Dynamic Adverse Selection:
A Theory of Illiquidity, Fire Sales, and Flight to Quality*

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Abstract

We develop a dynamic equilibrium model of asset markets affected by adverse selection. There exists a unique equilibrium where better assets trade at higher prices but in less liquid markets. Sellers of high-quality assets can separate because they are more willing to accept a lower trading probability. As a result, the emergence of adverse selection generates a drop in liquidity. It may also lead to a decline in the price-dividend ratio—a fire sale—and a flight to quality. Subsidies to purchasing assets may be Pareto improving and can reverse the fire sale and flight to quality.

1 Introduction

This paper develops a dynamic equilibrium model of asset markets with adverse selection. Individuals choose prices at which to trade heterogeneous durable assets for homogeneous perishable consumption goods. Their choice depends on their expectations about the difficulty of trading and the type of asset that is available at each price. In equilibrium, asset sellers are rationed by a shortage of buyers at all prices above some threshold, and it is increasingly time-consuming to sell an asset at higher prices. This keeps the owners of low quality assets from trying to sell them at high prices. The owners of high quality assets are willing to set a high price despite the low sale probability because holding a better asset gives a higher continuation value in the event that they fail to sell it. The owners of low quality assets opt for a low price and a high sale probability.

We work in a deliberately stylized dynamic general equilibrium framework in which the distribution of asset holdings evolves endogenously over time as individuals trade. Assets are perfectly durable and pay a constant dividend, some amount of a perfectly perishable consumption good. Better quality assets pay a higher dividend but only the asset's current owner observes the dividend. This is the source of private information and the root of the adverse selection problem, as in Akerlof (1970). Individuals are risk-neutral and have a discount factor that changes over time, independently across individuals, creating gains from trade. The only permissible trades are between the consumption good and the asset. Finally, discount factors are observable, which ensures that patient individuals never sell assets since there are no gains from trade. We believe this framework is useful for capturing our main idea that illiquidity may serve to separate high and low quality assets in markets with private information.

We define a competitive equilibrium in this environment and prove that it is unique. Key to our equilibrium concept is that buyers' beliefs about the quality of asset purchased at a particular price must respect sellers' incentive to sell at that price. More precisely, if buyers anticipate getting a particular type of asset with positive probability for a given price, it must be weakly optimal for some seller to offer that type asset at that price.

In equilibrium, higher quality assets trade at a higher price but the expected revenue from selling an asset, the product of its price and per-period trading probability, is decreasing in the quality of the asset. In the case where the support of the distribution of asset qualities is convex, we derive simple closed-form expressions for the relationship between prices, dividends, and sale probabilities. We also find that the existence of illiquidity does not hinge on assumptions about the frequency of trading opportunities. More precisely, we prove that even in the limit with continuous trading opportunities, there are not enough buyers in the

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market for high quality assets and so it takes a real amount of calendar time to sell at a high price. From the perspective of a seller, selling opportunities arrive at a Poisson arrival rate. While this may seem similar to the predictions of search theoretic models of illiquidity in asset markets (e.g. Duffie, Gârleanu and Pedersen, 2005; Weill, 2008; Lagos and Rocheteau, 2009), there are important differences. For example, the difficulty of finding a buyer depends primarily on the extent of private information rather than on the availability of trading opportunities. This is because real trading delays are essential for separating the good assets from the bad ones. Of course, in reality adverse selection and search frictions may coexist in a market, and it is indeed straightforward to introduce search into our framework (Guerrieri, Shimer and Wright, 2010; Chang, 2011).

Although our model is abstract, we believe it may be useful for understanding and quantifying the importance of adverse selection for market liquidity. To be concrete, consider the market for AAA-rated asset-backed securities during the 2007–2008 financial crisis. Prior to the crisis, market participants viewed these securities as a safe investment, nearly indistinguishable from a Treasury bond. In the early stages of the crisis, investors started to recognize that some of these securities were likely to pay less than face value. Moreover, it was difficult to determine the exact assets that backed each individual security. Anticipating that she might later have to sell it, the owner of an asset has an incentive to learn its quality. On the other hand, it may not have been profitable for potential buyers to investigate the quality of all possible assets because they did not know which assets would later be for sale. Although we do not model the process of learning about an asset's quality, we view this world with private information and adverse selection as the starting point for our model.

If this view is correct, our model predicts that a seller should always be able to sell an asset at a sufficiently low price. However, within an asset class, such as AAA-rated asset-backed securities, the owners of good quality assets will choose to hold out for a higher price, recognizing that there will be a shortage of buyers at that price and so it will take time to sell the asset. Moreover, the price that buyers are willing to pay for a high quality asset will be depressed because the market is less liquid. That is, even if a buyer somehow understood that a particular asset would pay the promised dividends with certainty, he would pay less for it because he would anticipate having trouble reselling it to future buyers who don't have his information. Illiquidity therefore further depresses asset prices. In particular, sellers' knowledge of the quality of their assets depresses their liquidity and may depress the value of all securities even if the average quality is unchanged. This is why we view an event where sellers start to learn the quality of the assets in their portfolio as a fire sale.[1] During a fire

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[1] For a detailed description of the first phase of the crisis and an analysis of the source of the adverse selection problem, see Gorton (2008). This view of the crisis is consistent with Dang, Gorton and Holmström

sale, buyers still would like to reinvest their income in some asset, and so the decline in the demand for asset-backed securities will boost the demand for other assets that do not suffer from an adverse selection problem, such as Treasury bonds. Thus our model generates a flight-to-quality episode.

Markets naturally create solutions to adverse selection problems. One solution is reputation effects sustained through repeated interactions between buyers and sellers. In our model, all trade is anonymous so there is no possibility of sustaining a reputation for delivering only high quality assets. We view this as a reasonable description of a financial crisis, even if it is a poor description of the behavior of large financial intermediaries during normal times. When facing solvency constraints, sellers may be willing to sacrifice their long-run reputation for the short-run benefits of liquidating their portfolio.

A second market solution is paying a third party to evaluate the quality of assets. Indeed, this is one role that rating agencies are supposed to play. But during the financial crisis, the rating agencies lost their credibility and there was no one with the reputation and capabilities to take their place.

Absent market forces, there may be a potential role for policy interventions to boost asset prices, liquidity, and welfare. We consider one such program in this paper, a subsidy to purchasing assets at low prices, financed by a tax on dividends. Under some conditions, such a program can raise the price, liquidity, and value of all assets, even those that do not receive the subsidy. But typically these policies have distributional effects in general equilibrium, raising the value of some assets and lowering the value of others. For example, an individual who only owns Treasury bonds suffers from a policy that ameliorates the adverse selection problem in the asset-backed securities markets and so moderates the flight-to-quality.

A large theoretical literature argues that adverse selection may be important in financial markets. Most papers in this literature look at a different market structure in which all trades must take place at one price (e.g. Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010; Chiu and Koeppl, 2011; Tirole, forthcoming). These papers can also generate endogenous illiquidity because sellers may choose not to sell high quality assets if the equilibrium price is too low. The nature of illiquidity is different in our model: sellers try to sell all their assets at optimally chosen prices, recognizing that sales will be rationed at most prices. As we discuss in the conclusion, the equilibrium in our framework appears to be more sensitive to the presence of a small amount of private

(2009), who conclude, "Systemic crises concern debt. The crisis that can occur with debt is due to the fact that the debt is not riskless. A bad enough shock can cause information insensitive debt to become information sensitive, make the production of private information profitable, and trigger adverse selection. Instead of trading at the new and lower expected value of the debt given the shock, agents trade much less than they could or even not at all. There is a collapse in trade. The onset of adverse selection is the crisis."

information and to the support of the asset quality distribution. This may be relevant for understanding why a realistically small amount of private information can incapacitate secondary markets.

A third approach to adverse selection assumes random matching between uninformed buyers and informed sellers and allows the buyers to make take-it-or-leave-it offers to sellers. Some buyers offer higher prices than others and the owners of high quality assets only sell when they are offered a high price. This generates an endogenous composition of sellers, which mitigates the adverse selection problem in that environment (Inderst, 2005; Camargo and Lester, 2011). Our approach to generating a separating equilibrium is fundamentally different in that it does not depend on an endogenous composition of sellers. We highlight this by assuming in our simplest model that the fraction of individuals who are sellers and the fraction of assets owned by those individuals are constant and exogenous.

Whether adverse selection is important for financial markets is ultimately an empirical question. In practice, it is difficult to measure the extent of adverse selection in any market simply because the data demands are acute. In one of the more successful efforts, Finkelstein and Poterba (2004) find a correlation between characteristics of annuity contracts and characteristics of annuity buyers that are unobserved by annuity sellers. Our model would suggest a similar test in securities markets, a correlation between the frequency that an asset is resold and the asset's terminal payoff conditional on observable characteristics. While our reading of the existing evidence, e.g. Downing, Jaffee and Wallace (2009), suggests that the extent of adverse selection in asset markets is small but positive, it is worth stressing that even a small amount of private information generates some illiquidity in our environment. In particular, the product of the price of an asset and its sale probability must always be a *decreasing* function of the privately-observed dividend.

Another potential argument against the relevance of adverse selection in secondary markets is that neither buyers nor sellers knew what they were trading. For example, Arora, Barak, Brunnermeier and Ge (2011) claim that the structure of collateralized debt obligations made it computationally infeasible for anyone but the original issuer to measure the quality of the underlying assets. This is important since in our framework, symmetric lack of information is not a barrier to trade. Indeed, the best evidence on asymmetric information indicates that mortgage originators hold on to mortgages that ex post perform better than the mortgages they sell (Downing, Jaffee and Wallace, 2009). We are unaware of any direct evidence for (or against) private information in the secondary market. But our model gives us confidence that the problem may be real, despite the computational complexity of unraveling the underlying securities. In equilibrium, prices transmit information from sellers to buyers. Even if the owner of an asset cannot observe an asset's dividend, he knows what

he paid for the asset and therefore he knows what value the seller assigned to the asset. Here our model gives a different perspective than models in which all sales occur at a single price.

This paper builds on our previous work with Randall Wright (Guerrieri, Shimer and Wright, 2010). It also complements a contemporaneous paper by Chang (2011). There are a number of small differences between that paper and this one. For example, we look at an environment in which individuals may later want to resell assets that they purchase today. This means that buyers care about the liquidity of the asset and so liquidity affects the equilibrium price-dividend ratio. It follows that interventions in the market which boost liquidity may also raise asset prices. We allow individuals to hold multiple assets. We also focus explicitly on a general equilibrium environment, allowing for the possibility that buyers may be driven to a corner in which they do not consume anything. This is essential for our model to generate a flight to quality. As we discuss in the conclusion, it is also essential to a model in which individuals' discount factors are unobservable. Still, both papers leverage our earlier research to study separating equilibria in a dynamic adverse selection environment.

Our notion of liquidity also builds on DeMarzo and Duffie (1999), who study optimal security design by an issuer with private information. That paper shows that the issuer may commit to retain some ownership of the security in order to signal that it is of high quality. We show that in an equilibrium environment, there is no need for sellers to make such commitments. Instead, when the seller of a high quality asset demands a high price, the market ensures that the seller retains ownership with some probability by rationing sales at that price.

This paper proceeds as follows. Section 2 describes our basic model. Section 3 describes the individual's problem and shows how to express it recursively. Section 4 defines equilibrium and establishes existence and uniqueness. Section 5 provides closed-form solutions for a version of the model with a continuum of assets. Section 6 extends the model to have persistent preference shocks and then shows that the frictions survive in the continuous time limit. Section 7 discusses how our model can generate fire sales following the revelation of some information and how illiquidity and insolvency can be alleviated through an asset purchase program, although the program necessarily loses money. Section 8 concludes.

2 Model

There is a unit measure of risk-neutral individuals. In each period t, they can be in one of two states, $s_t \in \{l, h\}$, which determines their discount factor β_{s_t} between periods t and $t + 1$. We assume $0 < \beta_l < \beta_h < 1$. The preference shock is independent across individuals, which potentially allows for gains from trade. For now we assume that the preference shock

is also independent over time. Thus π_s denotes the probability that an individual is in state $s \in \{l, h\}$ in any period, and it is also the fraction of individuals who are in state s in any period. For any particular individual, let $s^t \equiv \{s_0, \ldots, s_t\}$ denote the history of states through period t.

There is a finite number of different types of assets, indicated by $j \in \{1, \ldots, J\}$. Assets are perfectly durable and their supply is fixed; let K_j denote the measure of type j assets in the economy. Each type j asset produces δ_j units of a homogeneous, nondurable consumption good each period, and so aggregate consumption $\sum_{j=1}^{J} \delta_j K_j$ is fixed. Without loss of generality, assume that higher type assets produce more of the consumption good, $0 \leq \delta_1 < \cdots < \delta_J$. The assumption that there is a finite number of asset types simplifies our notation, but in Section 5 we discuss the limiting case with a continuum of assets.

We are interested in how a market economy allocates consumption across individuals. For the remainder of the paper, we refer to the assets as "trees" and the consumption good as "fruit." The timing of events within period t is as follows:

1. each individual i owns a vector $\{k_{i,j}\}_{j=1}^{J}$ of trees which produce fruit;

2. each individual's discount factor between periods t and $t+1$ is realized;

3. individuals trade trees for fruit in a competitive market;

4. individuals consume the fruit that they hold.

We require that each individual's consumption and holdings of each type of tree are nonnegative in every period and we do not allow any other trades, e.g. contingent claims against shocks to the discount factor. In addition, we assume that only the owner of a tree can observe its quality, creating an adverse selection problem; however, we assume that individuals' discount factors are observable. Key to our equilibrium concept, which we discuss below, is that the buyer of a tree may be able to infer its quality from the price at which it is sold.

With observable discount factors, a version the Milgrom and Stokey (1982) "no trade theorem" implies that high discount factor individuals never sell trees and low discount factor individuals never buy trees in any equilibrium despite the presence of private information.[2] For this reason, we refer to individuals with low discount factors as "sellers" and those with

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[2]This is not necessarily true with unobservable discount factors. In the conclusion we discuss such an environment and argue that despite this, it may still be the case that in equilibrium high discount factor individuals do not want to sell trees and low discount factor individuals do not want to buy trees. Our equilibrium is therefore unaffected by this additional source of private information for an open set of parameter values.

high discount factors as "buyers." Trade in trees for fruit therefore transfers consumption from patient individuals to impatient ones.

We now describe the market structure more precisely. After trees have borne fruit, a continuum of markets distinguished by their positive price $p \in \mathbb{R}_+$ may open up. Each buyer may take his fruit to any market (or combination of markets), attempting to purchase trees in that market. Each seller may take his trees to any market (or combination of markets) attempting to sell trees in that market. However, each piece of fruit and each tree may only be brought to one market.

All individuals have rational beliefs about the ratio of buyers to sellers in all markets. Let $\Theta(p)$ denote the ratio of the amount of fruit brought by buyers to a market p, relative to the cost of purchasing all the trees in that market at a price p. If $\Theta(p) < 1$, there is not enough fruit to purchase all the trees offered for sale in the market, while if $\Theta(p) > 1$, there is more than enough. A seller believes that if he brings a tree to a market p, it will sell with probability $\min\{\Theta(p), 1\}$. That is, if there are excess trees in the market, the seller believes that his sale may be rationed. Likewise, a buyer who brings p units of fruit to market p believes that he will buy a tree with probability $\min\{\Theta(p)^{-1}, 1\}$. If there is excess fruit in the market, he may be rationed. A seller who is rationed keeps his tree until the following period, while a buyer who is rationed must eat his fruit.

Individuals also have rational beliefs about the types of tree sold in each market. Let $\Gamma(p) \equiv \{\gamma_j(p)\}_{j=1}^J \in \Delta^J$ denote the probability distribution over trees available for sale in a market p, where Δ^J is the J-dimensional unit simplex.[3] Buyers expect that, conditional on buying a tree at a price p, it will be a type j tree with probability $\gamma_j(p)$. Buyers only learn the quality of the tree that they have purchased after giving up their fruit. They have no recourse if unsatisfied with the quality.

Although trade does not happen at every price p, the functions Θ and Γ are not arbitrary. Instead, if $\Theta(p) < \infty$ (the buyer-seller ratio is finite) and $\gamma_j(p) > 0$ (a positive fraction of the trees for sale are of type j), sellers must find it weakly optimal to sell type j trees at price p. Without this restriction on beliefs, there would be equilibria in which, for example, no one pays a high price for a tree because everyone believes that they will only purchase low quality trees at that price. We define equilibrium precisely in Section 4 below.

We assume throughout this paper that the endogenous functions Θ and Γ are constant over time, so the environment is in a sense stationary. This restriction seems natural to us, and indeed we are able to prove existence and uniqueness of an equilibrium with this property. Key to this result is that, although the distribution of tree holdings across individuals evolves over time, the fraction of type j trees held by individuals with a high discount factor is

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[3]That is, $\gamma_j(p) \geq 0$ for all j and $\sum_{j=1}^J \gamma_j(p) = 1$.

necessarily a constant π_h at the start of every period because preferences are independently and identically distributed over time.

3 Individual's Problem

Each individual starts off at time 0 with some vector of tree holdings $\{k_j\}_{j=1}^J$ and preference state $s \in \{l, h\}$. In each subsequent period t and history of preference shocks s^t, he decides how many trees to attempt to buy or sell at every possible price p, recognizing that he may be rationed at some prices and that the price may affect the quality of the trees that he buys. Let $V_s^*(\{k_j\})$ denote the supremum of the individual's expected lifetime utility over feasible policies, given initial preferences s and tree holdings $\{k_j\}$. In an online Appendix, we characterize this value explicitly and prove that it is linear in tree holdings: $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for some positive numbers $v_{s,j}$. This is a consequence of the linearity of both the individual's objective function and the constraints that he faces.

In addition, we prove that the marginal value of tree holdings satisfies relatively simple recursive problems. A seller solves

$$v_{l,j} = \delta_j + \max_{p \in \mathbb{R}_+} \left(\min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l \bar{v}_j \right), \tag{1}$$

where

$$\bar{v}_j \equiv \pi_h v_{h,j} + \pi_l v_{l,j}. \tag{2}$$

The individual earns a dividend δ_j from the tree and also gets p units of fruit if he manages to sell the tree at the chosen price p. Otherwise he keeps the tree until the following period. Note that there is no loss of generality in assuming that a seller always tries to sell all his trees, since he can always offer them at a high price such that this is optimal, $p > \beta_l \bar{v}_j$. Of course, at such a high price, he may be unable to sell it, $\Theta(p) = 0$, in which case the outcome is the same as holding onto the tree.

Similarly, a buyer solves

$$v_{h,j} = \max_{p \in \mathbb{R}_+} \left(\min\{\Theta(p)^{-1}, 1\}\frac{\delta_j}{p}\beta_h \sum_{j'} \gamma_{j'}(p)\bar{v}_{j'} + (1 - \min\{\Theta(p)^{-1}, 1\})\delta_j \right) + \beta_h \bar{v}_j.$$

A type j tree delivers δ_j of fruit, which the buyer uses in an attempt to purchase trees at an optimally chosen price p. If he succeeds, he buys δ_j/p trees of unknown quality, type j' with probability $\gamma_{j'}(p)$, while if he fails he consumes the fruit. Finally, he gets the continuation value of the tree in the next period. Again, note that a buyer always finds it weakly optimal

to attempt to purchase a tree at a sufficiently low price p, rather than simply consuming the fruit without attempting to purchase a tree. We therefore do not explicitly incorporate this last option in the value function.

Since the maximand is multiplicative in δ_j, we can equivalently write the buyer's value function as

$$v_{h,j} = \delta_j \lambda + \beta_h \bar{v}_j, \tag{3}$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} + \left(1 - \min\{\Theta(p)^{-1}, 1\}\right) \right). \tag{4}$$

The variable λ is the endogenous value of a unit of fruit to a buyer, independent of the type of tree that produced the fruit. If $\lambda = 1$, a unit of fruit is simply worth its consumption value, and so buyers find it weakly optimal to consume their fruit. But we may have $\lambda > 1$ in equilibrium, so buyers strictly prefer to use their fruit to purchase trees.

Proposition 1 *Let $\{v_{s,j}\}$, $\{\bar{v}_j\}$, and λ be positive-valued numbers that solve the Bellman equations (1)–(4) for $s = l, h$. Then $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for all $\{k_j\}$.*

The proof is in an online appendix. Note that for some choices of the functions Θ and Γ, there is no positive-valued solution to the Bellman equations. In this case, the price of trees is so low that it is possible for an individual to obtain unbounded utility and there is no solution to the individual's problem. Not surprisingly, this cannot be the case in equilibrium.

4 Equilibrium

4.1 Partial Equilibrium

We are now ready to define equilibrium. We do so in two steps. First, we define an equilibrium where the buyer's value of fruit λ is fixed, which we call "partial equilibrium". Then, we turn to the complete definition of a competitive equilibrium, where the value of λ is endogenous and ensures that the fruit market clears.

Definition 1 *A partial equilibrium for fixed $\lambda \geq 1$ is a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ satisfying the following conditions:*

1. *Sellers' Optimality: for all $j \in \{1, \ldots, J\}$, $v_{l,j}$ solves (1) where $\bar{v}_j$ is defined in (2);*

2. *Equilibrium Beliefs: for all $j \in \{1, \ldots, J\}$ and for all p with $\Theta(p) < \infty$ and $\gamma_j(p) > 0$, p solves the maximization problem on the right-hand side of equation (1);*

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3. *Buyers' Optimality: for all $j \in \{1, \ldots, J\}$, $v_{h,j}$ solves (3) where λ is defined in (4) and $\bar{v}_j$ in (2);*

4. *Active Markets: $p \in \mathbb{P}$ only if it solves the maximization problem on the right-hand side of equation (4);*

5. *Consistency of Supply with Beliefs: for all $j \in \{1, \ldots, J\}$,*

$$\frac{K_j}{\sum_{j'} K_{j'}} = \int_{\mathbb{P}} \gamma_j(p) dF(p).$$

Sellers' Optimality requires that sellers choose an optimal price for selling each type of tree, given the ease of trade. *Equilibrium Beliefs* imposes that if individuals expect some type j trees to be for sale at price p, it must be weakly optimal to sell type j trees at that price. *Buyers' Optimality* states that buyers choose an optimal price to buy trees, given the ease of trade and the composition of trees for sale at each price. *Active Markets* imposes that if there is trade at a price p, this must be an optimal price for buying trees. Finally, *Consistency of Supply with Beliefs* imposes that the share of sellers' trees that are of type j is equal to the fraction of type j trees among those offered for sale, where F denotes the fraction of trees that are offered for sale at a price less than or equal to p.[4]

We characterize partial equilibria using the solution to a sequence of constrained optimization problems:

Definition 2 *For given λ, a solution to problem (P_j) is a vector $(v_{l,j}, \bar{v}_j, \theta_j, p_j)$ that solves the following Bellman equation*

$$v_{l,j} = \delta_j + \max_{p,\theta} \left(\min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_j \right)$$

$$\text{s.t. } \lambda \leq \min\{\theta^{-1}, 1\}\frac{\beta_h \bar{v}_j}{p} + \left(1 - \min\{\theta^{-1}, 1\}\right), \tag{5}$$

$$\text{and } v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_{j'} \text{ for all } j' < j \tag{6}$$

with

$$\bar{v}_j = \pi_h(\delta_j \lambda + \beta_h \bar{v}_j) + \pi_l v_{l,j}.$$

We are interested in solving the sequence of problems $(P) \equiv \{(P_1), \ldots, (P_J)\}$. To do so, start with Problem (P_1). Constraint (6) disappears from Problem (P_1), and so we can solve

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[4]The definition of partial equilibrium builds on our definition of equilibrium in Guerrieri, Shimer and Wright (2010); however, the sorting condition in that paper does not hold in this environment and so we cannot directly apply our earlier proofs.

directly for $v_{l,1}$ and $\bar{v}_1$, as well as the optimal policy p_1 and θ_1. Standard arguments ensure that the maximized value is unique if $\lambda \geq 1$. In general, for Problem (P_j), constraints (5) and (6) for $j' = j - 1$ bind, which uniquely determines p_j and θ_j as well as $v_{l,j}$ and $\bar{v}_j$ given $v_{l,j-1}$ and $\bar{v}_{j-1}$. Proceeding by induction yields the following Lemma:

Lemma 1 *For fixed $\lambda \in [1, \beta_h/\beta_l]$, the solution to the sequence of problem (P) has $v_{l,j+1} > v_{l,j}$, $\bar{v}_{j+1} > \bar{v}_j$, $p_{j+1} > p_j$, and $\theta_{j+1} \leq \min\{\theta_j, 1\}$ for all $j < J$. It is the unique such solution to the system of equations*

$$\lambda p_j = \beta_h \bar{v}_j \text{ for all } j, \tag{7}$$

$$v_{l,j} = \delta_j + \min\{\theta_j, 1\} p_j + (1 - \min\{\theta_j, 1\}) \beta_l \bar{v}_j \text{ for all } j, \tag{8}$$

$$v_{l,j} = \delta_j + \theta_{j+1} p_{j+1} + (1 - \theta_{j+1}) \beta_l \bar{v}_j \text{ for all } j < J, \tag{9}$$

$$\bar{v}_j = \pi_h(\delta_j \lambda + \beta_h \bar{v}_j) + \pi_l v_{l,j} \text{ for all } j, \tag{10}$$

and $\theta_1 \geq 1$ if $\lambda = 1$, $\theta_1 \leq 1$ if $\lambda = \beta_h/\beta_l$, and $\theta_1 = 1$ otherwise.

If $\delta_1 > 0$ and $\lambda < \beta_h/\beta_l$, this defines $\theta_j > 0$ for all j; otherwise $\theta_j = 0$ for all $j \geq 2$. We focus on values of λ between 1 and β_h/β_l because these are the relevant ones for equilibrium. One could, however, also characterize the solution to problem (P) for $\lambda > \beta_h/\beta_l$; it would have $\theta_j = 0$ for all j.

Proposition 2 *Fix $\lambda \in [1, \beta_h/\beta_l]$. There exists a partial equilibrium and any partial equilibrium is given by the solution to problem (P). More precisely:*

- *Existence: Take any $\{v_{l,j}, \bar{v}_j, \theta_j, p_j\}$ that solves problem (P). Then there exists a partial equilibrium $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ where $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $v_{h,j} = \delta_j \lambda + \beta_h \bar{v}_j$, and $dF(p_j) = K_j / \sum_{j'} K_{j'}$.*

- *Uniqueness: Take any partial equilibrium $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$. For all j, there exists a $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$. If also $\Theta(p_j) > 0$, then $(v_{l,j}, \bar{v}_j, \Theta(p_j), p_j)$ solves problem (P_j).*

The proof in the appendix gives a complete characterization of the partial equilibrium, including the entire functions Θ and Γ. Since we proved in Lemma 1 that the solution to problem (P) is unique, except possibly for the value of θ_1, this essentially proves uniqueness of the partial equilibrium.

Figure 1 illustrates a partial equilibrium for the case with $J = 2$.[5] The two upward-sloping curves indicate pairs of prices and resale probabilities such that buyers are willing to
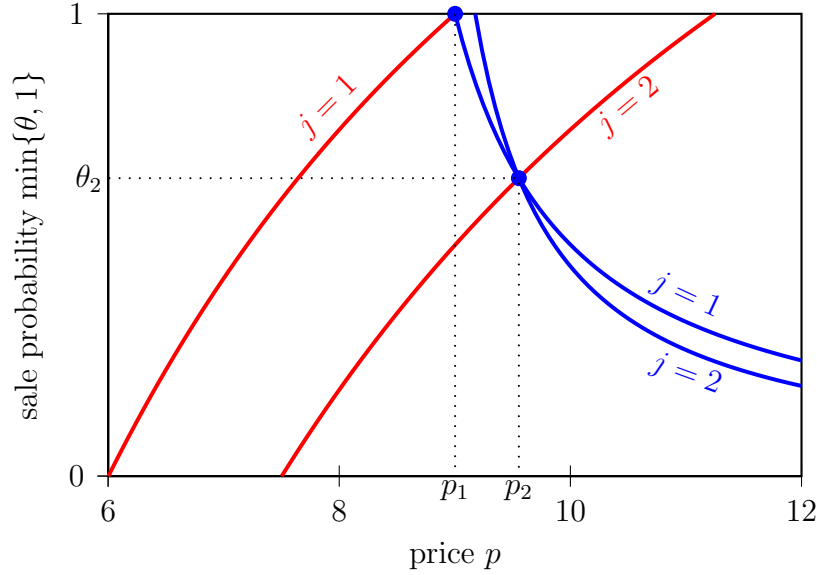

Figure 1: Illustration of problem (P) and partial equilibrium.

purchase each of the trees when the value of a unit of fruit is λ. A buyer is willing to pay a higher price for a tree if he anticipates being able to resell it with a higher probability when he becomes a seller at some future date. In terms of problem (P), these curves describe the relationship between θ_j and p_j implied by equations (7), (8), and (10) conditional on λ. This recognizes that the continuation value $\bar{v}_j$ accounts for the resaleability of the tree.

The two downward-sloping curves are the indifference curves for the seller of each of the trees evaluated at their equilibrium values. Each of them is downward sloping because a seller is willing to accept a lower sale probability if he receives a higher price conditional on a sale. The seller of tree $j = 1$ is not constrained by worse trees and so in equilibrium is able to sell the tree with probability 1. The indifference curve of this seller therefore intersects the buyers' indifference curve at a price that reflects the complete liquidity of this tree. To construct this indifference curve, first compute $\bar{v}_1$ from equations (7), (8), and (10) and the condition $\theta_1 = 1$. Then eliminate $v_{l,1}$ from equations (8) and (9) and solve for θ_2 as a function of p_2 given this value of $\bar{v}_1$.

The seller of tree $j = 2$ is constrained by the need to signal that he holds the high quality tree. The point (p_2, θ_2) leaves the seller of a type 1 tree indifferent between attempting to sell it for p_2 with probability θ_2 and selling it for sure at the lower price p_1. Moreover, buyers are willing to purchase type 2 trees at price p_2 when they recognize that they can resell them with probability θ_2. Buyers would only pay a higher price for type 2 trees if the resale probability were higher, but then the sellers of type 1 trees would attempt to sell at this

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higher price.

The figure also illustrates the indifference curve of a type 2 seller through the equilibrium price-sale probability pair (p_2, θ_2). We construct this in the same manner as a type 1 seller's indifference curve. Note that the sellers' indifference curves satisfy a single-crossing property, which is key to our separating equilibrium. The owner of a higher quality tree is willing to accept a greater reduction in the sale probability for a given increase in the price because the continuation value of holding a higher quality tree is higher. This illustrates how higher quality trees sell at a higher price but with a lower probability in equilibrium. Finally, if there are more types of trees, we can use a similar inductive procedure to construct the price and sale probability of each type.

4.2 Competitive Equilibrium

We now turn to a full competitive equilibrium in which λ is endogenous:

Definition 3 *A competitive equilibrium is a number $\lambda \in [1, \beta_h/\beta_l]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ satisfying the following conditions:*

1. $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ is a partial equilibrium for fixed λ; and

2. the fruit market clears: $\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \left(\sum_{j=1}^{J} K_j \right) \int_{\mathbb{P}} \Theta(p) p dF(p)$.

A competitive equilibrium is a partial equilibrium plus the market clearing condition that states that the fruit brought to market by buyers is equal to the value of trees brought to the market by sellers times the buyer-seller ratio. Recall from Proposition 2 that $dF(p_j) = K_j / \sum_{j'} K_{j'}$ in partial equilibrium, where p_j is the equilibrium price of type j trees. The market clearing condition therefore reduces to

$$\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \sum_{j=1}^{J} \Theta(p_j) p_j K_j. \tag{11}$$

The left hand side is the fruit held by buyers at the start of the period, while each term in the right hand side is the equilibrium cost of purchasing a particular type of tree multiplied by the buyer-seller ratio for that tree.

Proposition 3 *A competitive equilibrium $(\lambda, \{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ exists and is unique.*

The proof shows that an increase in the value of fruit to a buyer λ drives down the amount of fruit that sellers expect to get from selling any type j tree, that is, $p_j\Theta(p_j)$. Indeed, in the limit when $\lambda = \beta_h/\beta_l$, $\Theta(p_j) = 0$ for all $j > 1$, and so trade breaks down in all but the worst type of tree. At the opposite limit of $\lambda = 1$, buyers are indifferent about purchasing trees and so $\Theta(p_1) > 1$ and buyers are rationed. By varying λ, we find the unique value at which the fruit market clears.

In general, we can distinguish between three cases, each of which is generic in the parameter space. We show in the proof of Proposition 3 that if there are very few sellers, $\pi_l < \underline{\pi}$, the unique equilibrium has $\lambda = 1$ and buyers consume some of their fruit. Conversely, if there are many sellers, $\pi_l > \bar{\pi}$, the unique equilibrium has $\lambda = \beta_h/\beta_l$ and there is only a market in the worst type of tree. At intermediate values of π_l, $\beta_h/\beta_l > \lambda > 1$, there is a market for every type of tree, and buyers use all their fruit to purchase trees. The thresholds satisfy $1 > \bar{\pi} > \underline{\pi} > 0$ and depend on all the other model parameters.

5   Continuous Types of Trees

We have assumed so far that there are only a finite number of types of trees. It is conceptually straightforward to extend our analysis to an environment with a continuum of trees. This is useful because it shows that the behavior of the economy is not particularly sensitive to the number of types of trees, but rather to the support of the dividend distribution.

The only change in our environment is that we assume the dividend distribution is $(\underline{\delta}, \bar{\delta})$, where $0 \leq \underline{\delta} < \bar{\delta} \leq \infty$. Let $G(\delta)$ denote the cumulative distribution of trees on this support. We similarly let $v_l(\delta)$, $v_h(\delta)$, and $\bar{v}(\delta)$ denote the value to a seller, the value to a buyer, and the expected value of a tree that bears δ units of fruit per period. These satisfy the analogs of equations (1)–(3). Definition 1 (partial equilibrium) and Definition 3 (competitive equilibrium) change only to reflect this new notation.[6] We omit these formalities in the interest of space.

We find that in equilibrium, the price of the lowest quality tree is

$$\underline{p} = \frac{\underline{\delta}\beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{12}$$

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[6]One minor modification is the market clearing condition when $\lambda = 1$. In the economy with finitely many types of trees, we used $\Theta(p_1) \geq 1$ to ensure that buyers brought all their trees to the market even when $\lambda = 1$. Here it is easier to allow buyers to consume a positive fraction of their fruit and impose $\Theta(P(\underline{\delta})) = 1$.

For $p < \underline{p}$, $\Theta(p) = \infty$ and $\Gamma(p)$ is defined arbitrarily. For $p > \underline{p}$,

$$\Theta(p) = \left(\underline{p}/p\right)^{\frac{\beta_h}{\beta_h - \beta_l \lambda}}, \tag{13}$$

while a different type of tree $\delta = D(p)$ is offered at each price $p \geq \underline{p}$, where

$$D(p) = p\left(\frac{\lambda + (\beta_h - \lambda\beta_l)(1 - \Theta(p))\pi_l}{\beta_h(\pi_l + \lambda\pi_h)} - 1\right). \tag{14}$$

These equations hold as long as $D(p) \leq \bar{\delta}$. For higher prices, $\Theta(p)$ is pinned down by the indifference curve of the seller of a type $\bar{\delta}$ tree and $D(p) = \bar{\delta}$. This determines a partial equilibrium for fixed $\lambda \in [1, \beta_h/\beta_l]$.

We also find that the competitive equilibrium is unique and always has $\lambda < \beta_h/\beta_l$. It satisfies

$$\pi_h \int_{\underline{\delta}}^{\bar{\delta}} \delta dG(\delta) \geq \pi_l \int_{\underline{\delta}}^{\bar{\delta}} \Theta(P(\delta))P(\delta)dG(\delta) \text{ with equality if } \lambda > 1, \tag{15}$$

where $P(\delta)$ is the equilibrium price of a type δ tree, so $D(P(\delta)) \equiv \delta$. If this holds as an inequality, the difference is the measure of fruit consumed by buyers. We can prove directly from the functional form for Θ and P that an increase in λ reduces the right hand side of this inequality, ensuring that the competitive equilibrium is unique. Indeed, as λ converges to β_h/β_l, the right hand side converges to 0, ruling out the possibility of an equilibrium in which λ takes on this limiting value. We summarize these results in a proposition:

Proposition 4 *Equations (12)–(15) uniquely describe a competitive equilibrium when the support of the tree distribution is $(\underline{\delta}, \bar{\delta})$. This is the unique limit of the economy with a finite number of trees.*

We believe that this is also the unique equilibrium of the limiting economy, but our approach to establishing uniqueness—solving a sequence of problems (P)—does not easily extend to an economy with uncountably many types of trees.

6 Persistent Shocks and Continuous Time

Our model explains how adverse selection can generate illiquid assets that only sell with a certain probability each period. But suppose that the time between periods is negligible. Will the illiquidity become negligible as well? We argue in this section that it will not. Instead, equilibrium requires that a real amount of calendar time elapse before a high quality tree is sold.

To show this, we consider the behavior of the economy when the number of periods per unit of calendar time increases without bound. That is, we take the limit of the economy as the discount factors converge to 1, holding fixed the ratio of discount rates $(1 - \beta_h)/(1 - \beta_l)$ and the present value of dividends $\delta_j/(1-\beta_s)$. But as we take this limit, we also want to avoid changing the stochastic process of shocks. With i.i.d. shocks and very short time periods, there is almost no difference in preferences between high and low types of individuals and so the gains from trade become negligible. We therefore first introduce persistent shocks into the model and then prove that as the period length shortens, the probability of sale per period falls to zero, while the probability of sale per unit of calendar time converges to a well-behaved number.

6.1 Persistent Shocks

Assume now that $s_t \in \{l, h\}$ follows a first order stochastic Markov process and let $\pi_{ss'}$ denote the probability that the state next period is s' given that the current state is s. A partial equilibrium with a fixed value of $\lambda \geq 1$ is still characterized by a pair of functions $\{v_{s,j}\} \in \mathbb{R}_+^{2J}$ that represent the value of an individual who starts a period in preference state s holding a type j tree; a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ representing the buyer-seller ratio at an arbitrary price p; a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$ representing the distribution of tree types available at price p; and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ representing the share of trees available at a price less than or equal to p. The definition of partial equilibrium is analogous to definition 1 for the i.i.d. case, except for the obvious change in the continuation value:

$$v_{l,j} = \delta_j + \max_p \big(\min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})\big), \qquad (1')$$

$$v_{h,j} = \delta_j\lambda + \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})), \qquad (3')$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\}\frac{\beta_h \sum_{j=1}^J \gamma_j(p)(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{p} + \big(1 - \min\{\Theta(p)^{-1}, 1\}\big)\right). \quad (4')$$

We omit the formal definition, which simply substitutes these expressions for their i.i.d. analogs. The characterization of partial equilibrium and proof that it exists and is unique is similarly unchanged. In equilibrium, type j trees sell for a price p_j satisfying the buyers' indifference condition

$$p_j = \frac{\beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{\lambda},$$

while the condition for excluding type $j - 1$ trees from the market pins down the sale probability θ_j when $j \geq 2$

$$\theta_j\big(p_j - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big) = \min\{\theta_{j-1}, 1\}\big(p_{j-1} - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big).$$

These equations pin down the value functions, prices, and buyer-seller ratios given λ.

In the model with idiosyncratic shocks, we found that the value of fruit to a high discount factor individual, λ, always lies in the interval $[1, \beta_h/\beta_l]$. With persistent shocks, the lower bound, which ensures that high discount factor individuals are willing to buy trees, $p_j \leq \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})$, is unchanged. However, the upper bound, which ensures that low discount factor individuals are willing to sell trees, $p_j \geq \beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})$, is given by the larger root of

$$\beta_h(\lambda - (\lambda - 1)\pi_{hl}) = \beta_l\lambda(\lambda - (\lambda - 1)\pi_{ll}).$$

We denote this upper bound by $\bar{\lambda}$. It always exceeds 1 and $\bar{\lambda} > \beta_h/\beta_l$ if and only if shocks are persistent, $\pi_{ll} > \pi_{hl}$.

The definition of a competitive equilibrium with persistent shocks is also complicated by endogeneity of the distribution of tree holdings. In the i.i.d. case, high discount factor individuals start each period holding a fraction $\pi_h K_j$ type j trees, but this is not true with persistent shocks. Instead, let μ_j denote the measure of type j trees held by high discount factor individuals at the start of a period. In steady state, this satisfies

$$\mu_j = \pi_{hh}\left(\mu_j + \sigma_j\right) + \pi_{lh}\left(K_j - \mu_j - \sigma_j\right),$$

where σ_j is the measure of type j trees purchased by high discount factor individuals each period. High discount factor individuals hold $\mu_j + \sigma_j$ type j trees at the end of each period, while the rest are held by low discount factor individuals. Multiplying by the appropriate preference transition probabilities delivers the measure held by high discount factor individuals at the start of the following period. To solve for μ_j, we first need to compute the measure of trees sold each period, σ_j. This is the product of the measure of trees for sale times the average sale probability weighted by the fraction of trees that are of type j at an arbitrary price p:

$$\sigma_j = \left(\sum_{j'}(K_{j'} - \mu_{j'})\right)\int_{\mathbb{P}} \min\{\Theta(p), 1\}\gamma_j(p)dF(p).$$

Alternatively, consistency of supplies with beliefs implies

$$\frac{K_j - \mu_j}{\sum_{j'}(K_{j'} - \mu_{j'})} = \int_{\mathbb{P}} \gamma_j(p)dF(p),$$

and so we can rewrite the measure sold as

$$\sigma_j = (K_j - \mu_j)\frac{\int_{\mathbb{P}} \min\{\Theta(p), 1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)},$$

the product of the measure of trees for sale and the average sale probability. Use this to solve for μ_j:

$$\mu_j = \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh})\frac{\int_{\mathbb{P}} \min\{\Theta(p),1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)}}{1 - (\pi_{hh} - \pi_{lh})\left(1 - \frac{\int_{\mathbb{P}} \min\{\Theta(p),1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)}\right)}K_j. \tag{16}$$

If $\pi_{hh} = \pi_{lh}$, this reduces to $\mu_j = \pi_{lh}K_j = \pi_{hh}K_j$, but if shocks are persistent, $\pi_{hh} > \pi_{lh}$, then μ_j is increasing in the measure of type j trees that are sold each period.

We are now in a position to define equilibrium:

Definition 4 *A stationary competitive equilibrium with persistent shocks is a number $\lambda \in [1, \bar{\lambda}]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$, and measures $\mu_j \in [0, K_j]$ satisfying the following conditions:*

1. *$(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ is a partial equilibrium with persistent shocks for fixed λ;*

2. *the fruit market clears: $\sum_{j=1}^{J} \delta_j \mu_j = \left(\sum_{j=1}^{J}(K_j - \mu_j)\right)\int_{\mathbb{P}} \Theta(p)pdF(p)$; and*

3. *measures are consistent with trades: μ_j satisfies equation (16).*

If there are a continuum of types trees, we can again obtain closed-form solutions. In particular, arguments analogous to those in Proposition 4 imply

$$\Theta(p) = \frac{\lambda(1 - \beta_l(\pi_{ll} - \pi_{hl})) - (\lambda - 1)\pi_{hl}}{(\lambda - (\lambda - 1)\pi_{hl})(p/\underline{p})^{\frac{\beta_h(\lambda(1-\beta_l(\pi_{ll}-\pi_{hl}))-(\lambda-1)\pi_{hl})}{\beta_h(\lambda-(\lambda-1)\pi_{hl})-\beta_l\lambda(\lambda-(\lambda-1)\pi_{ll})}} - \beta_l\lambda(\pi_{ll} - \pi_{hl})}. \tag{13'}$$

Similarly, the type of tree sold at price p satisfies

$$D(p) = p\left(\frac{\lambda + (\beta_h\pi_{hl} - \lambda\beta_l\pi_{ll})(1 - \Theta(p))}{\beta_h(\lambda - (\lambda - 1)\pi_{hl} + (1 - \Theta(p))\beta_l\lambda(\pi_{hl} - \pi_{ll}))} - 1\right). \tag{14'}$$

These expressions generalize equations (13) and (14) to the model with persistent shocks. Finally, the share of trees that are of type δ or less and are held by high discount factor individuals is

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh})\Theta(P(\delta'))}{1 - (\pi_{hh} - \pi_{lh})(1 - \Theta(P(\delta')))} dG(\delta'), \tag{16'}$$

where again $P(\delta)$ is the inverse of $D(p)$.

We do not prove existence and uniqueness of equilibrium in this environment. For starters, extending the proof of Proposition 3 is cumbersome because the measures μ_j are endogenous and depend on λ. But this can easily be handled using the closed-form solutions when there are a continuum of types of trees. More importantly, such a proof would only establish existence and uniqueness of a stationary competitive equilibrium, not that there is a unique equilibrium for arbitrary initial conditions. The distinction is important because μ_j is a payoff-relevant state variable in the model with persistent shocks. Given an initial value of the vector $\{\mu_j\}$, subsequent trades determine the evolution of this vector, which in turn determines the evolution of the value of fruit to a buyer λ. We have not characterized a partial equilibrium with time-varying λ, indeed we have not even introduced notation that would allow us to do so. Therefore we cannot discuss the full set of potentially nonstationary equilibria in this environment. Nevertheless, we believe that our analysis of stationary equilibria is an important first step.

6.2   Continuous Time Limit

We are now in a position to consider the continuous time limit of this model. For a fixed period length $\Delta > 0$, define discount rates ρ_s and transition rates q_{hl} and q_{lh} as

$$\rho_s = \frac{1 - \beta_s}{\Delta}, \ q_{hl} = \frac{\pi_{hl}}{\Delta}, \text{ and } q_{lh} = \frac{\pi_{lh}}{\Delta}.$$

Also assume a type δ tree produces $\delta\Delta$ fruit per period. We interpret $1/\Delta$ as the number of periods within a unit of calendar time. With fixed values of ρ_s, q_{hl}, and q_{lh}, the limit as $\Delta \to 0$ (and so $\beta_s \to 1$ and π_{hl} and $\pi_{lh} \to 0$) then corresponds to the continuous time limit of the model. We find that in this limit, $\Theta(p) \to 0$ but the sale rate per unit of time converges to a number:

$$\alpha(p) \equiv \lim_{\Delta \to 0} \frac{\Theta(p)}{\Delta} = \frac{\rho_l + q_{lh} + q_{hl}/\lambda}{\left(p/\underline{p}\right)^{\frac{\rho_l + q_{lh} + q_{hl}/\lambda}{\rho_l - \rho_h - (\lambda - 1)(q_{lh} + q_{hl}/\lambda)}} - 1}$$

for all $p \geq \underline{p}$, while the type of tree sold at price p converges to

$$D(p) = p \left(\rho_h + \frac{q_{hl}((\lambda - 1)\alpha(p) + \lambda\rho_l - \rho_h)}{q_{hl} + \lambda(q_{lh} + \rho_l + \alpha(p))} \right).$$

In particular, the worst type of tree has dividend per unit of calendar time $\underline{\delta} = D(\underline{p})$ and no resale risk, $\alpha(\underline{p}) = \infty$. This pins down the lowest price,

$$\underline{p} = \frac{\underline{\delta}\lambda}{(\lambda - 1)q_{hl} + \lambda\rho_h}.$$

From the perspective of a seller, $\alpha(p)$ is the arrival rate of a Poisson process that permits her to sell at a price p. Equivalently, the probability that she fails to sell at a price $p > \underline{p}$ during a unit of elapsed time is $\exp(-\alpha(p))$, an increasing function of p that converges to 1 as p converges to infinity and is well-behaved in the limiting economy. One can also find the arrival rate of trading opportunities to a buyer; this is infinite if $p > \underline{p}$ and zero if $p < \underline{p}$.

To close the model, we can compute the measure of type δ trees held by high discount factor individuals, the limit of equation (16'). This gives

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{q_{lh} + \alpha(P(\delta'))}{q_{hl} + q_{lh} + \alpha(P(\delta'))} dG(\delta').$$

Substituting this into the fruit market clearing condition gives

$$\int_{\underline{\delta}}^{\bar{\delta}} \frac{\delta(q_{lh} + \alpha(P(\delta)))}{q_{hl} + q_{lh} + \alpha(P(\delta))} dG(\delta) \geq \int_{\underline{\delta}}^{\bar{\delta}} \frac{\alpha(P(\delta))P(\delta)q_{hl}}{q_{hl} + q_{lh} + \alpha(P(\delta))} dG(\delta),$$

with equality if $\lambda > 1$. The left hand side is the integral of the dividend per unit of time δ times the density $dG_h(\delta)$, i.e. the amount of fruit held by high discount factor individuals at the start of a period. The integrand on the right hand side is the product of the probability per unit of time of selling a type δ tree, $\alpha(P(\delta))$, times the price of the tree, $P(\delta)$, times the density of such trees held by low discount factor individuals, $dG(\delta) - dG_h(\delta)$. Integrating over the support of the dividend distribution gives the amount of fruit required to purchase the trees that are sold at each instant.

In equilibrium, there is a continuum of marketplaces, each distinguished by its price p. Sellers try to sell their trees in the appropriate market, while buyers bring their fruit to markets and possibly consume some of it. In all but the worst market, with price $\underline{p}$, there is always too little fruit to purchase all of the trees. That is, a stock of trees always remains in the market to be purchased by the gradual inflow of new fruit from buyers. Buyers are

able to purchase trees immediately, but sellers are rationed and get rid of their trees only at a Poisson rate. Of course, a seller could immediately sell her trees for the low price $\underline{p}$, but she chooses not to do so.

More generally, the illiquidity generated by adverse selection do not disappear when the period length is short. Intuitively, it must take a real amount of calendar time to sell a tree at a high price or the owners of low quality trees would misrepresent them as being of high quality. This is in contrast to models where trading is slow because of search frictions.[7] In such a framework, the extent of search frictions governs the speed of trading and as the number of trading opportunities per unit of calendar time increases, the relevant frictions naturally disappear.

7 Discussion

This section explores how our model can be used to understand a financial crisis characterized by a collapse in the liquidity and price of some assets and a flight to other high quality, liquid assets. We also ask how outside intervention may increase liquidity and prices of the first type of asset and restore normal prices for the second type. We focus on a discrete time model with i.i.d. preference shocks, obviating the need to discuss transitional dynamics.

7.1 Fire Sales

Consider an initial situation where everyone believes that all trees produce δ_0 fruit per unit of time. At time 0, everyone learns that there is dispersion in the quality of trees. For example, this may correspond to the development of a technology that tells sellers which of their trees produce more fruit. In this case, average fruit production is still δ_0 but there is now private information. Alternatively, the outbreak of a disease may reduce the productivity of some trees while leaving others unaffected, reducing average fruit production and creating private information. We are interested in understanding how the equilibrium responds to this one-time unanticipated shock.

We first consider a partial equilibrium exercise where the value of λ is held fixed. Naturally the price of trees with $\delta_j < \delta_0$ falls, since these trees are known to be of lower quality than before. Moreover, the market for all types of trees $j > 1$ becomes less liquid, pushing down their resale value. If average tree quality does not increase by too much, the price of an average tree must fall, reflecting the illiquidity in the tree market.

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[7]See, for example, Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009) for models where assets are illiquid because of search frictions.

The emergence of adverse selection also has a general equilibrium effect through the value of fruit to buyers, λ. If λ did not change, lower prices and lower liquidity would reduce the amount of fruit needed to purchase trees, given by the right hand side of equation (11). This is inconsistent with equilibrium if initially $\lambda > 1$, and so λ must fall. The logic behind the proof of uniqueness in Proposition 3 implies that this increases the product $\Theta(p_j)p_j$ for all j, restoring equilibrium in the fruit market.

Whether the equilibrium with adverse selection is Pareto inferior to the initial equilibrium depends on parameter values. At one extreme, suppose that $\delta_1 = 0$. In this case, the price of the worst tree is zero, $p_1 = 0$, and there is no market in any better tree, $\Theta(p) = 0$ for all $p > 0$. Absent any trade, individuals effectively live in autarky in the economy with adverse selection; in particular, buyers consume all their fruit so $\lambda = 1$. Since autarky was feasible but not optimal in the initial economy, an individual can only be better off under autarky if the quality of his trees increases sufficiently. We can rule this possibility out by assuming that δ_J is not too much larger than δ_0. More generally, continuity of equilibrium prices and liquidity in the distribution of dividends ensures that adverse selection is welfare reducing if the lower bound of the tree distribution is sufficiently close to zero and the upper bound is not too much larger than δ_0.

On the other hand, suppose buyers use all their fruit to purchase trees ($\lambda > 1$) both before and after the shock, sellers sell all their trees before the shock ($\lambda < \beta_h/\beta_l$), and average fruit production is unchanged at δ_0. Then on average tree prices are higher and sellers are better off with adverse selection. To see this, note that as long as $\lambda > 1$, buyers use all their fruit to buy trees. On the other hand, the emergence of adverse selection creates an illiquid market in trees, so sellers no longer sell all their trees. It follows that sellers' consumption is unchanged in period 0—they consume all the fruit in the economy both before and after the shock—yet they are left with a valuable tree in the economy with adverse selection. The claim follows immediately.

Perhaps more surprisingly, it is easy to construct examples in which adverse selection increases all prices and makes all sellers better off. Let $\pi_h = 0.9$, $\beta_h = 0.91$, and $\beta_l = 0.8$. If all trees produce $\delta_0 = 1$ unit of fruit, the price of trees is $p_0 = 9$, the value of a tree to a seller is $v_{l,0} = 10$, and the value of a tree to a buyer is $v_{h,0} = 11.23$. If instead ten percent of the trees produce $\delta_1 = 0.99$ and the rest produce $\delta_2 = 901/900 \approx 1.0011$, the price of bad trees increases to $p_1 = 9.55$, the price of good trees increases to 9.60, the value of tree sellers increases to $v_{l,1} = 10.54$ and $v_{l,2} = 10.55$, while the value of tree buyers falls to $v_{h,1} = 11.02$ and $v_{h,2} = 11.09$. This occurs because only a fraction $\theta_2 = 0.93$ of the type 2 trees sell, driving down the value of fruit from $\lambda = 1.12$ to 1.05 in the new economy with adverse selection. On the other hand, if $\beta_h = 0.89$, an adverse selection shock reduces all

prices, buyers' values, and sellers' values. This happens because $\lambda = 1$ both before and after the shock, shutting down the general equilibrium channels.

7.2 Flight to Quality

The emergence of adverse selection can generate a flight to quality. To explain what we mean by a flight to quality, we extend our model by introducing a safe asset. We suppose that in addition to the trees that we have already modeled—apple trees, to be concrete—there is another type of tree that is not subject to adverse selection, banana trees. Banana trees produce a known amount of fruit, apples and bananas are perfect substitutes in consumption, and either fruit can be used to purchase either type of tree. In particular, buyers value apples and bananas at a common level λ. It is straightforward to extend our definition of a competitive equilibrium to this environment.

Absent adverse selection in the apple tree market, buyers use all their fruit to purchase all the trees and both trees have the same price-dividend ratio. In particular, if in equilibrium $\lambda > 1$, buyers hold just enough fruit to purchase all the sellers' trees. The emergence of adverse selection in the apple tree market reduces the amount of fruit buyers need to purchase the apple trees at a given value of λ. Rather than consume that fruit, we have argued that the equilibrium value of fruit λ must fall to restore equilibrium. The excess fruit goes towards purchasing banana trees, driving up their price according to equation (12). This is a flight to quality.

Pushing this example further, suppose that initially some people own only apple trees and others own only banana trees, consistent with our definition of equilibrium.[8] Now the emergence of an adverse selection problem in the apple tree market means that the "natural buyers" of apple trees—those who already own apple trees—hold more apples than they need to purchase the trees that are sold each period. They therefore use some of their apples to purchase banana trees, driving up the price of those trees. The owners of apple trees may note that there are still enough apples to purchase all the trees available for sale at the old price-dividend ratio, but buyers still move towards the safe, liquid banana tree market.

More generally, the existence of a safe asset moderates the general equilibrium effects that we highlighted in the previous subsection. The smaller is the value of the apple tree market relative to the banana tree market, the more likely is it that all apple tree owners are made worse off by the emergence of adverse selection in the apple tree market. On the other hand, by driving up the value of banana trees, the same shock increases the welfare of individuals who are selling those safe assets.

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[8]A small amount of heterogeneity in the taste for apples versus bananas would make this the unique equilibrium.

7.3 Asset Purchase Program

We believe our model may be useful for understanding the potential impact of an asset purchase program, such as the original vision of the Troubled Asset Relief Program in 2008 or the Public-Private Investment Program for Legacy Assets in 2009. Both of these programs were designed to alleviate the adverse selection problem in the market for troubled assets, thereby improving also the solvency of financial institutions exposed to these assets. According to the U.S. government, this would occur not only because of the direct subsidy but also through the equilibrium effects on the price and liquidity of assets that were not sold to the government. We show that this is consistent with the predictions of our model.

To be concrete, we consider an economy with two types of trees, $j = 1, 2$, selling at prices $p_1 < p_2$ with buyer-seller ratios $\theta_1 > \theta_2$. We analyze an unexpected permanent market intervention consisting of a subsidy $\sigma(p)$ to anyone selling a tree for price p, financed by a tax of $\tau \leq \delta_1$ units of fruit per tree held by a seller at the beginning of the period. If there were only one type of tree, this policy would not distort the equilibrium allocation, but with two types of trees it distorts the equilibrium by changing the value of different types of trees.

The definition of equilibrium is unchanged except for the introduction of taxes and subsidies in the sellers' Bellman equation (1). In particular, sellers still set optimal prices for their trees given the sale probability $\Theta(p)$, internalizing the fact that they get a subsidy $\sigma(p)$ if they sell a tree for p and that in any case they pay the tax τ per tree. Let p'_j and θ'_j denote equilibrium prices and buyer-seller ratios for type j trees after the intervention. We assume the subsidy schedule is $\sigma(p) = \bar{\sigma} > 0$ for $p \leq \bar{p}$ and $\sigma(p) = 0$ otherwise. Moreover, we focus on cases where $p'_1 < \bar{p} < p'_2$, so that the sale of bad trees is subsidized but not constrained by the price cap $\bar{p}$ and the sale of good trees is not subsidized.

To start, assume that buyers consume some fruit both before and after the policy intervention, $\lambda = \lambda' = 1$.[9] The subsidy for bad trees naturally raises their price, $p'_1 > p_1$, and their value to both buyers and sellers, $v'_{s,1} > v_{s,1}$, $s = l, h$. This makes it easier to exclude bad trees from the good tree market, and so one can prove that both the resale probability for good trees and the total amount of fruit paid for good trees increase, $\theta'_2 > \theta_2$ and $\theta'_2 p'_2 > \theta_2 p_2$. As a result, the intervention unambiguously increases liquidity in the sense that both the amount of fruit transferred from buyers to sellers and the amount of trees transferred from sellers to buyers increase. In addition, it is easy to construct examples in which this simple intervention is Pareto improving, $v'_{s,j} > v_{s,j}$, including the last example in Section 7.1. It is also possible to construct examples in which the taxes drive down the price of good trees and hence the values of those trees, $p'_2 < p_2$ and $v'_{s,2} < v_{s,2}$; change the example to assume

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[9]In the absence of asymmetric information, this would be the case when $\beta_h < \pi_h$. With asymmetric information, this is the case when β_h is sufficiently small relative to π_h.

that 90 percent of trees produce $\delta_1 = 0.99$ and the rest produce $\delta_2 = 1.09$ fruit.

On the other hand, suppose $\lambda > 1$, so in the pre-intervention equilibrium buyers use all their fruit to buy trees. If after the intervention $\lambda' < \beta_h/\beta_l$, so sellers would prefer to sell all their trees, then liquidity cannot unambiguously increase: either $p_1 \geq p_1'$ or $\theta_2 p_2 \geq \theta_2' p_2'$ or both. This follows immediately from the fruit market clearing condition (11) and the restriction that $\theta_1 \leq 1$ since $\lambda > 1$ and $\theta_1' \geq 1$ since $\lambda' < \beta_h/\beta_l$. Moreover, a large subsidy that completely eliminates the illiquidity of good trees, so $\theta_2' = 1$, cannot be a Pareto improvement. Prior to the intervention, buyers use all their fruit to purchase some of the sellers trees while after the intervention they purchase all of the sellers' trees. It follows that the intervention can only reduce sellers' current consumption and it eliminates their future wealth. At least one type of seller must be worse off.

This logic hinges on the structure of our general equilibrium economy but we believe it is a general feature of an exchange economy. The logic is robust to allowing for more general policy interventions. It is also robust to allowing for a safe asset, banana trees. The subsidy that completely eliminates illiquidity in the apple tree market may then make apple tree sellers better off by raising the amount of fruit that they consume. However, if $\lambda > 1$ and $\lambda' < \beta_h/\beta_l$, this necessarily comes at the expense of the owners of banana trees, who get less fruit for their trees.

8   Conclusion

We have developed a dynamic model of asset trading in the presence of adverse selection. There always exists a unique separating equilibrium in which better assets sell for a higher price but in a less liquid market. The emergence of adverse selection causes a liquidity crisis in the sense that the volume of asset sales declines. It may also cause a decline in prices beyond the underlying decline in average dividends—i.e., a fire sale—and a flight to safe assets.

The equilibrium outcome is sensitive to the support of the distribution and especially to the lower bound of the support. For example, even a small measure of assets that do not generate a dividend is enough to shut down all trade. This recalls the behavior of markets in the presence of Knightian uncertainty, in which traders behave as if they anticipate purchasing the worst possible asset (Caballero and Krishnamurthy, 2008; Routledge and Zin, 2009; Easley and OHara, 2010). The emergence of Knightian uncertainty can similarly cause a collapse in asset prices and trading volumes, although the source of this fragility is very different in our environment.

Sensitivity to the support of the distribution distinguishes our model from an environment

in which all trade must occur at a single price (e.g. Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010; Chiu and Koeppl, 2011; Tirole, forthcoming). In this case, a moderate mean-preserving spread in dividends will not affect either prices or trading volumes. Although a quantitative evaluation of the role of adverse selection is beyond the scope of this paper, this suggests to us that our model may be useful for understanding why even a small amount of asymmetric information may have a big impact on equilibrium outcomes.[10]

Finally, we have assumed throughout our analysis that individuals' discount factors are observable. It seems natural to ask what would happen if both asset quality and trading motives were private information. In this case, patient individuals might have incentive to sell their low quality assets at a high price. We can prove that if $\lambda p_1 \geq p_J$ in our equilibrium, then the equilibrium allocation is unaffected by this additional source of private information.[11] Intuitively, an unobservable discount factor gives a patient individual an opportunity to buy a bad tree for p_1 and attempt to resell it for $p_J > p_1$. The reason that this trade might not be profitable is that the individual must use beginning-of-period fruit, which is worth λ to him, to purchase the tree and he only gets back fruit at the end of the period, which must be consumed and so is worth 1. If $\lambda p_1 < p_J$ in our equilibrium, then unobservable discount factors must change the equilibrium allocation. We are currently exploring the possibility that there exists a semi-pooling equilibrium in which individuals with different discount factors sell different types of trees at a common price.[12]

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[10]In practice, the extent of asymmetric information may be small. For example, Downing, Jaffee and Wallace (2009) argue that mortgages that originators do not resell earn a 4 to 6 basis point premium over mortgages that are bundled and resold.

[11]It is straightforward to extend our definition of equilibrium to this environment. The only change in equilibrium involves beliefs about buyer-seller ratios at very high prices: for $p > \lambda p_1$, $\Theta(p) = 0$ and $\Gamma(p)$ is arbitrary. This implies that there is no price at which a patient individual can and would sell any of his trees.

[12]The existence of a semi-pooling equilibrium is related to Chang (2011), which develops a related model with two sources of private information. Sellers know both the quality of the asset they are selling and their cost of holding the asset. However, she assumes that sellers' holding costs always exceed buyers' so there are gains from trade, that an individual's identity as a buyer or seller is known, and that buyers have excess fruit ($\lambda = 1$). We are interested in the case where there may be no gains from trade since an individual's preferences are unknown, yet scarcity of fruit ($\lambda > 1$) can sustain some trade.

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Appendix

Omitted Proofs

Proof of Lemma 1. Consider problem (P_1). Given that there is no $j' < 1$, the only constraint is (5). If such a constraint were slack, we could increase p and hence raise the value of the objective function, which ensures the constraint binds. Eliminating the price by substituting the binding constraint into the objective function gives

$$v_{l,1} = \delta_1 + \max_\theta \left(\min\{\theta, 1\} \frac{\beta_h \min\{\theta^{-1}, 1\}}{\lambda - 1 + \min\{\theta^{-1}, 1\}} + (1 - \min\{\theta, 1\})\beta_l \right) \bar{v}_1.$$

If $\lambda = 1$, any $\theta_1 \geq 1$ attains the maximum. If $\lambda = \beta_h/\beta_l$, any $\theta_1 \in [0, 1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1 = 1$. Substituting back into the original problem gives $v_{l,1} = \delta_1 + p_1$ and $p_1 = \beta_h \bar{v}_1/\lambda$, establishing the result for $j = 1$.

For $j \geq 2$ we proceed by induction. Assume for all $j' \in \{2, \ldots, j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, $v_{l,j'}$ and $\bar{v}_{j'}$ in the statement of the lemma. We first prove that $\bar{v}_j > \bar{v}_{j-1}$. To do this, consider the policy (θ_{j-1}, p_{j-1}). If this solved problem (P_j), combining the objective function and the definition of $\bar{v}_j$ gives

$$\begin{aligned}
\bar{v}_j &= \frac{\delta_j(\pi_h \lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\}p_{j-1}}{1 - \pi_h \beta_h - \pi_l \beta_l(1 - \min\{\theta_{j-1}, 1\})} \\
&> \frac{\delta_{j-1}(\pi_h \lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\}p_{j-1}}{1 - \pi_h \beta_h - \pi_l \beta_l(1 - \min\{\theta_{j-1}, 1\})} = \bar{v}_{j-1}.
\end{aligned}$$

The inequality uses the fact that the denominator is positive together with $\delta_j > \delta_{j-1}$; and the last equality comes from the objective function and the definition of $\bar{v}_{j-1}$ in problem (P_{j-1}). Since the proposed policy satisfies all of the constraints in problem (P_{j-1}) and $\bar{v}_j > \bar{v}_{j-1}$, it also satisfies all the constraints in problem (P_j). The optimal policy must deliver a weakly higher value, proving $\bar{v}_j > \bar{v}_{j-1}$.

Next we prove that at any solution to problem (P_j) the constraint (5) is binding. If there were an optimal policy (θ, p) such that it was slack, consider a small increase in p to $p' > p$ and a reduction in θ to $\theta' < \theta$ so that $\min\{\theta, 1\}(p - \beta_l \bar{v}_{j-1}) = \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j-1})$ while constraint (5) is still satisfied. Now suppose for some $j' \neq j - 1$, $\min\{\theta, 1\}(p - \beta_l \bar{v}_{j'}) < \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Subtracting the inequality from the preceding equation gives

$$\big(\min\{\theta, 1\} - \min\{\theta', 1\} \big)(\bar{v}_{j'} - \bar{v}_{j-1}) > 0.$$

Given that $\theta' < \theta$, the above inequality yields $\bar{v}_{j'} > \bar{v}_{j-1}$ and hence $j' \geq j$. This implies that the change in policy does not tighten the constraints (6) for $j' < j$, while it raises the value of the objective function in problem (P_j), a contradiction. Therefore constraint (5) must bind at the optimum.

We now show that the binding constraint (5) implies that $\theta_j \leq 1$ for all $j \geq 2$. By contradiction, assume that the solution to problem (P_j) is some (θ, p) with $\theta > 1$. In this case, the objective function reduces to $v_{l,j} = \delta_j + p$, while the constraint (6) for $j' = 1$ imposes $v_{l,1} \geq \delta_1 + p$. Since we have shown that $v_{l,1} = \delta_1 + p_1$, this implies $p \leq p_1$. Moreover, $\bar{v}_j > \bar{v}_1$ implies $\beta_h \bar{v}_j / \lambda > \beta_h \bar{v}_1 / \lambda = p_1$ and hence $\beta_h \bar{v}_j / p > \lambda$. Now a change to the policy $(1, p)$ relaxes the constraint (5) without affecting any other piece of the problem (P_j) and is therefore weakly optimal. But this cannot be optimal because (5) is slack, a contradiction. This proves that $\theta_j \leq 1$ for all $j \geq 2$ and hence, using the binding constraint (5), $p_j = \beta_h \bar{v}_j / \lambda$.

Next, we prove that if $\lambda < \beta_h / \beta_l$, the constraint (6) is binding at $j' = j-1$. We break our proof into two parts. First, consider $j = 2$ and, to find a contradiction, assume that there is a solution (θ, p) to problem (P_2) such that constraint (6) is slack for $j' = 1$. Then problem (P_2) is equivalent to problem (P_1) except for the value of the dividend $\delta_2 > \delta_1$. Following the same argument used for problem (P_1), we can show that $\theta_2 \geq 1$ and so constraint (6) reduces to $v_{l,1} \geq \delta_1 + p_2$. But since $p_1 = \beta_h \bar{v}_1 / \lambda < p_2 = \beta_h \bar{v}_2 / \lambda$, this contradicts $v_{l,1} = \delta_1 + p_1$. Constraint (6) must bind when $j = 2$.

Next consider $j > 2$ and again assume by contradiction that there is a solution (θ, p) to problem (P_j) such that constraint (6) is slack for $j' = j-1$. Then problem (P_j) is equivalent to problem (P_{j-1}) except in the value of the dividend δ. Since constraint (6) is binding in the solution to problem (P_{j-1}) and $\theta_{j-1} \leq 1$, we have

$$v_{l,j-2} = \delta_{j-2} + \theta_{j-1} p_{j-1} + (1 - \theta_{j-1}) \beta_l \bar{v}_{j-2} = \delta_{j-2} + \theta p + (1 - \theta) \beta_l \bar{v}_{j-2},$$

and hence

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2}) = \theta(p - \beta_l \bar{v}_{j-2}). \tag{17}$$

Since $p = \beta_h \bar{v}_j / \lambda$ and $p_{j-1} = \beta_h \bar{v}_{j-1} / \lambda$, $p - \beta_l \bar{v}_{j-2} > p_{j-1} - \beta_l \bar{v}_{j-2} > 0$ and so $\theta_{j-1} > \theta > 0$. But now combine equation (17) with $\theta_{j-1} > \theta$ and $\bar{v}_{j-1} > \bar{v}_{j-2}$ to get

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-1}) < \theta(p - \beta_l \bar{v}_{j-1}).$$

This implies that constraint (6) for $j' = j - 1$ is violated, a contradiction. This proves that constraint (6) must bind whenever $\lambda < \beta_j / \beta_l$ and establishes all the equations in the statement of the lemma.

Alternatively, suppose $\lambda = \beta_h/\beta_l$. Since $p_j = \beta_h \bar{v}_j/\lambda = \beta_l \bar{v}_j$, the objective function in problem (P_j) reduces to $v_{l,j} = \delta_j + \beta_l \bar{v}_j$, while constraint (6) imposes

$$v_{l,j'} = \delta_{j'} + \beta_l \bar{v}_{j'} \geq \delta_{j'} + \beta_l \big(\theta \bar{v}_j + (1-\theta)\bar{v}_{j'}\big)$$

for all $j' < j$. Since $\bar{v}_j > \bar{v}_{j'}$, this implies $\theta = 0$ in the solution to the problem. It is easy to verify that this is implied by the equations in the statement of the lemma.

Finally, we need to prove that there is a unique value of $\bar{v}_j > \bar{v}_{j-1}$ that solves the four equations in the statement of the lemma. Combining them we obtain

$$(1 - \pi_h \beta_h - \pi_l \beta_l)\bar{v}_j = \delta_j(\pi_l + \lambda \pi_h) + \pi_l \min\{\theta_{j-1}, 1\} \frac{(\beta_h - \beta_l \lambda)^2 \bar{v}_{j-1}\bar{v}_j}{(\beta_h \bar{v}_j - \beta_l \lambda \bar{v}_{j-1})\lambda}. \tag{18}$$

If $\lambda = \beta_h/\beta_l$, the last term is zero and so this pins down $\bar{v}_j$ uniquely. Otherwise we prove that there is a unique solution to equation (18) with $\bar{v}_j > \bar{v}_{j-1}$. In particular, the left hand side is a linearly increasing function of $\bar{v}_j$, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $\bar{v}_j \to \infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side exceeds the left hand side.

First assume $j = 2$ so $\theta_{j-1} = \theta_1 \geq 1$. Then we seek to prove that

$$(1 - \pi_h \beta_h - \pi_l \beta_l)\bar{v}_1 < \delta_2(\pi_l + \lambda \pi_h) + \pi_l \frac{(\beta_h - \beta_l \lambda)\bar{v}_1}{\lambda}.$$

Since $\bar{v}_1 = (\delta_1 \lambda(\pi_l + \lambda \pi_h))/(\lambda - \beta_h(\pi_l + \lambda \pi_h))$ and $\delta_1 < \delta_2$, we can confirm this directly. Next take $j \geq 3$. In this case, in the limit with $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side of (18) converges to

$$\delta_j(\pi_l + \lambda \pi_h) + \pi_l \theta_{j-1} \frac{(\beta_h - \beta_l \lambda)\bar{v}_{j-1}}{\lambda} > \delta_{j-1}(\pi_l + \lambda \pi_h) + \pi_l \min\{\theta_{j-2}, 1\} \frac{(\beta_h - \beta_l \lambda)^2 \bar{v}_{j-2}\bar{v}_{j-1}}{(\beta_h \bar{v}_{j-1} - \beta_l \lambda \bar{v}_{j-2})\lambda},$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2}, 1\}(p_{j-2} - \beta_l \bar{v}_{j-2}) = \theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2})$$

and the assumption $\delta_{j-1} < \delta_j$. The right hand side of the inequality is the same as the right hand side of equation (18) for type $j-1$. The desired inequality then follows by comparing the left hand side of the inequality to the left hand side of equation (18) for type $j-1$. This

completes the proof. ∎

Proof of Proposition 2.

We first prove that the solution to problem (P) describes a partial equilibrium and then prove that there is no other equilibrium.

Existence. As described in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $dF(p_j) = K_j/\sum_{j'} K_{j'}$, and $v_{s,j}$ solves problem (P_j). Also for notational convenience define $p_{J+1} = \infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p < p_1$, $\Theta(p) = \infty$ and $\Gamma(p)$ can be chosen arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \ldots, J\}$ and $p \in (p_j, p_{j+1})$, $\gamma_j(p) = 1$ and $\Theta(p)$ satisfies sellers' indifference condition $v_{l,j} = \delta_j + \min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l \bar{v}_j$; equivalently, $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$. To prove that this is a partial equilibrium, we need to verify that the five equilibrium conditions hold.

To show that the third and fourth equilibrium conditions—Buyers' Optimality and Active Markets—are satisfied, it is enough to prove that the prices $\{p_j\}$ solve the optimization problem in equation (4). Lemma 1 implies that $p_j = \beta_h \bar{v}_j / \lambda$ for all λ and j; and $\Theta(p_j) \leq 1$ if $\lambda > 1$. Together these conditions imply that any price p_j achieves the maximum in this optimization problem. For any price $p \in (p_j, p_{j-1})$, $\gamma_j(p) = 1$ by construction, and so the right hand side of equation (4) is smaller than when evaluate at p_j. Moreover, for any $p < p_1$, $\Theta(p) = \infty$ and so the right hand side is $1 \leq \lambda$.

Next we prove that $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all j and p, with equality if $p \in [p_j, p_{j+1})$. The first and second equilibrium conditions—Sellers's Optimality and Equilibrium Beliefs— follow immediately from this. The equality holds by construction. Let us now focus on the inequalities.

First take any $j' \in \{2, \ldots, J\}$, $j < j'$, and $p \in [p_{j'}, p_{j'+1})$. By the construction of Θ,

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_{j'}).$$

Then $p_{j'} \leq p$ implies that $\min\{\Theta(p_{j'}), 1\} \geq \min\{\Theta(p), 1\}$. Since $j < j'$, Lemma 1 implies that $\bar{v}_{j'} > \bar{v}_j$ and so $\min\{\Theta(p_{j'}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$. Also condition (6) in problem $(P_{j'})$ implies $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j)$. Combining the last two inequalities gives $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all $p \in [p_{j'}, p_{j'+1})$ and $j < j'$.

Similarly, take any $j' \in \{1, \ldots, J-1\}$, $j > j'$, and $p \in [p_{j'}, p_{j'+1})$. The construction of Θ

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implies $\min\{\Theta(p_{j'}),1\}(p_{j'}-\beta_l\bar{v}_{j'})=\min\{\Theta(p),1\}(p-\beta_l\bar{v}_{j'})$, while Lemma 1 together with $\Theta(p_j)=\theta_j$ implies $\min\{\Theta(p_{j'}),1\}(p_{j'}-\beta_l\bar{v}_{j'})=\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_{j'})$. The two equalities together imply

$$\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_{j'})=\min\{\Theta(p),1\}(p-\beta_l\bar{v}_{j'})$$

Then $p_{j'+1}>p$ implies $\min\{\Theta(p_{j'+1}),1\}\leq\min\{\Theta(p),1\}$. Since $j>j'$, Lemma 1 implies that $\bar{v}_j>\bar{v}_{j'}$ and so $\min\{\Theta(p_{j'+1}),1\}(\bar{v}_{j'}-\bar{v}_j)\geq\min\{\Theta(p),1\}(\bar{v}_{j'}-\bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_j)\geq\min\{\Theta(p),1\}(p-\beta_l\bar{v}_j)$. Also, since $(\Theta(p_{j'+1}),p_{j'+1})$ is a feasible policy in problem (P_j), $\min\{\Theta(p_j),1\}(p_j-\beta_l\bar{v}_j)\geq\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_j)$. Combining inequalities gives $\min\{\Theta(p_j),1\}(p_j-\beta_l\bar{v}_j)\geq\min\{\Theta(p),1\}(p-\beta_l\bar{v}_j)$ for all $p\in[p_{j'},p_{j'+1})$ and $j>j'$.

Finally, consider $p<p_1$. Since $\Theta(p)=\infty$, $\min\{\Theta(p),1\}(p-\beta_l\bar{v}_j)=p-\beta_l\bar{v}_j<p_1-\beta_l\bar{v}_j\leq\min\{\Theta(p_1),1\}(p_1-\beta_l\bar{v}_j)$, where the first inequality uses $p<p_1$ and the second uses the fact that $\Theta(p_1)<1$ only if $\lambda=\beta_h/\beta_l$; but in this case, $p_1=\beta_l\bar{v}_1\leq\beta_l\bar{v}_j$. Since we have already proved that $\min\{\Theta(p_1),1\}(p_1-\beta_l\bar{v}_j)\leq\min\{\Theta(p_j),1\}(p_j-\beta_l\bar{v}_j)$, this establishes the inequality for $p<p_1$.

The last piece of the definition of equilibrium is Consistency of Supplies with Beliefs. This holds by the construction of the distribution function F in the statement of the Proposition.

Uniqueness. Now take any partial equilibrium $\{\{v_{h,j}\},\{v_{l,j}\},\Theta,\Gamma,F\}$. We first claim that $\bar{v}$ is increasing in j. Take $j>j'$ and let $p_{j'}$ denote the price offered by j'. Type j Sellers' Optimality implies

$$v_{l,j}\geq\delta_j+\min\{\Theta(p_{j'}),1\}p_{j'}+(1-\min\{\Theta(p_{j'}),1\})\beta_l\bar{v}_j,$$

and so combining with type j Buyers' Optimality, equation (3), and solving for $\bar{v}_j$ gives

$$\bar{v}_j\geq\frac{\delta_j(\pi_l+\pi_h\lambda)+\pi_l\min\{\Theta(p_{j'}),1\}p_{j'}}{\pi_l(1-\min\{\Theta(p_{j'}),1\})\beta_l+\pi_h\beta_h}>\frac{\delta_{j'}(\pi_l+\pi_h\lambda)+\pi_l\min\{\Theta(p_{j'}),1\}p_{j'}}{\pi_l(1-\min\{\Theta(p_{j'}),1\})\beta_l+\pi_h\beta_h}=\bar{v}_{j'},$$

where the second inequality uses $\delta_j>\delta_{j'}$ and the equality solves the same equations for $\bar{v}_{j'}$.

Consistency of Supplies with Beliefs implies that for each $j\in\{1,\dots,J\}$, there exists a price $p_j\in\mathbb{P}$ with $\gamma_j(p_j)>0$.

Now in the remainder of the proof, assume also that $\theta_j\equiv\Theta(p_j)>0$. First we prove that the constraint $\lambda\leq\min\{\theta_j^{-1},1\}\beta_h\bar{v}_j/p_j+(1-\min\{\theta_j^{-1},1\})$ is satisfied. Second we prove that the constraint $v_{l,j'}\geq\delta_{j'}+\min\{\theta_j,1\}p_j+(1-\min\{\theta_j,1\})\beta_l\bar{v}_{j'})$ is satisfied for all $j'<j$. Third we prove that the pair (θ_j,p_j) delivers value $v_{l,j}$ to sellers of type j trees. Fourth we

prove that (θ_j, p_j) solves (P_j).

Step 1. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. Active Markets implies that the expected value of a unit of fruit to a buyer who pays p_j must equal λ and so there must be a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. If $\theta_j = \infty$, $\min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\} = 1 \leq \lambda$, which is impossible; therefore $\theta_j < \infty$. Then Equilibrium Beliefs implies p_j is an optimal price for type j' sellers and so for all p' and $\theta' \equiv \Theta(p')$, $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Since $\theta_j > 0$, $\min\{\theta_j, 1\}(p' - \beta_l \bar{v}_{j'}) > \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'})$ for all $p' > p_j$, and so the two inequalities imply $\min\{\theta_j, 1\} > \min\{\theta', 1\}$.

Now take any $j'' < j'$, so $\bar{v}_{j''} < \bar{v}_{j'}$. Then since $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$, $\min\{\theta_j, 1\} > \min\{\theta', 1\}$, and $\bar{v}_{j''} < \bar{v}_{j'}$,

$$\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j''}) > \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j''}).$$

Type j'' Sellers' Optimality condition implies $\bar{v}_{j''} \geq \delta_{j''} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j''}$ and so the previous inequality gives $\bar{v}_{j''} > \delta_{j''} + \min\{\theta', 1\}p' + (1 - \min\{\theta', 1\})\beta_l \bar{v}_{j''}$. Rational beliefs implies $\gamma_{j''}(p') = 0$. That is, any $p' > p_j$ attracts only type j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1}, 1\}\beta_h \bar{v}_{j'}/p' + (1 - \min\{\theta'^{-1}, 1\})$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Sellers' Optimality implies $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j', p_j, and $\theta_j = \Theta(p_j)$.

Step 3. Equilibrium Beliefs implies $v_{l,j} = \delta_j + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j, p_j, and $\theta_j = \Theta(p_j) < \infty$ with $\gamma_j(p_j) > 0$.

Step 4. Suppose there is a policy (θ, p) that satisfies the constraints of problem (P_j) and delivers a higher payoff. That is,

$$v_{l,j} < \delta_j + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_j$$
$$\lambda \leq \min\{\theta^{-1}, 1\}\beta_h \bar{v}_j/p + 1 - \min\{\theta^{-1}, 1\}$$
$$v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_{j'} \text{ for all } j' < j.$$

If these inequalities hold with $\theta > 1$, then the same set of inequalities holds with $\theta = 1$, and so we may assume $\theta \leq 1$ without loss of generality. Choose $p' < p$ such that

$$v_{l,j} < \delta_j + \theta p' + (1 - \theta)\beta_l \bar{v}_j \tag{19}$$
$$\lambda < \beta_h \bar{v}_j/p' \tag{20}$$
$$v_{l,j'} > \delta_{j'} + \theta p' + (1 - \theta)\beta_l \bar{v}_{j'} \text{ for all } j' < j. \tag{21}$$

The previous inequalities imply that this is always feasible by setting p' close enough to p. Now sellers' optimality implies $v_{l,j} \geq \delta_j + \min\{\Theta(p'), 1\}p' + (1 - \min\{\Theta(p'), 1\})\beta_l\bar{v}_j)$, which, together with inequality (19), implies $\Theta(p') < \theta$. This together with inequality (21) implies that

$$v_{l,j'} > \delta_{j'} + \Theta(p')p' + (1 - \Theta(p'))\beta_l\bar{v}_{j'} \text{ for all } j' < j,$$

and so, due to Equilibrium Beliefs, $\gamma_{j'}(p') = 0$ for all $j' < j$. But then, using inequality (20), we obtain

$$\lambda < \frac{\beta_h\bar{v}_j}{p'} \leq \frac{\beta_h\sum_{j'=1}^J \gamma_{j'}(p')\bar{v}_{j'}}{p'} = \min\{\Theta(p')^{-1}, 1\}\frac{\beta_h\sum_{j'=1}^J \gamma_{j'}(p')\bar{v}_{j'}}{p'} + \left(1 - \min\{\Theta(p')^{-1}, 1\}\right),$$

where the second inequality uses monotonicity of $\bar{v}_j$ and $\gamma_{j'}(p') = 0$ for $j' < j$; and the last equation uses $\Theta(p') < \theta \leq 1$. This contradicts Buyers' Optimality condition and completes the proof. ∎

Proof of Proposition 3. To prove that there exists a unique competitive equilibrium, it is enough to prove that there exists a unique $\lambda \in [1, \beta_h/\beta_l]$ such that the partial equilibrium associated to that λ clears the fruit market.

For given $\lambda \in [1, \beta_h/\beta_l]$, let $x_j(\lambda) \equiv \theta_j(\lambda)p_j(\lambda)$, where $\theta_j(\lambda)$ and $p_j(\lambda)$ are the partial equilibrium sale probability and price for trees of type j. For all $j > 1$ and given $x_{j-1}(\lambda)$, define

$$f_j(x_j, \lambda) \equiv x_j\left[1 - \frac{\beta_l}{\beta_h}\lambda\frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - x_{j-1}(\lambda)\left[1 - \frac{\beta_l}{\beta_h}\lambda\right],$$

where, with some abuse of notation,

$$p_j(x_j, \lambda) = \frac{\delta_j\beta_h(\pi_l + \lambda\pi_h) + x_j\pi_l[\beta_h - \beta_l\lambda]}{\lambda(1 - \bar{\beta})}. \tag{22}$$

For given $\lambda \in [1, \beta_h/\beta_l]$, Proposition 2 and Lemma 1 ensure that $p_j(x_j(\lambda), \lambda)$ is the equilibrium price for type-j trees with $x_j(\lambda)$ being implicitly defined by $f_j(x_j, \lambda) = 0$ for all $j > 1$. Moreover, for $\lambda \in (1, \beta_h/\beta_l)$

$$x_1(\lambda) = p_1(x_1(\lambda), \lambda) = \frac{\delta_1\beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{23}$$

Lemma 1 also implies that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ for all $j > 1$. From $f_j(x_j, \lambda) = 0$ for all $j > 1$ immediately follows that $x_j(\lambda) < x_{j-1}(\lambda)$ for all $j > 1$.

Next, define $M(\lambda)$ as

$$M(\lambda) \equiv \sum_{j=1}^{J} [\pi_h \delta_j - \pi_l x_j(\lambda)] K_j.$$

Market clearing requires $M(\lambda) = 0$. Now we show that $x_j'(\lambda) < 0$ and hence $M'(\lambda) > 0$ for all $\lambda \in (1, \beta_h/\beta_l)$. For $j = 1$ we can directly calculate

$$x_1'(\lambda) = -\frac{\delta_1 \beta_h \pi_l}{[\lambda - \beta_h(\pi_l + \lambda \pi_h)]^2} < 0.$$

For all $j > 1$, given $x_{j-1}'(\lambda) < 0$ we can proceed recursively as follows. Applying the implicit function theorem to $f_j(x_j, \lambda) = 0$, we obtain

$$x_j'(\lambda) = -\frac{\partial f_j(x_j, \lambda)/\partial \lambda}{\partial f_j(x_j, \lambda)/\partial x_j}.$$

First, we can calculate

$$\frac{\partial f_j(x_j, \lambda)}{\partial x_j} = 1 - \frac{\beta_l}{\beta_h} \lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} + x_j \frac{\beta_l}{\beta_h} \lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)^2} \frac{\partial p_j(x_j, \lambda)}{\partial x_j}.$$

It is easy to show that $\partial f_j(x_j, \lambda)/\partial x_j > 0$ given that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ and

$$\frac{\partial p_j(x_j, \lambda)}{\partial x_j} = \frac{\pi_l[\beta_h - \beta_l \lambda]}{\lambda(1 - \bar{\beta})} > 0.$$

Second, we can calculate

$$\begin{aligned}
\frac{\partial f_j(x_j, \lambda)}{\partial \lambda} = &\frac{\beta_l}{\beta_h} \left[x_{j-1}(\lambda) - x_j \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} \right] - \left(1 - \frac{\beta_l}{\beta_h} \lambda \right) x_{j-1}'(\lambda) \\
&- \frac{\beta_l}{\beta_h} \lambda \frac{x_j}{p_j(x_j, \lambda)} \frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial x_{j-1}(\lambda)} x_{j-1}'(\lambda) \\
&- \frac{\beta_l}{\beta_h} \lambda \frac{x_j}{p_j(x_j, \lambda)} \left[\frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial \lambda} - \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} \frac{\partial p_j(x_j, \lambda)}{\partial \lambda} \right]
\end{aligned}$$

where the first term is positive because $x_j(\lambda) < x_{j-1}(\lambda)$ and $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$, the second term is positive because $\lambda \in (1, \beta_h/\beta_l)$ and $x_{j-1}'(\lambda) < 0$, and the third term is positive because of the last inequality together with $\partial p_j(x_j, \lambda)/\partial x_j > 0$. Finally, to show that the last term is also positive we need to show that the term in square bracket is negative where

$$\frac{\partial p_j(x_j, \lambda)}{\partial \lambda} = -\frac{\beta_h \pi_l(\delta_j + x_j)}{\lambda^2(1 - \bar{\beta})}.$$

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Using expression (22) for $p_j(x_j, \lambda)$ and $f_j(x_j, \lambda) = 0$ for all j, after some algebra, one can show that this is always the case given that $\lambda \in (1, \beta_h/\beta_l)$. This implies that $x'_j(\lambda) < 0$ for all j and hence $M'(\lambda) > 0$.

Finally, define

$$\underline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(0)]K_j} \text{ and } \overline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(\beta_h/\beta_l - 1)]K_j},$$

where $x_1(\lambda)$ is given in equation (23) and $x_j(\lambda)$ solves $f_j(x_j, \lambda) = 0$ for all $j > 1$. It is easy to see that $\underline{\pi} < \overline{\pi}$ given that $x'_j(\lambda) < 0$. Moreover, $M(0) < 0$ iff $\pi_l > \underline{\pi}$ and $M(\beta_h/\beta_l - 1) > 0$ iff $\pi_l < \overline{\pi}$. Given that $M'(\lambda) > 0$, it follows that if $\pi_l \in (\underline{\pi}, \overline{\pi})$, there exists a unique equilibrium with $\lambda \in (1, \beta_h/\beta_l)$. If instead $\pi_l \leq \underline{\pi}$, then both $M(0)$ and $M(\beta_h/\beta_l - 1)$ are larger than zero, while if $\pi_l \geq \overline{\pi}$, they are both smaller than zero. Lemma 1 implies that $x_1(\lambda) \geq p_1(\lambda)$ if $\lambda = 1$ and $x_1(\lambda) \leq p_1(\lambda)$ if $\lambda = \beta_h/\beta_l$. This implies that if $\pi_l \leq \underline{\pi}$, there exists a unique equilibrium with $\lambda = 1$, where $x_1(0) \geq p_1(0)$ is pinned down by market clearing. If instead $\pi_l \geq \overline{\pi}$, then there exists a unique equilibrium with $\lambda = \beta_h/\beta_l$, where $x_1(0) \leq p_1(0)$ is pinned down by market clearing. This completes the proof. ∎

Proof of Proposition 4. We start by establishing that equations (12)–(15) describe an equilibrium. First, in any competitive equilibrium, Sellers' Optimality and Buyers' Optimality imply

$$v_l(\delta) = \delta + \Theta(P(\delta))P(\delta) + (1 - \Theta(P(\delta)))\beta_l \bar{v}(\delta),$$
$$v_h(\delta) = \delta\lambda + \beta_h \bar{v}(\delta).$$

Adding π_l times the first equation to π_h times the second and solving for $\bar{v}(\delta)$ gives

$$\bar{v}(\delta) = \frac{\delta(\pi_l + \pi_h\lambda) + \pi_l\Theta(P(\delta))P(\delta)}{1 - \pi_h\beta_h - \pi_l(1 - \Theta(P(\delta)))\beta_l}.$$

Then substitute for $\delta = D(P(\delta))$ using equation (14) and simplify to get $P(\delta) = \beta_h \bar{v}(\delta)/\lambda$, consistent with Lemma 1 in the discrete-type economy. Next, Equilibrium Beliefs implies $P(\delta)$ maximizes $\Theta(p)(p - \beta_l \bar{v}(\delta))$. Using equation (13) for $\Theta(p)$, differentiate this expression to show that it is increasing in p when $p < P(\delta)$ and decreasing when $p > P(\delta)$, where $P(\delta)$ is given by the previous paragraph. The uniquely optimal price for a type δ tree is $P(\delta)$. Thus these prices and this value function satisfy Equilibrium Beliefs. Next, any $p > \underline{p}$ delivers value λ to a buyer by construction, satisfying Active Markets. Consistency of Supply with

Beliefs pins down the amount of trees available at each price, $F(P(\delta)) = G(\delta)$ for all δ. With this, the fruit market clearing condition reduces to condition (15).

To show that this is the unique limit of the economy with a finite number of trees, start with the condition that the seller of a type $j \geq 2$ tree must be indifferent about representing it as a type $j+1$ tree. Since $\Theta(p_j) < 1$, $\Theta(p_{j+1})(p_{j+1} - \beta_l \bar{v}_j) = \Theta(p_j)(p_j - \beta_l \bar{v}_j)$, or equivalently

$$\frac{\Theta(p_{j+1}) - \Theta(p_j)}{p_{j+1} - p_j} = -\frac{\Theta(p_j)}{p_{j+1} - \beta_l \bar{v}_j}.$$

Now eliminate $\bar{v}_j$ using the buyer's indifference condition $\bar{v}_j = p_j \lambda / \beta_h$ and take the limit as $\delta_{j+1} \to \delta_j$, so $p_{j+1} \to p_j$. This gives

$$\Theta'(p_j) = -\frac{\beta_h \Theta(p_j)}{p_j(\beta_h - \beta_l \lambda)}.$$

If $\lambda = \beta_h / \beta_l$, this implies $\Theta(p) = 0$ for all $p > \underline{p}$. Otherwise, solve this differential equation using the terminal condition $\Theta(\underline{p}) = 1$ to get equation (13). The remaining expressions follow immediately from the Bellman equations. ∎

Online Appendix

Individual's Problem: Details

For any period t, history s^{t-1}, and type $j \in \{1, \ldots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote individual i's beginning-of-period t holdings of type j trees. For any period t, history s^t, type $j \in \{1, \ldots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of type j trees at a price $p \in P$. The individual chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t).$$

This simply states that the individual maximizes the expected discounted value of consumption, given the stochastic process for the discount factor. The individual faces a standard budget constraint,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp,$$

for all t and s^t. The left hand side is the fruit produced by the trees he owns at the start of period t. The right hand side is consumption plus the net purchase of trees at nonnegative prices p. He also faces a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t),$$

for all $j \in \{1, \ldots, J\}$. This states that the increase in his tree holdings is given by his net purchase of that type of tree. Finally, the individual faces a set of constraints that depends on whether his discount factor is high or low.

If the individual has a high discount factor, $s_t = h$, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he must have enough fruit to purchase trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp.$$

If the individual wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $\Theta(p)pq$ fruit to the market to make this purchase. This constrains his ability to buy trees in markets with excess demand. Together with the budget constraint, this also ensures consumption is nonnegative. Finally, he can only purchase type j trees at a price p if individuals are selling them at that price, that is

$$q_{i,j,t}(P; s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\}; s^t) \right) dp$$

for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of type j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of type j.

If the individual has a low discount factor, $s_t = l$, he is a seller, which implies $q_{i,j,t}(P; s^t)$ is nonpositive for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq - \int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_{i,j,t}(\{p\}; s^t) dp,$$

for all $j \in \{1, \ldots, J\}$. Each tree only sells with probability $\min\{\Theta(p), 1\}$ at price p, so if $\Theta(p) < 1$, an individual must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $\bar{V}^*(\{k_j\})$ be the supremum of the individuals' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function $\bar{V}^*$ satisfies the following functional equation:

$$\bar{V}(\{k_j\}) = \pi_h V_h(\{k_j\}) + \pi_l V_l(\{k_j\}), \tag{24}$$

where

$$V_h(\{k_j\}) = \max_{\{q_j, k'_j\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h \bar{V}(\{k'_j\}) \right) \qquad (25)$$

subject to $k'_j = k_j + q_j(\mathbb{R}_+)$ for all $j \in \{1, \ldots, J\}$

$$\sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp,$$

$$q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+$$

$$q_j(P) \geq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,$$

and

$$V_l(\{k_j\}) = \max_{\{q_j, k'_j\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l \bar{V}(\{k'_j\}) \right) \qquad (26)$$

subject to $k'_j = k_j + q_j(\mathbb{R}_+)$ for all $j \in \{1, \ldots, J\}$

$$k_j \geq - \int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\},$$

$$q_j(P) \leq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,$$

We now prove Proposition 1 working with the recursive version of the individuals' problem.

Let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_{s,j}\}$ and λ that solve the Bellman equations (1), (3), and (4) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling type j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p) \left(p - \beta_l \bar{v}_j \right)$$

for all δ. We seek to prove that $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^{J} \bar{v}_j k_j$ where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.

If $\lambda = 1$, equations (1) and (3) imply

$$\bar{v}_j = \pi_h \left(\delta_j + \beta_h \bar{v}_j \right) + \pi_l \left(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j \right).$$

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for all δ. Equivalently,

$$\bar{v}_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j}) p_{l,j}}{1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 1$, the same equations imply

$$\bar{v}_j = \pi_h \left(\delta_j \left((1 - \bar{\Theta}(p_h)^{-1}) + \bar{\Theta}(p_h)^{-1} \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h} \right) + \beta_h \bar{v}_j \right)$$

$$+ \pi_l \left(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j \right)$$

for all δ. Since $v_{l,j}$ and $v_{h,j}$ are positive by assumption so is $\bar{v}_j$, and equivalently we can write

$$\bar{v}_j \left(1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j})) - \pi_h \beta_h \bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h \bar{v}_j} \right)$$

$$= \pi_h \delta_j \left(1 - \bar{\Theta}(p_h)^{-1} \right) + \pi_l \left(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} \right).$$

The right hand side of this expression is positive for all j. Once again since $\bar{v}_j > 0$, with $\lambda > 1$, this holds if and only if

$$1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j})) > \pi_h \beta_h \bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h \bar{v}_j}. \tag{27}$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for an individual to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $\bar{V}(\{k_j\}) = \sum_{j=1}^{J} \bar{v}_j k_j$ and $V_s(\{k_j\}) \equiv \sum_{j=1}^{J} v_{s,j} k_j$ for $s = l, h$. It is straightforward to prove that $\bar{V}$ and $\bar{V}_s$ solve equations (24), (25), and (26) and that the same policy is optimal.

Finally, we adapt Theorem 4.3 from Werning (2009), which states the following: suppose $\bar{V}(k)$ for all k satisfies the recursive equations (24), (25), and (26) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k_{i,j,t}^*(s^{t-1})\}$ satisfying

$$\lim_{t \to \infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \bar{V}(\{k_{i,j,t}^*(s^{t-1})\}) = 0. \tag{28}$$

Then, $\bar{V}^* = \bar{V}$.

If $\lambda = 1$, an optimal plan is to sell type j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of $\bar{V}$, condition (28) holds trivially.

If $\lambda > 1$, it is still optimal to sell type j trees at price $p_{l,j}$ when impatient, but patient individuals purchase trees at price p_h and do not consume. Thus

$$k'_{h,j} = k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^J \delta_{j'}k_{j'}}{p_h}$$

$$k'_{l,j} = \big(1 - \underline{\Theta}(p_{l,j})\big)k_j.$$

Using linearity of the value function, the expected discounted value next period of an individual with tree holdings $\{k_j\}$ this period is

$$\sum_{j=1}^J \bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k'_{l,j}\big)$$

$$= \sum_{j=1}^J \bar{v}_j\left(\pi_h\beta_h\left(k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^J \delta_{j'}k_{j'}}{p_h}\right) + \pi_l\beta_l\big(1 - \underline{\Theta}(p_{l,j})\big)k_j\right)$$

$$= \sum_{j=1}^J \bar{v}_j k_j\left(\pi_h\beta_h + \pi_l\beta_l\big(1 - \underline{\Theta}(p_{l,j})\big) + \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^J \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right),$$

where the second equality simply rearranges terms in the summation. Equation (27) implies that each term of this sum is strictly smaller than $\bar{v}_j k_j$. This implies that there exists an $\eta < 1$ such that

$$\eta > \frac{\sum_{j=1}^J \bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k_{l,j}\big)}{\sum_{j=1}^J \bar{v}_j k_j} = \frac{\pi_h\beta_h\bar{V}(\{k'_{h,j}\}) + \pi_l\beta_l\bar{V}(\{k'_{l,j}\})}{\bar{V}(\{k_j\})},$$

and so condition (28) holds.