Optimal Regulation in the Presence of Reputation Concerns*

Andrew Atkeson† Christian Hellwig‡ Guillermo Ordoñez§

April 2010

Abstract

We study a market with free entry and exit of firms that can costly invest in their quality. If the investment is observable, the first best is characterized by high quality and a large output. If the investment is non observable, free entry creates adverse selection, blocking market existence. If buyers learn over time about firms' quality, reputation formation allows firms to recover their investment in expectation and the market exists. However quality and output are low.

We show that, if the government has the same information as the market and can commit to a schedule of taxes and subsidies, it can achieve an allocation arbitrarily close to the first best. Moreover, even if the government has less information than the market, and for example only observes entry, it can still improve welfare by imposing entry costs. However, in this last case there is a trade-off between quality and output that prevents being close to the first-best.

*We thank Fernando Alvarez, Larry Samuelson, Jean Tirole and seminar participants at Paris school of Economics, Stockholm IIES, Toulouse School of Economics, Yale and the 2009 SED Meetings at Istambul for their comments. We also thanks the Minneapolis Fed for its hospitality. The usual waiver of liability applies. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

†UCLA, Federal Reserve Bank of Minneapolis, and NBER.

‡UCLA and CEPR.

§Yale University.

1 Introduction

The need for government intervention to impose discipline in markets has been frequently discredited by the argument that firms' reputation concerns are enough to self-discipline their behavior. A recent example of this discussion sparked in the aftermath of the 2008 financial crisis. In 1963, the former Federal Reserve chairman, Alan Greenspan wrote "Reputation, in an unregulated economy, is a major competitive tool...Left to their own devices, it is alleged, businessmen would attempt to sell unsafe food and drugs, fraudulent securities, and shoddy buildings...but it is in the self-interest of every businessman to have a reputation for honest dealings and a quality product".[1] Forty five years later, in his remarks before the House of Representatives he declared "Those of us who have looked to the self-interest of lending institutions to protect shareholders equity, myself included, are in a state of shocked disbelief".[2]

In this paper we argue that regulation can in fact leverage on market learning to foster reputation incentives and improve welfare. If the government knows the same as the market, it can achieve an allocation arbitrarily close to the informational unconstrained first best by committing to a scheme of taxes and subsidies that reward firms with good reputations more than what the market is able to without commitment. Furthermore, even if the government knows less than the market, still it can improve welfare by imposing taxes and benefits that move the market forces towards a better exploitation of statistical discrimination across firms that behave differently.

More specifically, we study a market with free entry and exit of firms, where entering firms decide whether to make a costly investment to become high quality or not. High quality firms generate better products on average. When buyers pay the expected utility from consumption, prices increase with reputation, which is defined as the probability buyers assign to the firm being of high quality.

When the investment is observable, the equilibrium provides a first best benchmark in which all entrants invest to be high quality and there is a large output. At the other extreme, when investment is non observable and there is no learning, no entrant invests and the market does not exist. However, if buyers learn about firms' quality over time, for example by experiencing consumption, firms can recover the invest-

[1] "The Assault on Integrity". The Objectivist Newsletter, August 1963

[2] New York Times, "Greenspan Concedes Error on Regulation", October 24th, 2008

ment cost in the future from an expected increase in reputation. In this case there is a market, but characterized by inefficient low quality and low output.

When learning happens, we show that market inefficiency is not generated by the asymmetric information from unobservable investments, but rather because buyers cannot commit to make payments that provide the right incentives. Learning provides the force to restore efficiency, but the market may not be able to take full advantage of it. For example, the payments good reputation firms obtain may not be enough to compensate the time needed to build that reputation. We show that a government with access to a commitment technology, such as the faculty to imposing taxes to firms with poor reputation and subsidies to those with good reputation, can achieve an allocation arbitrarily close to the first best. This payment scheme has the property of introducing stronger incentives at high reputation levels, where they are the most needed, and weaker incentives at lower reputation levels, where they are less needed.

We then show that even when the government knows less than the market, its intervention can still improve welfare. If the government only observes entry, then positive entry costs improve welfare. If additionally the government observes firms' continuation, it can improve welfare further by front-loading costs and back-loading subsidies. Since high quality firms do not exit the market, while low quality firms exit in equilibrium, this policy rewards more in expectation firms that do invest. However, even when this intervention improves welfare, it is still characterized by less quality and less output than the first best benchmark.

Small positive entry costs always increase both quality and output, improving welfare. Contrarily, larger entry costs still increase quality, but at the expense of an output decline. The intuition of this result relies on three properties of firms' expected profits. First, firms' expected profits increase with reputation. Second, the expected profits for high quality firms are higher than for low quality firms, at any reputation level. This is because the reputation of high quality firms is always more likely to increase. Finally, firms' expected profits increase with the aggregate price level in the market, or, which is the same, decline with production.

When entry costs increase, both the firms that invest and the firms that do not invest, require higher expected profits at entry to compensate those costs. Two channels raise expected profits: higher reputation assigned to entrants - in equilibrium higher quality of entrants - and higher aggregate prices - in equilibrium lower output. Any increase in entry costs should be compensated by an increase in initial reputation, or quality of entrants. Initial improvements in initial reputation raises expected profits more for high quality firms than for low quality ones. This is because learning becomes easier and high quality firms gain more from easier learning. This gives room for lower aggregate prices - or higher output - to compensate investment. Additional improvements in initial reputation increase expected profits less for good firms than for bad ones. This is because learning becomes more difficult and bad firms gain relatively more from learning difficulties. Then, only an increase in aggregate prices - or an output reduction - can compensate firms to keep investing.

A technical contribution of the paper is the analytical derivation of the firms' value functions in continuous time when exit is an endogenous choice and firms know their type. We do this for three different processes of signals arrival: Bad news, good news and Brownian motion. This result allows the analytical comparison between entry conditions for high and low quality firms, providing tractability in welfare comparisons across different regulation policies. Furthermore, since in this paper types are not assumed but are the result of investment decisions, we are able to obtain the reputation assigned to entrants and the extent of adverse selection in the market endogenously from entry conditions.

This paper is related to two strands of literature that have not been systematically connected: reputation and regulation. With respect to the reputation strand, Mailath and Samuelson (2001) discusses a reputation model where firms enter at an exogenous reputation to replace those that exogenously die. The work of Tadelis (1999 and 2002) studies the market for names and the endogenous value of reputation as a tradeable asset. Among models with exit of firms, our paper is related to Horner (2002) and Bar-Isaac (2003). However, none of these papers consider a reputation model with both entry and exit decision by firms nor use it to discuss regulation implications. Furthermore our paper does not rely on exogenous types but on initial unobservable choices that determines the extent of adverse selection in the market. From a technical viewpoint, ours is the first paper that fully characterizes value functions with exit decisions in continuous time, when firms know their type and have the option to exit. Related papers of reputation in continuous time are Faingold and Sannikov (2007) and Board and Meyer-ter Vehn (2010) .

With respect to the regulation strand, this paper contributes to Leland (1979), extended later by Shaked and Sutton (1981) and Shapiro (1896), who introduce moral hazard and investment decisions. It also complements von Weizsacker (1980), who discusses how barriers to entry may increase welfare once we consider economies of scale and differentiated products. More recently, Albano and Lizzeri (2001) analyze the efficiency effects of certification intermediaries, but without making reference to reputation concerns and Garcia-Fontes and Hopenhayn (2000) focus on entry restrictions, while we allow for more general regulation possibilities and taxing schemes.

In the next Section we describe the economy and solve the informational unconstrained first best allocation. In Section 3 we discuss the unregulated market solution. In Section 4 we show the role of regulation in filling the gap between the market solution and the first best. Finally, in Section 5 we make some final remarks.

2 The Model

In this Section we start describing endowments, technologies, preferences and market characteristics. Then we characterize two benchmarks determined by opposite information assumptions about firms' initial investment. First, we obtain the optimal allocation in the economy when investment is publicly observable and we show this is the solution of unregulated markets in a decentralized equilibrium. This result provides the full information benchmark. Second, we discuss the opposite situation with asymmetric information, in which only firms observe their own investment and buyers do not have memory, preventing learning about firms' decisions. The solution is a complete shut down of the market, generated by an extreme endogenous adverse selection problem.

2.1 The Economy

Time is continuous and denoted by $t \in [0, \infty)$. There is a representative household in the economy. At each moment, this household derives utility from the consumption of two final goods: one which we term the "experience" good and one which we term the "numeraire" good. Let Y_t denote the household's consumption of the experience good and N_t its consumption of the numeraire good at t. The household's utility is given by

$$\int_t e^{-\hat{r}t}\left[\frac{1}{1-\eta}Y_t^{1-\eta}+N_t\right] \tag{1}$$

where $\hat{r}$ is the discount factor and $\eta \in (0,1)$. At each time t, there is an endowment of 1 unit of the numeraire good. This good is not storable.

The experience good is produced using intermediate goods generated by a continuum of firms, each of which can produce one unit of this intermediate good at each time t at zero cost. The contribution of any single unit of the intermediate good to the production of the experience good at t is stochastic depending on its quality. If a unit of the intermediate good produced at t is "good" (g) it contributes one unit to the production of the experience good at t. If it is "bad" (b) it subtracts from, or destroys, κ units of the experience good produced at t.[3]

Each of the firms producing the intermediate good at t can be one of two types — High quality (H) or Low Quality (L). The probability that the output of a given firm at t is good depends on the firm's type, $Pr(g|H) = \alpha_H > Pr(g|L) = \alpha_L$. Let ϕ denote the public belief regarding the probability that a given intermediate goods firm is high quality, which we call its reputation. The expected contribution to the production of the experience good of a unit of the intermediate good produced by that firm is denoted $y(\phi)$ and is given by the affine function $y(\phi) = [\alpha_H\phi + \alpha_L(1-\phi)] + [(1-\alpha_H)\phi + (1-\alpha_L)(1-\phi)](-\kappa)$.

$$y(\phi) = a_1\phi - a_0 \tag{2}$$

where $a_1 = (\alpha_H - \alpha_L)(1+\kappa)$ and $a_0 = \kappa - \alpha_L(1+\kappa)$, that we assume positive. Hence the expected contribution to the production of the experience good of a unit of the intermediate good produced by a high quality firm is positive ($y(1) > 0$) while the expected contribution to the production of the experience good from a unit of the intermediate good produced by a low quality firm is negative ($y(0) < 0$).

For $\phi \in [0,1]$, let $m_t(\phi)$ denote the measure of units of the intermediate good produced at t by firms that are high quality with probability ϕ, or which is the same, the stock of intermediate goods firms having a reputation ϕ. The resource constraint for the

[3]The assumptions of zero production costs and negative contribution of "bad" units are just a normalization that simplify the exposition. Assuming positive production costs and contribution of "bad" units deliver the same results but make equations less straightforward.

experience good is then given by

$$Y_t = \int_\phi y(\phi) m_t(\phi) d\phi \tag{3}$$

There are free entry and exit of intermediate goods firms. In order to cease production, intermediate goods firms choose a rate of exit $\omega_t^i(\phi) \in [\delta, \bar{\omega}]$, where $\delta > 0$ is an exogenous exit probability and $\bar{\omega}$ is a parameter to keep the evolution of the measure of firms well defined.

At each moment t, new intermediate goods firms can enter and start production. To create a new, high quality, intermediate goods firm, an investment of C units of the numeraire good is required. New low quality intermediate goods firms can be created at zero cost. We denote $m_t^e \in [0, \bar{m}^e]$ the flow of new intermediate goods firms at t and $\phi_t^e \in [0, 1]$ the fraction of those new firms that invest to be high quality. Since firms are homogenous before the investment decision, the reputation assigned to entrants at moment t is ϕ_t^e. Then the resource constraint for the numeraire good is

$$N_t = 1 - C\phi_t^e m_t^e. \tag{4}$$

Finally, with respect to the market structure, we assume the experience and intermediate goods are transacted at spot prices in each moment t. The price of the experience good is then its marginal utility $Y_t^{-\eta}$. We also assume that producers of the experience good compete to purchase a unit of intermediate good offered by each intermediate goods firm so that the price that each intermediate goods firm charges for its output reflects the full surplus that this unit of the intermediate good contributes to the output of the experience good. Specifically, the price at t in units of the numeraire good, for a unit of the intermediate good produced by a firm that is believed to be of high quality with probability ϕ is given by its expected value of the marginal product at t,

$$p_t(\phi) = y(\phi) Y_t^{-\eta} \tag{5}$$

2.2 Full Information Benchmark

Here we assume consumers observe the initial investment, this is, whether firms pays C to become high quality or not. First, we describe the optimal allocation in this economy as a planning problem. Then we show the solution of the planner's problem is also the solution of an unregulated market decentralized equilibrium.

2.2.1 The planner's problem

When investment is observable, the measure of intermediate goods firms $m_t(\phi)$ at each t is described by two numbers, $m_t(1)$ and $m_t(0)$. We assume that an initial stock of high quality firms $m_0(1) > 0$ is given.

At each t, the planner chooses the measure of entering firms $m_t^e \geq 0$, the fraction $\phi_t^e \in [0,1]$ of those entering firms that invest C to become high quality, the continuation probabilities $\omega_t^H(1)$ and $\omega_t^L(0)$, the production of intermediate goods $m_t(1) \geq 0$ and $m_t(0) \geq 0$ consumption of the experience and numeraire goods Y_t and N_t to maximize welfare (1) subject to the following constraints: (3), (4),

$$dm_t(1) = [-\omega_t^H(1)m_t(1) + \phi_t^e m_t^e]dt \tag{6}$$

and

$$dm_t(0) = [-\omega_t^L(0)m_t(0) + (1-\phi_t^e)m_t^e]dt \tag{7}$$

Clearly, since the output of a firm known to be low quality is expected to subtract from production of the experience good ($y(0) < 0$), it is optimal to set $\omega_t^L(0) = \bar{\omega}$ and $\phi_t^e = 1$. Likewise, since an existing firm known to be of high quality can contribute $y(1)$ to production of the experience good at zero cost as long as it continues, it is optimal to set $\omega_t^H(1) = \delta$, its minimum value.

The cost, in terms of utility, of creating a new firm at t with probability $\phi_t^e = 1$ of being high quality is given by C while the benefit is given by

$$\int_{s\geq 0} e^{-rs} Y_{t+s}^{-\eta} y(1) ds$$

where $r = \hat{r} + \delta$ is the effective discount rate. Hence, an allocation with constant consumption of the experience good at level $Y_t = \bar{Y}$ where

$$y(1)\bar{Y}^{-\eta} = Cr \tag{8}$$

is optimal. Then, there is an optimal stock of intermediate good producers determined by equation (3), $\bar{m}(1) = \bar{Y}/y(1)$

The optimal choice of entry m_t^e is a dynamic choice. If $y(1)m_0(1)$ is less than this optimal level $\bar{Y}$, the planner creates new high quality firms at the maximum rate possible ($\bar{m}^e$) until the stock $\bar{m}(1)$ of high quality firms is attained. If $y(1)m_0(1)$ exceeds this optimal level, the planner creates no new intermediate goods firms until the stock of existing high quality firms has depreciated down to this level at rate δ. Once this stock of high quality intermediate goods firms $\bar{Y}$ is attained, the planner imposes a flow of new firms $m^e = \delta\bar{m}(1)$ to maintain the stock at a constant level.

2.2.2 Decentralized equilibrium

In what follows we focus on a decentralized stationary equilibrium, with constant consumption of the experience good $Y_t = Y$ for all t. Firms choose whether to invest or not upon entry and when to exit, decisions that are publicly observable. If a new firm invests, it will be able to charge a price $p_t(1) = y(1)Y_t^{-\eta} > 0$ for its products as long as it remains in the market. If the firm does not invest, it will be able to charge at most a price $p_t(0) = y(0)Y_t^{-\eta} < 0$ at each t.

Given these prices, and since production costs are zero, high quality firms remains in the market as much as possible, exiting only exogenously at rate δ. Contrarily, low quality firms exit as fast as possible, with probability $\bar{\omega}$. Since there is a probability $1 - \bar{\omega}$ at each t low quality firms remain in the market making loses, no firm would enter without investing, then $\phi^e = 1$. The total production of the experience good in steady state is determined by entry until the point in which the free entry condition for firms is fulfilled with equality, which is also given by equation 8.

Hence, the decentralized equilibrium for an unregulated economy when investment is observable achieves the social planner allocation, characterized by high investment, $\phi^e = 1$, and high production, $\bar{Y}$.

2.3 No Learning Benchmark

As an opposite benchmark, in the extreme case of non-observable investment and no learning (outcomes and investments are not publicly observable and recorded, for example), free entry blocks entirely the existence of the market for intermediate goods and hence consumption of the experience good.

The reason is simple. In order to recover the initial investment, new intermediate goods firms that invest should expect discounted profits for at least C. Since investment is non observable and there is no learning, all entering firms have the same expected profits, regardless of their investment decisions, naturally preferring not to invest. Forecasting this outcome, the experience good producers are willing to buy only at negative prices ($p(0) < 0$). The only equilibrium is given by no entry of intermediate goods producers and $Y = 0$. This is an extreme version of 1970 Akerlof's lemons problems, in which the fraction of high quality firms is not given exogenously but endogenously determined by investment decisions at the moment of entry.

3 Reputation in an Unregulated Market

In this section we assume that public signals about each intermediate goods firm's quality are revealed over time as long as the firm continues in operation, so that it is possible for experience goods producers to learn over time whether an intermediate goods firm has invested or not upon entry.

In a stationary equilibrium, consumption aggregates Y_t and N_t are constant, as is entry flows m_t^e, the fraction ϕ_t^e of entrants who invest and become high quality, and the stock $m_t(\phi)$ of intermediate goods with belief ϕ about their quality. As above, profits are linear in ϕ and scale with $Y^{-\eta}$. This is

$$\pi(\phi) = p(\phi) = y(\phi)Y^{-\eta} = (a_1\phi - a_0)Y^{-\eta}$$

Total discounted profits in steady state are then

$$\Pi_i(Y, \phi^e) = Y^{-\eta} E_{i|\phi^e} \int_{s=0}^{\tau_i(\phi^e)} e^{-rs} y(\phi_{t+s}) ds = Y^{-\eta} V_i(\phi^e)$$

where $i \in \{L, H\}$ and $\tau_i(\phi^e)$ represents the expected exiting moment for a firm i that starts with reputation ϕ^e.

As can be seen, the effects of the reputation prior ϕ^e (that determines the future evolution of reputation) and the effects of production Y (that determined the level of profits), can be separated in the discounted expected profits. This convenient property allows us to focus on the derivation of value functions when $Y = 1$ (i.e., $\Pi_i(1, \phi^e) = V_i(\phi^e)$) which summarizes the expected evolution of reputation, learning that depends only on the true type of the firm, $i \in \{L, H\}$, and on the initial prior assigned to the firm, ϕ^e.

To compute these value functions we should consider both learning properties and optimal stopping decisions. Before deriving them formally under different information structures, we briefly describe two of their main properties and discuss the optimal stopping process.

Regardless the learning structure, two intuitive properties of the value functions are: i) They are increasing in ϕ^e, since profits are increasing in reputation and ii) value functions for firms H are always larger than those for firms L for any reputation $\phi \in (0, 1)$. This is because high quality firms expect their reputation to grow while low quality firms expect their reputation to decline.

The stopping point is characterized by a lower bound reputation level $\bar{\phi}$ at which low quality firms are indifferent between exiting or not, randomizing exiting, while high quality firms strictly prefer to continue (given the second property mentioned above). The reason is the following: if the reputation of a low quality firm drifts below $\bar{\phi}$, it does not want to continue, however it is not an equilibrium to exit. This is because the market would infer in this case all continuing firms with reputation below $\bar{\phi}$ are high quality, in which case low quality firms would like to continue. The only equilibrium is such that low quality firms randomize for their reputation to stay exactly at $\bar{\phi}$, point at which they are indeed willing to randomize.

This equilibrium implies that $\bar{\phi}$ is indeed the lowest reputation sustained by the market and observed in the market, being characterized by the following equation.

$$V_L(\bar{\phi})Y^{-\eta} = 0 \tag{9}$$

Now, we show the solution of the unregulated market, which is completely characterized by two free entry conditions, and show this solution delivers and allocation strictly between the full information and the no-learning benchmarks discussed previously. Next, we formally obtain the value functions $V_i(\phi)$ with optimal stopping for general signal structures and general profit functions. The properties of these value functions are important both to analyze the distance between the market and the first best under different specifications of the information structure and to discuss what a government can do to make this distance shorter.

3.1 Free Entry and the Solution of the Unregulated Market

The quality of entrants ϕ^e and the production of the experience good Y in the stationary equilibrium are determined by the following free entry conditions.

$$\begin{aligned} V_H(\phi^e)Y^{-\eta} &\geq C \\ V_L(\phi^e)Y^{-\eta} &\geq 0 \end{aligned} \tag{10}$$

It is clear that both conditions bind. If the free entry condition for high quality firms is not binding, $\phi^e = 1$. Since $V_H(1) = V_L(1) = \frac{y(1)}{r}$, the condition for low quality firms is not binding either. Hence $\phi^e = 1$ is not an equilibrium. If the free entry condition for low type firms is not binding, $\phi^e = 0$. However, since $y(0) < 0$ there is no entry and $\phi^e = 0$ is not an equilibrium either. Hence the free entry conditions are characterized by both equations in expression (10) binding and $\phi^e \in (0, 1)$.

We obtain the average quality of entrants ϕ^e from the entry condition for the low type. By construction, from equation (9), $\phi^e = \bar{\phi}$. Hence, without regulation, new firms enter with the lowest possible reputation sustainable in the market, this is, the reputation at which low quality firms are willing to exit.

The stationary consumption of the experience good in the unregulated market economy Y_M, is determined by the difference between free entry conditions,

$$Y_M = \left[\frac{V_H(\bar{\phi}) - V_L(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} \tag{11}$$

while the optimum stationary production of the experience good in the full information benchmark $\bar{Y}$ from equation (8) is,

$$\bar{Y} = \left[\frac{y(1)}{rC}\right]^{\frac{1}{\eta}} = \left[\frac{V_H(1)}{C}\right]^{\frac{1}{\eta}} > Y_M \tag{12}$$

The market solution is clearly suboptimal when compared to the unconstrained benchmark, with less quality of entrants ($\phi^e = \bar{\phi} < 1$) and less production of the experience good ($Y_M < \bar{Y}$).

3.2 Value Functions for Different Information Structures

We consider stationary equilibria under three alternative assumptions about the structure of signals of quality: *bad news, good news, and Brownian Motion*. In the bad news case, if the firm is of low quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the firm is high quality. In the good news case, the assumption is reversed — if the firm is of high quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the firm is low quality. Finally, in the Brownian Motion case, signals about firm quality arrive continuously. Specifically,

$$dS_t = \mu_i dt + \sigma dZ_t$$

where $i = \{H, L\}$, S_t is a Brownian motion with drifts that depend on the firm's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

In the Brownian motion case, if learning only arises from experiencing the intermediate inputs, $\mu_H = \alpha_H$ and $\mu_L = \alpha_L$ and σ is interpreted as noise in production. For example, high quality firms are expected to produce a fraction α_H of good products, but this production is hit by shocks with variance σ. However, in what follows we allow for more general sources of information. Signals can arrive not only from experience production but also from rumors about investment, from filtration of firms' accounting data, from noisy observations about firms' activities, etc.

Similarly, in the good news case, if learning only happens from experiencing the intermediate inputs, we can assume buyers can only observe good products generated by high quality firms. In this case $\lambda = \alpha_H$. In the bad news case, for example, we can assume buyers can only observe bad products generated by low quality firms. In this case $\lambda = 1 - \alpha_L$. However, as mentioned above, we obtain the results for a general λ, allowing for more general sources of information.

Since production Y only scales the expected discount profits in the stationary equilibrium, we focus on obtaining the value functions for $Y = 1$, this is $V_i(\phi^e)$.

3.2.1 Bad News

In this case $dS_t \in \{0, 1\}$, which means either there is a signal or not at each t. The bad news case is defined by $Pr(dS_t = 1|H) = 0$ and $Pr(dS_t = 1|L) = \lambda$, which means there is a positive Poisson arrival only for low quality firms. When a signal arrives the producer of intermediate goods is revealed to be of low quality and hence the public belief about its quality jumps down to $\phi = 0$. With this reputation, the firm would never be able to sell its output at a non-negative price. Thus, following this event, it is optimal for the firm to cease production and exit as quickly as possible.

It is convenient in this case for us to use a transformed variable $l = (1-\phi)/\phi : [0, 1] \rightarrow (\infty, 0]$ to summarize the reputation level of an intermediate goods firm. The evolution of l is determined by,

$$\frac{dl_t}{dt} = \left[\frac{Pr(dS_t|L) - Pr(dS_t|H)}{Pr(dS_t|H)}\right] l_t$$

When bad news arrive (i.e., $dS_t = 1$)

$$\frac{dl_t}{dt} = \left[\frac{\lambda - 0}{0}\right] l_t = \infty$$

and reputation jumps immediately to $l = \infty$. Since $\phi = \frac{1}{1+l}$, this means reputation drops down immediately to $\phi = 0$.

While there are no news (i.e., $dS_t = 0$), reputation increases. After t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_0 = .. = S_t = 0|L)}{Pr(S_0 = .. = S_t = 0|H)}\right] l_0 = \left[\frac{(1-\lambda)^t}{1}\right] l_0 = (1-\lambda)^t l_0$$

where $l_0 = \frac{1-\phi^e}{\phi^e}$. This means l_t is decreasing (reputation is increasing) over time at a rate $1 - \lambda$.

While there are no news, the evolution of reputation for firms with high and low quality is the same. After bad news, a firm exits and obtains zero thereafter.[4] Then, the value functions for both types only differ in their discount factor.

[4]For simplicity, since in this case exit only occurs after bad news, we assume $\bar{\omega} = 1$.

Proposition 1 *Value functions for general profit functions and bad news*

A value function for a low quality firm with reputation l, for a general $\pi(l(\phi))$, is

$$V_L(l) = \int_{s=0}^{\infty} e^{-(r+\lambda)s}\pi((1-\lambda)^s l)ds$$

and the value function for a high quality firm with reputation l is

$$V_H(l) = \int_{s=0}^{\infty} e^{-rs}\pi((1-\lambda)^s l)ds$$

Solving explicitly the integrals for the case of linear payoffs and no marginal costs, $\pi(\phi) = a_1\phi - a_0$ (hence $\pi(l) = \frac{a_1}{1+l} - a_0$),

$$V_L(l) = \frac{1}{r+\lambda}\left[a_1 \Upsilon_{m_{r+\lambda}}(-l) - a_0\right] \tag{13}$$

$$V_H(l) = \frac{1}{r}\left[a_1 \Upsilon_{m_r}(-l) - a_0\right] \tag{14}$$

where $\Upsilon_m(-l) =_2 F_1(1, m; m+1, -l)$ is an hypergeometric function,

$$m_r = -\frac{r}{ln(1-\lambda)} > 0 \qquad \texttt{and} \qquad m_{r+\lambda} = -\frac{r+\lambda}{ln(1-\lambda)} > m_r > 0$$

The hypergeometric function has well defined properties when $m > 0$. In particular, it is monotonically increasing in ϕ from 0 to 1.

$$\Upsilon_m\left(-\frac{1-\phi}{\phi}\right) : [0,1] \to [0,1]$$

Now we denote $V_i(\phi) = V_i(l(\phi))$ for all ϕ and $i \in \{L, H\}$. Since $\lim_{\phi\to 0} V_L(\phi) = -\frac{a_0}{r+\lambda} < 0$, there is a $\phi = \bar{\phi}$ such that $V_L(\bar{\phi}) = 0$. This is the reputation at which low quality firms are indifferent between exiting or not. As discussed above, exiting strategies imply that in equilibrium, no firm ha sa reputation below $\bar{\phi}$. Hence we just discuss properties of value functions in the range $[\bar{\phi}, 1]$.

$$V_L(\phi) \;:\; [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r+\lambda}]$$
$$V_H(\phi) \;:\; [\bar{\phi}, 1] \to [V_H(\bar{\phi}), \frac{a_1 - a_0}{r}]$$

where $V_H(\bar{\phi}) = \frac{1}{r}\left[a_1 \Upsilon_{m_r}\left(-\frac{1-\phi}{\phi}\right) - a_0\right] > 0$.

Three properties of these value functions (directly from the properties of hypergeometric functions) are summarized in the following proposition,

Proposition 2 *Properties of value functions with linear profits and bad news.*

- $V_H(\phi) > V_L(\phi)$ *for all* ϕ.
- $\frac{\partial \frac{V_L(\phi)}{V_H(\phi)}}{\partial \phi} > 0$*, so the ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ.
- $\frac{\partial (V_H(\phi) - V_L(\phi))}{\partial \phi} > 0$*, so the difference* $V_H(\phi) - V_L(\phi)$ *is increasing in* ϕ.

3.2.2 Good News

In this case $Pr(dS_t = 1|H) = \lambda$ and $Pr(dS_t = 1|L) = 0$. When a signal arrives the intermediate goods firm is revealed to be of high quality and hence the public belief ϕ regarding this firm jumps up to $\phi = 1$. After good news the firm does not lose its reputation anymore.

Again, we use the variable $l = (1-\phi)/\phi$. When good news arrive (i.e., $dS_t = 1$)

$$\frac{dl}{dt} = \left[\frac{0-\lambda}{\lambda}\right] l_t = -l_t$$

and reputation jumps immediately to $l = 0$, or $\phi = 1$.

While there are no news (i.e., $dS_t = 0$), reputation decreases. After t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_0 = .. = S_t = 0|L)}{Pr(S_0 = .. = S_t = 0|H)}\right] l_0 = \left[\frac{1}{(1-\lambda)^t}\right] l_0 = (1-\lambda)^{-t} l_0$$

which means l_t is increasing (reputation is decreasing) over time at a rate $\frac{1}{1-\lambda}$.

Denoting $\pi(l(1))$ the payoffs for a firm with $\phi = 1$, the value function for a firm that generates good news is,

$$V(l(1)) = \frac{\pi(l(1))}{r}$$

There is a key difference between good news and bad news. Under bad news, reputation only increases, which means exit never occurs, unless a bad signal is revealed. Under good news, reputation continuously decrease and firms that hit $\bar{\phi}$ will exit with probability λ such that ϕ is never below $\bar{\phi}$.

Proposition 3 *Value functions for general profit functions and good news*

The value function for a low quality firm with reputation l is

$$V_L(l) = \int_{s=0}^{T(l)} e^{-rs} \pi((1-\lambda)^{-s} l) ds \tag{15}$$

The value function for a high quality firm with reputation l is

$$V_H(l) = \int_{s=0}^{T(l)} e^{-(r+\lambda)s} \left[\pi((1-\lambda)^{-s} l) + \lambda \frac{\pi(l(1))}{r} \right] ds + \int_{s=T(l)}^{\infty} e^{-(r+\lambda)s} \lambda \frac{\pi(l(1))}{r} ds \tag{16}$$

where $T(l)$ is the time required for l to increase up to $\bar{l} = \frac{1-\bar{\phi}}{\bar{\phi}}$.

$$T(l) = \frac{log(l) - log(\bar{l})}{log(1-\lambda)} \tag{17}$$

In the case of linear payoffs and no marginal costs, the reputation at which low quality firms are willing to exit is given by $\pi(\bar{l}) = \frac{a_1}{1+\bar{l}} - a_0 = 0$. In this case, $\bar{l}$ is given by the reputation above which profits are negative. Then $\bar{l} = \frac{a_1 - a_0}{a_0}$ and $T(l)$ is given following equation (17). Value functions are,

$$\begin{aligned} V_L(l) &= \frac{1}{r}\left[a_1 \left(1 - \Upsilon_{m_r}\left(-\frac{1}{l} \right) \right) - a_0 \right] \\ &\quad - \frac{e^{-rT(l)}}{r} \left[a_1 \left(1 - \Upsilon_{m_r}\left(-\frac{a_0}{a_1 - a_0} \right) \right) - a_0 \right] \end{aligned} \tag{18}$$

$$\begin{aligned} V_H(l) &= \frac{1}{r+\lambda}\left[a_1 \left(1 - \Upsilon_{m_{r+\lambda}}\left(-\frac{1}{l} \right) \right) - a_0 + \lambda \frac{a_1 - a_0}{r} \right] \\ &\quad - \frac{e^{-(r+\lambda)T(l)}}{r+\lambda} \left[a_1 \left(1 - \Upsilon_{m_{r+\lambda}}\left(-\frac{a_0}{a_1 - a_0} \right) \right) - a_0 \right] \end{aligned} \tag{19}$$

Now we denote $V_i(\phi) = V_i(l(\phi))$ for all ϕ and $i \in \{L, H\}$. Since $T(l(1)) = \infty$, using the previously discussed properties of the hypergeometric functions,

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r}] \\ V_H(\phi) &: [\bar{\phi}, 1] \to [\frac{\lambda}{r+\lambda}\frac{a_1 - a_0}{r}, \frac{a_1 - a_0}{r}] \end{aligned}$$

Proposition 4 *Properties of value functions with linear profits and good news.*

- $V_H(\phi) > V_L(\phi)$ *for all* $\phi < 1$ *and* $V_H(1) = V_L(1)$.
- $\frac{\partial(V_L(\phi)/V_H(\phi))}{\partial\phi} > 0$, *so the ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ.
- $\frac{\partial(V_H(\phi)-V_L(\phi))}{\partial\phi} < 0$, *so the difference* $V_H(\phi) - V_L(\phi)$ *is decreasing in* ϕ.

3.2.3 Brownian Motion

Assume now the signal process follows a Brownian motion

$$dS_t = \mu_i dt + \sigma dZ_t$$

where $i = \{H, L\}$, drifts depend on the firm's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

The following Proposition shows reputation, both for high and low quality firms, also follows a Brownian motion. The proof is in the Appendix.

Proposition 5 *The reputation process high quality firms expect is a submartingale*

$$d\phi_t^H = \frac{\lambda^2(\phi_t)}{\phi_t}dt + \lambda(\phi_t)dZ_t \tag{20}$$

and the reputation process low quality firms expect is a supermartingale

$$d\phi_t^L = -\frac{\lambda^2(\phi_t)}{(1-\phi_t)}dt + \lambda(\phi_t)dZ_t \tag{21}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ *and* $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ *is the signal to noise ratio.*

Four clear properties arise from inspecting equations (20) and (21). First, high quality firms expect a positive drift in their evolution of reputation while low quality firms expect a negative drift. Second, when reputation ϕ_t is either 0 or 1, drifts and volatilities are zero, which means at those points reputation do not change, both for high and low quality firms. Third, reputation varies more at intermediate levels of ϕ_t, and volatilities are larger. Finally, the drift for high quality firms is higher than for low quality firms for bad reputations and lower for good reputations, since ϕ_t is in the denominator of the drift for high quality firms, while $(1-\phi_t)$ is in the denominator of the drift for low quality firms.

We can now write the ordinary differential equations that characterize the value functions for high and low quality firms. The discussion about the determination of these ODEs is in the Appendix.

$$r\rho V_L\left(\phi\right) = \rho\pi\left(\phi\right) - \phi^2\left(1-\phi\right)V_L'\left(\phi\right) + \frac{1}{2}\phi^2\left(1-\phi\right)^2 V_L''\left(\phi\right) \tag{22}$$

$$r\rho V_H\left(\phi\right) = \rho\pi\left(\phi\right) + \phi\left(1-\phi\right)^2 V_H'\left(\phi\right) + \frac{1}{2}\phi^2\left(1-\phi\right)^2 V_H''\left(\phi\right) \tag{23}$$

where

$$\rho = \frac{\sigma^2}{\left(\mu_H - \mu_L\right)^2}$$

Solving these ODEs, we can obtain the value functions for high and low quality firms. The discussion about the determination of these value functions is in the Appendix.

Proposition 6 *Value functions for general profit functions and Brownian motion*

The value function of low quality firms with reputation l is

$$V_L\left(l\right) = K\left\{\int_0^1 \theta^{\gamma-1}\pi\left(\theta l\right)d\theta - \int_{\chi/l}^1 \theta^{-\gamma}\pi\left(\theta l\right)d\theta\right\} \tag{24}$$

The value function of high quality firms with reputation l is

$$V_H\left(l\right) = K\left\{\int_0^1 \theta^{\gamma-2}\pi\left(\theta l\right)d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi\left(\theta l\right)d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\} \tag{25}$$

where $\theta = l'/l$,

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \texttt{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}$$

Boundary conditions (value matching and smooth-pasting for low and high types) must be satisfied at $\bar{l}$. These conditions jointly determine $\bar{l}$, χ and ψ:[5]

$$V_L\left(\bar{l}\right) = V_L'\left(\bar{l}\right) = V_H'\left(\bar{l}\right) = 0$$

Using the formal expressions of the value functions and derivatives (in the Appendix),

$$\begin{aligned} V_L(\bar{l}) &= 0 \Rightarrow \int_{\chi/\bar{l}}^{1} \theta^{-\gamma}\pi\left(\theta\bar{l}\right) d\theta = \int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right) d\theta \\ \bar{l}V_L'(\bar{l}) &= 0 \Rightarrow (1-\gamma)\int_{\chi/\bar{l}}^{1} \theta^{-\gamma}\pi\left(\theta\bar{l}\right) d\theta = \gamma\int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right) d\theta \end{aligned}$$

Combining the two conditions, we find the equation that pins down $\bar{l}$:

$$\int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right) d\theta = 0 \tag{26}$$

and the equation that pins down χ

$$\int_{\chi/\bar{l}}^1 \theta^{-\gamma}\pi\left(\theta\bar{l}\right) d\theta = 0 \tag{27}$$

Finally, the condition that pins down ψ is

$$(1-\gamma)\int_0^1 \theta^{\gamma-2}\pi(\theta\bar{l})d\theta = \gamma\left[\int_{\psi/\bar{l}}^1 \theta^{-\gamma-1}\pi(\theta\bar{l})d\theta - \frac{\pi(0)}{\gamma}\left(\frac{\psi}{\bar{l}}\right)^{-\gamma}\right] \tag{28}$$

These expressions completely characterized value functions and the reputation at which low quality firms exit. The next Proposition lists the main properties. The

[5]Value matching and smooth pasting conditions for low quality firms arises from optimal exiting decisions and the smooth pasting condition for high quality firms arises from a belief process that is reflecting at $\bar{\phi}$

proof is in the Appendix.

Proposition 7 *Properties of value functions with linear profits and Brownian motion.*

- $V_H(\phi) > V_L(\phi)$ *for all* $\phi \in [\bar{\phi}, 1)$ *and* $V_H(1) = V_L(1)$.
- $\frac{\partial (V_L(\phi)/V_H(\phi))}{\partial \phi} > 0$, *so the ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ.
- $\frac{\partial (V_H(\phi) - V_L(\phi))}{\partial \phi} > 0$ *for* $\phi \in (\bar{\phi}, \phi^*)$ *and* $\frac{\partial (V_H(\phi) - V_L(\phi))}{\partial \phi} < 0$ *for* $\phi \in (\phi^*, 1]$. *So, the gap* $V_H(\phi) - V_L(\phi)$ *is increasing from* $\bar{\phi}$ *to* ϕ^* *and decreasing from* ϕ^* *to 1, where* ϕ^* *is characterized by* $\frac{\partial V_H(\phi^*)}{\partial \phi} = \frac{\partial V_L(\phi^*)}{\partial \phi}$. *By construction,* $\frac{\partial V_H(\bar{\phi})}{\partial \phi} = \frac{\partial V_L(\bar{\phi})}{\partial \phi} = 0$

3.3 Numerical Exercise

In this section we provide a numerical illustration of the value functions derived above under linear profits for the bad news, good news and Brownian motion cases.

We assume the following parameters $r = \hat{r} + \delta = 0.1$, $\alpha_H = 0.9$, $\alpha_L = 0.7$ and $\kappa = 3$ (hence $a_1 = 0.8$ and $a_0 = 0.2$). We also assume $C = 1$ and $\eta = 0.5$.

The first best is characterized by $\phi^e = 1$ and

$$\bar{Y} = \left[\frac{a_1 - a_0}{rC} \right]^{\frac{1}{\eta}} = 6^2 = 36$$

3.3.1 Bad and Good news

We assume $\lambda = 0.1$ in both cases. With bad news, the market result is characterized by the value functions in Figure 1, $\phi^e = \bar{\phi} = 0.16$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{C} \right]^{\frac{1}{\eta}} = 0.9^2 = 0.8$$

With good news, the market result is characterized by the value functions in Figure 2, $\phi^e = \bar{\phi} = 0.25$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{C} \right]^{\frac{1}{\eta}} = 3^2 = 9$$

Figure 1: Value Functions - Bad News

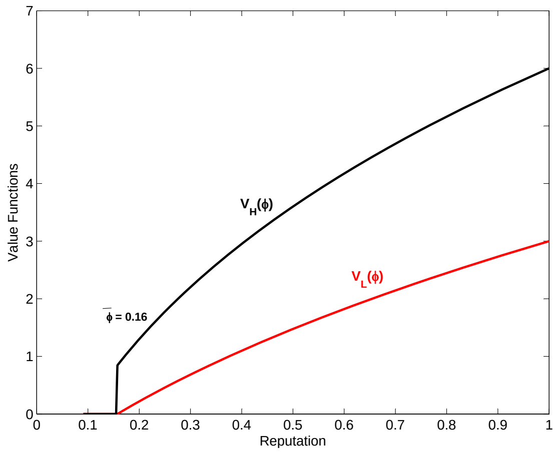
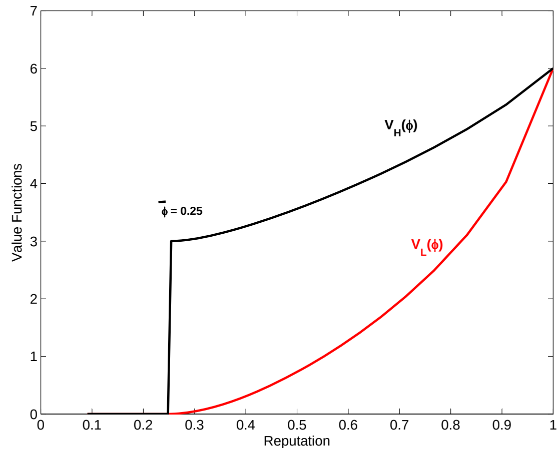


Figure 2: Value Functions-Good News



When a market is characterized by good news, it achieves an allocation closer to the first best than a market characterized by bad news. Since quality and production under good news are higher, markets characterized by this signalling structure have an unequivocally higher welfare.

In the case of good news, firms enter with reputation $\phi^e = 0.25$. High quality firms operate at that reputation until they generate a good signal and jumps to operate at reputation $\phi = 1$. Low quality firms always operate at reputation $\phi = 0.25$ and randomizes exiting with probability λ. In an unregulated economy, firms enter at the reputation with the maximum gap between value functions.

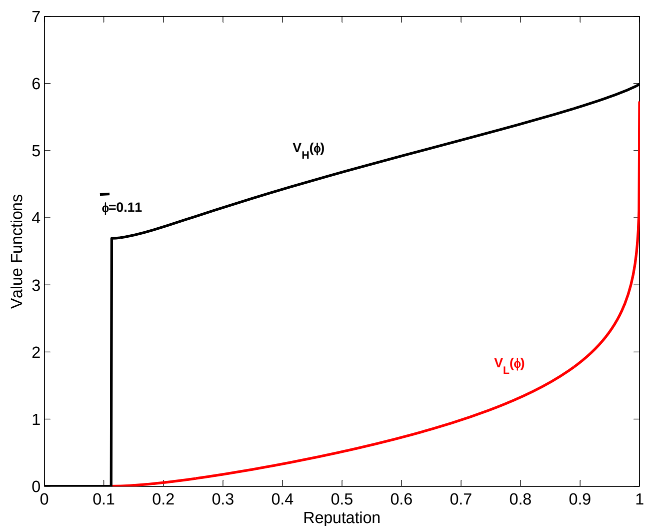
In the case of bad news, firms enter with a lower reputation $\phi^e = 0.16$. This is because reputation increases with time, allowing bad firms to be able to reap some profits in the future and motivating them to start at lower reputation. High quality firms experience an always increasing reputation and want to stay. Low quality firms also experience this increase, but with the probability λ of generating a bad news and exiting. Since entry happens at a reputation level with the smallest gap between value functions, the production in the market is also very low.

3.3.2 Brownian motion

We assume $\sigma = 0.2$, $\mu_H - \mu_L = 0.2$ (hence $\rho = 1$). The market result is characterized by the value functions in Figure 3. In this case $\phi^e = \bar{\phi} = 0.11$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{rC} \right]^{\frac{1}{\eta}} = 3.7^2 = 14$$

Figure 3: Market Value Functions



The results on this case cannot be compared with bad and good news cases so directly. However we can see that value functions combine those cases. As in the bad news cases, the gap between value functions increases at lower reputation levels. As in the good news cases, the gap between value functions decreases at higher reputation levels and are the same at $\phi = 1$. As in those cases, the unregulated market generates an inferior allocation when compared to the full information first best.

4 Optimal Regulation

Now we assume there is a government that can impose taxes or give subsidies to each intermediate goods firm, which are naturally only conditional on information available to the government. First, we show that if the government knows the same as the market, and hence can condition taxes and subsidies on reputation, then it can achieve an allocation arbitrarily close to the full information first best. Second, we show that even if the government knows much less than the market and only observes entry, still it can improve welfare by imposing an optimal entry cost.

4.1 The Government Knows the Same as the Market

First, we assume the government has the same information than the market. In particular it can observe the reputation of each firm. It is not necessary it observes the evolution of the public signal, since it can infer the reputation, for example, just by observing firms' revenues. We show the government can achieve an allocation arbitrarily close to the full information benchmark by committing to a scheme of payments that distort revenues via taxes and subsidies, paying less than the market to bad reputation firms and more to good reputation firms. To abstract from budget balance considerations we assume the resources to pay subsidies are obtained by lump sum taxes and that taxes to firms go back to consumers as lump sum subsidies.

4.1.1 Bad and Good News

In the case of bad news, we assume the government has two instruments - entry costs F and a per period subsidy f_B - to target $\phi^e = 1$ and $\bar{Y}$. From entry conditions,

$$\frac{V_L(1|f_B)}{V_H(1|f_B)} = \frac{r}{r+\lambda} = \frac{F}{C+F} \tag{29}$$

$$\bar{Y}^{-\eta} V_L(1|f_B) = F \tag{30}$$

From the first condition $F = \frac{r}{\lambda}C$. From the second condition $\frac{rC}{a_1-a_0}\left[\frac{a_1-a_0+f_B}{r+\lambda}\right] = \frac{r}{\lambda}C$, Hence

$$f_B = \frac{r}{\lambda}(a_1 - a_0)$$

In the case of good news, the government can be arbitrarily close to the first best, but not achieve it perfectly. The reason is that, once a low quality firm has a perfect reputation, there is no possibility to distinguish it from a high quality firm even after observing good news. We also assume the regulator has two instruments - entry costs F and a reward for having a perfect reputation (i.e., $\phi = 1$), f_G - to target a

$\hat{\phi}^e$ arbitrarily close to 1 and $\bar{Y}$. From entry conditions, F is optimally derived from targeting $\hat{\phi}^e$.

$$\bar{Y}^{-\eta} V_L(\hat{\phi}^e) = F$$

and f_G is determined by the ratio

$$\frac{V_L(\hat{\phi}^e)}{V_H(\hat{\phi}^e | f_G)} = \frac{F}{C+F}$$

4.1.2 Brownian motion

We propose the government to charge entry costs F and to commit to a payments scheme (after taxes and subsidies, again for $Y = 1$), $\pi(x) = c - d(x - \hat{x})$, where $x = log(\frac{1-\phi}{\phi}) : [0,1] \rightarrow (\infty, -\infty)$ and c, d and $\hat{x}$ are parameters optimally determined to achieve an allocation arbitrarily close to first best.[6]

Transforming the ordinary differential equations as a function of x we obtain,

$$\begin{aligned} r\rho V_L(x) &= \rho\pi(x) + \frac{1}{2} V_L'(x) + \frac{1}{2} V_L''(x) \\ r\rho V_H(x) &= \rho\pi(x) - \frac{1}{2} V_H'(x) + \frac{1}{2} V_H''(x) \end{aligned}$$

Solving these ODEs we can obtain the value functions. The proof is the Appendix.

Proposition 8 *Value functions for profits $\pi(x) = c - d(x - \hat{x})$ (where $x = log(\frac{1-\phi}{\phi})$) and Brownian motion*

The value functions for low and high quality firms, with reputation x are

$$\begin{aligned} rV_L(x) &= c - \frac{d}{2r\rho} - d(x - \hat{x}) + K_L e^{\nu_L x} \\ rV_H(x) &= c + \frac{d}{2r\rho} - d(x - \hat{x}) + K_H e^{\nu_H x} \end{aligned}$$

where

$$\nu_L = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} > 0 \qquad \text{and} \qquad \nu_H = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} > 0$$

[6] The reason we do not just assume $\pi(x) = c - dx$ will be clear later, but specifically it allows to make c and d only a function of $\bar{Y}$

From value matching and smooth pasting conditions, $V_L(\bar{x}) = V_L'(\bar{x}) = V_H'(\bar{x})0$*,*

$$K_L = \frac{d}{\nu_L e^{\nu_L \bar{x}}} \qquad \text{and} \qquad K_H = \frac{d}{\nu_H e^{\nu_H \bar{x}}}$$

and the exit point $\bar{x}$ *is*

$$\bar{x} = \frac{c}{d} - \frac{1}{2r\rho} + \frac{1}{\nu_L} + \hat{x} \tag{31}$$

Now we can describe an optimal taxation the government can impose to achieve an allocation arbitrarily close to first best,

Proposition 9 *A government can achieve an allocation arbitrarily close to first best,* $\bar{Y}$*,* $\phi^e = 1 - \epsilon$ *and* $\bar{\phi} = \epsilon$*, with* $\epsilon \to 0$*, by committing to a payment scheme after taxes and subsidies, which is a function of reputation.*

A particular payment scheme that achieves an allocation arbitrarily close to first best, as described in the Proposition, is $\pi(x) = c - d(x - \hat{x})$ where $x = log\left(\frac{1-\phi}{\phi}\right)$ and $\hat{x} = x^e = log\left(\frac{\epsilon}{1-\epsilon}\right)$. The government has access to three instruments (c^*, d^* and F^*) to achieve the three targets, $\bar{Y}$, $x^e \to -\infty$ (or $\phi^e \to 1$), $\bar{x} \to \infty$ (or $\bar{\phi} \to 0$).[7]

The optimal ratio c^*/d^* is obtained from equation (31). This ratio together with the ratio of entry conditions determine optimal entry costs F^*

$$\frac{2r\rho[c^* + \frac{d^*}{\nu_L}e^{\nu_L(x^e - \bar{x})}] - d^*}{2r\rho[c^* + \frac{d^*}{\nu_H}e^{\nu_H(x^e - \bar{x})}] + d^*} = \frac{F^*}{C + F^*}$$

Finally, a target $\bar{Y}$ together with c^*/d^* and F^* from above, jointly determine c^* (hence d^*) from the low quality firms' entry condition,

$$\bar{Y}^{-\eta}\left[\frac{c^*}{r} + \frac{d^*}{r\nu_L}e^{\nu_L(x^e - \bar{x})} - \frac{d^*}{r2r\rho}\right] = F^*$$

4.1.3 Numerical Exercise

In this section we use the same parameters as in the previous section, in which we illustrated the value functions and the market solution under different information structures.

[7] If the government achieves the target that almost all firms entering invest, then it is optimal it targets that almost none of them exit.

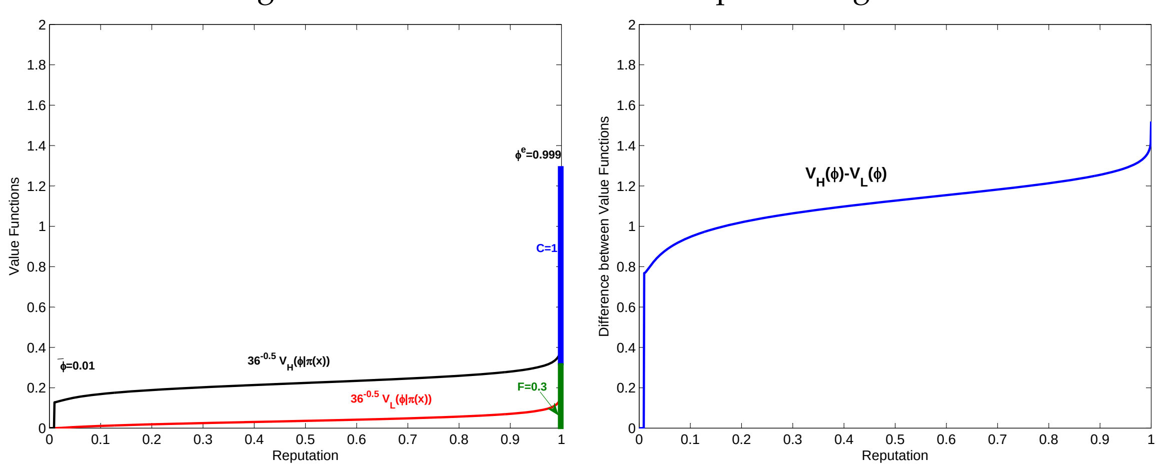
In the case of bad news, $F = \frac{r}{\lambda}C = 1$ (to achieve $\phi^e = 1$) and $f_B = \frac{r}{\lambda}(a_1 - a_0) = 0.6$ at each moment t, such that,

$$Y_M = \left[\frac{V_H(1|f_B) - V_L(1|f_B)}{C}\right]^{\frac{1}{\eta}} = \left[\frac{\frac{\lambda}{r(r+\lambda)}\left(\frac{r+\lambda}{\lambda}(a_1 - a_0)\right)}{C}\right]^{\frac{1}{\eta}} = 6^2 = 36 = \bar{Y}$$

In the case of good news, the government can distort payments to obtain an allocation arbitrarily close to the first best. We assume a target of $\hat{\phi}^e = 0.99$ and $\bar{Y}$. Evaluated at $\bar{Y}$, the entry condition for low quality firms determines $F = \frac{1}{6}5.6 = 0.93$. The reward for generating good signals f_G that implements $\hat{\phi}^e = 0.99$ at $F = 0.93$ is $f_G = 17.4$.

In the case of Brownian motion we assume the government can impose entry costs F and commit to a scheme of payments as a function of reputation. The government can reach the first best $\bar{Y} = 36$, almost all entering firms investing (here we target $\phi^e = 0.999$) and in expectation almost no firm exiting (here we target $\bar{\phi} = 0.01$). Optimal regulation in this case is characterized by payments $\pi(x) = c - d(x - x^e)$, where $x = log\left(\frac{1-\phi}{\phi}\right)$, $x^e = log\left(\frac{1-0.999}{0.999}\right)$, $c = 0.16$ and $d = 0.02$. This payment scheme implies that firms are taxed until they reach a reputation $\phi = 1 - 1^{e-15}$, above which they are subsidized. The entry costs needed to achieve this allocation are $F = 0.29$.

Figure 4: Value Functions under Optimal Regulation



This policy is able to closely implement the full information benchmark because its payments schedule generates value functions that do not approach each other for high reputation levels (as shown in Figure 4), allowing firms to compensate investment without relying in a production decline.

4.2 The Government Knows Less than the Market

Assume the regulator can only observe the entry of intermediate good firms. In a sense the government has much less information about firms than the market. Still it can improve welfare from the market situation just imposing the right entry costs F. Recall firms' payoffs in the absence of per period taxes and subsidies are linear in reputation $\pi(\phi) = (a_1\phi - a_0)Y^{-\eta}$. In this case, entry conditions are,

$$\begin{aligned} V_H(\phi^e)Y^{-\eta} &= C + F \\ V_L(\phi^e)Y^{-\eta} &= F \end{aligned} \tag{32}$$

First, we show there is a quality of entrants $\phi^{e*} \in [\bar{\phi}, 1]$ that maximizes the production of Y achievable by the market. Then, we show there is an entry fee F^* that uniquely implements ϕ^{e*}.

Proposition 10 *There is a ϕ^{e*} that maximizes production Y in the market.*

Proof Take the difference between entry conditions (32), such that,

$$\left(V_H(\phi^e) - V_L(\phi^e)\right) Y^{-\eta} = C.$$

Taking derivatives with respect to ϕ^e

$$\frac{\partial Y}{\partial \phi^e} = \frac{\left[\frac{\partial V_H(\phi^e)}{\partial \phi^e} - \frac{\partial V_L(\phi^e)}{\partial \phi^e}\right] Y}{\eta \left[V_H(\phi^e) - V_L(\phi^e)\right]}$$

The maximum output of the experience good is determined by the following condition

$$\frac{\partial V_H(\phi^e)}{\partial \phi^e} = \frac{\partial V_L(\phi^e)}{\partial \phi^e} \tag{33}$$

which is fulfilled for some $\phi \in (\bar{\phi}, 1)$ when firms' payoffs are linear. Recall that both derivatives are zero at $l(\bar{\phi})$ (by construction of smooth pasting conditions) and at $l(1)$ (this is straightforward to check by considering that ${}_2F_1(1, m; m+1, 0) = 1$ for all m). Also by construction both derivatives are positive for all $\phi \in [\bar{\phi}, 1]$. Q.E.D.

The next Proposition shows a uniquely positive relation between entry costs F and the quality of entrants ϕ^e. The proof is in the Appendix.

Proposition 11 *If firms' payoffs are linear in reputation ϕ, regardless of the information structure, there is a unique positive mapping between entry costs $F \in [0, \infty]$ and the reputation assigned to entrants, $\phi^e \in [\bar{\phi}, 1]$*

From entry conditions 32,

$$\frac{V_L(\phi^e)}{V_H(\phi^e)} = \frac{F}{C+F}, \tag{34}$$

which is independent of Y. To prove the Proposition it is enough to prove the ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing for bad news, good news and Brownian motion cases when profits are linear in ϕ, which are the second points in Propositions 2, 4 and 7. As F goes from 0 to ∞, the ratio goes from 0 at $\bar{\phi}$ to 1 at $\phi = 1$. This is relevant because it means ϕ^{e*} can be uniquely implemented by a F^*.

Since entry costs do not modify the exit decision, because the exit point $\bar{\phi}$ is independent of ϕ^e and Y, a direct implication of this Proposition is that, when entry costs exist, firms enter with a higher reputation than the reputation at which they exit (this is $\phi^e > \bar{\phi}$). Furthermore, higher entry costs imply larger gaps between ϕ^e and $\bar{\phi}$.

In the next sections we analyze what is ϕ^{e*} and hence F^* for the cases of bad news, good news and Brownian motion.

4.2.1 Bad and Good News

When the regulator only observes entry, it can use F to improve both quality and production in the case of bad news (since the difference between value functions increases with reputation, $\phi^{e*} = 1$), but not in the case of good news (since that difference decreases with reputation, $\phi^{e*} = \bar{\phi}$).

More specifically, in the case of bad news, it is possible to find an entry cost such that

$$\frac{V_L(1)}{V_H(1)} = \frac{r}{r+\lambda} = \frac{F^*}{C+F^*}$$

This implies an optimal $F^* = \frac{r}{\lambda}C$ allows the regulator to achieve $\phi^e = 1$. In this case the production in the regulated market, Y_F, is just determined by the entry condition for the low type. The first best and the market solution are identically defined as above,

$$\bar{Y} = \left[\frac{a_1 - a_0}{C}\right]^{\frac{1}{\eta}} > Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}} > Y_M = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}}$$

Contrarily, under good news entry costs can only decrease production. More specifically,

$$\bar{Y} = \left[\frac{a_1 - a_0}{C}\right]^{\frac{1}{\eta}} > Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}} = Y_M$$

A market characterized by good news and no entry fees achieves the same production than a market characterized by bad news and high entry fees. However, $\phi^e = \bar{\phi}$ in the good news case and $\phi^e = 1$ in the bad news case. Since the average quality of intermediate good firms is lower under good news, the total investment required to achieve the same production level is higher and the consumption of the numeraire good is lower. Hence, when the government only observes entry it can achieve a higher welfare when signals are characterized by bad news than when signals are based on good news. This result contrasts with the welfare comparisons between bad and good news under unregulated markets.

4.2.2 Brownian motion

In this case, the maximum gap $V_H(\phi) - V_L(\phi)$ is obtained at some intermediate $\phi^{e*} \in (\bar{\phi}, 1)$. By imposing a $F \in [0, F^*]$, the government can unequivocally achieve a higher welfare than in absence of entry costs. The maximum unequivocal increase in welfare is obtained at $\phi^{e*} > \bar{\phi}$ at which

$$\bar{Y} > Y_F = \left[\frac{V_H(\phi^{e*}) - V_L(\phi^{v})}{C}\right]^{\frac{1}{\eta}} > Y_M \tag{35}$$

since $V_H(\phi^{e*}) - V_L(\phi^{e*})$ is the maximum achievable by in the market given linear profits. Even when the government has information restrictions when compared to the market, it is still able to increase welfare by fostering the incentives coming from reputation concerns. Specifically, entry costs F help the market to reach an initial quality of entrants at an intermediate level where learning happens faster. This intervention helps statistical discrimination, and high quality firms have the chance to distinguish themselves from low quality firms at a faster rate. Hence the market allows for more production and a lower aggregate price to compensate firms to invest in quality.

However, regulation in this case is not even close to achieve the first best $\bar{Y}$ in equation (12). When the initial quality of entrants is very high, the reputation assigned to entrants is also very high. This introduces high incentives for firms to enter without investing. The way to compensate firms to really invest in quality is a higher price for the experience goods and hence a lower production of them in equilibrium. This is the main reason the effectiveness of entry costs to improve welfare is limited.

Even when F^* maximizes Y while still increasing ϕ^e, it does not maximize welfare. Increasing entry costs from 0 to F^* allows an increase both in quality ϕ^e and production Y, having an unequivocally positive impact on welfare. Increasing entry costs above F^* keeps increasing quality but at the cost of reducing production. This creates a trade-off that determine an optimum $F^{**} > F^*$ that maximizes welfare.

Proposition 12 *There are optimal entry costs $F^{**} > F^*$ that maximizes welfare, which are characterized by the following condition,*

$$(V_H'(\phi^{e**}) - V_L'(\phi^{e**})) = -\frac{\eta\delta C^{\frac{1}{\eta}} a_0 g'(\phi^{e**})[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{1-\frac{1}{\eta}}}{[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{\frac{1}{\eta}-1} - \delta g(\phi^{e**})} < 0$$

where $g(\phi^e) = \frac{(\delta+\varrho)\phi^e}{(\delta+\varrho)\phi^e+\delta(1-\phi^e)}$, $g'(\phi^e) = \frac{\delta}{(\delta+\varrho)\phi^e} g^2(\phi^e) > 0$ and $\varrho = \omega^L(\bar{\phi})m(0,\bar{\phi})$ is the endogenous exiting probability of low quality firms. This implies $\phi^{e**} \in (\phi^{e*}, 1)$.

The proof is in the Appendix. At the one hand, increasing quality reduces the mass of firms required to produce a given Y and hence liberates numeraire good for consumption. At the other hand, a reduction in equilibrium Y reduces welfare directly.

4.2.3 Numerical Exercise

Here we use the same parameters as in previous sections. In the case of bad news $F = \frac{r}{\lambda}C = 1$, such that $\phi^e = 1$ and hence

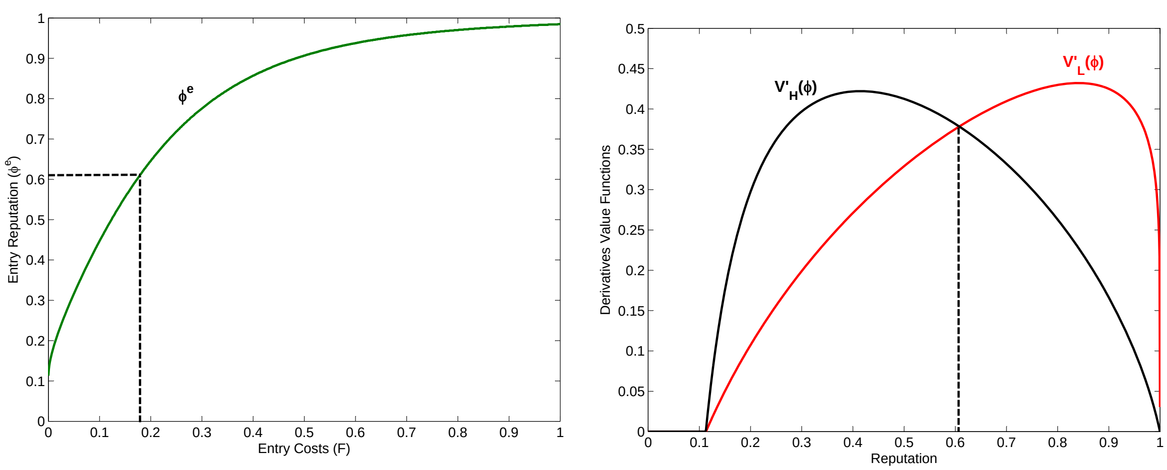
$$Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}} = 3^2 = 9$$

In the case of good news, $F^* = 0$ and it is optimal to have an unregulated market. Even when production is also 9, there is less welfare since quality is lower and then it is more costly to achieve the same production in terms of the numeraire good.

In the Brownian motion case, by increasing F, the regulator can both increase quality and production. The first panel of Figure 5 shows how ϕ^e increases with F (coming from an increasing ratio $\frac{V_L(\phi)}{V_H(\phi)}$). Entry costs F^* are determined to achieve the point at which the derivatives of value functions are the same (condition 33). The second panel of Figure 5 also shows these derivatives. Production is maximized by imposing $F^* = 0.18$ (this is, 18% of the cost to become a high quality firm) to reach $\phi^{e*} = 0.61$, at which derivatives of value functions are equalized. In this case,

$$Y_F = \left[\frac{V_H(0.61) - V_L(0.61)}{C}\right]^{\frac{1}{\eta}} = 4.2^2 = 18$$

Figure 5: Optimal Entry Costs F



5 Conclusions

We argue there may be room for regulation in markets with fully functioning reputation incentives but limited commitment. In this case we show learning is underexploited by the market in providing incentives. If the government has access to the same information as the market, but additionally it has access to a commitment technology, it can construct a scheme of payments based on reputation that foster reputation incentives for firms to invest in quality and achieve the unconstrained first best. Moreover, if the government has less information than the market, still can improve welfare by imposing positive entry costs.

Entry costs are typically criticized for reducing production and the market size. The main logic is clearly exposed in Hopenhayn (1992): Higher entry costs should be compensated by higher aggregate prices, hence by less total output. This argument has been widely used by the economic literature - from supporting trade liberalization to explaining TFP differences across countries - and by international organisms in proposing policy reforms to underdeveloped countries. Still, as shown by Djankov et al. (2002), there is a heavy regulation of entry of start up firms around the world, under the main justification of discouraging the entry of bad firms. In this paper we provide a unifying framework to study the trade off that entry costs create between production and quality. Interestingly we show there is a range of entry costs that increase both quality and total output and we characterize the optimal level of entry costs that maximize welfare by leveraging on market provided reputation incentives.

From a technical viewpoint we contribute in providing analytical solutions in continuous time for a model of reputation with free entry and exit of firms that know their type, because they know their own initial decisions. This explicit solution allows a complete welfare comparison across different regulation policies. We also endogeneize the reputation assigned to entrants in a market, since the adverse selection is generated by a decision of ex ante identical firms.

An important next step in understanding optimal regulation in the presence of reputation concerns is considering moral hazard problems in each period. This will extend the set of policies the government can use to maximize the incentives that learning provides to markets. This paper suggests moral hazard may not be a problem in itself when there is learning, the problem being a lack of commitment to take full advantage from reputation concerns generated by that learning.

Another natural extension is to study mechanisms and institutions the market can endogenously create to reduce commitment problems and align learning and reputation compensations to improve welfare. Possible institutions are financial intermediaries (horizontal integration of intermediate goods producers) that commit to cross subsidize member firms with different reputation (as in Biglaiser and Friedman (1994)). An alternative institution arises from vertical integration between experience goods producers and intermediate goods producers to relax informational problems.

An alternative way markets can simulate entry costs is to burn money at the moment of entry as a signal of investment. There are multiple equilibria introducing this possibility but the only robust equilibrium, from an evolutionary perspective, is the one we characterize without money burning. Furthermore, money burning is an inefficient way to replace entry fees, unless that money goes back to the economy, as we assume the market does with entry fees.

Finally it is important to mention that most of the literature that studies the effects of certifications to enter into a market, focuses in the informational element of certificates as screening of the initial investment (see Lizzeri (1999) and Albano and Lizzeri (2001)). Our case is more extreme, and suggests that even if certification does not add any particular information about the quality of new firms, it may still be welfare improving.

References

Abramowitz, Milton, and Irene Stegun. 1972. *Handbook of Mathematical Functions*. 10. Washington DC: National Bureau of Standards.

Akerlof, George. 1970. "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism." *Quarterly Journal of Economics*, no. 84:488–500.

Albano, Gian Luigi, and Alessandro Lizzeri. 2001. "Strategic Certification and Provision of Quality." *International Economic Review* 42:267–283.

Bar-Isaac, Heski. 2003. "Reputation and Survival: Learning in a Dynamic Signalling Model." *Review of Economic Studies* 70 (2): 231–251.

Biglaiser, Gary, and James Friedman. 1994. "Middlemen as Guarantors of Quality." *The International Journal of Industrial Organization*, no. 12:509–531.

Board, Simon, and Moritz Meyer-ter Vehn. 2010. "Reputation for Quality." mimeo, UCLA.

Djankov, Simeon, Rafael La Porta, Florencio Lopez-de Silanes, and Andrei Shleifer. 2002. "The Regulation of Entry." *Quarterly Journal of Economics*, no. 117:1–37.

Faingold, Eduardo, and Yuliy Sannikov. 2007. "Reputation Effects and Equilibrium Degeneracy in Continuous-Time Games." Cowles Foundation Discussion Paper 1624, Yale University.

Garcia-Fontes, Walter, and Hugo Hopenhayn. 2000. "Entry Restrictions and the Determination of Quality." *Spanish Economic Review*, no. 2:105–127.

Hopenhayn, Hugo. 1992. "Entry, Exit and Firms Dynamics in Long Run Equilibrium." *Econometrica*, no. 60:1127–1150.

Horner, Johannes. 2002. "Reputation and Competition." *American Economic Review* 92 (3): 644–663 (June).

Laurincikas, Antanas, and Ramunas Garunkstis. 2003. *The Lerch-zeta function*. Kluwer Academics Publishers.

Leland, Hayne E. 1979. "Quacks, Lemons, and Licensing: A Theory of Minimum Quality Standards." *Journal of Political Economy* 87:1328–1346.

Lizzeri, Alessandro. 1999. "Information revelation and Certification Intermediaries." *The RAND Journal of Economics* 30:214–231.

Mailath, George, and Larry Samuelson. 2001. "Who Wants a Good Reputation?" *Review of Economic Studies* 68:415–441.

Shaked, Avner, and John Sutton. 1981. "The Self-Regulating Profession." *Review of Economic Studies* 48:217–234.

Shapiro, Carl. 1896. "Investment, Moral Hazard, and Occupational Licensing." *Review of Economic Studies* 53:843–862.

Tadelis, Steven. 1999. "What's in a Name? Reputation as a Tradeable Asset." *American Economic Review* 89:548–563.

———. 2002. "The Market for Reputations as an Incentive Mechanism." *Journal of Political Economy* 110 (4): 854–882.

von Weizsacker, C.C. 1980. "Welfare Analysis of Barriers to Entry." *Bell Journal of Economics* 11:399–420.

A Appendix

A.1 Proof Proposition 5

The activities of the two types of firms induce two different probability measures over the paths of the signal S_t. Reputation evolves following the equation. Fix a prior ϕ^e.

$$\phi_t = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e)Pr(S_t|L)}$$

or

$$\phi_t = \frac{\phi^e \xi_t}{\phi^e \xi_t + (1-\phi^e)} \tag{36}$$

where ξ_t is the ratio between the likelihood that a path $S_s : s \in [0, t]$ arises from type H and the likelihood that it arises from type L. As in Faingold and Sannikov (2007), from Girsanov's Theorem, this ratio follows a Brownian motion characterized by $\mu_\xi = 0$ and $\sigma_\xi = \xi_t \zeta$,

$$d\xi_t = \xi_t \zeta dZ_s^L \tag{37}$$

where $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ and $dZ_s^L = \frac{dS_t - \mu_L dt}{\sigma}$ is a Brownian motion under the probability measure generated by type L.[8]

By Ito's formula,

$$d\phi = [\phi' \mu_\xi + \frac{1}{2}\phi'' \sigma_\xi^2]dt + \phi' \sigma_\xi dZ_s^L$$

$$d\phi_t = -\frac{1}{2}\frac{2\phi^{e2}(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^3}\xi_t^2 \zeta^2 dt + \frac{\phi^e(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^2}\xi_t \zeta dZ_s^L$$

and from equation (36) we can express it in terms of ϕ_t rather than ϕ^e

$$d\phi_t = -\phi_t^2(1-\phi_t)\zeta^2 dt + \phi_t(1-\phi_t)\zeta dZ_s^L$$

$$d\phi_t = \phi_t(1-\phi_t)\zeta[dZ_s^L - \phi_t \zeta dt]$$

replacing by the definition of dZ_s^L,

$$d\phi_t = \lambda(\phi_t) dZ_t^\phi \tag{38}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ and $dZ_t^\phi = \frac{1}{\sigma}[dS_t - (\phi_t \mu_H + (1-\phi_t)\mu_L)dt]$.

Conversely, suppose that ϕ_t is a process that solves equation (38). Define ξ_t using equation (36),

$$d\xi_t = -\frac{1-\phi^e}{\phi^e}\frac{\phi_t}{1-\phi_t}$$

By applying Ito's formula again, ξ_t satisfies equation (37). This implies ξ_t is the ratio between the likelihood that a path $S_s : s \in [0, t]$ arises from type H and the likelihood it arises from type L. Hence ϕ_t is determined by Bayes rule.

Finally, consider that, for different types will have different paths, that in expectation will move their reputation. Replacing dS_t^i in dZ_t^ϕ in equation (38) for the two different types of firms, deliver equations (20) and (21).

A.2 Ordinary Differential Equations with Brownian motion

Here we obtain the differential equations that characterizes the values functions.

[8] It is also possible to solve the problem defining $\xi_t = \frac{Pr(S_t|L)}{Pr(S_t|H)}$ such that $\phi_t = \frac{\phi^e}{\phi^e + (1-\phi^e)\xi_t}$, where $d\xi_t = \xi_t \zeta dZ_s^H$

Proposition 13 *Define* Ψ *the space of progressively measurable processes* ψ_t *for all* $t \geq 0$ *with* $E[\int_0^T \psi_t^2 dt] < \infty$ *for all* $0 < T < \infty$. *A bounded process* W_t^i *for all* $t \geq 0$ *is the continuation value for type* $i = \{H, L\}$ *if and only if, for some process* ψ_t^i *in* Ψ *we have,*

$$dW_t^i = [rW_t^i - r\pi(\phi_t)]dt + \psi_t^i dZ_t \tag{39}$$

Proof The flow continuation value W_t^i for type i is the expected payoff at time t,

$$W_t^i = rE_t^i \left[\int_t^\infty e^{-r(s-t)} \pi(\phi_s) ds \right]$$

Denote U_t^i the discounted sum of payoffs for type i conditional on the public information available at time t,

$$U_t^i = rE_t^i \left[\int_0^\infty e^{-rs} \pi(\phi) ds \right] = \int_0^t e^{-rs} r\pi(\phi_s) ds + W_t^i \tag{40}$$

Since U_t^i is a martingale, by the Martingale Representation Theorem, there exist a process ψ_t^i in Ψ such that,

$$dU_t^i = e^{-rt} \psi_t^i dZ_t \tag{41}$$

Differentiating (40) with respect to time

$$dU_t^i = e^{-rt} r\pi(\phi_t) dt - re^{-rt} W_t^i dt + e^{-rt} dW_t^i \tag{42}$$

Combining (41) and (42), we can obtain (39). Q.E.D.

In a Markovian equilibrium, we know $W_t^i = V_i(\phi_t)$. Since this continuation value depends on the reputation, which follows a Brownian motion, using Ito's Lemma,

$$dV_i(\phi) = \left[\mu_{i,\phi} V_i'(\phi) + \frac{1}{2} \sigma_\phi^2 V_i''(\phi) \right] dt + \sigma_\phi V_i'(\phi) dZ \tag{43}$$

where $\mu_{H,\phi} = \frac{\lambda^2(\phi)}{\phi}$, $\mu_{L,\phi} = -\frac{\lambda^2(\phi)}{(1-\phi)}$ and $\sigma_\phi = \lambda(\phi)$ from Proposition 5.

Matching drifts of equations (39) and (43) for each type i, yields the linear second order differential equation that characterizes continuation values $V_H(\phi)$ and $V_L(\phi)$,

$$\frac{1}{2} \lambda^2(\phi) V_L''(\phi) - \frac{\lambda^2(\phi)}{(1-\phi)} V_L'(\phi) - rV_L(\phi) + \pi(\phi) = 0 \tag{44}$$

and

$$\frac{1}{2} \lambda^2(\phi) V_H''(\phi) + \frac{\lambda^2(\phi)}{\phi} V_H'(\phi) - rV_H(\phi) + \pi(\phi) = 0 \tag{45}$$

A.3 Proof Proposition 6

A.3.1 Solving the ODE's

The second order differential equations can be rewritten as

$$\begin{aligned} r\rho V_L(\phi) &= \rho\pi(\phi) - \phi^2(1-\phi)V_L'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_L''(\phi) \\ r\rho V_H(\phi) &= \rho\pi(\phi) + \phi(1-\phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_H''(\phi) \end{aligned}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}$$

Changing variables to $l = (1-\phi)/\phi$ these ODE's can be written as

$$\begin{aligned} r\rho V_L(l) &= \rho\pi(l) + lV_L'(l) + \frac{1}{2}l^2 V_L''(l) \\ r\rho V_H(l) &= \rho\pi(l) + \frac{1}{2}l^2 V_H''(l) \end{aligned}$$

a) Solving for $V_L(l)$**:** We conjecture a solution of the form:

$$V_L(l) = K\left[l^{-\gamma}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - l^{\gamma-1}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

for some parameters γ and K. With this, we have

$$V_L'(l) = K\left[(-\gamma)\,l^{-\gamma-1}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)\,l^{\gamma-2}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

$$\begin{aligned} V_L''(l) &= K\left[(-\gamma)(-\gamma-1)\,l^{-\gamma-2}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)(\gamma-2)\,l^{\gamma-3}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right] \\ &\quad K(1-2\gamma)\frac{\pi(l)}{l^2} \end{aligned}$$

$$lV_L'(l) + \frac{1}{2}l^2 V_L''(l) = \frac{\gamma(\gamma-1)}{2}V_L(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)$$

which solves the ODE when $2r\rho = \gamma(\gamma-1)$ and $K(1-2\gamma) = -2\rho$, or

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \texttt{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}$$

Recall $\gamma(\rho) : [0,\infty] \to [1,\infty]$ and $K(\rho) > 0$. The parameters χ_1 and χ_2 will be determined later from boundary conditions.

b) Solving for $V_H(l)$: Define: $\Delta_H(l) = \pi(0) - V_H(l)$, $\bar{\pi}(l) = \pi(0) - \pi(l)$. Notice $\bar{\pi}(l)$ is increasing in l.

Rewriting the ODE for the high type as

$$\rho\Delta_H(l) = \rho\bar{\pi}(l) + \frac{1}{2}l^2\Delta_H''(l)$$

Proceeding as above we conjecture a solution of the form:

$$\Delta_H(l) = K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right]$$

for the same parameters γ and K defined previously. With this, we have

$$\begin{aligned}
\Delta_H'(l) &= K\left[(1-\gamma)l^{-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma l^{\gamma-1}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
\Delta_H''(l) &= K\left[-\gamma(1-\gamma)l^{-\gamma-1}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma(\gamma-1)l^{\gamma-2}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\quad + K(1-2\gamma)\frac{\bar{\pi}(l)}{l^2} \\
\frac{1}{2}l^2\Delta_H''(l) &= \frac{\gamma(\gamma-1)}{2}\Delta_H(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)
\end{aligned}$$

that fulfill the ODE by construction with the parameters γ and K defined above. The parameters ψ_1 and ψ_2 will be determined later also from boundary conditions.

A.3.2 Dealing with the boundary conditions at $l = 0$

Notice that we need $\lim_{l\to 0} V_L(l) = \lim_{l\to 0}\pi(l) = \pi(0)$, and $\lim_{l\to 0}\Delta_H(l) = \lim_{l\to 0}\bar{\pi}(l) = \lim_{l\to 0}\pi(l) - \pi(0) = 0$. The two limiting properties hold if and only if $\chi_1 = 0$ and $\psi_1 = 0$ (we then relabel $\chi_2 = \chi$ and $\psi_2 = \psi$).

We will proceed with the proof for the high type. The proof for the low type is related. Using Lipschitz continuity of $\bar{\pi}(l)$, assuming $\bar{\pi}(l) \leq \Lambda l$, and $\psi_2 \leq \infty$:

$$\begin{aligned}
\Delta_H(l) &= K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\leq \Lambda K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}dl'\right] \\
&= \Lambda K\left[l^{1-\gamma}\left(\frac{l^{\gamma}}{\gamma} - \frac{\psi_1^{\gamma}}{\gamma}\right) + l^{\gamma}\left(\frac{\psi_2^{1-\gamma}}{1-\gamma} - \frac{l^{1-\gamma}}{1-\gamma}\right)\right] \\
&= \Lambda K\left[l\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)\right] \\
&= \Lambda l
\end{aligned}$$

if and only if $\psi_1 = 0$ and assuming $\psi_2 = \infty$. Hence, $\lim_{l\to 0}\Delta_H(l) = 0$ if and only if $\psi_1 = 0$. A similar analysis delivers $\lim_{l\to 0} V_L(l) = \pi(0)$ if and only if $\chi_1 = 0$

A.3.3 Simplifying Value Functions

Changing variables inside the integrals: $\theta = l'/l$, so $l d\theta = dl'$ and the limits of integration. We start from obtaining $V_H(l)$.

$$\Delta_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\bar{\pi}(\theta l)\, d\theta + \int_1^{\psi/l} \theta^{-\gamma-1}\bar{\pi}(\theta l)\, d\theta\right\}$$

Since $\bar{\pi}(\theta l) = \pi(0) - \pi(\theta l)$ and $V_H(l) = \pi(0) - \Delta_H(l)$

$$\begin{aligned}
V_H(l) &= \pi(0)\left(1 - K\int_0^1 \theta^{\gamma-2}d\theta - K\int_1^{\psi/l}\theta^{-\gamma-1}d\theta\right) \\
&\quad + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\, d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\, d\theta\right\} \\
V_H(l) &= \pi(0)\left[\frac{K}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right] + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\, d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\, d\theta\right\}
\end{aligned}$$

Hence

$$V_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\, d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\, d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\} \tag{46}$$

Similarly, the low type's value function can be written as

$$V_L\left(l\right)=K\left\{\int_0^1\theta^{\gamma-1}\pi\left(\theta l\right)d\theta-\int_{\chi/l}^1\theta^{-\gamma}\pi\left(\theta l\right)d\theta\right\} \tag{47}$$

In reduced form

$$V_L\left(l\right) = K[B_L(l)-A_L(l)] \ \texttt{and} \tag{48}$$

$$V_H\left(l\right) = K[B_H(l)-A_H(l)] \tag{49}$$

where

$$B_L(l)=\int_0^1\theta^{\gamma-1}\pi\left(\theta l\right)d\theta \qquad \texttt{and} \qquad A_L(l)=\int_{\chi/l}^1\theta^{-\gamma}\pi\left(\theta l\right)d\theta$$

$$B_H(l)=\int_0^1\theta^{\gamma-2}\pi\left(\theta l\right)d\theta \qquad \texttt{and} \qquad A_H(l)=\int_{\psi/l}^1\theta^{-\gamma-1}\pi\left(\theta l\right)d\theta-\frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}$$

A.3.4 Derivatives

Taking derivatives of $V_L(l)$ components and multiplying by l,

$$l\frac{\partial A_L(l)}{\partial l}=\int_{\chi/l}^1\theta^{-\gamma}\pi'\left(\theta l\right)\theta ld\theta-\left(\frac{\chi}{l}\right)^{-\gamma}\pi(\chi)\left(-\frac{\chi}{l^2}\right)l$$

Integrating the first term by parts

$$\begin{aligned}\int_{\chi/l}^1\theta^{1-\gamma}\pi'\left(\theta l\right)ld\theta &= \theta^{1-\gamma}\pi(\theta l)|_{\chi/l}^1-\int_{\chi/l}^1(1-\gamma)\theta^{-\gamma}\pi(\theta l)d\theta \\ &= \pi(l)-\left(\frac{\chi}{l}\right)^{1-\gamma}\pi(\chi)-(1-\gamma)\int_{\chi/l}^1\theta^{-\gamma}\pi(\theta l)d\theta\end{aligned}$$

Then

$$l\frac{\partial A_L(l)}{\partial l}=\pi(l)-(1-\gamma)\int_{\chi/l}^1\theta^{-\gamma}\pi(\theta l)d\theta=\pi(l)-(1-\gamma)A_L(l)$$

Similarly

$$\begin{aligned}l\frac{\partial A_H(l)}{\partial l} &= \pi(l)+\gamma\int_{\psi/l}^1\theta^{-\gamma-1}\pi(\theta l)d\theta-\frac{\pi(0)}{\gamma}(-\gamma)\left(\frac{\psi}{l}\right)^{-\gamma-1}\left(-\frac{\psi}{l^2}\right)l \\ &= \pi(l)+\gamma\int_{\psi/l}^1\theta^{-\gamma-1}\pi(\theta l)d\theta-\gamma\frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}=\pi(l)+\gamma A_H(l)\end{aligned}$$

$$l\frac{\partial B_L(l)}{\partial l} = \pi(l) - \gamma \int_0^1 \theta^{\gamma-1}\pi(\theta l)d\theta = \pi(l) - \gamma B_L(l)$$

$$l\frac{\partial B_H(l)}{\partial l} = \pi(l) - (\gamma-1) \int_0^1 \theta^{\gamma-2}\pi(\theta l)d\theta = \pi(l) - (\gamma-1) B_H(l)$$

The derivatives can then be simplified as follows,

$$lV_L'(l) = K[-\gamma B_L(l) + (1-\gamma)A_L(l)] \text{ and} \tag{50}$$

$$lV_H'(l) = K[(1-\gamma)B_H(l) - \gamma A_H(l)] \tag{51}$$

A.4 Proof Proposition 7

The first part of the Proposition is a general property of value functions with learning.

Now we prove the second part of the Proposition: the ratio $\frac{V_L(\phi^e)}{V_H(\phi^e)}$ is an increasing function of ϕ^e, or which is the same $\frac{V_L(l_0)}{V_H(l_0)}$ is an decreasing function of l_0, that maps from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

First, we define the domain and image of the function. The lower reputation in the market at moment 0 is $\bar{l}$, where $V_L(\bar{l}) = 0$ and $V_H(\bar{l}) = 0 > 0$. We also know that $V_L(1) = V_H(1) > 0$. Finally, $0 < V_L(l) < V_H(l)$ for all other $l_0 \in [0, \bar{l})$. This implies $\frac{V_L(l_0)}{V_H(l_0)}$ is a mapping from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

We show the ratio $\frac{V_L(l)}{V_H(l)}$ is monotonically decreasing in $l \in [0, \bar{l}]$. This is the case if

$$\frac{lV_L'(l)}{V_L(l)} < \frac{lV_H'(l)}{V_H(l)}$$

$$\frac{-\gamma B_L(l) - (\gamma-1)A_L(l)}{B_L(l) - A_L(l)} < \frac{-(\gamma-1)B_H(l) - \gamma A_H(l)}{B_H(l) - A_H(l)}$$

$$\frac{B_L(l) - A_L(l)}{B_H(l) - A_H(l)} < \frac{\gamma B_L(l) + (\gamma-1)A_L(l)}{(\gamma-1)B_H(l) + \gamma A_H(l)}$$

After some algebra, dropping the argument l, this condition implies,

$$B_H\left[(B_L - A_L) + (2\gamma-1)A_L\right] > A_H\left[2\gamma(B_L - A_L) + (2\gamma-1)A_L\right]$$

or

$$B_H\left[\left(1 - \gamma\frac{A_H}{B_H}\right)(B_L - A_L) + (2\gamma-1)A_L\right] > A_H\left[\gamma(B_L - A_L) + (2\gamma-1)A_L\right] \tag{52}$$

We show the left hand side of (52) is positive and the right hand side of (52) is negative for all $l \in [0, \bar{l}]$, hence the condition is always satisfied and the ratio of value functions decreasing in that range.

1. $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

   First, we develop the integrals $B_L(l)$ and $B_H(l)$.
   Recall the profit function is linear in ϕ, ($y(\phi) = a_1\phi - a_0$) and $\phi = \frac{1}{1+l}$, For $Y = 1$,

$$\pi(\theta l) = \frac{a_1}{1+\theta l} - a_0$$

   and consider the general solution to the following integral (see Abramowitz and Stegun (1972)),

$$\int \theta^m \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = a_1 \theta^{m+1} \Phi(-\theta l, 1, m+1) - \frac{\theta^{m+1}}{m+1} a_0$$

   where $\Phi(-\theta l, 1, m+1)$ is a Hurwitz Lerch zeta-function.
   Applying this result to B_L,

$$B_L(l) = \int_0^1 \theta^{\gamma-1} \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = \left[a_1 \theta^{\gamma} \Phi(-\theta l, 1, \gamma) - \frac{\theta^{\gamma}}{\gamma} a_0 \right]_0^1$$

$$B_L(l) = a_1 \Phi(-l, 1, \gamma) - \frac{a_0}{\gamma}$$

   and similarly,

$$B_H = a_1 \Phi(-l, 1, \gamma - 1) - \frac{a_0}{\gamma - 1}$$

   Our strategy is to prove first $B_L(l) > 0$ for all $l \in [0, \bar{l}]$ an then to prove $B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$.
   Important properties of Herwitz Lerch zeta functions for the parameters we are considering ($\gamma \in [1, 2]$) are (see Laurincikas and Garunkstis (2003)):

   - $\Phi(0, 1, \gamma) = \frac{1}{\gamma}$
   - $\frac{\partial \Phi(-l,1,\gamma)}{\partial l} = \frac{1}{l} \left[\frac{1}{l+1} - \gamma \Phi(-l, 1, \gamma) \right] < 0$
   - $(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \gamma \Phi(-l, 1, \gamma)$

   By construction, $B_L(\bar{l}) = 0$, hence $\Phi(\bar{l}, 1, \gamma) = \frac{a_0}{\gamma a_1}$. Given the properties above

$$B_L(l) : [0, \bar{l}] \rightarrow [\frac{a_1 - a_0}{\gamma}, 0]$$

Furthermore, $B_L(l)$ is monotonically decreasing in the range
$B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$ if

$$\gamma(\gamma-1)[\Phi(-l,1,\gamma-1)-\Phi(-l,1,\gamma)] > \frac{a_0}{a_1}$$

Considering the third property above,

$$(\gamma-1)\Phi(-l,1,\gamma-1) > \Phi(-l,1,\gamma)+(\gamma-1)\Phi(-l,1,\gamma) > \frac{a_0}{\gamma a_1}+(\gamma-1)\Phi(-l,1,\gamma)$$

and hence, $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

2. $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

We develop the integral $A_L(l)$ and $A_H(l)$ following the steps above.

$$A_L(l) = \int_{\chi/l}^{1} \theta^{-\gamma}\left(\frac{a_1}{1+\theta l} - a_0\right) d\theta = \left[a_1 \theta^{1-\gamma}\Phi(-\theta l, 1, 1-\gamma) - \frac{\theta^{1-\gamma}}{1-\gamma} a_0\right]_{\chi/l}^{1}$$

$$A_L(l) = a_1\left[\Phi(-l,1,1-\gamma) - (\chi/l)^{1-\gamma}\Phi(-\chi,1,1-\gamma)\right] + \frac{a_0}{\gamma-1}\left(1-(\chi/l)^{1-\gamma}\right)$$

and,

$$A_H(l) = a_1\left[\Phi(-l,1,-\gamma) - (\psi/l)^{-\gamma}\Phi(-\psi,1,-\gamma)\right] + \frac{a_0}{\gamma} - \frac{a_1}{\gamma}(\psi/l)^{-\gamma}$$

Consider $A_H(0) = A_H(\psi) = -\frac{a_1-a_0}{\gamma} < 0$. We show that, if the function grows, the maximum is still negative. This is, we prove that $A_H(\hat{l}) < 0$ where $\hat{l} = argmax A_H(l)$ (hence $\frac{\partial A_H(l)}{\partial l}|_{l=\hat{l}} = 0$).

$$\frac{\partial A_H(l)}{\partial l} = \frac{a_1}{l}\left[\left(\frac{1}{1+l} + \gamma\Phi(-l,1,-\gamma)\right) - \gamma(l/\psi)^{\gamma}\Phi(-\psi,1,-\gamma)\right] - \frac{a_1}{l}(l/\psi)^{\gamma}$$

The condition satisfied at $l\frac{\partial A_H(l)}{\partial l} = 0$ is,

$$\left[\Phi(-l,1,-\gamma) - (\psi/l)^{-\gamma}\Phi(-\psi,1,-\gamma)\right] = \frac{1}{\gamma}(l/\psi)^{\gamma} - \frac{1}{1+l}$$

Evaluating $A_H(\hat{l})$ considering that condition,

$$a_1\left[\frac{1}{\gamma}(l/\psi)^\gamma - \frac{1}{1+l}\right] + \frac{a_0}{\gamma-1}\left(1-(\chi/l)^{1-\gamma}\right) < 0$$

since

$$\gamma a_1 \frac{1}{1+l} > a_0$$

Hence, $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

Finally, just for completeness, $A_L(0) = -\frac{a_1-a_0}{\gamma-1} < 0$, $A_L(\chi) = 0$ because $\chi/l = 1$ and $A_L(\bar{l}) = 0$ by construction. It can be further shown that $A_L(l) < 0$ for all $l \in (0, \chi)$ and $A_L(l) > 0$ for all $l \in (\chi, \bar{l})$.

3. $\gamma(B_L(l) - A_L(l)) + (2\gamma-1)A_L(l) > 0$ for all $l \in [0, \bar{l}]$

   Recall $\gamma(B_L - A_L) + (2\gamma-1)A_L = \gamma B_L + (\gamma-1)A_L = -\frac{lV_L'(l)}{K}$.

   By construction $\gamma B_L + (\gamma-1)A_L = 0$ at $l = 0$ and $l = \bar{l}$.

   For $l \in (\chi, \bar{l})$, since $A_L(l) \geq 0$ and $B_L(l) > 0$, $\gamma B_L + (\gamma-1)A_L > 0$. In particular, at $\bar{l}$, $A_L(\chi) = 0$ and $\gamma B_L(\chi) > 0$.

   As shown above, for $l \in [0, \chi]$, $B_L(l) > 0$ monotonically increasing and $A_L(l) < 0$ monotonically increasing. This implies $\gamma B_L + (\gamma-1)A_L$ goes monotonically from 0 at $l = 0$ to $\gamma B_L(\bar{l}) > 0$, and hence positive in the whole range.

4. $\left[\left(1-\gamma\frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma-1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$

   First, recall $(\gamma-1)B_H + \gamma A_H = -\frac{lV_H'(l)}{K}$. Hence, as in the point above, $(\gamma-1)B_H + \gamma A_H = 0$ at $l = 0$ and $l = \bar{l}$ by construction, which we can rewrite as $1-\gamma\frac{A_H(0)}{B_H(0)} = 1-\gamma\frac{A_H(\bar{l})}{B_H(\bar{l})} = \gamma$. Hence at these two extreme points, the term in the left hand side is 0, the same as the one in the right hand side.

   More generally $(\gamma-1)B_H + \gamma A_H > 0$ (and then $1 < 1-\gamma\frac{A_H(l)}{B_H(l)} < \gamma$). Since $A_L(\chi) = 0$, $\left(1-\gamma\frac{A_H(l)}{B_H(l)}\right)B_L(l) > 0$. By the same monotonicity arguments above, $\left[\left(1-\gamma\frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma-1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$.

Following a similar proof, we can show the third part of the Proposition. First, $V_L'(\bar{\phi}) = V_H'(\bar{\phi}) = 0$ by construction and $V_L'(1) = V_H'(1) = 0$, from the expressions above. Second $V_L'(\phi)$ and $V_H'(\phi)$ are positive for all $\phi \in (\bar{\phi}, 1)$. Third, these derivatives are single peaked and the reputation that maximizes $V_H'(\phi)$ is lower than the reputation that maximizes $V_L'(\phi)$. Finally, $V_H''(\bar{\phi}) > V_L''(\bar{\phi})$ and $V_H''(1) < V_L''(1)$, which means the two derivatives cross only one time, at ϕ^*. These properties arise from inspection of the derivatives of linear profits value functions and from properties of the hypergeometric functions that characterize them.

A.5 Proof of Proposition 8

From the ODEs, we can obtain the homogeneous solutions from

$$\begin{aligned} \frac{1}{2}V_L''(x) + \frac{1}{2}V_L'(x) - r\rho V_L(x) &= 0 \\ \frac{1}{2}V_H''(x) - \frac{1}{2}V_H'(x) - r\rho V_H(x) &= 0 \end{aligned}$$

For low quality firms: $\nu_1^L = -\frac{1}{2} - \sqrt{\frac{1}{4} + 2r\rho} < 0$ and $\nu_2^L = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} > 0$.

For high quality firms: $\nu_1^H = \frac{1}{2} - \sqrt{\frac{1}{4} + 2r\rho} < 0$ and $\nu_2^H = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} > 0$.

Conjecturing $V_L(x) = C - D(x - \hat{x})$, then $V_L'(x) = -D$ and $V_L''(x) = 0$. The particular solution for the low type can be obtained from rewriting the ODE as,

$$-\frac{1}{2}D - r\rho[C - D(x - \hat{x})] + \rho[c - d(x - \hat{x})] = 0$$

Equating the coefficient of x separately to zero (since the equation must hold as an identity in x), we obtain that $rD = d$ and $rC = c - \frac{d}{2r\rho}$. Similarly for $V_H(x)$. Adding the particular and homogenous solutions we obtain the value functions as,

$$\begin{aligned} rV_L(x) &= c - \frac{d}{2r\rho} - d(x - \hat{x}) + K_1^L e^{\nu_1^L x} + K_2^L e^{\nu_2^L x} \\ rV_H(x) &= c + \frac{d}{2r\rho} - d(x - \hat{x}) + K_1^H e^{\nu_1^H x} + K_2^H e^{\nu_2^H x} \end{aligned}$$

Assuming the government cannot pay more than $\hat{\pi}$, $\lim_{x \to -\infty} V_i(x) = \bar{\pi}$ and $K_1^L = K_1^H = 0$. Then we relabel K_2^i as K_i and $\nu_2^i = \nu_i$. K_2^L is determined from smooth pasting condition for the low quality firm ($V_L'(\bar{x}) = 0$),

$$K_L = \frac{d}{\nu_L e^{\nu_L \bar{x}}}$$

The exiting point $\bar{x}$ is determined by value matching $V_L(\bar{x}) = 0$

$$c - \frac{d}{2r\rho} - d(\bar{x} - \hat{x}) + \frac{d}{\nu_L} = 0$$

or

$$\bar{x} = \frac{c}{d} - \frac{1}{2r\rho} + \frac{1}{\nu_L} + \hat{x} \tag{53}$$

Finally, K_H is determined by the smooth pasting condition for the high quality firm ($V'_H(\bar{x}) = 0$).

$$K_H = \frac{d}{\nu_H e^{\nu_H \bar{x}}}$$

A.6 Proof of Proposition 12

First, we describe the stationary distribution of firms, this is the steady state mass of high quality firms $m(1)$ and of low quality firms $m(0)$. Firms H leave the market with probability δ. Firms L leave the market with probability $\delta + \varrho$, where $\varrho = \omega^L(\bar{\phi})m(0, \bar{\phi})$ is the endogenous exiting probability. For example, under bad and good news $\varrho = \lambda$.

From the evolution of firms H and L,

$$\begin{aligned} (\delta + \varrho)(m - m(1)) + (1 - \phi^e)m^e &= 0 \\ \delta m(1) + \phi^e m^e &= 0 \end{aligned}$$

Then

$$\begin{aligned} m(1) &= \frac{(\delta + \varrho)\phi^e}{(\delta + \varrho)\phi^e + \delta(1 - \phi^e)} m = g(\phi^e)m \\ m^e &= \frac{\delta}{\phi^e} m(1) \end{aligned}$$

Since $Y(\phi^e, m) = (a_1 g(\phi^e) - a_0)m$

$$\max_{m,\phi^e} \frac{1}{1-\eta} Y(\phi^e, m)^{1-\eta} + 1 - C\phi^e m^e + \varphi[\Pi_H(\phi^e) - \Pi_L(\phi^e) - C]$$

First order conditions are

$$\begin{aligned} Y^{-\eta} a_1 g'(\phi^e)m - C\delta g'(\phi^e)m + \varphi \left[\frac{\partial \Pi_H(\phi^e)}{\partial \phi^e} - \frac{\partial \Pi_L(\phi^e)}{\partial \phi^e} \right] &= 0 \\ Y^{-\eta}(a_1 g(\phi^e) - a_0) - C\delta g(\phi^e) + \varphi \left[\frac{\partial \Pi_H(\phi^e)}{\partial m} - \frac{\partial \Pi_L(\phi^e)}{\partial m} \right] &= 0 \\ \Pi_H(\phi^e) - \Pi_L(\phi^e) &= C \end{aligned}$$

where $g'(\phi^e) = \frac{\delta}{(\delta+\varrho)\phi^e} g^2(\phi^e) > 0$

Taking derivatives

$$
\begin{aligned}
\frac{\partial \Pi_H(\phi^e)}{\partial \phi^e} - \frac{\partial \Pi_L(\phi^e)}{\partial \phi^e} &= -\eta \frac{a_1 g'(\phi^e)}{a_1 g(\phi^e) - a_0} C + Y^{-\eta}(V_H' - V_L') \\
\frac{\partial \Pi_H(\phi^e)}{\partial m} - \frac{\partial \Pi_L(\phi^e)}{\partial m} &= -\eta \frac{C}{m} \\
\varphi &= \frac{Y^{1-\eta} - C\delta g(\phi^e) m}{\eta C}
\end{aligned}
$$

Evaluate the first order conditions at $V_H' - V_L' = 0$

$$
C\delta m g'(\phi^e)\left[\frac{a_1 g(\phi^e)}{a_1 g(\phi^e) - a_0} - 1\right] > 0
$$

This means the quality of entrants that maximizes welfare ϕ^{e**} is greater than the one that maximizes production ϕ^{e*}. From Proposition 11, this implies $F^{**} > F^*$. We can further characterize the optimal quality of entrants to maximize welfare in terms of the difference between value functions.

$$
(V_H'(\phi^{e**}) - V_L'(\phi^{e**})) = -\frac{\eta \delta C^{\frac{1}{\eta}} a_0 g'(\phi^{e**})[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{1-\frac{1}{\eta}}}{[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{\frac{1}{\eta}-1} - \delta g(\phi^{e**})} < 0
$$

which implies $\phi^{e**} \in (\phi^{e*}, 1)$. Naturally $F^{**} = F^*$ is optimum only if $a_0 = 0$, $\delta = 0$ or $\eta = 0$.