t)$. The bond supply adjusts in each node z^t to ensure that the bond/equity ratio equals ϕ:

$$b(z^t) \quad = \quad \phi\left[\varpi(z^t) - b(z^t)\right]$$

for all z^t. The diversified trader invests a fraction $\phi/(1+\phi)$ in bonds and the remainder in equity.

All households are endowed with a claim to their per capita share of both diversifiable and non-diversifiable income. Households cannot directly trade away their claim to non-diversifiable risk, though households can hedge this risk to the extent that they can trade a sufficiently rich menu of securities. For example, the complete households can hedge both their idiosyncratic and their aggregate risk. We also assume that households that cannot hold the equity claim, the bonds-only traders, trade away their claim to diversifiable capital in exchange for a claim that they can hold.

Finally, the households face exogenous limits on their net asset positions. The value of the household's net assets must always be greater than $-\psi$ times the value of their non-diversifiable income, where $\psi \in (0,1]$. We allow households to trade away or borrow up to 100% of the value of their claims to diversifiable capital.

All households are infinitely lived and rank stochastic consumption streams $\{c(z^t,\eta^t)\}$ according to the following criterion

$$U(c) = E\left\{\sum_{t\geq 1}^{\infty} \beta^t \pi(z^t,\eta^t) \frac{c(z^t,\eta^t)^{1-\alpha}}{1-\alpha}\right\}, \tag{2.2}$$

where $\alpha > 0$ denotes the coefficient of relative risk aversion, and $c(z^t,\eta^t)$ denotes the household's consumption in state (z^t,η^t).

2.2 Asset Trading Technologies

All of the households have access to *only* one of four asset trading technologies. We assume households cannot switch between technologies. It is straightforward to extend the methodology we develop to allow for exogenous transitions between trading technologies. The probability of these transitions could even be contingent on the household's realized shocks. The first technology we consider gives households access to a complete menu of assets.

Complete Traders We start with the household in the first asset partition who can trade both a complete set of securities as well as claims to diversifiable income. The budget constraint for this trader in the spot market in state (z^t,η^t) as

$$\begin{aligned} &\gamma Y(z^t)\eta_t + a(z^t,\eta^t) + \sigma(z^{t-1},\eta^{t-1})\left[(1-\gamma)Y(z^t) + \varpi(z^t)\right] - c(z^t,\eta^t) \\ \geq &\sum_{z^{t+1}\succ z^t} q(z_{t+1},z^t) \sum_{\eta^{t+1}\succ \eta^t} a(z^{t+1},\eta^{t+1})\pi(\eta^{t+1}|z^{t+1},\eta^t) + \sigma(z^t,\eta^t)\varpi(z^t)\ \forall (z^t,\eta^t), \end{aligned} \tag{2.3}$$

where $a(z^t,\eta^t)$ denotes the number of unit claims to the final good held for state (z^t,η^t), $\sigma(z^{t-1},\eta^{t-1})$ denotes the number of claims on diversifiable income acquired in state (z^{t-1},η^{t-1}), where $(z^t,\eta^t) \succ (z^{t-1},\eta^{t-1})$. The period 0 spot budget constraint is given by

$$\varpi(z^0)\left[1 - a(z_0,\eta_0)\right] \geq q(z_1,z^0)\sum_{z_1,\eta_1} a(z^1,\eta^1)\pi(\eta^{t+1}|z^{t+1},\eta^t), \tag{2.4}$$

where z^0 and η^0 are degenerate states representing the initial position in the planning state at time 0 before any of the shocks have been realized, and where $\varpi(z^0)$ denotes the price of capital in the planning stage and $q(z_1,z^0)$ denotes the price in this stage of a claim to consumption in period 1. In addition to their spot budget constraint, these traders also face a lower bound on the value of their net asset position. Let $\underline{M}(\eta^t, z^t)$ be defined as

$$\underline{M}(\eta^t, z^t) = -\psi \sum_{\{z^\tau \succeq z^t, \eta^\tau \succeq \eta^t\}} \gamma Y(z^\tau)\eta_\tau \frac{\pi(z^\tau, \eta^\tau)P(z^\tau, \eta^\tau)}{\pi(z^{t+1}, \eta^{t+1})P(z^{t+1}, \eta^{t+1})} \tag{2.5}$$

The lower bound is given by:

$$a(z^{t+1}, \eta^{t+1}) + \sigma(z^t, \eta^t)\left[d(z^{t+1}) + \varpi(z^{t+1})\right] \geq M(\eta^t, z^t). \tag{2.6}$$

The complete trader's problem is to choose $\{c(z^t, \eta^t), a(z^{t+1}, \eta^{t+1}), \sigma(z^t, \eta^t)\}$, $a(z^1, \eta^1)$ and $\sigma(z^0, \eta^0)$ so as to maximize (2.2) subject (2.3-2.6).

z-complete Traders The households in the second asset partition have a budget constraint in the spot market in state (z^t, η^t) given by

$$\begin{aligned} & \gamma Y(z^t)\eta_t + a(z^t, \eta^{t-1}) + \sigma(z^{t-1}, \eta^{t-1})\left[(1-\gamma)Y(z^t) + \varpi(z^t)\right] - c(z^t, \eta^t) \\ \geq & \sum_{z^{t+1} \succ z^t} q(z_{t+1}, z^t)a(z^{t+1}, \eta^t) + \sigma(z^t, \eta^t)\varpi(z^t)\ \forall (z^t, \eta^t), \end{aligned} \tag{2.7}$$

where $a(z^{t+1}, \eta^t)$ denotes the number of claims acquired in state (z^t, η^t) that payoff one unit if the aggregate state tomorrow is z^{t+1}, and $a(z^t, \eta^{t-1})$ is such that $\eta^t \succ \eta^{t-1}$. The period 0 spot budget constraint is given by

$$\varpi(z^0)\left[1 - \sigma(z_0, \eta_0)\right] \geq \sum_{z_1, \eta_1} q(z_1, z^0)a(z^1, \eta^0)\pi(\eta^{t+1}|z^{t+1}, \eta^t). \tag{2.8}$$

The z-complete traders face bound on their net asset position which is given by

$$a(z^{t+1}, \eta^t) + \sigma(z^t, \eta^t)\left[d(z^{t+1}) + \varpi(z^{t+1})\right] \geq M(\eta^t, z^t) \tag{2.9}$$

for each $(z^{t+1}, \eta^{t+1}) \succ (z^t, \eta^t)$. Note here that for each aggregate state tomorrow, z^{t+1}, the magnitude of the bound is determined by the idiosyncratic state η^{t+1} in which the present value of non-diversifiable income is smallest.

The z-complete trader's problem is to choose $\{c(z^t, \eta^t), a(z^{t+1}, \eta^t), \sigma(z^t, \eta^t)\}$, $a(z^1, \eta^0)$ and $\sigma(z^0, \eta^0)$ so as to maximize (2.2) subject (2.7-2.9).

Diversified investors We think of diversified investors as trading a claim to financial wealth, broadly defined, not just equity. These households in the third asset partition have a budget constraint in the spot market in state (z^t, η^t) given by

$$y(z^t, \eta_t) + \sigma(z^{t-1}, \eta^{t-1}) \left[d(z^t) + \varpi(z^t)\right] - c(z^t, \eta^t) \geq \sigma(z^t, \eta^t)\varpi(z^t) \ \forall (z^t, \eta^t), \tag{2.10}$$

a degenerate period 0 constraint

$$\varpi(z^0) \left[1 - \sigma(z_0, \eta_0)\right] \geq 0, \tag{2.11}$$

and a net asset position bound

$$\sigma(z^t, \eta^t) \left[d(z^{t+1}) + \varpi(z^{t+1})\right] \geq \underline{M}(\eta^t, z^t), \tag{2.12}$$

for each $(z^{t+1}, \eta^{t+1}) \succ (z^t, \eta^t)$. The equity trader's problem is to choose $\{c(z^t, \eta^t), \sigma(z^t, \eta^t)\}$ and $\sigma(z^0, \eta^0)$ so as to maximize (2.2) subject (2.10-2.12).

Non-participants The households in the fourth and final partition have a spot budget constraint in state (z^t, η^t) given by

$$\gamma Y(z^t)\eta_t + a(z^{t-1}, \eta^{t-1}) - c(z^t, \eta^t) \geq \sum_{z^{t+1} \succ z^t} q(z_{t+1}, z^t) a(z^t, \eta^t), \tag{2.13}$$

where $z^t \succ z^{t-1}$ and $\eta^t \succ \eta^{t-1}$, for states other than the first, and a first period budget constraint given by

$$\varpi(z^0) \geq a(z^0, \eta^0) \sum_{z_1, \eta_1} q(z_1, z^0) \pi(\eta^{t+1} | z^{t+1}, \eta^t) \tag{2.14}$$

The asset bound for non-participants is given by

$$a(z^t, \eta^t) \geq \underline{M}(\eta^t, z^t) \tag{2.15}$$

The non-participant's problem is to choose $\{c(z^t, \eta^t), a(z^t, \eta^t)\}$ and $a(z^0, \eta^0)$ so as to maximize (2.2) subject to (2.13-2.15).

2.3 Equilibrium

The market clearing condition in the bond market is given by:

$$\sum_{\eta^t} \left[\mu_1 a^c(z^t, \eta^t) + \mu_2 a^z(z^t, \eta^{t-1}(\eta^t)) + \mu_4 a^{np}(z^{t-1}(z^t), \eta^{t-1}(\eta^t))\right] \pi(\eta^t | z^t) = 0,$$

where a^c, a^z, a^{div}, and a^{np} denote the bond holdings of the complete-markets, z-complete, equity-only, and bonds-only traders respectively. The market clearing condition in the output claim market is given by

$$\sum_{\eta^t} \left[\mu_1 \sigma^c(z^t, \eta^t) + \mu_2 \sigma^z(z^t, \eta^t) + \mu_3 \sigma^{div}(z^t, \eta^t)\right] \pi(\eta^t | z^t) = 1.$$

An equilibrium for this economy is defined in the standard way. It consists of a list of bond and output claim holdings, a consumption allocation and a list of bond and tradeable output claim prices such that: (i) given these prices, a trader's asset and consumption choices maximizer her expected utility subject to the budget constraints, the solvency constraints and the measurability constraints, and (ii) the asset markets clear.

The next section analytically characterizes the household consumption function and the equilibrium pricing kernel in terms of the distribution of the household's stochastic multipliers.

3 Solving for Equilibrium Allocations and Prices

This section reformulates the household's problem in terms of a present-value budget constraint, and sequences of measurability constraints and solvency constraints. These measurability constraints capture the restrictions imposed by the different trading technologies of households. We show how to use the cumulative multipliers on these constraints as stochastic weights that fully characterize equilibrium allocations and prices. We are not the first to do so. Cuoco and He (1994) were the first to use this stochastic weighting scheme in a continuous-time setup. Basak and Cuoco (1998) also make use of stochastic weights to characterize equilibrium prices and allocations, but in a two-agent economy without idiosyncratic risk, in which one agent is excluded from the stock market. The approach we adopt in this paper can handle idiosyncratic risk, several trading technologies and a large number of agents. Lustig (2005) uses similar methods to solve a *complete* markets problem with solvency constraints, and he derives a similar aggregation result.

3.1 Measurability Conditions

We begin by recursively substituting into the spot budget constraints, in order to derive an expression in terms of future consumption sequences and the initial asset position in state (z^t, η^t).

Complete Traders For example, start from the complete traders constraint (2.3), and assume it holds with equality. Then we can substitute for future $a(z^{t+i}, \eta^{t+i})$, while using the equity no-arbitrage condition

$$\varpi(z^t) = \sum_{(\eta_{t+1}, z_{t+1})} \left[d(z^{t+1}) + \varpi(z^{t+1})\right] q(z_{t+1}, \eta_{t+1}, z^t),$$

to obtain the following budget constraint in terms of present value prices:

$$a(z^t, \eta^t) + \sigma(z^{t-1}, \eta^{t-1}) \left[(1-\gamma)Y(z^t) + \varpi(z^t)\right] = \sum_{\{z^\tau \succeq z^t, \eta^\tau \succeq \eta^t\}} \left[c(z^\tau, \eta^\tau) - y(z^\tau, \eta_\tau)\right] \frac{\pi(z^\tau, \eta^\tau)P(z^\tau, \eta^\tau)}{\pi(z^t, \eta^t)P(z^t, \eta^t)}.$$

Rather than carry around both a and σ, we will find it convenient to define net wealth as

$$\hat{a}(z^t, \eta^t) \equiv a(z^t, \eta^t) + \sigma(z^{t-1}, \eta^{t-1}) \left[(1-\gamma)Y(z^t) + \varpi(z^t)\right].$$

The borrowing constraint in terms of $\hat{a}$ is given by

$$\hat{a}(z^t, \eta^t) \geq \underline{M}(\eta^t, z^t). \tag{3.1}$$

Requiring that condition (3.1) hold for each (z^t, η^t) is equivalent to the spot budget constraints (2.3) and borrowing constraints (2.6) for the complete traders for all $t \geq 1$. In addition we have the period 0 gross present value constraint

$$\varpi(z^0) = \sum_{t>0} \sum_{(z^t, \eta^t)} \left[c(z^t, \eta^t) - \gamma Y(z^t)\eta_t\right] \pi(z^t, \eta^t)P(z^t, \eta^t). \tag{3.2}$$

It is straightforward to show that the spot budget and debt bound constraints for the other types of traders imply condition (3.1) hold for each (z^t, η^t) and that condition (3.2) holds. However, they also imply additional measurability constraints which reflect the extent to which their gross asset position can vary with the realized state (z^t, η^t).

z-complete Traders The z-complete traders face the additional constraint that $a(z^t, \eta^t)$ is measurable with respect to (z^t, η^{t-1}). Since the payoff of the stock $\sigma(z^{t-1}, \eta^{t-1}) \left[(1-\gamma)Y(z^t) + \varpi(z^t)\right]$ is measurable with respect to(z^t, η^{t-1}), requiring that $a(z^t, \eta^t) = a(z^t, \tilde{\eta}^t)$ for all z^t, and $\tilde{\eta}^t, \eta^t$ such that $\eta^{t-1}(\tilde{\eta}^t) = \eta^{t-1}(\eta^t)$ is equivalent to requiring that

$$\hat{a}(z^t, \left[\eta^{t-1}, \eta_t\right]) = \hat{a}(z^t, \left[\eta^{t-1}, \tilde{\eta}_t\right]), \tag{3.3}$$

for all z^t, η^{t-1}, and η_t, $\tilde{\eta}_t \in N$.

Diversified investors For the diversified investors, $a(z^t, \eta^t) = 0$ and hence the present value of net borrowing in (3.1) is equal to $\sigma(z^{t-1}, \eta^{t-1}) \left[(1-\gamma)Y(z^t) + \varpi(z^t)\right]$. Thus their additional measurability constraints take the form

$$\frac{\hat{a}([z^{t-1}, z_t], [\eta^{t-1}, \eta_t])}{(1-\gamma)Y(z^{t-1}, z_t) + \varpi(z^{t-1}, z_t)} = \frac{\hat{a}([z^{t-1}, \tilde{z}_t], [\eta^{t-1}, \tilde{\eta}_t])}{(1-\gamma)Y(z^{t-1}, \tilde{z}_t) + \varpi(z^{t-1}, \tilde{z}_t)}, \tag{3.4}$$

for all z^{t-1}, η^{t-1}, z_t, $\tilde{z}_t \in Z$, and η_t, $\tilde{\eta}_t \in N$.

Non-participants For the non-participants, the payoff in state (z^t, η^t) is supposed to be measurable with respect to (z^{t-1}, η^{t-1}), and hence their additional measurability constraints take the form:

$$\hat{a}(\left[z^{t-1}, z_t\right], \left[\eta^{t-1}, \eta_t\right]) = \hat{a}(\left[z^{t-1}, \tilde{z}_t\right], \left[\eta^{t-1}, \tilde{\eta}_t\right]), \tag{3.5}$$

for all z^{t-1}, η^{t-1}, z_t, $\tilde{z}_t \in Z$, and η_t, $\tilde{\eta}_t \in N$.

Summary Let $R^{port}(z^{t+1})$ denote the return on the passive trader's total portfolio. In general, for "passive" traders, we can state the measurability condition as:

$$\frac{\hat{a}([z^{t-1}, z_t], [\eta^{t-1}, \eta_t])}{R^{port}(z^t, z_t)} = \frac{\hat{a}([z^{t-1}, \tilde{z}_t], [\eta^{t-1}, \tilde{\eta}_t])}{R^{port}(z^t, \widetilde{z}_t)}, \tag{3.6}$$

for all z^{t-1}, η^{t-1}, z_t, $\tilde{z}_t \in Z$, and η_t, $\tilde{\eta}_t \in N$. For the non-participant, $R^{port}(z^{t+1}) = R^f(z^t)$ is the risk-free rate, for the diversified trader, $R^{port}(z^{t+1}) = R(z^{t+1})$ is the return on the market –the tradeable income claim. Of course, a similar condition holds for any investor with fixed portfolios in the riskless and risky assets.

Given these results, we can restate the household's problem as one of choosing an entire consumption plan from a restricted budget set. Next, we turn to examining a household's problem given this reformulation.

Because the complete traders do not face any measurability constraints, we start a z-complete trader's problem. The central result is a martingale condition for the stochastic multipliers. We also discuss the same problem for the other traders, and we derive an aggregation result. Finally, we conclude this section by providing an overview.

3.2 Martingale Conditions

To derive the martingale conditions that govern household consumption, we consider the household problem in a time zero trading setup. Markets open only once at time zero. The household chooses a consumption plan and a net wealth plan subject to a single budget constraint at time zero, as well as an infinite number of solvency constraints and measurability constraints. These measurability constraints act as direct restrictions on the household budget set. We start off by considering the active traders.

3.2.1 Active Traders

Let γ denote the multiplier on the present-value budget constraint, let $\nu(z^t, \eta^t)$ denote the multiplier on the measurability constraint in node (z^t, η^t), and, finally, let $\varphi(z^t, \eta^t)$ denote the multiplier on the debt constraint. The saddle point problem of a **z-complete trader** can be stated as:

$$
\begin{aligned}
L = \quad & \min_{\{\gamma,\nu,\varphi\}} \max_{\{c,\hat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{(z^t,\eta^t)} u(c(z^t,\eta^t))\pi(z^t,\eta^t) \\
& + \gamma \left\{ \sum_{t\geq 1} \sum_{(z^t,\eta^t)} \widetilde{P}(z^t,\eta^t) \left[\gamma Y(z^t)\eta_t - c(z^t,\eta^t) \right] + \varpi(z^0) \right\} \\
& + \sum_{t\geq 1} \sum_{z^t,\eta^t} \nu(z^t,\eta^t) \left\{ \sum_{\tau \geq t} \sum_{(z^\tau,\eta^\tau)\succeq(z^t,\eta^t)} \widetilde{P}(z^\tau,\eta^\tau) \left[\gamma Y(z^\tau)\eta_\tau - c(z^\tau,\eta^\tau)\right] - \widetilde{P}(z^t,\eta^t)\hat{a}(z^t,\eta^{t-1}) \right\} \\
& + \sum_{t\geq 1} \sum_{(z^t,\eta^t)} \varphi(z^t,\eta^t) \left\{ \underline{M}_t(z^t,\eta^t)\widetilde{P}(z^t,\eta^t) - \sum_{\tau\geq t} \sum_{(z^\tau,\eta^\tau)\succeq(z^t,\eta^t)} \widetilde{P}(z^\tau,\eta^\tau) \left[\gamma Y(z^\tau)\eta_\tau - c(z^\tau,\eta^\tau)\right] \right\}.
\end{aligned}
$$

Following Marcet and Marimon (1999), we can construct new weights for this Lagrangian as follows. First, we define the initial cumulative multiplier to be equal to the multiplier on the budget constraint: $\zeta_0 = \gamma$. Second, the multiplier evolves over time as follows for all $t \geq 1$:

$$
\zeta(z^t,\eta^t) = \zeta(z^{t-1},\eta^{t-1}) + \nu\left(z^t,\eta^t\right) - \varphi(z^t,\eta^t). \tag{3.7}
$$

Substituting for these cumulative multipliers in the Lagrangian, we recover the following expression for the constraints component of the Lagrangian:

$$
\begin{aligned}
+ \sum_{t\geq 1} \sum_{z^t,\eta^t} \widetilde{P}(z^t,\eta^t) \left\{ \zeta(z^t,\eta^t)\left(y(z^t,\eta^t) - c(z^t,\eta^t)\right) - \nu\left(z^t,\eta^t\right)\hat{a}(z^t,\eta^{t-1}) + \varphi(z^t,\eta^t)\underline{M}(z^t,\eta^t) \right\} \\
+ \gamma\varpi(z^0).
\end{aligned}
$$

This is a standard convex programming problem –the constraint set is still convex, even with the measurability conditions and the solvency constraints. The first order conditions are necessary and sufficient. The first-order condition with respect to consumption is given by:

$$
\beta^t u'(c(z^t,\eta^t))\pi(z^t,\eta^t) - \left[\zeta(z^{t-1},\eta^{t-1}) + \nu\left(z^t,\eta^t\right) - \varphi\left(z^t,\eta^t\right)\right] P(z^t)\pi(z^t,\eta^t) = 0 \tag{3.8}
$$

and the first order condition with respect to net wealth $\widehat{a}(z^t, \eta^{t-1})$ is given by:

$$\sum_{\eta^{t+1} \succ \eta^t} \nu\left(z^{t+1}, \eta^{t+1}\right) \pi(z^{t+1}, \eta^{t+1}) P(z^{t+1}) = 0 \tag{3.9}$$

This implies that the average measurability multiplier across idiosyncratic states η^{t+1} is zero since $P(z^{t+1})$ is independent of η^{t+1}. As we will show, the first order condition for consumption is the same for all households regardless of their trading technology, but the first order condition for net wealth is not. The first order condition for consumption in (3.8) implies that the cumulative multiplier measures the household's discounted marginal utility relative to the state price $P(z^t)$:

$$\frac{\beta^t u'(c(z^t, \eta^t))}{P(z^t)} = \zeta(z^t, \eta^t). \tag{3.10}$$

The marginal utility of households is proportional to their cumulative multiplier, regardless of their trading technology.

Active Trading Strategy Next, we use this condition to characterize the z-complete trader's consumption and trading choices.

Lemma 3.1. *The trader's cumulative multiplier is a super-martingale:*

$$\zeta(z^t, \eta^t) \geq \sum_{\eta^{t+1} \succ \eta^t} \zeta(z^{t+1}, \eta^{t+1}) \pi(\eta^{t+1} | z^{t+1}, \eta^t). \tag{3.11}$$

The cumulative multiplier is a martingale if the solvency constraints do not bind for any $\eta^{t+1} \succ \eta^t$ given z^{t+1}.

This result follows directly from the measurability condition in equation (3.9):

$$E\{\nu(z^{t+1}, \eta^{t+1}) | z^{t+1}\} = 0.$$

In each aggregate node z^{t+1}, the household's marginal utility innovations not driven by the solvency constraints have to be white noise. The trader has high marginal utility growth in low η states and low marginal utility growth in high η states, but these innovations to marginal utility growth average out to zero in each node (z^t, z_{t+1}). If the solvency constraints do bind, then the cumulative multipliers decrease on average for any given z-complete trader:

$$E\{\zeta(z^{t+1}, \eta^{t+1}) | z^{t+1}\} \leq \zeta(z^t, \eta^t).$$

Hence our multipliers are a bounded super-martingale.

As we are about to show, this trader takes advantage of the spread in state prices. Using this relationship and condition (3.9), we recover the standard Euler inequality:

$$u'(c(z^t,\eta^t)) \geq \beta E\left\{u'(c(z^{t+1},\eta^{t+1}))\frac{P(z^t)\pi(z^{t+1},\eta^{t+1})}{P(z^{t+1})\pi(z^t,\eta^t)}|z^{t+1}\right\}, \tag{3.12}$$

This condition holds as an equality if the borrowing constraint does not bind, i.e. $\varphi(z^{t+1},\eta^{t+1}) = 0$ for all $\eta^{t+1} \succ \eta^t$. If there is such an unconstrained z-complete trader in equilibrium, then equation (3.12) implies that the SDF equals the highest expected IMRS, averaging across η states tomorrow:

$$\frac{P(z^{t+1})}{P(z^t)} = \max_{z-traders} \beta E\left\{\frac{u'(c(z^{t+1},\eta^{t+1}))}{u'(c(z^t,\eta^t))}\frac{\pi(z^{t+1},\eta^{t+1})}{\pi(z^t,\eta^t)}|z^{t+1},\eta^t\right\}.$$

Unconstrained z-complete traders adjust their consumption growth upwards when state prices are low and downwards when state prices are high.

For the **complete traders**, there is no measurability constraint, and hence the constraints portion of the recursive Lagrangian is given simply by:

$$+\sum_{t\geq 1}\sum_{z^t,\eta^t}\widetilde{P}(z^t,\eta^t)\left\{\zeta(z^t,\eta^t)\left(y(z^t,\eta^t)-c(z^t,\eta^t)\right)-\nu\left(z^t,\eta^t\right)\hat{a}(z^t,\eta^t)+\varphi(z^t,\eta^t)\underline{M}(z^t,\eta^t)\right\}$$
$$+\gamma\varpi(z^0).$$

The first order condition with respect to $\widehat{a}(z^t,\eta^t)$ is given by:

$$\nu\left(z^{t+1},\eta^{t+1}\right)\pi(z^{t+1},\eta^{t+1})P(z^{t+1}) = 0, \tag{3.13}$$

which implies that $\nu\left(z^{t+1},\eta^{t+1}\right)$ is equal to zero for all z^{t+1},η^{t+1}. All of the other conditions, including the first-order condition with respect to consumption (3.8) and the recursive multiplier condition (3.7) are unchanged. This leads to the following recursive formulation of the cumulative multipliers:

$$\zeta(z^t,\eta^t) = \zeta(z^{t-1},\eta^{t-1}) - \varphi(z^t,\eta^t),$$

The multipliers decrease if the solvency constraint binds in node (z^t,η^t); if not, they remain unchanged. The history of a complete household η^t only affects today's consumption and asset accumulation, as summarized in ζ, through the binding solvency constraints. As a result, when state prices are high, the consumption share of the complete trader decreases if the solvency constraint does not bind, not only on average, across η' states, but state-by-state.

To recover the complete trader's Euler equation, we can simply drop all the multipliers ν on the measurability constraints, and we get that

$$u'(c(z^t,\eta^t)) \geq \beta \left\{ u'(c(z^{t+1},\eta^{t+1})) \frac{P(z^t)\pi(z^{t+1},\eta^{t+1})}{P(z^{t+1})\pi(z^t,\eta^t)} \right\}, \tag{3.14}$$

If there is an unconstrained complete trader in equilibrium –with $\varphi(z^t,\eta^t) = 0$ for all $\eta^t \succ \eta^{t-1}$–, then equation (3.14) implies that the SDF equals the highest IMRS; no averaging across η states tomorrow:

$$\frac{P(z^{t+1})}{P(z^t)} = \max_{c-traders} \beta \left\{ \frac{u'(c(z^{t+1},\eta^{t+1}))}{u'(c(z^t,\eta^t))} \frac{\pi(z^{t+1},\eta^{t+1})}{\pi(z^t,\eta^t)} \right\}.$$

The common characteristic for all active traders is that their marginal utility innovations are orthogonal to any aggregate variables, because we know that $E[\nu_{t+1}|z^{t+1}] = 0$ in each node z^{t+1}. Below, we explore the implications of this finding, but first, we show that diversified traders and non-participants satisfy the same martingale condition, but with respect to a different measure.

The next section derives the martingale condition for the passive traders.

3.2.2 Passive Traders

We start by looking at the diversified traders.

For the **diversified** investors, the constraints portion of the Lagrangian is given by

$$+\sum_{t\geq 1}\sum_{z^t,\eta^t} \widetilde{P}(z^t,\eta^t) \left[\begin{array}{c} \zeta(z^t,\eta^t)\,(y(z^t,\eta^t) - c(z^t,\eta^t)) - \nu\,(z^t,\eta^t)\,\sigma(z^{t-1},\eta^{t-1}) \\ [(1-\gamma)Y(z^t) + \varpi(z^t)] + \varphi(z^t,\eta^t)\underline{M}(z^t,\eta^t) \end{array} \right] + \gamma\varpi(z^0).$$

The other components of the Lagrangian are unchanged. The first order condition with respect to $\varpi(z^{t-1},\eta^{t-1})$ is given by:

$$\sum_{z^{t+1}\succ z^t,\eta^{t+1}\succ\eta^t} \nu\left(z^{t+1},\eta^{t+1}\right)\left[(1-\gamma)Y(z^{t+1}) + \varpi(z^{t+1})\right]\pi(z^{t+1},\eta^{t+1})P(z^{t+1}) = 0. \tag{3.15}$$

The other conditions are identical. The measurability condition in (3.7) can be stated as:

$$\zeta(z^t,\eta^t)E\left\{m(z^{t+1}|z^t)R(z^{t+1})|z^t\right\} \geq \sum_{z^{t+1}\succ z^t,\eta^{t+1}\succ\eta^t} \zeta(z^{t+1},\eta^{t+1})\widetilde{\pi}(z^{t+1},\eta^{t+1}|z^t,\eta^t), \tag{3.16}$$

where $R(z^{t+1})$ is the return on the tradeable income claim and the twisted probabilities are defined as:

$$\widetilde{\pi}(z^{t+1},\eta^{t+1}|z^t,\eta^t) = \frac{m(z^{t+1}|z^t)R(z^{t+1})}{E\left\{m(z^{t+1}|z^t)R(z^{t+1})\right\}}\pi(z^{t+1},\eta^{t+1}|z^t,\eta^t),$$

So, the diversified traders' multipliers satisfy the martingale condition with respect to the these "risk-neutral" probabilities, whenever the borrowing constraints do not bind. Moreover, when ever the debt constraints do bind, their multipliers are pushed downwards in order to satisfy the constraint. So, relative to these twisted probabilities, the equity traders multipliers are a supermartingale.

When z and η are independent, only the aggregate transition probabilities are twisted:

$$\widetilde{\pi}(z^{t+1}, \eta^{t+1}|z^t, \eta^t) = \widetilde{\phi}(z^{t+1}|z^t)\varphi(\eta^{t+1}|\eta^t) \tag{3.17}$$

The same is true of the non-participant's multipliers, however the twisting factor is different.

If there is an unconstrained stock trader in equilibrium –with $\varphi(z^t, \eta^t) = 0$ for all $(z^t, \eta^t) \succ (z^{t-1}, \eta^{t-1})$– then equation (3.14) implies that the stock price equals:

$$\varpi(z^t) = \max_{div-traders} \beta E\left\{\left[(1-\gamma)Y(z^{t+1}) + \varpi(z^{t+1})\right] \frac{u'(c(z^{t+1}, \eta^{t+1}))}{u'(c(z^t, \eta^t))} \frac{\pi(z^{t+1}, \eta^{t+1})}{\pi(z^t, \eta^t)} |z^t, \eta^t\right\}.$$

This restriction on the diversified trader's IMRS is of course much weaker than that for the z-complete traders. Diversified traders cannot respond to the spread in state prices tomorrow.

Non-participants Finally, for the non-participants, the constraints portion of the recursive Lagrangian is given by

$$+\sum_{t\geq 1}\sum_{z^t,\eta^t} \widetilde{P}(z^t, \eta^t)\left\{\zeta(z^t, \eta^t)\left(y(z^t, \eta^t) - c(z^t, \eta^t)\right) - \nu\left(z^t, \eta^t\right)\hat{a}(z^{t-1}, \eta^{t-1}) + \varphi(z^t, \eta^t)\underline{M}(z^t, \eta^t)\right\}$$
$$+\gamma\varpi(z^0).$$

The first order condition with respect to $\widehat{a}(z^{t-1}, \eta^{t-1})$ is given by:

$$\sum_{z^{t+1}\succ z^t, \eta^{t+1}\succ \eta^t} \nu\left(z^{t+1}, \eta^{t+1}\right)\pi(z^{t+1}, \eta^{t+1})P(z^{t+1}) = 0. \tag{3.18}$$

This implies that non-participants' multipliers have the supermartingale property:

$$\zeta(z^t, \eta^t)E\left\{m(z^{t+1}|z^t)|z^t\right\} \geq \sum_{z^{t+1}\succ z^t, \eta^{t+1}\succ \eta^t} \zeta(z^{t+1}, \eta^{t+1})\widetilde{\pi}(z^{t+1}, \eta^{t+1}|z^t, \eta^t) \tag{3.19}$$

with respect to the twisted probabilities

$$\widetilde{\pi}(z^{t+1}, \eta^{t+1}|z^t, \eta^t) = \frac{m(z^{t+1}|z^t)}{E\left\{m(z^{t+1}|z^t))\right\}}\pi(z^{t+1}, \eta^{t+1}|z^t, \eta^t),$$

whenever the borrowing constraints do not bind. The unconstrained non-participant's Euler equation is given by:

$$1 = R_t^f \beta \max_{np-traders} E\left\{ \frac{u'(c(z^{t+1}, \eta^{t+1}))}{u'(c(z^t, \eta^t))} \frac{\pi(z^{t+1}, \eta^{t+1})}{\pi(z^t, \eta^t)} | z^t, \eta^t \right\}.$$

As we have shown, all households share the same first order condition for consumption, regardless of their trading technology. This implies that we can derive a consumption sharing rule and an aggregation result for prices.

3.3 Aggregate Multiplier

We can characterize equilibrium prices and allocations using the household's multipliers and the aggregate multipliers.

Proposition 3.1. *The household consumption share, for all traders is given by*

$$\frac{c(z^t, \eta^t)}{C(z^t)} = \frac{\zeta(z^t, \eta^t)^{\frac{-1}{\alpha}}}{h(z^t)}, \text{ where } h(z^t) = \sum_{\eta^t} \zeta(z^t, \eta^t)^{\frac{-1}{\alpha}} \pi(\eta^t | z^t). \tag{3.20}$$

The SDF is given by the Breeden-Lucas SDF and a multiplicative adjustment:

$$m(z^{t+1}|z^t) \equiv \beta \left(\frac{C(z^{t+1})}{C(z^t)} \right)^{-\alpha} \left(\frac{h(z^{t+1})}{h(z^t)} \right)^{\alpha}. \tag{3.21}$$

The consumption sharing rule follows directly from the ratio of the first order conditions and the market clearing condition. Condition (3.8) implies that

$$c(z^t, \eta^t) = u'^{-1} \left[\zeta(z^t, \eta^t) P(z^t) \right].$$

In addition, the sum of individual consumptions aggregate up to aggregate consumption

$$C(z^t) = \sum_{\eta^t} c(z^t, \eta^t) \pi(\eta^t | z^t).$$

This implies that the consumption share of the individual with history (z^t, η^t) is

$$\frac{c(z^t, \eta^t)}{C(z^t)} = \frac{u'^{-1} \left[\zeta(z^t, \eta^t) P(z^t) \right]}{\sum_{\eta^t} u'^{-1} \left[\zeta(z^t, \eta^t) P(z^t) \right] \pi(\eta^t | z^t)}.$$

With CRRA preferences, this implies that the consumption share is given by

$$\frac{c(z^t, \eta^t)}{C(z^t)} = \frac{\zeta(z^t, \eta^t)^{\frac{-1}{\alpha}}}{h(z^t)}, \text{ where } h(z^t) = \sum_{\eta^t} \zeta(z^t, \eta^t)^{\frac{-1}{\alpha}} \pi(\eta^t | z^t).$$

Hence, the $-1/\alpha^{\text{th}}$ moment of the multipliers summarizes risk sharing within this economy. We refer to this moment of the multipliers simply as **the aggregate multiplier**. The equilibrium SDF is the standard Breeden-Lucas SDF times the growth rate of the aggregate multiplier. This aggregate multiplier reflects the aggregate shadow cost of the measurability and the borrowing constraints faced by households.

The expression for the SDF can be recovered directly by substituting for the consumption sharing rule in the household's first order condition for consumption (3.10). This aggregation result extends the complete market result in Lustig (2005) to the case of incomplete markets and heterogeneous trading technologies.

This proposition directly implies that an equilibrium for this class of incomplete market economies can be completely characterized by a process for these cumulative multipliers $\{\zeta(\eta^t, z^t)\}$, and by the associated aggregate multiplier process $\{h_t(z^t)\}$. Section 4 describes a method to solve for these multipliers. In the next subsection, we use the consumption sharing rule and the martingale condition to highlight the effect of the heterogeneity in trading strategies on savings and investment behavior.

3.4 Savings Behavior of Active Traders

Complete traders do not have a precautionary motive to save, while z-complete traders do. As a result, when interest rates are low, complete traders invariably de-cumulate assets, while z-complete traders do not necessarily choose to do so.

Corollary 3.1. *The unconstrained complete trader's consumption share increases at a rate* $-h_{t+1}(z^{t+1}) / h_t(z^t)$ *in each* η' *state in the next period.*

If $h_{t+1}(z^{t+1})/h_t(z^t) > 1$ on average, and hence the risk-free rate is lower than in a representative agent economy, the complete trader's consumption share decreases on average, because he is dissaving. Complete traders have no precautionary motive to save – as reflected in the absence of measurability constraints–, and hence they run down their assets in each η state, when state prices are high, until they hit the binding solvency constraints. This is an "aggressive" trading strategy. This is not true for the z-complete trader.

Corollary 3.2. *If the state price is low and* $h_{t+1}(z^{t+1})/h_t(z^t) \leq 1$*, the unconstrained z-complete trader's consumption share increases on average across* η' *states in the next period. If the state price is high and* $h_{t+1}(z^{t+1})/h_t(z^t) > 1$*, her consumption share can increase or decrease.*

Because of the market incompleteness, the z-complete trader may still accumulate assets in equilibrium even if the state price is high (or expected returns are low), and choose an increasing consumption path over time, as long as his borrowing constraint does not bind. This reflects his precautionary motive to save. Another way to illustrate the difference in savings behavior is by fixing the household's consumption level in this period, and considering the household's consumption choice for next period depending on its trading technology.

Corollary 3.3. *Fix a cumulative multiplier ζ. In the absence of binding solvency constraints, a z-complete trader consumes less on average in the next period than a complete trader, strictly less if marginal utility is strictly convex.*

If marginal utility is strictly convex, conditional on having the same consumption today, the z-complete trader always saves more on average tomorrow, regardless of the state prices. This of course implies she will accumulate more wealth. In fact, in a calibrated version of this economy, we will show that z-traders accumulate 4 times more wealth than complete traders. The next subsection consider the implications for investment behavior.

3.5 Investment Behavior

The martingale condition for active traders puts tight restrictions on the joint distribution of returns and consumption growth. Using the SDF expression in (3.21), we can state the martingale condition as $E_t[m_{t+1}\nu_{t+1}] = 0$ for non-participants, z-complete traders and complete traders. This gives rise to the following expression for marginal utility growth of an unconstrained trader:

$$E_t\left[\frac{\zeta_{t+1}}{\zeta_t}\right] = 1 - E_t[m_{t+1}]^{-1} cov_t\left[\frac{\zeta_{t+1}}{\zeta_t}, m_{t+1}\right] \tag{3.22}$$

The *covariance term* drops out for active traders (complete and z-complete traders) because $E[\nu_{t+1}|z^{t+1}] = 0$ in each node z^{t+1}. Hence the martingale property. This orthogonality condition is the hallmark of an "active trading" strategy. Using the consumption sharing rule, this implies the following orthogonality condition:

$$cov_t\left[\Delta \log c_{t+1} - \Delta \log C_{t+1} + \Delta \log h_{t+1}, X(z^{t+1})\right] \simeq 0$$

where $X(z^{t+1})$ is any random payoff (including m itself). This condition is trivially satisfied for the complete trader, whose consumption growth is $\Delta \log c_{t+1} = \Delta \log C_{t+1} - \Delta \log h_{t+1}$ in each node. Active traders increase their consumption growth when state prices are lower than in the representative agent model, and they decrease consumption growth when state prices are higher than in the representative agent model.

However, non-participants are not hedged against aggregate shocks and this induces a strong negative correlation between m and the growth rate of the multiplier. This imputes a *positive drift* to the growth rate of their multipliers, implying that the unconstrained non-participants "oversave":

$$u'(c(z^t,\eta^t)) \leq \beta E \left\{ u'(c(z^{t+1},\eta^{t+1})) \frac{P(z^t)\pi(z^{t+1},\eta^{t+1})}{P(z^{t+1})\pi(z^t,\eta^t)} \right\}$$

This follows by combining the drift for the multipliers with the first order condition for consumption.

Similarly, for diversified investors, z-complete traders and complete traders, the equivalent condition is $E_t\left[m_{t+1}R_{t+1}\nu_{t+1}\right] = 0$. If the probability of binding solvency constraints at $t+1$ is zero, then:

$$E_t\left[\frac{\zeta_{t+1}}{\zeta_t}\right] = 1 - cov_t\left[\frac{\zeta_{t+1}}{\zeta_t}, R_{t+1}m_{t+1}\right] \tag{3.23}$$

The second term always drops out for active traders. It is zero for the diversified traders only if h_{t+1}/h_t is deterministic. This turns out to be the case if there no non-participants. We refer to this case as complete separation of aggregate and idiosyncratic risk. Section 4.5 provides a precise statement of this result. In general, the diversified traders over-save as well.

Example We show a time series for the consumption growth and the multiplier growth generated dy simulating a calibrated version of the model (see section 5 for details).The z-complete traders in the top panel in figure 1 adjust their consumption growth rate, plotted against the $\{h_{t+1}/h_t\}$ shocks, to line up their IMRS with the SDF in each aggregate state of the world, allowing them to earn a high return on total wealth. They sell aggregate consumption claims for the high h_{t+1}/h_t states and buy claims for the low h_{t+1}/h_t states. On the other hand, the consumption growth rates of the diversified trader does not respond to h_{t+1}/h_t shocks. To understand this, it is helpful to examine the underlying multiplier processes. Figure 2 plots the multiplier growth rates of the different traders, against the h_{t+1}/h_t shocks. As we explained, the multipliers of the complete and z-complete traders do not respond to the aggregate shocks (see martingale result) –this is the hallmark of the sophisticated investment strategy–, while the multipliers of the diversified traders and non-participants actually increase when there is a large h_{t+1}/h_t shock. If not, they would violate their measurability constraints.

[Figure 1 about here.]

[Figure 2 about here.]

In a separate appendix, available from the authors' web sites[4], we show that these results go through even if agents have non-expected utility as in Epstein and Zin (1989). The next section

[4] `http://www.econ.ucla.edu/people/faculty/Lustig.html`

derives a recursive set of updating rules for these multipliers, and we show under what conditions this separation result obtains.

4 Computation

This section describes a computational method that builds on the recursive saddle point problem. Our algorithm proceeds in three steps. We start by conjecturing an $\{h_t\}$ process for the aggregate multiplier. Then we compute the new updating functions. Finally, we back out the new guess for the aggregate multiplier process:

1. guess an $\{h_t^0(z^t)\}$, starting with $\{h_t^0(z^t)\} = 1$
2. solve system of equations for household multiplier updating functions $T_0^j, j \in \{z, c, div, np\}$
3. household multiplier updating functions define new aggregate multiplier function $\{h^1(z^t)\}$:

$$h_{t+1}^1(z^{t+1}) = \sum_{j \in T} \int \sum_{\eta^{t+1} \succ \eta^t} \left[T^j(z^{t+1}, \eta^{t+1} | z^t, \eta^t)(\zeta(z^t, \eta^t)) \right]^{\frac{-1}{\alpha}} \frac{\pi(\eta^{t+1}, z^{t+1} | \eta^t, z^t)}{\pi(z^{t+1} | z^t)} d\Phi_t^j,$$

4. iterate until convergence of $\{h^k(z^t)\}$

In a first step, this method involves computing the updating rule for the cumulative multipliers by *solving a system of simultaneous equations*, rather than solving an optimization problem, for a given aggregate multiplier process. These equations are (1) the Kuhn-Tucker conditions on the measurability constraints, (2) the martingale conditions on the multipliers, and (3) the Kuhn-Tucker conditions on the solvency constraints. In a second step, we back out a new aggregate multiplier process, computed using the new updating functions. Finally, we iterate on this process until it converges. The optimality condition for consumption and the market clearing conditions are already satisfied by the consumption sharing rule we devised, so there is no need for a separate algorithm to clear the markets.

The first subsection describes how to compute the updating function for the individual multiplier, while the second subsection describes the updating operator for the aggregate multiplier process.

4.1 Updating function for household multipliers

To allow us to compute equilibrium allocations and prices for a calibrated version of this economy, we recast our optimality conditions in recursive form. To do so, we define a new accounting variable: the recursive savings function. This accounting variable serves the same purpose as the promised utility in recursive contracts. Making use of the consumption sharing rule, we can express the household's present discounted value of future savings or "promised savings" as a function of the individual's multiplier:

$$S(\zeta(z^t,\eta^t);z^t,\eta^t) \quad = \quad \left[\gamma\eta_t - \frac{\zeta(z^t,\eta^t)^{\frac{-1}{\alpha}}}{h(z^t)}\right]C(z^t) + \sum_{z^{t+1},\eta^{t+1}} \frac{\pi(z^{t+1},\eta^{t+1})P(z^{t+1})}{\pi(z^t,\eta^t)P(z^t)} S(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}).$$

This recursive expression for promised savings holds for all of our different asset traders.

We start by showing how to compute the updating functions for the cumulative multiplier, while taking the aggregate multiplier process as given. The updating rule for $\zeta(z^t,\eta^t)$ is determined by the borrowing restrictions, the measurability restrictions and the martingale conditions. The Kuhn-Tucker condition on the *borrowing constraint* reads as:

$$\varphi(\eta^{t+1},z^{t+1})\left[S(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) - \underline{M}(z^{t+1},\eta^{t+1})\right] = 0. \tag{4.1}$$

This condition is common to all traders, regardless of the trading technology. However, the measurability and optimality conditions depend upon the trading technology. We start with the z-complete trader's version of these constraints.

Let $S^z(\cdot)$ denote the **z-complete trader**'s savings function. Our *measurability constraint* requires that the discounted value of the future surpluses be equal for each future η^{t+1}, or

$$S^z(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) = S^z(\zeta(z^{t+1},\tilde{\eta}^{t+1});z^{t+1},\tilde{\eta}^{t+1}) \text{ for all } \eta^{t+1},\ \tilde{\eta}^{t+1} \text{ and } z^{t+1}.$$

This implies the following Kuhn-Tucker condition for the measurability constraints:

$$\left[S^z(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) - S^z(\zeta(z^{t+1},\tilde{\eta}^{t+1});z^{t+1},\tilde{\eta}^{t+1}) \text{ for all } \eta^{t+1}\right]\nu(\eta^{t+1},z^{t+1}), \tag{4.2}$$

for all η^{t+1}, $\tilde{\eta}^{t+1}$ and z^{t+1}. Conditions (4.1-4.2) and the martingale condition (3.9), reproduced here,

$$\sum_{\eta^{t+1}\succ\eta^t} \nu\left(z^{t+1},\eta^{t+1}\right)\pi(z^{t+1},\eta^{t+1})P(z^{t+1}) = 0$$

determine the *multiplier updating function*:

$$T^z(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t)) = \zeta(z^{t+1},\eta^{t+1}).$$

T^z is determined by solving a simple set of simultaneous equations. Using the martingale condition, note that in each node z_{t+1}, we have $\#Y-1$ measurability equations to be solved for $\#Y-1$

multipliers $\nu(\eta^t, \eta_{t+1}, z^{t+1})$. In addition, in each node z_{t+1}, we have $\#Y-1$ Kuhn-Tucker conditions to be solved for $\#Y-1$ multipliers $\varphi(\eta^t, \eta_{t+1}, z^{t+1})$. Finally, the law of motion for the cumulative multiplier ζ is given in (3.7).

In the recursive formulation, the **complete trader**'s *measurability condition* collapses to:

$$S^c(\zeta(z^{t+1}, \eta^{t+1}); z^{t+1}, \eta^{t+1}) = S^c(\zeta(z^{t+1}, \eta^{t+1}); z^{t+1}, \eta^{t+1}) \text{ for all } z^{t+1} \text{ and } \eta^{t+1}, \tag{4.3}$$

which is vacuous and hence can be dropped. Also in the recursive formulation, we know that the martingale condition reduces to:

$$\nu(z^{t+1}, \eta^{t+1}) = 0 \tag{4.4}$$

for all $z^{t+1} \succ z^t$ and $\eta^{t+1} \succ \eta^t$ in which the debt constraint doesn't bind. This condition implies that this trader's multipliers are a degenerate bounded super-Martingale.

Conditions (4.1), (4.1), (4.3), and (4.4) determine the complete trader's multiplier updating function

$$T^c(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) = \zeta(z^{t+1}, \eta^{t+1}).$$

Again, even though the form of the promised savings function, $S^c(\zeta(z^{t+1}, \eta^{t+1}); z^{t+1}, \eta^{t+1})$, is the same, because the multipliers are being updated differently with the complete structure, the value of function will in general be different from that of the z-complete securities market recursive savings function.

Diversified investors: For diversified investors, the *measurability condition* is given by:

$$\frac{S^{div}(\zeta(z^t, z_t, \eta^{t+1}); z^t, z_t, \eta^{t+1})}{[(1-\gamma)Y(z^t, z_t) + \varpi(z^t, z_t)]} = \frac{S^{div}(\zeta(z^t, \tilde{z}_{t+1}, \eta^t, \tilde{\eta}_{t+1}); z^t, \tilde{z}_{t+1}, \eta^t, \tilde{\eta}_{t+1})}{[(1-\gamma)Y(z^t, \tilde{z}_{t+1}) + \varpi(z^t, \tilde{z}_{t+1})]} \tag{4.5}$$

for all $\eta^{t+1}, \eta^t, \tilde{\eta}_{t+1}, z^{t+1}$ and $z^t, \tilde{z}_{t+1}$ and, making use of the above first-order condition.

Conditions (4.1), the Kuhn-Tucker condition for the measurability constraint in (4.5) and the martingale condition in (3.15) determine the diversified investor's multiplier updating function

$$T^{div}(z^{t+1}\eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)),$$

which in turn will determine the net savings function

$$S^{div}(\zeta(z^{t+1}, \eta^{t+1}); z^{t+1}, \eta^{t+1})$$

Note that we have $\#Z \times \#Y - 1$ measurability equations to be solved for $\#Z \times \#Y - 1$ multipliers $\nu(\eta^t, \eta_{t+1}, z^{t+1})$. (Recall that they average out to zero). In addition, in each node z_{t+1}, we have $\#Y-1$ Kuhn-Tucker conditions to be solved for $\#Y-1$ multipliers $\varphi(\eta^t, \eta_{t+1}, z^{t+1})$. Finally, the law of motion for the cumulative multiplier ζ is given in (3.7).

Fixed Portfolio Traders This approach can easily be generalized to deal with households who hold a fixed portfolio of securities (e.g. in the market and the riskless asset). Let R_t^{port} denote the total portfolio return from this strategy. For these investors, the *measurability condition* is given by:

$$\frac{S^{port}(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1})}{R^{port}(z^t,z_{t+1})} = \frac{S^{port}(\zeta(z^t,\tilde{z}_{t+1},\eta^t,\tilde{\eta}_{t+1});z^t,\tilde{z}_{t+1},\eta^t,\tilde{\eta}_{t+1})}{R^{port}(z^t,\tilde{z}_{t+1})}$$

for all $\eta^{t+1},\eta^t,\tilde{\eta}_{t+1},z^{t+1}$ and $z^t,\tilde{z}_{t+1}$ and, the martingale condition becomes:

$$\sum_{z^{t+1}\succ z^t,\eta^{t+1}\succ\eta^t} \nu\left(z^{t+1},\eta^{t+1}\right) R^{port}(z^t,\tilde{z}_{t+1})\pi(z^{t+1},\eta^{t+1})P(z^{t+1}) = 0,$$

Finally, we turn to the non-participants.

Non-participants: For non-participants, we get the following *measurability condition*:

$$S^{np}(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) = S^{np}(\zeta(\tilde{z}^{t+1},\tilde{\eta}^t);\tilde{z}^{t+1},\tilde{\eta}^{t+1}) \text{ for all } \eta^{t+1},\ \tilde{\eta}^{t+1},z^{t+1} \text{ and } \tilde{z}^{t+1} \quad (4.6)$$

Conditions (4.1) , (4.1), and the Kuhn-Tucker condition for the measurability constraint in (4.6) determine the updating function for the non-participants:

$$T^{np}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))$$

and

$$S^{np}(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}).$$

4.2 Aggregate multiplier updating operator

Finally, these updating functions for each of the trading technologies $T^j(\cdot), j \in \{c,z,div,np\}$ determine the law of motion for the aggregate multiplier:

$$h_{t+1}(z^{t+1}) = \sum_{j\in T}\int \sum_{\eta^{t+1}\succ\eta^t}\left\{\left[T^j(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right]^{\frac{-1}{\alpha}}\frac{\pi(\eta^{t+1},z^{t+1}|\eta^t,z^t)}{\pi(z^{t+1}|z^t)}\right\}d\Phi_t^j,$$

where Φ_t^j is the joint distribution of multipliers and endowments and $j \in \{c,z,div,np\}$. These aggregate multiplier dynamics govern the dynamics of the SDF, and hence of risk premia and asset prices. Clearly, this defines an aggregate multiplier updating operator $\{h_t^1(z^t)\} = T^h\{h_t^0(z^t)\}$

that maps the initial multiplier function $\{h_t(z^t)\}$ into a new aggregate multiplier function. We are looking for a fixed point of this operator.

4.3 Algorithm

In the next section, we develop some conditions under which aggregate and idiosyncratic risk separate. In the case of separation, $h(z^{t+1})/h(z^t)$ is deterministic, independent of the aggregate history z^t. However in general, the growth rate of the aggregate multiplier process $\{g_t(z^t)\}$ depends on the entire history. Of course, in an infinite horizon economy, we cannot record the entire aggregate history of shocks in the state space. To actually compute equilibria in a calibrated version of this economy, we propose an algorithm that only uses the last n shocks, following Verarcierto (1998), and we use s to denote a truncated aggregate history in Z^n. The algorithm we apply is:

1. conjecture a function $g_0(s, s') = 1$.
2. solve for the equilibrium updating functions $T_0^j(s', \eta'|s, \eta)(\zeta)$ for all trader groups $j \in \{c, z, div, np\}$.
3. By simulating for a panel of $N = 5000$ households for $T = 1000$, we compute a new aggregate weight forecasting function $g_1(s, s')$.
4. We continue iterating until $g_k(s, s')$ converges.

The computational algorithm is discussed in detail in the appendix (see section B). Using the recursive savings function, we can characterize the aggregate multiplier dynamics analytically under some assumption. First, we derive some bounds on the growth rate of $\{h_t\}$. Second, we conditions under which the growth rate is constant and hence the aggregate risk premium is not affected by limited participation. Finally, we show the growth rate converges to zero when agents become more patient.

4.4 Aggregate Multiplier Dynamics

Using the properties of the household multiplier updating function, we can establish some bounds on the growth rate of the aggregate multiplier process. If there are only complete traders or z-complete traders, $\{h_t\}$ is a non-decreasing (over time) stochastic process.

Proposition 4.1. *Suppose there are only complete or z-complete traders. The equilibrium stochastic process $\{h_t(z^t)\}$ is non-decreasing:*

$$\left(\frac{h_{t+1}(z^{t+1})}{h_t(z^t)}\right) \geq 1$$

If there are no binding solvency constraints, this holds with equality. Similarly, if there are only z-complete traders, we can establish the same property. If marginal utility is convex, this holds with strict inequality. Of course, this also implies that $\{h_t\}$ is non-decreasing if we have a mix of complete traders and z-complete traders. This not only means that binding measurability and solvency constraints lower the risk-free rate, but it also implies that any time variation in the volatility of the aggregate weight growth imputes volatility to the risk-free rate. This changes once we introduce diversified traders and non-participants into the mix.

Proposition 4.2. *Suppose there are only diversified investors. In the case of independence, the equilibrium stochastic process $\{h_t(z^t)\}$ is non-decreasing on average under the risk-neutral measure:*

$$\sum_{z^{t+1} \succ z^t} \widetilde{\phi}(z^{t+1}|z^t) \left(\frac{h_{t+1}(z^{t+1})}{h_t(z^t)} \right) \geq 1$$

In this case, in some states of the world the aggregate weight (gross) growth rate could be less than one. As we will show, this dampens the volatility of the risk-free rate, even when the aggregate weight growth contributes substantial volatility to the SDF. The same result holds for non-participants, but under a different measure.

Consumption Distribution How is our SDF related to how the consumption distribution evolves over time? There is a tight connection between the aggregate weight growth rate and the growth rate of the $-\alpha$-th moment of the consumption distribution. We define C_i^* as the $-\alpha^{\text{th}}$ moment of the consumption distribution in trader segment i.

Corollary 4.1. *If there are only complete and z-complete traders, then the SDF is bounded below by the growth rate of the $-\alpha^{th}$ moment of the consumption distribution:*

$$\beta \left(C_i^*(z^{t+1}) / C_i^*(z^t) \right) \leq m(z^{t+1}|z^t).$$

This follows directly from the martingale condition and the consumption sharing rule. If the borrowing limits never bind in equilibrium (e.g. in the case of natural borrowing limits), then these two SDF's coincide:

$$\beta \left(C^*(z^{t+1}) / C^*(z^t) \right) = m(z^{t+1}|z^t).$$

Finally, in the case of diversified traders, then the following inequality holds for the return on a claim to tradeable output:

$$E_t\left[\beta\left(C^*_{div}(z^{t+1})/C^*_{div}(z^t)\right)R(z^{t+1})\right] \leq E_t\left[m(z^{t+1}|z^t)R(z^{t+1})\right] = 1.$$

Kocherlakota and Pistaferri (2005) derive this exact aggregation result with respect to the $-\alpha^{\text{th}}$ moment of the consumption distribution directly from the household's Euler equation in an environment where all agents trade the same assets. We show this is a lower bound in the case of binding borrowing or solvency constraints. Before we conclude this section, we derive conditions under which the aggregate weight growth rate is constant and/or equal to one.

4.5 The Separability of Aggregate and Idiosyncratic Risk

In this section, we will show that if the following two conditions are satisfied and if all agents can trade a claim to all diversifiable income, the equilibrium distribution of the household multipliers does not depend on the realization of the aggregate shocks. This result is an extension of Krueger and Lustig (2006) to the case of segmented markets. In the absence of non-participants, the degree of consumption smoothing within and among different trading groups only affects the risk-free rate, not the risk premium. To prove this result, all we need to show is that the multiplier updating functions T^i do not depend on the aggregate history z^t.

Condition 4.1. *The aggregate shocks are i.i.d. : $\phi(z_{t+1}|z_t) = \phi(z_{t+1})$.*

Condition 4.2. *The idiosyncratic shocks are independent of the aggregate shocks:*

$$\pi(\eta_{t+1}, z_{t+1}|\eta_t, z_t) = \varphi(\eta_{t+1}|\eta_t)\phi(z_{t+1}|z_t).$$

We start out by noting the borrowing constraints are proportional to aggregate income. From our definiton (2.5) and our asset pricing result (3.21), it follows that

$$\underline{M}(\eta^t, z^t) = -\psi \sum_{\{z^\tau \succeq z^t, \eta^\tau \succeq \eta^t\}} \gamma Y(z^\tau)\eta_\tau \frac{\pi(z^\tau, \eta^\tau)\beta^\tau Y(z^\tau)^{-\alpha} h(z^\tau)^\alpha}{\pi(z^{t+1}, \eta^{t+1})\beta^{t+1} Y(z^{t+1})^{-\alpha} h(z^{t+1})^\alpha}.$$

Since the growth rate of $Y(z^t)$ is i.i.d. by assumption, it follows that $\underline{M}(\eta^t, z^t)/Y(z^t)$ is independent of z^t, and hence

$$\underline{M}(z^t, \eta^t) = \underline{M}(\eta^t)Y(z^t).$$

Then, we define the ratio of savings to aggregate consumption $\tilde{S}$ as follows:

$$S(\zeta(z^t,\eta^t);z^t,\eta^t) = Y(z^t)\tilde{S}(\zeta(z^t,\eta^t);z^t,\eta^t). \tag{4.7}$$

Our recursive relationship for $S(\zeta(z^t,\eta^t);z^t,\eta^t)$ implies that

$$\tilde{S}(\zeta(z^t,\eta^t);z^t,\eta^t) = \gamma\eta_t - \frac{\zeta(z^t,\eta^t)^{\frac{-1}{\alpha}}}{h(z^t)} + \beta\sum_{z_{t+1}}\widehat{\phi}(z_{t+1}|z^t)\sum_{\eta_{t+1}}\varphi(\eta_{t+1}|\eta_t)\tilde{S}(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}).$$

where

$$\widehat{\phi}(z_{t+1}|z^t) = \phi(z_{t+1})\left[\frac{h(z^{t+1})}{h(z^t)}\right]^{\gamma} e^{(1-\gamma)z_{t+1}}.$$

In addition, our debt constraint in terms of $\tilde{S}$ is simply given by:

$$\tilde{S}(\zeta(z^{t+1},\eta^{t+1});z^t,\eta^t) \leq \underline{M}(\eta^{t+1}). \tag{4.8}$$

Proposition 4.3. *If condition (4.2) and (4.1) are satisfied, in any economy without non-participants the equilibrium values of the multipliers ζ and the equilibrium consumption shares are independent of z^t.*

The reason behind the independence result is straightforward. Start by conjecturing that $h(z^{t+1})/h(z^t)$ does not depend on z^{t+1}, and conjecture that the savings/consumption ratio $\tilde{S}(\zeta(z^t,\eta^t);z^t,\eta^t)$ does not depend on z^t. This being the case, nothing else in the recursive equation depends on the realization of the aggregate shock z_t , because $\widehat{\phi}(z_{t+1})$ does not depend on z^t, in the measurability constraints z-complete traders or in the debt constraint. That versifies our conjecture about the savings consumption ratio So, the measurability constraint for the z-complete traders is independent of z_t:

$$\tilde{S}^z(\eta^{t+1}),\eta^{t+1}) = \tilde{S}^z(\zeta(\tilde{\eta}^{t+1});\tilde{\eta}^{t+1}) \text{ for all } \eta^{t+1},\ \tilde{\eta}^{t+1} \text{ and } z^{t+1}, \tag{4.9}$$

and this implies that the updating function does not depend on z^t either:

$$T^z(\eta^{t+1}|\eta^t)(\zeta(\eta^t)) = \zeta(\eta^{t+1}).$$

What about the diversified investors? Let pd_t denote the price/dividend ratio on a claim to consumption. For the diversified investors, the measurability constraint reads as:

$$\frac{\tilde{S}^{div}(\zeta(\eta^{t+1}),\eta^{t+1})}{[(1-\gamma)+pd_{t+1})]} = \frac{\tilde{S}^{div}(\zeta(\eta^t,\tilde{\eta}_{t+1});\eta^t,\tilde{\eta}_{t+1})}{[(1-\gamma)+pd_{t+1}]}$$

for all $\eta^{t+1},\eta^t,\tilde{\eta}_{t+1},z^{t+1}$ and $z^t,\tilde{z}_{t+1}$. Since the pd_t can only evolve deterministically, given the i.i.d. shocks and the conjecture about h_{t+1}/h_t, the diversified trader faces the same measurability constraints as the z-complete traders. Hence, the diversified investor's updating function does not depend on z^{t+1}:

$$T^{div}(\eta^{t+1}|\eta^t)(\zeta(\eta^t)) = \zeta(\eta^{t+1}).$$

This being the case, it easy to show that h_{t+1}/h_t does not depend on z^{t+1} either, as long as there are no non-participants, simply because nothing on the right hand side depends on z^{t+1}:

$$h_{t+1} - h_t = \sum_{j \in T} \int \sum_{\eta^{t+1} \succ \eta^t} \left\{ \left[T^j(\eta^{t+1}|\eta^t)(\zeta(\eta^t)) \right]^{\frac{-1}{\alpha}} \varphi(\eta_{t+1}|\eta_t) - \zeta(\eta^t)^{\frac{-1}{\alpha}} \right\} d\Phi_t^j \tag{4.10}$$

where $T = \{c, z, div\}$.

Corollary 4.2. *Independent of the market segmentation, if all households can trade a claim to diversifiable income, the (conditional) equity risk premium is the Breeden-Lucas one.*

When $\{h_{t+1}/h_t\}$ is non-random, market incompleteness only affects the risk-free rate, not the risk premium (see Krueger and Lustig (2006) for a formal proof). The consumption shares of all households do not depend on the aggregate shocks. There is no time variation in expected returns, and households only want to trade a claim to aggregate consumption to hedge against aggregate risk. All the asset market participants face the same measurability condition if $\{h_{t+1}/h_t\}$ is non-random. The distinction between active and passive traders is irrelevant, because there is no spread between state prices other than that in a representative agent model. Households all hold fixed portfolios (i.e. the market) in equilibrium, and there exists a stationary equilibrium with an invariant wealth distribution. This result implies that the multipliers are not affected by the aggregate shocks.

Non-participants This independence with respect to the value of z_{t+1} is not true for the non-participants, since the measurability condition in terms of $\tilde{S}$ is given by

$$\frac{\tilde{S}_{t+1}(\zeta(z^{t+1}, z_{t+1}, \eta^{t+1}); z^{t+1}, \eta^{t+1})}{e^{z_{t+1}}} = \frac{\tilde{S}_{t+1}(\zeta(, \tilde{z}^{t+1}, \tilde{\eta}^{t+1}); \tilde{z}_{t+1}, \tilde{\eta}^{t+1})}{e^{\tilde{z}_{t+1}}}, \tag{4.11}$$

for all $(\eta^{t+1}), (\eta^t, \tilde{\eta}_{t+1}), (z^{t+1})$ and $(z^t, \tilde{z}_{t+1})$. Clearly, this household's multiplier updating function will depend on the aggregate history. This measurability condition implies that the ratio of non-participant household net wealth to aggregate consumption needs to be counter-cyclical.

The inclusion of a positive measure of non-participants causes a breakdown in the separation of aggregate and idiosyncratic risk. There no longer is an equilibrium with a stationary distribution of wealth; $\{h_{t+1}/h_t\}$ depends on the entire history of aggregate shocks. This drives a wedge between the martingale condition of the active investors and the diversified investors. We explore the quantitative importance of this in the rest of the paper.

4.6 Shifting Aggregate Risk

We can define the aggregate promised savings function for each group of traders $j \in \{c, z, div, np\}$:

$$S_a^j(z^t) = \left[\gamma\mu^j - \frac{h^j(z^t)}{h(z^t)}\right] C(z^t) + \sum_{z^{t+1}} \frac{\pi(z^{t+1})P(z^{t+1})}{\pi(z^t)P(z^t)} S_a^j(z^{t+1}),$$

by aggregating across all the households in segment j, and exploiting the linearity of the pricing functional. Finally, the sum of the aggregate savings functions is (minus) a claim to diversifiable income:

$$\sum_j S_a^j(z^t) = -[\varpi(z^t) + (1-\gamma)Y(z^t)] \tag{4.12}$$

This follows directly from market clearing. The measurability restrictions on the household savings function, and this in turn generates restrictions on the aggregate consumption share of each trader group.

The **diversified traders** do not bear any of the residual aggregate risk, created by non-participants.

Proposition 4.4. *The aggregate consumption share $\widehat{C}_t^{div}(z^t) = h_t^{div}(z^t)/h(z^t)$ of diversified traders cannot depend on z_t.*

Since the measurability constraints are satisfied for the individual household's savings function, they also need to be satisfied for the aggregate savings function. So by the LLN:

$$\frac{S_a^{div}(z^t, z_{t+1})}{[(1-\gamma)Y(z^t, z_t) + \varpi(z^t, z_t)]} = \frac{S_a^{div}(z^t, \tilde{z}_{t+1})}{[(1-\gamma)Y(z^t, \tilde{z}_{t+1}) + \varpi(z^t, \tilde{z}_{t+1})]}$$

where we have used the fact that the denominator is measurable w.r.t. z^t. The household measurability condition implies that the aggregate savings of the diversified traders be proportional to the tradeable income claim in all the aggregate states. Let R^{trad} denote the return on the tradeable income claim and let R^{div} denote the return on the promised savings of the diversified traders. The measurability condition implies that in each z^t:

$$\frac{R^{div}(z^t, z_{t+1})}{R^{trad}(z^t, z_{t+1})} = \frac{R^{div}(z^t, \widetilde{z}_{t+1})}{R^{trad}(z^t, \widetilde{z}_{t+1})}$$

for all z_{t+1} and $\widetilde{z}_{t+1}$. The aggregate savings function of the diversified traders is a claim to $\{C(z^t)\left[\gamma\mu^j - \frac{h^j(z^t)}{h(z^t)}\right]\}$, while the tradeable income claim is a claim to $\{C(z^t)(\gamma - 1)\}$. Now, since the returns cannot depend on the realization of z_{t+1}, this implies that the probability distribution over normalized cash flows from node z^{t+1} onwards $\{\left[\gamma\mu^j - \frac{h^j(z^{t+\tau})}{h(z^{t+\tau})}\right]\}_{\tau=1}$ cannot depend on the realization z_{t+1} either.

By the same logic, the aggregate consumption share of non-participants $h_t^{np}(z^t)/h(z^t)$ has to be counter-cyclical, because their measurability condition implies that $S_a^{np}(z^t, z_{t+1})$ cannot depend on z_{t+1}.

Proposition 4.5. *The aggregate consumption share of non-participants $\widehat{C}^{np}(z^t) = h_t^{np}(z^t)/h(z^t)$ is inversely proportional to aggregate endowment growth rate*

This follows directly from the measurability condition of the non-participant households.

Since the diversified traders have constant consumption shares, and the non-participant traders have counter-cyclical consumption shares, regardless of the $\{h\}$ process, there cannot be an equilibrium without active traders. The market simply cannot be cleared without active traders, if there are non-participants. This holds because we have assumed that there is a single stochastic discount factor, but when there are no active traders, then diversified and nonparticipant traders are effectively completely segmented. In this case, there would exist an equilibrium with two separate stochastic discount factors; one for each group.

Table 1 summarizes the main effects of heterogeneity in trading technologies on asset prices and portfolio composition. These results rely on the absence of predictability of aggregate consumption growth and the independence of idiosyncratic and aggregate shocks. In the first panel, we summarize the effect on the equity premium. In the absence of non-participants, the composition of the other trader segments has no effect on the equity premium; the Breeden-Lucas risk premium obtains (Krueger and Lustig (2006)). However, as soon as there is a positive fraction of non-participants, this irrelevance result disappears. In the second panel, we look at the portfolio effects. All traders hold the market portfolio in the absence of non-participants. However, when there are non-participants, the active traders decide to increase their exposure to market risk.

[Table 1 about here.]

Next, we solve a calibrated version of this economy numerically, to examine the quantitative importance of heterogeneous trading opportunities for asset prices.

5 Quantitative Results

This section evaluates a calibrated version of the model. The first subsection discusses the calibration of the parameters and the endowment processes. The second subsection considers a benchmark calibration without aggregate consumption growth predictability (IID economy) designed to highlight that the model with heterogeneous trading opportunities manages to reconcile the low volatility of the risk free rate with the large and counter-cyclical volatility of the stochastic discount factor. We use this economy to explore the impact of changes in the active trader's segment composition. The last subsection presents the benchmark version of the model that matches asset pricing moments. This section explores the model's implications for the distribution of wealth and asset shares across households.

5.1 Calibration

The model is calibrated to annual data. We choose a coefficient of relative risk aversion α of five and a time discount factor β of .95. These preference parameters allow us to match the collaterizable wealth to income ratio in the data when the tradeable or collateralizable income share $1-\gamma$ is 10%, as discussed below. Non-tradeable income includes both labor income and entrepreneurial income, among other forms.

IID Economy In the benchmark calibration, there is no predictability in aggregate consumption growth, as in Campbell and Cochrane (1999) –we impose condition 4.1. We refer to this as the IID economy. This is a natural benchmark case because the statistical evidence for consumption growth predictability is weak. Moreover, in this case, all the equilibrium dynamics in risk premia flow from the binding borrowing and measurability constraints, not from the dynamics of the aggregate consumption growth process itself.

The IID experiment is designed to show that the heterogeneous trading technologies also generate similar dynamics endogenously.[5] The other moments for aggregate consumption growth are taken from Mehra and Prescott (1985). The average consumption growth rate is 1.8 %. The standard deviation is 3.15 %. Recessions are less frequent: 27% of realizations are low aggregate consumption growth states.

In addition, we impose independence of the idiosyncratic risk from aggregate shocks on the labor income process –condition 4.2 is satisfied. By ruling out counter-cyclical cross-sectional variance of labor income shocks, we want to focus on the effects of concentrating aggregate risk among a small section of households, as opposed to concentrating income risk in recessions. The Markov process for $\log \eta(y,z)$ is taken from Storesletten, Telmer, and Yaron (2003) (see page 28). The variance is XXX, and the autocorrelation is 0.89. We use a 4-state discretization. The elements of the process for $\log \eta$ are $\{0.38, 1.61\}$.

Finally, given conditions 4.1 and 4.2, the risk premium and portfolio irrelevance result we derived for the case without non-participants applies. This will provide us with a natural benchmark for the asset pricing and wealth distribution results.

[5] Campbell and Cochrane (1999)'s model is designed to demonstrate that the external habit process endogenously generates the right dynamics in risk premia without creating risk-free rate volatility.

Collateralizable Wealth The average ratio of household wealth to aggregate income in the US is 4.30 between 1950 and 2005. The wealth measure is total net wealth of households and non-profit organizations (Flow of Funds Tables). We choose a collateralizable income ratio α of 10 %. The implied ratio is 4.88 in the model's benchmark calibration. Finally, we set the solvency constraint equal to zero: $\underline{M} = 0$.

Dividend Process We report asset pricing results for two equity-like securities. First, we consider a simple leveraged claim to tradeable income. In the Flow of Funds, the ratio of debt-to-net worth is around 0.65, suggesting a leverage parameter ψ of 2. However, Cecchetti, Lam, and Mark (1990) report that standard deviation of the growth rate of dividends is at least 3.6 times that of aggregate consumption, suggesting that the appropriate leverage level is over 3. Following Abel (1999) and Bansal and Yaron (2004) , we choose to set the leverage parameter ψ to 3. The returns on this security are denoted R_{lc}. In addition, we consider a second security with more complex dividend dynamics. Log dividends are stochastically co-integrated with log consumption. Following Bansal, Dittmar, and Lundblad (2006), dividend growth is a function of aggregate consumption growth and the change in the dividend/consumption ratio q_t:

$$\begin{aligned} \Delta d_{t+1} &= \delta + \psi \Delta c_{t+1} + \Delta q_{t+1} \\ q_{t+1} &= \rho_q q_t + \varphi_d \sigma u_{t+1} \end{aligned} \tag{5.1}$$

u is white noise with mean zero and variance 1. σ is the standard deviation of aggregate consumption growth. The second equation imposes stationarity on the log dividend/consumption ratio. This seems like a natural restriction. We choose $\rho_q = 0.8$ for the autocorrelation of the consumption/dividend ratio, the leverage parameter ψ in the dividend growth process is set to 3 , and $\varphi_d = 4.5$. Equity is a claim to this dividend process. This dividend process has a correlation with aggregate consumption growth of .64, much closer to that in the data. We chose σ to match the standard deviation of dividend growth in the data (16 %). The returns on this security are denoted R_{eq}. We also consider the returns on a perpetuity (denoted R_b).

Composition In our benchmark model, 70 % of households only trade the riskless asset. The remaining 30 % is split between diversified investors, z-complete traders and complete traders. We begin by discussing the asset pricing implications of heterogeneous trading opportunities in the IID version of our economy. In the next subsection, we will show that this composition of traders allows for a close match of the wealth and asset share distribution.

5.2 IID Economy

We use the IID economy as a laboratory for understanding the interaction between active and passive traders and its effect on asset prices. This interaction generates counter-cyclical state price volatility without risk-free rate volatility, unlike other heterogeneous agent models (see e.g. Lustig (2005), Alvarez and Jermann (2001), and Guvenen (2003)).

5.2.1 Asset Pricing in the IID Economy

The asset pricing statistics for the IID economy were generated by drawing 10,000 realizations from the model, simulated with 3000 agents. Table 2 reports the asset pricing results in our baseline experiment. As a benchmark, the first column in the table also reports the corresponding numbers for the RA (representative agent) economy. We consider three cases in the HTT economy. In all cases the fractions of active traders (10%), diversified traders (20 %) and non-participants are constant (20%), but we change the composition of the active trader segment. The first column in table 2 reports the results for 10 % z-complete traders (case 1). In this case, there are no complete traders. The second column has 5% z-complete and 5% complete traders (case 2), and the last column has 10 % complete traders (case 3). The fractions of traders can be interpreted as fractions of human wealth (or labor income), rather than fractions of the population. Finally, the last column reports the moments in the data.

[Table 2 about here.]

Representative Agent Economy We start by listing some properties of returns in the RA economy. In the **RA economy**, the maximum Sharpe ratio is .19 and the equity risk premium ($E\left[R_{eq} - R_f\right]$) is 2.3 %. The conditional market price of risk $[\sigma_t[m]/E_t[m]]$ is constant, because the shocks are i.i.d. hence, the risk premia are constant as well. Finally, the risk-free rate in the RA economy is 12 % and it is also constant. As a result, there is no risk in bond returns ($E[R_b - R_f] = 0$).

All of the moments of risk premia reported in column 1 are identical in the HTT economy **without** non-participants, regardless of the composition of the pool of participants.[6] As long as all households can trade a claim to tradeable income, the lack of consumption smoothing has no bearing on risk premia, and its only effect is to lower the equilibrium risk-free rate (not reported in the table).

Heterogeneous Trading Technologies Economy In the **HTT economy**, the interaction between active and passive traders generates volatile state prices and a stable risk-free rate. We

[6]see Proposition 4.3.

start by considering case 1 –no complete traders. In case 1 of the HTT economy, the maximum Sharpe ratio, ($\sigma[m]/E[m]$), is .44, while the equity premium, ($E\left[R_{eq}-R_f\right]$), is 5.03 %. The risk premium on the leveraged consumption claim is higher (6.7%) ($E\left[R_{lc}-R_f\right]$), as a result of the higher volatility of returns ($std\left[R_{lc}-R_f\right]$). But, it is still well below the average realized excess return in post-war US data of 7.5 %. However, the average price/dividend ratio ($E[PD_{eq}]$) in the data is 33, substantially higher than that in the model. As others have suggested (Fama and French (2002)), a decrease in the risk premium over the last part of the sample may have causes higher realized returns.

The risk-free rate R_f is low (1.73 %) and essentially constant. The standard deviation of the risk-free rate is .06 %. There is also substantial time variation in expected excess returns; the standard deviation of the conditional market price of risk $Std\left[\sigma_t[m]/E_t[m]\right]$ is 3.3 %, comparable to that in Campbell and Cochrane (1999) 's model. It varies between .4 and .8. Since the risk-free rate is essentially constant in the IID economy, bond returns (a perpetuity in the model) are essentially equal to the risk-free rate ($E[R_b-R_f]$). In the data, long-run bonds yielded an average excess return of 1 % with a Sharpe ratio of .09.

Finally, we look at the autocorrelation of stock returns ($\rho[R_{eq}(t),R_{eq}(t-1)]$). This is close to zero in the model, as a result of the IID aggregate shocks, while the autocorrelation is around -.2 in the data. The correlation of returns with the risk-free rate in the data is around .2, compared to zero in the model ($\rho[R_{eq},R_f]$). The next subsection introduces some moderate autocorrelation in aggregate consumption growth. This allows for a better match of the time-series properties of returns in the data.

Complete Traders As we increase the fraction of complete traders in the active traders segment, the market price of risk increases from .44 to .51, but more significantly, the standard deviation of the conditional market price of risk $Std\left[\sigma_t[m]/E_t[m]\right]$ increases from 3.3 % to 5.8 %. These complete traders adopt a more aggressive trading strategy and are more levered in equity. This creates more counter-cyclical variation in the market price of risk. However, this does not come at the cost of introducing more volatility in the risk-free rate. The standard deviation of the risk-free rate increases from 3 to 29 basis points, still well below the standard deviation in the data.

Time Variation To understand the time variation, we focus on a specific case–the one with 5% complete and 5% z-complete traders. Figure 3 plots a simulated path of 100 years for the $\{h'/h\}$ shocks to the aggregate multiplier process in the top panel, the conditional risk premium on equity in the middle panel and the conditional market price of risk in the bottom panel. The shaded areas in the graph indicate low aggregate consumption growth states. As is clear from the top panel in figure 3, $[h'/h]$ is large in recessions -low aggregate consumption growth states- to induce the active traders to consume less in that state of the world, because the passive traders consume "too much" in those states. Similarly, $[h'/h]$ needs to be small in high aggregate consumption growth states, to induce the active traders to consume more in those states. The volatility in state prices induces the small segment of active traders to reallocate consumption across aggregate states and absorb the residual aggregate risk from the non-participants.

The middle panel plots the expected excess return on equity $E\left[R_{eq} - R_f\right]$. Clearly, the IID economy produces counter-cyclical variation in the risk premium. The underlying mechanism is shown in the bottom panel. As is clear from the bottom panel, the interaction between active and passive traders generates counter-cyclical variation in the conditional market price of risk $[\sigma_t[m]/E_t[m]]$. In high $[h'/h]$ states, active traders realize low portfolio returns. The wealth of active traders decreases as a fraction of total wealth. This means, that in order to clear the market, the future $[h'/h]$ -shocks need to be larger (in absolute value), and this in turn increases the conditional volatility of the stochastic discount factor. As a result, the conditional market price of risk $[\sigma_t[m]/E_t[m]]$ increases after each low aggregate consumption growth realization. The driving force behind the time variation is the time-varying exposure of active traders to equity risk. We explore this in the next subsection.

[Figure 3 about here.]

5.2.2 Portfolio and Consumption Choice in the IID Economy

The reason for the heterogeneity in portfolio choice is not only the heterogeneity in trading technologies, but also the presence of non-participants. In the case without non-participants, all households, complete, z-complete and diversified traders would choose the same market portfolio: 25 % equity and 75 % bonds. However, in the case of non-participation, the fraction active traders invest in equity varies over time and across traders. On average, the equity share is 93 % for the z-complete trader and about 160 % for the complete traders. These fractions are highly volatile as well. The standard deviation is 60 % for the complete trader and 30 % for the -complete trader.

Not surprisingly, the heterogeneity in portfolio choice shows up in portfolio returns. Table 3 reports the average portfolio returns realized by all traders in a segment. We take case 2 as our benchmark. We start with the complete investors. Their investment strategy delivers an average excess return on their portfolio of 12 % ($E\left[R_c^W - R_f\right]$) or roughly twice the equity premium. The average excess return on the z-complete traders' portfolio ($E\left[R_z^W - R_f\right]$) is lower at 6.2 %. The z-complete trader earns about the equity risk premium on his portfolio. Finally, the diversified investor earns excess returns of around .8 % while the non-participants realize negative excess returns. As a result, these investors do not manage to accumulate wealth. Importantly, the z-complete traders not only take on more risk; they are better compensated per unit of risk as well. The Sharpe ratio on their portfolio $\left(E[R_z^W - R_f]/\sigma[R_z^W - R_f]\right)$ is about .47, compared to .11 for the diversified investors.

Figure 4 plots the wealth (top panel), the equity share (share of total portfolio invested in leveraged consumption claim's) and the conditional market price of risk (bottom panel) for the z-complete trader. The sequence of aggregate shocks (shaded area) is the same as in figure 3. These z-traders invest a much larger portfolio share in equity than the diversified trader, but more importantly, the share varies substantially over time, between 50 and 150 %. Their equity exposure (middle panel) tracks the variation in the conditional market price of risk (bottom panel) and the equity premium perfectly.

Since the active traders are highly leveraged, their share of total wealth (see top panel) declines substantially after a low aggregate shock, and their "market share" declines. As a result, the conditional volatility of the aggregate multiplier shocks increases; larger shocks are needed to get the active traders to clear the markets. In response to the increase in the conditional market price of risk, the active traders increase their leverage. This also explains why increasing the size of the complete traders imputes more time variation to the conditional market price of risk, since these traders are more levered.

[Figure 4 about here.]

[Table 3 about here.]

Consumption This heterogeneity in portfolio choice shows up in household consumption and aggregate consumption for each trader segment as well. The left panel in table 4 reports the correlation of stock returns and household consumption growth as well as the standard deviation of household consumption growth. The hatted variables denote shares of aggregate consumption. The panel on the right report moments for consumption aggregated across all households in a trader segment.

As a benchmark, consider the case without non-participants. Household consumption shares do not depend on aggregate shocks z^t, regardless of their trading technology, and the correlation of consumption share growth with returns is zero $\rho\left[R_s, (\Delta \log(\widehat{c}_i)\right]$ for all participants.

However, in the economy with non-participants, the correlation of consumption share growth with stock returns is highest for complete traders (.64), and decreases to .58 for z-complete traders and 0 for diversified traders. The overall correlation for the participants $\rho\left[R_s, (\Delta \log(\widehat{c}_p)\right]$ is only about .20. The correlation of household consumption growth for all participants $\rho\left[R_s, (\Delta \log(c_i)\right]$ is .43. So an econometrician with data on all market participants would estimate the coefficient of relative risk aversion from the Euler equation for stock returns to be much higher than 5. The standard deviation of household consumption growth can be ranked according to the trading technology, from 5.6 % for the complete traders to 12 % for the non-participants. Note that the standard self-insurance mechanism breaks down for non-participants and diversified traders; they fail to accumulate enough assets.

Of course, the z-complete and complete traders absorb the residual of aggregate risk created by the passive traders. The second panel in table 4 reports the correlation of returns with *aggregate* consumption share growth and standard deviation of aggregate consumption growth for each group of traders. This is the growth rate of total consumption in each segment $\widehat{C}^j(z^t) = h_t^j(z^t)/h(z^t)$. The z-complete traders and the complete traders bear all of the aggregate risk. The aggregate consumption share growth of this trader segment has a correlation of .95 with stock returns. The same correlation for diversified investors is -.08, while the correlation for bond holders is -.9.

The IID economy cannot match the autocorrelation properties of stock returns. The next subsection drops the independence assumption.

[Table 4 about here.]

5.2.3 Robustness

Finally, we examined the impact of relaxing the borrowing limits or increasing the tradeable income share. We find this mainly increases the risk-free rate, but has a small effect on risk premia. First, we increased the fraction of the present-value of labor income that households can borrow against, which is parameterized by ϕ. Starting from our benchmark value of 0, risk premia fall by almost 1% for both our levered claim and the dividend security as we increase ϕ from 0 to 0.05. However, further increases in ϕ have no effect. At $\phi = .25$, the risk premium on the levered security is 1.1% lower than at $\phi = 0.$. At the same time, the market price of risk, $\sigma[m]/E[m]$, falls from an average of 0.47 down to an average of 0.40, while the standard deviation of the conditional market price of risk $Std[\sigma_t[m]/E_t[m]]$ decreases from 0.05 to 0.03. However, the risk-free rate increases by 160 basis points. Thus, risk premia remained relatively high and volatile even in this extreme case; the tightness of the borrowing limits mainly impacts the risk-free rate. This points to the offloading of aggregate risk on active traders as the main driving force behind the volatile and counter-cyclical state prices, not the borrowing limits. Second, we also examined the impact of increasing the tradeable share of income. If we decrease γ from 0.90 to 0.70, the average market price of risk dropped from 0.47 to 0.42, and the standard deviation of the conditional market price of risk decreases from 0.05 to 0.03. At the same time, the average risk premium on the levered output claim falls from 6.44 % to 6.36%. However, the risk free rate increases from 1.92% to 6.53%.

5.2.4 Accuracy

To assess the accuracy of the approximation method, we report the highest coefficient of variation for the actual simulated realizations of $[h'/h]_t$, conditioning on the truncated history of length 5. These are reported in the upper panel of 5. If the method were completely accurate, this statistic would be zero because the actual realizations would not vary in a truncated history. This coefficient

(CV) varies between .57 % and .28 %. So, the forecasting errors are small. The truncated aggregate history explains approximately all of the variation in $[h'/h]_t$.[7] In addition, we checked how well we would have done simply by conditioning on the first moment of the wealth distribution. In the lower panel of 5, we report the R^2 in a regression of the log SDF on the first moment of the wealth distribution. The R^2 are vary between 3% and 60 %. Clearly, approximate aggregation does not hold, in the sense that more moments of the wealth distribution are necessary to forecast the SDF.

[Table 5 about here.]

5.3 MP Economy

This section considers a version of this economy with negative autocorrelation in aggregate consumption growth that can match the behavior of asset returns in the data, and we explore its implications for the wealth and asset class share distribution.

We drop condition 4.1, and we simply adopt Mehra and Prescott's (MP) calibration of the aggregate consumption growth process . The first order autocorrelation of aggregate consumption growth is -.15. This calibration allows for a better match of the time series properties of returns. We adopt the case with only z-complete traders in the active traders segment as our benchmark. These make up 10 % of the population. The remaining 90 % is split between diversified traders (20 %) and non-participants (70 %). The model's market segmentation was calibrated to match asset prices. As an out-of-sample check of the model, the next subsection compares the implications of these choices for the wealth distribution and the asset class share distribution against the data.

5.3.1 Asset Pricing in the MP economy

The asset pricing moments for the MP economy are reported in Table 6. In the RA version of the MP economy (column 1), the risk-free and the conditional market price of risk are no longer constant. However, the heterogeneity in trading technologies increases risk premia, creates more time-variation in risk premia without increasing the volatility of the risk-free rate. The maximum Sharpe ratio is .48, while the Sharpe ratio on equity in post-war US data is .45. The conditional market price of risk has a standard deviation of 5.4%. The model produces a low risk-free rate of .86 %, a large risk premium on equity of 5.8 %. The risk-free rate in the MP economy is more volatile (2.8 %), but not more so than the RA risk-free rate (3%). The standard deviation of the conditional market price of risk is 5.4 %, compared to 1.1 % in the data. The additional risk-free rate variation brings the volatility of equity returns in line with the data. In addition, the MP economy comes close to matching the autocorrelation properties of the returns we observe in the

[7] The implied R^2 is approximately $1 - CV^2$.

data. The autocorrelation is -.19 in the model, as in the data. The contemporaneous correlation of returns on equity and the risk-free rate is .2 in the model, compared to .27 in the data.

[Table 6 about here.]

5.3.2 Portfolio Choice and Consumption

The top panel of Table 7 reports the moments of portfolio returns and the wealth distribution. On average, the z-complete trader invests 69 % in equity, but the fraction is highly volatile (19 %). The z-trader realizes an average excess return of 5.7 % ($E[R_i^W - R_f]$), compared to 1.9 % for the diversified trader and -1 % for the non-participant. Not only does the z-trader take on more risk; his compensation per unit of risk is much higher. The z-trader's Sharpe ratio on his portfolio is .37, compared to .22 for the diversified trader and -.16 for the non-participant.

In addition, the z-trader accumulates 3.2 times the average wealth level ($E[W_i/W]$), while the diversified trader is right at the average. Non-participants fail to accumulate wealth; on average, their holdings amount to only 69 % of the average. This will severely limit the amount of self-insurance these non-participant households can achieve. On average, the z-trader accumulates 4.6 times more wealth than the non-participant. Because the z-trader invests a large fraction of his wealth in the risky asset, his wealth share is highly volatile. The coefficient of variation for the z-trader's wealth share is 45 %. However, most of this reflects aggregate rather than idiosyncratic risk. On the other hand, these coefficient of variation for the passive traders are higher, but that reflects mostly idiosyncratic risk.

The welfare costs of being a passive trader are quite large. Figure plots the fraction of lifetime consumption a fixed portfolio trader would be willing to give up to become a z-complete trader against the fraction he invests in equity. The full line shows the welfare costs if the trader invested a fixed fraction in the dividend claim; the dotted line does the same for the levered consumption claim. The fixed portfolio trader needs leverage of 80 % (levered claim) or 180 % (dividend claim) to reduce the welfare cost to less than 1.5 % of lifetime consumption. This graph also shows that the diversified trader, who holds 25 % in equity, would be willing to give up between 10 and 15 % of consumption to become an active trader.

[Figure 5 about here.]

The middle panel of Table 7 reports the moments of household consumption growth and aggregate consumption share growth. We report the ratio of the standard deviation of household consumption growth and the standard deviation of aggregate consumption growth to make the numbers comparable to recent studies of household consumption growth; the standard deviation of aggregate consumption growth in our model is much higher than the same standard deviation in recent decades.

The z-trader's consumption growth has the lowest volatility (7.7 %) -2.8 times the volatility of aggregate consumption growth-, but most of this variation is common across z-traders; the volatility of their aggregate share growth rate is 4.6 %. The z-traders exploit the variation in state prices. On the other hand, the diversified traders's volatility is 11.5 % (3.4 times the volatility of aggregate consumption growth), and virtually none of this volatility is common (only .4 %). This not surprising given the result in section 4.6. The non-participant's consumption growth, expressed in shares of aggregate consumption, is the highest at 13.2 % (3.8 times the volatility of aggregate consumption growth), almost all of which is due to idiosyncratic risk. Their failure to accumulate enough assets after good y^t histories prevents them from self-insuring. As we discussed in section 4.6, the consumption share of active traders is highly pro-cyclical, while the consumption share of the non-participants is counter-cyclical.

Note that the overall correlation of consumption growth with returns for all participants is about .4 and .16 for non-participants. However, for the diversified traders, this correlation is .75. So, if an econometrician with access to data generated by our model were to limit his sample to wealthier households, the risk aversion estimate from the Euler equation for stocks would decrease, even though households have the same preferences. This is exactly what Mankiw and Zeldes (1991) and Brav, Constantinides, and Geczy (2002b) have documented. We estimated the EIS off the household Euler equation for bond returns and stock returns. We followed the procedure outlined by Vissing-Jorgensen (2002). We find similar evidence of preference heterogeneity. First, both the estimates obtained from the bond and stock Euler equation are biased upwards. All these households have EIS of .2, but we find estimates between $[.92, 1.1]$ using the bond returns and between $[.48, .78]$ for stock returns. Vissing-Jorgensen (2002) reports estimates in the range$[.3, .4]$ for stock returns and $[.8, 1]$ for bond returns. Our EIS estimates are highest for the most sophisticated investors, as has been documented in the data. Also note that the estimates are upward biased for all households.[8]

Finally, we also compared the equilibrium stochastic discount factor to the growth rate of the $-\alpha$-th moment of the consumption distribution for all the households $\beta\,(C_i^*(z^{t+1})/C_i^*(z^t))$. In section 4.4, we showed this growth rate is a lower bound on the actual SDF. The standard deviation of this growth rate is less than half of the actual volatility of the SDF. This is consistent with the empirical findings of Kocherlakota and Pistaferri (2005) who tested $\beta\,(C_i^*(z^{t+1})/C_i^*(z^t))$ on the Euler equation for stocks and bonds using household consumption data; they found large Euler equation errors.

[Table 7 about here.]

The next subsection considers the model's implications for the wealth distribution and the asset

[8] The source of the bias is the time variation in the second moments of household consumption growth and its correlation with the instruments.

class share distribution. Since we calibrated the market segmentation to match asset prices, we regard these as over-identifying restrictions on our model.

5.4 Wealth and Asset Class Share Distribution

We consider two versions of the benchmark model. In the version labeled "standard", households are ex ante identical. In the version labeled "twisted", we introduce permanent income differences to match the income distribution, while keeping the fraction of human wealth in each trader segment constant. This way, the asset pricing implications of the model are not affected.[9] Essentially, the heterogeneity in trading opportunities makes the rich richer and the poor poorer. However, the middle class in our model accumulates too much assets.

Table 8 reports the summary statistics and the percentile ratios for the standard and twisted version of the model in the first panel. We contrast these with the same ratios from the 2004 SCF for US households. The Gini coefficient in the data is .727 (SCF, 2001). Our model produces a Gini coefficient of .59. The model without heterogeneous trading opportunities produces a Gini coefficient of .48. So, the heterogeneity in trading opportunities bridges half of the gap with the data, by producing fatter tails and a more skewed distribution. The skewness of the wealth distribution increases from .8 to 3.1 (compared to 3.6 in the data) while the kurtosis increases from 2.8 to 15.8. (compared to 15.9 in the data).

First, consider the standard version of the model (column 1). Households in the 75 -th percentile accumulate 5 times as much wealth as households in the 25-th percentile, while households in the 80-th percentile accumulate 8.7 times as much wealth as households in the 20-th. The effect of the heterogeneity in trading technologies is most visible in the tails. The 90/10 ratio is 220 in the standard model. This ratio is only 45 in a version of the model with only diversified traders.

The second column reports the same statistics for the version of the model that is calibrated to match the income distribution. The 75/25 ratio increase to 7.6 while the 80/20 ratio increases to 15.12. The 90/10 ratio increases to 1731. The twisted version of the model still falls well short of the data. The poor households still accumulate too much wealth in the model compared to the data. This is clear from figure 6. The full line shows the model's wealth distribution, while the dotted line shows the distribution of wealth among US households in 2004. This discrepancy is not surprising given that these households have no life-cycle motive for borrowing and saving. However, the model does quite well in matching the right tail of the wealth distribution in the data.

The second panel focusses on the left tail of the wealth distribution. The 50/10 ratio in the twisted version of our model is 134, compared to 100 in the data. However, the 90/50 ratio is only 3.5 in our model, compared to 9.5 in the data. This discrepancy is partly due to the fact that

[9] Table 10 in the Appendix lists the percentile ratios in the twisted version of the model and the data.

the twisted income distribution in our model does not quite match that in the data in the highest income percentiles.

[Table 8 about here.]

[Figure 6 about here.]

Finally, we turn to the asset class share distribution, and we check whether our model can replicate the distribution of asset shares in the data. Table 9 shows the equity share (as a fraction of the household portfolio) at different percentiles of the wealth distribution in the model and the data. In the data, we rank households in terms of net worth and we backed out their equity holding as a fraction of net wealth less private business holdings – the latter is non-tradeable (like labor income). Because there is quite some time variation in these shares, we report the 2001 and 2004 numbers. Overall, the standard model tends to under-predict equity shares between the 50 and 80th percentile, but it does rather well in the left and the right tail.

[Table 9 about here.]

6 Conclusion

We calibrated a model with heterogeneity in trading technologies to match some moments of asset prices in the data. In the quantitative section of the paper, we showed that this type of heterogeneity brings the standard model much closer to matching the asset class share and wealth distribution in the data. The passive traders in our model accumulate much less wealth than the active traders, even though they have identical preferences, simply because they are handsomely compensated for bearing the residual aggregate risk created by the non-participants. In fact, we showed that the data generated by our model can reproduce some of the evidence for preference heterogeneity in the literature. To solve the model, we developed a new solution method that, at least in this class of models, substantially simplifies the computations.

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A Proofs

- Proof of Lemma 3.1:

  Proof. Our optimality conditions (3.7, 3.8, 3.9) imply that if the borrowing constraint does not bind, then
  $$\zeta(z^t, \eta^t) = \sum_{\eta^{t+1} \succ \eta^t} \zeta(z^{t+1}, \eta^{t+1}) \pi(\eta^{t+1} | z^{t+1}, \eta^t). \tag{A.1}$$
  Hence, when the borrowing constraint doesn't bind for any possible η^{t+1} given z^{t+1}, the multipliers are a Martingale. □

- Proof of Corollary 3.2:

  Proof. We know that $E\{\zeta(z^{t+1}, \eta^{t+1}) | z^{t+1}\} \leq \zeta(z^t, \eta^t)$. This implies that
  $$E\{\zeta^{-1/\alpha}(z^{t+1}, \eta^{t+1}) | z^{t+1}\} \geq E\{\zeta(z^{t+1}, \eta^{t+1}) | z^{t+1}\}^{-1/\alpha} = \zeta(z^t, \eta^t)^{-1/\alpha}.$$
  Assume $h_t(z^{t+1} \leq h_t(z^t)$. Then the risk-sharing rule in (A.3) implies the unconstrained z-complete trader's consumption share increases over time.

  □

- Proof of Corollary 3.1:

  Proof. Follows directly from the risk sharing rule. □

- *Proof.* Proof of Corollary 3.3: In fact, Corollary (3.2) and (3.1) imply that if we take two traders with the same initial ζ, the complete trader will always choose higher average consumption than the z-complete trader, irrespective of $\{h\}$:
  $$E\{\zeta^{-\alpha}(z^{t+1}, \eta^{t+1}) | z^{t+1}\} < E\{\zeta(z^{t+1}, \eta^{t+1}) | z^{t+1}\}^{-\alpha} = \zeta(z^t, \eta^t)^{-\alpha}.$$

  □

- Proof of Proposition 3.1:

  Proof. Condition (3.8) implies that
  $$c(z^t, \eta^t) = u'^{-1}\left[\zeta(z^t, \eta^t) P(z^t)\right].$$
  In addition, the sum of individual consumptions aggregate up to aggregate consumption
  $$C(z^t) = \sum_{\eta^t} c(z^t, \eta^t) \pi(\eta^t | z^t).$$
  This implies that the consumption share of the individual with history (z^t, η^t) is
  $$\frac{c(z^t, \eta^t)}{C(z^t)} = \frac{u'^{-1}\left[\zeta(z^t, \eta^t) P(z^t)\right]}{\sum_{\eta^t} u'^{-1}\left[\zeta(z^t, \eta^t) P(z^t)\right] \pi(\eta^t | z^t)}. \tag{A.2}$$

With CRRA preferences, this implies that the consumption share is given by

$$\frac{c(z^t,\eta^t)}{C(z^t)} = \frac{\zeta(z^t,\eta^t)^{\frac{-1}{\alpha}}}{h(z^t)}, \text{ where } h(z^t) = \sum_{\eta^t} \zeta(z^t,\eta^t)^{\frac{-1}{\alpha}} \pi(\eta^t|z^t). \tag{A.3}$$

Hence, the $-1/\alpha^{\text{th}}$ moment of the multipliers summaries risk sharing within this economy. And, with this moment we get a simple linear risk sharing rule with respect to aggregate consumption.

Making use of (A.2) and the individual first-order condition, we get that

$$\beta^t u' \left[\frac{u'^{-1}\left[\zeta(z^t,\eta^t)P(z^t)\right]}{\sum_{\eta^t} u'^{-1}\left[\zeta(z^t,\eta^t)P(z^t)\right]\pi(\eta^t|z^t)} C(z^t) \right] = P(z^t)\zeta(z^t,\eta^t).$$

If u'^{-1} is homogeneous, which it is with CRRA preferences, then this expression simplifies to

$$\beta^t u' \left[\frac{C(z^t)}{\sum_{\eta^t} u'^{-1}\left[\zeta(z^t,\eta^t)\right]\pi(\eta^t|z^t)} \right] = P(z^t),$$

which implies that the ratio of the state prices is given by

$$\frac{\beta u' \left[\frac{C(z^{t+1})}{\sum_{\eta^t} u'^{-1}[\zeta(z^{t+1},\eta^{t+1})]\pi(\eta^{t+1}|z^{t+1})} \right]}{u' \left[\frac{C(z^t)}{\sum_{\eta^t} u'^{-1}[\zeta(z^t,\eta^t)]\pi(\eta^t|z^t)} \right]} = \frac{P(z^{t+1})}{P(z^t)}. \tag{A.4}$$

Given that we are assuming CRRA preferences, this implies the following proposition. $\square$

- Proof of Corollary 4.1:

Proof. To see this, note that if we use the risk sharing rule in equation (A.3), we obtain that the $-\alpha$-th power of consumption for an individual household is:

$$c(z^t,\eta^t)^{-\alpha} = \frac{\zeta(z^t,\eta^t)}{h(z^t)^{-\alpha}} C_t(z^t)^{-\alpha}.$$

Next, we define C^* as the $-\alpha^{\text{th}}$ moment of the consumption distribution, or

$$C^*(z^t) = \sum_{\eta^t} c(z^t,\eta^t)^{-\alpha} \frac{\pi(z^t,\eta^t)}{\pi(z^t)} = \frac{C_t(z^t)^{-\alpha}}{h(z^t)^{-\alpha}} \sum_{\eta^t} \zeta(z^t,\eta^t) \frac{\pi(z^t,\eta^t)}{\pi(z^t)},$$

and, we compute the growth rate of the $-\alpha$-th power of consumption:

$$\beta\left(C^*(z^{t+1})/C^*(z^t)\right) = \frac{\pi(z^{t+1})}{\pi(z^t)} \frac{\beta\left(\frac{C(z^{t+1})}{h(z^{t+1})}\right)^{-\alpha}}{\left(\frac{C(z^t)}{h(z^t)}\right)^{-\alpha}} \left(\frac{\frac{\sum_{\eta^{t+1}} \pi(z^{t+1},\eta^{t+1})}{\pi(z^{t+1})} \zeta_{t+1}}{\frac{\sum_{\eta^t} \pi(z^t,\eta^t)}{\pi(z^t)} \zeta_t} \right),$$

where the last term is equal to one if the borrowing constraints do not bind, and smaller than one otherwise. This follows from the martingale condition for z-complete and complete traders. For the diversified traders, we know that the last term is one if we sum across aggregate states and multiply by the diversifiable income claim return

$$\zeta(z^t,\eta^t)E\left\{m(z^{t+1}|z^t)R(z^{t+1})|z^t\right\} = \sum_{z^{t+1}\succ z^t,\eta^{t+1}\succ\eta^t} \zeta(z^{t+1},\eta^{t+1})\widetilde{\pi}(z^{t+1},\eta^{t+1}|z^t,\eta^t)$$

This in turn implies that

$$\beta\left(C^*(z^{t+1})/C^*(z^t)\right) \leq m(z^{t+1}|z^t).$$

for complete and z-complete traders and that:

$$E_t\left[\beta\left(C^*(z^{t+1})/C^*(z^t)\right)R(z^{t+1})\right] \leq E_t\left[m(z^{t+1}|z^t)R(z^{t+1})\right] = 1.$$

for diversified traders. □

- Proof of Proposition 4.1:

Proof. If the solvency constraints do not bind anywhere, then we know that on average

$$\zeta\left(z^{t+1},\eta^{t+1}\right)\pi(\eta^{t+1},z^{t+1}|\eta^t,z^t) = \zeta(z^t,\eta^t),$$

from equation (3.11). In that case, $h_{t+1}(z^{t+1}) = h_t(z^t)$ for all z^t. This implies that

$$\frac{h_{t+1}(z^{t+1})}{h_t(z^t)} > 1 \text{ for all } z^t$$

To see why, note that

$$h_{t+1}(z^{t+1}) - h_t(z^t)$$

$$= \int \sum_{\eta^{t+1}\succ\eta^t} \left\{\left[T^{com}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right]^{\frac{-1}{\alpha}} \frac{\pi(\eta^{t+1},z^{t+1}|\eta^t,z^t)}{\pi(z^{t+1}|z^t)} - \zeta(z^t,\eta^t)^{\frac{-1}{\alpha}}\right\} d\Phi$$

Now, we know that

$$\left[T^{com}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right] \leq \zeta(z^t,\eta^t),$$

with strict inequality if the debt bounds bind. From Jensen's inequality, since this a strictly convex function, this implies the following inequality

$$\left[T^{com}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right]^{-\frac{1}{\alpha}}$$
$$\geq \left[\zeta(z^t,\eta^t)\right]^{-\frac{1}{\alpha}},$$

with strict inequality if the debt bounds bind. This implies that, piece-by-piece, the elements in the integrand are non-negative, which implies that $h_{t+1}(z^{t+1}) - h_t(z^t) > 0$. □

- Proof of Proposition 4.1:

Proof. If the solvency constraints do not bind anywhere, then we know that on average

$$\sum_{\eta'} \zeta\left(z^{t+1},\eta^{t+1}\right)\pi(\eta^{t+1},z^{t+1}|\eta^t,z^t) = \zeta(z^t,\eta^t),$$

from equation (3.11). In that case, $h_{t+1}(z^{t+1}) = h_t(z^t)$ for all z^t. This implies that

$$\frac{h_{t+1}(z^{t+1})}{h_t(z^t)} > 1 \text{ for all } z^t$$

To see why, note that

$$h_{t+1}(z^{t+1}) - h_t(z^t)$$
$$= \int \sum_{\eta^{t+1} \succ \eta^t} \left\{ \left[T^z(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right]^{\frac{-1}{\alpha}} \frac{\pi(\eta^{t+1}, z^{t+1}|\eta^t, z^t)}{\pi(z^{t+1}|z^t)} - \zeta(z^t, \eta^t)^{\frac{-1}{\alpha}} \right\} d\Phi$$

Now, we know that

$$\sum_{\eta^{t+1} \succ \eta^t} \frac{\pi(\eta^{t+1}, z^{t+1}|\eta^t, z^t)}{\pi(z^{t+1}|z^t)} \left[T^z(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right] \leq \zeta(z^t, \eta^t),$$

with strict inequality if the debt bounds bind. From Jensen's inequality, since this a strictly convex function, this implies the following inequality

$$\sum_{\eta^{t+1} \succ \eta^t} \frac{\pi(\eta^{t+1}, z^{t+1}|\eta^t, z^t)}{\pi(z^{t+1}|z^t)} \left[T^z(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right]^{-\frac{1}{\alpha}}$$
$$> \left[\sum_{\eta^{t+1} \succ \eta^t} \frac{\pi(\eta^{t+1}, z^{t+1}|\eta^t, z^t)}{\pi(z^{t+1}|z^t)} \left[T^z(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right] \right]^{-\frac{1}{\alpha}}$$
$$\geq \left[\zeta(z^t, \eta^t) \right]^{-\frac{1}{\alpha}},$$

with strict inequality if the debt bounds bind. This implies that, piece-by-piece, the elements in the integrand are non-negative, which implies that $h_{t+1}(z^{t+1}) - h_t(z^t) > 0$. □

- Proof of Proposition 4.2:

Proof. Note that:

$$h_{t+1}(z^{t+1}) - h_t(z^t)$$
$$= \int \sum_{\eta^{t+1} \succ \eta^t} \left\{ \left[T^{eq}(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right]^{\frac{-1}{\alpha}} \varphi(\eta^{t+1}|\eta^t) - \zeta(z^t, \eta^t)^{\frac{-1}{\alpha}} \right\} d\Phi$$

Now, we know that

$$\sum_{z^{t+1} \succ z^t} \widetilde{\phi}(z^{t+1}|z^t) \sum_{\eta^{t+1} \succ \eta^t} \varphi(\eta^{t+1}|\eta^t) \left[T^{eq}(z^{t+1}, \eta^{t+1}|z^t, \eta^t)(\zeta(z^t, \eta^t)) \right] \leq \zeta(z^t, \eta^t),$$

with strict inequality if the debt bounds bind. From Jensen's inequality, since this a strictly convex function, this implies the following inequality

$$\sum_{z^{t+1}\succ z^t} \widetilde{\phi}(z^{t+1}|z^t) \sum_{\eta^{t+1}\succ\eta^t} \varphi(\eta^{t+1}|\eta^t) \left[T^{eq}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right]^{-\frac{1}{\alpha}}$$

$$> \left[\sum_{z^{t+1}\succ z^t} \widetilde{\phi}(z^{t+1}|z^t) \sum_{\eta^{t+1}\succ\eta^t} \varphi(\eta^{t+1}|\eta^t) \left[T^{eq}(z^{t+1},\eta^{t+1}|z^t,\eta^t)(\zeta(z^t,\eta^t))\right]\right]^{-\frac{1}{\alpha}}$$

$$\geq \left[\zeta(z^t,\eta^t)\right]^{-\frac{1}{\alpha}},$$

with strict inequality if the debt bounds bind. This implies that, piece-by-piece, the elements in the integrand are non-negative, which implies that $\sum_{z^{t+1}\succ z^t} \widetilde{\phi}(z^{t+1}|z^t)h_{t+1}(z^{t+1}) - h_t(z^t) > 0$. □

- Proof of Proposition 4.3:

Proof. Conjecture that $\frac{h(z^{t+1})}{h(z^t)} = g_{t+1}$ is a non-random sequence. Normalize h_t to one. Conjecture that $S(\zeta(z^t,\eta^t); z^t,\eta^t)$ does not depend on z^t. Given conditions (4.1) and (4.2), we know that

$$\tilde{S}_t(\zeta(\eta^t);\eta^t) = \left[\gamma\eta_t - \zeta(\eta^t)^{\frac{-1}{\alpha}}\right] + \widehat{\beta}_t \sum_{\eta_{t+1}} \varphi(\eta_{t+1}|\eta_t)\tilde{S}_{t+1}(\zeta(\eta^{t+1});\eta^{t+1}), \tag{A.5}$$

where $\widehat{\beta}_t = \beta\sum_{z_{t+1}}\phi(z_{t+1})g_{t+1}^{\gamma}\exp((1-\gamma)z_{t+1})$ and $\lambda(z_{t+1})$ is defined as the growth rate $\frac{Y_{t+1}}{Y_t}$. In addition, our debt constraint in terms of $\tilde{S}$ is simply

$$\tilde{S}_t(\zeta(\eta^t);\eta^t) \leq \underline{M}(\eta^t). \tag{A.6}$$

Note that neither the recursion (A.5) or the debt constraint (A.6) depend upon the value of the realization of z_t. For z-complete traders, the measurability condition is given by

$$\tilde{S}_t(\zeta(\eta^{t+1});\eta^{t+1}) = \tilde{S}_t(\zeta(\tilde{\eta}^{t+1});\tilde{\eta}^{t+1}) \tag{A.7}$$

for all η^{t+1}, $\tilde{\eta}^{t+1}$ and z^{t+1} where $\eta^t(\eta^{t+1}) = \eta^t(\tilde{\eta}^{t+1})$. Their optimality condition is still (A.1). Hence, none of the equations determining either $\tilde{S}$ or the updating rule for ζ depend on z_{t+1}. This is also true for the complete traders, since their measurability condition is degenerate, and their optimality condition is (4.4). The dynamics of the multipliers on the measurability constraints and the solvency constraints do not depend on z^t, only on η^t. This confirms that $\{h_t\}$ does not depend on the aggregate history of shocks $\{z^t\}$, and hence is a non-random sequence.

This independence is also true for the diversified investors. The reason is that their measurability condition is given by

$$\frac{\tilde{S}_{t+1}(\zeta(z^{t+1},z_{t+1},\eta^{t+1});z^{t+1},\eta^{t+1})}{[(1-\gamma)+\varpi_{t+1}(z^{t+1})/Y(z^{t+1})]} = \frac{\tilde{S}_{t+1}(\zeta(,\tilde{z}^{t+1},\tilde{\eta}^{t+1});\tilde{z}^{t+1},\tilde{\eta}^{t+1})}{[(1-\gamma)+\varpi_{t+1}(\tilde{z}^{t+1})/Y(\tilde{z}^{t+1})]}, \tag{A.8}$$

for all for all η^{t+1} and $\tilde{\eta}^{t+1}$, z^{t+1} and $\tilde{z}^{t+1}$ where $\eta^t(\eta^{t+1}) = \eta^t(\tilde{\eta}^{t+1})$ and $z^t(z^{t+1}) = z^t(\tilde{z}^{t+1})$. Hence, the independence holds iff $\varpi_{t+1}(z^{t+1})/Y(z^{t+1})$ is deterministic, i.e. does not depend on z^{t+1}. Given conditions (4.1) and (4.2), and given our conjecture that $\{h_t\}$ is deterministic, it is easy to show that $\tilde{\varpi}_t$ is deterministic as well, because no arbitrage implies that:$\tilde{\varpi}_t = 1 + \widehat{\beta}\tilde{\varpi}_{t+1}$.

□

- Proof of proposition 4.4:

Proof. First, since the measurability constraints are satisfied for the individual household's savings function, they also need to be satisfied for the aggregate savings function. So by the LLN:

$$\frac{S_a^{div}(z^t, z_t)}{[(1-\gamma)Y(z^t, z_t) + \varpi(z^t, z_t)]} \quad = \quad \frac{S_a^{div}(z^t, \tilde{z}_{t+1})}{[(1-\gamma)Y(z^t, \tilde{z}_{t+1}) + \varpi(z^t, \tilde{z}_{t+1})]}$$

where we have used the fact that the denominator is measurable w.r.t. z^t. Note that

$$\sum_k S_a^k(z^t) = -\left[(1-\gamma)Y(z^t, \tilde{z}_{t+1}) + \varpi(z^t, \tilde{z}_{t+1})\right].$$

Hence the ratio

$$S_a^{div}(z^t)/\sum_k S_a^k(z^t)$$

cannot not depend on z_t, because of the *measurability* condition. This implies that $\frac{h^j(z^t)}{h(z^t)}$ cannot depend on z_t either. This can easily be shown by starting in a finite horizon case, and working backwards from T. In period T, the measurability constraint implies that:

$$C(z^T)\left[\gamma\mu^j - \frac{h^j(z^T)}{h(z^T)}\right]$$

is proportional to

$$C(z^T)(\gamma - 1)$$

for all z^T. This follows from the definition of the aggregate savings function and the aggregate tradeable income claim. Since this condition holds for all z_T, this implies that $\frac{h^j(z^T)}{h(z^T)}$ cannot depend on z_t. Next, consider period $T-1$. The same argument implies that $\frac{h^j(z^{T-1})}{h(z^{T-1})}$ cannot depend on z_{T-1}. This follows from $S_a^j(z^{T-1})$ being proportional to $S_a(z^{T-1})$ in all z_{T-1}. This requirement together with the previous result implies that $\frac{h^j(z^{T-1})}{h(z^{T-1})}$ cannot depend on z_{T-1}. By backward induction, this implies that $\frac{h^j(z^t)}{h(z^t)}$ cannot depend on z_t. If we take the limit of T to ∞, this result also applies to the infinite horizon economy. □

- Proof of proposition 4.5:

Proof. For non-participant traders $j = np$, $S_a^j(z^t)$ cannot not depend on z_t, because of the *measurability* condition. This implies that $\left[\gamma - \frac{h^j(z^t)}{h(z^t)}\right] C(z^t)$ cannot depend on z_t. This can easily be shown by starting in a finite horizon case, and working backwards from T. In period T, we know that:

$$\left[\gamma\mu^j - \frac{h^j(z^T)}{h(z^T)}\right] C(z^T)$$

cannot depend on z_T. Since this condition holds for all z_t, this implies that $\left[\gamma - \frac{h^j(z^T)}{h(z^T)}\right] C(z^T) = k(z^{T-1})$. The same argument implies that $\left[\gamma - \frac{h^j(z^{T-1})}{h(z^{T-1})}\right] C(z^{T-1}) = k(z^{T-2})$. By backward induction, this implies the result. If we take the limit of T to ∞, this result also applies to the infinite horizon economy. □

B Computational Algorithm

We use a finite history of length n of the aggregate shocks to (reasonably) accurately compute the equilibrium. The variable n determines the set of aggregate finite histories $S(n)$ that we are keeping track of, and $s \in S(n)$ denotes a generic member. The number of elements of $S(n)$ is given by $n^{\#Z}$, where $\#Z$ is the number of aggregate states. The individual state is then given by his multiplier, the finite aggregate history, and his individual shock; besides his multiplier, there are $n^{\#Z} * \#N$ states for the individual.

The algorithm works as follows. Assume that we have a matrix $g(s, s')$, which gives the value of our moment $h(z^{t+1})/h(z^t)$ in the case where the transition is from finite history s to finite history s'. Given this matrix we can compute the aggregate state price in the stationary version of the economy, which we will denote by $\hat{P}(s')/\hat{P}(s)$. In computing the equilibrium, we find it more convenient to keep track of agents by their consumption share c rather than their (normalized) multiplier ζ. Note that $c^{-\alpha} = \zeta$.

To compute $\hat{D}(c, s, \eta)$, we first assume that c is unchanged and we simply use the recursive savings equation to compute $\hat{D}_0$. Then, to compute $\hat{D}_{j+1}$ given $\hat{D}_j$ we do the following algorithm:

1. We start with a savings grid where the highest savings level is the debt/savings limit. Note that since this is a fraction of the net present value of income, we can compute this directly given g.

2. For savings grid point S_i, we can compute the associated consumption shares $c'(s', \eta')$, where $S_i = \hat{D}_j(c'(s', \eta'), s', \eta')$. Since $\hat{D}_j$ is piecewise linear, it is trivial to invert this function.

3. Given S_i and $\tilde{c}'(s', \eta')$, we can compute the consumption share today from the optimality condition for state today (s, η). This is given by

$$E\left\{\tilde{c}'(s', \eta')^{-\alpha} | s, \eta\right\} g(s, s')^{\alpha} = c(s, \eta)^{-\alpha},$$

   If we do this for every grid point savings grid tomorrow, fixing the state today (s, η), this yields a vector of current consumption shares $\mathbf{c}$ and their future associated net savings levels $\mathbf{S}'$ for each possible transition (η, s, s'). We can then fit linear piecewise linear functions to the $[\mathbf{c}, \mathbf{S}']$ for each transition (η, s, s'). Hence we have constructed $S'(c; \eta, s, s')$.

4. Given these piecewise linear functions $S'(c; \eta, s, s')$, we can compute trivially compute $\hat{D}_{j+1}(c, s, \eta)$ for each current consumption share c in our grid by our recursive saving equation since c is the consumption share today and we have already computed the future savings levels via our piecewise linear function for each possible future transition (η, s, s'). In this way, we can compute a vector of current consumption shares $\mathbf{c}$ and their associated current net savings levels $\hat{\mathbf{D}}_{j+1}$. We can then fit linear piecewise linear functions to the $\left[\mathbf{c}, \hat{\mathbf{D}}_{j+1}\right]$ for each (s, η). In so doing we have constructed the function $\hat{D}_{j+1}(c, s, \eta)$.

5. The iterations continues until the $\hat{D}_j$ functions converge. As one of the products of this computation we have the vectors $\mathbf{c}$ and $\mathbf{c}'(\eta')$ for each transition (η, s, s'). We store these vectors in an array and use them in our simulation step when we update the values of $g(s, s')$ implied by our transition functions for consumption shares.

6. To simulate our economy and update H, we take a single panel draw of aggregate and idiosyncratic shocks. We then compute the updated consumption shares, where each period we normalize the consumption shares to average 1, and use the normalization factor to generate a revised estimate $g'(s, s')$. Given this revised estimate we repeat the iterations until the estimate of H' converges.

C Additional Tables

[Table 10 about here.]

Table 1: Asset Pricing and Portfolio Implications

	Market Segmentation		
complete	μ_1	μ_1	μ_1
z-complete	μ_2	μ_2	μ_2
diversified	μ_3	0	μ_3
non-part	0	0	μ_4
	Asset Prices		
R^e	RA	RA	$\neq RA$
R^f	$< RA$	$< RA$	$< RA$
	Portfolios		
complete	$Market$	$Market$	$Levered$
z-complete	$Market$	$Market$	$Levered$
diversified	$Market$	$Market$	$Market$
non-part	/	/	$Bonds$

Table 2: Asset Pricing in the IID Economy

	RA Economy	**HTT Economy**			**Data**
		Case 1	*Case 2*	*Case 3*	
complete		0%	5%	10%	
z-complete		10%	5%	0%	
diversified		20%	20%	20%	
non-part		70%	70%	70%	
$E[R_f]$	12.96	1.737	1.922	2.185	1.049
$\sigma[R_f]$	0.000	0.066	0.237	0.292	1.560
$\sigma[m]/E[m]$	0.193	0.440	0.467	0.510	
$Std[\sigma_t[m]/E_t[m]]$	0.000	0.033	0.045	0.058	
$E[R_{eq} - R_f]$	2.328	5.036	5.305	5.882	7.531
$\sigma[R_{eq} - R_f]$	12.74	11.66	11.67	11.98	16.94
$E[R_{eq} - R_f]/\sigma[R_{eq} - R_f]$	0.182	0.431	0.454	0.490	0.444
$E[R_{lc} - R_f]$	3.081	6.702	6.435	6.874	7.531
$\sigma[R_{lc} - R_f]$	15.94	15.27	13.89	13.69	16.94
$E[R_{lc} - R_f]/\sigma[R_{lc} - R_f]$	0.193	0.438	0.463	0.502	0.444
$E[W^{Coll}/C]$	0.855	5.960	4.889	6.458	3.870
$E[PD_{eq}]$	7.936	20.98	18.02	23.16	33.87
$\sigma[PD_{eq}]$	13.09	15.92	15.59	15.20	16.78
$E[R_b - R_f]$	0.000	-0.271	-0.046	-0.437	1.070
$\sigma[R_b - R_f]$	0.000	0.604	0.143	0.935	9.366
$E[R_b - R_f]/\sigma[R_b - R_f]$	/	-0.324	-0.467	-0.449	.1145
$\rho[R_{eq}(t), R_{eq}(t-1)]$	0.000	-0.015	-0.010	-0.010	-0.191
$\rho[R_{lc}(t), R_{lc}(t-1)]$	0.000	0.003	0.012	-0.005	-0.191
$\rho[R_{eq}, R_f]$	0.000	-0.024	-0.014	-0.020	0.272

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration idiosyncratic shocks and IID calibration of aggregate shocks. Reports the moments of asset prices for the RA (Representative Agent) economy, for the HTT (Heterogeneous Trading Technology) economy and for the data. We use post-war US annual data for 1946-2005. The market return is the CRSP value weighted return for NYSE/NASDAQ/AMEX. We use the Fama risk-free rate series from CRSP (average 3-month yield). To compute the standard deviation of the risk-free rate, we compute the annualized standard deviation of the ex post real monthly risk-free rate. The return on the long-run bond is measured using the Bond Total return index for 30-year US bonds from Global Financial Data.

Table 3: Household Portfolio Returns in the IID Economy

	Case 1	*Case 2*	*Case 3*
complete	0%	5%	10%
z-complete	10%	5%	0%
diversified	20%	20%	20%
non-part	70%	70%	70%
$E[R_c^W - R_f]$	NA	0.125	0.126
$E[R_z^W - R_f]$	0.056	0.062	NA
$E[R_{div}^W - R_f]$	0.008	0.008	0.009
$E[R_{np}^W - R_f]$	−0.012	−0.0120	−0.010
$E[R_c^W - R_f]/\sigma[R_c^W - R_f]$	NA	0.064	0.116
$E[R_z^W - R_f])/\sigma[R_z^W - R_f]$	0.429	0.476	NA
$E[R_{div}^W - R_f]/\sigma[R_{div}^W - R_f]$	0.123	0.119	0.145
$E[R_{np}^W - R_f]/\sigma[R_{np}^W - R_f]$	−0.167	−0.172	−0.171

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and IID calibration of aggregate shocks. Reports the moments of average returns on the portfolio of each trader. These are the moments of *average* portfolio returns for all the traders in a segment.

Table 4: Consumption in the IID Economy

	Case 1	*Case 2*	*Case 3*		*Case 1*	*Case 2*	*Case 3*
complete	0%	5%	10%		0%	5%	10%
z-complete	10%	5%	0%		10%	5%	0%
diversified	20%	20%	20%		20%	20%	20%
non-part	70%	70%	70%		70%	70%	70%
	Household Consumption				**Aggregate Consumption**		
$\sigma\,[\Delta\log(c_c)]$	NA	5.641	5.417	$\sigma[\Delta\log(\widehat{C}_c)]$	NA	3.840	4.873
$\sigma\,[\Delta\log(c_z)]$	7.892	7.131	NA	$\sigma[\Delta\log(\widehat{C}_z)]$	3.972	4.402	NA
$\sigma\,[\Delta\log(c_{div})]$	11.44	11.35	11.07	$\sigma[\Delta\log(\widehat{C}_{div})]$	0.334	0.329	0.368
$\sigma\,[\Delta\log(c_{np})]$	12.62	12.50	12.35	$\sigma[\Delta\log(\widehat{C}_{np})]$	1.062	1.037	1.071
$\rho\,[R_s,(\Delta\log(\widehat{c}_p)]$	0.163	0.204	0.283				
$\rho\,[R_s,(\Delta\log(\widehat{c}_c)]$	NA	0.649	0.857	$\rho[R_s,\Delta\log(\widehat{C}_c)]$	NA	0.949	0.951
$\rho\,[R_s,(\Delta\log(\widehat{c}_z)]$	0.482	0.588	NA	$\rho[R_s,\Delta\log(\widehat{C}_z)]$	0.965	0.956	NA
$\rho\,[R_s,(\Delta\log(\widehat{c}_{div})]$	0.003	−0.002	−0.003	$\rho[R_s,\Delta\log(\widehat{C}_{div})]$	0.119	−0.083	−0.117
$\rho\,[R_s,(\Delta\log(\widehat{c}_{np})]$	−0.071	−0.070	−0.073	$\rho[R_s,\Delta\log(\widehat{C}_{np})]$	−0.965	−0.959	−0.948
$\sigma(\Delta\log(c_c))$	8.652	8.411	NA	$\sigma(\Delta\log(C_c))$	8.310	7.280	NA
$\sigma(\Delta\log(c_z))$	NA	9.72	10.17	$\sigma(\Delta\log(C_z))$	NA	7.852	7.438
$\sigma(\Delta\log(c_{div}))$	11.83	12.10	12.21	$\sigma(\Delta\log(C_{div}))$	3.545	3.556	3.622
$\sigma(\Delta\log(c_{np}))$	12.83	13.00	13.11	$\sigma(\Delta\log(C_{np}))$	2.560	2.582	2.554
$\rho\,[R_s,(\Delta\log(c_p)]$	0.431	0.463	0.503				
$\rho\,[R_s,(\Delta\log(c_c)]$	NA	0.843	0.920				
$\rho\,[R_s,(\Delta\log(c_z)]$	0.712	0.784	NA				
$\rho\,[R_s,(\Delta\log(c_{div})]$	0.291	0.287	0.291				
$\rho\,[R_s,(\Delta\log(c_{np})]$	0.200	0.203	0.209				

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and IID calibration of aggregate shocks. The first panel reports the moments for average household consumption share (share of aggregate endowment) growth in each trader segment. The second panel reports the moments for the growth rate of the aggregate consumption share of each trader segment. The third panel reports the moments of average returns on the portfolio of each trader. These are the moments of *average* portfolio returns for all the traders in a segment. Hatted variables denote shares of the aggregate endowment.

Table 5: Approximation in the IID Economy

	Case 1	*Case 2*	*Case 3*
complete	0%	5%	10%
z-complete	10%	5%	0%
diversified	20%	20%	20%
non-part	70%	70%	70%
$z' = l, z = l$	31.5	57.5	3.1
$z' = h, z = l$	32.2	53.1	1.0
$z' = l, z = h$	15.7	22.5	4.5
$z' = h, z = h$	27.9	18.3	9.5
$\sup \frac{std([h'/h])}{E([h'/h])} (\%)$	0.579	0.309	0.287

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration idiosyncratic shocks and IID calibration of aggregate shocks. The first panel reports the R^2 in a regression of the $\log SDF_t$ on the mean of the wealth distribution $E(\log W)_t$. The second panel reports the maximal coefficient of variation across all aggregate truncated histories of the actual aggregate multiplier growth rate $[h'/h]$ in percentages.

Table 6: Asset Pricing in the MP Economy

	RA	**HTT**	**Data**
$E[R_f]$	13.04	0.866	1.049
$\sigma[R_f]$	3.144	2.897	1.560
$\sigma[m]/E[m]$	0.193	0.481	
$Std[\sigma_t[m]/E_t[m]]$	0.011	0.054	
$E[R_{eq} - R_f]$	2.324	5.861	7.531
$\sigma[R_{eq} - R_f]$	13.34	12.49	16.94
$E[R_{eq} - R_f]/\sigma[R_{eq} - R_f]$	0.174	0.469	0.444
$E[R_{lc} - R_f]$	4.397	10.87	7.531
$\sigma[R_{lc} - R_f]$	23.07	22.87	16.94
$E[R_{lc} - R_f]/\sigma[R_{lc} - R_f]$	0.190	0.475	0.444
$E[PD]_{eq}$	7.989	18.72	33.87
$\sigma[PD]_{eq}$	12.81	15.20	16.78
$\rho[R_{eq}, R_f]$	0.204	0.204	0.272
$\rho[R_{eq}(t), R_{eq}(t-1)]$	-0.193	-0.199	-0.191
$\rho[R_{lc}(t), R_{lc}(t-1)]$	-0.103	-0.134	-0.191
$E[R_b - R_f]$	0.449	-0.604	1.070
$\sigma[R_b - R_f]$	2.337	1.297	9.366
$[E(R_b - R_f)]/[\sigma(R_b - R_f)]$	0.192	-0.466	0.114

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks. Reports the moments of asset prices for the RA (Representative Agent) economy, for the HTT (Heterogeneous Trading Technology) economy and for the data. We use post-war US annual data for 1946-2005. The market return is the CRSP value weighted return for NYSE/NASDAQ/AMEX. We use the Fama risk-free rate series from CRSP (average 3-month yield). To compute the standard deviation of the risk-free rate, we compute the annualized standard deviation of the ex post real monthly risk-free rate. The return on the long-run bond is measured using the Bond Total return index for 30-year US bonds from Global Financial Data.

Table 7: Household Wealth and Consumption in the MP Economy

	Returns		Wealth	
	$E[R_i^W - R_f]$	$E[R_i^W - R_f]/\sigma[R_i^W - R_f]$	$E[W_i/W]$	$\sigma[W_i/W]$
$z-complete$	0.075	0.368	3.213	1.592
div	0.019	0.225	0.979	0.686
np	-0.013	-0.162	0.689	0.522
	Household Consumption		**Aggregate Consumption**	
	$\sigma[\Delta\log(\widehat{c}_i)]$	$\rho[R_s, (\Delta\log(\widehat{c}_i)]$	$\sigma[\Delta\log(\widehat{C}_i)]$	$\rho[R_s, (\Delta\log(\widehat{C}_i)]$
p	/	0.191		
$z-complete$	7.7641	0.559	4.616	0.944
div	11.589	0.007	0.406	0.202
np	13.299	-0.091	1.471	-0.951
	$\sigma[\Delta\log(c_i)]/\sigma[\Delta\log(C)]$	$\rho[R_s, (\Delta\log(c_i)]$	$\sigma[\Delta\log(C_i)]$	$\rho[R_s, (\Delta\log(C_i)]$
p		0.447		
$z-complete$	2.874	0.758	8.0403	1.000
div	3.465	0.291	3.6702	1.000
np	3.839	0.169	2.1895	1.000
	EIS_{R_b}	$EIS_{R_{eq}}$		
p	xxx	xxx		
z	1.097	0.784		
div	0.983	0.559		
np	0.928	0.484		

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks. Hatted variables denote shares of aggregate consumption. EIS_{R_b} and $EIS_{R_{eq}}$ are the EIS estimates obtained by regressing average household consumption growth on bond returns and stock return using data generated by the MP model. We use the log dividend-price ratio, the lagged real stock return and the bond risk premium as instruments.

Table 8: Household Wealth Distribution

	Bewley Model		HTT Model		US Data 2004	
	Wealth		**Wealth**		**Net Worth**	**Total Assets**
	Standard	Twisted	Standard	Twisted		
kurtosis	1.956	2.842	10.43	15.78	15.87	48.85
skewness	0.231	0.882	2.398	3.189	3.616	6.250
Gini	0.405	0.486	0.513	0.587	0.793	0.697
W_{75}/W_{25}	4.124	5.529	5.030	6.967	25.09	10.64
W_{80}/W_{20}	6.623	9.409	8.803	13.31	65.41	33.42
W_{85}/W_{15}	13.34	19.12	21.78	36.15	211.9	55.75
W_{90}/W_{10}	54.35	82.20	252.0	472.6	999.1	580.5
W_{50}/W_{10}	26.10	25.33	103.4	134.8	105.0	91.00
W_{90}/W_{50}	2.082	3.245	2.436	3.505	9.510	6.378

Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks. The wealth data are from the 2004 SCF. The HTT model has 10% z-complete traders, 20 % diversified traders and 70 % non-participants. The Bewley model has 100 % diversified traders.

Table 9: Equity Share Distribution

	Data		**Model**	
Percentile	2001	2004	Standard	Twisted
15 %	4.512	2.633	5.694	3.942
25 %	15.40	6.797	6.617	3.293
35 %	6.057	6.669	7.331	3.722
50 %	8.077	2.762	6.817	3.115
65 %	11.09	10.16	6.572	8.207
75 %	19.04	10.12	7.962	11.02
80 %	14.45	17.34	9.204	10.08
85 %	24.16	16.56	13.11	9.263
90 %	32.59	18.94	27.50	12.78
95 %	34.30	25.37	52.02	41.86
100 %	42.67	34.19	59.02	59.80

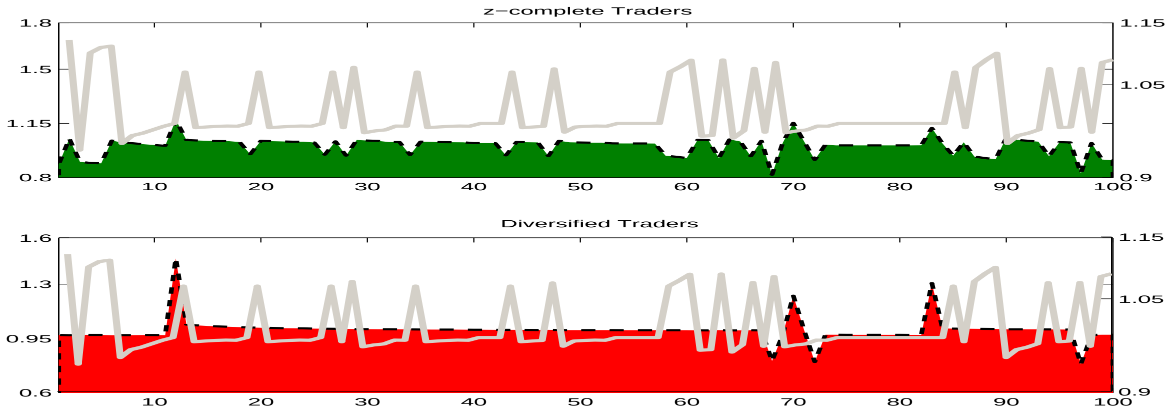
Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks. The wealth data are from the 2001 and 2004 SCF. The equity share reported is the share of equity as a fraction of net worth less private business holdings.

Table 10: Matching Income Distribution

	Model	**US Data**
75/50	2.739	1.785
80/50	2.893	2.041
85/50	3.062	2.414
90/50	3.353	2.908
75/25	4.136	3.449
80/25	4.369	3.943
85/25	4.624	4.663
90/25	5.063	5.618
80/20	4.613	4.710
85/15	6.537	7.024
90/10	11.42	11.64

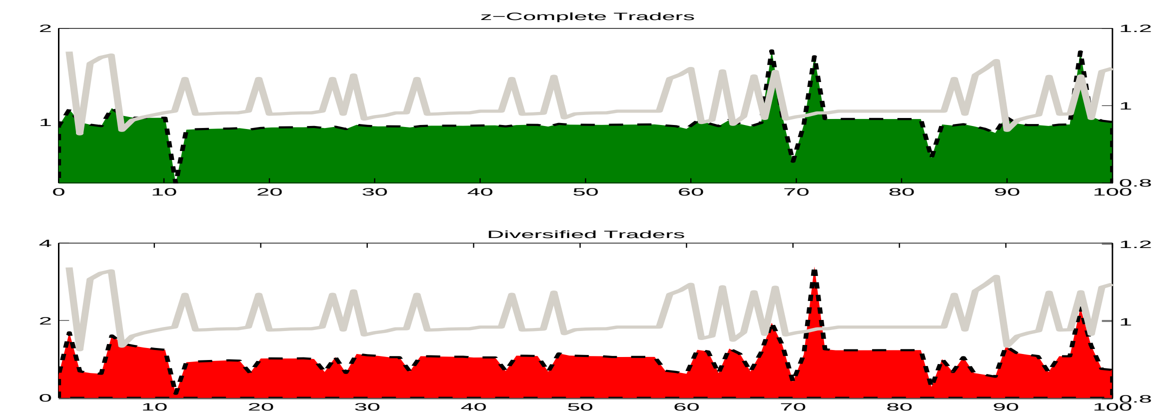
Notes: Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 10000 draws from an economy with 3000 agents. Benchmark calibration of aggregate and idiosyncratic shocks. The income data are from the 2004 SCF.

Figure 1: Consumption Growth



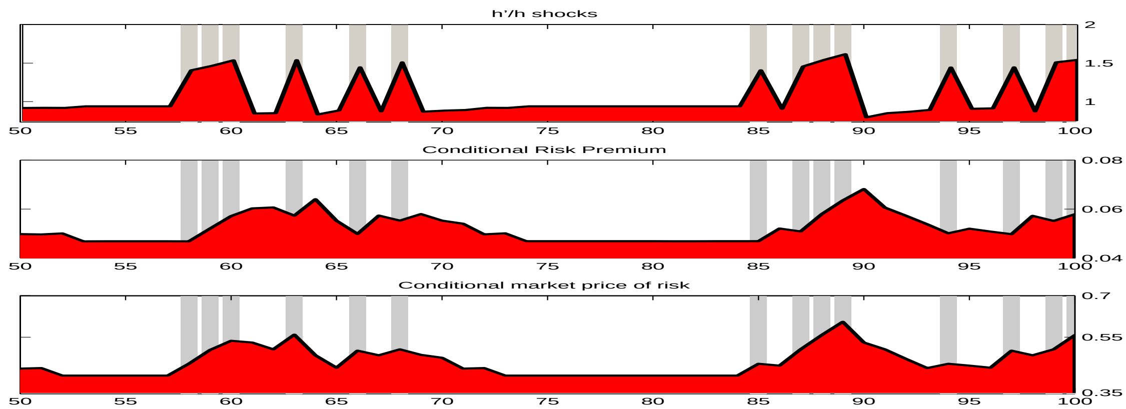
Notes: Growth Rate of Household Consumption Shares. Market Segmentation: 5% complete, 5% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 100 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and IID calibration of aggregate shocks. The thick dashed line is the aggregate weight growth rate h_{t+1}/h_t.

Figure 2: Multiplier Growth



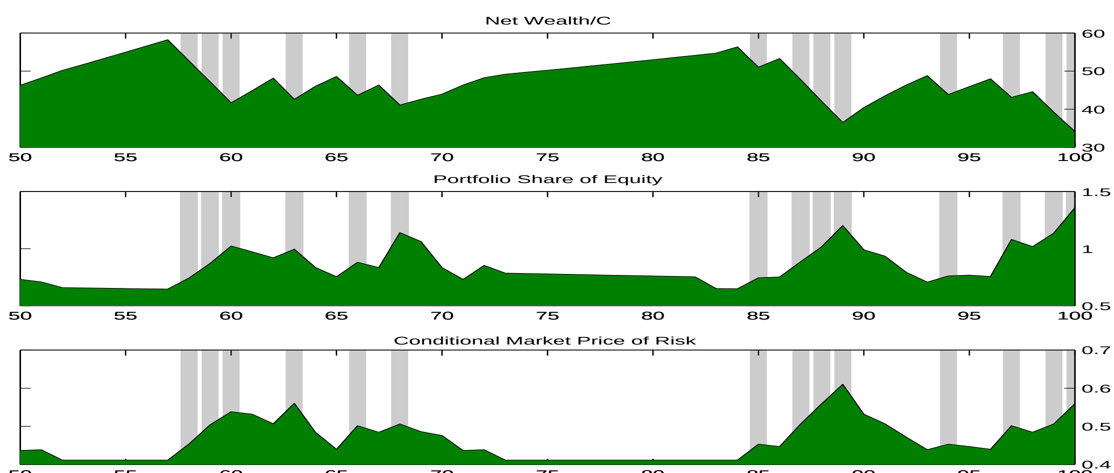
Notes: Market Segmentation: 5% complete, 5% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 100 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and IID calibration of aggregate shocks. The thick dashed line is the aggregate weight growth rate h_{t+1}/h_t.

Figure 3: Conditional Risk Premium and Market Price of Risk



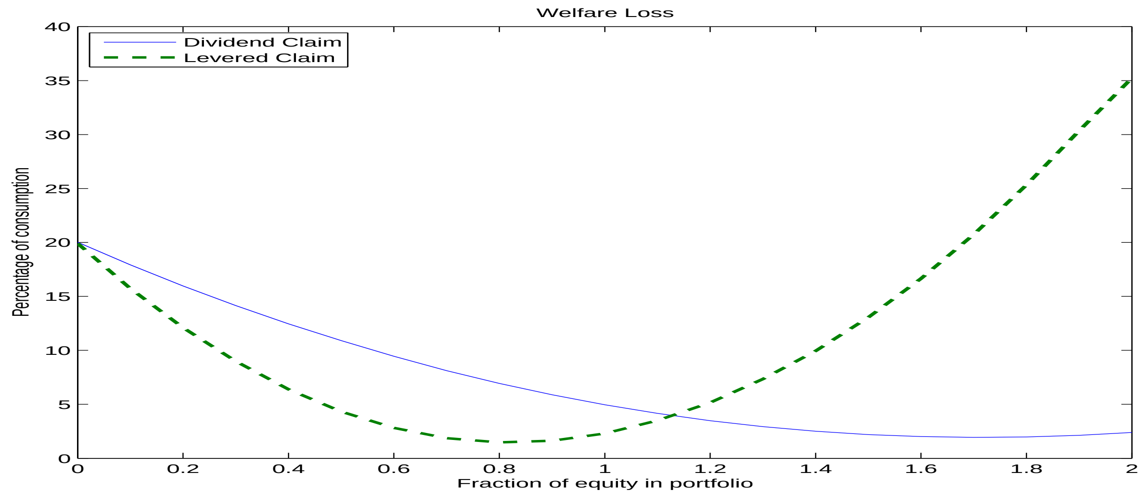
Notes: Market Segmentation: 5% complete, 5% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. Plot of 50 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and IID calibration of idiosyncratic shocks. The shaded are indicates low aggregate consumption growth states.

Figure 4: Equity Share



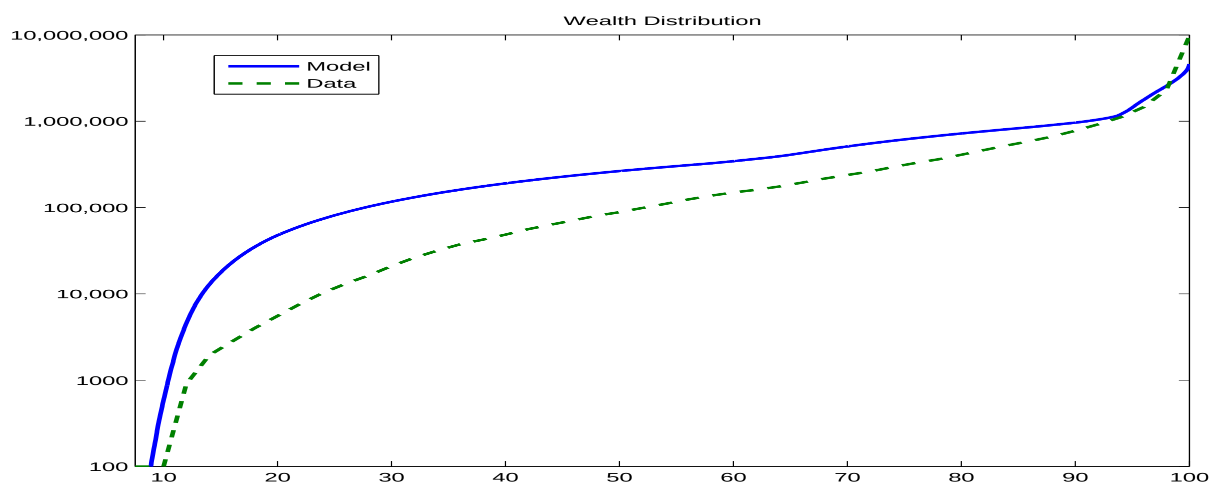
Notes: Market Segmentation: 5% complete, 5% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 100 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic and IID calibration of aggregate shocks. The shaded areas indicate low aggregate consumption growth states.

Figure 5: Equity Share



Notes: Market Segmentation: 0% complete, 10% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by 100 draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks.

Figure 6: Wealth Distribution



Notes: Market Segmentation: 5% complete, 5% in z-complete, 20% diversified and 70% non-participants. Parameters setting: $\gamma = 5$, $\beta = 0.95$, collateralized share of income is 0.1. The simulation moments are generated by draws from an economy with 3000 agents. Benchmark calibration of idiosyncratic shocks and MP calibration of aggregate shocks. The wealth data is from the 2004 SCF. Data shown in \$ 2004.

D Separate Appendix

E Ex Ante Heterogeneity

Suppose there are some differences in permanent income and initial endowments of financial wealth. Let x_y index the permanent component, meaning that a household with label x_y receives x_y times the labor income process and the initial endowment of financial wealth of the average household. The only part that affects the stationary equilibrium is the labor income part.

Lemma E.1. *If the borrowing constraints are proportional to x_y, then optimal consumption is proportional to x_y as well.*

This lemma implies that the fraction μ_i can be interpreted as the fraction of human wealth (not financial wealth) held by households in segment i. For example, if μ_z is calibrated to 5 %, that really means 5 % of human wealth is held by z-complete traders (not 5 % of the population).

Proof of Lemma E.1:

Proof. We use $\widetilde{P}(z^t,\eta^t)$ to denote $P(z^t)\pi(z^t,\eta^t)$. Let γ denote the multiplier on the present-value budget constraint, let $\nu(z^t,\eta^t)$ denote the multiplier on the measurability constraint in node (z^t,η^t), and, finally, let $\varphi(z^t,\eta^t)$ denote the multiplier on the debt constraint. We consider the case in which the borrowing constraint is $x_y\underline{M}_t$ – proportional in x_y. We consider the case in which the initial endowment of the tradeable income claim is proportional to x_y as well. Let $\{c,\hat{a}\}$ denote the optimal consumption and asset choices for a household with $x_y=1$. Using the proportionality assumptions and the consumption conjecture, the saddle point problem for a household with permanent income x_y can be stated as:

$$\begin{aligned}
L(x_y) = \quad & \sum_{t=1}^{\infty}\beta^t\sum_{(z^t,\eta^t)}u(c(z^t,\eta^t))\pi(z^t,\eta^t)x_y^{1-\alpha} \\
& +x_y\widehat{\gamma}_0\left\{\sum_{t\geq 1}\sum_{(z^t,\eta^t)}\widetilde{P}(z^t,\eta^t)\left[\gamma Y(z^t)\eta_t-c(z^t,\eta^t)\right]+\varpi(z^0)\right\} \\
& +x_y\sum_{t\geq 1}\sum_{z^t,\eta^t}\widehat{\nu}(z^t,\eta^t) \\
& \left\{\sum_{\tau\geq t}\sum_{(z^\tau,\eta^\tau)\succ(z^t,\eta^t)}\widetilde{P}(z^\tau,\eta^\tau)\left[\gamma Y(z^\tau)\eta_\tau-c(z^\tau,\eta^\tau)\right]-\widetilde{P}(z^t,\eta^t)\hat{a}(z^t,\eta^{t-1})\right\} \\
& +x_y\sum_{t\geq 1}\sum_{(z^t,\eta^t)}\widehat{\varphi}(z^t,\eta^t) \\
& \left\{\underline{M}_t(z^t,\eta^t)\widetilde{P}(z^t,\eta^t)-\sum_{\tau\geq t}\sum_{(z^\tau,\eta^\tau)\succ(z^t,\eta^t)}\widetilde{P}(z^\tau,\eta^\tau)\left[\gamma Y(z^\tau)\eta_\tau-c(z^\tau,\eta^\tau)\right]\right\}.
\end{aligned}$$

Let $\{\gamma,\nu,\varphi\}$ denote the saddle point multipliers for a household with $x_y=1$. Then it is easy to see that $\{\widehat{\gamma},\widehat{\nu},\widehat{\varphi}\}= x_y^{-\alpha}\{\gamma,\nu,\varphi\}$ and $x_y\{c,\hat{a}\}$ is a saddle point as well.

□

E.1 Other preferences

Our analytic framework extends readily to the case of Epstein and Zin (1989)'s recursive preferences since these preferences also feature the homogeneity of the inverse of marginal utility over consumption. To show this, assume that preferences are defined by the following recursion:

$$V_t = \left[(1-\beta)c_t^{1-\rho} + \beta(\mathcal{R}_t V_{t+1})^{1-\rho}\right]^{1/(1-\rho)},$$

where $\mathcal{R}$ is a twisted expectations operator:

$$\mathcal{R}_t V_{t+1} = \left(E_t\left[V_{t+1}^{1-\gamma}\right]\right)^{1/(1-\gamma)}.$$

We define the following adjusted cumulative multiplier:

$$\widetilde{\zeta}(z^t,\eta^t) = \frac{\zeta(z^t,\eta^t)}{M^t(z^t,\eta^t)}.$$

and the $-1/\rho$-th moment of these weights:

$$h(z^t) = \sum_{\eta^t} \widetilde{\zeta}(z^t,\eta^t)^{-\frac{1}{\rho}}\pi(\eta^t|z^t).$$

Proposition E.1. *In the case of Epstein-Zin preferences, the trader's consumption satisfies the following rule:*

$$c(z^t,\eta^t) = \frac{\widetilde{\zeta}(z^t,\eta^t)^{-\frac{1}{\rho}}}{h(z^t)}C(z^t) \tag{E.1}$$

and the pricing kernel is given by:

$$\frac{P(z^{t+1})}{P(z^t)} = \beta\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\rho}\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\rho},$$

The consumption sharing rule and the main aggregation result go through in the case of recursive preferences.

Proof of Proposition E.1:

Proof. This change in preferences would change the first-order condition with respect to consumption $c(z^t,\eta^t)$ (3.8) (which is common to all our asset structures) to

$$\frac{\partial V_0}{\partial c_t} = \zeta(z^t,\eta^t)P(z^t)\pi(z^t,\eta^t) \tag{E.2}$$

where $\zeta(z^t,\eta^t)$ satisfies our multiplier recursion (3.7).

To derive an expression for $\partial V_0/\partial c_t$, note first that

$$\frac{\partial V(z^t,\eta^t)}{\partial c(z^t,\eta^t)} = V(z^t,\eta^t)^{\rho}(1-\beta)c(z^t,\eta^t)^{-\rho},$$

and

$$\frac{\partial V(z^t,\eta^t)}{\partial c(z^{t+1},\eta^{t+1})} = \beta\frac{\partial V(z^t,\eta^t)}{\partial c(z^t,\eta^t)}\left[\frac{V(z^{t+1},\eta^{t+1})}{E_t\left(V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right]^{\rho-\gamma}\left[\frac{c(z^{t+1},\eta^{t+1})}{c(z^t,\eta^t)}\right]^{-\rho}\pi(z^{t+1},\eta^{t+1}|\, z^t,\eta^t).$$

Using the chain rule, these expression imply that

$$\frac{\partial V(z^{t-1},\eta^{t-1})}{\partial c(z^{t+1},\eta^{t+1})} = \left(\frac{V(z^{t-1},\eta^{t-1})}{c(z^{t+1},\eta^{t+1})}\right)^{\rho}\beta^2(1-\beta)M(z^t,\eta^t)M(z^{t+1},\eta^{t+1})\pi(z^{t+1},\eta^{t+1}|z^{t-1},\eta^{t-1}),$$

where

$$M(z^t,\eta^t) = \left[\frac{V_t(z^t,\eta^t)}{\mathcal{R}_{t-1}V_t(z^t,\eta^t)}\right]^{\rho-\gamma}.$$

By backward induction we get that

$$\frac{\partial V_0}{\partial c(z^t,\eta^t)} = V_0^{\rho}\beta^t(1-\beta)M^t(z^t,\eta^t)c(z^t,\eta^t)^{-\rho}\pi(z^t,\eta^t), \tag{E.3}$$

where

$$M^t(z^t,\eta^t) = \Pi_{\tau=0}^t M(z^\tau,\eta^\tau).$$

These results imply that our first-order condition (E.2) can be expressed as

$$V_0^{\rho}\beta^t(1-\beta)M^t(z^t,\eta^t)c(z^t,\eta^t)^{-\rho} = \zeta(z^t,\eta^t)P(z^t).$$

To derive the new expression for the household consumption share which replaces (A.3), note that our first-condition implies that

$$\frac{M^t(z^t,\widetilde{\eta}^t)}{M^t(z^t,\eta^t)}\frac{c(z^t,\widetilde{\eta}^t)^{-\rho}}{c(z^t,\eta^t)^{-\rho}} = \frac{\zeta(z^t,\widetilde{\eta}^t)}{\zeta(z^t,\eta^t)}.$$

This in turn implies our new consumption rule

$$c(z^t,\eta^t) = \frac{\widetilde{\zeta}(z^t,\eta^t)^{-\frac{1}{\rho}}}{h(z^t)}C(z^t) \tag{E.4}$$

where

$$\widetilde{\zeta}(z^t,\eta^t) = \frac{\zeta(z^t,\eta^t)}{M^t(z^t,\eta^t)}.$$

and where

$$h(z^t) = \sum_{\eta^t}\widetilde{\zeta}(z^t,\eta^t)^{-\frac{1}{\rho}}\pi(\eta^t|z^t).$$

To see how this changes the pricing kernel, note that our first-order condition (E.3) implies that

$$\begin{aligned}
\frac{P(z^{t+1})}{P(z^t)} &= \beta\frac{\widetilde{\zeta}(z^t,\eta^t)}{\widetilde{\zeta}(z^{t+1},\eta^{t+1})}\left[\frac{c(z^{t+1},\eta^{t+1})}{c(z^t,\eta^t)}\right]^{-\rho} \\
&= \beta\frac{\widetilde{\zeta}(z^t,\eta^t)}{\widetilde{\zeta}(z^{t+1},\eta^{t+1})}\left[\frac{\widetilde{\zeta}(z^{t+1},\eta^{t+1})^{-\frac{1}{\rho}}}{\widetilde{\zeta}(z^t,\eta^t)^{-\frac{1}{\rho}}}\frac{C(z^{t+1})}{C(z^t)}\frac{h(z^t)}{h(z^{t+1})}\right]^{-\rho} \\
&= \beta\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\rho}\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\rho},
\end{aligned}$$

where we use (E.4). □