A Common Value Auction with Bidder Solicitation*

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We study a common-value, first-price auction in which the number of bidders is endogenous: the seller (auctioneer) knows the value and solicits bidders at a cost. The number of bidders, which is unobservable to them, may thus depend on the true value, giving rise to a *solicitation effect* – being solicited already conveys information. The solicitation effect is a key difference from standard common value auctions. In contrast to standard auctions, the equilibrium bid distribution may exhibit atoms. We also discuss information aggregation in the case of small bidder solicitation cost. We show that there is a type of equilibrium that aggregates information well when the most favorable signals are informative. However, there may also be an equilibrium that fails to aggregate any information.

This paper considers a single good, common values, first-price auction in which the number of bidders is endogenous: the seller (auctioneer) knows the value and solicits bidders at a constant cost per sampled bidder. The bidders do not know the value and do not observe the number of solicited bidders. Each bidder only learns that he was summoned to the auction and observes a private noisy signal of the true value. In equilibrium, bidders bid optimally given their signals, the bidding strategies of others, and the solicitation strategy of the seller. The seller chooses optimally how many bidders to solicit given the true value and the bidders' strategies.

The novel feature is the endogenous solicitation of the bidders by the informed seller. It implies that the number of bidders may vary across the different value states, which gives rise to a *solicitation effect:* The mere fact of being summoned

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by the seller to bid conveys information to the bidder. This effect is a key difference between the analysis of the present paper and that of standard common value auctions. The relationship between the underlying value and the number of bidders is not always in the same direction. It is possible to have equilibria in which a seller of a high value good samples more bidders (solicitation is "good news," a solicitation *blessing*) and equilibria in which a seller of a low value good samples more bidders (solicitation is "bad news," a solicitation *curse*).

This analysis has two objectives. First, to develop an understanding of the new model introduced here and, in particular, the nature of its equilibria. As will be seen later, the endogenous solicitation produces some peculiarities in the structure of equilibrium. Second, an exploration of the question of information aggregation—the relation between the expected winning bid (price) and the true value—when the cost of soliciting bidders is small. The question of information aggregation by markets is of central importance in economics and probably does not require justification. One of our main observations is that sometimes a market of the sort we consider may totally fail to aggregate the information, even under conditions that normally are viewed as conducive to successful aggregation (participation of many bidders and existence of highly informative signals).

Although the model features an auction augmented by a preliminary solicitation stage, we think of it as a model of a market with adverse selection in which an informed agent contacts simultaneously potential partners rather than a model of a formal auction mechanism. In line with this, the analysis does not adopt the perspective of mechanism design. In particular, it does not endow the auctioneer with the power to commit to an optimal solicitation policy. It is possible of course to explore this side as well, inquiring about optimal solicitation by a seller who has full or partial commitment power, but we chose here to focus mainly on the other aspects.

We did not construct the model with a specific market in mind, but many markets share the feature that an informed agent contacts a number of partially informed agents for a potential transaction. For example, a potential borrower who contacts several lenders in an attempt to obtain funding for a project on which she has private information. The lenders obtain noisy signals and offer terms, while being aware that the borrower might be applying to other lenders. The borrower is the counterpart of the seller in our model (selling a bond) and the lenders are the bidders.[1] Of course, in actual markets of this sort the contact is sometimes indeed simultaneous,

[1] Broecker (1990) and Riordan (1993) have modeled this situation as an ordinary common value auction. However, accounting for endogenous (and unobservable) solicitation of terms by the borrower is natural in this environment and might produce new insights.

sometimes sequential and sometimes a combination of the two. We have explored the sequential scenario in a separate paper (Lauermann and Wolinsky (2012)) and focus here on the simultaneous case.

There are two states: in state h the value of the good is high for all bidders; in state l it is low for all bidders. The seller knows the true value and selects the desired number of bidders randomly from a large population of potential bidders. Solicited bidders obtain conditionally independent signals and participate in a first-price auction. We characterize symmetric equilibria in which all bidders use the same bidding strategies.

The analysis yields the following insights. The equilibrium bid distributions may exhibit atoms–bidders with different signals submit the same bid. This observation contrasts with the standard intuition that atoms induce bidders to overbid them slightly. This intuition fails when winning in the atom is profitable only in the high value state, but on average more bids fall in the atom in the low value state. The atom then protects bidders in the low value state: Overbidding it would increase the probability of winning more significantly when winning involves a loss and hence is unprofitable. One may think that a similar consideration could give rise to atoms in an ordinary common value auction, in which the number of bidders is constant across states. However, it turns out that this is not the case. In an ordinary common value auction, an atom may arise only under special circumstances (only at the bottom of the bid distribution and only if there is a mass of the lowest signals that share the same information content).

When the solicitation cost is small, there are at most two kinds of distributions of the winning bid that may arise in equilibrium: a nearly atomless distribution and a distribution that places nearly all the mass on a common price below the ex-ante expected value.

The nearly atomless outcome qualitatively resembles the equilibrium outcome of an ordinary common value auction. It is partially revealing in the sense that the expected winning bid differs across the states. If the most favorable signal is very informative, it aggregates information well, in the sense that the expected winning bid is close to the true value.

The outcome that exhibits the atom–the "pooling" outcome–fails to aggregate information. It is interesting to note that this outcome may arise even when the sampling cost is small and the most favorable signals are very informative. Moreover, since the atom occurs at a price that is strictly below the ex-ante expected value, the price not only fails to approach the value in each state but also the expected revenue is strictly below the expected ex-ante value despite the fact that many bidders participate and signals may be very informative.

Existence of equilibrium is established for the case in which the set of feasible bids is any sufficiently fine grid. We show that a partially revealing equilibrium always exists. The pooling equilibrium exists under additional assumptions on the distribution that generates the signal. We do not know whether a pooling equilibrium exists for all signal distributions.

As a by-product of the analysis, we derive explicitly the distribution of the winning bid of the ordinary auction when the number of bidders goes to infinity. We are not familiar with such derivation in the relevant literature.

Three strands of related literature should be mentioned. First, there is an obvious relation to the literature on common value auctions.[2] This paper complements that literature by adding the endogenous solicitation of bidders, which may change the nature of the equilibrium in a significant way. In particular, it complements the discussion within that literature of information aggregation by the price when the number of bidders becomes exogenous large (Wilson (1977) and Milgrom (1979, 1981)). Our results imply that exogenously large auctions may aggregate information quite differently from the endogenously large auctions studied here.

Second, Lauermann and Wolinsky (2012) studies the sequential search counterpart of the present paper (the counterpart of the auctioneer samples the counterparts of the bidders sequentially rather than simultaneously). The results of the present paper on information aggregation fall between the results of the two literatures just mentioned. Under the informational assumptions of the present paper, in the common values auction the equilibrium is partially revealing and, with a large number of bidders, aggregates the information well when there are highly informative signals (i.e., the likelihood ratio associated with the most favorable signal is large). In contrast, still under the same conditions on the informativeness of signals, in the sequential search environment information aggregation always completely fails. The present paper exhibits results of these two types in different equilibria.

Finally, the present model can be interpreted as a simultaneous ("batch-")search model like Burdett-Judd (1983), with the added feature of adverse selection. Section 9 explains further this connection.

1 Model

Basics.—This is a single-good, common value, first-price auction environment with two underlying states, h and l. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_l and v_h, with $0 \leq v_l < v_h$. The seller's cost is zero.

[2] This literature is too voluminous to mention an arbitrary selection of names.

Nature draws a state $w \in \{l, h\}$ with prior probabilities $\rho_l > 0$ and $\rho_h > 0$, $\rho_l + \rho_h = 1$. The seller learns the realization of the state w and invites n_w bidders, $n_w \leq N$. If $n_w < N$, the seller selects the invitees randomly with equal probability. We use $\mathbf{n}$ to denote the vector (n_l, n_h).

The seller incurs a solicitation cost $s > 0$ for each invited bidder. We assume that $N \geq \frac{v_h}{s}$. Therefore, N does not constrain the seller.

Each invited bidder observes a private signal $x \in [\underline{x}, \bar{x}]$ and submits a bid b from a set of feasible bids P_Δ. Conditional on the state, signals are independently and identically distributed according to a cumulative distribution G_w, $w \in \{h, l\}$. A bidder does not observe w nor how many other bidders are invited to bid.

The invited bidders bid simultaneously: The highest bid wins and ties are broken randomly with equal probabilities.

If in state $w \in \{h, l\}$ the winning bid is p, then the payoffs are $v_w - p$ for the winning bidder and zero for all others. The seller's payoff is $p - n_w s$.

Further Details.—The set of feasible bids P_Δ may either be the full interval $[0, v_h]$ or a grid

$$P_\Delta = \begin{cases} [0, v_h] & \text{if} \quad \Delta = 0, \\ [0, v_l] \cup \{v_l + \Delta, v_l + 2\Delta, \cdots, v_h - \Delta, v_h\} & \text{if} \quad \Delta > 0. \end{cases}$$

Here, Δ is the step size of the grid. Notice that we leave the continuum of prices on $[0, v_l]$ even when $\Delta > 0$. This avoids some irrelevant distinctions between the case in which the bottom equilibrium bid is v_l and the case in which it is $v_l - \Delta$. Much of the following analysis holds for both of the cases $\Delta > 0$ and $\Delta = 0$. We will mention it explicitly when the discussion focuses on just one of these cases.

The signal distributions G_w, $w \in \{l, h\}$, have identical supports, $[\underline{x}, \bar{x}] \subset \mathbb{R}$, and strictly positive densities g_w. The likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is non-decreasing, $\frac{g_h(\underline{x})}{g_l(\underline{x})} = \lim_{x \to \underline{x}} \frac{g_h(x)}{g_l(x)}$, and $\frac{g_h(\bar{x})}{g_l(\bar{x})} = \lim_{x \to \bar{x}} \frac{g_h(x)}{g_l(x)}$. Thus, larger values of x indicate a higher likelihood of the higher value. The signals are not trivial and boundedly informative,

$$0 < \frac{g_h(\underline{x})}{g_l(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_l(\bar{x})} < \infty.$$

The boundedness of the likelihood ratios implies that G_h and G_l are mutually absolutely continuous, i.e., letting $G_w(A)$ denote the measure of a set $A \subset [\underline{x}, \bar{x}]$, $G_h(A) = 0 \Leftrightarrow G_l(A) = 0$.

The prior likelihood ratio and the likelihood ratio at the most favorable signal $\bar{x}$

appear often in the analysis. We therefore dedicate to them special symbols,

$$\rho = \frac{\rho_h}{\rho_l} \qquad \text{and} \qquad \overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}.$$

Expected Payoffs and Equilibrium.—The posterior probability of state $w \in \{l, h\}$ in the eyes of a bidder conditional on being solicited and receiving signal x is

$$\Pr[w|x, \mathbf{n}] = \frac{\rho_w g_w(x) \frac{n_w}{N}}{\rho_l g_l(x) \frac{n_l}{N} + \rho_h g_h(x) \frac{n_h}{N}} = \frac{\rho_w g_w(x) n_w}{\rho_l g_l(x) n_l + \rho_h g_h(x) n_h}.$$

The terms $g_w(x)$ reflect the information contained in the signal, the terms $\frac{n_w}{N}$ reflect the information that is conveyed to the bidder by being invited, and the ρ_w reflect the prior information. Since the signals accrue only to bidders who were sampled, we do not need a separate piece of notation for the information that this bidder was sampled. Notice that N cancels out and hence does not play any role in the analysis. The posterior likelihood ratio,

$$\frac{\Pr[h|x, \mathbf{n}]}{\Pr[l|x, \mathbf{n}]} = \frac{\rho_h}{\rho_l} \frac{g_h(x)}{g_l(x)} \frac{n_h}{n_l},$$

is thus a product of three likelihood ratios: The prior likelihood ratio $\frac{\rho_h}{\rho_l}$, the signal likelihood ratio $\frac{g_h(x)}{g_l(x)}$, and the sampling likelihood ratio $\frac{n_h}{n_l}$.

We study pure and symmetric bidding strategies $\beta : [\underline{x}, \bar{x}] \to P_\Delta$ that are measurable. When there are n bidders who employ a bidding strategy β, the cumulative distribution of the *winning bid* in state w is

$$F_w(p|\beta, n) = G_w(\{x|\beta(x) \leq b\})^n = \left[\int_{x|\beta(x)\leq p} g_w(x)\, dx\right]^n.$$

The expected winning bid with n bidders in state w is

$$\mathbb{E}_w[p|\beta, n] = \int_{\underline{x}}^{\bar{x}} p dF_w(p|\beta, n).$$

Let $\pi_w(b|\beta, n)$ be the probability of winning with bid b, given state w, bidding strategy β employed by the other bidders, and n bidders. The expected payoff to a bidder who bids b, conditional on being solicited and observing the signal x, given the bidding strategy β and the solicitation strategy $\mathbf{n} = (n_l, n_h)$, is

$$U(b|x, \beta, \mathbf{n}) = \frac{\rho_l g_l(x) n_l \pi_l(b|\beta, n_l)(v_l - b) + \rho_h g_h(x) n_h \pi_h(b|\beta, n_h)(v_h - b)}{\rho_l g_l(x) n_l + \rho_h g_h(x) n_h}. \quad (1)$$

Denote by $\Gamma_0(N, \mathbf{n}, P_\Delta)$ the *bidding game* when the auctioneer is known to invite $\mathbf{n} = (n_l, n_h)$ bidders and the set of possible bids is P_Δ. The ordinary common value auction is a special case of the bidding game with $n_l = n_h$.

A *bidding equilibrium* of $\Gamma_0(N, \mathbf{n}, P_\Delta)$ is a strategy β such that, for all x, $b = \beta(x)$ maximizes $U(b|x, \beta, \mathbf{n})$ over P_Δ.

Denote by $\Gamma(s, P_\Delta)$ the overall game in which the potential number of bidders is $N = \lceil \frac{v_h}{s} \rceil$, the smallest natural number larger than $\frac{v_h}{s}$.

A *pure equilibrium* of $\Gamma(s, P_\Delta)$ consists of a bidding strategy β and a solicitation strategy $\mathbf{n} = (n_l, n_h)$ such that (i) β is a bidding equilibrium of $\Gamma_0\left(\lceil \frac{v_h}{s} \rceil, \mathbf{n}, P_\Delta\right)$, and (ii) the solicitation strategy is optimal for the seller,

$$n_w \in \underset{n \in \{1,2,\cdots,N\}}{\arg\max}\ \mathbb{E}_w[p|\beta, n] - ns \qquad \text{for} \qquad w \in \{l, h\}.$$

Since a pure equilibrium might not exist, we allow for mixed solicitation strategies. Let $\boldsymbol{\eta} = (\eta_l, \eta_h)$ denote a mixed solicitation strategy, where $\eta_w(n)$ is the probability with which $n = 1, \cdots, N$ bidders are invited in state w. Let $\overline{n}_w(\eta_w) = \sum_{n=1}^{N} n\eta_w(n)$ and $\overline{\pi}_w[b|\beta, \eta] = \sum_{n=1}^{N} \eta_w(n)\, n\pi_w(b|\beta, n)/\overline{n}_w$. These are the expected number of bidders and the weighted average probability of winning in state w and are analogous to n_w and $\pi_w[b|\beta, n]$ in the deterministic solicitation case. To make the expressions less dense we omit here and later the argument of $\overline{n}_w(\eta_w)$ and write just $\overline{n}_w$ instead. The expected payoff to a bidder who bids b, conditional on being solicited and observing the signal x, given the common bidding strategy β and the solicitation strategy $\boldsymbol{\eta} = (\eta_l, \eta_h)$ is

$$U(b|x, \beta, \boldsymbol{\eta}) = \frac{\rho_l g_l(x)\, \overline{n}_l \overline{\pi}_l[b|\beta, \eta_l](v_l - b) + \rho_h g_h(x)\, \overline{n}_h \overline{\pi}_h[b|\beta, \eta_h](v_h - b)}{\rho_l g_l(x)\, \overline{n}_l + \rho_h g_h(x)\, \overline{n}_h}. \tag{2}$$

In a complete analogy to the above definitions (for pure strategies), $\Gamma_0(N, \boldsymbol{\eta}, P_\Delta)$ is the bidding game given $\boldsymbol{\eta} = (\eta_l, \eta_h)$ and $\Gamma(s, P_\Delta)$ is the full game.

A bidding equilibrium of $\Gamma_0(N, \boldsymbol{\eta}, P_\Delta)$ is a strategy β such that, for all x, $b = \beta(x)$ maximizes $U(b|x, \beta, \boldsymbol{\eta})$ over P_Δ. The strategy profile $(\beta, \boldsymbol{\eta})$ is an *equilibrium* of $\Gamma(s, P_\Delta)$ if (i) β is a bidding equilibrium of $\Gamma_0\left(\lceil \frac{v_h}{s} \rceil, \boldsymbol{\eta}, P_\Delta\right)$ and (ii) the solicitation strategy is optimal,

$$\eta_w(n) > 0 \Rightarrow n \in \underset{n \in \{1,2,\ldots.\}}{\arg\max}\ \mathbb{E}_w[p|\beta, n] - ns.$$

Before proceeding let us note that the paper contains the proofs of the formally stated results. Some proofs are in the body of the paper and the rest are relegated to the **appendix**. We will not repeat it each time. If the proof of a formally stated claim is not in its immediate vicinity, then it is in the appendix.

2 Bidding Equilibrium: Single Crossing, Bertrand, and Monotonicity of Bids

This section derives some properties of a bidding equilibrium strategy β. The main property is the monotonicity of the bidding equilibrium β when at least two bidders are invited in each of the states. If the likelihood ratio $\frac{g_h}{g_l}$ is strictly increasing everywhere, a bidding equilibrium β is necessarily non-decreasing. If the likelihood ratio is constant over some interval, all signals in this interval contain the same information and, if β is not constant over this interval, the bids need not be monotonic. Nevertheless, there is an equivalent bidding equilibrium that is monotonic and that is obtained by reordering the bids over such intervals.

A bidding equilibrium $\widetilde{\beta}$ is said to be *equivalent* to a bidding equilibrium β if the implied joint distributions over bids and states are identical.

Proposition 1 **(Monotonicity of Bidding Equilibrium)** *Suppose $\boldsymbol{\eta}$ is such that $\eta_l(1) = \eta_h(1) = 0$, and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta(x')|x', \beta, \boldsymbol{\eta}) \geq U(\beta(x)|x, \beta, \boldsymbol{\eta})$. The inequality is strict if and only if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$.*

2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is non-decreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_l}$ is strictly increasing.*

The proof of Proposition 1 relies on two lemmas.

Lemma 1 **(Single-Crossing)** *Given any bidding strategy β, any solicitation strategy $\boldsymbol{\eta}$ and any bids $b' > b \geq v_l$.*

1. *If $\overline{\pi}_w[b'|\beta, \eta_w] > 0$ for some $w \in \{l, h\}$ then, for all $x' > x$,*

$$U(b'|x, \beta, \boldsymbol{\eta}) \geq U(b|x, \beta, \boldsymbol{\eta}) \Rightarrow U(b'|x', \beta, \boldsymbol{\eta}) \geq U(b|x', \beta, \boldsymbol{\eta});$$

   where the second inequality is strict if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$.

2. *If $\overline{\pi}_w[b'|\beta, \eta_w] = 0$ for some $w \in \{l, h\}$, then $\overline{\pi}_w[b|\beta, \eta_w] = 0$ for both w, and $U(b'|x, \beta, \boldsymbol{\eta}) = U(b|x, \beta, \boldsymbol{\eta}) = 0$ for all x.*

The following lemma collects a number of additional properties of a bidding equilibrium β. One of them is a straightforward Bertrand property: when the seller solicits two or more bids in both states, then $\beta(x) \geq v_l$, for all x.

Lemma 2 **(Bertrand and Other Properties)** *Suppose* $\eta_l(1) = \eta_h(1) = 0$ *and* β *is a bidding equilibrium.*

1. $\overline{\pi}_w[\beta(x)|\beta, \eta_w] > 0$ *if* $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$.
2. $\beta(x) \in [v_l, v_h)$ *for almost all* x.
3. $U(\beta(x')|x', \beta, \boldsymbol{\eta}) \geq U(\beta(x)|x, \beta, \boldsymbol{\eta})$ *if* $x' > x$. *The inequality is strict if and only if* $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$.
4. *If* $P_\Delta = [0, v_h]$, *then* $\beta(x) \in (v_l, v_h)$ *for all* $x > \underline{x}$ *for which* $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$.

The proof of the lemma utilizes that the set of feasible bids is dense below v_l. If the price grid is finite below v_l as well, equilibrium may involve bids just below v_l—just like in the usual Bertrand pricing game with price grid—but such equilibria would not add anything important.

The single crossing condition in Lemma 1 does not require that other bidders use a monotone (non-decreasing) bidding strategy. Furthermore, the proof also holds if other bidders use mixed strategies. It follows that, when there are at least two bidders in each state, every symmetric equilibrium is in monotone strategies and hence the restriction to pure bidding strategies is without loss of generality. In contrast, some existing single crossing conditions for auctions, such as the condition in Athey (2001), require monotonicity of the strategy of other bidders.

The proof of the single crossing condition avoids assuming monotonicity by using the two state assumption: The condition that $b \geq v_l$ implies that (i) the low state is unambiguously bad (profit is negative because the bid is higher than the value) and that (ii) the higher bid must be worse in the low state (because the increased probability of winning decreases profits in the low state). With more than two states, such a strong result may not hold and single crossing may require stronger assumptions (such as monotonicity) on the strategies of other bidders.

Observe that, when the number of solicited bidders depends on the state, monotonicity is not immediately obvious. Signals inform bidders not only about the expected value but also about the number of competitors. If fewer buyers are solicited when $w = h$, a higher signal implies both a higher value and less competition. The following example illustrates this consideration. It also clarifies why the assumption that at least two bidders are solicited in both states is needed for establishing monotonicity.

Example of a Non-Monotone Bidding Equilibrium: Let $[\underline{x}, \bar{x}] = [0, 1]$, with $g_h(x) = 2x$ and $g_l(x) = 2 - 2x$. Thus, the signals $x = 1$ and $x = 0$ reveal the state.[3] Suppose that $v_l > 0$, $n_h = 1$ and $n_l = 100$. It follows that $\pi_h[b|\beta, 1] = 1$ for all $b \geq 0$. Hence, $\beta(1) = 0$ in every bidding equilibrium. So, if β were weakly increasing, then $\beta(x) = 0$ for all x. However, this strategy cannot be an equilibrium. At $x = 0$ the expected payoff from bidding $b = 0$ is $\frac{1}{100} v_l$ while the expected payoff from bidding $b' = \varepsilon$ is $v_l - \varepsilon$. Because $v_l > 0$, a deviation to b' is profitable for small ε.[4] Thus, in this example no bidding equilibrium strategy is weakly increasing.

In light of Proposition 1, from now on, whenever $\eta_w(1) = 0$, attention will be confined only to monotone bidding equilibria (whether or not $\frac{g_h(x)}{g_l(x)}$ is strictly increasing).

3 Bidding Equilibrium: Atoms

One significant consequence of the endogenous solicitation of bidders is the emergence of atoms in the bidding equilibrium. In auctions with private values, a standard argument involving slight overbidding (or undercutting) precludes atoms in which bidders get positive payoffs. This argument does not apply directly to common value auctions, since overbidding the atom may have different consequences in different underlying states owing to possibly different frequency of bids that are tied in the atom in the different states. Still, as is shown below, a somewhat more subtle argument still precludes atoms in an ordinary common value auction ($n_l = n_h = n$), except at the lowest equilibrium bid. However, when $n_l > n_h$ atoms may arise in a bidding equilibrium.

Example of an Atom in a Bidding Equilibrium.—Suppose that $v_l = 0$ and $v_h = 1$, with uniform prior $\rho_h = \rho_l = \frac{1}{2}$. Let $[\underline{x}, \bar{x}] = [0, 1]$, $g_h(x) = 0.8 + 0.4x$ and $g_l(x) = 1.2 - 0.4x$. Thus, $\frac{g_h(x)}{g_l(x)}$ is increasing as required. Let $\bar{b}$ be any number in $[\frac{1}{3}, \frac{4}{10}]$.

Claim: Suppose $n_l = 6$ and $n_h = 2$. There is a bidding equilibrium in which

$$\beta(x) = \bar{b} \qquad \forall x \in [\underline{x}, \bar{x}].$$

[3] The example violates the assumption that likelihood ratios are bounded. This simplifies the argument but it is easily possible to change the example so that signals are boundedly informative while the equilibrium bids are still decreasing.

[4] In fact, one can show that β must be strictly decreasing on $[0, 1]$, using arguments analogous to the proof of Proposition 1.

Proof: The expected value conditional on x and winning with bid b is

$$\mathbb{E}[v|x,\text{ win at } b;\, \beta,\, \mathbf{n}] = \frac{1}{1+\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}} v_l + \frac{\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}}{1+\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}} v_h.$$

At the atom, the assumption that ties are broken randomly implies that $\pi_h\left[\bar{b}|\beta, n_h\right] = \frac{1}{n_h} = \frac{1}{2}$ and $\pi_l\left[\bar{b}|\beta, n_l\right] = \frac{1}{n_l} = \frac{1}{6}$. Further, $\frac{\rho_h}{\rho_l} = 1$, $v_l = 0$, $v_h = 1$. Thus,

$$\mathbb{E}[v|x,\text{ win at } \bar{b};\, \beta,\, \mathbf{n}] = \frac{\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}}{1+\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}} = \frac{\frac{g_h(x)}{g_l(x)}}{1+\frac{g_h(x)}{g_l(x)}} \geq \frac{4}{10}.$$

The inequality is from $\frac{g_h(x)}{g_l(x)} \geq \frac{g_h(0)}{g_l(0)} = \frac{2}{3}$ for all x. Because $\bar{b} \leq \frac{4}{10}$, when bidding $\bar{b}$, almost all buyers expect strictly positive payoffs whereas undercutting $\bar{b}$ yields zero payoff. Consequently, for almost all signals (except possibly $x = \underline{x}$ if $\bar{b} = 0.4$), buyers strictly prefer bidding $\bar{b}$ to any $b < \bar{b}$.

There is also no incentive for any bidder to overbid $\bar{b}$. The expected value conditional on winning when overbidding $\bar{b}$ is

$$\mathbb{E}[v|x,\text{ win at } b > \bar{b};\, \beta,\, \mathbf{n}] = \frac{\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{1}{1}}{1+\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{1}{1}} \leq \frac{1}{3}.$$

The inequality is from $\frac{g_h(x)}{g_l(x)} \leq \frac{3}{2}$ for all x. Any bid above $\bar{b}$ is sure to win. Hence, because $\bar{b} \geq \frac{1}{3}$, bidding $b > \bar{b}$ yields strictly negative payoffs. However, when bidding $\bar{b}$, expected payoffs are positive. Therefore, for all signals, buyers strictly prefer bidding $\bar{b}$ to any $b > \bar{b}$. ∎

The key to the atom's immunity to deviations is the fact that $n_l > n_h$. Slightly overbidding the atom would result in a discontinuous increase in payoff in state h, but an even more significant decrease in state l. In other words, given the uniform tie-breaking rule, bidding in an atom provides insurance against winning too frequently ("hiding in the crowd") in state l where the payoff is negative.[5]

A later result (Lemma 9) implies that, if $n_l = 3n_h$ and n_h is sufficiently large, there exists no equilibrium in strictly increasing strategies. Thus, atoms may be "unavoidable" if the number of bidders depends on the state.

Finally, observe that bidding equilibria discussed here are not full equilibria. The

[5] Atakan and Ekmekci (2012) find that atoms may occur when after the auction the winning bidders have to take an action whose payoff depends on the unknown state. Winning at the atom may inform the bidders about the state owing to the differential probability of this event across states. Consequently, bidders may be reluctant to overbid if the value of information for the subsequent decision problem is sufficiently high.

seller's solicitation strategy is obviously not optimal. Optimal solicitation in a face of a single atom would be $n_l = n_h = 1$. We return in Section 8.1 to the existence of a full equilibrium with an atom similar to the previous example.

Winning Probability at Atoms.—To continue the discussion of atoms, the following lemma derives an expression for the winning probability in the case of a tie. Define the generalized inverse of β by

$$\begin{aligned} x_-(p) &= \inf\{x \in [\underline{x}, \bar{x}] \,|\beta(x) \geq p\}, \\ x_+(p) &= \sup\{x \in [\underline{x}, \bar{x}] \,|\beta(x) \leq p\}, \end{aligned}$$

with $\underline{x} = \sup \emptyset$ and $\bar{x} = \inf \emptyset$. When there is no danger of confusion we will omit the arguments and write x_- and x_+.

Lemma 3 *Suppose β is non-decreasing and, for some b, $x_-(b) < x_+(b)$. Then,*

$$\pi_w(b|\beta, n) = \frac{G_w(x_+)^n - G_w(x_-)^n}{n(G_w(x_+) - G_w(x_-))}.$$

Building on this result, we can obtain useful bounds on the likelihood ratio $\frac{\pi_h(b|\beta,n_h)}{\pi_l(b|\beta,n_l)}$ that play an important role in a few points in the subsequent analysis including the next proposition.

Lemma 4 *Suppose β is non-decreasing. If $n_h \geq n_l \geq 2$, then $\frac{G_h(x)^{n_h}}{G_l(x)^{n_l}}$ is weakly increasing and*

$$\frac{G_h(x_-)^{n_h-1}}{G_l(x_-)^{n_l-1}} \leq \frac{\pi_h[b|\beta, n_h]}{\pi_l[b|\beta, n_l]} \leq \frac{G_h(x_+)^{n_h-1}}{G_l(x_+)^{n_l-1}}.$$

The inequalities are strict unless $n_l = n_h$ and $\frac{g_h(x_+)}{g_l(x_+)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$.

The Case of $n_h \geq n_l$.— In this case the bidding equilibrium is essentially free of atoms. Atoms may arise only if $n_h = n_l$ and $\frac{g_h}{g_l}$ is constant at the bottom of the signal distributions on some interval $[\underline{x}, \hat{x}]$, and then only at the lowest possible bid. Thus, if either $n_h > n_l$ or $\frac{g_h}{g_l}$ is strictly increasing, then the bidding equilibrium in the case of $n_h \geq n_l$ cannot have an atom at all. The case of $n_h \geq n_l$ includes of course the ordinary common value auction $n_l = n_h = n$ as a special case.[6]

[6] For the standard common value auction ($n_l = n_h = n$), the absence of atoms when $\frac{g_h}{g_l}$ is strictly increasing is well known. The second part of the proposition is related to results from Rodriguez (2000), which imply that when $n_l = n_h = 2$ and $\frac{g_h}{g_l}$ is not strictly increasing, then atoms may occur only at the bottom of the bid distribution. In fact, one can easily show if $\frac{g_h}{g_l}$ is constant on some interval $[\underline{x}, x']$, then β is constant on that interval as well; see the remark at the end of the proof.

Proposition 2 (**No Atoms if** $n_h \geq n_l$) *Suppose that β is a bidding equilibrium given $\mathbf{n}$, with $n_h \geq n_l \geq 2$, and $P_\Delta = [0, v_h]$.*

1. *β is strictly increasing if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$ for all $x' > \underline{x}$ and/or $n_h > n_l$.*
2. *If $\beta(x) = \beta(x')$ for some $x \neq x'$, then $\frac{g_h(x')}{g_l(x')} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$, $n_h = n_l$, and $U(\beta(x')|x', \beta, \mathbf{n}) = 0$.*

4 Optimal Solicitation: Characterization

The seller's payoff when sampling n bidders who use bidding strategy β is $\mathbb{E}_w[p|\beta, n] - ns$. This expression is strictly concave in n whenever the bidding strategy is not constant. Consequently, either there is a unique optimal number of sampled bidders or the optimum is attained at two adjacent integers.

Lemma 5 Optimal Solicitation *Given any symmetric bidding function β, there is a number n_w^* such that*

$$\{n_w^*, n_w^* + 1\} \supseteq \underset{n \in \{1,2,\cdots,N\}}{\arg\max} \; \mathbb{E}_w[p|\beta, n] - ns.$$

The lemma is an immediate consequence of the concavity of the expectation of the first-order statistic in the number of trials.

Proof of Lemma 5: The probability that the winning bid is below p is $\left(G_w\left(\beta^{-1}([0,p])\right)\right)^n$, where $\beta^{-1}([0,p]) = \{x : \beta(x) \in [0,p]\}$. Recall that the expected value of a positive random variable can be expressed by the integral of its decumulative distribution function,

$$\mathbb{E}_w[p|\beta, n] = \int_0^{v_h} \left(1 - \left(G_w\left(\beta^{-1}([0,t])\right)\right)^n\right) dt.$$

The incremental benefit of soliciting one more bidder is therefore

$$\mathbb{E}_w[p|\beta, n+1] - \mathbb{E}_w[p|\beta, n] = \int_0^{v_h} \left(G_w\left(\beta^{-1}[0,t]\right)\right)^n \left(1 - \left(G_w\left(\beta^{-1}[0,t]\right)\right)\right) dt. \quad (3)$$

The lemma is immediate whenever β is degenerate: If all buyers bid the same, the uniquely optimal number is $n^* = 1$. If β is not degenerate, then inspection of the incremental benefit of soliciting one more bidder shows that it is strictly decreasing in n. Thus, the objective function is strictly concave, which implies the lemma. ∎

Given the lemma, we will restrict attention in the following to mixed strategies η that have support on at most two adjacent integers. In addition, can represent any such mixed strategy η_w by $n_w \in \{1, ..., N\}$ and $\gamma_w \in (0, 1]$, where $\gamma_w = \eta_w(n_w) > 0$

and $1-\gamma_w = \eta_w(n_w+1) \geq 0$. A solicitation strategy is pure if $\gamma_w = 1$. Thus, from here on, when we talk about n_w in the context of a strategy η_w, we mean the bottom of the support of η_w. In fact, since our characterization results pertain to the case of small sampling costs and many bidders, they are not affected by whether or not the equilibrium strategies are actually pure or mixed. Mixed solicitation strategies matter only for the existence arguments.

Relative Solicitation Incentives: To understand how the incentive to solicit bidders depends on the seller's type, observe that for a non-decreasing β (and recalling that $x_+(p) = \sup\{x|\beta(x) \leq p\}$),

$$\mathbb{E}_w[p|\beta, n+1] - \mathbb{E}_w[p|\beta, n] = \int_0^{v_h} (G_w(x_+(p)))^n (1-(G_w(x_+(p)))) \, dp.$$

Thus, the incremental benefit of sampling another bidder depends on two terms, $(G_w(x_+(p)))^n$—the probability that all n buyers bid below p—and $(1-(G_w(x_+(p))))$—the probability that the additional buyer bids higher. The monotone likelihood ratio property implies that $(G_l(x_+(p)))^n \geq (G_h(x_+(p)))^n$ while $1-(G_l(x_+(p))) \leq 1-(G_h(x_+(p)))$. Intuitively, when $w = l$, the highest bid of the already sampled n sellers is likely to be lower than when $w = h$, making further sampling more desirable. However, at $w = l$, there is a lower probability that an additional bid is high, rendering further sampling less beneficial. The incentive to solicit bidders depends on the relative magnitudes of these two countervailing terms. As we demonstrate in examples, for given signal distribution, there can simultaneously be an equilibrium in which the high type samples more bidders than the low type (*solicitation blessing*) and an equilibrium in which the low type samples more bidders (*solicitation curse*).

5 Large Numbers: Basic Results

Our main characterization results and the associated insights into the question of information aggregation are derived for an environment in which the solicitation cost is small and the numbers of solicited bidders are large. This section obtains basic results that are used in the subsequent characterization of equilibria with large numbers of bidders. Recall the shorthand $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$.

Lemma 6 (**Utilizing Poisson approximation for Binomial Distribution**) *Consider some sequence* $\{(x^k, \mathbf{n}^k)\}$ *with* $\min\{n_l^k, n_h^k\} \to \infty$ *and* $\lim_{k\to\infty} \frac{n_h^k}{n_l^k} = r \in (0,\infty)$*. If* $\lim_{k\to\infty} (G_l(x^k))^{n_l^k} = q$ *then*

$$\lim_{k\to\infty} \left(G_h(x^k)\right)^{n_h^k} = q^{\bar{g}r}.$$

For a given w and n_w^k, the number of signals above any cutoff x^k is binomially distributed, with n_w^k independent trials and success probabilities $1 - G_w(x^k)$. As is well-known, when the number of trials is large and probability of success is proportionately small, the binomial distribution is approximated by a Poisson distribution. Specifically, if $\lim n_w^k[1 - G_w(x^k)] = \delta_w \in (0,\infty)$, then the number $m = \#$(signals weakly above x^k) is Poisson distributed with parameter δ_w in the limit as $n_w^k \to \infty$. Therefore, $\lim G_w(x^k)^{n_w^k} = \Pr[m = 0] = e^{-\delta_w}$. Now, when $\delta_w \in (0,\infty)$, it must be that $x^k \to \bar{x}$, and hence, $\frac{1-G_h(x^k)}{1-G_l(x^k)} \to \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Therefore, $\frac{\delta_h}{\delta_l} = \frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k} = \bar{g}r$. Thus, if $\lim (G_l(x^k))^{n_l^k} = q$, then

$$\lim_{k\to\infty} \left(G_h(x^k)\right)^{n_h^k} = e^{-\delta_l\left(\frac{\delta_h}{\delta_l}\right)} = \left(\lim_{k\to\infty} G_l^{n_l^k}(x^k)\right)^{\left(\frac{\delta_h}{\delta_l}\right)} = q^{\bar{g}r}.$$

Note that the lemma also implies that if β^k is non-decreasing and $\lim F_l(p|\beta^k, n_l^k) = q$, then

$$\lim_{k\to\infty} F_h(p \mid \beta^k, n_h^k) = q^{\bar{g}r}. \tag{4}$$

This is because for non-decreasing β, $F_w(p|\beta^k, n_w^k) = G_w(x_+^k(p))^{n_w^k}$.

As the number of bidders grows, the interim expected payoff for each bidder vanishes to zero.

Lemma 7 (**Zero Profit in the Limit**) *For every* ε *there is an* $M(\varepsilon)$ *such that, if* $n_l > M(\varepsilon)$ *and* $n_h > M(\varepsilon)$*, then* $U(\beta(x)|x, \beta, \boldsymbol{\eta}) < \varepsilon$ *for all* x *in every bidding equilibrium* β*.*

Consider a sequence of solicitation strategies $\boldsymbol{\eta}^k$ such that $\min\{n_h^k, n_l^k\} \to \infty$ and a corresponding sequence β^k of bidding equilibria. Lemma 7 implies that for any sequence of bids $\{b^k\}$, if $\lim b^k = b$ and $\lim \bar{\pi}_l[b^k; \beta^k, \eta_l^k] > 0$, then

$$\limsup_{k\to\infty} \mathbb{E}[v|x, \text{ win at } b^k; \beta^k, \boldsymbol{\eta}^k] \leq b. \tag{5}$$

In addition, for any sequence of signals $\left\{x^k\right\}$ for which $\lim \beta^k\left(x^k\right) = b$, individual rationality requires that

$$\liminf_{k\to\infty} \mathbb{E}[v|x^k, \text{ win at } \beta^k(x^k); \beta^k, \boldsymbol{\eta}^k] \geq b. \tag{6}$$

Therefore, $\lim \bar{\pi}_l\left[\beta^k\left(x^k\right); \beta^k, \eta_l^k\right] > 0$ requires

$$\lim_{k\to\infty} \mathbb{E}[v|x^k, \text{ win at } \beta^k(x^k); \beta^k, \boldsymbol{\eta}^k] = b. \tag{7}$$

Observations (4) and (7) together will imply a tight characterization (in Proposition 3 below) of the limiting distribution of the winning bid if there are no atoms in the limit.

The last lemma provides a condition to be satisfied in the limit by the seller's optimal solicitation strategy.

Lemma 8 (**Total Solicitation Costs**) *Consider a sequence $s^k \to 0$ and a sequence of bidding strategies β^k. Suppose that η_w^k is an optimal solicitation strategy given β^k in state w and the winning bid distribution $F_w\left(\cdot|\beta^k, \eta_w^k\right)$ converges pointwise. Then,*

$$\lim_{k\to\infty} n_w^k s^k = -\int_0^{v_h}\left(\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)\right)\ln\left(\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)\right) dp.$$

The lemma allows us to characterize the total solicitation costs in the limit as a function of the distribution of the winning bid. In particular, it immediately implies

Corollary 1 *If $\lim_{k\to\infty} F_w\left(\cdot|\beta^k, \eta_w^k\right)$ is non-degenerate, then $\lim n_w^k s^k > 0$.*

To understand the lemma intuitively, observe that the optimality of n^k means that it is more profitable for the seller to solicit n^k than to solicit αn^k bidders instead, for $\alpha \neq 1$. With αn^k bidders, the distribution of the winning bid changes to $F_w\left(p|\beta^k, \alpha n^k\right) = \left(F_w\left(p|\beta^k, n^k\right)\right)^\alpha$, since $F_w\left(p|\beta^k, \alpha n^k\right) = (G_w(\beta^{k-1}([0,p])))^{\alpha n^k}$. The expected payoff from soliciting αn^k bidders is therefore

$$\int_0^{v_h}\left(1 - F_w\left(p|\beta^k, n^k\right)^\alpha\right) dp - \alpha\left(n^k s^k\right).$$

Ignoring integer constraints, soliciting n^k bidders is optimal if the derivative of the expected payoff with respect to α vanishes at $\alpha = 1$,

$$-\int_0^{v_h}\left(F_w\left(p|\beta^k, n^k\right)\right)\ln\left(F_w\left(p|\beta^k, n^k\right)\right) dp - n^k s^k = 0. \tag{8}$$

The lemma follows from the observation that when $s^k \to 0$ either the number of solicited bidders is so large that the integer constraints can indeed be ignored or the number of optimally solicited bidders is bounded. In the latter case, the Lemma is shown to hold trivially because the distribution of the winning bids must become degenerate.

6 Characterization of Equilibria with Small Sampling Costs

This section studies the nature of the equilibrium bid distribution when the sampling cost is small. In particular, it inquires about the extent of information aggregation by the equilibrium winning bid – whether the winning bid is near the true value when the sampling costs are small and many bidders may be sampled. Wilson (1977) and Milgrom (1979, 1981) studied the latter question in the context of ordinary common value auctions, without the solicitation element.

Overall, the analysis implies that there are at most two kinds of equilibrium outcomes when the sampling cost is negligible: A partially revealing outcome that qualitatively resembles the equilibrium outcome of an ordinary common value auction and a degenerate "pooling" outcome that is qualitatively different.

Consider a sequence of games $\Gamma\left(s^k, P_{\Delta^k}\right)$ indexed by k where $s^k \to 0$, $\Delta^k \geq 0$ and $\Delta^k \to 0$. (Recall that s^k is the sampling cost and Δ^k the step size of the price grid.) Let β^k and $\boldsymbol{\eta}^k = (\eta_l^k, \eta_h^k)$ be equilibrium bidding and solicitation strategies for $\Gamma\left(s^k, P_{\Delta^k}\right)$. Recall that $F_w\left(\cdot|\beta^k, \eta_w^k\right)$ denotes the cumulative distribution function of the winning bid. We study $\lim_{k\to\infty} F_w\left(\cdot|\beta^k, \eta_w^k\right)$ thinking of it as an approximation for $F_w\left(\cdot|\beta^k, \eta_w^k\right)$ when s^k and Δ^k are small.

From here on the term "limit" (and the operator lim) refers to a limit over a subsequence such that all the magnitudes of interest are converging.[7] We will not repeat this qualification each time, but it is always there.

All the characterization results that we are about to report hold both for the case of a finite price grid ($\Delta^k > 0$, $\Delta^k \to 0$) and for the case of continuum of prices ($\Delta^k \equiv 0$). To help the reading, we first present the results and prove them for the continuum case and only later explain that they also hold when the limit is taken over a sequence of finite grids.

The following theorem characterizes the set of possible equilibrium outcomes when solicitation costs are negligible. It is perhaps the main result of this paper.

[7] By Helly's selection theorem, every sequence of cumulative distribution functions has a pointwise everywhere convergent subsequence. This is immediate from the monotonicity of F_w; see Kolmogorov and Fomin (1970, p. 372).

Recall the shorthand notation $\rho = \frac{\rho_h}{\rho_l}$, $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Given $\rho > 0$, $\overline{g} > 1$ and a parameter $r > 0$ ($r \neq \overline{g}$), define the functions $\phi_w\left(\cdot|\rho, \overline{g}, r\right)$ by

$$\phi_l\left(p|\rho, \overline{g}, r\right) = \begin{cases} 1 & \text{if} \quad p \geq \frac{v_l + \rho\overline{g}rv_h}{1+\rho\overline{g}r}, \\ \left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}} & \text{if} \quad v_l < p \leq \frac{v_l+\rho\overline{g}rv_h}{1+\rho\overline{g}r}, \\ 0 & \text{if} \quad p \leq v_l, \end{cases} \tag{9}$$

and

$$\phi_h\left(\cdot|\rho, \overline{g}, r\right) = \left(\phi_l\left(\cdot|\rho, \overline{g}, r\right)\right)^{\overline{g}r}.$$

Observe that if $\overline{g}r > 1$, then $\phi_w\left(\cdot|\rho, \overline{g}, r\right)$ is a cumulative distribution function.[8]

Theorem 2 (**Equilibrium Characterization**) *Consider a sequence of games $\Gamma\left(s^k, P_0\right)$, such that $s^k \to 0$, and a corresponding sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$.*

(i). There exists a unique number $r^* \equiv r^*\left(\rho, \overline{g}\right) > \frac{1}{\overline{g}}$ such that if $\overline{g}\lim_{k\to\infty}\frac{n_h^k}{n_l^k} > 1$ and $\min\left\{n_l^k, n_h^k\right\} \to \infty$, then $\lim_{k\to\infty}\frac{n_h^k}{n_l^k} = r^*$ and

$$\lim_{k\to\infty} F_w(\cdot|\beta^k, \eta_w^k) \equiv \phi_w\left(\cdot|\rho, \overline{g}, r^*\right). \tag{10}$$

(ii). Otherwise, $\lim_{k\to\infty} F_w\left(\cdot|\beta^k, \eta_w^k\right)$ is a degenerate distribution with probability mass 1 on some number $C \leq \rho_l v_l + \rho_h v_h$.

Since $r^*\overline{g} > 1$, the functions $\lim_{k\to\infty} F_w\left(\cdot|\beta^k, \eta_w^k\right)$ in (10) are indeed distribution functions. They are partially revealing in the sense that they have the same support (hence, not perfectly revealing). However, $\lim_{k\to\infty} F_h\left(\cdot|\beta^k, \eta_h^k\right)$ stochastically dominates $\lim_{k\to\infty} F_l\left(\cdot|\beta^k, \eta_l^k\right)$ and, hence, gives rise to a higher expected price.

The theorem says that in the limit an equilibrium winning bid distribution takes one of two forms. Either it is the unique partially revealing function described in Part (i), or it is a mass point below the ex-ante expected value. Furthermore, the partially revealing outcome arises if and only if the sampling behavior satisfies $\min\left\{n_l^k, n_h^k\right\} \to \infty$ and $\overline{g}\lim\frac{n_h^k}{n_l^k} > 1$. Uniqueness is not claimed, so that equilibria of both types may coexist, as verified later.

The proof of Theorem 2 is split into three propositions that are presented over the following two subsections. The first subsection characterizes the limit of *bidding equilibria* given any sampling behavior (not necessarily optimal) such that

[8] For the statement of the theorem, recall that by Lemma 5 we represent a mixed equilibrium strategy η_w by n_w and γ_w, where $\gamma_w = \eta_w\left(n_w\right)$ and $1 - \gamma_w = \eta_w\left(n_w + 1\right)$.

$\min\left\{n_l^k, n_h^k\right\} \to \infty$ and $\frac{n_h^k}{n_l^k}$ converges. The second subsection completes the characterization for the case of an optimal sampling strategy. That subsection shows in particular how the number r^* is determined. Since r^* is uniquely determined by $(\rho, \overline{g})$, the partially revealing limit outcome depends only on ρ and $\overline{g}$.

6.1 Bidding Equilibria

Consider first bidding equilibria alone, without the optimal solicitation requirement. Recall $\rho = \frac{\rho_h}{\rho_l}$, $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$ and the functions $\phi_w(\cdot|\rho, \overline{g}, r)$ defined in (9).

Proposition 3 *Consider a sequence of bidding games $\Gamma_0\left(N^k, \boldsymbol{\eta}^k, P_0\right)$ such that $\min\{n_l^k, n_h^k\} \to \infty$ and $\lim_{k\to\infty} \frac{n_h^k}{n_l^k} = r$, and a corresponding sequence of bidding equilibria β^k.*

(i). If $r\overline{g} > 1$, then

$$\lim_{k\to\infty} F_w(\cdot|\beta^k, \eta_w^k) = \phi_w(\cdot|\rho, \overline{g}, r).$$

(ii). If $r\overline{g} \leq 1$, then $\lim_{k\to\infty} F_w\left(\cdot|\beta^k, \eta_w^k\right)$ is a degenerate distribution with probability mass 1 on some number $C \leq \rho_l v_l + \rho_h v_h$.

The special case of $r = 1$ is of course the ordinary CV auction. Thus, the Proposition implies that $\phi_w(p|\rho, \overline{g}, 1)$ is the limiting winning bid distribution of the ordinary auction as $n \to \infty$.

More importantly, the proposition identifies $\overline{g}r = \frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k}$ as a key magnitude in the nature of the equilibrium distribution of the winning bid. When $\overline{g}r > 1$, the limiting distribution is atomless. When $\overline{g}r \leq 1$, the limiting distribution is degenerate. Notice that the relationship of $\overline{g}r$ to 1 determines whether being solicited and observing the most favorable signal $\bar{x}$ is "good news" or "bad news" for a bidder, in the sense of making the bidder more or less optimistic than the prior. If $r\overline{g} > 1$, this is "good news" – for large enough k, the compound likelihood ratio $\frac{\rho_h}{\rho_l}\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{\bar{n}_h^k}{\bar{n}_l^k}$ of a bidder who observed the most favorable signal $\bar{x}$ is larger than the prior likelihood ratio $\frac{\rho_h}{\rho_l}$. Conversely, if $r\overline{g} < 1$, being solicited and observing $\bar{x}$ is "bad news."

The formal proof of the proposition is in the appendix. Roughly speaking, it proceeds as follows. Part (i) is proved in three steps. First, it is established that if $r\overline{g} > 1$, then $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$ has no atoms. This means that in the limit it can be identified with the probability of winning with bid p,

$$\lim_{k\to\infty} \bar{\pi}_w[p|\beta^k, \eta_w^k] = \lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right).$$

At a price p such that $\lim F_w\left(p|\beta^k, \eta_w^k\right) > 0$ the zero-profit plus IR condition (7) is equivalent to

$$\rho\bar{g}r\frac{\lim_{k\to\infty}\bar{\pi}_h[p|\beta^k, \eta_h^k]}{\lim_{k\to\infty}\bar{\pi}_l[p|\beta^k, \eta_l^k]} = \frac{p - v_l}{v_h - p},$$

where we used the shorthands $\rho = \frac{\rho_h}{\rho_l}$, $\bar{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$, $r = \lim_{k\to\infty}\frac{n_h^k}{n_l^k}$. Substituting for $\lim_{k\to\infty}\bar{\pi}_w[p|\beta^k, \eta_w^k]$ from the previous step and using the Poisson approximation from Lemma 6 and Equation (4), we get two equations for $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$,

$$\rho\bar{g}r\frac{\lim_{k\to\infty} F_h\left(p|\beta^k, \eta_h^k\right)}{\lim_{k\to\infty} F_l\left(p|\beta^k, \eta_l^k\right)} = \frac{p - v_l}{v_h - p},$$

and

$$\lim_{k\to\infty} F_h\left(p|\beta^k, \eta_h^k\right) = \left[\lim_{k\to\infty} F_l\left(p|\beta^k, \eta_l^k\right)\right]^{\bar{g}r}.$$

Solving the two equations yields $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right) = \phi_w\left(p|\rho, \bar{g}, r\right)$ for $w \in \{l, h\}$, as stated in Part (i) of the proposition.

The proof of Part (ii) observes that if $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$ is strictly increasing on any interval (p', p''), then the argument of Part (i) applies, and over this interval $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$ should coincide with $\phi_w\left(p|\rho, \bar{g}, r\right)$. However, when $r\bar{g} < 1$ then $\phi_w\left(p|\rho, \bar{g}, r\right)$ is decreasing—which implies a contradiction. Therefore, $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$ should consist of atoms. The final argument rules out the possibility of multiple atoms by pointing out a profitable downward deviation from a prospective atom that is not the bottom atom.

Proposition 3 does not tell us whether the atom that arises in the case of $r\bar{g} < 1$ is a limit of atoms in winning bid distributions along the sequence or that whether the atom emerges only in the limit. The following lemma shows that it is the former case—significant atoms are already present along the sequence.

Lemma 9 ***(Tieing at the Top)*** *Consider a sequence of bidding games* $\Gamma_0\left(N^k, \boldsymbol{\eta}^k, P_0\right)$ *such that* $\min\{n_l^k, n_h^k\} \to \infty$ *and* $\lim_{k\to\infty}\frac{n_h^k}{n_l^k} = r$, *and a corresponding sequence of bidding equilibria* β^k. *Suppose that* $r\bar{g} < 1$. *Then there is a sequence* $\left\{\left(x_-^k, x_+^k\right)\right\}$, $x_-^k \le x_+^k$ *such that:*

1. β^k *is constant on* $(x_-^k,\ x_+^k)$.

2. $\lim_{k\to\infty} n_w^k\left(G_w\left(x_+^k\right) - G_w\left(x_-^k\right)\right) = \infty$ *for* $w \in \{l, h\}$.

3. $\lim_{k\to\infty}\left(G_w\left(x_-^k\right)\right)^{n_w^k} = 0$ *and* $\lim_{k\to\infty}\left(G_w\left(x_+^k\right)\right)^{n_w^k} = 1$ *for* $w \in \{l, h\}$.

6.2 Full Equilibrium (Including Solicitation)

We return now to the full model: A sequence of games $\Gamma\left(s^k, P_0\right)$, where β^k and $\boldsymbol{\eta}^k = (\eta_l^k, \eta_h^k)$ are corresponding equilibrium bidding and solicitation[9] strategies.

The first proposition states that with equilibrium solicitation if there exists a partially revealing equilibrium of the sort described by Theorem 2-(i), then $\lim \frac{n_h^k}{n_l^k}$ is unique, which means that $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$ is unique as well by Proposition 3.

Proposition 4 *Consider a sequence of games* $\Gamma\left(s^k, P_0\right)$*, such that* $s^k \to 0$*. Suppose that* $(\beta^k, \boldsymbol{\eta}^k)$ *is a corresponding sequence of equilibria such that* $\min\left\{n_l^k, n_h^k\right\} \to \infty$*. There exists a unique number* $r^* = r^*(\rho, \overline{g}) \in (\frac{1}{\overline{g}}, \infty)$ *such that if* $\overline{g} \lim \frac{n_h^k}{n_l^k} > 1$ *then* $\lim \frac{n_h^k}{n_l^k} = r^*$*.*

The proof will use the following technical lemma. It is stated outside the proof, since it contains more material than needed for the proof and will be referenced in the subsequent analysis as well. Define the function J by

$$J(r;\rho,\overline{g}) = \int_0^1 (x - \frac{1}{\overline{g}}) x^{\frac{1}{\overline{g}r - 1}} \frac{\ln x}{(1 + x\rho\overline{g}r)^2} dx. \tag{11}$$

Lemma 10 *For the function defined in (11):*

(i). For any $\rho > 0$*,* $\overline{g} > 1$*, there is a unique number* $r^* = r^*(\rho, \overline{g}) \in (\frac{1}{\overline{g}}, \infty)$ *s.t.* $J(r^*;\rho,\overline{g}) = 0$*.*

(ii). $J(r;\rho,\overline{g}) < 0$ *for* $r \in (\frac{1}{\overline{g}}, r^*)$ *and* $J(r;\rho,\overline{g}) > 0$ *for* $r \in (r^*, \infty)$*.*

Proof of Proposition 4: Let $r = \lim \frac{n_h^k}{n_l^k}$ and suppose that $r < \infty$ and $\overline{g}r > 1$. By Proposition 3 $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right) = \phi_w\left(\cdot|\rho, \overline{g}, r\right)$. This together with Lemma 8 imply that

$$\lim_{k\to\infty} n_w^k s^k = -\int_{v_l}^{\frac{v_l + \rho\overline{g}rv_h}{1+\rho\overline{g}r}} \left(\phi_w\left(p|\rho,\overline{g},r\right)\right) \ln\left(\phi_w\left(p|\rho,\overline{g},r\right)\right) dp.$$

Since $\lim\left(n_h^k s^k\right) = r \lim\left(n_l^k s^k\right)$, it follows that

$$\frac{1}{r}\int_{v_l}^{\frac{v_l + \rho\overline{g}rv_h}{1+\rho\overline{g}r}} \left(\phi_h\left(p|\rho,\overline{g},r\right)\right) \ln\left(\phi_h\left(p|\rho,\overline{g},r\right)\right) dp = \int_{v_l}^{\frac{v_l + \rho\overline{g}rv_h}{1+\rho\overline{g}r}h} \left(\phi_l\left(p|\rho,\overline{g},r\right)\right) \ln\left(\phi_l\left(p|\rho,\overline{g},r\right)\right) dp. \tag{12}$$

[9] Recall that by Lemma 5 we represent a mixed equilibrium strategy η_w by n_w and γ_w, where $\gamma_w = \eta_w\left(n_w\right)$ and $1 - \gamma_w = \eta_w\left(n_w + 1\right)$.

Recall the function J from (11) above.

Lemma 11 *If r satisfies equation (12), then $J(r;\rho,\overline{g})=0$.*

This lemma together with Lemma (10) imply that (12) is satisfied by the unique r named $r^*=r^*(\rho,\overline{g})$ by Lemma (10). Therefore, for any sequence of equilibria such that $\overline{g}\lim\frac{n_h^k}{n_l^k}>1$ and $\lim\frac{n_h^k}{n_l^k}<\infty$, it must be that $\lim\frac{n_h^k}{n_l^k}=r^*(\rho,\overline{g})$.

To complete the proof, it remains to show that indeed $\lim\frac{n_h^k}{n_l^k}<\infty$. Suppose to the contrary that $\lim\frac{n_h^k}{n_l^k}=\infty$. Then Proposition 3 implies that $\lim F_w\left(\cdot|\beta^k,\eta_w^k\right)$ is a degenerate distribution with support v_w. Lemma 8 implies that $\lim n_w^k s^k=0$, so that seller type w's equilibrium payoff converges to v_w.

By Lemma 6, if $\lim F_h\left(p|\beta^k,\eta_h^k\right)=0$ then $\lim F_l\left(p|\beta^k,\eta_h^k\right)=0$. Therefore, if seller type l solicits n_h^k bidders, $\lim\mathbb{E}_l\left[p|\beta^k,n_h^k\right]\geq v_h$. Since $\lim n_h^k s^k=0$, for large k, seller type l's payoff with this strategy is near v_h which is larger than her equilibrium payoff near v_l—contradiction. ∎

All of the above characterization results deal with the case of $\min\left\{n_l^k,n_h^k\right\}=\infty$. We show next that if $\min\left\{n_l^k,n_h^k\right\}\nrightarrow\infty$, then the limit distribution of the winning bid has probability mass 1 on some price C below the ex-ante expected value.

Proposition 5 *Consider a sequence of games $\Gamma\left(s^k,P_0\right)$ such that $s^k\to 0$. Suppose that $(\beta^k,\boldsymbol{\eta}^k)$ is a corresponding sequence of equilibria such that $\min\left\{n_l^k,n_h^k\right\}\nrightarrow\infty$. Then $\lim_{k\to\infty}F_w\left(p|\beta^k,\eta_w^k\right)$ has probability mass 1 on some $C\leq\rho_l v_l+\rho_h v_h$, for both $w=l$ and $w=h$.*

Propositions 3-5 complete the proof of Theorem 2.

6.3 Finite Grid

Theorem 2 and its proof were stated for the case of a continuum set of possible bids P_0. These results also hold for the case in which bids are restricted to finite grids that become finer along the sequence. It is important to know this for the subsequent discussion of existence.

The required modifications are fairly small. Everywhere in Theorem 2 and Propositions 3-5 where it says "Consider a sequence of games $\Gamma\left(s^k,P_0\right)$, such that $s^k\to 0$..." (or "...a sequence of bidding games $\Gamma_0\left(N^k,\boldsymbol{\eta}^k,P_0\right)$...") it will say instead "Consider a sequence of games $\Gamma\left(s^k,P_{\Delta^k}\right)$, such that $\Delta^k\geq 0$ and $\lim\left(s^k,\Delta^k\right)=(0,0)$..." (and analogously for bidding games). The proofs go through almost verbatim. The only changes are in the places where the proofs use a slight undercutting argument. There are two such places in the proof of Proposition 3. In those places we have to make sure that, for a sufficiently fine grid, there exist such undercutting bids that belong to the grid. This is done in the appendix. Each of these instances is followed by a remark that provides the needed argument for the case $\Delta^k > 0$, $\Delta^k \to 0$.

7 Remarks on the Limit Equilibria

The Role of $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k}$.—Theorem 2 and Proposition 3 expose the relationship between the inequalities $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k} \gtrless 1$ and whether the equilibrium is partially revealing. To understand this recall that competition drives bidders' payoffs to zero when $\min\left\{n_l^k, n_h^k\right\} \to \infty$ for some sequence of solicitation strategies $\mathbf{n}^k$. This could happen either through convergence of bids to the expected value conditional on winning or through a vanishing probability of winning. That is—recalling that $\mathbb{E}\left[v|x,\, x_{(1)} \leq x,\, \beta^k,\, \mathbf{n}^k\right]$ denotes the expected value for a bidder whose signal x is the highest, with $x_{(1)}$ being the highest signal of the competitors—we have that, for a large k and a signal x such that Pr(winner's signal $\leq x$) is significant, either $\beta^k(x) \approx \mathbb{E}\left[v|x,\, x_{(1)} \leq x,\, \beta^k,\, \mathbf{n}^k\right]$ or there is an atom at $\beta^k(x)$ that makes the probability of winning very small. Now, when $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \mathbf{n}^k\right]$ is strictly increasing in x, it must be the former case. To see that an atom at bid p (with positive probability of winning) could not survive, consider a bidder with a signal x' close to $x_+(p)$ (the sup of x's at that atom). This bidder would benefit from slightly overbidding p since p is necessarily sufficiently below $\mathbb{E}\left[v|x',\, x_{(1)} \leq x_+(p);\, \beta^k,\, \mathbf{n}^k\right]$ by virtue of being profitable for signals at the bottom of the atom's range (i.e., $p \leq \mathbb{E}\left[v|x_-(p),\, x_{(1)} \leq x_+(p);\, \beta^k,\, \mathbf{n}^k\right]$).[10] Conversely, when $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \mathbf{n}^k\right]$ is strictly decreasing, there is an atom, since the monotonicity of β^k rules out the former case.

Now, for sufficiently large x and k, $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \mathbf{n}^k\right]$ is strictly increasing (decreasing) in x if $\lim \frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k} > 1$ (< 1). To see this recall that $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \mathbf{n}^k\right]$ is increasing iff

$$\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}\frac{G_h(x)^{n_h^k-1}}{G_l(x)^{n_l^k-1}}$$

is increasing and notice that $\frac{G_h(x)^{n_h^k-1}}{G_l(x)^{n_l^k-1}}$ is strictly increasing iff $\frac{g_h(x)}{g_l(x)}\frac{n_h^k-1}{n_l^k-1} > \frac{G_h(x)}{G_l(x)}$. For large k and x near $\bar{x}$—which are the only ones with significant probability of winning when k is large—this is equivalent to $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k} > 1$.

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[10] The argument relies on the bounds on $\frac{G_h(x)^{n_h^k-1}}{G_l(x)^{n_l^k-1}}$ given by Lemma 4 and its extension in Lemma 18.

A somewhat different way to present the above relationship is to note that $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k} > 1$ implies $\frac{\rho_h}{\rho_l}\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{n_h^k}{n_l^k} > \frac{\rho_h}{\rho_l}$, which means that a sampled bidder who observes the highest possible signal is more optimistic about h than he would be based on the prior alone. This implies that an atom is impossible in this case, since, for large k, the value conditional on winning at such atom would necessarily be smaller or equal to the ex-ante expected value, $\rho_l v_l + \rho_h v_h$. Hence, a bidder with high enough signal would benefit from slightly overbidding and winning with certainty, since in this case the expected value conditional on winning coincides with the expected value conditional on being sampled, which exceeds $\rho_l v_l + \rho_h v_h$ by the argument above.

Seller's Revenue–Consider a sequence of bidding games $\Gamma_0\left(N^k, \boldsymbol{\eta}^k, P_0\right)$ such that $\lim \frac{n_h^k}{n_l^k} = r$ and a corresponding sequence of bidding equilibria β^k. As we know, when $\bar{g}r > 1$, the distribution of the winning bid converges to the partially revealing limit $\phi_w\left(\cdot|\rho, \bar{g}', r\right)$ and, for large k and for any x that has a meaningful probability of winning, $\beta^k(x) \approx \mathbb{E}\left[v|x,\, x_{(1)} \leq x,\, \beta^k,\, \boldsymbol{\eta}^k\right]$. Letting $y_{(1)}$ denote the highest signal among all sampled bidders, the winning bid is close to $\mathbb{E}\left[v|x = y_{(1)},\, x_{(1)} \leq x,\, \beta^k,\, \boldsymbol{\eta}^k\right]$. Therefore, the law of iterated expectations implies that the seller's ex-ante expected revenue is approximately

$$\mathbb{E}_{y_{(1)}}\left[\mathbb{E}\left[v|y_{(1)},\, x_{(1)} \leq y_{(1)};\, \beta^k,\, \boldsymbol{\eta}^k\right]\right] = \rho_l v_l + \rho_h v_h.$$

It follows that in the limit the seller extracts the whole ex-ante surplus.

Inspection of ϕ_h shows that for $\bar{g}' > \bar{g}$, $\phi_h\left(\cdot|\rho, \bar{g}', r\right)$ stochastically dominates $\phi_h\left(\cdot|\rho, \overline{g}, r\right)$. Consequently, at $w = h$ the expected revenue of the seller increases in $\bar{g} \equiv \frac{g_h(\bar{x})}{g_l(\bar{x})}$—the maximal signal likelihood ratio. Since the ex-ante expected revenue equals the ex-ante expected value, this implies that the expected revenue of the seller decreases in $\bar{g}$ at $w = l$.[11]

When $\bar{g}r \leq 1$, the limit distribution of the winning bid has an atom with probability mass one on some price $C \leq \rho_l v_l + \rho_h v_h$. The latter inequality may be strict. This means that seller's revenue may be strictly below the ex-ante value $\rho_l v_l + \rho_h v_h$. The bidders' interim expected payoffs are still zero in the limit, since the probability of winning converges to zero.

Total Solicitation Costs—The seller's revenue discussed above is gross of the solicitation costs. Lemma 8 and Theorem 2 provide a complete characterization of the total solicitation costs $n_w^k s^k$ in the limit. Let $(\beta^k, \boldsymbol{\eta}^k)$ be a sequence of

[11] See Figure 1 for an illustration. The straight black lines of the right panel show the expected revenue for each state as a function of $\bar{g}$ when $\rho = 1$.

equilibria corresponding to $s^k \to 0$. Either the limit distribution of the winning bid, $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)$, is partially revealing and the total solicitation costs converge to

$$\lim_{k\to\infty} n_w^k s^k = -\int_0^{v_h} \left(\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)\right) \ln\left(\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right)\right) dp,$$

or it is a mass point with probability 1 and the total solicitation costs vanish to zero, $\lim_{k\to\infty} n_w^k s^k = 0$.

Thus, in the partially revealing limit, the seller does not enjoy the entire surplus extracted from the bidders since the total solicitation cost is positive. Because the total solicitation costs are positive in the partially revealing limit, the seller's ex-ante expected payoff may be higher in the pooling equilibrium than in the partially revealing equilibrium—especially if the atom is close to the ex-ante expected payoff, meaning the seller can extract almost the entire surplus in the pooling equilibrium.

The Large Ordinary Common Value Auction.—As a by-product, Proposition 3 also characterizes the limit distribution of the winning bid for the large ordinary common value auction. This distribution is given by $\phi_w\left(p|\rho, \overline{g}, r=1\right)$, $w = l, h$. The characterization shows in particular that the limit distribution is continuous in $\bar{g}$. When $\bar{g}$ is large, then the winnings bids are close to the true values in probability: When $\bar{g} \to \infty$, inspection of ϕ_w at $r = 1$ shows that the distribution becomes degenerate with all its weight on v_h and v_l, respectively. To the best of our knowledge, this complete characterization of the equilibrium outcome of the large common value auction is new to the literature.[12] The key step towards this characterization is the Poisson approximation from Lemma 6.

Comparison of Prices and Revenue.—Let us compare the outcome of the large ordinary common value auction with the outcome in the partially revealing equilibrium with bidder solicitation. Consider a sequence of bidding games $\Gamma_0\left(N^k, \boldsymbol{\eta}^k, P_0\right)$ such that $\lim \frac{n_h^k}{n_l^k} = r$ and $\bar{g}r > 1$, and a corresponding sequence of bidding equilibria β^k. Thus, the distributions of the winning bid are given by $\phi_w\left(p|\rho, \overline{g}, r\right)$, $w = l, h$. Let $\bar{E}_w\left[p|\rho, \overline{g}, r\right]$ denote the limit of the seller's expected revenue at w,

$$\bar{E}_w\left[p|\rho, \overline{g}, r\right] = \int p d\phi_w\left(p|\rho, \overline{g}, r\right).$$

Of course, the limit revenue for the ordinary CV auction is obtained by plugging

[12] We provide a closed form solution to the limit distribution of the winning bid. Kremer and Skrzypacz (2005) establish that the winning bid distribution is not degenerate for all $k+1$-price auctions for k goods when $\bar{g} \in (1, \infty)$. Milgrom (1979) shows that the winning bid converges in probability to the true value if and only if $\bar{g} = \infty$.

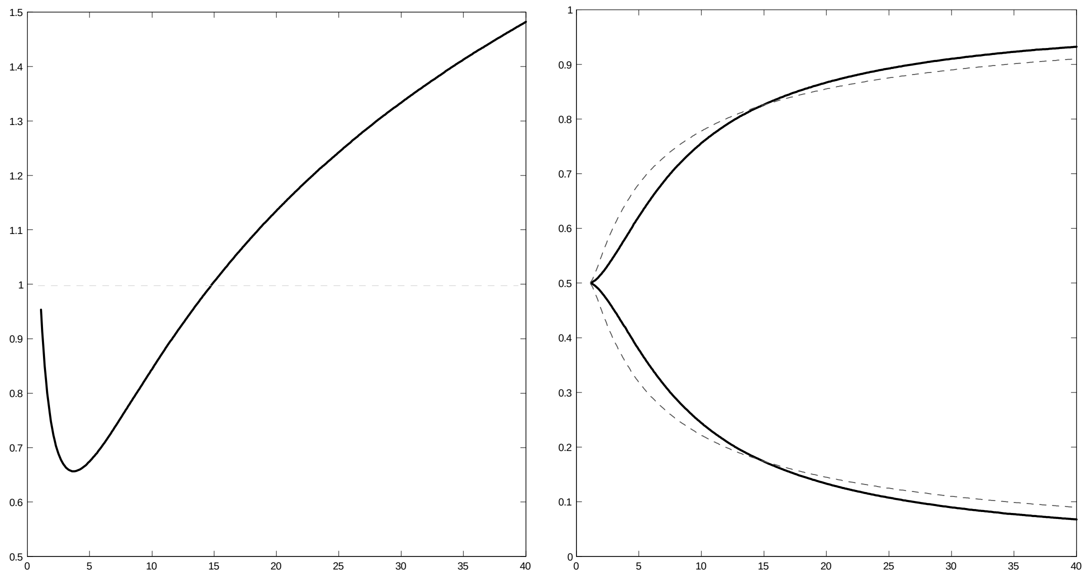

Figure 1: Left Panel: The ratio of the number of sampled bidders, $r^*(\bar{g}, 1)$, as a function of $\bar{g}$. Right Panel: Expected revenue as functions of $\bar{g}$. Straight black lines are expected revenues with solicitation, $\bar{E}_h[p|1, \overline{g}, r^*]$ (top) and $\bar{E}_l[p|1, \overline{g}, r^*]$ (bottom); dashed grey lines are the expected revenues of the ordinary common value auction, $\bar{E}_h[p|1, \overline{g}, 1]$ (top) and $\bar{E}_l[p|1, \overline{g}, 1]$ (bottom).

in $r = 1$, while the limit revenue for the equilibrium with endogenous solicitation is obtained with $r = r^*(\rho, \overline{g})$. Since $\phi_h(p|\rho, \overline{g}, r)$ is decreasing in r, $\bar{E}_h[p|\rho, \overline{g}, 1] > \bar{E}_h[p|\rho, \overline{g}, r^*]$, if $r^*(\rho, \overline{g}) < 1$ (i.e., when there is a *solicitation curse*), while the inequality is reversed if $r^*(\rho, \overline{g}) > 1$ (i.e., when there is a *solicitation blessing*). That is, when $r^*(\rho, \overline{g}) < 1$, there is less information revelation with bidder solicitation than in the ordinary auction; when $r^*(\rho, \overline{g}) > 1$, there is more information revelation with bidder solicitation. Figure 1 illustrates the shape of the ratio $r^*(1, \overline{g})$ and compares the expected revenue of each type of seller with and without solicitation. As shown, when $\bar{g}$ is small, $r^*(1, \overline{g}) < 1$ and when $\bar{g}$ is large, $r^*(1, \overline{g}) > 1$.[13]

8 Existence of Equilibrium with Grid

This section studies the existence of non-trivial equilibria.[14]

Theorem 3 *Consider a sequence of games* $\Gamma\left(s^k, P_{\Delta^k}\right)$*, such that* $\Delta^k > 0$ *and* $\lim\left(s^k, \Delta^k\right) = (0, 0)$.

[13] We conjecture that one can find for all ρ a cutoff $\hat{g}(\rho)$ such that $r^*(\rho, \overline{g}) \gtrless 1$ if $\overline{g} \gtrless \hat{g}(\rho)$, but we have not been able to verify this conjecture analytically.

[14] There is a trivial equilibrium where the auctioneer invites only one bidder and all summoned bidders bid 0.

(i). There always exists a sequence of equilibria that converges to the partially revealing outcome of Part (i) of Theorem 2.

(ii). Under certain conditions on the distribution of signals, there also exists a sequence of nontrivial equilibria that converges to the pooling outcome of Part (ii) of Theorem 2.

Observe that the grid of prices is finite in every step ($\Delta^k > 0$). This enables us to adapt familiar techniques to prove existence. We comment on this point later. Part (ii) means that we are able to demonstrate the existence of pooling equilibria under certain conditions. However, it does not mean that such equilibria exist only under those circumstances. We do not know whether pooling equilibria exist for all specifications of the model.

The Theorem is proved over the next two subsections. The first constructs a sequence of pooling equilibria. The second proves the existence of partially revealing equilibria.

8.1 Existence of Pooling Equilibria

We start with restrictions on the structure of signals that will be used in the construction of a sequence of nontrivial pooling equilibria.

1. Discrete Signals. The range of the signal values $[\underline{x}, \bar{x}]$ is divided into m subintervals

$$[\underline{x}, \epsilon], (\epsilon, 2\epsilon], \cdots, (\bar{x} - \epsilon, \bar{x}] \, .$$

The density functions g_w are step functions that are constant over each of these intervals and jump upwards at the boundaries.

2. Strengthening MLRP. The likelihood ratios satisfy

$$\frac{1}{G_l(\bar{x} - \epsilon)} < \frac{g_h\left(\bar{x}\right)}{g_l\left(\bar{x}\right)}, \tag{13}$$

and

$$\frac{g_h(\bar{x} - \epsilon)}{g_l(\bar{x} - \epsilon)} \frac{G_l(\bar{x} - \epsilon)}{G_h(\bar{x} - \epsilon)} \leq \frac{g_h(\bar{x})}{g_l(\bar{x})}. \tag{14}$$

Condition 1 means that, as far as the information is concerned, this is a discrete signal structure with m values. Consequently, the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is a step function as well, so there are at most m different likelihood ratios.[15] The continuum is kept only for purification purposes.

[15] The important assumption is the finiteness of the set of values that the likelihood ratio takes on. Density functions that are also step functions are consistent with that assumption but are not necessary.

Condition 2 can be thought of as a strengthening of the increasing likelihood ratio requirement at the top.[16] The first part is naturally satisfied if ϵ is not too large, since then $G_l(\bar{x}-\epsilon)$ is near 1, while $\frac{g_h(\bar{x})}{g_l(\bar{x})} > 1$. It is satisfied for example when $G_w(x) = x^{z_w}$, with $z_h > z_l$.

Proposition 6 Existence of Pooling Equilibrium *Consider a sequence of games* $\Gamma\left(s^k, P_{\Delta^k}\right)$ *such that* $\Delta^k > 0$, $\lim\left(s^k, \Delta^k\right) = (0,0)$ *and* $P_{\Delta^k} \subseteq P_{\Delta^{k'}}$, *for* $k < k'$. *Suppose that the signals are discrete and satisfy conditions (13) and (14) above. There exist bids* $\underline{b} < \bar{b} < \rho_l v_l + \rho_h v_h$ *and a sequence of equilibria* $(\beta^k, \boldsymbol{\eta}^k)$ *such that* $\min\left\{n_l^k, n_h^k\right\} \to \infty$ *and*

$$\beta^k(x) \begin{cases} = \bar{b} & \text{if} \quad x > \bar{x}-\epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x}-\epsilon, \end{cases} \qquad \text{for sufficiently large } k.$$

Thus, for this sequence of equilibria, the winning bid converges to $\bar{b}$ almost surely. The requirement that $P_{\Delta^k} \subset P_{\Delta^{k'}}$ is needed only to assure that $\bar{b} \in P_{\Delta^k}$, for all sufficiently large k's. However, essentially the same result can be proved without this assumption by looking at a sequence of $\bar{b}^k$'s.

Before turning to the formal proof, let us discuss some key steps of this construction. First, it is immediate from the form of β^k that $s^k \to 0$ implies $\min\left\{n_l^k, n_h^k\right\} \to \infty$. Next, given the strategies β^k, the probability $\pi_w[\bar{b}|\beta^k, n_w^k]$ of winning with bid $\bar{b}$ is approximately $1/n_w^k\left[1 - G_w(\bar{x}-\epsilon)\right]$ when k is large. This implies that the compound likelihood ratio $\frac{\rho_h}{\rho_l}\frac{n_h^k}{n_l^k}\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{\pi_h[\bar{b}|\beta^k,n_h^k]}{\pi_l[\bar{b}|\beta^k,n_l^k]}$ approaches $\frac{\rho_h}{\rho_l}$. That is, the sampling likelihood ratio $\frac{n_h^k}{n_l^k}$ and signal likelihood ratio $\frac{g_h(\bar{x})}{g_l(\bar{x})}$ exactly offset in the limit the winning likelihood ratio $\frac{\pi_h[\bar{b}|\beta^k,n_h^k]}{\pi_l[\bar{b}|\beta^k,n_l^k]}$. Therefore,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } \bar{b};\, \beta^k,\, \mathbf{n}^k\right] = \rho_l v_l + \rho_h v_h,$$

independently of $\underline{b}$ and $\bar{b}$.

Thus, bidding $\bar{b}$ yields positive payoff to a bidder with signal $x > \bar{x}-\epsilon$. To verify that bidding $\bar{b}$ is indeed optimal for such bidder, we have to consider all possible deviations. The deviation that requires a relatively more subtle argument is overbidding $\bar{b}$ by a bidder with signal $x > \bar{x}-\epsilon$. The payoff of such a bidder at $\bar{b}$ approaches 0 when k is large and slight overbidding assures a win. It turns out that optimality of the equilibrium sampling strategy assures that $\mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \mathbf{n}^k\right] < \bar{b}$.

[16] In fact, for the existence proof we only need the implication that $\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\frac{G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} \leq 1$. This inequality is implied by $\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\frac{G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)} \leq \frac{g_h(\bar{x})}{g_l(\bar{x})}$ since $\frac{g_h(\bar{x})}{g_l(\bar{x})} = \frac{1-G_h(\bar{x}-\varepsilon)}{1-G_l(\bar{x}-\varepsilon)} < \frac{\ln G_h(\bar{x}-\varepsilon)}{\ln G_l(\bar{x}-\varepsilon)}$ because $\frac{1-z}{\ln z}$ is decreasing in z.

To see the essence of this argument, suppose that the bidding strategy is simply

$$\beta^k(x) = \begin{cases} \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon, \end{cases}$$

the optimal solicitation strategy is pure, and allow us to ignore integer problems. Ignoring integer constraints, optimal solicitation implies the equality of the marginal benefit of an additional bidder to its cost in each state,

$$\begin{aligned} \left(G_l(\bar{x} - \epsilon)\right)^{n_l^k} \left(1 - G_l(\bar{x} - \epsilon)\right) \left(\bar{b} - \underline{b}\right) &= s^k, \\ \left(G_h(\bar{x} - \epsilon)\right)^{n_h^k} \left(1 - G_h(\bar{x} - \epsilon)\right) \left(\bar{b} - \underline{b}\right) &= s^k. \end{aligned}$$

Substituting out s^k, making a logarithmic transformation, rearranging and then taking limits we get

$$\lim_{k \to \infty} \frac{n_h^k}{n_l^k} = \frac{\ln G_l(\bar{x} - \epsilon)}{\ln G_h(\bar{x} - \epsilon)}.$$

This ratio is smaller than one, so that being solicited is bad news. Moreover, $\frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$, which follows from $\frac{g_h(\bar{x})}{g_l(\bar{x})} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_l(\bar{x}-\epsilon)}$ and $\frac{1-z}{\ln z}$ being decreasing in z. Hence, if solicitation is optimal given β^k then

$$\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim_{k \to \infty} \frac{n_h^k}{n_l^k} < 1. \tag{15}$$

Note that the limiting ratio of the number of solicited bidders is independent of the choice of $\bar{b}$ and $\underline{b}$.

Since a bid $b > \bar{b}$ wins with certainty,

$$\mathbb{E}\left[v | \bar{x}, \text{ win at } b > \bar{b};\, \beta^k, \mathbf{n}^k\right] = \frac{v_l + \frac{\rho_h}{\rho_l} \frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{n_h^k}{n_l^k} v_h}{1 + \frac{\rho_h}{\rho_l} \frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{n_h^k}{n_l^k} v_h}.$$

Therefore, by (15) for large enough k,

$$\mathbb{E}\left[v | \bar{x}, \text{ win at } b > \bar{b};\, \beta^k, \mathbf{n}^k\right] < \rho_l v_l + \rho_h v_h.$$

Choosing $\bar{b}$ sufficiently close to $\rho_l v_l + \rho_h v_h$ assures that this upward deviation is unprofitable.

The formal proof deals with the above deviation without the special simplifying assumptions, addresses the other potential deviations and shows how to choose $\underline{b}$ and $\bar{b}$ to assure immunity against all the deviations simultaneously. However, the more special argument that is tied to the endogenous sampling is contained in the above discussion.

Proof of Proposition 6:

Auxiliary Game A: Let $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ be an auxiliary game in which $\underline{b} < \bar{b}$ and the bidding strategies are constrained to satisfy

$$\beta(x)\begin{cases} = \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon. \end{cases} \tag{16}$$

A strategy profile $(\beta, \boldsymbol{\eta})$ is an equilibrium of $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ if $\boldsymbol{\eta}$ is an optimal solicitation strategy for the seller given β, and given $\boldsymbol{\eta}$, the strategy $\beta(x)$ is a best response subject to (16).

The heart of the proof consists of three lemmas on the equilibrium of the auxiliary game that are proved in the appendix. The first establishes existence when $\Delta > 0$. In this case, the auxiliary game is a finite Bayesian game.

Lemma 12 *If $\Delta > 0$, $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ has an equilibrium.*

The second lemma collects implications of the optimal sampling

Lemma 13 *Consider a sequence of auxiliary games $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$ such that $s^k \to 0$. Let β^k satisfy (16) and $\boldsymbol{\eta}^k$ be an optimal solicitation strategy given β^k, then:*

1. $\lim_{k\to\infty} n_w^k = \infty,\ w \in \{\ell, h\}$.
2. $\lim_{k\to\infty} \frac{n_h^k}{n_l^k} = \frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$.
3. $\lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_l(\bar{x}-\epsilon)^{n_l^k-1}} \leq \frac{(1-G_l(\bar{x}-\epsilon))}{(1-G_h(\bar{x}-\epsilon))}\frac{1}{G_h(\bar{x}-\epsilon)}$.

The third lemma utilizes the previous lemma to calculate limiting expected values conditional on winning.

Lemma 14 *Consider a sequence of auxiliary games $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$ such that $s^k \to 0$. Let β^k satisfy (16) and $\boldsymbol{\eta}^k$ be an optimal solicitation strategy given β^k. Then there are numbers v_1^*, v_2^*, v_3^* independent of $\underline{b}$, $\bar{b}$ such that*

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ win\ at\ \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &= \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ win\ at\ \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_1^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ win\ at\ b > \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_2^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ win\ at\ b \in \left(\underline{b}, \bar{b}\right);\ \beta^k, \boldsymbol{\eta}^k\right] &\leq v_3^* < \rho_l v_l + \rho_h v_h.
\end{aligned}$$

Select any $\underline{b}$ and $\bar{b}$ that satisfy

$$\max\{v_1^*, v_2^*, v_3^*\} < \underline{b} < \bar{b} < \rho_l v_l + \rho_h v_h. \tag{17}$$

By Lemma 12, the auxiliary game $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$, $\Delta^k > 0$, has an equilibrium $(\beta^k, \boldsymbol{\eta}^k)$. We show next that $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for s^k sufficiently small by proving that the constraints (16) do not bind if (17) holds.

From Lemma 13, $\min\left\{n_l^k, n_h^k\right\} \to \infty$.

Step 1. Bidding $\beta^k = \bar{b}$ is optimal if $x > \bar{x} - \epsilon$.

(i) Bidding $b > \bar{b}$ is unprofitable. By the choice of $\bar{b} > v_2^*$ and Lemma (14),

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] < \bar{b}. \tag{18}$$

Thus, there is some K_1 such that bidding $b > \bar{b}$ is strictly unprofitable for all $k \geq K_1$.

(ii) Bidding $b < \bar{b}$ is unprofitable. First, by Lemma 14 and the choice of $\bar{b}$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \text{win at } \bar{b};\ \beta^k, \boldsymbol{\eta}^k\right] = \rho_l v_l + \rho_h v_h > \bar{b}. \tag{19}$$

For any $b < \bar{b}$, Lemma 3 implies

$$\lim_{k\to\infty} \frac{\bar{\pi}_w\left[\bar{b}|\beta^k, \eta_w^k\right]}{\bar{\pi}_w\left[b|\beta^k, \eta_w^k\right]} \geq \lim_{k\to\infty} \frac{\frac{1}{n_w^k}\frac{1}{1-G_w(\bar{x}-\epsilon)}\left(1-\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k}\right)}{\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k-1}} = \infty. \tag{20}$$

where the last equality follows from $n_w^k \to \infty$. By (19), the payoff conditional on winning at $\bar{b}$ is bounded away from 0. It now follows from (20) that there is some K_2 such that for all $k \geq K_2$, the payoff from bidding $b < \bar{b}$ is an arbitrarily small fraction of the payoff of bidding $\bar{b}$, so that undercutting $\bar{b}$ is unprofitable for $x > \bar{x} - \epsilon$. □

Step 2. Bidding $b > \underline{b}$ is unprofitable for $x \leq \bar{x} - \epsilon$.

By Lemma 14, the choice of $\bar{b} > \max\{v_1^*, v_2^*, v_3^*\}$ and MLRP, for all $x \leq \bar{x} - \epsilon$:

(i) Bidding $b > \bar{b}$ is unprofitable. For $x \leq \bar{x} - \epsilon$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|x, \text{ win at } b > \bar{b};\ \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right].$$

Hence, (18) implies that bidding $b > \bar{b}$ is strictly unprofitable for $x \leq \bar{x} - \epsilon$ and all $k \geq K_1$.

(ii) Bidding $\bar{b}$ is unprofitable, since for all $x \leq \bar{x} - \epsilon$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|x, \text{ win at } \bar{b};\ \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} \mathbb{E}\left[v|\bar{x} - \epsilon, \text{ win at } \bar{b};\ \beta^k, \boldsymbol{\eta}^k\right] \leq v_1^* < \bar{b}.$$

Thus, there is some K_3, such that bidding $\bar{b}$ is unprofitable for $x \leq \bar{x} - \epsilon$ when $k \geq K_3$.
(iii) Bidding $b \in \left(\underline{b}, \bar{b}\right)$ is unprofitable, since for all $x \leq \bar{x} - \epsilon$,

$$\lim_{k \to \infty} \mathbb{E}\left[v | x, \text{ win at } b \in \left(\underline{b}, \bar{b}\right); \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k \to \infty} \mathbb{E}\left[v | \bar{x} - \epsilon, \, b \in \left(\underline{b}, \bar{b}\right); \beta^k, \boldsymbol{\eta}^k\right] \leq v_3^* < \underline{b}.$$

Thus, there is some K_4 such that for all $k \geq K_4$ bidding any $b \in \left(\underline{b}, \bar{b}\right)$ is unprofitable for all $x \leq \bar{x} - \epsilon$. □

Let $K = \max\{K_1, \cdots, K_4\}$. Step 1 and Step 2 imply that the additional constraints of the auxiliary game do not bind when $k \geq K$ and (17) holds. Thus, $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for $k \geq K$. For $k < K$, we can pick any equilibrium. By construction, $\beta^k(x) = \bar{b}$ for all $x > \bar{x} - \epsilon$ and $k \geq K$. ∎

Proposition 6 establishes Part (ii) of Theorem 3.

8.2 Existence of Partially Revealing Equilibria

Proposition 7 *Consider a sequence of games $\Gamma\left(s^k, P_{\Delta^k}\right)$, $s^k > 0$, $\Delta^k > 0$ and $\left(s^k, \Delta^k\right) \to 0$. There exists a sequence of equilibria $(\beta^k, \boldsymbol{\eta}^k)$ that converges to the partially revealing outcome of Part (i) of Theorem 2.*

Proof of Proposition 7:

Auxiliary Game B. We define a second auxiliary game $\Gamma^B(s, P_\Delta | \underline{n}_l, \underline{r})$ as follows: (i) the two types are represented by separate players who choose n_l and n_h simultaneously; (ii) the chosen numbers (n_l, n_h) determine the actual numbers of solicited bidders as $\hat{n}_l = \max\{n_l, \underline{n}_l\}$ and $\hat{n}_h = \max\{\underline{r}\hat{n}_l, n_h\}$; (iii) everything else is just as before. An equilibrium (β, η_l, η_h) of the auxiliary game is defined as usual: β is a bidding equilibrium given the distribution of solicited bidders implied by (η_l, η_h); given β, η_w maximizes seller w's profit evaluated at the corresponding $\hat{n}_w$'s, i.e., $\eta_w(n) > 0 \Rightarrow n \in \arg\max_{n \in \{1,2,\ldots.\}} \mathbb{E}_w[p | \beta, \hat{n}_w(n)] - \hat{n}_w(n)s$. Note that the sets of equilibria of $\Gamma^B(s, P_\Delta | 1, 0)$ and $\Gamma(s, P_\Delta)$ are identical since the constraints do not bind.

Lemma 15 *If $\Delta > 0$, $\Gamma^B(s, P_\Delta | \underline{n}_l, \underline{r})$ has an equilibrium.*

The proof is analogous to the proof of Lemma 12 and omitted.

Next, we show that for sufficiently large k, all the equilibria of a certain sequence $\Gamma^B\left(s, P_{\Delta^k} | \underline{n}_l^k, \underline{r}\right)$ of auxiliary games are partially revealing.

Lemma 16 *Consider a sequence of auxiliary games* $\Gamma^B\left(s, P_{\Delta^k}|\underline{n}_l^k, \underline{r}\right)$ *such that* $\left(s^k, \Delta^k\right) \to (0,0)$, $\underline{n}_l^k = \frac{1}{\sqrt{s^k}}$ *and* $\underline{r} \in (\frac{1}{\overline{g}}, r^*(\rho, \overline{g}))$. *For any sequence of equilibria* $(\beta^k, \eta_l^k, \eta_h^k)$ *of* $\Gamma^B\left(s^k, P_{\Delta^k}|\underline{n}_l^k, \underline{r}\right)$: $n_l^k > \underline{n}_l^k$ *for large* k, $\lim \frac{n_h^k}{n_l^k} = r^*(\rho, \overline{g})$ *and* $\lim F_w\left(p|\beta^k, \eta_w^k\right) = \phi_w\left(\cdot|\rho, \overline{g}, r^*\right)$, *with* ϕ_w *defined by (9).*

Proof: Given the sequence of equilibria, let $r = \lim_{k\to\infty} \frac{\hat{n}_h^k}{\hat{n}_l^k}$.

Step 1. $\lim F_w\left(p|\beta^k, \hat{n}_w^k\right) = \phi_w\left(p|\rho, \overline{g}, r\right)$.

Proof of Step 1: The choice of $\underline{n}_l^k$ and $\underline{r}$ implies $\min\left\{\hat{n}_l^k, \hat{n}_h^k\right\} \to \infty$ and $\overline{g}\lim\left(\hat{n}_h^k/\hat{n}_l^k\right) > 1$. Hence, Proposition 3 (in its extension to the case of $\Delta^k \geq 0$) implies that $\lim F_w\left(p|\beta^k, \hat{n}_w^k\right) = \phi_w\left(p|\rho, \overline{g}, r\right)$, for all p and $w = l, h$. □

Step 2. For k sufficiently large, $n_l^k > \underline{n}_l^k$ and $n_h^k > \underline{r} n_l^k$

Proof of Step 2: By choice of $\underline{r}$ and by the argument from Lemma 11 (in the proof of Proposition 4), $\frac{1}{\overline{g}} < \lim_{k\to\infty} \frac{\hat{n}_h^k}{\hat{n}_l^k} < \infty$. Hence, ϕ_l is not degenerate. Let m_w^k denote an unconstrained optimal solicitation for type w given β^k. By Lemma 8, m_l^k satisfies

$$\lim_{k\to\infty} m_l^k s^k = -\int_0^{v_h} \phi_l\left(p|\rho, \overline{g}, r\right) \ln\left(\phi_l\left(p|\rho, \overline{g}, r\right)\right) dp > 0.$$

Since $\underline{n}_l^k s^k = \sqrt{s^k} \to 0$, $\lim m_l^k s^k > 0$ implies $\lim \frac{\underline{n}_l^k}{m_l^k} = 0$, so that $m_l^k > \underline{n}_l^k$ for sufficiently large k. Thus, $n_l^k = m_l^k > \underline{n}_l^k$, as claimed.

Suppose to the contrary that $m_h^k \leq \underline{r}\hat{n}_l^k$. Then, the strict concavity of the seller's optimization implies $\hat{n}_h^k = \underline{r}\hat{n}_l^k$. By $\frac{1}{\overline{g}} < \underline{r} < r^*(\rho, \overline{g})$ and Lemma 10, $J(\underline{r}; \rho, \overline{g}) < 0$.

From the proof of Lemma 11—especially Equation (55)—$J(\underline{r}; \rho, \overline{g}) < 0$ implies

$$\lim_{k\to\infty} \hat{n}_h^k \left(\mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k + 1\right] - \mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k\right]\right) > \lim_{k\to\infty} \hat{n}_h^k s^k.$$

Hence, for sufficiently large k,

$$\mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k + 1\right] - \mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k\right] > s^k.$$

That is, at $\hat{n}_h^k$ sampling an additional bidder is strictly profitable for type h. Therefore, $n_h^k = m_h^k > \underline{r}\hat{n}_l^k$, as claimed. □

Step 3. $\lim \hat{n}_h^k/\hat{n}_l^k = \lim n_h^k/n_l^k = r^*(\rho, \overline{g})$.

Proof of Step 3: By Step 2, $\hat{n}_h^k$ and $\hat{n}_l^k$ are both unconstrained optimal given β^k. Therefore, Lemma 11 requires that $\lim \frac{\hat{n}_h^k}{\hat{n}_l^k} = r^*(\rho, \overline{g})$. □

Steps 1 and 3 together establish the lemma. ∎

Lemma 16 implies that, for suitably chosen $\left(\underline{n}_l^k, \underline{r}\right)$ and for sufficiently large k, all equilibria of the auxiliary game $\Gamma^B\left(s^k, P_{\Delta^k}|\underline{n}_l^k, \underline{r}\right)$ are also equilibria of $\Gamma\left(s^k, P_{\Delta^k}\right)$

and are close to the partially revealing outcome. Lemma 15 implies that $\Gamma^B\left(s^k, P_{\Delta^k}|\underline{n}_l^k, \underline{r}\right)$ has an equilibrium when $\Delta^k > 0$. Therefore, there exists a sequence of equilibria $(\beta^k, \boldsymbol{\eta}^k)$ for $\Gamma\left(s^k, P_{\Delta^k}\right)$ that converges to the partially revealing outcome of Part (i) of Theorem 2. ∎

8.3 Existence without Grid

We use the finiteness of the relevant set of feasible bids to prove existence of equilibrium in the auxiliary games. This is the only place where we use the grid. The characterization results in Lemmas 14 and 16 hold also without the grid. Therefore, if we could prove existence of equilibrium for the auxiliary games without the grid, then we could also drop the requirement that $\Delta^k > 0$ from Propositions 6 and 7.

The difficulty for showing existence without a grid is the presence of atoms in equilibrium, which implies that buyers' *equilibrium* payoffs can be discontinuous in their bids. In particular, we cannot argue that the limit of a sequence of equilibria for a vanishingly small grid is an equilibrium of the continuum case. The reason is that there may be atoms in the limit that are absent in the sequence. To illustrate the problem, consider a sequence of games with grid P_{Δ^k} and suppose that along the sequence bidders bid either b or $b + \Delta^k$, depending on whether their signal is below or a above some threshold $\hat{x}$. The pointwise limit strategy as $\Delta^k \to 0$ would be that all bidders bid the constant b. However, this bidding strategy would imply a strictly lower winning probability for buyers who bid $b + \Delta^k$ along the sequence and a strictly higher winning probability for buyers bidding b. Thus, the limit strategy may not be an equilibrium of the game with a continuum of bids, even though the elements of the sequence may have been.[17]

A *possible* solution to the existence problem without a grid is to change the tie-breaking rule, as suggested by Jackson, Simon, Swinkels, and Zame (2002). Specifically, consider the following extension: Buyers submit two numbers, the first interpreted as a bid (just as before) and the second number interpreted as eagerness to trade. If there is a unique highest bid, the seller chooses to buy from that bidder. When several bids are tied, the seller may choose among the buyers based on their expressed eagerness. Extending our model in this way solves the existence problem, because the limit of a sequence of equilibrium strategies for a vanishingly small grid corresponds to an equilibrium of the extended game with a continuum of bids. For instance, in the example from the last paragraph, one may specify as the limit strategy of the extended game that buyers bid b for all signals. Buyers with signals above

[17] There is no such problem for the seller's strategy because of the continuity of the seller's payoffs in β and η. If $\left(\beta^k, \eta^k\right)$ converge pointwise to (β^*, η^*), and if η^k is an optimal solicitation strategy given β^k, then η^* is an optimal solicitation strategy given β^*.

the threshold (who bid $b + \Delta^k$ along the sequence) all express the same eagerness, say e_h, and buyers with signals below the threshold (who bid b along the sequence) express a different eagerness, say e_l. If multiple bidders are tied at b, then the seller picks first among those bidders who express e_h, choosing randomly if there are multiple such bidders; if no bidder expressed e_h, the seller chooses randomly among bidders expressing e_l (and, finally, choosing bidders who expressed anything else last). This limit strategy preserves the winning probabilities, and, hence, the payoffs in a continuous way. Thus, if the elements of the described sequence of bidding strategies each constitute an equilibrium, so would the limit.

9 Discussion and Conclusion

9.1 Information Aggregation

For a common values auction environment, Wilson (1977) and Milgrom (1979) derived conditions on the informativeness of the signals under which the price aggregates information when the number of bidders becomes large. In their environment, the known number of bidders is exogenous and independent of the state of nature. They show that the winning bid approaches the true value when the number of bidders becomes large if and only if there are unboundedly informative, favorable signals, $\overline{g} \equiv \frac{g_h(\overline{x})}{g_l(\overline{x})} = \infty$. If $\overline{g} < \infty$, then our results imply that the limit equilibrium of the standard common value auction is partially revealing, but it becomes continuously more revealing as $\overline{g}$ increases.[18]

In a related sequential search version of that model that differs mainly in that the seller searches sequentially for buyers, Lauermann and Wolinsky (2012) show that, when the search cost is negligible, nearly perfect information aggregation requires stronger conditions on the informativeness of the most favorable signals: Not only $\overline{g} = \infty$, but also the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ has to increase at a sufficiently fast rate when x approaches $\overline{x}$. If $\overline{g} < \infty$, the equilibrium is complete pooling and both types trade at a price equal to the ex-ante expected value.[19]

The present model combines elements from both of these environments. It is an auction in which the buyers compete directly in prices, but the endogenous state dependent solicitation of buyers is reminiscent of the search model. Indeed, in terms of information aggregation, the current model exhibits both patterns of information aggregation. The partially revealing equilibria resemble the equilibria of the standard auction. In particular, when $\overline{g}$ is large, the aggregation of information

[18] See the remarks on large ordinary common value auctions in Section 7.

[19] In Lauermann and Wolinsky (2012), the roles of buyers and sellers are reversed, so that the buyer is the informed and the sellers are the uninformed agents.

is nearly perfect. To see this, recall that $r^*(\rho, \overline{g})$ is the solution to $J(r; \rho, \overline{g}) = 0$.

Lemma 17 $\lim_{\overline{g} \to \infty} \overline{g} r^*(\rho, \overline{g}) = \infty$.

Proof: Inspection of $J(r; \rho, \overline{g}) = \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r - 1}} \frac{\ln x}{(1 + x\rho\overline{g}r)^2} dx$ reveals that, if $\overline{g}r$ is bounded, then $J(r; \rho, \overline{g}) < 0$ for large $\overline{g}$. Therefore, it must be that $\overline{g} \to \infty$ implies $\overline{g}r^*(\rho, \overline{g}) \to \infty$ ∎

Now, it can be observed from Equations (9) and (10) that when $\overline{g}r^*$ becomes large, the limiting distribution of the winning bid $\phi_w(\cdot|\rho, \bar{g}, r^*)$ puts almost all its weight on v_w. Thus, large $\overline{g}$ implies nearly perfect information aggregation in the partially revealing equilibrium.

In contrast, the pooling equilibrium of Section 8.1 aggregates no information—the winning bid is at or even strictly below the ex-ante expected value and such equilibria may exist independently of how large is $\overline{g}$. In this sense it resembles the equilibrium of the corresponding sequential search model.

9.2 Information Aggregation and Efficiency

This paper devotes much attention to the question of information aggregation. The reader may wonder whether the aggregation of information is of importance in a common values environment. The answer is that it may have significant efficiency consequences. First, even in the model in its present form, the total solicitation cost is tied to the degree of information aggregation. It is negligible when the information is nearly perfectly revealed (in the partially revealing equilibrium in the case of large $\overline{g}$) and when no information is revealed (in the pooling equilibrium). However, it is not negligible when the information is partially revealed.

Second, for simplicity, we have assumed that the seller's cost is zero and that the value is determined exogenously. Therefore, trade is always efficient and the extent of information revelation has no efficiency consequences in this respect. However, straightforward enrichments of the model will introduce such efficiency consequences. For example, if the seller's cost is $c \in (v_l, v_h)$, efficiency requires that trade takes place only in state h. In this case, failure of information aggregation implies allocative inefficiencies. Alternatively, if the seller has an opportunity to invest in quality improvements prior to trade, failure of information aggregation could imply inefficiently weak investment incentives.

9.3 Unbounded Likelihood Ratio

The boundedness of the signal likelihood ratio, $\bar{g} < \infty$, is important for our characterization argument. Here, we report two additional results about limiting equilibrium outcomes when the signal likelihood ratio is unbounded and discuss directions for future research on this topic.[20]

First, the characterization result of Theorem 2 extends to unbounded likelihood ratios, that is, we can allow for $\bar{g} = \infty$ in the statement of the theorem in the following sense: for every sequence of equilibria for vanishing solicitation costs in which at least two bidders are solicited, either the limit is perfectly revealing or there is a common atom at a price below the ex-ante expected value. This is a natural continuity implication of the theorem for $\bar{g} = \infty$, since we have already argued that the theorem implies nearly fully revealing prices when $\bar{g} < \infty$ but sufficiently large in Section 9.1.

Second, for some distribution functions G_w, $w = l, h$, with $\bar{g} = \infty$ and some sequence $s^k \to 0$, there exists both a sequence of equilibria along which prices converge to the true values (complete revelation) and a sequence of equilibria along which there remains an arbitrarily large atom at the top (pooling).

We conclude that the characterization and the existence of limit outcomes with atoms do not depend on the assumption that the likelihood ratio is bounded. The identification of general sufficient conditions for atoms to persist in the limit, however, is left for future research. It would likely require the use of different techniques than the one used in our analysis and is beyond the scope of the current paper.

9.4 Signaling: Observable Number of Bidders

If the number of solicited bidders is observable, it may signal the seller's information.[21] Consider a variation on our model in which the buyers observe the total number of solicited bidders before submitting their bids, while everything else remains unchanged. This variation has two types of pure strategy equilibria – separating and pooling. In the pooling equilibrium, both types of the seller solicit the same number of bidders. Multiple pooling equilibria can be supported by specifying that buyers believe that a seller who solicits an out-of-equilibrium number of bidders must be of the low type, consequently bidding at most v_l. Bidding in the pooling equilibria is the same as in the standard common value auction, because the number of bidders is independent of the state. In the separating equilibrium $n_l = 2$

[20] We state these results informally without proofs because they would have further increased the length of the paper. Complete results and their proofs are available online in a supplementary note at www.sites.google.com/site/slauerma.

[21] Our interest is in analyzing a specific trading environment in which the seller cannot verifiable communicate the number of solicited sellers. We discuss this variation as an exercise to provide further insight into the mechanism of the model. One may also be interested in the mechanism that is optimal for the seller. This mechanism likely resembles a full-rent extraction mechanism as in Cremer and McLean (1988) because the seller may utilize the correlation of buyers' signals, with the added difficulty that the seller has private information; see Severinov (2008).

and $n_h > 2$. Bidders bid v_l if two bidders are solicited and bid v_h if n_h bidders are solicited. To ensure incentive compatibility, it must be that $v_h - n_h s = v_l - 2s$. Thus, in the separating equilibrium, the payoff of each type of the seller is $v_l - 2s$. Therefore, in this equilibrium, if s is small, the seller's revenue is lower than it is when the number of bidders is not observable, as in the model of this paper.[22]

9.5 Seller's Commitment

Suppose that the seller can commit ex-ante to a solicitation strategy, with the rest of the game remaining unchanged. If the seller can commit ex-ante to a solicitation strategy, she can extract nearly the entire surplus when s is small: For example, the auctioneer may commit to solicit $1/\sqrt{s}$ bidders in both states. This would induce an ordinary auction. When s is small, the number of solicited bidders is large. Hence, the expected revenue is approximately equal to the ex-ante expected value while the total solicitation cost is just $\sqrt{s}$. The resulting profit is strictly higher than the seller's profit in the partially revealing equilibrium of the original model without commitment where the total solicitation cost might be significant. Thus, relative to the sellers' preferred number of bidders, in the absence of commitment the auction is "too large" in both states.

Since the commitment described above is the same across the states, this argument also holds when the seller does not know the true state.

9.6 Uninformed Seller

Suppose now that the seller is *uninformed* about the state, again with the rest of the game remaining unchanged. Therefore, $n_l = n_h = n(s)$. When s is small, $n(s)$ is large and the distributions of the winning bid are close to $\phi_w\left(\cdot|\rho, \overline{g}, r = 1\right)$. Lemma 8 implies that $n(s)s > 0$. Thus, despite the fact that the ex-ante expected revenue would be near the ex-ante expected surplus, the uninformed seller's expected profit would be smaller than that.[23]

It may be surprising that the seller incurs non-vanishing total solicitation costs even when uninformed. Intuitively, the seller is expected to solicit "too many" bidders which induces bidders to act more cautiously to mitigate the winner's curse.

[22] There are additional partially separating equilibria in mixed strategies. For example, the high type may randomly choose either 2 or n_h bidders. If 2 bidders are chosen, bidders bid as in the corresponding common value auction in which the priors are adjusted appropriately for the seller's strategy.

[23] It is not obvious whether the expected profit of an uninformed seller is lower or higher than its counterpart in the partially revealing equilibrium of the informed seller's case—the ex-ante expected revenue is equal but the total solicitation cost may differ.

This in turn induces the seller to indeed solicit a large number and incur significant total cost even when the marginal solicitation cost is small.[24]

This teaches us that non-vanishing total solicitation costs with an informed seller are not only attributable to the interplay of the separating and pooling incentives of the two types, but that these costs are also due to a commitment problem that arises already when the seller is uninformed.

9.7 Simultaneous Search

Although this paper is couched in the terminology of auctions, it could be equivalently thought of as a simultaneous search model along the lines of Burdett and Judd (1983) with an added element of adverse selection. The seller in our model is the counterpart of the buyer in their model.[25] The important difference is in the private information that the sampling agent has in our model. The private information implies both additional substantive insights and some additional analytical challenges. Together, the current paper and Lauermann and Wolinsky (2012) span the two common modes of search, sequential and simultaneous.

9.8 About the Assumptions

The assumption of a binary state is used in the proof of the monotonicity of the bidders' best response. In this sense, the assumption plays an important role. Nevertheless, the assumption buys us more than we need—the best response to any strategy is monotone. Therefore, it may be possible to obtain similar results for monotone equilibria with more than two states. This seems to be an interesting extension that continued work on this subject may address.

We have assumed that the seller is fully informed about the state. It is hard to see that anything substantial would change if the seller observed a noisy signal of the state instead. Of course, if the signal were not binary, then the model would be like a multi-state world.

[24] As noted above, with commitment the uninformed seller's profit can also be near the total surplus when s is small.

[25] The roles of the seller and the buyers in our model can be reversed to make the models exactly parallel.

10 Appendix—For Online Publication

The proofs are grouped according to the section in which they appear in the main text.

10.1 Bidding Equilibrium Characterization

Proof of Lemma 1: $b' > b \geq v_l$ implies $(v_l - b') < (v_l - b)$ and $\overline{\pi}_l(b'|\beta,\eta_l) \geq \overline{\pi}_l(b|\beta,\eta_l)$. These together with the hypothesis $\overline{\pi}_l(b'|\beta,\eta_l) > 0$ and $b' > b \geq v_l$ imply

$$\overline{\pi}_l(b'|\beta,\eta_l)\left(v_l - b'\right) < \overline{\pi}_l(b|\beta,\eta_l)\left(v_l - b\right). \tag{21}$$

Hence, $U(b'|x,\beta,\boldsymbol{\eta}) \geq U(b|x,\beta,\boldsymbol{\eta})$ requires

$$\overline{\pi}_h(b'|\beta,\eta_h)\left(v_h - b'\right) > \overline{\pi}_h(b|\beta,\eta_h)\left(v_h - b\right). \tag{22}$$

Rewriting $U(b'|x,\beta,\boldsymbol{\eta})$ yields

$$\frac{\rho_l g_l(x)\,\bar{n}_l}{\rho_l g_l(x)\,\bar{n}_l + \rho_h g_h(x)\,\bar{n}_h}\left[\overline{\pi}_l(b|\beta,\eta_l)(v_l - b) + \frac{\rho_h g_h(x)\,\bar{n}_h}{\rho_l g_l(x)\,\bar{n}_l}\overline{\pi}_h(b|\beta,\eta_h)\left(v_h - b\right)\right]. \tag{23}$$

It follows from $U(b'|x,\beta,\boldsymbol{\eta}) \geq U(b|x,\beta,\boldsymbol{\eta})$ and (21) that

$$\begin{aligned} & \frac{\rho_h g_h(x)\,\overline{n}_h}{\rho_l g_l(x)\,\overline{n}_l}\left[\overline{\pi}_h(b'|\beta,\eta_h)\left(v_h - b'\right) - \overline{\pi}_h(b|\beta,\eta_h)\left(v_h - b\right)\right] \\ \geq\ & \overline{\pi}_l(b|\beta,\eta_l)(v_l - b) - \overline{\pi}_l(b'|\beta,\eta_l)(v_l - b') > 0. \end{aligned}$$

Since $x' > x$ and $\frac{g_h(x)}{g_l(x)}$ is non-decreasing,

$$\begin{aligned} & \frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_l g_l(x')\,\overline{n}_l}\left[\overline{\pi}_h(b'|\beta,\eta_h)\left(v_h - b'\right) - \overline{\pi}_h(b|\beta,\eta_h)\left(v_h - b\right)\right] \\ \geq\ & \overline{\pi}_l(b|\beta,\eta_l)(v_l - b) - \overline{\pi}_l(b'|\beta,\eta_l)(v_l - b') > 0. \end{aligned} \tag{24}$$

which implies

$$\begin{aligned} & U(b'|x',\beta,\boldsymbol{\eta}) \\ =\ & \frac{\rho_l g_l(x')\,\overline{n}_l}{\rho_l g_l(x')\,\overline{n}_l + \rho_h g_h(x')\,\overline{n}_h}\left[\overline{\pi}_l(b'|\beta,\eta_l)(v_l - b') + \frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_l g_l(x')\,\overline{n}_l}\overline{\pi}_h(b'|\beta,\eta_h)\left(v_h - b'\right)\right] \\ \geq\ & \frac{\rho_l g_l(x')\,\overline{n}_l}{\rho_l g_l(x')\,\overline{n}_l + \rho_h g_h(x')\,\overline{n}_h}\left[\overline{\pi}_l(b|\beta,\eta_l)(v_l - b) + \frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_l g_l(x')\,\overline{n}_l}\overline{\pi}_h(b|\beta,\eta_h)\left(v_h - b\right)\right] \quad (25) \\ =\ & U(b|x',\beta,\boldsymbol{\eta}). \end{aligned}$$

If $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$, then (24) and (25) hold with strict inequalities.

The last part of the lemma is immediate because G_h and G_l are mutually absolutely continuous, so that $G_h(\{x|\beta(x) \le b\}) = 0 \Leftrightarrow G_l(\{x|\beta(x) \le b\}) = 0$. ∎

Proof of Lemma 2:

Step 0: If $\pi_w(b|\beta, n) > 0$ for some $n \ge 2$ and $w = l$ or h, then $\overline{\pi}_w(b|\beta, \eta_w) > 0$ for both w and any η_w.

Proof of Step 0: $\pi_w(b|\beta, n) > 0$ for some n and w implies that $G_w(\{x|\beta(x) \le b\}) > 0$. Since G_h and G_l are mutually absolutely continuous, it follows that $G_{w'}(\{x|\beta(x) \le b\}) > 0$ also for $w' \ne w$. Therefore, $\overline{\pi}_w(b|\beta, \eta_w) > 0$ for both w and any η_w. □

Step 1. $\beta(x) \ge v_l$ for almost all x.

Proof of Step 1: Let $\underline{b} \equiv \inf\{b|\pi_w(b|\beta, n) > 0$ for some n and $w\}$. Suppose $\underline{b} < v_l$. It may not be that β has an atom at $\underline{b}$ (i.e., $\int_{\{x:\beta(x)=\underline{b}\}} g_w(x)dx > 0$) since by a standard Bertrand argument $U(\underline{b} + \varepsilon|x, \beta, \boldsymbol{\eta}) > U(\underline{b}|x, \beta, \boldsymbol{\eta})$ for sufficiently small $\varepsilon \in (0, v_l - \underline{b})$. Therefore, there exists a sequence of x^k such that $\beta(x^k) \to \underline{b}$ and $\overline{\pi}_w(\beta(x^k)|\beta, \eta_w) \to 0$ (owing to $\eta_w(0) = \eta_w(1) = 0$). Hence, equilibrium payoffs $U(\beta(x^k)|x^k, \beta, \boldsymbol{\eta}) \to 0$. However, by the definition of $\underline{b}$ and monotonicity of $\overline{\pi}_w$, $\overline{\pi}_w[b|\beta, \eta_w]$ is strictly positive for all $b \in (\underline{b}, v_l)$. Thus, for all $b \in (\underline{b}, v_l)$, the payoff $U(b|x, \beta, \boldsymbol{\eta}) > 0$. This contradicts the optimality of $\beta(x^k)$ for sufficiently large k, a standard Bertrand argument. Thus, $\underline{b} \ge v_l$. Finally, $\pi_w(b|\beta, n) = 0$ for all $b < v_l$ implies that $G_w(\{x|\beta(x) \ge v_l\}) = 1$, proving the step. □

Step 2. $\beta(x) < v_h$ for all x. □

Proof of Step 2: It clearly cannot be that $G_w(\{x|\beta(x) > v_h\}) = 1$ for any w, since this would imply that bidders have strictly negative payoffs in expectations. Suppose that $\beta(x') \ge v_h$ for some x'. From $G_l(\{x|\beta(x) > v_h\}) < 1$, $\beta(x') \ge v_h$ implies $\overline{\pi}[\beta(x')|\beta, \eta_l] > 0$ and $U(\beta(x')|x', \beta, \boldsymbol{\eta}) < 0$, a contradiction to optimality of $\beta(x')$. □

Step 3. $\overline{\pi}_w(\beta(x)|\beta, \eta_w) > 0$ for almost all x for $w \in \{l, h\}$.

Proof of Step 3: Fix $w \in \{l, h\}$. Let $X = \{x|\overline{\pi}_w(\beta(x)|\beta, \eta_w) = 0\}$. The probability that in state w all bidders are from that set is $\Sigma_n \eta_w(n)[G_w(X)]^n$. Since in that event some bidder has to win, we have $\Sigma_n \eta_w(n)[G_w(X)]^n \le \Pr[\{$Winning bidder has signal $x \in X\}|w] \le \overline{n}_w \int_{x \in X} \overline{\pi}_w(\beta(x)|\beta, \eta_w) g(x) dx = 0$. Hence, $G_w(X) = 0$. □

Step 4. For any $x' > x$, $U(\beta(x')|x', \beta, \boldsymbol{\eta}) \ge U(\beta(x)|x, \beta, \boldsymbol{\eta})$. The inequality is strict if and only if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$. Thus, $\frac{g_h(x')}{g_l(x')} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$ implies that $U(\beta(x')|x', \beta, \boldsymbol{\eta})$ is strictly positive.

Proof of Step 4: From (2) it follows (after dividing the numerator and denominator by $g_l(x)$) that

$$U(b|x,\beta,\boldsymbol{\eta}) = \frac{\rho_l \overline{n}_l \overline{\pi}_l(b|\beta,\eta_l)(v_l - b) + \rho_h \frac{g_h(x)}{g_l(x)} \overline{n}_h \overline{\pi}_h(b|\beta,\eta_h)(v_h - b)}{\rho_l \overline{n}_l + \rho_h \frac{g_h(x)}{g_l(x)} \overline{n}_h}. \tag{26}$$

Therefore, for any $x' > x$,

$$U(\beta(x')|x',\beta,\boldsymbol{\eta}) \geq U(\beta(x)|x',\beta,\boldsymbol{\eta}) \geq U(\beta(x)|x,\beta,\boldsymbol{\eta}) \geq 0, \tag{27}$$

where the first and last inequalities are equilibrium conditions; the second inequality owes to $\frac{g_h(x')}{g_l(x')} \geq \frac{g_h(x)}{g_l(x)}$ and $\overline{\pi}_h(\beta(x)|\beta,\eta_h)(v_h - \beta(x)) \geq 0 \geq \overline{\pi}_l(\beta(x)|\beta,\eta_l)(v_l - \beta(x))$, which follows from Steps 1 and 2.

Suppose $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$. Now, either $\overline{\pi}_w(\beta(x)|\beta,\eta_w) > 0$, in which case $\overline{\pi}_h(\beta(x)|\beta,\eta_h)(v_h - \beta(x)) > 0$ and it follows from (26) and $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$ that the second inequality in (27) is strict, or $\overline{\pi}_w(\beta(x)|\beta,\eta_w) = 0$ and hence $U(\beta(x)|x,\beta,\boldsymbol{\eta}) = 0$. In the latter case, by Step 3, there is some $y \in (\underline{x}, x')$ such that $\overline{\pi}_w(\beta(y)|\beta,\eta_w) > 0$. We can choose y such that $\frac{g_h(x')}{g_l(x')} > \frac{g_h(y)}{g_l(y)}$ (recall that $\frac{g_h(\underline{x})}{g_l(\underline{x})} = \lim_{x\to\underline{x}} \frac{g_h(x)}{g_l(x)}$). By Steps 1 and 2, $\overline{\pi}_h(\beta(y)|\beta,\eta_h)(v_h - \beta(y)) > 0$. Since $\frac{g_h(x')}{g_l(x')} > \frac{g_h(y)}{g_l(y)}$, it follows from (26) and the fact that β is a bidding equilibrium that

$$U(\beta(x')|x',\beta,\boldsymbol{\eta}) \geq U(\beta(y)|x',\beta,\boldsymbol{\eta}) > U(\beta(y)|y,\beta,\boldsymbol{\eta}) \geq 0 = U(\beta(x)|x,\beta,\boldsymbol{\eta}).$$

Conversely, $\frac{g_h(x')}{g_l(x')} = \frac{g_h(x)}{g_l(x)}$ implies

$$U(\beta(x')|x',\beta,\boldsymbol{\eta}) = U(\beta(x')|x,\beta,\boldsymbol{\eta}) \leq U(\beta(x)|x,\beta,\boldsymbol{\eta}) = U(\beta(x)|x',\beta,\boldsymbol{\eta}) \leq U(\beta(x')|x',\beta,\boldsymbol{\eta}),$$

where the inequalities are equilibrium conditions while the equalities owe to the fact that x and x' contain the same information. Therefore, $U(\beta(x')|x',\beta,\boldsymbol{\eta}) = U(\beta(x)|x,\beta,\boldsymbol{\eta})$. □

Step 5. The strict positivity of $U(\beta(x)|x,\beta,\boldsymbol{\eta})$ implies immediately that $\overline{\pi}_w(\beta(x)|\beta,\eta_w) > 0$ for any x for which $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$. (Step 3 established this only for almost all x). This proves Part (1) of the Lemma.

Step 6. If $P_\Delta = [0, v_h]$, then $\beta(x) > v_l$ for any x for which $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$ (as opposed to just $\geq$ established in Step 1).

Proof of Step 6: The same standard Bertrand argument used in the proof of Step 1 implies that there cannot be mass point at v_l. Therefore, $U(v_l|x,\beta,\boldsymbol{\eta}) = 0$. Since Step 4 implies that $U(\beta(x)|x,\beta,\boldsymbol{\eta}) > 0$ for all $x > \underline{x}$, it must be that $\beta(x) > v_l$ for all $x > \underline{x}$. □

This completes the proof of the lemma. ■

Proof of Proposition 1:

Part (1): Proved by Lemma 2.

Part (2)**:** Suppose that $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Since β is a bidding equilibrium, $U(\beta(x)|x,\beta,\boldsymbol{\eta}) \geq U(\beta(x')|x,\beta,\boldsymbol{\eta})$. By Lemma 2, $\overline{\pi}_w[\beta(x')|\beta,\eta_w] > 0$ and $\beta(x') \geq v_l$. Therefore, by Lemma 1, $U(\beta(x)|x',\beta,\boldsymbol{\eta}) > U(\beta(x')|x',\beta,\boldsymbol{\eta})$, contradicting the optimality of $\beta(x')$ for x'. Thus, the supposition $\beta(x') < \beta(x)$ is false. Hence, $\beta(x') \geq \beta(x)$.

Next, suppose that $\frac{g_h(x')}{g_l(x')} = \frac{g_h(x)}{g_l(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Then there is some interval containing x and x' over which $\frac{g_h(x)}{g_l(x)}$ is constant, say, C. Let $[x_-, x_+]$ be the closure of this interval. By the above argument, $\beta(x'') \leq \beta(x)$ whenever $x'' < x_- < x$ and $\beta(x) \leq \beta(x''')$ whenever $x < x_+ < x'''$. Define $\widetilde{\beta}_1(x)$ by

$$\widetilde{\beta}_1(x) = \ \inf\{b : G_h(x) \leq G_h(\{t|\beta(t) \leq b\})\} \quad \text{if} \quad x \in [x_-, x_+]$$

Thus, on $[x_-, x_+]$ the signals are essentially "reordered" to make $\widetilde{\beta}_1(x)$ monotone. Outside $[x_-, x_+]$, $\widetilde{\beta}_1(x)$ coincides with $\beta(x)$. Note that $\tilde{\beta}(x') \leq \tilde{\beta}(x) \leq \tilde{\beta}(x'')$ for all $x' < x_-$ and $x_+ < x''$. With this definition,

$$G_h(\{x|\widetilde{\beta}_1(x) \leq b\}) = G_h(\{x|\beta(x) \leq b\}),$$

for all b. That is, the distribution of bids induced by $\widetilde{\beta}_1$ is equal to the distribution of bids induced by β in state h. It is also the same in state l because $\widetilde{\beta}_1 = \beta$ outside $[x_-, x_+]$ and because the distributions G_l and G_h conditional on $x \in (x_-, x_+)$ are identical (owing to the constant $\frac{g_h(x)}{g_l(x)}$).

The equality of the distributions of bids under $\widetilde{\beta}_1$ and β implies that, for any $x \notin \{x_-, x_+\}$, $\tilde{\beta}_1(x)$ is optimal: for $x \notin [x_-, x_+]$ this follows immediately from $\widetilde{\beta}_1(x) = \beta(x)$; for $x \in (x_-, x_+)$ this follows from $\tilde{\beta}_1(x) = \beta(y)$ where y is some value of the signal such that $\frac{g_h(y)}{g_l(y)} = \frac{g_h(x)}{g_l(x)}$. For $x \in \{x_-, x_+\}$, note that we can represent the distribution of signals by an equivalent pair of densities that is equal to the original densities almost everywhere, so that the resulting equilibrium still corresponds to the same distributional strategy. Here, $\tilde{\beta}_1$ can be rationalized at $\{x_-, x_+\}$ by changing the densities at the points $x \in \{x_-, x_+\}$. At x_-, if $\tilde{\beta}_1(x_-) = \tilde{\beta}_1(x_- + \varepsilon)$ for some ε (an atom), $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_w(x_-) = \lim_{\varepsilon \to 0} g_w(x_- + \varepsilon)$. Otherwise, $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_w(x_-) = \lim_{\varepsilon \to 0} g_w(x_- - \varepsilon)$. Similarly for x_+. It follows that $\widetilde{\beta}_1$ is monotone on $[x_-, x_+]$ and that it is equivalent to β.

Repeating this construction for all intervals over which $\frac{g_h(x)}{g_l(x)}$ is constant, we get a sequence of bidding strategies (constructing the sequence by starting with the longest interval of signals on which $\frac{g_h(x)}{g_l(x)}$ is constant). Let $\tilde{\beta}$ be the pointwise limit of this sequence on $(\underline{x}, \bar{x}]$ and let $\tilde{\beta}(\underline{x}) = \lim_{\varepsilon \to 0} \beta(\underline{x} + \varepsilon)$. Then, $\tilde{\beta}$ is an equivalent bidding equilibrium that is monotone on $[\underline{x}, \bar{x}]$, as claimed. ∎

10.2 Bidding Equilibrium: Atoms

Proof of Lemma 3: Since β is non-decreasing, $G_w(\{x|\beta(x) < b\}) = G_w(x_-(b))$ and $G_w(\{x|\beta(x) > b\}) = 1 - G_w(x_+(b))$. We rewrite the winning probability at b:

$$\begin{aligned}
& \pi_w(b|\beta, n) \\
= & \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{1}{i+1} G_w(x_-(b))^{n-i-1} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^i \\
= & \sum_{i=0}^{n-1} \frac{(n-1)!}{(n-1-i)!i!} \frac{1}{i+1} G_w(x_-(b))^{n-i-1} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^i \\
= & \frac{1}{n} \sum_{i=0}^{n-1} \frac{n!}{(n-1-i)!i!} \frac{1}{i+1} G_w(x_-(b))^{n-i-1} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^i \\
= & \frac{1}{n} \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} \frac{1}{k} G_w(x_-(b))^{n-k} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^{k-1} \\
= & \frac{\sum_{k=1}^{n} \frac{n!}{(n-k)!k!} G_w(x_-(b))^{n-k} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^k}{n\left[G_w(x_+(b)) - G_w(x_-(b))\right]} \\
= & \frac{\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} G_w(x_-(b))^{n-k} \left[G_w(x_+(b)) - G_w(x_-(b))\right]^k - G_w(x_-(b))^n}{n\left[G_w(x_+(b)) - G_w(x_-(b))\right]} \\
= & \frac{\left(G_w(x_-(b)) + G_w(x_+(b)) - G_w(x_-(b))\right)^n - G_w(x_-(b))^n}{n\left[G_w(x_+(b)) - G_w(x_-(b))\right]} \\
= & \frac{G_w(x_+(b))^n - G_w(x_-(b))^n}{n\left[G_w(x_+(b)) - G_w(x_-(b))\right]}.
\end{aligned}$$

The critical step is to apply the binomial theorem, $\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k = (a+b)^n$.
∎

Proof of Lemma 4: Part (i) of Lemma 18 below appears in the text as Lemma 4. The proof of Lemma 18 follows its statement. The second and third parts of this lemma are used in later proofs.

Lemma 18 *(i) If $n_h \geq n_l$, then $\frac{G_h(x)^{n_h}}{G_l(x)^{n_l}}$ is increasing and*

$$\frac{G_h(x_-)^{n_h-1}}{G_l(x_-)^{n_l-1}} \leq \frac{\pi_h[b|\beta, n_h]}{\pi_l[b|\beta, n_l]} \leq \frac{G_h(x_+)^{n_h-1}}{G_l(x_+)^{n_l-1}},$$

with strict inequalities unless $n_l = n_h$ *and* $\frac{g_h(x_+)}{g_l(x_+)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$*.*

(ii) If $\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}$ *is increasing on* $[x_-, x_+]$*, then*

$$\left(\frac{\frac{g_h(x_-)}{g_l(x_-)}}{\frac{g_h(x_+)}{g_l(x_+)}}\right)\frac{G_h\left(x_-\right)^{n_h-1}}{G_l\left(x_-\right)^{n_l-1}} \leq \frac{\pi_h\left[b|\beta,n_h\right]}{\pi_l\left[b|\beta,n_l\right]} \leq \frac{G_h\left(x_+\right)^{n_h-1}}{G_l\left(x_+\right)^{n_l-1}}\left(\frac{\frac{g_h(x_+)}{g_l(x_+)}}{\frac{g_h(x_-)}{g_l(x_-)}}\right),$$

with strict inequalities if $\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}$ *is strictly increasing or* $\frac{g_h(x_+)}{g_l(x_+)} > \frac{g_h(x_-)}{g_l(x_-)}$*.*

(iii) If $\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}$ *is decreasing on* $[x_-, x_+]$*, then*

$$\left(\frac{\frac{g_h(x_+)}{g_l(x_+)}}{\frac{g_h(x_-)}{g_l(x_-)}}\right)\frac{G_h\left(x_-\right)^{n_h-1}}{G_l\left(x_-\right)^{n_l-1}} \geq \frac{\pi_h\left[b|\beta,n_h\right]}{\pi_l\left[b|\beta,n_l\right]} \geq \frac{G_h\left(x_+\right)^{n_h-1}}{G_l\left(x_+\right)^{n_l-1}}\left(\frac{\frac{g_h(x_-)}{g_l(x_-)}}{\frac{g_h(x_+)}{g_l(x_+)}}\right),$$

with strict inequalities if $\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}$ *is strictly decreasing or* $\frac{g_h(x_+)}{g_l(x_+)} > \frac{g_h(x_-)}{g_l(x_-)}$*.*

Proof of Lemma 18: Note that

$$\begin{aligned}\frac{d}{dx}\left(\frac{G_h(x)^{n_h}}{G_l\left(x\right)^{n_l}}\right) &= \frac{n_h\frac{g_h(x)}{G_h(x)}G_h(x)^{n_h}G_l\left(x\right)^{n_l} - n_l\frac{g_l(x)}{G_l(x)}G_h(x)^{n_h}G_l\left(x\right)^{n_l}}{\left(G_l\left(x\right)^{n_l}\right)^2} \\ &\geq n_l\left(\frac{g_h\left(x\right)}{G_h(x)} - \frac{g_l\left(x\right)}{G_l(x)}\right)\frac{G_h(x)^{n_h}}{G_l\left(x\right)^{n_l}},\end{aligned}$$

where the inequality is from $n_h \geq n_l$. Hence, $\frac{G_h(x)^{n_h}}{G_l(x)^{n_l}}$ is weakly increasing since the MLRP implies $\frac{g_h(x)}{G_h(x)} - \frac{g_l(x)}{G_l(x)} \geq 0$. In fact, it is strictly increasing unless both $n_h = n_l$ and $\frac{g_h(x)}{g_l(x)} - \frac{G_h(x)}{G_l(x)} = 0$, which requires $\frac{g_h(x)}{g_l(x)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$.

Rewriting,

$$\frac{\pi_h\left[b|\beta,n_h\right]}{\pi_l\left[b|\beta,n_l\right]} = \frac{n_l}{n_h}\frac{G_h\left(x_-\right)^{n_h-1} + G_h\left(x_-\right)^{n_h-2}G_h\left(x_+\right) + ... + G_h\left(x_+\right)^{n_h-1}}{G_l\left(x_-\right)^{n_l-1} + G_l\left(x_-\right)^{n_l-2}G_l\left(x_+\right) + ... + G_l\left(x_+\right)^{n_l-1}}.$$

Divide through by $\frac{G_h(x_+)^{n_h-1}}{G_l(x_+)^{n_l-1}}$ to obtain

$$\left(\frac{\pi_h\left[b|\beta,n_h\right]}{\pi_l\left[b|\beta,n_l\right]}\right)/\left(\frac{G_h\left(x_+\right)^{n_h-1}}{G_l\left(x_+\right)^{n_l-1}}\right) = \frac{\left[1+\frac{G_h(x_-)}{G_h(x_+)}+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2+...+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_h-1}\right]/n_h}{\left[1+\frac{G_l(x_-)}{G_l(x_+)}+\left(\frac{G_l(x_-)}{G_l(x_+)}\right)^2+...+\left(\frac{G_l(x_-)}{G_l(x_+)}\right)^{n_l-1}\right]/n_l}$$

$$\leq \frac{\left[1+\frac{G_h(x_-)}{G_h(x_+)}+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2+...+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_l-1}\right]/n_l}{\left[1+\frac{G_l(x_-)}{G_l(x_+)}+\left(\frac{G_l(x_-)}{G_l(x_+)}\right)^2+...+\left(\frac{G_l(x_-)}{G_l(x_+)}\right)^{n_l-1}\right]/n_l} \leq 1$$

The first inequality follows from the fact that, since $\frac{G_h(x_-)}{G_h(x_+)} < 1$, the numerator after the inequality is an average of the largest n_l terms out of the n_h terms that are averaged on the numerator before the inequality sign. The second inequality follows from $\frac{G_h(x_-)}{G_h(x_+)} \leq \frac{G_l(x_-)}{G_l(x_+)}$ which in turn follows from $\frac{G_h(x_-)}{G_l(x_-)} \leq \frac{G_h(x_+)}{G_l(x_+)}$ which holds by MLRP.

Analogously, dividing through by $\frac{G_h(x_-)^{n_h-1}}{G_l(x_-)^{n_l-1}}$,

$$\left(\frac{\pi_h\left[b|\beta,n_h\right]}{\pi_l\left[b|\beta,n_l\right]}\right)/\left(\frac{G_h\left(x_-\right)^{n_h-1}}{G_l\left(x_-\right)^{n_l-1}}\right) = \frac{\left[1+\frac{G_h(x_+)}{G_h(x_-)}+\left(\frac{G_h(x_+)}{G_h(x_-)}\right)^2+...+\left(\frac{G_h(x_+)}{G_h(x_-)}\right)^{n_h-1}\right]/n_h}{\left[1+\frac{G_l(x_+)}{G_l(x_-)}+\left(\frac{G_l(x_+)}{G_l(x_-)}\right)^2+...+\left(\frac{G_l(x_+)}{G_l(x_-)}\right)^{n_l-1}\right]/n_l}$$

$$\geq \frac{\left[1+\frac{G_h(x_+)}{G_h(x_-)}+\left(\frac{G_h(x_+)}{G_h(x_-)}\right)^2+...+\left(\frac{G_h(x_+)}{G_h(x_-)}\right)^{n_l-1}\right]/n_l}{\left[1+\frac{G_l(x_+)}{G_l(x_-)}+\left(\frac{G_l(x_+)}{G_l(x_-)}\right)^2+...+\left(\frac{G_l(x_+)}{G_l(x_-)}\right)^{n_l-1}\right]/n_l} \geq 1$$

where the inequalities are explained by noting that $\frac{G_h(x_+)}{G_h(x_-)} > 1$ and reversing the previous arguments.

In both cases the two inequalities hold as equalities iff $n_h = n_l$ and $\frac{G_h(x_-)}{G_h(x_+)} = \frac{G_l(x_-)}{G_l(x_+)}$. However, the last equality is equivalent to $\frac{G_h(x_-)}{G_l(x_-)} = \frac{G_h(x_+)}{G_l(x_+)}$ which holds iff $\frac{g_h(x)}{g_l(x)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$ for all $x < x_+$.

(ii) & (iii)

$$\frac{n_l}{n_h}\frac{G_l(x_+)-G_l(x_-)}{G_h(x_+)-G_h(x_-)}\frac{G_h(x_+)^{n_h}-G_h(x_-)^{n_h}}{G_l(x_+)^{n_l}-G_l(x_-)^{n_l}} = \frac{\frac{\int_{x_-}^{x_+} G_h(x)^{n_h-1} g_h(x)dx}{\int_{x_-}^{x_+} g_h(x)dx}}{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)dx}{\int_{x_-}^{x_+} g_l(x)dx}}$$

$$= \frac{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}\frac{g_h(x)}{g_l(x)}dx}{\int_{x_-}^{x_+} g_l(x)\frac{g_h(x)}{g_l(x)}dx}}{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)dx}{\int_{x_-}^{x_+} g_l(x)dx}}$$

Now

$$\left(\frac{\frac{g_h(x_-)}{g_l(x_-)}}{\frac{g_h(x_+)}{g_l(x_+)}}\right)\frac{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}dx}{\int_{x_-}^{x_+} g_l(x)dx}}{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)dx}{\int_{x_-}^{x_+} g_l(x)dx}} \leq \frac{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}\frac{g_h(x)}{g_l(x)}dx}{\int_{x_-}^{x_+} g_l(x)\frac{g_h(x)}{g_l(x)}dx}}{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)dx}{\int_{x_-}^{x_+} g_l(x)dx}}$$

$$\leq \left(\frac{\frac{g_h(x_+)}{g_l(x_+)}}{\frac{g_h(x_-)}{g_l(x_-)}}\right)\frac{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}dx}{\int_{x_-}^{x_+} g_l(x)dx}}{\frac{\int_{x_-}^{x_+} G_l(x)^{n_l-1} g_l(x)dx}{\int_{x_-}^{x_+} g_l(x)dx}}$$

The results for decreasing $\frac{G_h(x)^{n_h-1}}{G_l(x)^{n_l-1}}$ follow immediately. ∎

Proof of Proposition 2: Suppose that $n_h \geq n_l \geq 2$ and β is a bidding equilibrium given (n_l, n_h). Since $n_h \geq n_l \geq 2$, by Proposition 1, we restrict attention to monotone β. Assume that $\beta(\underline{x}) = \lim_{\varepsilon\to 0}\beta(x+\varepsilon)$. The following claim implies the proposition. A remark after the proof demonstrates that atoms must occur under the stated conditions.

Claim 4 *If for some p, $x_+(p) > x_-(p)$ (an atom), then (i) $\frac{g_h(x_+(p))}{g_l(x_+(p))} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$, (ii) $n_h = n_l$, (iii) $x_-(p) = \underline{x}$, and (iv) $U(p|\underline{x}, \beta, \mathbf{n}) = 0$.*

Proof of Claim: We prove the claim in a sequence of steps. First, by Lemma 2, $v_l \leq p < v_h$.

Rewrite (1) slightly to get the expected payoff $U(p|x, \beta, \mathbf{n})$ of a bidder with signal x who bids at this atom,

$$U(p|x,\beta,\mathbf{n}) = \frac{\rho_l g_l(x) n_l \pi_l(p|\beta, n_l)}{\rho_l g_l(x) n_l + \rho_h g_h(x) n_h}\left[(v_l - p) + \frac{\rho_h g_h(x) n_h}{\rho_l g_l(x) n_l}\frac{\pi_h(p|\beta, n_h)}{\pi_l(p|\beta, n_l)}(v_h - p)\right]. \tag{28}$$

Step 1: Recall from Lemma 18-(i) that $\frac{\pi_h(b|\beta,n_h)}{\pi_l(b|\beta,n_l)} \leq \frac{G_h(x_+)^{n-1}}{G_l(x_+)^{n-1}}$, with equality iff $n_l = n_h$ and $\frac{g_h(x_+)}{g_l(x_+)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$.

Step 2: $\frac{g_h(x)}{g_l(x)} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$ for all $x \leq x_+$, $n_h = n_l$, and $U(p|x,\beta,\mathbf{n}) = 0$ for $x \in (x_-, x_+)$.

Proof of Step 2: The expected payoff of a bidder with signal $x \in (x_-, x_+)$ who bids "just above" p is approximately,

$$\begin{aligned} &\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} U(p+\varepsilon|x,\beta,\mathbf{n}) \qquad (29)\\ = &\frac{\rho_l g_l(x) n_l G_l(x_+)^{n_l-1}}{\rho_l g_l(x) n_l + \rho_h g_h(x) n_h}\left[[v_l - p] + \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{G_h(x_+)^{n_h-1}}{G_l(x_+)^{n_l-1}}(v_h - p)\right]. \end{aligned}$$

By optimality, $U(p|x,\beta,\mathbf{n}) \geq 0$ for all $x \in (x_-, x_+)$ and, by Lemma 3, $G_l(x_+)^{n_l-1} > \pi_l(p|\beta,n_l)$. Therefore, it follows from (28), (29), the MLRP, and Step 1 that $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} U(p+\varepsilon|x,\beta,\mathbf{n}) \geq U(p|x,\beta,\mathbf{n})$ for all x. The inequality is strict if $U(p|x,\beta,\mathbf{n}) > 0$ or $n_h > n_l$ or $\frac{g_h(x_+)}{g_l(x_+)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$ (or any combination). Therefore, optimality implies the step. □

Step 3: $x_- = \underline{x}$.

Proof of Step 3: Suppose not and let $x' \in [\underline{x}, \bar{x}]$ be such that $\underline{x} < x' < x_-$ and $\beta(x') < p$. This and the monotonicity of β imply $\pi_w(\beta(x')|\beta,n_w) > 0$ for $w \in \{l,h\}$.

Observe that this and $U(p|x,\beta,\mathbf{n}) = 0$ for all $x \in (x_-, x_+)$ implies

$$(v_l - p) + \frac{\rho_h g_h(x) n_h}{\rho_l g_l(x) n_l}\frac{\pi_h(p|\beta,n_h)}{\pi_l(p|\beta,n_l)}(v_h - p) = 0. \qquad (30)$$

Since by Step 2, $\frac{g_h(x)}{g_l(x)}$ is constant on $[\underline{x}, x_+)$, we have $\frac{\pi_h(\beta(x')|\beta,n)}{\pi_l(\beta(x')|\beta,n)} = \frac{\pi_h(p|\beta,n)}{\pi_l(p|\beta,n)} = \frac{G_h(x_+)^{n-1}}{G_l(x_+)^{n-1}}$; see Lemma 18. This together with (30) and $\beta(x') < p$ imply that, for $x \in (x_-, x_+)$,

$$\begin{aligned} &(v_l - \beta(x')) + \frac{\rho_h g_h(x') n_h}{\rho_l g_l(x') n_l}\frac{\pi_h(\beta(x')|\beta,n)}{\pi_l(\beta(x')|\beta,n)}(v_h - \beta(x'))\\ > \; &(v_l - p) + \frac{\rho_h g_h(x') n_h}{\rho_l g_l(x') n_l}\frac{\pi_h(\beta(x')|\beta,n)}{\pi_l(\beta(x')|\beta,n)}(v_h - p)\\ = \; &(v_l - p) + \frac{\rho_h g_h(x) n_h}{\rho_l g_l(x) n_l}\frac{\pi_h(p|\beta,n_h)}{\pi_l(p|\beta,n_l)}(v_h - p) = 0. \end{aligned}$$

However, this together with $\pi_l(\beta(x')|\beta,n_l) > 0$ inferred above implies $U(\beta(x')|x',\beta,\mathbf{n}) > 0$ contradicting Step 2. Therefore, $x_- = \underline{x}$. □

Step 2 and Step 3 imply the Claim. ∎

Remark (Existence of Equilibrium with Atoms): Suppose that $n_h = n_l \geq 2$ and suppose that there is no atom at $p' = \beta(x')$ for some x' with $\frac{g_h(x')}{g_l(x')} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$. Let $p = \beta(x'')$ for some $x'' \in (\underline{x}, x')$.

Note that $U(\beta(\underline{x})|\underline{x}, \beta, \mathbf{n}) = 0$ in every bidding equilibrium. If β does not have an atom at $\underline{x}$, this is immediate from monotonicity of β and if β does have an atom at $\underline{x}$, this follows from the claim. Thus, $U(\beta(x'')|x'', \beta, \mathbf{n}) = 0$.

Now,

$$\begin{aligned} 0 &= (v_l - p) + \frac{\rho_h g_h(\underline{x}) n_h}{\rho_l g_l(\underline{x}) n_l} \frac{\pi_h(p|\beta, n_h)}{\pi_l(p|\beta, n_l)} (v_h - p) \\ &= (v_l - p) + \frac{\rho_h g_h(x') n_h}{\rho_l g_l(x') n_l} \frac{\pi_h(p'|\beta, n_h)}{\pi_l(p'|\beta, n_l)} (v_h - p) \\ &> (v_l - p') + \frac{\rho_h g_h(x') n_h}{\rho_l g_l(x') n_l} \frac{\pi_h(p'|\beta, n_h)}{\pi_l(p'|\beta, n_l)} (v_h - p'), \end{aligned}$$

where the second equality is from Lemma 18-(i) and the inequality from $p' > p$. Thus, $0 > U(p'|x', \beta, \mathbf{n})$, a contradiction. Thus, it must be $\beta(x') = \beta(\underline{x})$, as claimed. This finishes the proof of the second part of the proposition. ∎

10.3 Large Numbers: Basic Results

Proof of Lemma 6: Suppose $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = q \in (0,1)$. If $q > 0$, then $\lim x^k = \bar{x}$ (for $\lim x^k < \bar{x}$ implies $q = 0$). The claim is immediate if $x^k = \bar{x}$ for all k large enough since then $q = 1 = \left(G_l\left(x^k\right)\right)^{n_l^k} = \left(G_h\left(x^k\right)\right)^{n_h^k}$ for k large. So, suppose $x^k < \bar{x}$ for all k large enough but $\lim x^k = \bar{x}$. From $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = q$,

$$\lim_{k\to\infty} (1 - \frac{1 - G_l\left(x^k\right)}{n_l^k} n_l^k)^{n_l^k} = q.$$

This implies $\lim \left(1 - G_l\left(x^k\right)\right) n_l^k = -\ln q$. Therefore,

$$\begin{aligned} \lim_{k\to\infty} \left(G_h\left(x^k\right)\right)^{n_h^k} &= \lim_{k\to\infty} \left(1 - \left(1 - G_h\left(x^k\right)\right) n_l^k \frac{1}{n_l^k}\right)^{n_h^k} \\ &= \lim_{k\to\infty} \left(1 - \left(1 - G_l\left(x^k\right)\right) n_l^k \frac{1 - G_h\left(x^k\right)}{1 - G_l\left(x^k\right)} \frac{1}{n_l^k}\right)^{n_l^k \frac{n_h^k}{n_l^k}} \\ &= \left[\lim_{k\to\infty} \left(1 + (\ln q)\,\overline{g} \frac{1}{n_l^k}\right)^{n_l^k}\right]^{\lim\left(\frac{n_h^k}{n_l^k}\right)} \\ &= e^{\ln q^{\overline{g}} \lim \frac{n_h^k}{n_l^k}} = q^{\overline{g} \lim \frac{n_h^k}{n_l^k}}. \end{aligned}$$

The lemma now follows analogously if $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = q \in \{0,1\}$ and $\lim x^k = \bar{x}$. If $\lim x^k < \bar{x}$, then $\lim \left(G_w\left(x^k\right)\right)^{n_w^k} = 0$ for $w \in \{l,h\}$. ∎

Proof of Lemma 7: By contradiction. Suppose that there is some $\varepsilon > 0$ and a sequence $\boldsymbol{\eta}^k = (n_l^k, n_h^k)$ such that $\min\left\{n_h^k, n_l^k\right\} \to \infty$ and a corresponding sequences of bidding equilibria β^k and signals x^k such that $U(\beta(x^k)|x^k, \beta, \boldsymbol{\eta}^k) > \varepsilon$, for all k. By Lemma 2, $U(\beta(x)|x, \beta, \boldsymbol{\eta}^k)$ increases in x, it follows that $U(\beta^k(\bar{x})|\bar{x}, \beta^k, \boldsymbol{\eta}^k) > \varepsilon$ for all k. Rewriting the expected payoffs (analogously to (23)), observe that

$$U(\beta^k(\bar{x})|\bar{x}, \beta^k, \boldsymbol{\eta}^k) - U(\beta^k(\bar{x})|x, \beta^k, \boldsymbol{\eta}^k) < (v_h - v_l)\left(1 - \frac{\frac{g_h(x)}{g_l(x)}}{\frac{g_h(\bar{x})}{g_l(\bar{x})}}\right).$$

By assumption, $\lim_{x \to \bar{x}} \frac{g_h(x)}{g_l(x)} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Therefore, there exists $x' < \bar{x}$ such that the right side above is smaller than $\frac{\varepsilon}{2}$. Hence, for all k, we have $\frac{\varepsilon}{2} < U(\beta^k(\bar{x})|x', \beta^k, \boldsymbol{\eta}^k) \leq U(\beta^k(x')|x', \beta^k, \boldsymbol{\eta}^k)$ where the last inequality follows from the optimality of $\beta^k(x')$. However, $x' < \bar{x}$ implies that $1 - G_w(x') > 0$ and so the number of bidders with signals above x' goes to infinity as $k \to \infty$. Hence, we conclude that there are an unboundedly large number of bidders who each expect a payoff of $\frac{\varepsilon}{2}$. However, the total available surplus for the bidders is bounded by v_h; a contradiction. ∎

Proof of Lemma 8: We prove the lemma for the case of a pure solicitation strategy. The proof extents directly to mixed strategies. Let

$$X^k(p) = \left(\beta^k\right)^{-1}([0,p]) = \left\{x|\beta^k(x) \leq p\right\}.$$

Case 1: n_w^k is bounded as $s^k \to 0$. From $s^k \to 0$, it must be that $\lim_{k\to\infty} \mathbb{E}_w\left[p|\beta^k, n_w^k + 1\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k\right] = 0$, that is,

$$\lim_{k\to\infty} \int_0^{v_h} \left(G_w\left(X^k(p)\right)\right)^{n_w^k}\left(1 - \left(G_w\left(X^k(p)\right)\right)\right) dp = 0.$$

Thus, the integrand converges to zero almost everywhere. From the monotonicity of G_w, this requires that for some $C \in [0, v_h]$,

$$\lim_{k\to\infty} G_w\left(X^k(p)\right) = \begin{cases} 1 & \text{if } p > C, \\ 0 & \text{if } p < C. \end{cases}$$

Hence, $F_w\left(p|\beta^k, n_w^k\right) = G_w\left(X^k(p)\right)^{n_w^k}$ implies that $-\int_0^{v_h}\left(F_w\left(p|\beta^k, n_w^k\right)\right)\ln\left(F_w\left(p|\beta^k, n_w^k\right)\right)dp$ converges to zero, as claimed (recall $\lim_{a\to 0} a \ln a = 0$).

Case 2: $n_w^k \to \infty$ as $s^k \to 0$. From optimality of n_w^k

$$\begin{aligned} & n_w^k \int_0^{v_h} \left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k-1}\left(1-\left(G_w\left(X^k\left(p\right)\right)\right)\right) dp \\ \geq \quad & n_w^k s^k \geq n_w^k \int_0^{v_h} \left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k}\left(1-\left(G_w\left(X^k\left(p\right)\right)\right)\right) dp. \end{aligned}$$

From $\lim F_w\left(p|\beta^k, n_w^k\right) = \lim\left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k}$ it follows that

$$\lim_{k\to\infty} n_w^k \left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k}\left(1-\left(G_w\left(X^k\left(p\right)\right)\right)\right) = -\left(\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)\right)\ln\left(\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)\right).$$

In particular, if $\lim\left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k} \in (0,1)$, then $\lim\left(1-\frac{\left(1-G_w\left(X^k(p)\right)\right)n_w^k}{n_w^k}\right)^{n_w^k} = e^{-\lim\left(1-G_w\left(X^k(p)\right)\right)n_w^k}$ implies

$$-\lim_{k\to\infty}\left(1-G_w\left(X^k\left(p\right)\right)\right)n_w^k = \ln\left(\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)\right).$$

If $\lim\left(G_w\left(X^k\left(p\right)\right)\right)^{n_w^k} \in \{0,1\}$, the claim follows from $\lim_{a\to 0} a\ln a = 0$. Thus, the lemma holds. ∎

10.4 Characterization: Partially Revealing Equilibrium

Proof of Proposition 3: Consider a sequence of bidding games $\Gamma_0\left(N^k, \mathbf{n}^k, P_0\right)$ such that $\min\left\{n_l^k, n_h^k\right\} \to \infty$, $\lim_{k\to\infty}\frac{n_h^k}{n_l^k} = r$ and $\overline{g}r > 1$. Let β^k be a corresponding sequence of bidding equilibria. By Proposition 1, we may assume that each bidding strategy β^k is monotone.

We prove the proposition for a sequence of pure solicitation strategies $\mathbf{n}^k$ to shorten the algebra. However, the fact that $\min\left\{n_l^k, n_h^k\right\} \to \infty$ implies that the proof immediately extents to sequence of mixed solicitation strategies $\boldsymbol{\eta}^k$ that have support on at most two adjacent integers.

Part (i) is proved in three steps presented by the following three lemmas. The first establishes that there are no atoms in the equilibrium distribution of the winning bid in the limit. Recall that $x_+\left(p\right) = \sup\left\{x|\beta\left(x\right) \leq p\right\}$ and that $\pi_w[p|\beta, n_w]$ is the probability of winning.

Lemma 19 **(No Atoms in Limit)** *Given $r\overline{g} > 1$, for any sequence $\{x^k\}$, and $w \in \{l,h\}$,*

$$\lim_{k\to\infty}\pi_w[\beta^k\left(x^k\right)|\beta^k, n_w^k] = \lim_{k\to\infty} G_w(x^k)^{n_w^k-1}$$

The proof of this lemma comes immediately after the end of the present proof. By definition of $x_+(p)$,

$$F_w(p|\beta^k, n_w^k) = G_w(x_+^k(p))^{n_w^k}. \tag{31}$$

It follows that $\lim F_h\left(p|\beta^k, n_h^k\right) \in (0,1)$ implies that $x_+^k(p) \to \bar{x}$ for otherwise the RHS would go to 0. Recall that $x_{(1)}$ denotes the highest signal among the competitors of a fixed bidder. The next lemma uses the zero-profit plus IR condition (7) to obtain an equation holding for prices in the support of $\lim F_w\left(p|\beta^k, n_w^k\right)$.

Lemma 20 *For every price for which* $\lim F_w\left(p|\beta^k, n_w^k\right) \in (0,1)$,

$$p = \lim_{k\to\infty} \mathbb{E}[v|x_+^k(p)\,,\ x_{(1)} \le x_+^k(p)\,,\ \beta^k,\ \mathbf{n}^k].$$

Proof of Lemma 20: By the zero-profit plus IR condition (7), by $\lim F_h\left(p|\beta^k, n_h^k\right) \in (0,1)$ and by the definition of $x_+^k(p)$

$$\lim_{k\to\infty} \mathbb{E}[v|x_+^k(p)\,,\ x_{(1)} \le x_+^k(p)\,,\ \beta^k,\ \mathbf{n}^k] = \lim_{k\to\infty} \beta^k\left(x_+^k(p)\right) \ge p \tag{32}$$

It may not be that $\lim_{k\to\infty} \beta^k\left(x_+^k(p)\right) > p$, since by Lemma 19 there is no atom at $\lim_{k\to\infty} \beta^k\left(x_+^k(p)\right)$ and hence far enough in the sequence a bidder with $x_+^k(p)$ has a profitable downward deviation from $\beta^k\left(x_+^k(p)\right)$. Therefore, $\lim_{k\to\infty} \beta^k\left(x_+^k(p)\right) = p$ and the result follows from (32). □

Finally, the last lemma combines the insights of the two previous lemmas to solve for $\lim F_w\left(p|\beta^k, n_w^k\right)$ for any p in its support.

Lemma 21 $\lim F_w\left(p|\beta^k, n_w^k\right) = \phi_w\left(p|\rho, \overline{g}, r\right)$ *for* $w \in \{l, h\}$ *and* $p \in [0, v_h]$.

Proof of Lemma 21: Consider p such that $\lim F_w\left(p|\beta^k, n_w^k\right) \in (0,1)$. The result of Lemma 20 is equivalent to

$$\frac{\rho_h}{\rho_l} \lim_{k\to\infty} \frac{g_h\left(x_+^k(p)\right)}{g_l\left(x_+^k(p)\right)} \frac{n_h^k}{n_l^k} \frac{\lim_{k\to\infty} \pi_h[p|\beta^k, n_h^k]}{\lim_{k\to\infty} \pi_l[p|\beta^k, n_l^k]} = \frac{p - v_l}{v_h - p}$$

Using Lemma 19 and $x_+^k(p) \to \bar{x}$, it can be rewritten as

$$\frac{\rho_h}{\rho_l} \frac{g_h(\bar{x})}{g_l(\bar{x})} \lim_{k\to\infty} \frac{n_h^k}{n_l^k} \frac{G_h\left(x_+^k(p)\right)^{n_h^k}}{G_l\left(x_+^k(p)\right)^{n_l^k}} = \frac{p - v_l}{v_h - p}, \tag{33}$$

The RHS is positive and finite since $\lim F_w\left(p|\beta^k, n_w^k\right) \in (0,1)$ implies $p \in (v_l, v_h)$.

By Lemma 6 and (4),

$$\lim F_h\left(p|\beta^k, n_l^k\right) = \left[\lim F_l\left(p|\beta^k, n_h^k\right)\right]^{\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim\left(\frac{n_h^k}{n_l^k}\right)}. \tag{34}$$

Using (31) and (34) to substitute for $\lim G_w\left(x_+^k(p)\right)^{n_w^k}$ in (33) we get

$$\lim_{k\to\infty} F_l\left(p|\beta^k, n_l^k\right) = \left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}}.$$

where $r = \lim\frac{n_h^k}{n_l^k}$, $\rho = \frac{\rho_h}{\rho_l}$ and $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Hence, if $\lim F_w\left(p|\beta^k, n_w^k\right) > 0$, then $\lim F_l\left(p|\beta^k, n_l^k\right) = \phi_l\left(p|\rho, \overline{g}, r\right)$ and using again (34) $\lim F_h\left(p|\beta^k, n_h^k\right) = \phi_h\left(p|\rho, \overline{g}, r\right)$. Finally, the monotonicity of F_w and the definition of ϕ_w imply that $\lim F_w\left(p|\beta^k, n_w^k\right) \in \{0,1\} \Leftrightarrow \phi_w\left(p|\rho, \overline{g}, r\right) \in \{0,1\}$. □

Part (ii). Assume $r\overline{g} \leq 1$. Suppose to the contrary that $\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)$ is strictly increasing over some interval (p', p''). Then the above analysis applies to $p \in (p', p'')$: First, because there is no atom at p, the conclusion of Lemma 19 holds. Second, given this conclusion, the proofs of Lemmas 20 and 21 also apply for $r\overline{g} \leq 1$. Thus, if $r\overline{g} < 1$, then by Lemma 21, $\lim F_w\left(p|\beta^k, n_l^k\right) = \phi_w\left(p|\rho, \overline{g}, r\right)$; if $r\overline{g} = 1$, then by (34) $\lim F_h\left(p|\beta^k, n_l^k\right) = \lim F_l\left(p|\beta^k, n_h^k\right)$ and hence by (33) $p = \rho_l v_l + \rho_h v_h$. However, both of these observations lead to contradictions: $\phi_w\left(p|\rho, \overline{g}, r\right)$ is strictly decreasing when $r\overline{g} < 1$, and the constant p when $r\overline{g} = 1$ clearly cannot hold for all $p \in (p', p'')$. It follows that $\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)$ must consist of atoms. Suppose to the contrary that $\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)$ has at least two different atoms. Let p_1 and p_2, $p_1 < p_2$, be two adjacent atoms. In what follows, we show that there may not be a sequence of equilibrium bid distributions $F_w\left(p|\beta^k, n_w^k\right)$ (in the sequence of bidding games $\Gamma_0\left(N^k, \mathbf{n}^k, P_{\Delta^k}\right)$) that converge to the postulated limit distribution. First, observe that it may not be that far enough in the sequence $F_w\left(p_i|\beta^k, n_w^k\right)$ is strictly increasing in the neighborhoods of p_i for essentially the same argument that ruled out strict monotonicity of $\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)$. Therefore, for large enough k, $F_w\left(p|\beta^k, n_w^k\right)$ has an atom at $p_i^k \to p_i$. Observe that $r\overline{g} < 1$ implies that, for large k, $\frac{G_h(x)^{n_h}}{G_l(x)^{n_l}}$ is decreasing on $[x_-^k(p_i^k), x_+^k(p_i^k)]$. This is because $\frac{G_h(x)^{n_h}}{G_l(x)^{n_l}}$ is decreasing at x iff $\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k} < \frac{G_h(x)}{G_l(x)}$ and, since $x_-^k(p_i) \to \bar{x}$, it is implied for large k by $r\overline{g} < 1$. This together with Lemma 18-(III) imply that, for k large enough,

$$\frac{G_h\left(x_-^k(p_2^k)\right)^{n_h-1}}{G_l\left(x_-^k(p_2^k)\right)^{n_l-1}} \approx \left(\frac{\frac{g_h\left(x_+^k(p_2^k)\right)}{g_l\left(x_+^k(p_2^k)\right)}}{\frac{g_h\left(x_-^k(p_2^k)\right)}{g_l\left(x_-^k(p_2^k)\right)}}\right)\frac{G_h\left(x_-^k(p_2^k)\right)^{n_h-1}}{G_l\left(x_-^k(p_2^k)\right)^{n_l-1}} > \frac{\pi_h\left[p_2^k|\beta, n_h^k\right]}{\pi_l\left[p_2^k|\beta, n_l^k\right]}$$

It follows that, for a given $\varepsilon \in (0, p_2 - p_1)$ and a large k, the profit of a bidder with $x \in \left(x_-^k(p_2^k), x_+^k(p_2^k)\right)$ from a downward deviation to $p_2^k - \varepsilon$ is bounded away from 0. This is because the probability of winning after the deviation would still be bounded below by $\lim_{k\to\infty} F_w\left(p_1|\beta^k, n_w^k\right)$ and hence bounded away from 0, while the profit conditional on winning would also be bounded away from 0, since

$$\begin{aligned} \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}\frac{\pi_h[p_2^k-\varepsilon|\beta^k,n_h^k]}{\pi_l[p_2^k-\varepsilon|\beta^k,n_l^k]} &\approx \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}\frac{G_h\left(x_-^k(p_2^k)\right)^{n_h^k-1}}{G_l\left(x_-^k(p_2^k)\right)^{n_l^k-1}} > \\ \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}\frac{\pi_h[p_2^k|\beta^k,n_h^k]}{\pi_l[p_2^k|\beta^k,n_l^k]} &\approx \frac{p_2^k - v_l}{v_h - p_2^k} > \frac{p_2^k-\varepsilon-v_l}{v_h-(p_2^k-\varepsilon)} \end{aligned} \tag{35}$$

Therefore, for a large k, a bidder with $x \in \left(x_-^k(p_2^k), x_+^k(p_2^k)\right)$ would profit from a downward deviation to $p_2^k - \varepsilon$ in contradiction to equilibrium. Thus, the supposition that $\lim_{k\to\infty} F_w\left(p_i|\beta^k, n_w^k\right)$ has two (or more) atoms in its support is false, so there must be a single atom.

The proof for the case of $r\overline{g} = 1$ is almost identical except that the middle inequality in (35) should be replaced by "$\approx$". ∎

Remark: Adapting above Proof of Proposition 3-(ii) to finite price grid.

With finite grid $\Delta^k > 0$ the only change to the above proof is that we have to make sure that the undercutting argument of the second to last paragraph of the proof is compatible with the grid. That is, it is possible to find the required downward deviation $p_2^k - \varepsilon$ in the grid. However, this is obviously the case, since for k sufficiently large $\Delta^k << p_2 - p_1$. ∎

Proof of Lemma 19 (No Atoms in Limit): Let $x_-^k = x_-^k\left(\beta^k\left(x^k\right)\right)$ and $x_+^k = x_+^k\left(\beta^k\left(x^k\right)\right)$ so that $x_-^k = \inf\left\{x|\beta^k(x) \geq \beta^k\left(x^k\right)\right\}$ and $x_+^k = \sup\left\{x|\beta^k(x) \leq \beta^k\left(x^k\right)\right\}$. Monotonicity of β^k implies $\left(G_w\left(x_+^k\right)\right)^{n_w^k-1} \geq \pi_w\left[\beta^k\left(x^k\right)|\beta^k, n_w^k\right] \geq \left(G_w\left(x_-^k\right)\right)^{n_w^k-1}$.

Suppose to the contrary that $\lim_{k\to\infty}\pi_w[\beta^k\left(x^k\right)|\beta^k, n_w^k] \neq \lim_{k\to\infty} G_w(x^k)^{n_w^k-1}$. This means that $\pi_w[\beta^k(x)|\beta^k, n_w^k]$ has an atom on $\left(x_-^k, x_+^k\right)$ that persists to the limit. That is, $\lim\left(G_w\left(x_+^k\right)\right)^{n_w^k-1} > \lim\left(G_w\left(x_-^k\right)\right)^{n_w^k-1}$. So, let

$$\lim_{k\to\infty}\left(G_l\left(x_+^k\right)\right)^{n_l^k-1} = q > \lim_{k\to\infty}\left(G_l\left(x_-^k\right)\right)^{n_l^k-1} \geq 0 \tag{36}$$

and hence, by Lemma 6,

$$\lim_{k\to\infty}\left(G_h\left(x_+^k\right)\right)^{n_h^k-1} = q^{\frac{g_h(x)}{g_l(x)}\lim\frac{n_h^k}{n_l^k}} > 0 \tag{37}$$

Obviously, $\lim \left(G_w\left(x_+^k\right)\right)^{n_w^k-1} > 0$ requires that $x_+^k \to \bar{x}$.

Recall that individual rationality, (6), implies

$$\mathbb{E}\left[v|x, \text{ win at } \beta^k(x);\, \beta^k, \eta^k\right] \geq \beta^k(x) \qquad \forall k. \tag{38}$$

which in turn requires

$$\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}\frac{\pi_h\left[\beta^k(x)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k(x)|\beta^k, n_l^k\right]} \geq \frac{\beta^k(x)-v_l}{v_h-\beta^k(x)}. \tag{39}$$

Step 1. If (36), then

$$\lim_{k\to\infty}\mathbb{E}\left[v|x^k, x_{(1)} \leq x_+^k;\, \beta^k, \eta^k\right] > \lim_{k\to\infty}\beta^k\left(x^k\right). \tag{40}$$

Proof of Step 1. Suppose that $\lim x_-^k < \bar{x}$. This implies $\lim\left(G_w\left(x_-^k\right)\right)^{n_w^k} = 0$ and hence $\frac{\left(G_h\left(x_+^k\right)\right)^{n_h^k}-\left(G_h\left(x_-^k\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k}-\left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k}} \to \frac{\left(G_h\left(x_+^k\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k}} \in (0,\infty)$. Therefore, Lemma 3 implies

$$\lim_{k\to\infty}\frac{\pi_h\left[\beta^k\left(x^k\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(x^k\right)|\beta^k, n_l^k\right]} = \lim_{k\to\infty}\frac{n_\ell^k}{n_h^k}\frac{\left[1-G_\ell\left(x_-^k\right)\right]}{\left[1-G_h\left(x_-^k\right)\right]}\frac{\left(G_h\left(x_+^k\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k}}.$$

Substituting into (39) yields for $x = x_-^k$,

$$\lim_{k\to\infty}\frac{\rho_h}{\rho_l}\frac{g_h\left(x_-^k\right)}{g_l\left(x_-^k\right)}\frac{\left[1-G_\ell\left(x_-^k\right)\right]}{\left[1-G_h\left(x_-^k\right)\right]}\frac{G_h\left(x_+^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k}} \geq \frac{\lim_{k\to\infty}\beta^k\left(x^k\right)-v_l}{v_h-\lim_{k\to\infty}\beta^k\left(x^k\right)}.$$

The monotone likelihood ratio property implies that $\frac{g_h\left(x_-^k\right)}{g_l\left(x_-^k\right)}\frac{\left[1-G_\ell\left(x_-^k\right)\right]}{\left[1-G_h\left(x_-^k\right)\right]} \leq 1$. Hence, $\bar{g}\lim\frac{n_h^k}{n_l^k} > 1$ and $\frac{g_h\left(x_+^k\right)}{g_l\left(x_+^k\right)} \to \bar{g}$ imply

$$\lim_{k\to\infty}\frac{\rho_h}{\rho_l}\frac{g_h\left(x_+^k\right)}{g_l\left(x_+^k\right)}\frac{n_h^k}{n_l^k}\frac{G_h\left(x_+^k\right)^{n_h^k-1}}{G_\ell\left(x_+^k\right)^{n_\ell^k-1}} > \frac{\lim_{k\to\infty}\beta^k\left(x^k\right)-v_l}{v_h-\lim_{k\to\infty}\beta^k\left(x^k\right)}. \tag{41}$$

which is equivalent to (40).

Suppose that $\lim x_-^k = \bar{x}$. Observe that $\lim\frac{g_h\left(x_+^k\right)}{g_l\left(x_+^k\right)} = \lim\frac{g_h\left(x_-^k\right)}{g_l\left(x_-^k\right)} = \bar{g}$ and $\bar{g}\lim\frac{n_h^k}{n_l^k} > 1$ imply that $\frac{G_h(x)^{n_h^k-1}}{G_\ell(x)^{n_\ell^k-1}}$ is strictly increasing on $[x_-^k, x_+^k]$. Therefore, from Lemma **18-**(ii)

$$\frac{\pi_h\left(\beta^k\left(x^k\right)|\beta^k,n_h^k\right)}{\pi_l\left(\beta^k\left(x^k\right)|\beta^k,n_l^k\right)}<\frac{G_h\left(x_+^k\right)^{n_h^k-1}}{G_l\left(x_+^k\right)^{n_l^k-1}},$$

and since by (36) and Lemma 6 $\lim_{k\to\infty}\frac{G_h\left(x_+^k\right)^{n_h^k-1}}{G_l\left(x_+^k\right)^{n_l^k-1}}=q^{1-\frac{g_\ell(\bar{x})}{g_h(\bar{x})}\lim\frac{n_l^k}{n_h^k}}>0=\lim_{k\to\infty}\frac{G_h\left(x_-^k\right)^{n_h^k-1}}{G_l\left(x_-^k\right)^{n_l^k-1}}$, the strict inequality persists in the limit and

$$\lim_{k\to\infty}\frac{\pi_h\left(\beta^k\left(x^k\right)|\beta^k,n_h^k\right)}{\pi_l\left(\beta^k\left(x^k\right)|\beta^k,n_l^k\right)}<\lim_{k\to\infty}\frac{G_h\left(x_+^k\right)^{n_h^k-1}}{G_l\left(x_+^k\right)^{n_l^k-1}},$$

which implies the claim via (39). □

Step 2. For k large enough there is a bid $b^k>\beta^k\left(x^k\right)$ such that

$$U(b^k|x^k,\beta^k,\mathbf{n}^k)>U(\beta^k\left(x^k\right)|x^k,\beta^k,n\boldsymbol{\eta}^k).$$

Proof of Step 2:

Rearranging (1) shows that

$$U(b^k|x^k,\beta^k,\mathbf{n}^k)=\frac{\pi_l\left[\beta^k\left(x\right)|\beta^k,n_l^k\right]\left(v_h-\beta^k\left(x\right)\right)}{1+\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h^k}{n_l^k}}\left[\frac{\rho_h}{\rho_l}\frac{g_h\left(x\right)}{g_l\left(x\right)}\frac{n_h^k}{n_l^k}\frac{\pi_h\left[\beta^k\left(x\right)|\beta^k,n_h^k\right]}{\pi_l\left[\beta^k\left(x\right)|\beta^k,n_l^k\right]}-\frac{\beta^k\left(x\right)-v_l}{v_h-\beta^k\left(x\right)}\right].$$

Observe that, for large enough k, it is possible to choose a bid $b^k>\beta^k\left(x^k\right)$ that is arbitrarily close to $\beta^k\left(x^k\right)$ and such that $\pi_w\left[b^k|\beta^k,n_w^k\right]\approx G_w\left(x_+^k\right)^{n_w^k-1}$. This is because there is a small neighborhood above $\beta^k\left(x^k\right)$ at which there is no atom (If $\lim_{k\to\infty}\beta^k\left(x_+^k\right)=\lim_{k\to\infty}\beta^k\left(x^k\right)$, there is such a neighborhood just above $\beta^k\left(x_+^k\right)$; if $\lim_{k\to\infty}\beta^k\left(x_+^k\right)>\lim_{k\to\infty}\beta^k\left(x^k\right)$, then $\left(\beta^k\left(x^k\right),\beta^k\left(x_+^k\right)\right)$ is such a neighborhood). Therefore, for large enough k,

$$\begin{aligned}U(b^k|x^k,\beta^k,\mathbf{n}^k)&=\left(\pi_l\left[b^k|\beta^k,n_l^k\right]+\pi_h\left[b^k|\beta^k,n_h^k\right]\right)\left[\mathbb{E}\left[v|x^k,x_{(1)}\le b^k;\,\beta^k,\eta^k\right]-b^k\right]\\&\approx\left(G_l\left(x_+^k\right)^{n_l^k-1}+G_h\left(x_+^k\right)^{n_h^k-1}\right)\left[\mathbb{E}\left[v|x^k,x_{(1)}\le x_+^k;\,\beta^k,\eta^k\right]-\beta^k\left(x^k\right)\right]\end{aligned}$$

By (36) and (37), $\lim_{k\to\infty}\left[G_l\left(x_+^k\right)^{n_l^k-1}+G_h\left(x_+^k\right)^{n_h^k-1}\right]\approx q+q^{\frac{g_h(x)}{g_l(x)}\lim\frac{n_h^k}{n_l^k}}$. By Step 1, there is an $\varepsilon>0$ such that

$$\lim_{k\to\infty}\mathbb{E}\left[v|x^k,x_{(1)}\le x_+^k;\,\beta^k,\eta^k\right]>\lim_{k\to\infty}\beta^k\left(x^k\right)+\varepsilon.\qquad(42)$$

Therefore, for sufficiently large k, $U(b^k|x^k,\beta^k,\mathbf{n}^k)$ is bounded away from 0, and, by Lemma 7, $\lim_{k\to\infty} U(\beta^k\left(x^k\right)|x^k,\beta^k,\mathbf{n}^k)=0$, which establishes Step 2. □

Step 2 contradicts β^k being an equilibrium, buyers with signals sufficiently close to $\bar{x}$ can ensure strictly positive, nonvanishing profits in the limit. This contradicts the zero-profit condition from Lemma 7. Hence, in the limit there cannot be any atoms in the distribution of the winning bid. ∎

Remark: Adapting above Proof of Lemma 19 to finite price grid

With finite grid $\Delta^k>0$ the only change to the above proof is that we have to make sure that it is possible to select a bid b^k, as described in Step 2, that also belongs to the grid. Let $\varepsilon>0$ satisfy (42). If $\lim_{k\to\infty}\beta^k\left(x_+^k\right)>\lim_{k\to\infty}\beta^k\left(x^k\right)$, then there is no problem since, for sufficiently large k, $\Delta^k<\min\left\{\varepsilon,\left(\lim_{k\to\infty}\beta^k\left(x^k\right),\lim_{k\to\infty}\beta^k\left(x_+^k\right)\right)\right\}$ and therefore there exists a b^k in the grid such that $b^k\in\left(\lim_{k\to\infty}\beta^k\left(x^k\right),\lim_{k\to\infty}\beta^k\left(x_+^k\right)\right)$ and $b^k<\lim_{k\to\infty}\beta^k\left(x^k\right)+\varepsilon$, as needed for the proof. If $\lim_{k\to\infty}\beta^k\left(x_+^k\right)=\lim_{k\to\infty}\beta^k\left(x^k\right)$, let $\varepsilon>0$ be defined by (42). By Step 1, we can choose $\varepsilon>0$ such that

$$\lim_{k\to\infty}\mathbb{E}\left[v|x^k,x_{(1)}\leq x_+^k;\beta^k,\eta^k\right]>\lim_{k\to\infty}\beta^k\left(x^k\right)+\varepsilon.$$

Observe that, for all k with $\Delta^k<\varepsilon$, there exists some $b^k\in\left(\beta^k\left(x^k\right),\beta^k\left(x^k\right)+\varepsilon\right)$ such that

$$G_l\left(x_+^k\left(b^k\right)\right)^{n_l^k}-G_l\left(x_-^k\left(b^k\right)\right)^{n_l^k}\leq\frac{2\Delta^k}{\varepsilon}.$$

since otherwise,

$$G_l\left(x_+^k\left(\beta^k\left(x^k\right)+\varepsilon\right)\right)^{n_l^k}\geq\sum_{i=0}^{\lfloor\varepsilon/\Delta^k\rfloor-1}G_l\left(x_+^k\left(\beta^k\left(x^k\right)+i\Delta^k\right)\right)^{n_l^k}-G_l\left(x_-^k\left(\beta^k\left(x^k\right)+i\Delta^k\right)\right)^{n_l^k}>1.$$

Thus, there is no atom at b^k in the limit, hence

$$\lim_{k\to\infty}\pi_w\left[b^k|\beta^k,n_w^k\right]=\lim_{k\to\infty}G_w\left(x_+^k\left(b^k\right)\right)^{n_w^k}\geq\lim_{k\to\infty}G_w\left(x_+^k\right)^{n_w^k}. \tag{43}$$

which is what is required for the proof. □

Proof of Lemma 9 (Unavoidable Ties): We consider a sequence of bidding games $\Gamma_0\left(N^k,\boldsymbol{\eta}^k,P_0\right)$ such that $\min\{n_l^k,n_h^k\}\to\infty$, $\lim_{k\to\infty}\frac{n_h^k}{n_l^k}=r$ with $r\overline{g}<1$, and a corresponding sequence of bidding equilibria β^k.

Step 1. Suppose that for some sequence $\left\{x^k\right\}$, (i) $x^k\leq x_-^k\left(\beta^k\left(\bar{x}\right)\right)$ for all k

and (ii) $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} \in (0,1)$, then

$$\lim_{k\to\infty} \frac{G_h\left(x^k\right)^{n_h^k}}{G_l(x^k)^{n_l^k}} > \lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]}. \tag{44}$$

Proof: Let

$$\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = \hat{q}.$$

Then, Lemma 6, $\bar{g}r < 1$, and $\hat{q} \in (0,1)$ implies

$$\lim_{k\to\infty} \frac{G_h\left(x^k\right)^{n_h^k}}{G_l(x^k)^{n_l^k}} = \hat{q}^{\bar{g}r-1} > 1. \tag{45}$$

Abbreviate

$$\bar{x}_-^k = x_-^k\left(\beta^k\left(\bar{x}\right)\right)$$

and let $\lim \left(G_l\left(\bar{x}_-^k\right)\right)^{n_l^k} = q$.

Case 1. Suppose that $\lim \left(G_l\left(\bar{x}_-^k\right)\right)^{n_l^k} = q = 1$. Therefore, Lemma 6 implies

$$\lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]} = \frac{q^{\bar{g}r}}{q} = 1.$$

This and (45) implies (44).

Case 2. Suppose that $\lim \left(G_l\left(\bar{x}_-^k\right)\right)^{n_l^k} = q < 1$. By the hypothesis that $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} > 0$ and $x^k \leq \bar{x}_-^k$

$$\lim \left(G_l\left(\bar{x}_-^k\right)\right)^{n_l^k} = q \geq \hat{q} > 0.$$

From Lemma 3,

$$\lim_{k\to\infty} \pi_w\left[\beta^k\left(\bar{x}\right)|\beta^k, n_w^k\right] = \lim_{k\to\infty} \frac{1 - G_w\left(\bar{x}_-^k\right)^{n_w^k}}{n_w^k\left(1 - G_w\left(\bar{x}_-^k\right)\right)}.$$

From Lemma 6 and its proof,

$$\lim_{k\to\infty} \pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right] = \frac{1-q}{-\ln q} \text{ and } \lim_{k\to\infty} \pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right] = \frac{1-q^{\bar{g}r}}{-\ln q^{\bar{g}r}}.$$

Hence,

$$\lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]} = \frac{\frac{1-q^{r\bar{g}}}{-r\bar{g}\ln q}}{\frac{1-q}{-\ln q}} = \frac{1-q^{\bar{g}r}}{\bar{g}r\left(1-q\right)}. \tag{46}$$

Since $0 < \bar{g}r < 1$ and $0 < q < 1$, straightforward manipulation shows that

$$q^{\bar{g}r-1} > \frac{1-q^{\bar{g}r}}{\bar{g}r\,(1-q)}. \tag{47}$$

From $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = \hat{q} \leq q$ and $\bar{g}r < 1$, $\hat{q}^{\bar{g}r-1} \geq q^{\bar{g}r-1}$. The equality from (45), $\hat{q}^{\bar{g}r-1} \geq q^{\bar{g}r-1}$, the inequality (47), and the equality (46) imply

$$\lim_{k\to\infty} \frac{G_h\left(x^k\right)^{n_h^k}}{G_l(x^k)^{n_l^k}} > \lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]},$$

as claimed. □

Step 2. Consider any sequence $\left\{x^k\right\}$ for which $\lim \left(G_h\left(x^k\right)\right)^{n_h^k} \in (0,1)$. Let

$$x_-^k = x_-^k\left(\beta^k\left(x^k\right)\right) \text{ and } x_+^k = x_+^k\left(\beta^k\left(x^k\right)\right).$$

Then,

$$\lim \left(G_h\left(x_-^k\right)\right)^{n_h^k} = 0 \text{ and } \lim \left(G_h\left(x_+^k\right)\right)^{n_h^k} = 1. \tag{48}$$

Proof: It is sufficient for the claim to prove that $\lim \left(G_h\left(x_+^k\right)\right)^{n_h^k} = 1$ whenever $\lim \left(G_h\left(x^k\right)\right)^{n_h^k} \in (0,1)$, because one can choose $\lim \left(G_h\left(x^k\right)\right)^{n_h^k}$ arbitrarily small. If $x_+^k = \bar{x}$ for sufficiently large k, we are done. So, suppose that $x_+^k < \bar{x}$ for all k large. Therefore, one can find a bid b^k such that

$$\beta^k\left(x^k\right) < b^k < \beta^k\left(\bar{x}\right) \tag{49}$$

and there is no atom at b^k,

$$x_-^k\left(b^k\right) = x_+^k\left(b^k\right). \tag{50}$$

By (49) and monotonicity of β^k, $x_-^k\left(b^k\right) \geq x^k$. Therefore,

$$\lim_{k\to\infty} \pi_h[b^k|\beta^k, n_h^k] = \lim_{k\to\infty} \left(G_h\left(x_-^k\left(b^k\right)\right)\right)^{n_h^k} \geq \lim_{k\to\infty} \left(G_h\left(x^k\right)\right)^{n_h^k} > 0. \tag{51}$$

Thus, the zero profit condition (5) requires that

$$\rho\bar{g}r \frac{\lim_{k\to\infty} \bar{\pi}_h[b^k|\beta^k, \eta_h^k]}{\lim_{k\to\infty} \bar{\pi}_l[b^k|\beta^k, \eta_l^k]} = \rho\bar{g}r \frac{\lim_{k\to\infty} \left(G_h\left(x_-^k\left(b^k\right)\right)\right)^{n_h^k}}{\lim_{k\to\infty} \left(G_l\left(x_-^k\left(b^k\right)\right)\right)^{n_l^k}} \leq \lim_{k\to\infty} \frac{b^k - v_l}{v_h - b^k}, \tag{52}$$

while individual rationality of $\beta^k(\bar{x})$ requires that

$$\rho\overline{g}r \lim_{k\to\infty} \frac{\pi_h\left[\beta^k(\bar{x})\,|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k(\bar{x})\,|\beta^k, n_l^k\right]} \geq \lim_{k\to\infty} \frac{\beta^k(\bar{x}) - v_l}{v_h - \beta^k(\bar{x})}. \tag{53}$$

Since, $b^k \leq \beta^k(\bar{x})$ for all k, (52) and (53) imply

$$\lim_{k\to\infty} \frac{\pi_h\left[\beta^k(\bar{x})\,|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k(\bar{x})\,|\beta^k, n_l^k\right]} \geq \frac{\lim_{k\to\infty}\left(G_h\left(x_-^k\left(b^k\right)\right)\right)^{n_h^k}}{\lim_{k\to\infty}\left(G_l\left(x_-^k\left(b^k\right)\right)\right)^{n_l^k}}. \tag{54}$$

From Step 1, (54) requires $\lim_{k\to\infty}\left(G_l\left(x_-^k\left(b^k\right)\right)\right)^{n_l^k} \in \{0,1\}$. Since $\lim_{k\to\infty}\left(G_l\left(x_-^k\left(b^k\right)\right)\right)^{n_l^k} > 0$ by (51), $\lim_{k\to\infty}\left(G_l\left(x_-^k\left(b^k\right)\right)\right)^{n_l^k} = 1$ follows. Because this must be true for all choices of $\left\{b^k\right\}$ for which (49) and (50) hold, it must be true that $\lim\left(G_h\left(x_+^k\left(\beta^k\left(x^k\right)\right)\right)\right)^{n_h^k} = 1$. As outlined at the start, this implies (48). □

Step 2 implies Claim 3 of Lemma 9. Claim 3 is sufficient for Claims 1 and 2. ∎

10.5 Existence and Uniqueness of $r^*(\rho, \overline{g})$

Proof of Lemma 10:

Recall that $J(r;\rho,\overline{g}) = \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+x\rho\overline{g}r)^2} dx$.

Claim 1: For each $\overline{g} > 1$, there exists an r' (close to $\overline{g}^{-1}$) such that $J(r';\rho,\overline{g}) < 0$.

Proof: Write

$$\begin{aligned} J(r;\rho,\overline{g}) = &\int_0^{\frac{1}{\overline{g}}} \left(\frac{1}{\overline{g}} - x\right)\left\{(-\ln(x))\,(x)^{\left(\frac{1}{\overline{g}r-1}\right)}\,(1+\rho\overline{g}rx)^{-2}\right\} dx \\ &- \int_{\frac{1}{\overline{g}}}^1 \left(x - \frac{1}{\overline{g}}\right)\left\{(-\ln(x))\,(x)^{\left(\frac{1}{\overline{g}r-1}\right)}\,(1+\rho\overline{g}rx)^{-2}\right\} dx. \end{aligned}$$

The term in the brackets $\{\cdots\}$ is always nonnegative and therefore both integrals are positive.

Let $\sigma = \frac{1}{\overline{g}r-1}$. The first integral is

$$
\begin{aligned}
&\int_0^{\frac{1}{\overline{g}}}\left(\frac{1}{\overline{g}}-x\right)\left\{(-\ln(x))(x)^{\left(\frac{1}{\overline{g}r-1}\right)}(1+\rho\overline{g}rx)^{-2}\right\}dx\\
&=\int_0^{\frac{1}{\overline{g}}}\left(\frac{1}{\overline{g}}-x\right)\left\{(-\ln(x))x^{\sigma}\left(1+\rho\frac{\sigma+1}{\sigma}x\right)^{-2}\right\}dx\\
&\le\frac{1}{\overline{g}}\int_0^{\frac{1}{\overline{g}}}(-\ln(x))x^{\sigma}dx\\
&=\left(\frac{1}{\overline{g}}\right)^{\sigma+2}\left[\frac{1}{\sigma+1}-\ln\left(\overline{g}^{-1}\right)\right].
\end{aligned}
$$

Thus, the first integral vanishes to zero at a rate of at least $\left(\overline{g}^{-1}\right)^{\sigma}$ as σ approaches ∞ (or equivalently, $r\to\overline{g}^{-1}$).

The second integral is

$$
\begin{aligned}
&\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)\left\{-\ln(x)(x)^{\left(\frac{1}{\overline{g}r-1}\right)}(1+\rho\overline{g}rx)^{-2}\right\}dx\\
&=\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)\left\{(-\ln(x))x^{\sigma}\left(1+\rho\frac{\sigma+1}{\sigma}x\right)^{-2}\right\}dx\\
&\ge\left(1+\rho\frac{\sigma+1}{\sigma}\right)^{-2}\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)(-\ln(x))x^{\sigma}dx\\
&=\left(\frac{\sigma}{\sigma+\rho(\sigma+1)}\right)^{2}\left(\frac{1}{(\sigma+2)^2}-\frac{\overline{g}^{-1}}{(\sigma+1)^2}+\frac{\left(-\ln\left(\overline{g}^{-1}\right)\right)\left(\overline{g}^{-1}\right)^{\sigma+2}}{(\sigma+2)(\sigma+1)}+\frac{1}{\overline{g}}\frac{\left(\overline{g}^{-1}\right)^{\sigma+1}}{(\sigma+1)^2}-\frac{\left(\overline{g}^{-1}\right)^{\sigma+2}}{(\sigma+2)^2}\right).
\end{aligned}
$$

Thus, either the second integral stays positive or it vanishes at a rate of at most σ^{-2} as σ approaches ∞ (or equivalently, $r\to\overline{g}^{-1}$).

To sum up, $J(r;\rho,\overline{g})<0$ for $r\to\overline{g}^{-1}$. ∎

Claim 2: For sufficiently large r, $J(r;\rho,\overline{g})>0$.

Proof: We show that $\lim_{r\to\infty}r^2J(r;\rho,\overline{g})=\infty$. Let $\xi(x,r)$ denote the integrand of $r^2J(r;\rho,\overline{g})$. That is,

$$
\xi(x,r)\equiv\left(x-\frac{1}{\overline{g}}\right)\ln(x)\,x^{\frac{1}{\overline{g}r-1}}\left(\frac{r}{1+\rho\overline{g}rx}\right)^{2}.
$$

Observe that $\xi(x,r)$ is non-decreasing in r on the domain $x\in\left(0,\overline{g}^{-1}\right)$, and is non-increasing in r on the domain $x\in\left(\overline{g}^{-1},1\right)$. Therefore, by the monotone convergence theorem,

$$\lim_{r\to\infty} r^2 J(r;\rho,\overline{g}) \equiv \lim_{r\to\infty}\int_0^1 \xi(x,r)\,dx = \int_0^1 \lim_{r\to\infty}\xi(x,r)\,dx = \frac{1}{(\rho\overline{g})^2}\int_0^1\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx$$
$$= \frac{1}{(\rho\overline{g})^2}\left[\int_0^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx + \int_{\overline{g}^{-1}}^1\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx\right].$$

Now, letting $a \in \left(0, \overline{g}^{-1}\right)$,

$$\int_0^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx \geq \lim_{a\to 0}\int_a^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx$$
$$= \lim_{a\to 0}\left[\frac{1}{2}\left(\left(\ln\left(\overline{g}^{-1}\right)\right)^2 - (\ln(a))^2\right) + \frac{1}{\overline{g}}\left[\overline{g}\left(\ln\left(\overline{g}^{-1}\right)+1\right) - a^{-1}\left(1+\ln(a)\right)\right]\right] = \infty,$$

while $\int_{\overline{g}^{-1}}^1\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln(x)\,dx$ is obviously bounded. Therefore, $\lim_{r\to\infty} r^2 J(r;\rho,\overline{g}) = \infty$ hence $J(r;\rho,\overline{g}) > 0$ for large enough r. ∎

Claims 1 and 2 together with the continuity of $J(r;\rho,\overline{g})$ in r establish the existence of $r > 1/\overline{g}$ such that $J(r;\rho,\overline{g}) = 0$.

Claim 3: Fix a $\overline{g} > 1$. For $r > \overline{g}^{-1}$, if $J(r;\rho,\overline{g}) = 0$, then $J_r(r;\rho,\overline{g}) > 0$.
Proof: Recall that

$$J(r;\rho,\overline{g}) \equiv \int_0^1\left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+\rho\overline{g}rx)^2}dx = 0.$$

Since $x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+\rho\overline{g}rx)^2} < 0$ for all $x \in (0,1)$, the integrand is positive for all $x \in (0,\frac{1}{\overline{g}})$ and is negative for all $x \in (\frac{1}{\overline{g}},1)$. Therefore, at any $r > \overline{g}^{-1}$ that satisfies $J(r;\rho,\overline{g}) = 0$,

$$\int_0^{\frac{1}{\overline{g}}}\left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+\rho\overline{g}rx)^2}dx = -\int_{\frac{1}{\overline{g}}}^1\left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+\rho\overline{g}rx)^2}dx > 0.$$

Consider the function $r^2 J(r;\rho,\overline{g})$ and observe that

$$\frac{dr^2 J(r;\rho,\overline{g})}{dr} = r\int_0^1\left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+\rho\overline{g}rx)^2}\left[\frac{-\overline{g}r\ln x}{(\overline{g}r-1)^2}+\frac{2}{(1+\rho\overline{g}rx)}\right]dx.$$

The integrand is equal to the integrand of $J(r;\rho,\overline{g})$ times the term $\left[\frac{-\overline{g}r\ln x}{(\overline{g}r-1)^2}+\frac{2}{(1+\rho\overline{g}rx)}\right]$ which is non-negative and decreasing in x. Therefore, at r such that $J(r;\rho,\overline{g}) = 0$, the positive part over $(0,\frac{1}{\overline{g}})$ is weighted more heavily than the negative part over $(\frac{1}{\overline{g}},1)$ implying $\frac{dr^2 J(r;\rho,\overline{g})}{dr} > 0$. Now, at r such that $J(r;\rho,\overline{g}) = 0$, $sgn\left(J_r(r;\rho,\overline{g})\right) = sgn\left(\frac{dr^2 J(r;\rho,\overline{g})}{dr}\right)$. Therefore, $J_r(r;\rho,\overline{g}) > 0$ as required. □

Claim 3 concludes the proof of the Lemma, since $J_r(r;\rho,\overline{g}) > 0$ at any r such that $J(r;\rho,\overline{g}) = 0$, there can be only one such r. ∎

10.6 Proof of Lemma 11

Proof of Lemma 11: From $\lim n_h^k s^k = r \lim n_l^k s^k$, rewriting as in Lemma 8,

$$\begin{aligned}
&\lim_{k\to\infty} n_h^k \left(\mathbb{E}_h\left[p|\beta^k, n_h^k+1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]\right) - \lim_{k\to\infty} n_h^k s^k \\
= &-\int_{v_l}^{\bar{p}} \phi_h\left(p|\rho,\overline{g},r\right)\ln\phi_h\left(p|\rho,\overline{g},r\right)dp + r\int_{v_l}^{\bar{p}} \phi_l\left(p|\rho,\overline{g},r\right)\ln\phi_l\left(p|\rho,\overline{g},r\right)dp.
\end{aligned}$$

Using (9) to spell out $\phi_w\left(p|\rho,\overline{g},r\right)$, rearranging and dividing through by $\overline{g}r/(\overline{g}r-1)$, we get

$$\begin{aligned}
&\frac{\overline{g}r-1}{\overline{g}r}(\lim n_h^k \left(\mathbb{E}_h\left[p|\beta^k, n_h^k+1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]\right) - \lim n_h^k s^k) \\
= &-\int_{v_l}^{\bar{p}} \left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p} - \frac{1}{\overline{g}}\right)\left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}} \ln\left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)dp.
\end{aligned}$$

Changing the integration variable by substituting for p the function $\psi(x) = \frac{v_l + x\rho\overline{g}rv_h}{1+x\rho\overline{g}r}$ we get

$$\begin{aligned}
&\frac{\overline{g}r-1}{\overline{g}r}\left(\lim n_h^k \left(\mathbb{E}_h\left[p|\beta^k, n_h^k+1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]\right) - \lim n_h^k s\right) \qquad (55)\\
= &-\int_{v_l}^{\bar{p}} \left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p} - \frac{1}{\overline{g}}\right)\left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}} \ln\left(\frac{1}{\rho\overline{g}r}\frac{p-v_l}{v_h-p}\right)dp \\
= &-\int_{\psi^{-1}(v_l)}^{\psi^{-1}(\bar{p})} \left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_l}{v_h-\psi(x)} - \frac{1}{\overline{g}}\right)\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_l}{v_h-\psi(x)}\right)^{\frac{1}{\overline{g}r-1}} \ln\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_l}{v_h-\psi(x)}\right)\psi'(x)dx \\
= &-\int_0^1 \left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\ln(x)\frac{r\overline{g}\rho\left(v_h-v_l\right)}{\left(1+rx\overline{g}\rho\right)^2}dx \\
= &-r\overline{g}\rho\left(v_h-v_l\right)J(r;\rho,\overline{g}).
\end{aligned}$$

Lemma 8 requires that

$$\lim n_h^k \left(\mathbb{E}_h\left[p|\beta^k, n_h^k+1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]\right) - \lim n_h^k s = 0.$$

Hence, from $\overline{g}r > 1$, optimality requires that

$$J(r;\rho,\overline{g}) = J(\lim \frac{n_h^k}{n_l^k};\rho,\overline{g}) = 0.$$

∎

10.7 Bounded Number of Bidders

Proof of Proposition 5: From Lemma 8, whenever $\lim F_w\left(\cdot|\beta^k, n_w^k\right)$ is non-degenerate, $\lim n_w^k s^k > 0$. Thus, if n_w^k is bounded, the distribution of the winning bid must become degenerate with support on some number C in state w, $\lim F_w\left(C+\varepsilon|\beta^k, n_w^k\right) - F_w\left(C-\varepsilon|\beta^k, n_w^k\right) = 1$ for all ε. We argue that the distribution of the winning bid must be degenerate on C in the other state as well.

Suppose that $\lim_{k\to\infty} n_h^k = m < \infty$ (the case where n_l^k is bounded is argued below). If n_l^k is bounded as well, we are done: In both states, the distribution of the winning bid is degenerate in the limit on some numbers and these numbers must be the same because the boundedness of the likelihood ratio implies that $\lim F_h\left(\cdot|\beta^k, n_h^k\right)$ and $\lim F_l\left(\cdot|\beta^k, n_l^k\right)$ are mutually absolutely continuous if n_h^k and n_l^k are both bounded.

So, suppose $n_l^k \to \infty$. Now, by $n_h^k \to m < \infty$ and the bounded likelihood ratio, whenever $\lim F_h\left(p|\beta^k, n_h^k\right) = 0$ then $\lim F_l\left(p|\beta^k, n_l^k\right) = 0$, i.e., the lower bound on the support of $\lim F_l\left(\cdot|\beta^k, n_l^k\right)$ is weakly above C. If $C \geq v_l$, then buyers' individual rationality and the law of iterated expectations rule out that $\lim F_l\left(p|\beta^k, n_l^k\right) < 1$ for any $p > C \geq v_l$. Hence, if $C \geq v_l$, then the distribution of the winning bid is degenerate on C in the low state as well. If $C \leq v_l$, then $n_l^k \to \infty$ rules out a non-degenerate distribution of the winning bid below v_l in the low state (Bertrand competition). Thus, the distribution of the winning bid becomes degenerate in the low state with mass one on v_l. However, when the distribution of the winning bid is degenerate in the low state, Lemma 8 implies that $n_l^k s^k \to 0$. Hence, it must be that $C = v_l$: The boundedness of the likelihood ratio implies that when the high type samples n_l^k bidders, the winning bid is close to v_l as well by Lemma 6, while total solicitation costs are close to zero. Thus, in both state, the distribution of the winning bid must become degenerate if n_h^k remains bounded as $s^k \to \infty$.

Now, if $n_l^k \to m < \infty$, a similar argument applies. As before, the distribution of the winning bid must become degenerate on some number C in the low state. Also as before, if n_h^k is bounded as well, we are done. So, suppose $n_h^k \to \infty$. In this case, $n_h^k / n_l^k \to \infty$, so that the interim expected value converges to v_h for all signals. If there is some price p with $C \leq p$ such that $\lim F_h\left(p|\beta^k, n_h^k\right) > 0$, then the winner's curse at p is bounded and, hence, $\lim \mathbb{E}[v|x,\text{win at } p+\varepsilon; \beta^k, \eta^k] = v_h$ for all ε. Thus, the interim expected payoffs are bounded from below by $\lim F_h\left(p|\beta^k, n_h^k\right)(p - v_h)$ for almost all types. Feasibility requires therefore that $p = v_h$ whenever $\lim F_h\left(p|\beta^k, n_h^k\right) > 0$. Thus, it must be that the distribution of the winning bid becomes degenerate with mass one on v_h in the high state. By buyers' individual rationality, this requires $C \leq v_l$. Since the distribution of the winning bid is degenerate in the high state, Lemma 8 implies that $n_h^k s^k \to 0$. If the low type solicits n_n^k bidders, the low type would be sure to trade at v_h as well by Lemma 6, at almost no solicitation costs, contradicting $C \leq v_l$. Thus, if n_l^k is bounded, n_h^k must be bounded as well.

Therefore, in all case, the distribution of the winning bid must become degenerate on some number C. Of course, the number C must be below $\rho_l v_l + \rho_h v_h$ by individual rationality of buyers and the law of iterated expectations. This finishes the proof of the proposition. ∎

10.8 Existence of a Pooling Equilibrium

Proof of Lemma 12: The proof relies closely on Athey (2001). The existence of a bidding equilibrium (given the constraints on β) for a given $\boldsymbol{\eta}$ is an immediate corollary of Athey's Theorem 1 and our Proposition 1. We have to establish that an equilibrium exists also when $\boldsymbol{\eta}$ is part of the equilibrium.

Recall that $P_\Delta = [0, v_l] \cup \{v_l + \Delta, v_l + 2\Delta, ..., v_h - \Delta, v_h\}$. Let B_Δ denote the set of monotone bidding functions using bids from $\left\{v_l, v_l + \Delta, v_l + 2\Delta, ..., \underline{b}, \bar{b}\right\}$ and let $m = \| \{v_l, v_l + \Delta, v_l + 2\Delta, ..., \underline{b}\} \|$. Using Athey's idea, Σ_Δ is a set of vectors of dimension $m + 1$ whose coordinates belong to $[\underline{x}, \overline{x}]$

$$\Sigma_\Delta = \{\sigma = (\sigma_0, \sigma_1, ...\sigma_m) \in [\underline{x}, \bar{x} - \epsilon]^{m+1} \mid \underline{x} \equiv \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m \equiv \bar{x} - \epsilon\},$$

where σ determines a bidding strategy β_σ by $\beta_\sigma(x) = v_l + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, ..., m-1$. We set

$$\beta_\sigma(x) = \bar{b} \qquad \forall\ x > \bar{x} - \epsilon.$$

We say a bidding strategy β is a best response against $(\sigma, \boldsymbol{\eta})$ if for all $x \leq \bar{x} - \epsilon$,

$$\beta(x) \in \underset{b \in \left\{v_l, v_l + \Delta, v_l + 2\Delta, ..., \underline{b}, \bar{b}\right\}}{\arg\max} U(b | x, \beta_\sigma, \boldsymbol{\eta}).$$

Let D denote the set of probability distributions η over $\{1, ..., N\}$.

Define the correspondence Ψ from $\Sigma_\Delta \times D \times D$ into itself. For any $\sigma' \in \Sigma_\Delta$ and $\boldsymbol{\eta}' \in D \times D$,

$$\begin{aligned}
\Psi_1(\sigma', \boldsymbol{\eta}') &= \left\{\sigma \in \Sigma_\Delta \mid \beta_\sigma \text{ is best response against } (\sigma', \boldsymbol{\eta}') \text{ for } x \leq \bar{x} - \epsilon\right\}, \\
\Psi_2(\sigma', \boldsymbol{\eta}') &= \left\{\boldsymbol{\eta} = (\eta_l, \eta_h) \mid \text{if } \eta_w(n) > 0, \text{ then } n \in \underset{n \in \{1,..,N\}}{\arg\max} \mathbb{E}_w\left[p | \beta_{\sigma'}, n\right] - ns\right\}, \\
\Psi(\sigma', \boldsymbol{\eta}') &= \Psi_1(\sigma', \boldsymbol{\eta}') \times \Psi_2(\sigma', \boldsymbol{\eta}').
\end{aligned}$$

Both Σ_Δ and D are closed convex sets. Ψ_2 is convex valued and continuous by virtue of being the set of maximizers of a concave problem on a convex set (see Lemma 5). That Ψ_1 is convex valued and upper hemi-continuous is established by Athey (2001). To be precise, the convex valuedness is established directly by Lemma 2 of that paper, while Lemma 3 establishes the upper hemicontinuity of $\Psi_1(\sigma', \boldsymbol{\eta}')$ only with respect to σ' (since $\boldsymbol{\eta}$ is exogenously fixed in Athey's model). Nevertheless, since the bidder's payoff function in our model, $U(\cdot|x, \beta, \boldsymbol{\eta})$, is continuous in $\boldsymbol{\eta}$, Athey's original argument establishes that $\Psi_1(\sigma', \boldsymbol{\eta}')$ is continuous with respect to $(\sigma', \boldsymbol{\eta}')$ as well.

It follows that $\Psi = \Psi_1 \times \Psi_2$ is convex valued and upper hemicontinous. By Kakutani's Theorem, Ψ has a fixed point. Since the strategies in Σ_Δ are constrained to use only prices from $\left\{v_l, v_l + \Delta, v_l + 2\Delta, ..., \underline{b}, \bar{b}\right\}$, the bidding strategy determined by the fixed point satisfies

$$\beta_\sigma(x) \begin{cases} = \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon. \end{cases}$$

In order to claim that a fixed point of Ψ is indeed an equilibrium of the auxiliary game A, it only remains to argue that there is no profitable deviation to a price in $[0, v_l)$, which is also in P_Δ and which is allowed by the constraint $\beta(x) \leq \underline{b}$—but this obviously true and was also argued in Lemma **2**. ∎

Proof of Lemma 13. We want to show that whenever $s^k \to 0$, β^k satisfies (16) and $\boldsymbol{\eta}^k$ is an optimal solicitation strategy given β^k, then:

1. $\lim_{k\to\infty} n_w^k = \infty,\ w \in \{\ell, h\}$.
2. $\lim_{k\to\infty} \frac{n_h^k}{n_l^k} = \frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$.
3. $\frac{(1-G_l(\bar{x}-\epsilon))}{(1-G_h(\bar{x}-\epsilon))} \frac{1}{G_h(\bar{x}-\epsilon)} \geq \lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_l(\bar{x}-\epsilon)^{n_l^k-1}}$.

Given convexity of the seller's objective function, optimality requires that

$$\mathbb{E}_w\left[p|\ \beta^k, n_w^k\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k - 1\right] \geq s^k \geq \mathbb{E}_w\left[p|\beta^k, n_w^k + 1\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k\right]. \quad (56)$$

In particular, this implies

$$\begin{aligned} \mathbb{E}_h\left[p|\beta^k, n_h^k\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k - 1\right] \quad &\geq \quad \mathbb{E}_l\left[p|\beta^k, n_\ell^k + 1\right] - \mathbb{E}_l\left[p|\beta^k, n_\ell^k\right], \quad (57) \\ \mathbb{E}_l\left[p|\beta^k, n_\ell^k\right] - \mathbb{E}_l\left[p|\beta^k, n_\ell^k - 1\right] \quad &\geq \quad \mathbb{E}_h\left[p|\beta^k, n_h^k + 1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]. \quad (58) \end{aligned}$$

Observe that

$$\mathbb{E}_w\left[p|\beta^k, n\right] = (1 - G_w(\bar{x} - \epsilon)^n)\,\bar{b} + G_w(\bar{x} - \epsilon)^n\,\mathbb{E}_w(p|p \le \underline{b}, \beta^k, n).$$

where $\mathbb{E}_w(p|p \le \underline{b}, \beta^k, n)$ is the expected winning bid conditional on $p \le \underline{b}$. It follows that

$$\begin{aligned} \mathbb{E}_w\left[p|\beta^k, n+1\right] - \mathbb{E}_w\left[p|\beta^k, n\right] = \\ G_w(\bar{x} - \epsilon)^{n-1}(1 - G_w(\bar{x} - \epsilon))\,[\bar{b} - \mathbb{E}_w(p|p \le \underline{b}, \beta^k, n-1)] \\ + G_w(\bar{x} - \epsilon)^n\,[\mathbb{E}_w(p|p \le \underline{b}, \beta^k, n) - \mathbb{E}_w(p|p \le \underline{b}, \beta^k, n-1)]. \end{aligned} \tag{59}$$

Hence,

$$\mathbb{E}_w\left[p|\beta^k, n+1\right] - \mathbb{E}_w\left[p|\beta^k, n\right] \ge G_w(\bar{x} - \epsilon)^n\,(1 - G_w(\bar{x} - \epsilon))\,[\bar{b} - \mathbb{E}_w(p|p \le \underline{b}, \beta^k, n)]. \tag{60}$$

Step 1: $n_w^k \to \infty$ for $w \in \{l, h\}$.
Proof of Step 1: By (56) and (60),[26]

$$s^k \ge \mathbb{E}_w\left[p|\beta^k, n_w^k + 1\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k\right] \ge G_w(\bar{x} - \epsilon)^{n_w^k}\,(1 - G_w(\bar{x} - \epsilon))\,[\bar{b} - \mathbb{E}_w(p|p \le \underline{b}, \beta^k, n_w^k)].$$

It follows from $s^k \to 0$ that $G_w(\bar{x} - \epsilon)^{n_w^k} \to 0$ and, hence, $n_w^k \to \infty$. This establishes Part (1) of the Lemma. □

Step 2:

$$\begin{aligned} G_\ell(\bar{x} - \epsilon)\,\frac{1 - G_\ell(\bar{x} - \epsilon)}{1 - G_h(\bar{x} - \epsilon)} \lim_{k\to\infty} \frac{[\bar{b} - \mathbb{E}_l(p|p \le \underline{b}, \beta^k, n_\ell^k)]}{[\bar{b} - \mathbb{E}_h(p|p \le \underline{b}, \beta^k, n_h^k)]} \\ \le \lim_{k\to\infty} \frac{G_h(\bar{x} - \epsilon)^{n_\ell^k - 1}}{G_\ell(\bar{x} - \epsilon)^{n_h^k - 1}} \le \\ \frac{1}{G_h(\bar{x} - \epsilon)}\,\frac{1 - G_\ell(\bar{x} - \epsilon)}{1 - G_h(\bar{x} - \epsilon)} \lim_{k\to\infty} \frac{[\bar{b} - \mathbb{E}_l(p|p \le \underline{b}, \beta^k, n_\ell^k)]}{[\bar{b} - \mathbb{E}_h(p|p \le \underline{b}, \beta^k, n_h^k)]}. \end{aligned} \tag{61}$$

Proof of Step 2: Using (59) for $w = h$ and (60) for $w = l$ to rewrite (57) and

[26] The derivation of (60) does not assume monotonicity of β^k for $x \le \bar{x} - \epsilon$.

rearranging we get,

$$\frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}}\left\{\frac{(1-G_h(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k-1)\right]}{(1-G_\ell(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k)\right]}+\right.$$
$$\left.\frac{\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k)-\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k-1)}{(1-G_\ell(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k)\right]}\right\}\ge 1. \tag{62}$$

Similarly,

$$\frac{G_\ell(\bar{x}-\epsilon)^{n_\ell^k-1}}{G_h(\bar{x}-\epsilon)^{n_h^k}}\left\{\frac{(1-G_\ell(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k-1)\right]}{(1-G_h(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k)\right]}+\right.$$
$$\left.\frac{\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k)-\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k-1)}{(1-G_h(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k)\right]}\right\}\ge 1. \tag{63}$$

Now, $n_w^k\to\infty$ implies

$$\lim_{k\to\infty}\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k+1)=\lim_{k\to\infty}\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k)=\lim_{k\to\infty}\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k-1).$$

Hence,

$$\lim_{k\to\infty}\frac{\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k)-\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k-1)}{(1-G_w(\bar{x}-\epsilon))\left[\bar{b}-\mathbb{E}_w(p|p\le\underline{b},\beta^k,n_w^k)\right]}=0.$$

Therefore, taking limits in (62) and (63) and combining them to a single chain of inequalities we get (61). □

Step 3: $\lim_{k\to\infty}\frac{n_h^k}{n_l^k}=\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}<1.$

Proof of Step 3: From (61), $\lim\frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_l(\bar{x}-\epsilon)^{n_l^k}}\in(0,\infty)$. Therefore,

$$\lim_{k\to\infty}\left(\frac{G_h(\bar{x}-\epsilon)^{\frac{n_h^k}{n_l^k}}}{G_l(\bar{x}-\epsilon)}\right)^{n_l^k}\in(0,\infty),$$

Since $n_l^k\to\infty$, this requires that

$$\lim_{k\to\infty}\frac{G_h(\bar{x}-\epsilon)^{\frac{n_h^k}{n_l^k}}}{G_l(\bar{x}-\epsilon)}=1.$$

Therefore, $\ln G_h(\bar{x}-\epsilon)\lim\frac{n_h^k}{n_l^k}=\ln G_l(\bar{x}-\epsilon)$, hence,

$$\lim_{k\to\infty}\frac{n_h^k}{n_l^k}=\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}<1,$$

where the last inequality is a consequence of $G_l(\bar{x}-\epsilon)>G_h(\bar{x}-\epsilon)$. This establishes Part (2) of the lemma. □

Step 4: $\lim_{k\to\infty}\mathbb{E}_l(p|p\leq\underline{b},\beta^k,n_\ell^k)\geq\lim_{k\to\infty}\mathbb{E}_h(p|p\leq\underline{b},\beta^k,n_h^k)$

Proof of Step 4: Let $F_w(p|p\leq\underline{b},\beta^k,n_w^k)$ denote the distribution of the winning bid conditional on being $p\leq\underline{b}$. Recall $(x_-^k(p)=\inf\{x\mid\beta^k(x)\geq p\}$. Consider a subsequence over which the following limits exist and let $q_w=\lim F_w(p|p\leq\underline{b},\beta^k,n_w^k)$, $w=\ell,h$.

$$\begin{aligned}q_w\quad\equiv\quad&\lim_{k\to\infty}F_w(p|p\leq\underline{b},\beta^k,n_w^k)=\lim_{k\to\infty}\left(\frac{G_w(x_-^k(p))}{G_w(\bar{x}-\epsilon)}\right)^{n_w^k}=\\&\lim_{k\to\infty}\left(\frac{G_w(\bar{x}-\epsilon)-[G_w(\bar{x}-\epsilon)-G_w(x_-^k(p))]}{G_w(\bar{x}-\epsilon)}\right)^{n_w^k}\\&\lim_{k\to\infty}\left(1-\frac{\frac{G_w(\bar{x}-\epsilon)-G_w(x_-^k(p))}{G_w(\bar{x}-\epsilon)}n_w^k}{n_w^k}\right)^{n_w^k}.\end{aligned}$$

Observe that, for p such that $\limsup x_-^k(p)<\bar{x}-\epsilon$, we have $\liminf[G_w(\bar{x}-\epsilon)-G_w(x_-^k(p)]>0$ and hence $q_w=0$; for p such that from some point in the sequence $(\beta^k)^{-1}(p)=\bar{x}-\epsilon$, we have $q_w=1$; for p such that $x_-^k(p)<\bar{x}-\epsilon$ and $\lim x_-^k(p)=\bar{x}-\epsilon$, we have

$$\lim_{k\to\infty}\frac{G_w(\bar{x}-\epsilon)-G_w(x_-^k(p))}{G_w(\bar{x}-\epsilon)}n_w^k=-\ln q_w.$$

Therefore,

$$\frac{-\ln q_h}{-\ln q_\ell}=\lim_{k\to\infty}\frac{[G_h(\bar{x}-\epsilon)-G_h(x_-^k(p))]}{[G_\ell(\bar{x}-\epsilon)-G_\ell(x_-^k(p))]}\frac{G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\frac{n_h^k}{n_\ell^k}.$$

Now, since $\lim x_-^k(p)=\bar{x}-\epsilon$ and since g_w are step functions, it follows that, for k large enough, $G_w(\bar{x}-\epsilon)-G_w(x_-^k(p))=g_w(\bar{x}-\epsilon)[\bar{x}-\epsilon-x_-^k(p)]$. Using this observation and $\lim\frac{n_h^k}{n_l^k}=\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$

$$\frac{-\ln q_h}{-\ln q_\ell}=\lim\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}.$$

By condition (14), the right side is smaller than one, and so $\frac{\ln q_h}{\ln q_\ell}\leq 1$ (see the footnote). Therefore,

$$\lim_{k\to\infty} F_l(p|p \le \underline{b}, \beta^k, n_\ell^k) = q_\ell \le q_h = \lim_{k\to\infty} F_h(p|p \le \underline{b}, \beta^k, n_h^k).$$

Thus, $\lim F_l(p|p \le \underline{b}, \beta^k, n_\ell^k)$ stochastically dominates $\lim F_h(p|p \le \underline{b}, \beta^k, n_h^k)$. Hence, $\lim \mathbb{E}_l(p|p \le \underline{b}, \beta^k, n_\ell^k) \ge \lim \mathbb{E}_h(p|p \le \underline{b}, \beta^k, n_h^k)$. □

Step 5:

$$\lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k-1}} \le \frac{1}{G_h(\bar{x}-\epsilon)} \frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}.$$

Proof of Step 5: Step 4 implies that $\lim \frac{[\bar{b}-\mathbb{E}_l(p|p\le\underline{b},\beta^k,n_\ell^k)]}{[\bar{b}-\mathbb{E}_h(p|p\le\underline{b},\beta^k,n_h^k)]} \le 1$, which together with (61) implies the desired inequality. □

This proves Part (3) and concludes the proof of Lemma 13. ∎

Proof of Lemma 14. We want to show that there are numbers v_1^*, v_2^*,v_3^* independent of $\underline{b}, \bar{b}$ such that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &= \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &\le v_1^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &\le v_2^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}, \bar{b}\right);\, \beta^k, \boldsymbol{\eta}^k\right] &\le v_3^* < \rho_l v_l + \rho_h v_h.
\end{aligned}$$

Observe that

$$\begin{aligned}
\Pr\left[h|\bar{x}, \text{ win at } \bar{b},\, \beta^k,\, \eta^k\right] &= \frac{\Pr\left[h,\, \bar{x},\, \beta^k, \text{ win at } \bar{b}\right]}{\Pr\left[\bar{x},\, \beta^k, \text{ win at } \bar{b}\right]} \qquad (64)\\
&= \frac{\rho_h g_h(\bar{x}) \frac{n_h}{N} \frac{1}{n_h} \frac{1-(G_h(\bar{x}-\epsilon))^{n_h}}{1-G_h(\bar{x}-\epsilon)}}{\rho_h g_h(\bar{x}) \frac{n_h}{N} \frac{1}{n_h} \frac{1-(G_h(\bar{x}-\epsilon))^{n_h}}{1-G_h(\bar{x}-\epsilon)} + \rho_l g_l(\bar{x}) \frac{n_l}{N} \frac{1}{n_l} \frac{1-(G_l(\bar{x}-\epsilon))^{n_l}}{1-G_l(\bar{x}-\epsilon)}} \\
&= \frac{\rho_h \left(1-(G_h(\bar{x}-\epsilon))^{n_h}\right)}{\rho_h \left(1-(G_h(\bar{x}-\epsilon))^{n_h}\right) + \rho_l \left(1-(G_l(\bar{x}-\epsilon))^{n_l}\right)} \\
&\to {}_{k\to\infty} \rho_h.
\end{aligned}$$

For the second equality, note that $\rho_w g_w(\bar{x}) \frac{n_w}{N} = \Pr\left[w,\, \bar{x},\, \eta^k\right]$ and that by Lemma 3, $\Pr\left[\text{win at } \bar{b}|w,\, \beta^k,\, \eta^k\right] = \pi_w\left[\bar{b}|\beta^k, n_w^k\right] = \frac{1}{n_w} \frac{1-(G_w(\bar{x}-\epsilon))^{n_w}}{1-G_w(\bar{x}-\epsilon)}$. The third equality follows from $1-G_w(\bar{x}-\epsilon) = g_w(\bar{x})\epsilon$ and cancellation of terms. Finally, the convergence to ρ_h follows from $n_w^k \to \infty$ by Lemma 13. Therefore

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } \bar{b},\, \beta^k,\, \eta^k\right] = \rho_l v_l + \rho_h v_h,$$

that is, the first equation holds.

Replacing $g_w(\bar{x})$ by $g_w(\bar{x}-\epsilon)$ in (64), we obtain the corresponding expressions for $\Pr\left[h|\bar{x}-\epsilon,\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right]$. Then using $g_h(\underline{x})\,\epsilon = 1-G_h(\bar{x}-\epsilon)$ to substitute out $1-G_h(\bar{x}-\epsilon)$ and again canceling terms we get

$$\begin{aligned}\Pr\left[h|\bar{x}-\epsilon,\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right] &= \frac{\rho_h\frac{g_h(\bar{x}-\epsilon)}{g_h(\bar{x})}\left(1-G_h(\bar{x}-\epsilon)^{n_h}\right)}{\rho_h\frac{g_h(\bar{x}-\epsilon)}{g_h(\bar{x})}\left(1-G_h(\bar{x}-\epsilon)^{n_h}\right)+\frac{g_l(\bar{x}-\epsilon)}{g_l(\bar{x})}\rho_l\left(1-G_l(\bar{x}-\epsilon)^{n_l}\right)} \\ &\to_{k\to\infty} \frac{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_l(\bar{x})}\right)}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_l(\bar{x})}\right)+\rho_l} < \rho_h.\end{aligned}$$

The expression following the convergence sign is obtained by dividing through by $\frac{g_l(\bar{x}-\epsilon)}{g_l(\bar{x})}$ and noting that $G_w(\bar{x}-\epsilon)^{n_w}\to 0$ since $n_w^k\to\infty$ by Lemma 13. The last inequality owes to the increasing likelihood ratio $\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)} < \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Let

$$v_1^* = \frac{\rho_l}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_l(\bar{x})}\right)+\rho_l}v_l + \frac{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_l(\bar{x})}\right)}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_l(\bar{x})}\right)+\rho_l}v_h.$$

The second equation holds with

$$\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x}-\epsilon,\text{ win at }\bar{b};\,\beta^k,\,\boldsymbol{\eta}^k\right] = v_1^* < \rho_l v_l + \rho_h v_h.$$

The winner at $b>\bar{b}$ does not learn anything from winning. Hence,

$$\begin{aligned}\Pr\left[h|\bar{x},\text{ win at }b>\bar{b};\,\beta^k,\,\eta^k\right] &= \frac{\Pr\left[h,\,\bar{x},\text{ win at }b>\bar{b};\,\beta^k,\,\eta^k\right]}{\Pr\left[\bar{x},\text{ win at }b>\bar{b};\,\beta^k,\,\eta^k\right]} \\ &= \frac{\rho_h g_h(\bar{x})\frac{n_h}{N}}{\rho_h g_h(\bar{x})\frac{n_h}{N}+\rho_l g_l(\bar{x})\frac{n_l}{N}} \\ &= \frac{\rho_h\overline{g}\frac{n_h^k}{n_l^k}}{\rho_h\overline{g}\frac{n_h^k}{n_l^k}+\rho_l} \\ &\to_{k\to\infty} \frac{\rho_h\overline{g}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}}{\rho_h\overline{g}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_l},\end{aligned}$$

where the limit follows from $\frac{n_h^k}{n_l^k}\to\frac{\ln G_l(\hat{x})}{\ln G_h(\hat{x})}$ established in Lemma 13. Let

$$v_2^* = \frac{\rho_l}{\rho_h\overline{g}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_l}v_l + \frac{\rho_h\overline{g}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}}{\rho_h\overline{g}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_l}v_h.$$

From $\frac{g_h(\bar{x})\epsilon}{g_l(\bar{x})\epsilon} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_l(\bar{x}-\epsilon)}$, $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_l(\bar{x}-\epsilon)}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$. Because the function $\frac{1-z}{\ln z}$ is strictly increasing in $z \in (0,1)$ and because $G_l(\bar{x}-\epsilon) > G_h(\bar{x}-\epsilon)$,

$$\frac{1-G_h(\bar{x}-\epsilon)}{1-G_l(\bar{x}-\epsilon)}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1.$$

Therefore, $\Pr\left[h|\bar{x}, \text{ win at } b > \bar{b},\, \beta^k,\, \eta^k\right] < \rho_h$ and

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \eta^k\right] = v_2^* < \rho_l v_l + \rho_h v_h.$$

Define v_3^* by

$$v_3^* \equiv \frac{\rho_h \frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}}{\rho_h \frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)} + \rho_l} v_h + \frac{\rho_l}{\rho_h \frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)} + \rho_l} v_l. \tag{65}$$

Since by condition (13), $\frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)} < 1$, it follows that $v_3^* < \rho_l v_l + \rho_h v_h$. Now,

$$\begin{aligned}
& \Pr\left[v_h|\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}^k, \bar{b}\right);\, \beta^k, \eta^k\right] \qquad (66) \\
= \; & \frac{\rho_h \Pr[\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}^k, \bar{b}\right) |h;\, \beta^k, \eta^k]}{\sum_{w=h,\ell} \rho_w \Pr[\bar{x}-\epsilon,\, b \in \left(\underline{b}^k, \bar{b}\right) |w;\, \beta^k, \eta^k]} \\
= \; & \frac{\rho_h g_h(\bar{x}-\epsilon)\frac{n_h^k}{N^k} G_h(\bar{x}-\epsilon)^{n_h^k-1}}{\rho_h g_h(\bar{x}-\epsilon)\frac{n_h^k}{N^k} G_h(\bar{x}-\epsilon)^{n_h^k-1} + \rho_l g_l(\bar{x}-\epsilon)\frac{n_\ell^k}{N^k} G_\ell(\bar{x}-\epsilon)^{n_\ell^k-1}} \\
= \; & \lim_{k\to\infty} \frac{\rho_h \frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)} \frac{G_l(\bar{x}-\epsilon)\ln G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)} G_h(\bar{x}-\epsilon)^{n_h^k}}{\rho_h \frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)} \frac{G_l(\bar{x}-\epsilon)\ln G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)} G_h(\bar{x}-\epsilon)^{n_h^k} + \rho_l G_l(\bar{x}-\epsilon)^{n_l^k}} \\
\le \; & \lim_{k\to\infty} \frac{\rho_h \frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}}}{\rho_h \frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}} + \rho_l} \le \frac{\rho_h \frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}}{\rho_h \frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)} + \rho_l} < \rho_h.
\end{aligned}$$

The expression after the 2nd equality sign is explained by $\Pr[\text{win at } b \in \left(\underline{b}^k, \bar{b}\right) |w;\, \beta^k, \eta^k] = G_h(\hat{x})^{n_h^k-1}$. The expression following the convergence sign is obtained by dividing through by $\frac{g_h(\bar{x}-\epsilon)n_h^k}{G_l(\bar{x}-\epsilon)}$ and noting that $\lim \frac{n_h^k}{n_l^k} = \frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$. The following inequality follows from the assumption $\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\frac{G_l(\bar{x}-\epsilon)\ln G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)} \le 1$. The next inequality follows from Lemma 13. The final inequality follows from $\frac{1}{G_l(\hat{x})}\frac{1-G_l(\hat{x})}{1-G_h(\hat{x})} < 1$ which is implied by condition (13). The definition of v_3^* together with (66) imply

$$\lim_{k\to\infty} \Pr\left[v_h|\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}, \bar{b}\right);\, \beta^k, \eta^k\right] \le v_3^* < \rho_l v_l + \rho_h v_h.$$

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