Sequential Auctions with Synergy

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December 30, 2015

Job Market Paper

Abstract

This paper studies sequential auctions with synergy in which each bidder's values can be affiliated across auctions, and empirically assesses the revenue effects of bundling. Ignoring affiliation can lead to falsely detecting synergy where none exists. Motivated by data on synergistic pairs of oil and gas lease auctions, where the same winner often wins both tracts, I model a sequence in which a first-price auction is followed by an English auction. At the first auction, bidders know their first value and the distribution of their second value conditional on the first value. At the second auction, bidders learn their second value, which is affiliated with their first value and also affected by potential synergy if they won the first auction. Both synergy and affiliation take general functional forms. I establish nonparametric identification of the joint distribution of values, synergy function, and risk aversion parameter from observed bids in the two auctions. Intuitively, the effect of synergy is isolated by comparing the second-auction behavior of a first-auction winner and first-auction loser who bid the same amount in the first auction. Using the identification results, I develop a nonparametric estimation procedure for the model, assess its finite sample properties using Monte Carlo simulations, and apply it to the oil and gas lease data. I find both synergy and affiliation between adjacent tracts, though affiliation is primarily responsible for the observed allocation patterns. Bidders are risk averse. Counterfactual simulations reveal that bundled auctions would yield higher revenue, with a small loss to allocative efficiency.

I would like to thank the New Mexico State Land Office for generous assistance in collecting and interpreting the data. Special thanks are due to current and former chief geologists Stephen Wust, Joe Mraz, and Dan Fuqua for answering an endless stream of questions from me. Chris Barnhill, David Catanach, Kevin Hammit, Tracey Noriega, Philip White, and Lindsey Woods also provided valuable insight on the industry. I would like to thank Kei Kawai, Laurent Lamy, members of the NYU Econometrics Seminar, Stern Micro Friday Seminar, and NYU Micro Student Lunch for helpful comments and suggestions. I am indebted to Isabelle Perrigne and Quang Vuong for their invaluable guidance and support. All remaining errors are mine. Financial support from NYU-CRATE is gratefully acknowledged.

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1 Introduction

Synergy refers to when the value of a set of objects is greater than the sum of its parts. There are many instances in which synergistic or complementary objects are sold via auction. Geographically contiguous PCS licenses or adjacent bands of spectrum (Ausubel et al. (1997), Cramton (1997)), cable TV licenses (Gandal (1997)), electricity generation in adjacent time periods (Wolfram (1998)), adjacent school milk contracts (Marshall et al. (2006)), construction contracts (De Silva, Jeitschko, and Kosmopoulou (2005)), agri-environmental contracts (Saïd and Thoyer (2007)), and long-haul truckloads (Triki et al. (2014)) are some examples in which winning a set of objects accrues synergy to the bidder. Also, when competing localities pay recruitment subsidies to firms, there are benefits from agglomeration if multiple firms form an industrial cluster in the same area (Martin (1999)).

While there are a number of ways to organize synergistic auctions, it is often the case that these auctions are held sequentially, selling one object at a time. The Israeli cable TV licenses described in Gandal (1997), the construction contracts studied by De Silva et al. (2005), and the milk contracts sold by Georgia school districts (Marshall et al. (2006))[1] are of this type. Compared to simultaneous ascending or combinatorial auctions, sequential auctions may be the more approachable alternative for sellers and buyers with limited resources. Of course, the presence of synergy across a sequence creates an interesting dynamic problem for bidders. Strategic questions arise for the seller as well; for instance, in most cases it would be relatively easy to switch to auctions of bundles, in which the synergistic objects are sold together as a single package. Since a general comparison of sequential, bundled, and other types of auctions tends to be difficult, the optimal choice of method will often need to be assessed empirically through counterfactual analyses. For this reason, the ability to estimate bidders' synergy functions, value distributions, and other structural parameters is very useful.

In the empirical auction literature, structural estimation of sequential auction models has focused on independent objects linked by bidder constraints, or homogeneous goods with decreasing marginal values.[2] For instance, Jofre-Bonet and Pesendorfer (2003) estimate an infinite horizon model of first-price procurement auctions, where capacity constraints generate dynamics across the sequence. The construction contracts being auctioned are otherwise independent, so a firm's cost draws across auctions are also independent conditional on remaining capacity. On the other hand, Lamy (2010) studies two-stage sequential English auctions in which values across auctions are not independent. The model is for homogeneous goods like fish and tobacco, so bidders retain the same value in the first and second auction

[1] Marshall et al. (2006) model the Georgia milk auctions as simultaneous.

[2] Brendstrup and Paarsch (2007) estimate a joint distribution of values from two auctions, in which a bidder's values for the two different objects can be dependent. However, this is not a sequential model; the results of the first auction have no effect on the second, such that the timing of the auctions is irrelevant.

unless they win the first. For the first-auction winner, second-unit value is assumed to be lower than first-unit value, consistent with decreasing marginal values. Neither of these papers focuses on the topic of synergy.

This paper proposes a structural analysis of sequential auctions with synergy in which each bidder's values can be affiliated across auctions, and empirically assesses the revenue effects of bundling. Both synergy and affiliation take general functional forms. Motivated by data on synergistic pairs of oil and gas lease auctions, it seeks to address contexts that do not fit in either the independent objects or homogeneous goods framework.

In the oil and gas lease auctions of New Mexico, it is sometimes the case that two adjacent halves of a square mile are auctioned on the same day. By convention, one object is sold by a first-price sealed-bid auction, and the other object is sold later (but on the same day) using an English auction. I observe that the same winner often wins both tracts. The two auctions are likely to be linked by synergy, for two reasons. First, as noted in Sunnevåg (2000), equipment and crews will already be nearby, reducing the cost of moving them between disparate locations and possibly eliminating duplicates. Second, in recent years much of the drilling in New Mexico has been horizontal; with permission from government authorities, adjacent tracts can be put together to form a "project area" penetrated by a single horizontal well. As drilling a well is very expensive, the ability to access a larger area with one well rather than multiple wells can be valuable. For these reasons, there is likely to be extra value to winning two adjacent tracts beyond the sum of one's values for each individually.

Meanwhile, since the tracts in a pair are two adjacent halves of a square mile, a bidder's values for the two are likely to be affiliated, even after conditioning on covariates. For instance, a firm may like certain geological formations because its engineers are especially skilled in drilling that type of geology. If these geological features are geographically clustered, the firm's values for adjacent tracts will be affiliated. This is a concept distinct from synergy, as it concerns the correlation of values for individual tracts and has nothing to say about how they sum. Affiliation of this kind is likely to coexist with synergy in other contexts as well, as synergy often emerges from some sort of adjacency, which is conducive to affiliation. However, it is important to see that this creates a potential identification problem; if we observe that the winner of the first auction is more likely to win the second, it could be due to affiliation, synergy, or both. A misspecified model of independence across auctions would attribute the phenomenon entirely to synergy, even if synergy did not actually exist. This is troubling from both a seller's and a social planner's point of view, as affiliation can have different implications for revenue and welfare relative to synergy. As such, relaxing independence is especially meaningful when estimating auctions with synergy.

In light of the data, I model a sequence in which a first-price auction is followed by an

English auction. At the first auction, bidders know their first-auction value and the distribution of their second-auction value conditional on the first value. Uncertainty regarding the second value comes from noise - such as the outcome of other auctions being realized - in the time between the two auctions, which I do not explicitly model. At the second auction, bidders learn their second value, which is affiliated with their first value and also affected by synergy. I allow bidders to be risk averse.

I establish nonparametric identification of the auction model from observables, which include first-auction bids, second-auction prices, and bidder identities. Specifically, the distribution of second-auction value conditional on first-auction value is identified from second-auction prices conditional on first-auction bids. Then, the synergy function is identified by comparing the second-auction value distributions of a first-auction winner and first-auction loser *who bid the same amount in the first auction*. This conditioning on the first-auction bid neutralizes affiliation and allows us to isolate the effect of synergy. Using results from the previous steps, first-auction values are identified from the first-order condition for bidding. Finally, the risk aversion parameter is identified by exploiting variation in the number of bidders.

Based on the identification steps, I develop a nonparametric estimation procedure that recovers the structural parameters of the auction model. I assess its finite sample performance in a Monte Carlo study, and apply it to the oil and gas lease data. I find both synergy and affiliation between adjacent tracts, though affiliation is primarily responsible for the observed pattern in which the same bidder often wins both tracts. This result highlights the importance of allowing for affiliation across auctions. Bidders are risk averse, and this is key to explaining revenue patterns. Counterfactual simulations using the estimated structural parameters reveal that bundled auctions would yield higher revenue, with a small loss to allocative efficiency.

This paper contributes to the literature by analyzing sequential auctions of affiliated objects linked by synergy, a context frequently observed in practice but which has evaded structural econometric analysis to date. In doing so, it distinguishes synergy from affiliation across auctions. While the model and estimation procedure of this paper are tailored to the empirical application at hand, the main insight behind disentangling synergy from affiliation is adaptable to other contexts as long as all bids in the first auction are monotonic in values and observed. Structural parameters recovered by the estimation method are valuable inputs when comparing sequential auctions to bundled auctions or other alternatives.

The paper is organized as follows. The remainder of Section 1 provides an overview of the related literature. Section 2 describes the data and empirical evidence. Section 3 develops a model of sequential auctions with synergy. Section 4 establishes nonparametric identification of the model. Section 5 develops an estimation procedure and discusses a Monte Carlo study assessing finite sample performance. Section 6 describes estimation details specific to the data at hand, and discusses the estimation results. Section 7 performs counterfactual simulations of interest including those for bundled auctions. Section 8 concludes. The appendix collects all proofs.

Related literature

This paper is preceded by the empirical literature on sequential auctions, which begins with Ashenfelter (1989)'s study of wine auctions, and includes among others Gandal (1997) and De Silva et al. (2005), whose regression analyses find evidence of synergy in Israeli cable TV license auctions and Oklahoma DOT construction auctions, respectively. Within that literature, this paper is most closely related to the structural econometric work that starts with Jofre-Bonet and Pesendorfer (2003), which was described earlier. Balat (2013) builds on the model of Jofre-Bonet and Pesendorfer (2003) to include auction-level unobserved heterogeneity and endogenous participation. He finds that the accelerated release of procurement projects under the American Recovery and Reinvestment Act increased procurement prices by increasing firms' backlogs.

Donald, Paarsch, and Robert (2006) study sequential English auctions of homogeneous goods, in which a Poisson demand generation process imposes stationarity across auctions in the sequence. Also in the homogeneous goods framework, Brendstrup and Paarsch (2006) and Brendstrup (2007) study identification and estimation of sequential English auctions using only the last stage of the game, without specifying equilibria for the earlier stages. However, Lamy (2010) finds that identification actually fails in the context of Brendstrup and Paarsch (2006) and Brendstrup (2007). Building on an equilibrium for the whole two-stage game characterized in Lamy (2012), he establishes conditions under which the model is identified, and develops an estimation procedure that uses both stages of the game.

Meanwhile, specifically addressing synergy, Brendstrup (2006) proposes a nonparametric test for synergy that compares the price distribution of an object sold on its own to that of the same object sold second in a sequence; Groeger (2014) estimates a dynamic auction model to measure savings in bid preparation costs that come from having recently prepared bids on contracts of the same type; and Donna and Espin-Sanchez (2015) study sequential auctions of identical water units, which fall into a complements regime or a subtitutes regime depending on weather seasonality.

This paper also relates to empirical work on synergy in non-sequential auctions. Ausubel et al. (1997) and others discuss synergy in the simultaneous ascending PCS auctions run by the FCC. Marshall et al. (2006) model and estimate simultaneous first-price auctions with a specific form of synergy in the Georgia school milk market. Gentry, Komarova, and Schiraldi (2015) also study simultaneous first-price auctions with synergy, but take a more general approach, establishing nonparametric identification of the model under certain restrictions. They apply their framework to Michigan highway procurement auctions and find that bidders view small projects as complements but large projects as substitutes. Cantillon and Pesendorfer (2013) study combinatorial first-price auctions of London bus routes, where synergies could exist. They show conditions for nonparametric identification of the combinatorial auction model, and propose a two-stage estimation procedure to recover bidders' costs from bids. Upon applying the procedure, they find evidence of decreasing returns to scale rather than synergy.

The model introduced in this paper is of theoretical interest as well, as it has not been analyzed before. The theory of sequential auctions is more complete for the case of single-unit demand, where bidders demand at most one unit. A number of early papers explore equilibrium price trends when bidders have single-unit demand for identical goods. In particular, Milgrom and Weber (1999) show that the sequence of prices is a martingale under common assumptions, while McAfee and Vincent (1993), Engelbrecht-Wiggans (1994), Jeitschko (1999) and others offer explanations for declining prices. Budish and Zeithammer (2011) study single-unit demand for two non-identical goods. Meanwhile, when it comes to multi-unit demand, equilibrium analysis is challenging and often intractable. As such, most papers restrict their analysis to two auctions and assume either that a bidder's values for the two goods are the same, that all bidders share the same values, that bidders are represented by a single type variable, or that values are independent across auctions and learned one at a time. Examples include Ortega-Reichert (1968), Hausch (1986), and Caillaud and Mezzetti (2004), who study information revelation across a sequence of auctions, as well as Benoit and Krishna (2001) and Pitchik (2009), who study the effect of budget constraints. Exceptions include Katzman (1999) and Lamy (2012). They study sequential second-price auctions of two homogeneous goods with declining marginal values. Values are not independent since they are ordered, and bidders know both values at the start of the sequence. However, Katzman (1999) still restricts to bid functions that depend only on one value, while Lamy (2012) characterizes the set of equilibria more generally. Relative to this literature, the model used in this paper allows for a flexible relationship between values across auctions, but still retains tractability because bidders learn their second value at the second auction.

In the theory of sequential auctions addressing synergy in particular, price trends have been a topic of interest just as in the literature for single-unit demand. Branco (1997), Jeitschko and Wolfstetter (2002), and Menezes and Monteiro (2003) show how prices can decline for identical objects or when values have a two-point support, while Sørensen (2006) show how prices can increase for stochastically equivalent objects. Jofre-Bonet and Pesendorfer (2014) ask whether first-price or second-price auctions achieve lower procurement cost, and find that second-price auctions are better for complements given risk-neutral bidders and independence across auctions. The issue of whether to bundle in the presence of synergy has also been a topic of interest. Grimm (2007) finds that when bidders can subcontract, bundled auctions yield lower procurement costs than sequential auctions. Subramaniam and Venkatesh (2009) analyze a parametric model and suggest that bundling is better than sequential auctions when the number of bidders is low or synergy is strong. Papers that study simultaneous auctions with synergy also shed light on bundling. Levin (1997) finds that when bidders are symmetric and represented by a single type variable, bundling maximizes revenue over other simultaneous mechanisms. Benoit and Krishna (2001), though not principally focused on bundling, provide an example in which bundling decreases auction revenue in the presence of synergy.

2 Data

2.1 Overview

This paper examines pairs of adjacent leases sold on the New Mexico State Trust Lands during 2000-2014. A pair consists of two adjacent leases, 320 acres each, that together form a square-mile plot of land. Leases in a pair have the same lease terms, including the royalty rate, rental payments, and length of the lease. Also, they are very similar geologically, as they are adjacent halves of a square mile.

As mentioned in the introduction, one lease is sold by first-price sealed-bid auction, and the other lease is sold by English auction later in the day. For the first-price sealed bid auction, I observe all bids and bidder identities. For the English auction, I observe the transaction price and the identity of the winner only.

Table 1 displays the number of pairs observed for each N, which is the number of bidders in the first auction. Table 2 displays other basic statistics, including within-pair statistics that are particularly relevant for the study of pairs.

The auction prices of paired leases are highly correlated, confirming the geological similarity of adjacent leases. 93% of bidders winning the second auction ("A2") also participate in the first auction ("A1"), indicating that by and large, the same set of bidders are bidding on both items. Meanwhile, the probability that both leases will be won by the same bidder is higher than it would be if all A1 participants had an equal chance of winning A2. This suggests that the winner of A1 is more likely to win A2 than other bidders.

To check this correlation more formally, I perform a probit analysis where the unit of observation is a bidder-lease in a first auction, and the dependent variable is whether that bidder wins the paired second auction. Only auctions with two or more bidders are used. The results are displayed in Table 3. Columns (1)-(3) include number-of-bidders fixed effects,

Table 1: Number of pairs 2000-2014, by number of bidders N in the first auction

N	pairs
0	14
1	267
2	247
3	165
4	98
5	50
6	21
7	9
8	1

Table 2: Statistics for paired leases, 2000-2014

Mean winning bid per acre (2009 dollars)	\$239	
Reserve price per acre	\$15.625	
For $N \geq 2$:		
Correlation of final price in 1st and 2nd auction	0.91	
Probability that 2nd-auction winner also bid on 1st auction	93%	
Probability that pair is won by same bidder:	observed	even odds
$N = 2$	74%	50%
$N = 3$	62%	33%

Table 3: Probit regression results for probability of winning second auction

	(1) $N \geq 2$	(2) $N \geq 2$	(3) $N \geq 2$	(4) $N = 2$	(5) $N = 3$
Won first auction	1.562***	2.045***	2.041***	1.723***	1.771***
	(0.093)	(0.197)	(0.201)	(0.169)	(0.194)
Number of bidders fixed effects	Y	Y	Y	-	-
Bidder fixed effects	Y	N	N	Y	Y
Bidder-date fixed effects	N	Y	Y	N	N
Lease descriptive covariates	N	N	Y	N	N
Observations	1563	612	612	381	408

Standard errors in parentheses

* $p<0.10$, ** $p<0.05$, *** $p<0.01$

and columns (4) and (5) focus on $N = 2$ and $N = 3$, respectively. Columns (1), (4), and (5) control for bidder fixed effects, and columns (2) and (3) control for bidder-date-of-auction fixed effects.[3] Column (3) also controls for covariates describing the lease, which are listed in Table 7 and defined in section 6.1.

In every column, winning the first auction has a highly significant positive effect on the observed probability of winning the second auction. Using the column (1) specification, the probit coefficient can be interpreted as follows: winning A1 increases the observed probability of winning A2 from 0.17 to 0.72 if $N = 2$, and from 0.13 to 0.66 if $N = 3$, for an average tract and an average bidder. We can conclude that the winner of A1 is more likely to win A2 than other bidders. The cause, however, cannot be diagnosed without further investigation.

2.2 Evidence of synergy

Intuitively, synergy gives winners of the first auction ("A1") a boost in winning the second auction ("A2"). However, the mere observation that A1 winners are more likely to win A2 need not indicate synergy. Instead, the phenomenon can be due to affiliation of a bidder's values for the first (v_1) and second item (v_2), which is especially likely in this empirical context as the tracts in question are adjacent halves of a square mile. In order to confirm the presence of synergy, we need to account for the fact that even without synergy, the A1 winner is more likely to have the highest v_2 due to affiliation.

One way to perform such a test is to use a regression discontinuity design. For each bidder in the first auction, define

[3] There are 128 bidders in the sample, some of which bid very few times. Bidders or bidder-dates that do not bid enough to compute fixed effects are dropped from the regression.

$$z \equiv ln(b) - ln(\text{highest competing } b)$$

where b is his bid in the first auction. Then $z > 0$ indicates an A1 winner, and $z < 0$ indicates an A1 loser. A large $|z|$ indicates a large gap between the first and second highest bids in A1. If bidders' v_1 and v_2 are affiliated, a larger $|z|$ makes it more likely that the same bidder will win both A1 and A2. On the other hand, if $|z|$ is very small, this means the A1 winner just barely won. In the absence of synergy, such a bidder should not be much more likely to win A2 than if he just barely lost. This is the idea I exploit to detect synergy; I look for a discontinuity in the probability of winning A2 at $z = 0$. The test does not necessarily prove or disprove synergy, but can provide suggestive indications. As an earlier example of exploiting the idea of RD in the auctions literature, Kawai and Nakabayashi (2014) examine bidders who narrowly won the first round of a multi-round auction, and find evidence of collusion in their pattern of winning subsequent rounds.

Formally, I seek to measure

$$\beta = y^+ - y^-$$

where $y^+ \equiv lim_{z \to 0^+} E[y_i | z_i = z]$ and $y^- \equiv lim_{z \to 0^-} E[y_i | z_i = z]$. As proposed in Hahn, Todd, and Van der Klaauw (2001), I use local linear regression to estimate y^+ and y^-.

As different bidders may have more or less aggressive bidding strategies in A1, which is a first-price sealed-bid auction (unlike A2, which is English), it is best to compare the same bidder against himself in the two scenarios of $z \to 0^+$ and $z \to 0^-$. The results that follow are for the most frequent bidder, who allows the largest number of data points.[4]

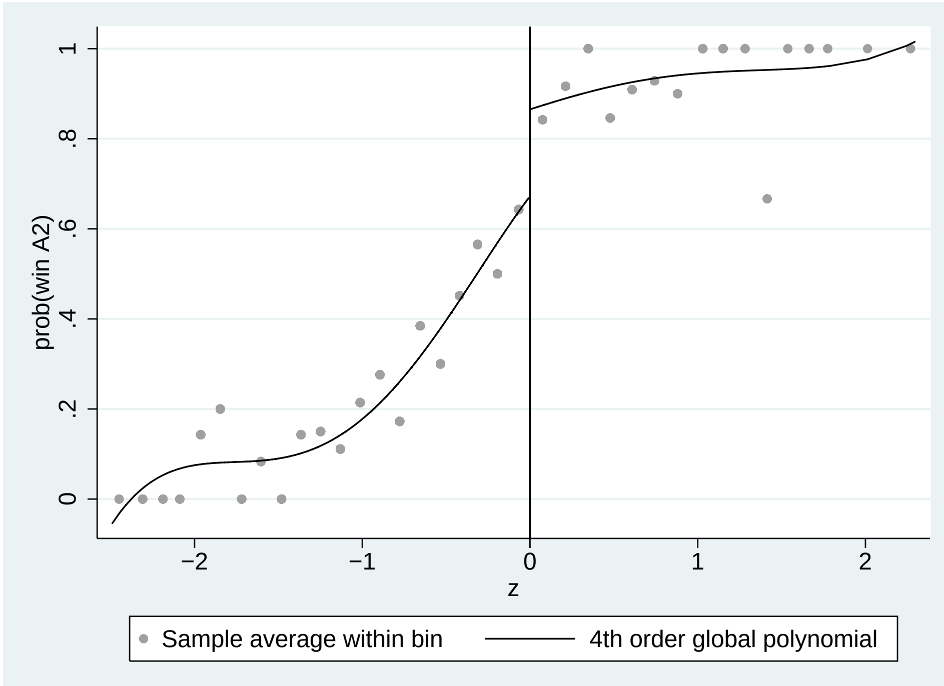
An RD-style plot of the data is displayed in Figure 1.[5] Two features of Figure 1 stand out. First, the probability of winning the second auction is increasing in z. This is consistent with affiliation of values across adjacent tracts, which makes the results of A1 predictive of A2. Second, there seems to be a discontinuity at $z = 0$, consistent with synergy between adjacent tracts.

The local linear regression results are shown in Table 4. The second row of Table 4 corrects for the bias in conventional RD estimates as discussed in Calonico, Cattaneo, and Titiunik (2014b), and the third row increases the standard error to account for the fact that this bias is itself estimated. The columns show different choices of bandwidth selectors: CV represents the cross-validation method proposed by Ludwig and Miller (2007), IK represents Imbens and Kalyanaraman (2012), and CCT represents Calonico et al. (2014b).

[4] The number of observations drops exponentially going down the ordered list of bidders. See the appendix for more on other bidders.

[5] Figure 1 and Table 4 are obtained using the software packages described in Calonico, Cattaneo, and Titiunik (2014a).

Figure 1: Regression discontinuity plot



Though the null of no synergy cannot be rejected with the robust confidence intervals in the third row, there are nonetheless suggestive indications of synergy, both in the plot of data and in the estimation results. The estimated jump in the probability of winning is roughly 0.2.

3 A model of sequential auctions with synergy

Motivated by the empirical setting, I build a model of sequential auctions with synergy. To fix ideas, I introduce the model in the context of risk-neutral, symmetric bidders. Afterwards, I extend the model to asymmetric bidders and risk aversion.

Private values paradigm

I develop the model within the private values paradigm. In common value models of oil and gas leases, the source of interdependence is a relatively large amount of uncertainty regarding value-relevant components of z, such as how much oil is underground. However, the Permian Basin in New Mexico is an area where knowledge of the geology is more complete due to a long history of development and production dating back to the 1920s. Seismic work done by the state is publicly available. Permits for new seismic surveys are no longer requested in the basin, as these are only done in areas that are not well known. Much of the basin has already been drilled in the past. And when land is drilled, electric wireline logs that

Table 4: Sharp RD estimates using local linear regression

Bandwidth selector:	CV	IK	CCT
Conventional	0.215***	0.211**	0.193
	(0.082)	(0.106)	(0.120)
Bias-corrected	0.191**	0.185*	0.176
	(0.082)	(0.106)	(0.120)
Robust	0.191	0.185	0.176
	(0.122)	(0.139)	(0.145)
Observations	545	545	545

Epanechnikov kernel
Standard errors in parentheses
* $p<0.10$, ** $p<0.05$, *** $p<0.01$

record geologic formations are submitted to the New Mexico Oil Conservation Division and made public. Conversations with agency staff and bidders suggest that, though the science is never exact and uncertainty remains, the industry has a fairly good idea of oil and gas potential in the basin, and bidders are working with the same, publicly available information when they assess the value of a tract to their firm. As one bidder put it,

> "Bear in mind that New Mexico has been producing oil and gas for over 80 years and there have been thousands of wells drilled. This provides us a lot of historical data. Most tracts that show up on a given monthly sale, are in an area with lots of production history and exploration success, or have the lack thereof."

Meanwhile, valuations of a lease can be highly idiosyncratic by bidder for firm-specific reasons. Different firms have different niches and areas of interest. They may be interested in different depths or layers of the same tract of land, and engineering teams may design different plans for how to drill it. Firms vary in their leaseholding strategies. In particular, winning a lease does not require the firm to drill; it grants the right, but not the obligation, for five years. As such, one firm may plan on drilling in the first year, while another firm may plan on the last year. The tract may not be drilled at all - this is very common - and different firms may have different probabilities of drilling for each tract. Leasing budgets, operating costs, and infrastructure also vary across firms. In light of the relatively small uncertainty regarding oil and gas potential, the private values paradigm is a decent approximation of our setting. It is also the more relevant paradigm for other data contexts. Nonetheless, it certainly simplifies the complexities of the real environment.

3.1 Setup

I introduce the full model first, and then discuss the merits and demerits of particular features in turn.

A pair of adjacent tracts is leased via auction on the same day. One tract is sold by a first-price sealed-bid auction, and the other is sold by an English auction, which happens later chronologically. Before bidding in the first auction, each bidder draws a value

$$v_1 \sim F^1(\cdot)$$

which is his private value for the first object.

Between the first auction (A1) and second auction (A2), there is noise that affects bidders' values for the second object. Therefore, bidders do not know their value for the second object (v_2) with certainty at the time of the first auction. However, they do know the distribution from which v_2 will be drawn:

$$v_2 \sim F^2(\cdot|v_1)$$

The distribution F^2 is conditional on v_1, allowing for affiliation between v_1 and v_2. v_2 is learned after the first but before the second auction.

The firm that won the first auction benefits from synergy if he also wins the second auction, so his ultimate value for the second object is not just v_2 but

$$s(v_1, v_2).$$

I allow the synergy function s to be a function of both v_1 and v_2 to be as general as possible. To simplify notation when expressing the idea that the A1-winner applies synergy to his v_2 when valuing the second half, I define

$$D(x|v_1) \equiv prob(s(v_1, v_2) \leq x|v_1)$$

and say winners of A1 draw their ultimate value for the second half from the distribution $D(\cdot|v_1)$. I assume that the same set of bidders participate in the first and second auction.

Now I discuss the ideas underlying specific parts of this model.

I do not explicitly model the noise between the first and second auction, but in the case of the oil and gas lease auctions, one source of noise is other auctions that take place in between the two sales. The type and number of tracts won and lost in these intervening auctions can lead to adjustments in bidders' values. In other data contexts, noise may come from the passage of time, often months, between the two auctions.[6]

[6]In Marshall et al. (2006), school milk procurements take place from May through August of each year,

The distribution of v_2 is conditional on v_1, but it is not conditional on any other signal. The underlying assumption is that v_1 is a sufficient statistic for anything known by a bidder at the time of A1 that his F^2 could depend on. This does not require the two objects to be identical; if each object has its own descriptive covariates, then the statement can be made conditional on these covariates. For the oil and gas lease pairs, which are adjacent halves of a square mile, the assumption is a reasonable approximation of reality. More generally, it is reasonable when, as in other contexts involving adjacency, the objects are related and determinants of private value "shocks" are likely to be similar. On the other hand, if the objects are not so related, the model may be too crude of an approximation. The alternative for those cases would be to have a separate signal for the second object at the time of the first auction. However, two-dimensional types introduce significant difficulties to characterizing equilibria, let alone estimating the model. In contexts where it is appropriate, the model introduced here provides a practical way forward.

It is helpful to compare this setup with other sequential or dynamic auction models. The literature on sequential auctions of homogeneous goods, such as Brendstrup and Paarsch (2006) and Lamy (2010), has employed models in which bidders' value for the second item remains v_1 if they do not win the first item. If they do win, the value of the second unit is always less than the value of the first unit. The empirical applications were in auctions of commodities, such as fish and tobacco. Letting v_2 be a draw from $F_2(\cdot|v_1)$ is less restrictive and encompasses the special case where it is exactly v_1. On the other hand, bidders in Lamy (2010) learn both (v_1, v_2) prior to bidding in the first auction. However, identification and estimation in Lamy (2010) still proceed from establishing that first auction bids depend only on v_1 under certain conditions.

This is also different from the dynamic auction model used in Jofre-Bonet and Pesendorfer (2003), where a bidder's values across auctions are independent conditional on covariates and state variables. Here, the distribution of a bidder's v_2 is directly dependent on his v_1, encompassing independence across auctions as a special case. Allowing for such correlation is critical when attempting to measure synergy. As discussed in section 2.2, ignoring affiliation could lead us to detect synergy where none exists. On the other hand, Jofre-Bonet and Pesendorfer (2003) consider an infinite horizon of auctions, while this paper models just two auctions.

Notation

It is useful to introduce some notation that simplifies long expressions in the expected profit function. Though the model is different, I follow the style of notation used by Lamy (2012).

The distribution of the highest competing bid in the second auction given that the bidder

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and in Gandal (1997), Israeli cable TV licenses are auctioned over a period spanning 1988-1991.

wins the first auction and the highest competing bid in A1 is t is[7]

$$H_1(u|t) = F^2(u|\beta \le t)^{N-2} F^2(u|\beta = t) \tag{1}$$

To explain, the probability that the highest competing bid in A2 is $\le u$ is equal to the probability that all bidders other than this bidder have values $\le u$ for the second item. Since the highest competing bid in A1 is t, the other bidders in A2 consist of one bidder who bid t in A1 and $N-2$ bidders who bid $\le t$ in A1. The right-hand side of (1) expresses the probability that all of these competing bidders have values $\le u$. The subscript 1 on H indicates the case where the bidder wins the first auction.

Next, the distribution of the highest competing bid in A2 given that the bidder loses A1 and the highest competing bid in A1 is t is

$$H_2(u|t) = F^2(u|\beta \le t)^{N-2} D(u|\beta = t) \tag{2}$$

The subscript 2 on H indicates the case where the bidder loses the first auction. The right-hand side of (2) is the same as that of (1) except that $D(u|\beta = t)$ replaces $F^2(u|\beta = t)$. Having lost A1, the bidder knows he will be competing against the winner of A1, who benefits from synergy. Therefore, H_2 is different from H_1 if synergy exists.

Assumptions

For now, assume all items are homogeneous for expositional ease. Section 5.2 will discuss how to work with heterogeneity across pairs.

AS1 (v_1, v_2) are independent across bidders: $(V_{1i}, V_{2i}) \perp (V_{1j}, V_{2j})$

AS2 $F_1(\cdot)$ is differentiable, with density $f_1 = F_1'$.

AS3 $F_2(\cdot|v_1)$ is stochastically ordered in v_1: $v_1' > v_1$ implies $F_2(\cdot|v_1') \le F_2(\cdot|v_1)$.

AS4 $F_2(\cdot|v_1)$ and $D(\cdot|v_1)$ are differentiable and have the same support, for every v_1.

AS5 By definition of synergy, $s(v_1, v_2) \ge v_2$.

AS6 $\frac{\partial s(v_1,v_2)}{\partial v_1} \ge 0$ and $\frac{\partial s(v_1,v_2)}{\partial v_2} \ge \frac{H_2(v_2|t)}{H_1(s(v_1,v_2)|t)}$, $\forall t$.

[7] H_1 and H_2 are conditional only on the highest competing bid t, and not on any other bids. This is because these expressions will be used in the expected profit function, which is computed by the bidder before the first auction happens. Before the first auction, all he knows is that if he wins with bid b, the highest competing bid must be less than b, and if he loses with bid b, the highest competing bid must be greater than b.

AS7 The reserve price r is not binding.

AS1 means that while values can be dependent across the first and second auction, values are independent across bidders. AS3 means that a bidder with higher v_1 is more likely to have a higher v_2. This makes sense given that the two tracts in a pair are located in the same square mile; even with intervening noise between the two auctions, v_1 and v_2 are likely to be positively correlated. This assumption also helps make bidding in the first auction monotonic in v_1. AS4, which says all bidders bidding in the second auction draw their values from the same support, is helpful for identifying the value distributions, as I will discuss in the identification section. AS5 states the idea that by definition of synergy, the value of winning both tracts is at least as great as the sum of its parts. In particular, this means that $F^2(u|\beta = t) \geq D(u|\beta = t)$ for any u, t, and hence $H_1(u|t) \geq H_2(u|t)$. AS6 places some restrictions on the form of $s(v_1, v_2)$, which is the synergy-included total value of the second tract to a firm that won the first tract. For one, AS6 says $s(\cdot, \cdot)$ is a nondecreasing function in v_1 and v_2; this is natural. The second part of AS6, $\frac{\partial s(v_1,v_2)}{\partial v_2} \geq \frac{H_2(v_2|t)}{H_1(s(v_1,v_2)|t)}$, is a more restrictive assumption on the form of s that helps ensure monotonic bidding strategies in the first auction. Note that if $s(v_1, v_2) > v_2$, $\frac{H_2(v_2|t)}{H_1(s(v_1,v_2)|t)} < 1$. Popular forms of synergy such as $s(v_1, v_2) = v_2 + \alpha$ used in Krishna and Rosenthal (1996) and $s(v_1, v_2) = \alpha v_2$ ($\alpha \geq 1$) satisfy AS6.

3.2 Bidding in the second auction

Working backwards, we discuss bidding in the second auction before thinking about the first auction. The second auction is an English auction. Under the private value paradigm, it is a dominant strategy for each bidder to bid up to his value for the tract. For the bidder who won the first auction, this is $s(v_1, v_2)$. For all other bidders who bid in the first auction, this is v_2.

3.3 Bidding in the first auction

Now we consider bidding in the first auction (A1), which is a first-price sealed-bid auction.

Expected profit at the time of the first auction

Using the notation just introduced, the expected profit from the two auctions at the time of the first auction, if the bidder bids b is

$$\pi(v_1,b) = \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ \int\limits_{t=\underline{b}}^{b} \left(v_1 - b + \int\limits_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1,v_2)-u) dH_1(u|t) \right) dG^{N-1}(t) + \int\limits_{t=b}^{\bar{b}} \int\limits_{u=\underline{v}}^{v_2} (v_2-u) dH_2(u|t) dG^{N-1}(t) \right\} dF^2(v_2|v_1)$$

The outer integral over v_2 represents the fact that at the time of the first auction, v_2 is unknown. The first expression inside the outer integral represents the case where the bidder wins the first auction, and the second expression represents the case where the bidder loses the first auction. Notice that when he wins the first auction, he benefits not only from $v_1 - b$, but also from the fact that the second item is now worth $s(v_1, v_2)$ to him rather than just v_2 due to synergy. $G(\cdot)$ is the distribution of sealed bids in the first auction, and $G^{N-1}(\cdot)$ is the distribution of the highest bid out of $N-1$ bidders.

First-order condition

A bidder will bid the b that maximizes his expected profit $\pi(v_1, b)$. Taking the derivative of $\pi(v_1, b)$ with respect to b and setting it equal to zero gives

$$\frac{G(b)}{(N-1)g(b)} = \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ v_1 - b + \int\limits_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1,v_2)-u) dH_1(u|b) - \int\limits_{u=\underline{v}}^{v_2} (v_2-u) dH_2(u|b) \right\} dF^2(v_2|v_1)$$

Using integration by parts, the first-order condition can be simplified to

$$b = v_1 + \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ \int\limits_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|b) du - \int\limits_{u=\underline{v}}^{v_2} H_2(u|b) du \right\} dF^2(v_2|v_1) - \frac{G(b)}{(N-1)g(b)} \qquad (3)$$

It is instructive to compare this FOC to the FOC of a stand-alone first-price auction. From Guerre, Perrigne, and Vuong (2000), we know that the FOC for a stand-alone first-price auction is $b = v_1 - \frac{G(b)}{(N-1)g(b)}$. In (3), there is an additional term on the right-hand side that represents the expected benefit due to synergy in the second auction from winning the first auction. In other words, the v_1 in the stand-alone first-price auction is replaced by v_1 plus the expected benefit of synergy. If there is no synergy, i.e. $s(v_1, v_2) = v_2$, then (3) collapses to $b = v_1 - \frac{G(b)}{(N-1)g(b)}$, as in Guerre et al. (2000).

Note that by construction, the bids in the first auction are a function only of v_1; only the distribution of v_2, not the realization, is known at the time of the first auction.

3.4 Equilibrium properties

Strictly increasing bid function

It can be shown that bids in the first auction are strictly increasing in v_1. As a preliminary step to doing so, Proposition 1 proves that if a low-value bidder bids more than a high-value bidder, both bids must be in the best-response set BR of both bidders.

Proposition 1. *Let $v_1' < v_1$, with $b \in BR(v_1)$ and $b' \in BR(v_1')$. If $b' > b$, then it implies that $b' \in BR(v_1)$ and $b \in BR(v_1')$.*

Then using Proposition 1, we can prove that bidding in the first auction is monotonic in v_1.

Proposition 2. *The bid function $b(v_1)$ in the first auction is strictly increasing in v_1.*

Uniqueness of equilibrium

Since the bid function $b(\cdot)$ is strictly increasing, we can revisit (1) and (2) to see that in a symmetric equilibrium,

$$H_1(u|b(x)) \equiv F^2(u|b(v_1) \leq b(x))^{N-2} F^2(u|b(v_1) = b(x)) = F^2(u|v_1 \leq x)^{N-2} F^2(u|v_1 = x)$$

$$H_2(u|b(x)) \equiv F^2(u|b(v_1) \leq b(x))^{N-2} D(u|b(v_1) = b(x)) = F^2(u|v_1 \leq x)^{N-2} D(u|v_1 = x)$$

Then in a given equilibrium, $H_1(u|b)$ and $H_2(u|b)$ in (3) can be replaced with $H_1(u|\xi(b))$ and $H_2(u|\xi(b))$, where $\xi(\cdot)$ is the inverse bid function for that equilibrium, and the following equation must be satisfied:

$$b(v_1) = v_1 + \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ \int\limits_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|v_1)du - \int\limits_{u=\underline{v}}^{v_2} H_2(u|v_1)du \right\} dF^2(v_2|v_1) - \frac{G(b)}{(N-1)g(b)} \tag{4}$$

Now, it is helpful to see that (4) can be represented by

$$b = T(v_1) - \frac{G(b)}{(N-1)g(b)} \tag{5}$$

where

$$T(v_1) \equiv v_1 + \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ \int\limits_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|v_1)du - \int\limits_{u=\underline{v}}^{v_2} H_2(u|v_1)du \right\} dF^2(v_2|v_1) \tag{6}$$

Hence, symmetric equilibria of the first auction are equivalent to that of a stand-alone first-price auction where bidders have values $T(v_1)$. From Riley and Samuelson (1981), we know there is a unique symmetric equilibrium for such an auction. This is stated in Proposition 3.

Proposition 3. *There is a unique symmetric Bayes-Nash equilibrium for the first auction.*

Monotonicity of bidding and uniqueness of equilibrium is useful when it comes to identifying the model from data. Also, this means $F^2(v_2|b(v_1))$ is a simple transformation of $F^2(v_2|v_1)$ that retains the stochastic ordering property. In an abuse of notation, I use the same "F^2" to denote both $F^2(\cdot|b(v_1))$ and $F^2(\cdot|v_1)$. In intermediate steps of estimation and other parts of the paper, it will often be easier to work with $F^2(v_2|b)$ rather than $F^2(v_2|v_1)$, since b is observed while v_1 is not. The same can be said of $s(b(v_1), v_2)$ versus $s(v_1, v_2)$. Now, we can rewrite the first-order condition in (3), replacing $F^2(v_2|v_1)$ with $F^2(v_2|b)$ and $s(v_1, v_2)$ with $s(b, v_2)$:

$$b = v_1 + \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{u=\underline{v}}^{s(b,v_2)} H_1(u|b)du - \int_{u=\underline{v}}^{v_2} H_2(u|b)du \right\} dF^2(v_2|b) - \frac{G(b)}{(N-1)g(b)}$$

Then rearranging this FOC, we can write the inverse bid function

$$v_1 = \xi(b) \equiv b + \frac{G(b)}{(N-1)g(b)} - \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{u=\underline{v}}^{s(b,v_2)} H_1(u|b)du - \int_{u=\underline{v}}^{v_2} H_2(u|b)du \right\} dF^2(v_2|b) \qquad (7)$$

The effect of synergy on revenue

In this section, I investigate the effect of positive synergy on auction revenue, relative to the case of no synergy. In order to do so, I first establish that $T(v_1)$ as defined in (6) is a monotonic function of v_1.

Proposition 4. *$T(v_1)$ is a monotonically increasing function of v_1.*

Now it can be shown that when positive synergy exists, first auction revenue increases relative to the case of no synergy. In the second auction, which is an English auction, synergy increases the final price when the A1-winner ends up being the second highest bidder in A2, or when synergy causes the A1-winner to win A2 where he would have lost otherwise. In all other cases, synergy leaves the final price unchanged. This leads to the next proposition.

Proposition 5. *When $s(v_1, v_2) > v_2$, revenue in the first auction is higher than it would be if $s(v_1, v_2) = v_2$. Revenue in the second auction is at least as high as it would be if $s(v_1, v_2) = v_2$.*

Proposition 5 tells us that positive synergy definitely increases A1 revenue and likely increases A2 revenue compared to the case of no synergy.

3.5 Asymmetric bidders

The model of sequential auctions with synergy can be extended to the case where bidders have asymmetric value distributions and synergy functions. In this section, I extend the model to the case of two asymmetric subgroups. Nothing prevents us from going to larger numbers of subgroups, though mathematical expressions will become increasingly long and complex.

The asymmetric model requires additional notation. First, a subscript m will indicate the subgroup to which value distributions and synergy functions belong, so

$$v_1 \sim F_m^1(\cdot)$$

$$v_2 \sim F_m^2(\cdot|v_1)$$

$$s_m(v_1, v_2)$$

$$D_m(x|v_1) \equiv prob(s_m(v_1, v_2) \leq x|v_1)$$

Then, the distribution of the highest competing bid in the second auction given that a bidder from subgroup m wins the first auction and the highest competing bid in the first auction is t from subgroup m is

$$H_1^{m,m}(u|t) = F_m^2(u|\beta_m \leq t)^{N_m-2} F_{-m}^2(u|\beta_{-m} \leq t)^{N_{-m}} F_m^2(u|\beta_m = t)$$

The distribution of the highest competing bid in the second auction given that a bidder from subgroup m wins the first auction and the highest competing bid in the first auction is t from subgroup $-m$ is

$$H_1^{m,-m}(u|t) = F_m^2(u|\beta_m \leq t)^{N_m-1} F_{-m}^2(u|\beta_{-m} \leq t)^{N_{-m}-1} F_{-m}^2(u|\beta_{-m} = t)$$

The distribution of the highest competing bid in the second auction given that a bidder from subgroup m loses the first auction and the highest competing bid in the first auction is t from subgroup m is

$$H_2^{m,m}(u|t) = F_m^2(u|\beta_m \leq t)^{N_m-2} F_{-m}^2(u|\beta_{-m} \leq t)^{N_{-m}} D_m(u|\beta_m = t)$$

Finally, the distribution of the highest competing bid in the second auction given that a bidder from subgroup m loses the first auction and the highest competing bid in the first auction is t from subgroup $-m$ is

$$H_2^{m,-m}(u|t) = F_m^2(u|\beta_m \leq t)^{N_m-1} F_{-m}^2(u|\beta_{-m} \leq t)^{N_{-m}-1} D_{-m}(u|\beta_{-m} = t)$$

Additionally, for a bidder from subgroup m, the probability that the highest competing bid in the first auction is $\leq t$ is $G_m(t)^{N_m-1}G_{-m}(t)^{N_{-m}}$, where G_m is the distribution of first auction bids from subgroup m.

Then, for a bidder from subgroup m, the probability that the highest competing bid in the first auction is $= t$ is $\frac{\partial G_m(t)^{N_m-1}G_{-m}(t)^{N_{-m}}}{\partial t}$, and can be expressed as $j_m(t) + k_m(t)$, where

$$j_m(t) \equiv (N_m - 1)G_m(t)^{N_m-2} g_m(t) G_{-m}(t)^{N_{-m}}$$

is the probability that the highest competing bid in the first auction is $= t$ and from subgroup m, and

$$k_m(t) \equiv N_{-m} G_{-m}(t)^{N_{-m}-1} g_{-m}(t) G_m(t)^{N_m-1}$$

is the probability that the highest competing bid in the first auction is $= t$ and from subgroup $-m$.

Using the above notation, the expected profit at the time of the first auction for a bidder from subgroup m is

$$\pi_m(v_1, b) = \int_{v_2=\underline{v}}^{\bar{v}} X_m(v_1, v_2, b) dF_m^2(v_2|v_1)$$

where

$$\begin{aligned} X_m(v_1, v_2, b) \equiv\ & \int_{t=\underline{b}}^{b} [v_1 - b + \int_{u=\underline{v}}^{s_m(v_1,v_2)} (s_m(v_1, v_2) - u) dH_1^{m,m}(u|t)] j_m(t) dt \\ & + \int_{t=\underline{b}}^{b} [v_1 - b + \int_{u=\underline{v}}^{s_m(v_1,v_2)} (s_m(v_1, v_2) - u) dH_1^{m,-m}(u|t)] k_m(t) dt \\ & + \int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2^{m,m}(u|t) j_m(t) dt \\ & + \int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2^{m,-m}(u|t) k_m(t) dt \end{aligned}$$

In the equation defining X_m, the first two parts account for the probability that the bidder wins the first auction and the last two parts account for the probability that he loses the first auction. There are two parts to each case because with asymmetry, the identity (subgroup) of the highest competing bidder in the first auction matters for the bidder's expected profit in the second auction.

Taking a derivative of the expected profit function $\pi_m(v_1, b)$ with respect to b yields the first-order condition for bidding.

$$\int_{v_2=\underline{v}}^{\bar{v}} \frac{\partial X_m(v_1, v_2, b)}{\partial b} dF_m^2(v_2|v_1) = 0$$

After simplifying and rearranging, the FOC for subgroup m can be rewritten as follows

$$\begin{aligned} G_m(b)^{N_m-1}G_{-m}(b) \quad = \quad & (v_1 - b)(j_m(b) + k_m(b)) + \\ & \textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ j_m(b)[\int_{u=\underline{v}}^{s_m(v_1,v_2)} H_1^{m,m}(u|b)du - \int_{u=\underline{v}}^{v_2} H_2^{m,m}(u|b)du] \\ & +k_m(b)[\int_{u=\underline{v}}^{s_m(v_1,v_2)} H_1^{m,-m}(u|b)du - \int_{u=\underline{v}}^{v_2} H_2^{m,-m}(u|b)du]\} dF_m^2(v_2|v_1) \end{aligned} \tag{8}$$

The FOC for asymmetric bidders is structurally similar to the FOC for symmetric bidders in (3), but breaks down terms to account for differences between subgroups. If $F_m^2 = F_{-m}^2$, $s_m = s_{-m}$, and $G_m = G_{-m}$, equation (8) reduces to (3).

The logic of Proposition 1 and 2 can still be applied in the asymmetric case, so bidding in the first auction is monotonic in v_1 within each subgroup. On the other hand, with asymmetry, uniqueness of the equilibrium is not guaranteed. When bidders are asymmetric, Maskin and Riley (2003) used the Fundamental Theorem of ordinary differential equations, along with upper and lower boundary conditions, to prove the existence of a unique equilibrium. However, when there is a second-stage auction following the first one, boundary conditions such as $b_m(\bar{v}) = b_{-m}(\bar{v})$ generally do not hold for more than two asymmetric bidders. As a simple example, consider the case where one subgroup derives strong synergies but the other does not derive any, and there are multiple bidders in each subgroup. Even if the two types of bidders are identical in all other respects and have the same $v_1 = \bar{v}$, the all-inclusive value of winning the first auction in light of the second auction is different for the two types. So in general the optimal bids $\bar{b}_m$ and $\bar{b}_{-m}$ would be different. Whether more can be said about boundary conditions and uniqueness remains to be studied.

3.6 Risk aversion

The model of sequential auctions with synergy can also be extended to the case where bidders are risk averse. Since the second auction is an English auction, it remains a dominant strategy for bidders to bid their value in the second auction. However, risk aversion does affect bidding in the first auction, which uses the first-price sealed-bid format.

With risk aversion, the expected profit at the time of the first auction is

$$
\begin{aligned}
\pi(v_1, b) \quad = \quad & \int\limits_{v_2=\underline{v}}^{\bar{v}} \Bigg\{ \int\limits_{t=\underline{b}}^{b} \int\limits_{u=\underline{v}}^{s(v_1,v_2)} U(v_1 - b + s(v_1, v_2) - u) dH_1(u|t) dG^{N-1}(t) \\
& + U(v_1 - b) \int\limits_{t=\underline{b}}^{b} \int\limits_{u=s(v_1,v_2)}^{\bar{v}} dH_1(u|t) dG^{N-1}(t) \\
& + \int\limits_{t=b}^{\bar{b}} \int\limits_{u=\underline{v}}^{v_2} U(v_2 - u) dH_2(u|t) dG^{N-1}(t) \Bigg\} dF^2(v_2|v_1)
\end{aligned} \tag{9}
$$

All profits now show up inside the utility function $U(\cdot)$. The first expression inside the outer integral represents the case where a bidder wins both auctions, the second expression is the case of winning only the first auction, and the third expression is the case of winning only the second auction.

Again, taking a derivative of $\pi(v_1, b)$ with respect to b yields the first-order condition for bidding. The mathematical expression is more complex in the risk averse case:

$$
\begin{aligned}
\frac{G(b)}{(N-1)g(b)} \quad = \quad & \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(v_1,v_2)} U(v_1 - b + s(v_1, v_2) - u) dH_1(u|b) \\
& + \int_{u=s(v_1,v_2)}^{\bar{v}} U(v_1 - b) dH_1(u|b) - \int_{u=\underline{v}}^{v_2} U(v_2 - u) dH_2(u|b) \} dF^2(v_2|v_1) / \\
& \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(v_1,v_2)} U'(v_1 - b + s(v_1, v_2) - u) dH_1(u|t \le b) \\
& + \int_{u=s(v_1,v_2)}^{\bar{v}} U'(v_1 - b) dH_1(u|t \le b) \} dF^2(v_2|v_1)
\end{aligned} \tag{10}
$$

What is the effect of risk aversion on first auction revenue compared to the risk neutral case? No general statement can be made, as there are two opposing forces. On the one hand, risk aversion pushes bidders to bid more in a first-price auction, as they want to buy insurance against the possibility of losing. On the other hand, uncertainty regarding v_2 at the time of the first auction means the second auction is like a lottery, of which the certainty equivalent decreases as bidders grow more risk averse. This would push risk averse bidders to bid less. The sign of the ultimate effect depends on the value distributions, the synergy function, and the amount of risk aversion.

4 Identification

In this section, I show that the model primitives, meaning the value distributions $F^1(\cdot)$, $F^2(\cdot|\cdot)$, and the synergy function $s(\cdot, \cdot)$, are identified from the observable data, which are the joint distribution of first auction bids and second auction prices, along with bidder identities. The key idea behind this identification result is as follows: suppose we observe two ex-ante symmetric bidders submit identical bids in the first auction, but one of them wins and the other loses. The fact that they bid the same means they had the same v_1 when they started. If winning the first auction has no effect on bidders' values for the second item, the winner should behave no differently from the loser in the second auction. By comparing the behavior of the winner and the loser, we can measure the synergy that comes from having two adjacent tracts.

As in the previous section, I begin with the case of risk-neutral, symmetric bidders, and then extend to asymmetry and risk aversion. I abstract away from auction-specific heterogeneity, the discussion of which is deferred to section 5.2.

4.1 Model restrictions

Before delving into whether the model is identified, it is useful to first consider what restrictions the model places on the data. The data is rationalized by the model if there exists a structure $[F^1(\cdot),\ F^2(\cdot|\cdot),\ s(\cdot,\cdot)]$ that yields the observed distribution of first auction bids, second auction prices, and bidder identities in equilibrium. Throughout the paper, b refers to sealed bids in the first auction. Only transaction prices are observed in the second auction, which is an English auction.

Proposition 6. *The observed distribution of first auction bids, second auction prices, and bidder identities are rationalized by the model if and only if there exist [$F^1(\cdot)$, $F^2(\cdot|\cdot)$, $s(\cdot,\cdot)$] satisfying assumptions AS2-AS6 such that:*

R1 $G(b_1,...,b_N) = \Pi_{i=1}^N G(b_i)$

R2 $\xi(b)$ as defined in (7) is strictly increasing in b.

R3 Conditional on all the first auction bids including the highest bid b_{w1} and the bid b_{w2} of the bidder who won the second auction, the probability that the second auction price is $\leq p$

... and the same winner wins both auctions is:

$$(1 - D(p|b_{w1}))\prod_{j\neq w1} F^2(p|b_j) + \int_{u=\underline{v}}^{p} \prod_{j\neq w1} F^2(u|b_j) dD(u|b_{w1})$$

... and bidder $i \neq w1$ wins the second auction is:

$$(1 - F^2(p|b_i))D(p|b_{w1})\prod_{j\neq i,w1} F^2(p|b_j) + \int_{u=\underline{v}}^{p} D(u|b_{w1})\prod_{j\neq i,w1} F^2(u|b_j) dF^2(u|b_i)$$

where $D(x|v_1) \equiv prob(s(v_1, v_2) \leq x|v_1)$.

Restrictions R1 and R2 are concerned with the first auction. R1 comes from the independent private values paradigm with symmetric bidders, and R2 comes from monotonic bidding. As pointed out in section 3.4, the first auction is observationally equivalent to a stand-alone first-price auction where bidders have values $T(v_1)$. Hence, these conditions for rationalizing the first auction data are not different from those listed in Guerre et al. (2000) for first-price auctions.

Restriction R3 is concerned with the second auction given the first auction, and this is where sequentiality comes into play. It states the probability of each event described, given the model. One easily graspable restriction on the data that comes from R3 and assumptions AS2-AS6 is that the probability of winning the second auction should be nondecreasing in a bidder's first auction bid b. This comes from the fact that a higher first auction bid means a higher v_1, and a higher v_1 leads to a stochastically dominant $F^2(v_2|v_1)$. Furthermore, the bidder with the highest b additionally benefits from potential synergy.

It is worth pointing out the ways in which the model does *not* restrict the data. Simulations show that the model does not restrict the revenue in the first auction to be higher than the second auction or vice versa, even if synergy is strictly positive. The intuition behind this is as follows: on the one hand, anticipating the benefits of synergy for the second auction leads to more aggressive bidding in the first auction; on the other hand, the winner of the first auction bidding in light of the synergy he has secured leads to higher prices in the second auction. The direction of the revenue relationship depends on the shape of the value distributions and the size of synergy. This is in line with the theory of Sørensen (2006), who finds that prices need not decrease in sequential second-price auctions of stochastically equivalent complementary objects. This is different from Branco (1997) and Menezes and Monteiro (2003), who consider different models of complementary objects with identical values and find that expected prices decline in the sequence.

Likewise, no general statement can be made about whether second auction prices are lower when the same winner wins versus when different winners win, even if synergy is strictly positive. Simulations show that depending on what the value distributions are, the model can generate both outcomes.

4.2 Identification

Now, to establish identification, I need to show that there is a unique structure $[F^1(\cdot), F^2(\cdot|\cdot), s(\cdot,\cdot)]$ that rationalizes the data. The identification strategy starts by looking at the second auction, and then proceeds back to the first auction.

Proposition 7. *The value distributions involved in the second auction, $F^2(\cdot|b)$ and $D(\cdot|b)$ are* *identified from the observables, which are all the bids in the first auction and the transaction price in the second auction, along with bidder identities.*

The identification argument, presented in the appendix, is based on Athey and Haile (2002). In their Theorem 2, Athey and Haile (2002) show that the value distributions of asymmetric IPV bidders are identified from transaction prices and winner identities. When it comes to the second auction in our model, the first auction induces asymmetry between bidders that were ex-ante symmetric. Specifically, the winner $w1$ of the first auction draws his value from $D(\cdot|b_{w1})$, and each loser i from the first auction draws from $F^2(\cdot|b_i)$. For a fixed set of first-auction bids $\{b_i\}$, we can apply Theorem 2 of Athey and Haile (2002), so each of these distributions is identified from transaction prices and winner identities in the second auction.

Having identified $F^2(\cdot|b)$ and $D(\cdot|b)$, the synergy function $s(\cdot,\cdot)$ is also identified. As mentioned at the beginning of the identification section, the intuition is to compare how a first-auction winner and first-auction loser behave differently in the second auction when they are otherwise identical, even to the point of having the same v_1. We can do just this by comparing $F^2(\cdot|b)$ and $D(\cdot|b)$; by conditioning on $b(v_1)$, we compare two bidders who only differ in that one of them won the first auction while the other did not. Therefore, the difference between $F^2(\cdot|b)$ and $D(\cdot|b)$ can be attributed to synergy. More precisely, recall that $F^2(\cdot|b)$ is the distribution of $v_2|b$ and $D(\cdot|b)$ is the distribution of $s(b,v_2)|b$. Since $s(b,v_2)$ is monotonically increasing in v_2, $s(b,\cdot)$ must map the α-quantile of $F^2(\cdot|b)$ to the α-quantile of $D(\cdot|b)$. Since $F^2(\cdot|b)$ and $D(\cdot|b)$ are identified, this mapping provides for nonparametric identification of $s(\cdot,\cdot)$. Figure 2 illustrates the idea graphically.

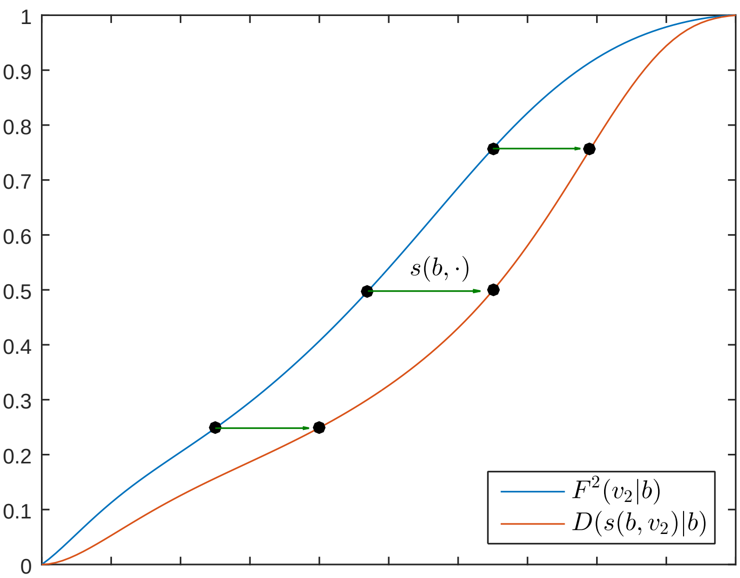
Finally, having identified $F^2(\cdot|b)$ and $s(b,\cdot)$, $F^1(\cdot)$ can be identified using the first-order condition for bidding in the first auction.

Proposition 8. *(i) If $F^2(\cdot|b)$ and $D(\cdot|b)$ are known, the synergy function $s(b,\cdot)$ is nonparametrically identified. (ii) Then, using $F^2(\cdot|b)$ and $s(b,\cdot)$, $F^1(\cdot)$ is identified nonparametrically from bids in the first auction.*

Once we identify $F^1(\cdot)$, we can tie the remaining loose ends: with some abuse of notation[8], $F^2(v_2|v_1(\alpha)) = F^2(v_2|b(v_1(\alpha))) = F^2(v_2|b(\alpha))$, and $s(v_1(\alpha),v_2) = s(b(v_1(\alpha)),v_2) = s(b(\alpha),v_2)$. Now all the primitives of the model, $F^1(\cdot)$, $F^2(\cdot|\cdot)$, and $s(\cdot,\cdot)$, are identified.

[8] To be notationally correct, I should write something like $F^2(v_2|v_1(\alpha)) = \tilde{F}^2(v_2|b(v_1(\alpha))) = \tilde{F}^2(v_2|b(\alpha))$. However, I abstract from notational correctness to avoid introducing more notation that is not central to the paper.

Figure 2: Nonparametric identification of $s(b, \cdot)$



4.3 Identification with asymmetric bidders

The model of sequential auctions with synergy is identified even if bidders are asymmetric. Now we must keep track of the subgroup of the first auction winner, as synergy may manifest itself differently depending on the subgroup of the bidder. The main idea for identification is to split the sample into subsamples depending on who won the first auction; for instance, if there are two subgroups of bidders, there would be one subsample of cases where subgroup 1 won the first auction, and another subsample where subgroup 2 won the first auction. Then $D_1(v_2|b)$ is identified from subsample 1, and $D_2(v_2|b)$ is identified from subsample 2. Of course, these subsamples are not random; by definition there is selection on the first auction bids. However, since the value distributions being identified from the subsamples are conditional on b (i.e. $F_m^2(\cdot|b)$ and $D_m(\cdot|b)$), that selection does not introduce problems.

Proposition 9. *The primitives of the asymmetric model, $F_m^1(\cdot)$, $F_m^2(\cdot|\cdot)$, and $s_m(\cdot,\cdot)$, are identified from the observables, which are all the bids in the first auction and the transaction price in the second auction, along with bidder identities.*

4.4 Identification with risk aversion

Is the model identified when bidders are risk averse? Propositions 7 and 8 apply even with risk averse bidders, since bidding strategies in the second auction, which is English, are unaffected by risk aversion. This means $F^2(\cdot|b)$ and $s(b, \cdot)$ are identified regardless of risk attitudes. $U(\cdot)$ and $F^1(v_1)$ remain to be identified.

Rewriting the first-order condition for risk averse bidders in (10), replacing $s(v_1, v_2)$ with $s(b, v_2)$ and $F^2(v_2|v_1)$ with $F^2(v_2|b)$, we get

$$\begin{aligned} \frac{G(b)}{(N-1)g(b)} = \; & \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(b,v_2)} U(v_1 - b + s(b, v_2) - u) dH_1(u|b) \\ & + \int_{u=s(b,v_2)}^{\bar{v}} U(v_1 - b) dH_1(u|b) - \int_{u=\underline{v}}^{v_2} U(v_2 - u) dH_2(u|b) \} dF^2(v_2|b) / \\ & \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(b,v_2)} U'(v_1 - b + s(b, v_2) - u) dH_1(u|t \leq b) \\ & + \int_{u=s(b,v_2)}^{\bar{v}} U'(v_1 - b) dH_1(u|t \leq b) \} dF^2(v_2|b) \end{aligned} \tag{11}$$

Every term in the right-hand side is observed or identified except for v_1 and $U(\cdot)$. And since $U'(\cdot) > 0$ and $U"(\cdot) \leq 0$ under risk aversion, the right-hand side is strictly increasing in v_1. This means that if we know $U(\cdot)$, we can use this FOC to uniqely back out the v_1 associated with any bid b. So the missing step is to identify $U(\cdot)$.

Appealing to ideas in Guerre et al. (2009), it is possible to identify $U(\cdot)$ if either the number of bidders varies exogenously, or if there is an instrument that affects the number of bidders but not the underlying private value distribution. I discuss each case in turn.

Exogenous participation

Suppose the number of bidders N varies exogenously in the data, such that $F^1(v_1; N') = F^1(v_1; N'')$, where $N' \neq N''$.[9] Let $\xi(b, U; N)$ represent the value of v_1 backed out from (11) as a function of b, $U(\cdot)$, and N. Then, the true $U(\cdot)$ must satisfy

$$\xi(b(\alpha|N'), U; N') = \xi(b(\alpha|N''), U; N'') \tag{12}$$

for all quantiles $\alpha \in [0, 1]$. These so-called compatibility conditions provide a basis for identifying $U(\cdot)$.

From (11), it appears that $\xi(\cdot)$ is a complicated function for which we do not have an explicit expression. As a result, it is difficult to provide a nonparametric identification strategy for $U(\cdot)$ the way Guerre, Perrigne, and Vuong (2009) did. I present a parametric alternative instead.

Suppose $U(\cdot)$ can be represented in parametric form, like the constant relative risk aversion utility $U(x) = x^{1-\rho}$. Then the compatibility conditions become a function of the single parameter ρ:

$$\xi(b(\alpha|N'), \rho; N') = \xi(b(\alpha|N''), \rho; N'') \tag{13}$$

[9]If there are covariates describing the auctioned object, then exogeneity here refers to exogenous variation of N conditional on observed covariates.

While I cannot show analytically that there is a unique ρ satisfying the compatibility conditions, it is possible to check this numerically as long as ρ is bounded. If numerical computations show that there is a unique best ρ in a bounded range known to contain the true parameter, then ρ is identified.

Endogenous participation

In real data settings, the number of bidders is often endogenously determined. For instance, there may be auction-specific unobserved heterogeneity u, of which higher realizations lead to more participation. Following Guerre et al. (2009), it is still possible to identify $U(\cdot)$ parametrically in this setting if the following conditions hold (assume items are observably homogeneous for ease of exposition):

1. There is an instrument x such that $N = N(x, u)$ and $F^1(v_1|x, u) = F^1(v_1|u)$
2. N is a sufficient statistic for u given x, e.g. $u = N - E[N|x]$

The second condition allows for recovery of u. After conditioning on u, all remaining variation in N comes from the instrument x, allowing us to return to the logic of the exogenous variation case. The compatibility conditions are the same as before except that they are now conditional on u:

$$\xi(b(\alpha|N', u), \rho; N', u) = \xi(b(\alpha|N'', u), \rho; N'', u) \tag{14}$$

for all quantiles $\alpha \in [0, 1]$. The risk aversion parameter is identified numerically if there is a unique value of ρ that best satisfies the compatibility condition.

5 Estimation

5.1 A multi-step estimation procedure

I develop a multi-step estimation procedure that closely follows the identification steps. Following the identification strategy in section 4.2, the first step of estimation is to estimate $D(\cdot|b)$ and $F^2(\cdot|b)$, which are the distributions of second auction values for the first-auction winner and first-auction loser, respectively, given the first-auction bids. For this task I propose a sieve maximum likelihood estimator using Bernstein polynomial bases, similar to the one used in Kong (2015). General properties of sieve maximum likelihood estimators are discussed in Chen (2008), and Komarova (2013) illustrates the use of Bernstein polynomials in sieve estimation.

In the second auction, we observe for each item the transaction price p, the identity of the winner, and the identity and first-auction bids of all bidders in the related first auction.

Taking the case of $N = 2$ (two bidders in the first auction) as an expositional example, the likelihood of the second-auction price and winner given the first-auction data can be expressed as follows for each item.

If the first-auction winner wins the second auction:

$$L = (1 - D(p|b_{w1}))f^2(p|b_{l1})$$

If the first-auction loser wins the second auction:

$$L = (1 - F^2(p|b_{l1}))d(p|b_{w1})$$

where d, f^2 are the derivatives with respect to the first argument of D, F^2 respectively; and b_{w1}, b_{l1} are the first-auction bids of the first-auction winner and loser, respectively. The log-likelihood of the observed second-auction data is then

$$\mathscr{L} = \sum_i log(L_i)$$

Now, to use sieve estimation, $D(\cdot|\cdot)$ and $F^2(\cdot|\cdot)$ can be approximated with Bernstein polynomials. Specifically, $D(v|b)$ and $F^2(v|b)$ can be approximated by bivariate Bernstein polynomials of the form

$$B(v,b) \equiv \sum_{i=0}^{m}\sum_{j=0}^{n} \gamma_{i,j} \begin{pmatrix} m \\ i \end{pmatrix} v^i(1-v)^{m-i} \begin{pmatrix} n \\ j \end{pmatrix} b^j(1-b)^{n-j} \tag{15}$$

where m and n are the polynomial degrees for v and b, respectively. This approximation does place a restriction that D and F^2 be continuous in b. Finally, D and F^2 are estimated by finding the polynomial parameters γ that maximize $\mathscr{L}$.

A benefit of using Bernstein polynomials is that they are easy to restrict to satisfy required properties. Since D and F^2 are cdf's, I restrict $B(v,b)$ to be weakly increasing in v by applying the restriction $\gamma_{i,j} \leq \gamma_{i',j}$ if $i < i'$. I also impose $\gamma_{0,0} = 0$ (i.e. $F^2(\underline{v}) = 0$) and $\gamma_{m,n} = 1$ (i.e. $F^2(\bar{v}) = 1$). Additionally, I restrict $D(v|b) \leq F^2(v|b)$, in keeping with the assumption $s(v_1, v_2) \geq v_2$ (AS5).[10]

The second step of the estimation procedure is to estimate $s(b, v_2)$. As the proof of identification for $s(b, v_2)$ is constructive, we can use it directly as an estimator as follows. In the identification section, we said that for a fixed b, $s(b,\cdot)$ maps the α-quantile of $F^2(\cdot|b)$ to the α-quantile of $D(\cdot|b)$, because $s(b, v_2)$ is monotonic in v_2. Therefore, given $\hat{F}(\cdot|\cdot)$ and $\hat{D}(\cdot|\cdot)$ from the first step of the estimation procedure, we obtain $\hat{s}(b,\cdot)$ nonparametrically

[10] I first perform the estimation without this restriction to see whether $s(v_1, v_2) \geq v_2$ generally holds, and estimate with the restriction only after confirming this. The restriction is placed to maintain consistency with the model in subsequent computations.

as the function that maps $\hat{F}^{2,-1}(\alpha|b) \rightarrow \hat{D}^{-1}(\alpha|b)$ for every quantile α on a grid over [0,1]. Since we can repeat this procedure for any b we choose, we have an estimator for $\hat{s}(\cdot,\cdot)$.

The third step of the estimation procedure is to estimate $F^1(\cdot)$, the distribution of v_1, using the inverse bid function $\xi(b)$ derived in (7):

$$\hat{v}_1 = \hat{\xi}(b) \equiv b + \frac{\hat{G}(b)}{(N-1)\hat{g}(b)} - \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{u=\underline{v}}^{\hat{s}(b,v_2)} \hat{H}_1(u|b)du - \int_{u=\underline{v}}^{v_2} \hat{H}_2(u|b)du \right\} d\hat{F}^2(v_2|b)$$

The cdf and pdf of first-auction bids $\hat{G}(\cdot)$ and $\hat{g}(\cdot)$ can be estimated from observed bids nonparametrically using a sieve estimator. $\hat{s}(\cdot,\cdot)$ and $\hat{F}^2(\cdot|\cdot)$ are known from estimation steps 1 and 2. $\hat{H}_1$ and $\hat{H}_2$, defined in (1) and (2), are functions of $\hat{F}^2(\cdot|\cdot)$ and $\hat{D}(\cdot|\cdot)$. Therefore, we are able to compute $\hat{\xi}(b)$. Since bids are monotonic in v_1, $F^{1,-1}(\alpha) \equiv v_1(\alpha) = \xi(b(\alpha))$ for any quantile α. Upon computing $\hat{\xi}(b(\alpha))$ for a grid of α over [0,1], we obtain $\hat{F}^1(\cdot)$ as the function that maps $\hat{\xi}(b(\alpha)) \rightarrow \alpha$.

If bidders are risk averse, $\hat{F}^1(\cdot)$ must be estimated conditional on a risk aversion parameter ρ, since the FOC for bidding depends on ρ. This ties into the fourth and final step of the estimation procedure.

The last step of the estimation procedure is to estimate the risk aversion parameter ρ. As discussed in section 4.4, ρ can be identified using the compatibility condition (13), $\xi(b(\alpha|N'),\rho;N') = \xi(b(\alpha|N''),\rho;N'')$. The idea is to find the ρ that satisfies this condition. For a given value of ρ, we can evaluate the condition by using the estimated inverse bid function $\hat{\xi}(\cdot)$ to compute the left-hand and right-hand sides of the equation. Specifically, $\hat{\xi}(b(\alpha|N'),\rho;N')$ takes the α-quantile of b conditional on the number of bidders $N = N'$ as an argument and returns the α-quantile of $\hat{v}_1$ conditional on $N = N'$. Likewise, $\hat{\xi}(b(\alpha|N''),\rho;N'')$ takes $b(\alpha|N'')$ as an argument and returns $\hat{v}(\alpha|N'')$. Given exogenous variation in the number of bidders, $\hat{v}(\alpha|N') = \hat{v}(\alpha|N'')$ should be true when computed using the true value of ρ. We can evaluate this condition for any value of ρ in an interval containing the true value, and find the ρ that satisfies is best, i.e.

$$\hat{\rho} = \arg\min_{\rho}(\hat{\xi}(b(\alpha|N'),\rho;N') - \hat{\xi}(b(\alpha|N''),\rho;N''))^2$$

If the number of bidders varies endogenously, we can use the modified compatibility condition as discussed in section 4.4. A choice remains of which quantile(s) α to evaluate the compatibility condition at. For instance, one could evaluate it at the median.

5.2 Auction heterogeneity

In the model and identification sections, auction-specific heterogeneity was suppressed for expositional ease. In the real data, there are characteristics z that differ across pairs, which must be accounted for in estimation.

Supposing we had a very large sample, the ideal way to deal with heterogeneity would be to estimate separate value distributions for every value of z. However, this approach is usually infeasible given the size of real datasets. As a result, a common approach in the empirical auction literature, as explained in Haile, Hong, and Shum (2003), has been to homogenize bids across auctions by "demeaning" them, i.e. transforming bids to residuals $\epsilon = b - z'\beta$ and working with the residuals in estimation. This allows one to "pool" all the data. The underlying assumptions are that $v = z'\beta + \mu$ (additive separability), and that the distribution of μ is invariant to z (homoskedasticity). Depending on the context, however, these assumptions may be quite strong.

In this paper, I need to homogenize bids in order to perform the first step of estimation, where I recover $F^2(\cdot|b)$ and $D(\cdot|b)$ using a sieve maximum likelihood estimator. As I do so, I seek to make the minimal assumptions that still allow me to pool heterogeneous objects for the estimation task at hand. Instead of transforming bids and prices to demeaned residuals, I transform bids and prices to quantiles conditional on $z'\beta$; that is, $b \to \tilde{b} \equiv G(b|z'\beta)$ and $p \to \tilde{p} \equiv J(p|z'\beta)$, where $G(\cdot)$ is the distribution of first auction bids and $J(\cdot)$ is the distribution of second auction prices. Both G and J are observed in the data. I then use these quantiles to perform the first step of estimation. Afterwards, the output from this step is transformed back to real values before proceeding with the other steps of estimation. Note that demeaning is a special case of taking quantiles; under assumptions of additive separability and homoskedasticity, the residuals $\epsilon = b - z'\beta$ map to quantiles of the bid distribution.

Although I would like to make as few assumptions as possible, this method is not without assumptions. The following assumptions underly the homogenizing procedure.

Assumptions

AS8 Single index assumption: $F^1(\cdot|z) = F^1(\cdot|z'\beta)$, $F^2(\cdot|\cdot, z) = F^2(\cdot|\cdot, z'\beta)$

Now define $\alpha_1 \equiv F^1(v_1|z'\beta)$, $\alpha_2 \equiv F^2(v_2|z'\beta)$, and $\alpha_s \equiv F^2(s(v_1, v_2)|z'\beta)$. Also define $\tilde{\alpha}_2 \equiv J(v_2|z'\beta)$ and $\tilde{\alpha}_s \equiv J(s(v_1, v_2)|z'\beta)$, where $J(\cdot)$ is the distribution of second-auction prices.

AS9 Quantile relationships are invariant to z:

1. $C(\alpha_1, \alpha_2|z) = C(\alpha_1, \alpha_2)$
2. $C(\alpha_1, \alpha_s|z) = C(\alpha_1, \alpha_s)$

In assumption AS9, the $C(\cdot, \cdot)$'s are copulas defined on $[0,1]^2 \to [0,1]$. AS9 says that the quantile of a bidder's v_1 implies a distribution for what the quantile of his v_2 will be, and that this quantile-to-quantile relationship, or copula, is invariant to $z'\beta$. A8.2 is the strongest part of the assumption, as it implicitly restricts the synergy function $s(\cdot, \cdot)$ to preserve a quantile relationship across different $z'\beta$. For instance, suppose that when $z'\beta = High$, synergy boosts a bidder's value for the second object from the 0.5-quantile to the 0.6-quantile for $z'\beta = High$. Then AS9.2 implies that synergy must also boost a bidder at the 0.5-quantile to the 0.6-quantile when $z'\beta = Low$.

Proposition 10. *Under assumptions AS8 and AS9, $C(\alpha_1, \tilde{\alpha}_2)$ and $C(\alpha_1, \tilde{\alpha}_s)$ are invariant to z.*

Given Proposition 10, it directly follows that the objects to be estimated - $F^2(\tilde{\alpha}_2|\alpha_1)$ and $D(\tilde{\alpha}_s|\alpha_1)$ - are invariant to $z'\beta$, since $F^2(\tilde{\alpha}_2|\alpha_1)$ is just a marginal of $C(\alpha_1, \tilde{\alpha}_2)$, for instance. Therefore, observations with different $z'\beta$ can be pooled in the first step of estimation once the bids and prices have been transformed to quantiles in this way.[11]

To provide a comparison, if we were to take the demeaning approach, we would need all of the assumptions made here and two more in addition: that v_1 and v_2 are additively separable functions of $z'\beta$ and a residual μ, and that these residuals have the same distribution regardless of $z'\beta$. The quantile approach, on the other hand, does not assume additive separability and allows marginal distributions to vary with $z'\beta$.

I restate that only the first step of estimation requires homogenized bids. After the first step, $\hat{F}^2(\tilde{\alpha}_2|\alpha_1)$ and $\hat{D}(\tilde{\alpha}_s|\alpha_1)$ are translated back to their real-valued versions $\hat{F}^2(v_2|v_1)$ and $\hat{D}(s(v_1, v_2)|v_1)$ before proceeding with the other steps of estimation.

5.3 Monte Carlo study

To evaluate the ability of the estimator to recover the synergy function, I simulate datasets of varying size and apply the estimator. The model underlying the simulated data is specified as follows:

[11] If, in addition, $F^2(\cdot|\cdot, z'\beta)$ does not vary with the number of bidders N, observations with different N can be pooled during the first step of estimation. In that case, b should be transformed to $\tilde{b} \equiv G(b|z'\beta, N)$, conditioning on N as well as $z'\beta$. This is necessary because, in a first-price auction, the bidding strategy changes when N changes. On the other hand, p should be transformed to $J(p|z'\beta, N = n)$, fixing n, regardless of the number of bidders. This is alright because, in the second auction which is English, the bidding strategy does not change with N. In fact, this is necessary for pooling; unlike $G(\cdot)$, $J(\cdot)$ is the distribution of the second highest out of N values, so the meaning of the statistic represented by J changes with N. If we transform each p to the quantile of $J(\cdot|z'\beta, N)$ for its own N, it would be like all the $\tilde{p}$'s are in different units, and it would make no sense to pool across N.

Table 5: Simulation sieve orders selected by AIC and BIC

	AIC		BIC	
sample size	m	n	m	n
250	4	2	3	2
500	5	2	3	2
1000	6	2	4	2

*m and n are defined in equation (15).

- $N = 2$
- $v_1 \sim U[0, 1]$
- $v_2 \sim \text{Triangular}(0, 1, v_1)$
- $s(v_1, v_2) = \min(v_2 + 0.1, 1)$

Triangular(0,1,v_1) is a triangular distribution with lower limit 0, upper limit 1, and peak at v_1. Synergy takes a simple form in which a constant 0.1 is added to v_2 up to the constraint that $v_2 \leq 1$.

Sieve orders (i.e. polynomial degrees) selected by the AIC (Akaike information criterion) and BIC (Bayesian information criterion) are displayed in Table 5. Although originally derived for parametric models in the asymptotic case, AIC and BIC are sometimes used to select sieve orders for sieve estimation, where established rules of thumb do not exist. Estimation results for 100 Monte Carlo runs are displayed in Figure 3 for both AIC and BIC. The figures depict estimated synergy functions against the known, true synergy function.

When the sample size is small, the estimated synergy function is biased upwards. This is because a small sample size leads to selection of small sieve orders, which may not control for v_1 tightly enough in estimates of $F^2(v_2|v_1)$ and $D(v_2|v_1)$. Since synergy is identified by comparing $F^2(v_2|v_1)$ and $D(v_2|v_1)$ for the same v_1, small sieve orders may allow the effects of affiliation to seep into the estimates of synergy, resulting in upward bias. However, as sample sizes and therefore sieve orders increase, the bias goes to zero, and the synergy function is estimated quite well.

AIC, which selects larger sieve orders than BIC, seems to result in smaller bias than BIC. It should be noted that the parameters being estimated are not free and independent. As the polynomials are approximating cdf's, all parameters are bounded by [0,1]. Furthermore, since we restrict F^2 and D to be nondecreasing in v_2, as all cdf's should be, $\gamma_{i,j}$ is bounded by [$\gamma_{i-1,j}$,1]. In light of this, BIC and even AIC may be penalizing the number of parameters more than is optimal.

In order to reduce the bias in smaller samples, I experiment with criteria that penalize the number of parameters less than do AIC and BIC. AIC seeks to minimize $2k - 2ln(L)$, and

Figure 3: Monte Carlo experiments using estimator with AIC (left) and BIC (right)

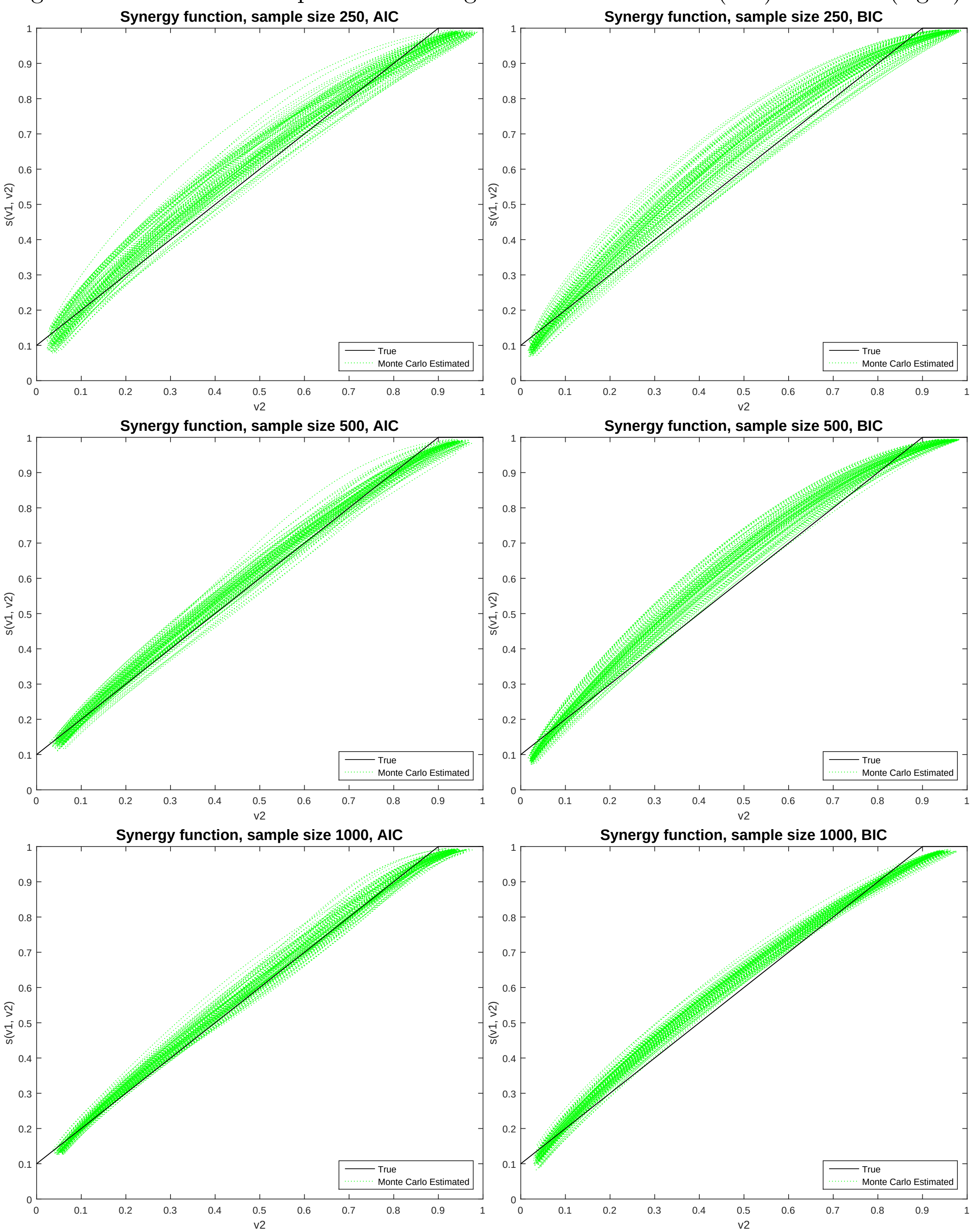


Table 6: Simulation sieve orders selected by IC3 and IC4

	IC3		IC4	
sample size	m	n	m	n
250	5	2	5	2
500	7	2	8	3
1000	7	3	10	3

*m and n are defined in equation (15).

BIC seeks to minimize $ln(n)k - ln(L)$, penalizing k by a multiple of 2 and $ln(n)$, respectively, where k is the number of estimated parameters, $ln(L)$ is the log likelihood of the data, and n is the sample size. I experiment with "IC3" and "IC4", which I define as IC3 $= k - 2ln(L)$ and IC4 $= 0.5k - 2ln(L)$. I repeat the simulations in Figure 3 with the newly defined criteria. Sieve orders selected by IC3 and IC4 are displayed in Table 6, and Monte Carlo results are displayed in Figure 4.

Compared to AIC and BIC, IC3 and IC4 seem to reduce bias at an acceptable cost to variance, and IC4 in turn seems favorable to IC3. I use the IC4 criterion to select sieve orders in my estimation.

6 Estimation using the paired leases in New Mexico

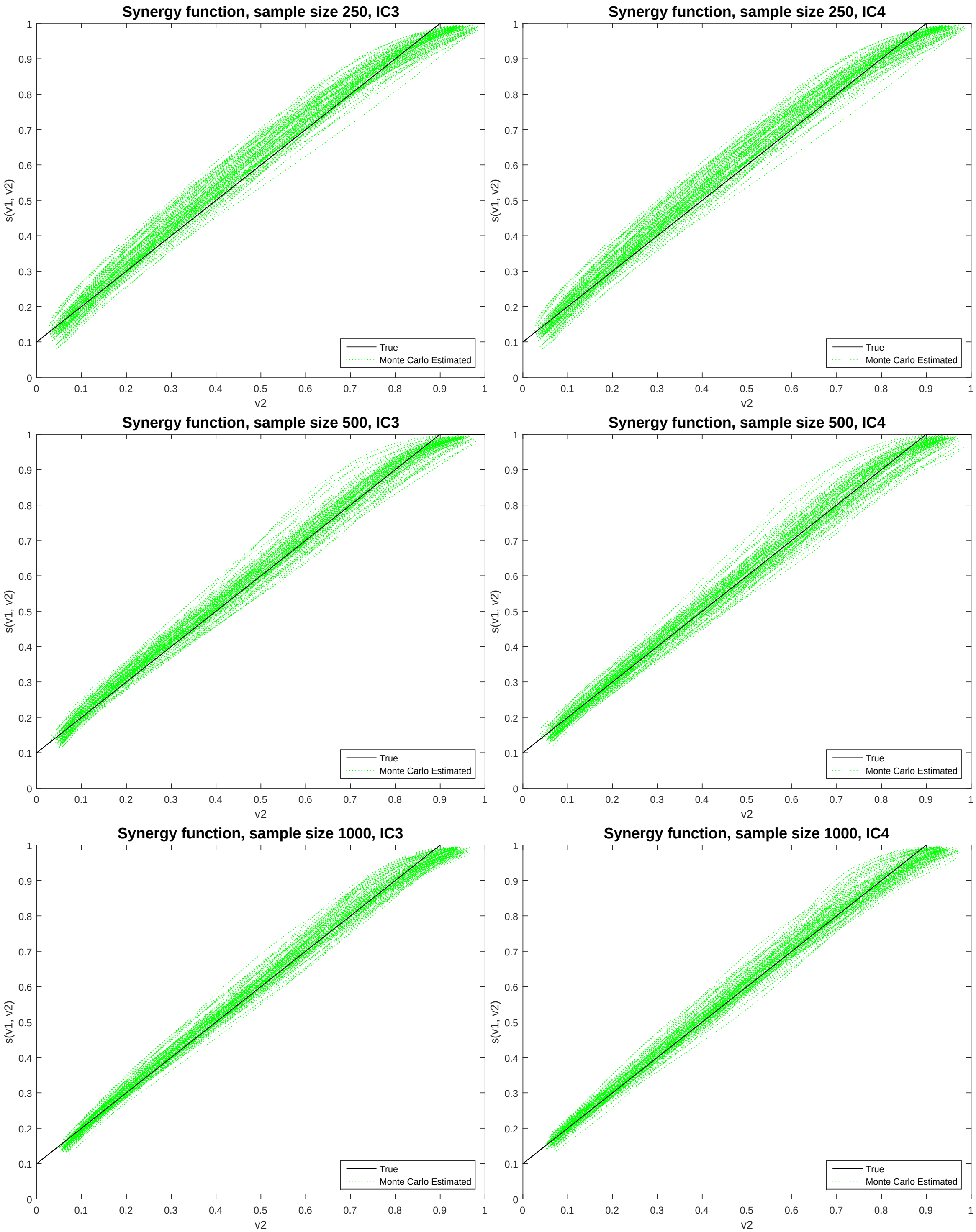
In light of the relatively small sample size (see Table 2), I estimate the model with symmetric bidders to reduce the burden on the estimator. Meanwhile, as Kong (2015) found risk aversion to be important in the New Mexico oil and gas lease auctions, I do allow bidders to be risk averse. Thus the primitives of the model are the v_1-distribution $F^1(\cdot)$, the conditional v_2-distribution $F^2(\cdot|\cdot)$, the synergy function $s(\cdot,\cdot)$, and the risk aversion parameter ρ. As the model assumes that the set of bidders participating in the first and second auction are the same, observations in which the second-auction winner did not bid in the first auction are dropped from the sieve maximum likelihood estimator. As shown in Table 2, 7% of all pairs with $N \geq 2$ fall into this category.

6.1 Covariates z

As discussed in section 5.2, lease characteristics z will be used to form a single index $z'\beta$ that controls for heterogeneity across pairs. This section explains the z's available in the data.

Observable characteristics of auctioned leases fall into three categories: lease terms (royalty rate, rental), time of auction (industry, economic, local conditions of that time), and location of the tract (encompassing geological features). The royalty rate is indicated by the lease prefix: VA (subregular), V0 (regular), or VB (premium). As the VA prefix was discon-

Figure 4: Monte Carlo experiments using estimator with IC3 and IC4



tinued in 2005, prefixes pre-2005 will be distinguished from prefixes post-2005. The rental rate (\$0.50 or \$1) is completely determined by whether the tract is located in a township[12] north (\$0.50) or south (\$1) of a horizontal geographic line; thus it is subsumed by the location variables. I choose year fixed effects to represent the time of auction, and supplement them with oil and gas prices.

Location contains important information and is observed to a very detailed level. To reduce this information to a smaller number of covariates while retaining flexibility, I first regress all submitted sealed bids on the lease term and time variables, along with township fixed effects, of which there are more than 200. I then sort the township coefficients into quartiles, and assign a dummy variable for each quartile. The townships in each quartile are mapped in Figure 5. Blank squares indicate townships with no data, and darker colors indicate higher quartiles. The map shows a pattern in which townships of the south-central area are highest value, and values decline as we move further out and away from this "center", in an almost concentric way.

To allow variation within township, I supplement the quartile dummies with a distance-to-center variable that is computed relative to own township: taking the average x-y coordinates of top-quartile townships as the "center" of high value, I compute how much farther each auctioned lease is from this "center" relative to the centroid of the township it is located in. It is expected that within each township, tracts that are located closer to the "center" will have relatively higher value. Also, this relative position within township is likely to matter only for townships within some radius of the center's influence, such as the townships in the upper quartiles. The location quartiles and distance-to-center variable are further supplemented with information on past drilling and production on the tract.

To ascertain which characteristics of the lease most affect its value to bidders, I regress the log of submitted sealed bids on these covariates. I also run a regression on lease fixed effects (fixed effects for each auction item) to assess explanatory power of the covariates by comparison. Table 7 shows the results.

The coefficient on the dummy variable for lease prefix VB is positive as expected, since the NMSLO assigns the VB prefix to premium tracts. Also as expected, the location quartile fixed effects are higher for higher quartiles, and being further away from the "center" leads to lower bids. Controlling for year fixed effects, gas prices seem to explain bids better than oil prices. The incremental explanatory power of the remaining variables on bids is negligible, perhaps because this information is subsumed in the lease prefixes, location variables, and year fixed-effects. As such, I take the covariates listed in column (3) to form the single index.

[12] A township is a 6 x 6 square mile plot of land in the Public Land Survey System (PLSS). It contains 36 sections.

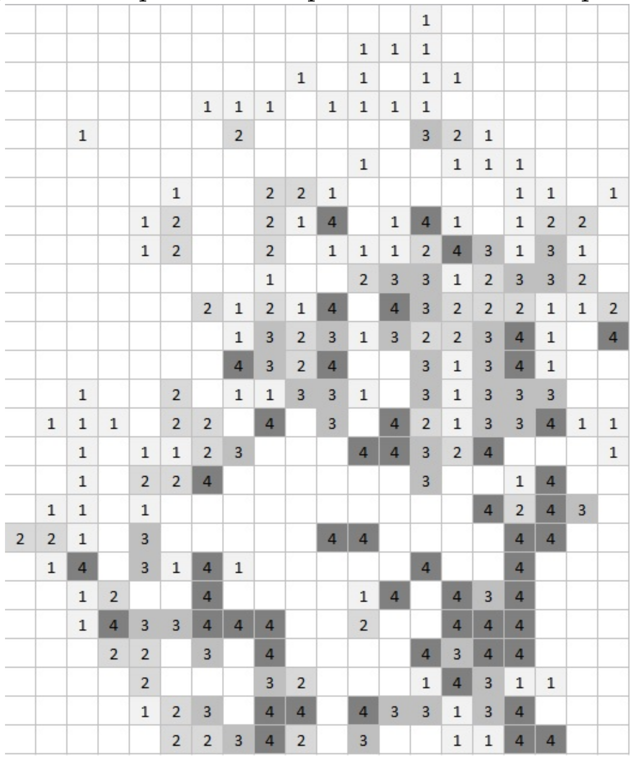
Table 7: Regression of ln(sealed bid) on observable characteristics

	(1)	(2)	(3)
lease prefix V0 pre-2005		0.087	0.099
		(0.073)	(0.073)
lease prefix VB pre-2005		0.215	0.226
		(0.254)	(0.254)
lease prefix VB post-2005		0.350***	0.368***
		(0.060)	(0.059)
location quartile 2		0.520***	0.517***
		(0.064)	(0.063)
location quartile 3		0.824***	0.826***
		(0.064)	(0.064)
location quartile 4		1.345***	1.343***
		(0.065)	(0.065)
relative dist. to center if upper qrtl		-0.037**	-0.036**
		(0.017)	(0.017)
nat gas 1 mo futures		0.045***	0.041***
		(0.015)	(0.014)
WTI oil price		-0.002	
		(0.003)	
drilled before		0.046	
		(0.046)	
log production 1970-auction date (boe)		0.007	
		(0.008)	
Constant	10.826***	9.253***	9.215***
	(0.499)	(0.148)	(0.137)
Lease fixed effects	Y	N	N
Year fixed effects	N	Y	Y
Observations	2090	2090	2090
R^2	0.561	0.316	0.315
Adjusted R^2	0.258	0.308	0.308

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Figure 5: Map of townships in each location quartile



6.2 Estimation of risk aversion

As explained in section 4.4, auction data at two different levels of N are needed to identify the risk aversion parameter. Since both winners and losers are needed in order to identify synergy, N must be at least 2. Table 1 shows that amongst $N \geq 2$, the sample size is largest at $N = 2$ and $N = 3$. Therefore, I use these two levels of N in my estimation, which gives a sample of roughly 400 pairs. Auctions with only one bidder or less are not useful, and auctions with $N \geq 4$ do not have enough observations to be usable.

As discussed in section 4.4, if N is endogenous we need an instrument(s) x that shifts N but satisfies $F^1(v_1|x) = F^1(v_1)$. Kong (2015) found that the amount of land offered for auction outside the Permian Basin qualifies as such an instrument. Bidders say that when there is a lot of acreage being auctioned on a single date, they have to filter down to the ones they can spend more time analyzing; therefore each individual item receives fewer bids. Meanwhile, the quantity of offerings outside the Permian Basin does not seem to affect values inside the Permian Basin.

To confirm this idea, regressions in Table 8 measure the effect of non-Permian acreage on the number and size of submitted sealed bids in our sample. Column (1) estimates a generalized linear model (GLM) where the dependent variable is the number of bidders N, and columns (2) and (3) are OLS regressions where the dependent variable is the log of submitted sealed bids (log deflated dollars). The effect of non-Permian acreage on N is significant and negative, supporting the anecdotal evidence that it leads to fewer bids on each item. Meanwhile, we may be concerned that increases in non-Permian supply also negatively affect bidders' values. If so, we would expect bid amounts to decrease in response to increases in non-Permian supply. However, columns (2) and (3) do not detect a significant negative effect, suggesting that non-Permian supply does not exert negative pressure on in-Permian values.

Also as discussed in section 4.4, we want to recover unobserved heterogeneity u in order to estimate risk aversion when N is endogenous. Under the assumption that N is a sufficient statistic for u given z and x, this involves computing $u = N - E[N|z,x]$.

The ideal way to compute $E[N|z,x]$ would be to do it nonparametrically. However, when z consists of many variables, or when the sample size is small, this can be impractical. As such, I take a parametric approach to computing $E[N|z,x]$, by estimating a generalized linear model where

$$N \sim Binomial\Big(\bar{N}, \frac{1}{1+e^{-(\gamma_0+\gamma_1 x+\gamma_2 z'\beta+\gamma_3(z'\beta)^2+\gamma_4(z'\beta)^3)}}\Big)$$

Squares and cubes of $z'\beta$ are included to allow for flexible forms of the relationship between $z'\beta$ and N. I let $\bar{N} = 24$, the maximum number of unique bidder names observed on a single auction date in the sample's time period. Then $E[N|z,x] = \bar{N}/[1 + e^{-(\gamma_0+\gamma_1 x+\gamma_2 z'\beta+\gamma_3(z'\beta)^2+\gamma_4(z'\beta)^3)}]$, and $u = N - E[N|z,x]$.

Once u is obtained in this way, the estimation steps described in section 5.1 can be performed conditional on (N,u). As conditioning on u nonparametrically is impractical given the sample size, I instead include unobserved heterogeneity u with the observable characteristics z to re-estimate the single index $\tilde{z}'\beta$, where $\tilde{z} \equiv [z\, u]$. This single index $\tilde{z}'\beta$ accounts for both observed and unobserved heterogeneity. I then perform all estimation steps using $\tilde{z}'\beta$ instead of $z'\beta$, and finally use the compatibility condition (13) to estimate ρ.

6.3 Empirical results

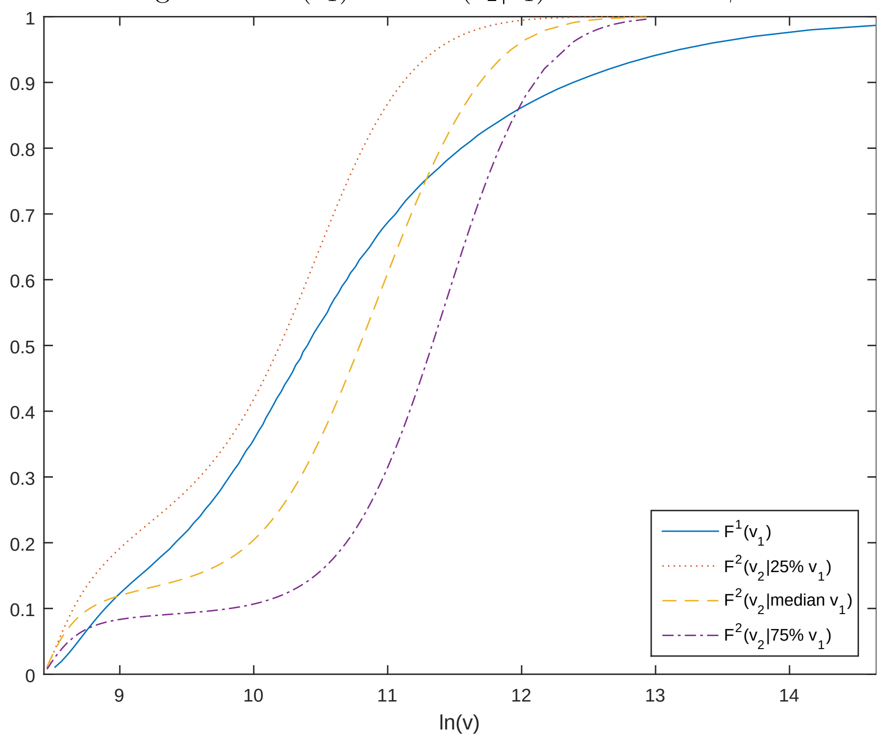
In this section I discuss the empirical results from applying the estimation procedure to the data. Figure 6 displays the estimated distributions $\hat{F}^1(v_1)$ and $\hat{F}^2(v_2|v_1)$. As expected, $\hat{F}^1$ is more dispersed than each $\hat{F}^2(\cdot|v_1)$. $\hat{F}^2(\cdot|v_1)$ is still quite dispersed; the interquartile range of $\hat{F}^2(\cdot|\text{median } v_1)$ is roughly 1 in logs. There are a number of possible explanations for dispersion in v_2 even after conditioning on v_1. As mentioned before, the outcomes of auctions that take place between A1 and A2 are one explanation. Bidders' own auction outcomes may affect v_2 through budget constraints or not wanting to "go home with nothing." Outcomes for competitors may also have an influence on v_2 for competitive reasons. Some sort of learning

Table 8: Effect of non-Permian acreage on N and sealed bids

	(1) GLM numbids	(2) OLS lnbid	(3) OLS lnbid
Non P. Basin acreage (1000s)	-0.007**	-0.001	-0.001
	(0.003)	(0.003)	(0.003)
lease prefix V0 pre-2005	0.138*	0.092	0.081
	(0.081)	(0.074)	(0.075)
lease prefix VB pre-2005	0.196	0.221	0.204
	(0.283)	(0.255)	(0.255)
lease prefix VB post-2005	0.098	0.350***	0.287***
	(0.068)	(0.060)	(0.062)
location quartile 2	0.196***	0.519***	0.490***
	(0.069)	(0.064)	(0.065)
location quartile 3	0.245***	0.823***	0.796***
	(0.070)	(0.064)	(0.065)
location quartile 4	0.420***	1.343***	1.275***
	(0.070)	(0.065)	(0.069)
relative dist. to center (mi) if upper qrtl	-0.032*	-0.037**	-0.033*
	(0.019)	(0.017)	(0.018)
nat gas 1 mo futures	-0.010	0.045***	0.048***
	(0.017)	(0.015)	(0.015)
WTI oil price	0.004	-0.002	-0.002
	(0.003)	(0.003)	(0.003)
drilled before	-0.000	0.046	0.047
	(0.051)	(0.046)	(0.046)
log production 1970-auction date (boe)	0.009	0.007	0.008
	(0.009)	(0.008)	(0.008)
Constant	-2.919***	9.262***	9.878***
	(0.160)	(0.149)	(0.204)
GLM detail	binomial with logit link		
Year fixed effects	Y	Y	Y
Number of bidders fixed effects	-	N	Y
Observations	864	2090	2090

GLM: generalized linear model
Standard errors in parentheses
* p<0.10, ** p<0.05, *** p<0.01

Figure 6: $\hat{F}^1(v_1)$ and $\hat{F}^2(v_2|v_1)$ at median $z'\beta$



may be taking place as well. On the other hand, it may be that $\hat{F}^2$ is more dispersed than the true F^2 due to approximations made in the estimated model. For instance, I estimate a model of symmetric bidders, but real bidders are not truly symmetric. Also, if the true F^2 is conditional on a second signal besides v_1, not accounting for that signal would result in a more dispersed $\hat{F}^2$.

The $\hat{F}^2(\cdot|v_1)$ in Figure 6 are stochastically ordered in v_1, indicating that v_1,v_2 are affiliated. Figure 7 visualizes the affiliation of v_1 and v_2 by plotting the conditional density $\hat{f}^2(v_2|v_1)$. Lighter colors indicate higher densities, and the tail ends have been trimmed to accentuate the shift of the central mode. We can see that as v_1 increases, so does the modal value of $v_2|v_1$.

Figure 8 plots the estimated synergy function. The estimator measures positive synergy, as $s(v_1, v_2) > v_2$. For median $z'\beta$ and median v_1, the added benefit of synergy, i.e. $s(v_1, v_2) - v_2$, is estimated to be on the order of \$14,000. To put this in context, the median value of v_2 conditional on median $z'\beta$ and median v_1 is roughly \$50,000.

In section 4.4 we saw that the CRRA parameter ρ is identified if there is a unique ρ that satisfies the compatibility condition $\xi(b(\alpha|N = 2), \rho; N = 2, \tilde{z}'\beta) = \xi(b(\alpha|N = 3), \rho; N = 3, \tilde{z}'\beta)$. Figure 9 displays the squared error $[\xi(b(\alpha|N = 2), \rho; N = 2, \tilde{z}'\beta) - \xi(b(\alpha|N = 3), \rho; N = 3, \tilde{z}'\beta)]^2$ evaluated at $\alpha = 0.5$ (median) as a function of ρ. ρ appears to be identified, as there is a unique value that satisfies the compatibility condition; $\hat{\rho} = 0.58$. As

Figure 7: $\hat{f}^2(v_2|v_1)$ at median $z'\beta$

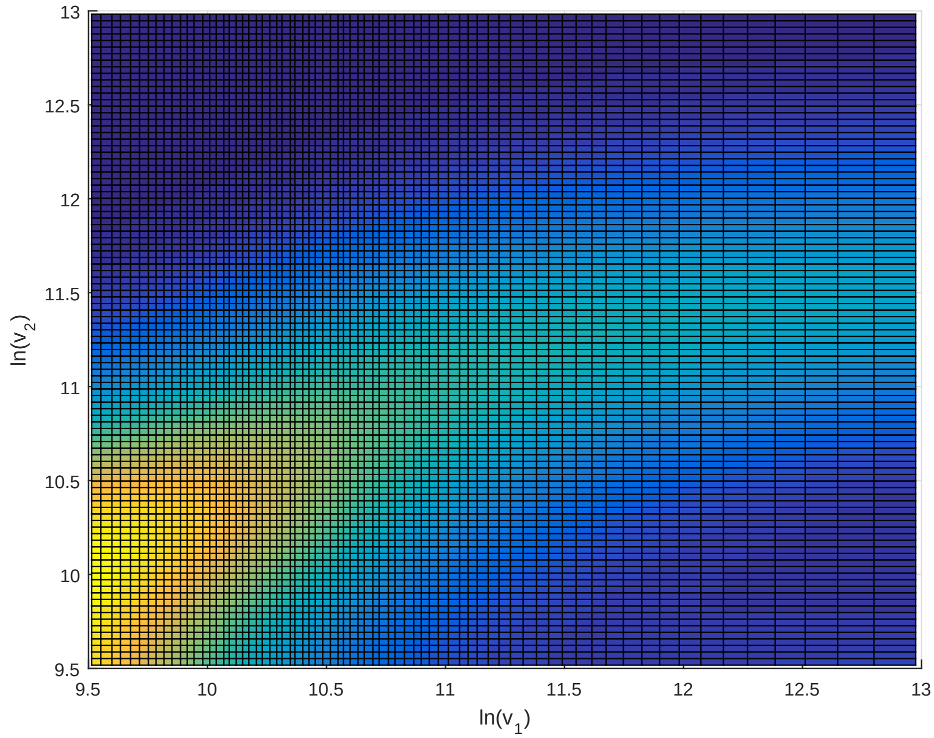


Figure 8: $\hat{s}(\text{median}\, v_1, v_2)$ at median $z'\beta$

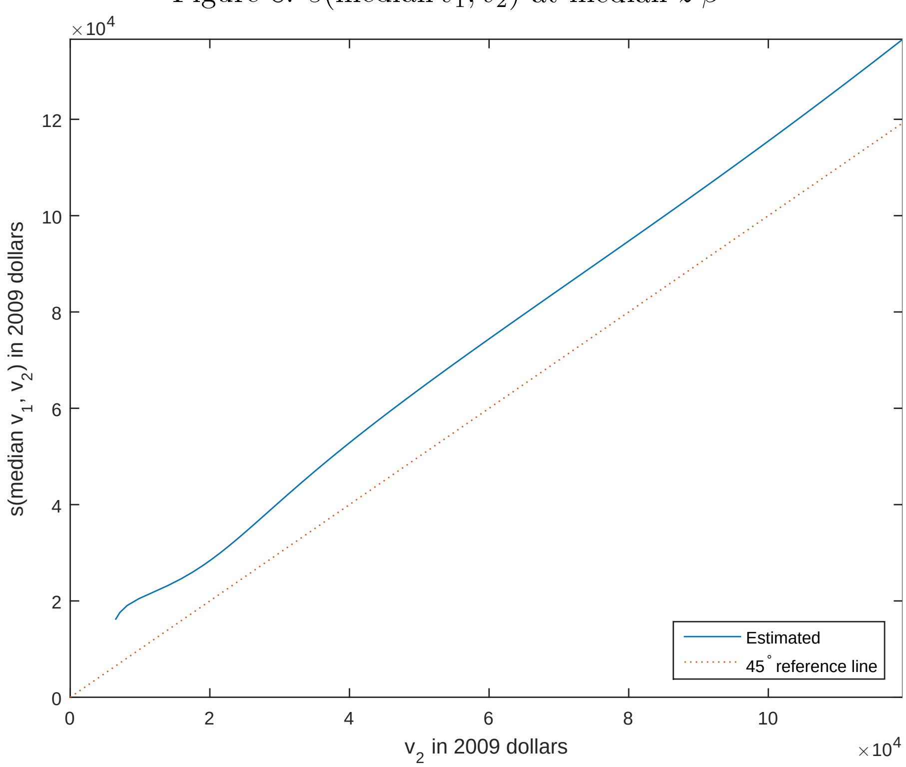
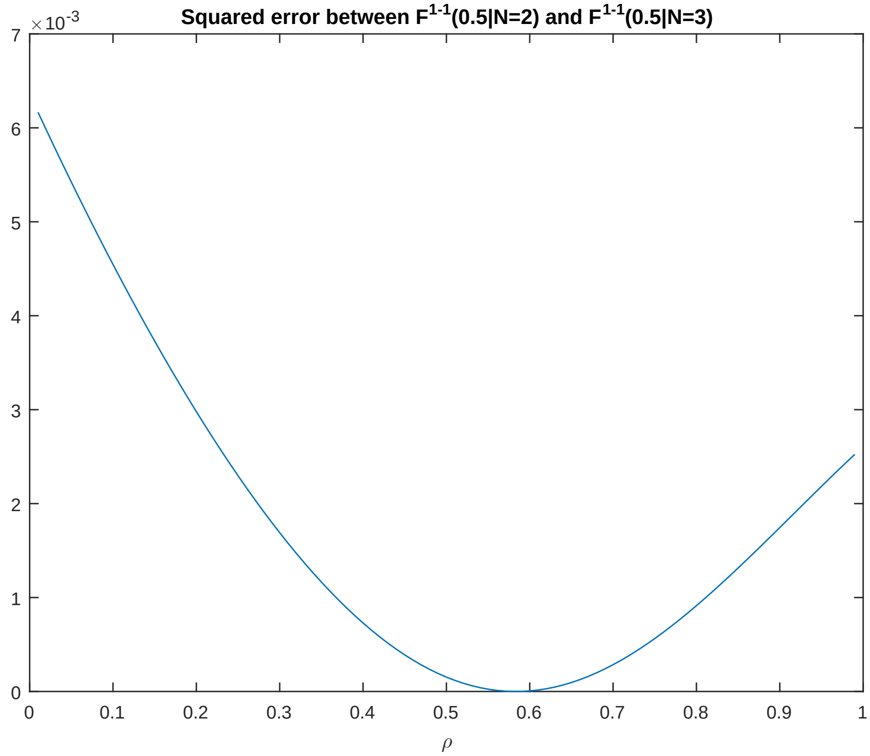


Figure 9: Squared error of compatibility condition, as a function of ρ



a comparison, Kong (2015) estimates CRRA parameters of 0.49 and 0.23 for two subgroups of bidders, Holt and Laury (2002) measure CRRA parameters centered around the 0.3-0.5 range in laboratory experiments, and Lu and Perrigne (2008) measure roughly 0.59 for the USFS timber auctions.

7 Counterfactuals

Given the structural estimates obtained, we can perform counterfactual simulations to understand the driving forces behind what we observe, and predict the outcome of alternative policies. Table 9 displays the results of counterfactual simulations performed with these objectives in mind.

The "observed" row shows what is observed in the data for pairs with $N = 2$ at median $z'\beta$.[13] Under the "simulated" heading, row (1) displays the expected revenue simulated using the full model; it is the simulated analog of the "observed" row. Subsequent rows show what revenue would be if selected elements of the full model were shut down. Columns (d)-(f) show counterfactual revenue when the pair is auctioned as a bundle, using the first-price sealed-bid format, the English auction format, and an even use of the two formats, respectively.

[13] Revenue "at" median $z'\beta$ is computed via kernel regression of revenue on $z'\beta$.

Table 9: Counterfactual revenues for a pair: median $z'\beta$, $N = 2$

		Sequential Revenue			Bundled Revenue			
	% same winner	(a) A1	(b) A2	(c) Total	(d) FPSB	(e) Eng	(f) $\frac{(d)+(e)}{2}$	(g) $\frac{(f)-(c)}{(c)}$
Observed	78%*	48,269	39,200	87469	-	-	-	-
Simulated								
(1) S + RA + A	77%	54,875	34,841	89,716	121,163	83,847	102,505	14%
(2) RA + A	71%	50,035	33,015	83,050	107,015	70,995	89,005	7%
(3) A	71%	31,078	33,015	64,093	70,609	70,995	70,802	10%
(4) S + RA	58%	54,074	37,348	91,422	128,219	92,830	110,525	21%
(5) S + A	77%	45,452	34,841	80,293	83,231	83,847	83,539	4%

*Excluding cases where A2-winner did not bid in A1, as these cases were not used in estimation.

"S" = synergy

"A" = affiliation

"RA" = risk aversion

Keeping in mind that first-price auctions yield higher revenue than English auctions when bidders are risk averse, column (f) is useful because it provides a fairer comparison with the sequential auctions, which use both formats. In addition, the State Land Office may have institutional reasons for using both formats, and column (f) respects that constraint. In each of these bundled auction simulations, I assume that v_1,v_2 are both known at the time of bidding, in order to provide a fair comparison with not bundling. The total value of the bundle is $v_1 + s(v_1, v_2)$.

Comparing the "observed" row to row (1) gives us a sense of model fit. The probability that the same bidder wins both tracts is 78% in the data and 77% when model-simulated, so the model fits that aspect of the data very well. Total revenue from the two auctions is $87,469 in the data and $89,716 as simulated by the model. Simulated revenues for the first auction and second auction separately do not fit the data as closely, though they are not far off. This may be due to the model assuming bidders are symmetric when there are some asymmetries in reality. Also, keep in mind that the observed revenue "at" median $z'\beta$ is itself an estimate obtained via kernel regression of observed auction revenue on $z'\beta$.

We observed in Table 2 that the A1-winner is more likely than other bidders to win A2, but until now we were unable to assess whether this was due to synergy or affiliation. Comparing rows (1) and (2) reveals that if synergy were eliminated, the proportion of cases in which the same bidder wins both tracts would drop from 77% to 71%. On the other hand, row (4) shows that if v_1,v_2 were not affiliated, that percentage would drop to 58%. We can conclude that both synergy and affiliation are responsible for the same-winner %, but affiliation is the primary explanation. This highlights the importance of allowing for and distinguishing affiliation from synergy.

Another phenomenon we observe in the pairs data is that revenue is higher in the first auction. Comparing rows (1)-(3) helps us understand the forces behind that observation. First, looking at row (1) relative to row (2), synergy seems to increase revenue in both auctions, but increases A1 revenue more, playing a part in the A1-A2 revenue difference. But second, comparing rows (2) and (3) reveals that the majority of the revenue gap is explained by risk aversion, which increases bidding strategies in A1 (first-price) but not in A2 (English). This is consistent with Kong (2015), which finds that risk aversion is primarily responsible for the dominance of first-price auctions over English auctions in the New Mexico setting overall.

Row (5) can be misleading as there is a substantial A1-A2 gap even without risk aversion. This is an artifact of A2 being a lottery to the bidder at the time of A1 (since v_2 is known only in distribution); for a given lottery, the certainty equivalent falls with risk aversion. Hence, when bidders are risk neutral the expected synergy is worth more than if they were risk averse. In this sense, the effective synergy in (5) is much larger than in (1). Row (5) should be interpreted with this in mind.

An obvious policy alternative in the presence of synergy would be to auction the pair as a bundle, as this guarantees that the winning bidder will realize synergy. A downside of bundling is that it forces a single bidder to take both tracts, even when the highest-value bidder for each tract is different. A general theoretical comparison of sequential versus bundled auctions that applies to this model does not exist. Ultimately, whether to bundle these tracts is an empirical question that depends on the primitives - including the size of synergy, the shape of the value distributions, the degree of affiliation, and degree of risk aversion - and their interaction. The size of synergy matters, because the computations of Subramaniam and Venkatesh (2009) suggest that the larger the synergy, the more likely that bundled auctions will increase revenue. Also, if synergy is large, a social planner may want to ensure that it is always realized (by using bundled auctions), while if it is small, he may prefer to award each tract separately to the highest-value bidder. Meanwhile, bundling tends to reduce heterogeneity in values (see Schmalensee (1984)) and more generally change the shape of the auction-relevant value distribution. How this will matter depends on the shape of the non-bundled value distributions and degree of affiliation between the two tracts. Risk aversion matters, because it affects bidding strategies in first-price auctions, and also because it changes the way bidders internalize the uncertainty surrounding the second auction and synergy when they bid in the first auction.

Comparing column (f) to column (c) indicates that bundling would increase auction revenue over sequential sales, assuming the State Land Office maintains its policy of using both the first-price sealed-bid and English auction formats evenly. Column (g) computes the percentage increase in auction revenue that would come from this bundling. Judging from

Table 10: Sequential versus bundled auctions, revenue and allocation

		Sequential	Bundled			
		(c)	(d) FPSB	(e) English	(f) $\frac{(d)+(e)}{2}$	(g) $\frac{(f)-(c)}{(c)}$
$N = 2$						
(1)	Revenue per pair	89,716	121,163	83,847	102,505	14%
(2)	Value of tracts to winner(s)	305,887	303,128			-1%
$N = 3$						
(3)	Revenue per pair	136,781	168,527	125,151	146,839	7%
(4)	Value of tracts to winner(s)	358,634	350,426			-2%

At median $z'\beta$

rows (1)-(5) of column (g), the benefit of bundling over not bundling seems to be greatest when both synergy and risk aversion are present. I conjecture that this is because bidders can depend on realizing synergy if they win the bundled auction - unlike in sequential auctions, where a bidder may lose the second auction even after winning the first - and this certainty is relatively more valuable when bidders are risk averse than when they are risk neutral.

Having used Table 9 to understand the forces at work, I focus exclusively on the question of whether to bundle in Table 10. Row (1) of Table 10 restates row (1) of Table 9. Meanwhile, since these auctions are run by a public institution, revenue considerations must be balanced against allocative efficiency, or the desire to award tracts to the firms that value them most. Row (2) addresses allocative efficiency by computing the total value derived from a pair of tracts by the winner(s). If a single bidder wins both - which is always the case for bundled auctions - this total value is inclusive of synergy. Row (2) shows that bundling leads to a small loss in this total value, of roughly 1%. One reason the loss is small is that, even in the sequential auctions currently being used, the same bidder often wins both tracts, leading to the same allocative outcome as bundled auctions. In the remaining cases where the allocative outcomes are different, cases that favor bundling and cases that favor sequential auctions seem to balance out. The gains to bundling come from synergy, and the losses come from not giving each tract to its respective highest-valuer. Now, I repeat this counterfactual exercise for three-bidder auctions, considering revenue and allocative efficiency in rows (3) and (4). As was the case for two-bidder auctions, bundling leads to higher revenue with a relatively small loss to allocative efficiency, though the revenue gains of bundling are smaller than in the two-bidder case.

As we saw in Table 1, a large majority of these auctions receive three bids or less. Relating to the bundling literature, the finding in Table 10 that bundling would be better in two-bidder and three-bidder auctions is consistent with the computations of Subramaniam and Venkatesh (2009), which suggest that the smaller the number of bidders, the more likely are bundled auctions to dominate sequential auctions in terms of revenue. This can be reversed for larger N, where it may be optimal to exploit competition twice by selling each tract separately. This helps explain why the revenue gains from bundling in row (3) are smaller than in row (1). The result is also generally consistent with papers that study bundling in contexts without synergy, such as Palfrey (1983) and Chakraborty (1999). Both of these papers find that the smaller the number of bidders, the more likely is bundling to increase revenue in Vickrey auctions.

8 Conclusion

This paper performs a structural analysis of two auctions that take place sequentially, are linked by synergy, and in which each bidder's values can be affiliated across auctions. It explains that ignoring affiliation can lead to falsely detecting synergy where none exists, and distinguishes synergy from affiliation in identifying and estimating the auction model. The model uses general functional forms for synergy and the joint distribution of v_1,v_2 while allowing for risk averse bidders. The paper establishes nonparametric identification of this model and develops a multi-step estimation procedure that recovers all model primitives. Applying the estimation method to oil and gas lease data, I find both synergy and affiliation between adjacent tracts. Affiliation is very important in explaining why the same bidder often wins both tracts. The model predicts, and counterfactual decomposition confirms, that synergy increases revenue in both auctions relative to the case of no synergy.

Meanwhile, bidders are risk averse, and this boosts first-auction revenue substantially, as the first auction is a first-price auction. Interestingly, it seems that the first-price sealed-bid auction - English auction sequence used in New Mexico strikes a balance between revenue and allocative efficiency. The first-price auction in the first stage takes advantage of higher bids generated by risk aversion, while in the second stage, where synergy induces endogenous asymmetry, the English auction maintains allocative efficiency. Counterfactual simulations reveal that bundled auctions would yield higher revenue, given the combination of synergy and risk aversion and the typically low number of bidders.

Groundwork for extending the procedure to asymmetric bidders is provided, but there are a number of issues to be considered when doing so. As the number of primitives multiplies, the sample size needs to grow. Stronger assumptions will be needed when homogenizing bid data. New channels of bias can arise when estimating asymmetric synergy functions with real data, and they should be assessed carefully.

The paper opens the door to analyzing synergy and affiliation in other types of sequential auctions. The main insight for distinguishing synergy from affiliation is adaptable to other auction formats, such as two second-price auctions, as long as first-auction bids are monotonic in values and observed. The equilibrium bidding strategy would have to be worked out separately for each type of sequence. Another very interesting possibility is that of extending the model to a longer sequence of affiliated items. Affiliation of a bidder's values across a longer sequence creates a challenge for analysis, but since bidders do not know future values ahead of time, the model retains hope of tractability, perhaps with the help of some well-placed assumptions. These questions remain open for future research.

Appendix

Evidence of synergy

In section 2.2, regression discontinuity results using local linear regression were shown for the most frequent bidder. Going down the list of bidders ordered by frequency of bids, the number of observations drops exponentially. For each of the remaining bidders, there were not enough observations near $z = 0$ to perform a meaningful regression discontinuity analysis. However, we can still examine some simple statistics for clues. For the 3 other bidders that had at least 5 observations on each side of $z = 0$ with $|z| < 0.2$, the following table shows the probability of winning the second auction when $z \in [-0.2, 0]$ (i.e. lost first auction with less than 20% bid difference) versus when $z \in [0, 0.2]$ (i.e. won first auction with less than 20% bid difference).

Table 11: Probability of winning the second auction, given $|z| < 0.2$

Bidder name:	CP	DG	DS
Lost first auction	22%	38%	0%
Won first auction	30%	57%	20%

Deriving the first-order condition in section 3.3

A bidder will bid the b that maximizes the expected profit $\pi(v_1, b)$. Taking the derivative of $\pi(v_1, b)$ with respect to b and setting it equal to zero gives

$$\begin{aligned}&\int_{v_2=\underline{v}}^{\bar{v}}\{\int_{t=\underline{b}}^{b}(-1)dG^{N-1}(t)\\&+[v_1 - b + \int_{u=\underline{v}}^{s(v_1,v_2)}(s(v_1,v_2)-u)dH_1(u|b)](N-1)G^{N-2}(b)g(b)\\&-\int_{u=\underline{v}}^{v_2}(v_2-u)dH_2(u|b)(N-1)G^{N-2}(b)g(b)\}dF^2(v_2|v_1) \quad = \quad 0\end{aligned}$$

This can be rewritten as

$$
\begin{aligned}
&G^{N-1}(b) \\
&+(N-1)G^{N-2}(b)g(b)\int_{v_2=\underline{v}}^{\bar{v}}\{v_1-b+\int_{u=\underline{v}}^{s(v_1,v_2)}(s(v_1,v_2)-u)dH_1(u|b) \\
&-\int_{u=\underline{v}}^{v_2}(v_2-u)dH_2(u|b)\}dF^2(v_2|v_1) \qquad = \quad 0
\end{aligned}
$$

Rearranging,

$$
\begin{aligned}
\frac{G(b)}{(N-1)g(b)} \quad = \quad & \int_{v_2=\underline{v}}^{\bar{v}}\{v_1-b+\int_{u=\underline{v}}^{s(v_1,v_2)}(s(v_1,v_2)-u)dH_1(u|b) \\
& -\int_{u=\underline{v}}^{v_2}(v_2-u)dH_2(u|b)\}dF^2(v_2|v_1)
\end{aligned}
$$

Some algebra using integration by parts shows that

$$
\int_{u=\underline{v}}^{s(v_1,v_2)}(s(v_1,v_2)-u)dH_1(u|b)=\int_{u=\underline{v}}^{s(v_1,v_2)}H_1(u|b)du
$$

$$
\int_{u=\underline{v}}^{v_2}(v_2-u)dH_2(u|b)=\int_{u=\underline{v}}^{v_2}H_2(u|b)du
$$

So the first-order condition can be simplified to

$$
b=v_1+\int_{v_2=\underline{v}}^{\bar{v}}\{\int_{u=\underline{v}}^{s(v_1,v_2)}H_1(u|b)du-\int_{u=\underline{v}}^{v_2}H_2(u|b)du\}dF^2(v_2|v_1)-\frac{G(b)}{(N-1)g(b)}
$$

Proposition 1

Proof. $b' \in BR(v_1')$ means $\pi(v_1',b')-\pi(v_1',b)\geq 0$ and $b\in BR(v_1)$ means $0\geq\pi(v_1,b')-\pi(v_1,b)$. Defining $\kappa(v_1)\equiv\pi(v_1,b')-\pi(v_1,b)$, this means $\kappa(v_1')\geq\kappa(v_1)$.

Writing out $\kappa(v_1)$ gives the following expression:

$$
\begin{aligned}
\int_{v_2=\underline{v}}^{\bar{v}}\{ \quad & v_1[G^{N-1}(b')-G^{N-1}(b)]-b'G^{N-1}(b')+bG^{N-1}(b) \\
& +\int_{t=b}^{b'}\int_{u=\underline{v}}^{s(v_1,v_2)}(s(v_1,v_2)-u)dH_1(u|t)dG^{N-1}(t) \\
& -\int_{t=b}^{b'}\int_{u=\underline{v}}^{v_2}(v_2-u)dH_2(u|t)dG^{N-1}(t)\ \}dF^2(v_2|v_1)
\end{aligned}
$$

If I take a derivative with respect to v_2 of the expression inside the outer integral (i.e. inside the squiggly brackets), I get

$$
\begin{aligned}
\frac{\partial\{\}}{\partial v_2} \quad &= \quad \int_{t=b}^{b'}[\int_{u=\underline{v}}^{s(v_1,v_2)}\frac{\partial s(v_1,v_2)}{\partial v_2}dH_1(u|t)-\int_{u=\underline{v}}^{v_2}dH_2(u|t)]dG^{N-1}(t) \\
&= \quad \int_{t=b}^{b'}[\frac{\partial s(v_1,v_2)}{\partial v_2}H_1(s(v_1,v_2)|t)-H_2(v_2|t)]dG^{N-1}(t)
\end{aligned}
$$

Since $\frac{\partial s(v_1,v_2)}{\partial v_2} \geq \frac{H_2(v_2|t)}{H_1(s(v_1,v_2)|t)}$ by assumption AS6, $\frac{\partial s(v_1,v_2)}{\partial v_2} H_1(s(v_1,v_2)|t) - H_2(v_2|t) \geq 0$ and hence $\frac{\partial\{\}}{\partial v_2} \geq 0$.

The derivative with respect to v_1 of the expression inside the outer integral is

$$\frac{\partial\{\}}{\partial v_1} = G^{N-1}(b') - G^{N-1}(b) + \int_{t=b}^{b'} \int_{u=\underline{v}}^{s(v_1,v_2)} \frac{\partial s(v_1,v_2)}{\partial v_1} dH_1(u|t) dG^{N-1}(t)$$

Since $b' > b$ and $\frac{\partial s(v_1,v_2)}{\partial v_1} \geq 0$, $\frac{\partial\{\}}{\partial v_1} \geq 0$. Then, given $\frac{\partial\{\}}{\partial v_2} \geq 0$, $\frac{\partial\{\}}{\partial v_1} \geq 0$, and the stochastic ordering of $F^2(v_2|v_1)$ in v_1, we obtain that κ is nondecreasing in v_1. So if $v' < v$ and $\kappa(v') \geq \kappa(v)$, it must be that $\kappa(v') = \kappa(v)$.

Then $\pi(v', b) = \pi(v', b')$ and $\pi(v, b') = \pi(v, b)$, so $b \in BR(v')$ and $b' \in BR(v)$. □

Proposition 2

Proof. In the RHS of (3), take a derivative of the expression inside the squiggly brackets with respect to v_2: $\frac{\partial\{\}}{\partial v_2} = \frac{\partial s(v_1,v_2)}{\partial v_2} H_1(s(v_1,v_2)|b) - H_2(v_2|b)$. Since $\frac{\partial s(v_1,v_2)}{\partial v_2} \geq \frac{H_2(v_2|t)}{H_1(s(v_1,v_2)|t)}$ by assumption AS6, $\frac{\partial\{\}}{\partial v_2} \geq 0$. Meanwhile, $\frac{\partial\{\}}{\partial v_1} = \frac{\partial s(v_1,v_2)}{\partial v_1} H_1(s(v_1,v_2)|b) \geq 0$. Given $\frac{\partial\{\}}{\partial v_2} \geq 0$, $\frac{\partial\{\}}{\partial v_1} \geq 0$, and the stochastic ordering of $F^2(v_2|v_1)$ in v_1, we obtain that the RHS of (3) is strictly increasing in v_1. Hence, for any given bid b, there can only be one v such that $b \in BR(v)$. So two different values cannot share the same best response b.

By Proposition 1, if $v_1' < v_1$ with $b \in BR(v_1)$ and $b' \in BR(v_1')$ and $b' > b$, then v and v' share b and b' as best responses, violating what we have just shown. Hence it cannot be that $b' > b$. Neither can $b' = b$, since this means two different values share the same best response. So it must be that $b' < b$ if $v_1' < v_1$. The bid function $\beta(v_1)$ in the first auction is strictly increasing in v_1. □

Proposition 4

Proof. From Proposition 2, we have that $b(v_1)$ is monotonically increasing in v_1. Meanwhile, from (5) we know that the first auction is equivalent to a stand-alone auction in which bidders have value $T(v_1)$. If bidding strategies in this hypothetical stand-alone auction are represented by the function $B(\cdot)$, then $b(v_1) = B(T(v_1))$. Now, since $b(\cdot)$ is strictly increasing in v_1, then $B(T(\cdot))$ must also be strictly increasing in v_1.

Meanwhile, $B(\cdot)$ is the bid function for a standard stand-alone first-price auction; hence $B(\cdot)$ is strictly increasing in its argument. Then in order for $B(T(v_1))$ to be strictly increasing in v_1, $T(v_1)$ must also be strictly increasing in v_1. □

Proposition 5

Proof. First, consider an auction in which there is no synergy, i.e. $s(v_1, v_2)$. Then the first auction is a stand-alone auction in which bidders have values $v_1 \sim F^1(\cdot)$. From Riley and Samuelson (1981), we know that for this auction the bid function is

$$\beta(v_1) = v_1 - \frac{1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} F^1(s)^{N-1} ds$$

Using integration by parts, we can rewrite this as

$$\beta(v_1) = \frac{N-1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} sF^1(s)^{N-2} f^1(s) ds$$

Next, consider the first auction out of a sequential pair in which there is positive synergy, i.e. $s(v_1, v_2) > v_2$. Let $\tilde{F}(\cdot)$ be the distribution of $T(v_1)$, which is defined in (6). From the formulation in (5), we can think of this auction as a transformed stand-alone auction, so the bid function is

$$B(T(v_1)) = \frac{N-1}{\tilde{F}(T(v_1))^{N-1}} \int_{T(\underline{v})}^{T(v_1)} u\tilde{F}(u)^{N-2} \tilde{f}(u) du$$

Now, since $T(v_1)$ is strictly increasing in v_1, T is invertible. Also, $\tilde{F}(T(v_1)) = F^1(v_1)$. I now rewrite $B(T(v_1))$ using a change of variables $s = T^{-1}(u)$, $u = T(s)$ and plugging in $\tilde{F}(T(\cdot)) = F^1(\cdot)$.

$$B(T(v_1)) = \frac{N-1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} T(s)F^1(s)^{N-2} f(s) \frac{T'(s)}{T'(s)} ds = \frac{N-1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} T(s)F^1(s)^{N-2} f(s) ds$$

But notice that if $s(v_1, v_2) > v_2$, $T(v_1) > v_1$, hence

$$\begin{aligned} B(T(v_1)) &= \tfrac{N-1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} T(s)F^1(s)^{N-2} f(s) ds \\ &\geq \tfrac{N-1}{F^1(v_1)^{N-1}} \int_{\underline{v}}^{v_1} sF^1(s)^{N-2} f(s) ds \qquad = \beta(v_1) \end{aligned}$$

Therefore, revenue in the first auction is higher when synergy is positive. □

Proposition 7

Proof. For a fixed set of first auction bids $\{b_i\}$, values in the second auction are drawn from $D(\cdot|b_{w1})$ for the A1-winner $w1$, and from $F^2(\cdot|b_i)$ each loser $i \neq w1$. These draws are independent across bidders. Furthermore, by assumption AS4, all value distributions involved are continuous and have the same support. Hence, we can apply Theorem 2 of Athey and Haile (2002), which establishes identification of asymmetric value distributions from transaction prices and bidder identities. Theorem 3 of Athey and Haile (2002) extends this to auctions with auction-specific covariates. □

Proposition 8

Proof. By assumption AS6, $s(b, v_2)$ is strictly increasing in v_2. So if we define $v_2(\alpha|b) \equiv F^{2,-1}(\alpha|b)$, i.e. the α-quantile of v_2 conditional on b, then $s(b, v_2(\alpha|b))$ must be the α-quantile of s conditional on b, $D^{-1}(\alpha|b)$. That is, for any quantile α,

$$s(b, F^{2,-1}(\alpha|b)) = D^{-1}(\alpha|b)$$

Since b is observed and $F^2(\cdot|b)$ and $D(\cdot|b)$ are identified from Proposition 7, we know the function $s(\cdot, \cdot)$. This proves part (i).

Next, consider (7), the inverse bid function. From Propositions 7 and 8, every component of the right-hand side is either observed or identified from data, so $\xi(b)$ can be computed. Since bids are monotonic in v_1, the α-quantile of v_1, $v_1(\alpha)$, corresponds to $\xi(b(\alpha))$. Now, since the distribution of b is observed and $\xi(b)$ can be computed for any b, we can compute $v_1(\alpha)$ for any quantile α. Hence, the distribution of v_1 is identified nonparametrically. This proves part (ii). □

Proposition 9

Proof. Split the data into two subsamples, one where the first auction winner is from subgroup m, and the other where the first auction winner is from subgroup $-m$. Take the first subsample. In the first subsample, bidders in the second auction are either the first auction winner from subgroup m, a first auction loser from subgroup m, or a first auction loser from subgroup $-m$. By Proposition 7, the value distributions from which each of these bidders draws their second auction values, $D_m(\cdot|b)$, $F^2_m(\cdot|b)$, and $F^2_{-m}(\cdot|b)$, are identified. Similarly, $D_{-m}(\cdot|b)$ is additionally identified from the second subsample.

Then, by Proposition 8, the synergy function $s_m(b, \cdot)$ is identified from $D_m(\cdot|b)$ and $F^2_m(\cdot|b)$, and $s_{-m}(b, \cdot)$ is identified from $D_{-m}(\cdot|b)$ and $F^2_{-m}(\cdot|b)$.

Finally, $F^1_m(v_1)$ and $F^1_{-m}(v_1)$ are identified using each subgroup's FOC for bidding in the first auction, equation (8). If we replace $s_m(v_1, v_2)$ with $s_m(b, v_2)$ and $F^2_m(v_2|v_1)$ with $F^2_m(v_2|b)$, every component of (8) other than v_1 is either observed or identified. Therefore, we can back out any quantile of v_1 by computing the equation using the same quantile of b_m.

Once $F^1_m(v_1)$ and $F^1_{-m}(v_1)$ are identified, we can convert $s_m(b, v_2)$ and $F^2_m(v_2|b)$ back to $s_m(v_1, v_2)$ and $F^2_m(v_2|v_1)$ by replacing $b(\alpha)$ with $v_1(\alpha)$. This completes the identification. □

Proposition 10

Proof. Consider the $N = 2$ case as an example. $J(\cdot|z'\beta)$ is the distribution of the second highest value out of $\{s(v_1, v_2),\, v_2\}$, which can be rewritten $\{F^{2,-1}(\alpha_s|z'\beta),\, F^{2,-1}(\alpha_2|z'\beta)\}$, where the -1 superscript indicates the inverse function. Now for any v, define $\alpha \equiv F^2(v|z'\beta)$ and $\tilde{\alpha} \equiv J(v|z'\beta)$. Then $\tilde{\alpha} \equiv J(v|z'\beta) = J(F^{2,-1}(\alpha|z'\beta)|z'\beta) = prob(\{F^{2,-1}(\alpha_s|z'\beta), F^{2,-1}(\alpha_2|z'\beta)\}_{(2)} \leq F^{2,-1}(\alpha|z'\beta)|z'\beta) = prob(\{\alpha_s, \alpha_2\}_{(2)} \leq \alpha|z'\beta)$.[14] From AS9, the distributions of α_s, α_2 are invariant to $z'\beta$, so we can simplify $\tilde{\alpha} = prob(\{\alpha_s, \alpha_2\}_{(2)} \leq \alpha|z'\beta)$ to $prob(\{\alpha_s, \alpha_2\}_{(2)} \leq \alpha)$. Hence $\tilde{\alpha}$ is a function only of α, invariant to z. Furthermore, since $C(\alpha_1, \alpha_2)$ is invariant to z according to AS9, and $\tilde{\alpha}$ is a function only of α, $C(\alpha_1, \tilde{\alpha}_2)$ is also invariant to z. The same applies for $C(\alpha_1, \tilde{\alpha}_s)$. □

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[14] The $\{\}_{(2)}$ subscript indicates the second order statistic out of the values in $\{\}$.

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