A Dynamic Model of Altruistically-Motivated Transfers

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Abstract

This paper studies a dynamic Markovian game of two infinitely-lived altruistic agents without commitment. Players can save, consume and give transfers to each other. We find a continuum of equilibria in which the poor player receives transfers until players effectively pool their wealth and tragedy-of-the-commons-type inefficiencies occur. The following type of equilibrium exists only if a shock is introduced: The donor withholds transfers until the recipient is constrained. We argue that more realistic environments should focus on the latter type of equilibrium since it is empirically more plausible and has desirable stability properties.

Dynamic economies with altruistic agents are an important class of models, but the literature has so far restricted itself to studying rather special cases. Our research agenda aims to fill this gap by providing a tractable theory for the behavior of imperfectly-altruistic agents in a fully-dynamic setting without commitment. In particular, we hope to provide a building-block model that is flexible and stable enough to be used in larger settings, such as heterogeneous-agents models in macroeconomics (as already done in Barczyk, 2011), but potentially also in microeconomic models of the family and development.

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We have found that in order to understand the fundamental workings and tensions in such frameworks (which so far seem to have obstructed progress in the literature) one has to begin by studying the simplest-possible setting. This is what we do in this paper: We study Markov-perfect equilibria in a deterministic environment inhabited by two infinitely-lived altruistic agents. Their only sources of income are a risk-free return on savings and voluntary transfers from the other player. A key contribution of the paper is that its insights point the way on how to attack more realistic settings (idiosyncratic earnings risk, overlapping generations etc., see our follow-up paper). Furthermore, there is a number of important contributions the paper makes in its own right.

We characterize dynamic incentives and distortions in consumption-savings decisions induced by strategic interactions between altruists in a fully-dynamic setting. Specifically, we are able to make three new points relative to the existing two-period models: First, in addition to the known possibility of over-consumption, agents under-consume in certain situations with respect to the efficient allocation; indeed, both under- and over-consumption are present in the class of equilibria we find in the deterministic setting. Second, the model predicts that inefficiencies occur long before transfers actually flow, a feature that the two-period models in the literature are necessarily silent upon. Third, we find that not only are the recipient's consumption-savings decisions distorted (a phenomenon known as the Samaritan's dilemma) but also the donor's.

Our analysis allows us to draw a sharp distinction between the Samaritan's dilemma, which in our setting is characterized by the *Party Theorem*, and what we call the *Prodigal-Son dilemma*. The latter says that under imperfect altruism no equilibrium exists in which a rich donor lifts a poor recipient out of poverty and both players are self-sufficient ever after. The potential donor realizes that the recipient would squander the transfer, come back, and ask for more. The squandering of the transfer is the Samaritan's dilemma (the *party*) whereas not providing transfers in anticipation of this is the Prodigal-Son dilemma. Both rest importantly on the assumption of no-commitment.

Finally, we analyze equilibria of the game. We find a continuum of *tragedy-of-the-commons-type equilibria* with the following features: When the asset distribution is imbalanced, the poor player receives an increasing schedule of transfers that give her incentives to save herself out of poverty. Once the asset distribution is sufficiently balanced, players essentially pool their assets and tragedy-of-the-commons-type inefficiencies occur. Agents consume at inefficiently high rates out of the common pool unless both players are perfectly altruistic. This is a novel type of equilibrium (it depends crucially on the infinite-horizon assumption) and exists despite well-defined property rights.

A *transfer-when-constrained equilibrium* exists only when uncertainty is added to the setting. In this equilibrium, the rich agent delays transfers until the recipient is constrained. By doing so, he can control the recipient's consumption once transfers flow. This equilibrium features a Samaritan's dilemma and savings inefficiencies that feed back over time. In regions where the asset distribution is balanced, players use strategies close to the ones they would use in a world where they are self-sufficient. We argue that future work in more complex environments should focus on this type of equilibrium because of its greater empirical relevance and superior stability properties.

On the technical side, we argue that it is useful to work in continuous time in dynamic-altruism models. Certain strategic interactions are of second order, which makes instantaneous best-response functions constant and eliminates multiplicity of equilibria in the Δt-stage games. Furthermore, our approach enables us to characterize equilibria by ordinary differential equations (ODEs) and a set of boundary conditions; the number of boundary conditions tells us if to expect no equilibrium, a finite number or a continuum of them for any given equilibrium type. Finally, we can study non-smooth equilibria in a dynamic (differential) Markov game[1] with the possibility of mass-type transfers.

In the model there are two infinitely-lived altruistic agents. One-sided altruism, perfect altruism (representative household), and selfish preferences are nested. Players decide about consumption, savings in a riskless asset (subject to a no-borrowing constraint), and a non-negative transfer to the other agent. They are endowed with an initial stock of assets but have no labor income; this assumption, together with

[1] The characteristic feature of a differential game is that the law of motion of the state vector is determined by differential equations in continuous time. The standard solution concept is Markov-perfect equilibrium (also referred to as *feedback* or *closed-loop* solution), but other concepts such as Stackelberg or Markov equilibrium (*open-loop*) are also common. For an introduction to the theory of differential games for economists see the book by Dockner et al. (2000) and chapter 13 in Fudenberg & Tirole (1993).

homotheticity of preferences, allows us to exploit the homogeneity of the setting and reduce the dimensionality of the state space to one.

The literature on altruism has analyzed a host of static and two-period models for transfers in the tradition of Becker's (1974) seminal paper. The analysis of Lindbeck & Weibull (1988) focuses on the Samaritan's dilemma and highlights the existence of multiple equilibria in a two-period model. Among others, Bernheim & Stark (1988) and Bruce & Waldman (1990) study two-period models that emphasize the Samaritan's dilemma and its consequences for economic policy. But, restricting the analysis to two periods carries limitations of what can be learned from a dynamic setting and in using the model in a more realistic setting.

In the macroeconomics literature, dynamic models often have to take a stance on how agents within a dynasty or family are connected. Two standard workhorse models highlight this: The infinitely-lived household is justified by the assumption that altruistic concerns connect subsequent generations as in Barro's (1974) seminal paper, whereas pure life-cycle OLG models are usually populated by households that act in complete isolation. While these two extremes are often convenient representations, there is a substantial literature which deems it important to employ a model which lies somewhere in-between. Abel (1987) studies under which conditions one of the two inter-generational transfer motives is operative that are necessary for Barro's (1974) neutrality result to hold. Laitner (1988) assesses the impact of a social-security system on capital accumulation in an overlapping-generations economy in which children and parents are imperfectly altruistic. However, while generations are allowed to interact strategically, they overlap for only one period. Altig & Davis (1988) study an array of inter- and intra-generational redistributive policies in an economy with altruistic agents who overlap for a large number of periods, but they assume commitment.

Furthermore, there are many computational studies in the macro literature in which altruistic agents overlap for more than one period. However, the authors make simplifying assumptions in order to circumvent the tensions that we analyze. Laitner (1992) provides a framework in which agents overlap for many periods, but agents are restricted to be perfectly altruistic. Fuster, Imrohoroglu & Imrohoroglu (2007) build on this framework to study pension systems. Nishiyama (2002) studies a setting with imperfect altruistic households in which generations overlap for at most two periods, but rules out the possibility that transfers are used for saving. In Kaplan (2010), imperfectly-altruistic parents and children interact strategically, but parents are not allowed to save.

In the applied microeconomics literature, many studies build on the *collective model*, which is due to Chiappori (1988). The key assumption of the model is that the family can always coordinate on efficient allocations. In reality, one would expect efficient outcomes within the household if agents have the ability to commit to future allocations (say at the point of marriage). Mazzocco (2007) employs an extension of the collective model that nests the possibilities of commitment and no-commitment and strongly rejects the assumption of commitment in the data. In light of this evidence, it is important to explore other, non-cooperative models for dynamic interaction between altruistic agents. This case is even stronger for *inter*-household interaction.

Finally, a note on the empirical evidence on inter-vivos transfers is in order. Transfers tend to flow from well-off to worse-off family members, and recipients are often liquidity-constrained (see for example Cox & Raines, 1985 and McGarry, 1999).

The remainder of the paper is structured as follows: Section 1 outlines the setting of the model and characterizes the set of Pareto-efficient allocations. Section 2 studies dynamic incentives and distortions in consumption-savings decisions induced by strategic interactions between imperfect altruists. Section 3 characterizes equilibria and presents our main results. Section 4 concludes and points out the way for future research.

1 Setting

1.1 Physical environment

Time t is continuous. There are two agents in the economy who are infinitely-lived. We will denote variables for the first agent, whom we will refer to as "she", as plain lower-case letters, e.g. c_t. Variables referring to the second agent, whom we will call "he", are denoted with prime-superscripts, e.g. c'_t. Both agents can hold a non-negative amount k_t in a riskless asset that pays a time-invariant rate of interest r.

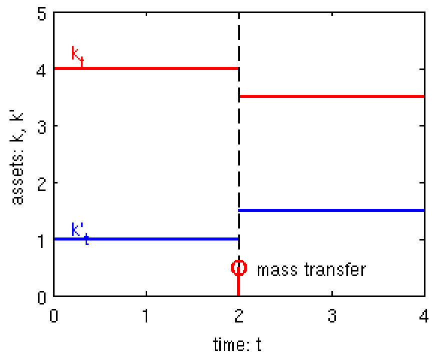

Figure 1

In each instant of time, agents choose a consumption rate $c_t \geq 0$ and a non-negative transfer rate g_t to the other agent (g stands for "gift"), so that their assets evolve according to

$$\dot{k}_t = rk_t - c_t - g_t + g'_t \tag{1}$$

$$\dot{k}'_t = rk'_t - c'_t - g'_t + g_t, \tag{2}$$

where dots denote the time-derivative of a variable. There is a no-borrowing constraint for both agents; when $k_t = 0$, we must have that she does not spend more than she receives, i.e. $c_t + g_t \leq g'_t$ (and equivalently for him, of course).[2]

We allow for transfers g that are of mass-point type, formally defined as a *Dirac delta* $\tilde{g}\delta$ in the evolution equation (1). This means that the players can transfer large amounts of resources instantaneously. A mass-point transfer induces a jump in the time path (trajectory) of the state of size $\tilde{g}$, see figure (1) for an example in which she provides a mass-point transfer of size $\frac{1}{2}$ to him. It is both reasonable to allow for this – after all, it is definitely feasible that large amounts of money change hands instantaneously – and it will also turn out convenient for our analysis in many respects.

Her preferences are given by

$$v_0 = \int_0^\infty e^{-\rho t}[u(c_t) + \alpha u(c'_t)]dt, \tag{3}$$

[2]This is the natural borrowing constraint in this setting if we assume that agents cannot borrow against future transfers by the other player.

where $\rho > 0$ is the discount rate and $0 \leq \alpha \leq 1$ the parameter which measures the intensity of altruism.[3] He is a mirror-symmetric copy of hers, but might have a different altruism parameter $0 \leq \alpha' \leq 1$. His preferences are

$$v'_0 = \int_0^\infty e^{-\rho t}[u(c'_t) + \alpha' u(c_t)]dt. \tag{4}$$

We assume that the agents have the same discount rate ρ; this is crucial for our analysis. We also assume that agents do not differ in form of the felicity function $u(\cdot)$. We choose logarithmic utility as the functional form: $u(c) = \ln c$.[4]

1.2 Equilibrium definition

As mentioned in the introduction, we focus on Markov-perfect equilibria. The payoff-relevant state is obviously (k, k'). A Markovian strategy is a pair of non-negative functions $\{c(k, k'), g(k, k')\}$ for her and a pair $\{c'(k, k'), g'(k, k')\}$ for him. We leave strategies unrestricted, in the sense that we impose no upper bound on consumption and transfer functions at any point in the state space. We enforce feasibility of consumption plans by setting "realized consumption" when she has zero assets to

$$c^*(0, k') = \min\left\{c(0, k'), g'(0, k')\right\}.$$

This says that she cannot eat more than he gives to her when she is broke, but she can announce plans to do so. In all other cases, realized consumption equals the announced strategy $c(k, k')$ because she faces no constraint.[5] We define realized consumption $c'^*(k, 0)$ for him in the same manner. Furthermore, we rule out that a broke player gives transfers, which means $g(0, k') = 0$ and $g'(k, 0) = 0$.

When the other player's strategy is Markov, the best-response problem of each player is a dynamic-programming problem and best responses will also be Markov. Just as in discrete time, continuous-time dynamic programming splits the agent's

[3] With this linearly separable formulation of altruistic preferences we are in line with the bulk of the literature.

[4] Many, but not all results in the paper can still be obtained for power utility.

[5] Note that in continuous time, an agent is never constrained in the choice of the flow rate of consumption (or also transfers in this setting) unless he is *directly* at the borrowing constraint: Given any small amount ε of assets, he can always choose an arbitrarily high consumption rate M for some short time interval $\Delta t < \varepsilon/M$.

problem into two parts: today versus the entire future. Consider the game at time t for a given state (k_t, k'_t). Suppose we know her equilibrium value function $v(k_t, k'_t)$. Given his equilibrium strategy $\{c'(k_t, k'_t), g(k_t, k'_t)\}$, ***Bellman's principle*** then says that we can write her problem over a short horizon Δt as follows:

$$\begin{aligned} v(k_t, k'_t) &= \max_{c,g} \Big\{ u(c)\Delta t + \alpha u\big(c'(k_t, k'_t)\big)\Delta t + e^{-\rho\Delta t} v(k_{t+\Delta t}, k'_{t+\Delta t}) \Big\}, \\ \text{s.t.} \quad k_{t+\Delta t} &= k_t + \big[rk_t - c - g + g'(k_t, k'_t)\big]\Delta t + o(\Delta t), \qquad (5) \\ k'_{t+\Delta t} &= k'_t + \big[rk'_t - c'(k_t, k'_t) - g'(k_t, k'_t) + g\big]\Delta t + o(\Delta t), \end{aligned}$$

where $o(\Delta t)$ denotes terms that are lower than order Δt.[6] We will refer to this problem as her *Δt-problem*, which is at the heart of how we define best-responding in this setting.

If the problem is well-behaved[7], this problem's limit as Δt goes to zero is equivalent to the Hamilton-Jacobi-Bellman (HJB) equation being fulfilled. To see this, take a first-order Taylor expansion of the term $e^{-\rho\Delta t} v(k_{t+\Delta t}, k'_{t+\Delta t})$ in Δt and simplify to obtain the HJB:

$$\rho v = \max_{c,g} \{u(c) + \alpha u(c') + (rk - c - g + g')v_k + (rk' - c' - g' + g)v_{k'}\}. \qquad (6)$$

Subscripts denote partial derivatives, e.g. $v_k = \frac{\partial v}{\partial k}$. We suppress the dependence of the functions v etc. on (k, k') for better readability. The HJB is a partial differential equation (PDE) that imposes restrictions on v and its partial derivatives v_k and $v_{k'}$ for all points (k, k') in the state space. His problem is characterized by a mirror-symmetric HJB; we denote his value function by v' and the partial derivatives by $v'_{k'}$ and v'_k. Throughout the main text we will state *her* equations, with the understanding that his equations are mirror-symmetric; the reader can find *his* equations in appendix A.1.

Wherever v is differentiable, the partial derivatives v_k and $v_{k'}$ are defined in the obvious manner. However, we will also be concerned with the case where the above problem is non-smooth in the following sense: One or both player's policies are

[6]Formally: $o(\Delta t)$ is such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

[7]"Well-behaved" means that we can use standard conditions for dynamic programming, which is here: The other player's strategy is Lipschitz continuous, which ensures that the ODEs for k_t and k'_t are solvable.

discontinuous and one or both value functions are not differentiable.[8] We restrict attention to the class of equilibria where strategies are piecewise continuously differentiable functions and the state space is partitioned into a finite number of smooth regions (short: *regions*). On the seams between regions, we will take directional derivatives of $v(\cdot)$, where the derivative's direction is governed by the strategies on the seam. Specifically, we will say that (c, g) are a best response to (c', g') at (k, k') if (c, g) maximizes the curly bracket on the right-hand side of (6), where the gradient (v_k, v'_k) is taken into the direction $(\dot{k}, \dot{k}')$ which is implied by the respective quadruple $(c, g; c', g')$. We will be more specific on this in section A.6 once we have reduced the state space to one dimension.[9]

A second special case arises if players use mass transfers. In this case, the economy will jump from the current state (k, k') to a distant point $(\bar{k}, \bar{k}')$. For a mass transfer of size $g\delta$, we have $\bar{k} = k - g$ and $\bar{k}' = k' + g$. The principle laid out in the Δt-problem (5) requires then that policies be optimal taking into account the (directional) derivatives at the new state $(\bar{k}, \bar{k}')$ when evaluating consumption decisions. An example is in order to illustrate why this is reasonable: Imagine a child owning 1$ who is deciding about her expenses on lollipops over a day, knowing that she will receive a 100$ gift in the evening of that day. She should clearly take into account the marginal value of assets at 101$ and not at 1$ when deciding about the number of lollipops to buy on that day. Again, we will be mathematically precise on this issue in section A.6.

We now have everything in place to define a recursive equilibrium.

Definition 1 *A* Markov-perfect equilibrium (MPE) *is a collection of functions* $\{v(\cdot), c(\cdot), g(\cdot)\}$ *for her and* $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ *for him such that*

1. $\{v(\cdot), c(\cdot), g(\cdot)\}$ *solve her problem given* $\{c'(\cdot), g'(\cdot)\}$*, i.e. they solve* (6) *in the*

[8] As Fudenberg & Tirole (1993) point out, restricting equilibria to continuously differentiable strategies (short: C_1) in a differential game may be too restrictive. Even if the other player's strategy is C_1, we need to look for the player's best response in the space of piecewise-C_1 functions, since this is the space of deviations we have to allow for when applying Pontryagin's maximum principle. Should a piecewise-C_1 function indeed be a best response, then the other player's response problem would already be ill-defined since Lipschitz-continuity for the law of motion breaks down. In this case, there may not exist a unique solution for the differential equation governing the evolution of the state.

[9] This directional approach to HJBs yields very stable results and is intimately related to viscosity solutions for HJBs, see the book by Bardi & Capuzzo-Dolcetta (2008). Viscosity solutions are the agreed-upon solution concept for HJBs in the general (non-smooth) case.

sense laid out above, and

2. $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ *solve his problem given* $\{c(\cdot), g(\cdot)\}$.

Since players' strategies are required to be optimal for all points in the state space, players have to be best-responding at any node of the game tree, even the ones off the equilibrium path. As is well known, Markov perfection thus implies subgame perfection.[10]

1.3 Pareto-optimal allocations

The set of Pareto-efficient allocations will serve as an important benchmark for the analysis throughout the paper. In later sections we will return to these and see that equilibrium allocations are Pareto-efficient only in certain circumstances.

To this end consider a benevolent planner who places a weight η on her life-time value and a weight $(1-\eta)$ on his. Given initial assets k_0 and k_0', he chooses optimal savings policies k_t, k_t' and consumption policies c_t, c_t' for $0 \leq t < \infty$ to maximize

$$J^\eta = \eta \int_0^\infty e^{-\rho t}\big[u(c_t) + \alpha u(c_t')\big]dt + (1-\eta)\int_0^\infty e^{-\rho t}\big[u(c_t') + \alpha' u(c_t)\big]dt. \quad (7)$$

Varying $\eta \in [0,1]$ yields all allocations on the Pareto frontier.

For the sake of the planner's problem it is valid to pool the individual assets of the players together. Defining $K_t = k_t + k_t'$ as the total resources in the economy at time t, the planner's Bellman equation is given by[11]

$$\rho V^\eta(K) = \max_{c,c'}\Big\{\big[\eta + \alpha'(1-\eta)\big]u(c) + \big[(1-\eta)+\alpha\eta\big]u(c') + (r-c-c')V_K^\eta(K)\Big\}. \quad (8)$$

Intra-temporally, the planner divides consumption between the two agents in order to equalize the margins, i.e.

$$\big[\eta + \alpha'(1-\eta)\big]u_c(c_t) = \big[(1-\eta)+\alpha\eta\big]u_c(c_t') \qquad \forall t.$$

[10] If $r > 0$, note that any state (k, k') can be reached in finite time if players are sufficiently frugal, so we cannot exclude any node from the analysis.

[11] Note that the planner's problem becomes a standard cake-eating problem but with interest payments.

Solving for $u_c(c_t)$ yields

$$u_c(c_t) = \frac{(1-\eta)+\alpha\eta}{\eta+\alpha'(1-\eta)} u_c(c_t') \qquad \forall t. \tag{9}$$

As in standard dynamic planning problems, marginal utilities are proportional over time. Here, the factor of proportionality is a function of the planner's weight η on her and the altruism parameters α and α'. It is instructive to consider the extreme cases where $\eta = 0$ or $\eta = 1$. Placing all weight on her yields $u_c(c_t) = \alpha u_c(c_t')$, whereas placing all weight on him yields $u_c(c_t) = \frac{1}{\alpha'} u_c(c_t')$. Thus, just as in the static altruism setting, the ratio of marginal utilities is restricted to the interval $[\alpha, \frac{1}{\alpha'}]$. The more altruistic both agents are, the smaller is the consumption inequality tolerated by the Pareto planner. These bounds approach zero and infinity as altruism goes to zero, until reaching the standard case with selfish agents. For perfect altruism ($\alpha = \alpha' = 1$) there is a unique Pareto-optimal allocation and both agents always consume the same amount.

We now proceed to the inter-temporal optimality conditions. Note that equation (9) gives us c_t' as a function of c_t,[12] so that the planner's problem collapses to a conventional consumption-savings problem with a modified objective function (to see this, substitute out c_t' in the objective (7) using (9)). It follows that the Euler equation from the standard one-person consumption-savings problem must hold. Otherwise, the planner would re-allocate resources inter-temporally for one agent maintaining the present value of resources allocated to this agent. We must have:[13]

$$\begin{aligned} \frac{d}{dt} u_c(c) &= (\rho - r) u_c(c), \\ \frac{d}{dt} u_c(c') &= (\rho - r) u_c(c'), \end{aligned} \tag{10}$$

where the second equation is already implied by the first, using the intra-temporal optimality condition (9).

[12]When $\eta = 1$ and $\alpha = 0$, equation (9) does not give us c_t' as a function of c_t any more – in this case, however, it is obviously optimal for the planner to set $c_t' = 0$ for all t. Analogously, $\eta = 0$ and $\alpha' = 0$ implies that $c_t = 0$.

[13]Technically, we obtain the same Euler equations in the standard manner from the HJB (8): Take the derivative with respect to K and use the first-order conditions $V_K = [\eta + \alpha'(1-\eta)] u_c(c) = [(1-\eta) + \alpha\eta] u_c(c')$ and the fact that $\frac{d}{dt} V_K = \dot{K} V_{KK} = (r - c - c') V_{KK}$.

With $u(c_t) = \ln c_t$, the Euler equations (10) imply that $d \ln c_t/dt = r - \rho$, i.e. consumption grows at rate $(r - \rho)$ for both agents. The closed-form solution for the consumption plans is then given by

$$c_t = \rho P_\eta K_t, \qquad c'_t = \rho(1 - P_\eta)K_t,$$
$$\text{where} \quad P_\eta = \frac{\eta + (1-\eta)\alpha'}{1 + \eta\alpha + (1-\eta)\alpha'}. \tag{11}$$

Intuitively, the sum of both agents consumption, $c_t + c'_t = \rho K_t$, is what a single agent with assets K_t would optimally consume. Here, the planner splits the amount ρK_t between the two players, where the splitting rule depends on the weight η as well as the altruism parameters α and α'.

Finally, it will be useful to establish a connection between the planner's problem and an equilibrium of our setting with the restriction that agents have the possibility to commit to future consumption and transfers at $t = 0$. Any Pareto-optimal allocation can then be implemented by first assigning wealth shares $k_0 = P_\eta K_0$ (to her) and $k'_0 = (1 - P_\eta)K_0$ (to him), followed by a transfer stage, but then shutting down the possibility for further transfers. If the initial wealth shares are outside the range of P_η's spanned by $\eta \in [0, 1]$, the richer agent gives an initial transfer to implement her/his preferred allocation, which is equivalent to the solution of the planner's problem with $\eta = 0$ or $\eta = 1$. This is also the type of equilibrium that obtains in the static altruism setting, in which a transfer stage precedes the consumption stage. In essence, commitment removes the dynamic component from the game.

2 Understanding players' incentives

This section sheds light on the agents' incentives when responding to the other player's consumption and transfer strategy. We highlight the strategic interactions between players as well as the inefficiencies that result.

In the maximization problem that she faces, which is characterized by the HJB (6), separate the max-operators to obtain:

$$\begin{aligned}\rho v =& \alpha u(c') + (rk + g')v_k + (rk' - g' - c')v_{k'} + \\ &+ \max_{g \geq 0} \Big\{ g \big[\underbrace{v_{k'} - v_k}_{\equiv \mu\text{: transfer motive}} \big] \Big\} + \max_{c \geq 0} \Big\{ u(c) - cv_k \Big\}.\end{aligned} \tag{12}$$

The first-order condition (FOC) for consumption is given by

$$u_c(c) = v_k, \tag{13}$$

which says that the marginal utility of current consumption is set equal to the marginal value of saving in the optimum.

Here we see the crucial simplification that continuous time gives us with respect to discrete time: His contemporaneous consumption decision c' does not affect her optimal choice c, nor does his transfer decision g'. In other words, her best-response function over a short amount of time is a constant. This means that we can obtain her optimal consumption as in a standard consumption-savings problem without calculating best responses for each action of the other player.[14] Furthermore, constant best responses ensure existence and uniqueness of equilibrium in the "stage games", i.e. the interactions of players over very short horizons Δt-games (more on this follows below). This is not the case in discrete time: Lindbeck & Weibull (1988) find multiple equilibria already in a two-period setting.

At first glance it seems striking that his current decisions should not matter to her. But this is only true for the decisions taken *at the same instant* of time. In general, his decisions do matter for her, which will become evident from her Euler equation in subsection 2.1. For now remember that the effects of his future decisions on her are all contained in the partial derivative v_k, which encodes the incentives stemming from the entire continuation of the game.

A second important simplification with respect to discrete time is that there is

[14]One could argue that this problem can be avoided in discrete time by having players move sequentially. However, this has the following disadvantages: First, assumptions on the timing protocol can influence the results. Second, discrete time periods are tantamount to assuming commitment over the period length. Note that the player who moves first cannot adjust his decision when observing the other's action, even though this might be in his best interest. The advantage of the continuous-time setting is that the timing protocol does not matter since agents can react infinitely fast – the planning horizon goes to zero.

no interaction between the decision problems of the consumption and the transfer choices: The two max-operators for c and g are separate.

The key to why these simplifications with respect to discrete time arise is that immediate strategic considerations are of second order. We can see this from the following first-order approximation of the marginal value of saving, which determines consumption through the FOC (13), in a small neighborhood of the current state (k_t, k'_t):

$$v_k(k_{t+\Delta t}, k'_{t+\Delta t}) = v_k(k_t, k'_t) + \underbrace{v_{kk'}(k_t, k'_t)\big[rk'_t - c'_t - g'_t \ldots\big]\Delta t}_{\text{of second order}} + \ldots$$

Clearly, as Δt becomes small, the changes in the marginal value of saving induced by his policies c'_t and g'_t over the planning period Δt become negligible. So even when he chooses a very high c'_t, there is a Δt small enough so that v_k is (almost) equal to its current value. It is therefore valid to disregard the interaction effects between the consumption decisions when letting $\Delta t \to 0$. Intuitively, if she reconsiders her savings decisions on a daily basis, there is no need to worry about his daily savings decisions (his control, a flow variable), since the impact it has on his assets (his state, a stock variable) is small – in order to be sufficiently informed it is enough to keep an eye on his bank account.

Equation (12) shows that choosing a transfer g is a linear optimization problem. The term $\mu \equiv (v_{k'} - v_k)$ is the marginal benefit of transfering an additional unit of resources from her to him. We will refer to μ as her *transfer motive*. Whenever μ is negative, transfers are set to zero – after all, she cannot force him to give transfers to her. If $\mu = 0$, then any transfer flow is consistent with optimality; in this case she is (locally) indifferent with regard to the distribution of assets between him and her. Should the transfer motive be *positive*, however, then the agent wants to choose g as large as possible. In fact, since the agent is allowed to make mass-point transfers, she would choose to follow the vector $(-1, 1)$ in (k, k')-space as long as the directional derivative $v_{k'} - v_k = \mu$ is positive. As section 3 will show this actually makes it impossible that $\mu > 0$ in the first place.[15]

[15] Note that in this case, the HJB ceases to be a valid characterization for his problem since – as a PDE – it only contains *local* information on the value function. When making a mass-point transfer, however, it is crucial to consider the continuation value v globally.

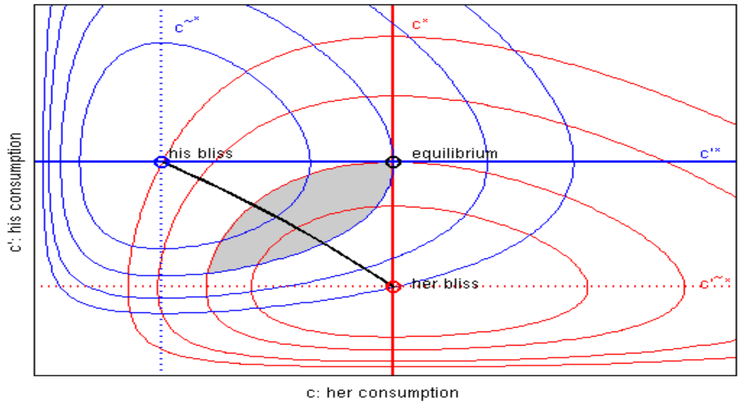

Figure 2: Equilibrium over Δt.

We will now study how players' joint consumption decisions lead to inefficiencies. This is because players do not fully internalize the effect of their actions on the other.

Consider the game at time t for a given state (k_t, k'_t), where $k_t > 0$ and $k'_t > 0$. Suppose that after a short period Δt the equilibrium policies and the continuation values are solved for. How would the agents feel about different consumption rates (c, c') over a short time interval Δt, assuming that afterwards the equilibrium allocation is played? Assume that no transfers are given over Δt. In the spirit of her Δt-problem in (5), we write today's value $v(k_t, k'_t)$ as a sum of flow utility collected over Δt and the continuation value, which is $e^{-\rho \Delta t} v(k_{t+\Delta t}, k'_{t+\Delta t}) \simeq v(k_t, k'_t) - \rho v \Delta t + v_k \dot{k} \Delta t + v_{k'} \dot{k}' \Delta t$:

$$v(k_t, k'_t) \simeq v(k_t, k'_t) - \rho v \Delta t + \max_c \left\{ u(c) \Delta t + \alpha u(c') \Delta t + v_k \dot{k} \Delta t + v_{k'} \dot{k}' \Delta t \right\}.$$

Define the term inside the curly bracket on the right-hand side divided by Δt (the Hamiltonian) as

$$H(c, c') = u(c) + \alpha u(c') + v_k \dot{k} + v_{k'} \dot{k}'.$$

$H(c, c')$ tells us how she feels about consumption tuples (c, c'), taking into account both flow utility over Δt and their repercussions on the continuation value of the game.

Figure (2) plots the contours of the functions H and H' in (c, c')-space, where we fix some derivatives $v_k > v_{k'}$ and $v'_{k'} > v'_k$. The vertical solid line indicates $c^*(c') \equiv \arg\max_c H(c, c')$, which is her best-response function; the horizontal solid line is his best response $c'^*(c) \equiv \arg\max_{c'} H'(c, c')$. As discussed above, best responses are constant in the other player's action. The equilibrium Δt-allocation occurs at the intersection of the two best responses. The horizontal dotted line indicates $\tilde{c}'^*(c) \equiv \max_{c'} H(c, c')$, which is the consumption rate that she would choose for him if she could do so; the vertical dotted line indicates his preferred consumption for her, $\tilde{c}^*(c') \equiv \arg\max_c H'(c, c')$. Note that all of these are constant functions due to the separability of the instantaneous utility function.

An immediate observation is that the indifference curves of the two players are crossing each other, illustrating that they disagree about allocations. Only in the case of perfect altruism ($\alpha = \alpha' = 1$) will the indifference curves coincide. The indifference curves become straight lines that cross each other in the case of selfishness.

The crucial feature that figure 2 highlights is the inefficiency which usually arises in this environment. Consider the problem of a Δt-planner who assigns consumption tuples (c, c') using a weight $\eta \in [0, 1]$ on her utility in the following problem:

$$\max_{c,c'} \{\eta H(c, c') + (1 - \eta) H'(c, c')\}. \tag{14}$$

Note that this problem is different from the planner's problem in (8): The Δt-planner sets consumption only over a horizon Δt, taking as given that the equilibrium policies will be placed thereafter. The Pareto planner, instead, uses the marginal value of assets V_K^η, which is calculated using the planner's policy after $t + \Delta t$.

In order to trace out the Pareto frontier of the Δt-game, vary $\eta \in [0, 1]$ in the Δt-planner's problem (14). The resulting curve connects her and his *bliss points*, which are obtained by setting $\eta = 1$ and $\eta = 0$, respectively. The shaded area emanating from the equilibrium allocation contains the allocations corresponding to Pareto improvements over the time interval Δt. Committing to one of these policies over Δt, would raise both player's welfare. However, both players would be tempted to break the agreement and revert to the best-response strategy.

In the example of figure 2, both players are over-consuming: It would be Pareto-improving if they coordinated on lower consumption rates. This is because both players would prefer the other to consume less. In the figure this is apparent since $\tilde{c}'^*$

(the consumption she would choose for him) is below c'^* (the consumption he indeed chooses), and $\tilde{c}^*$ is to the left of c^*. However, there could also be under-consumption by one (or both) players. Pareto improvements would then be associated with one (or both) players consuming more than in the Δt-equilibrium. Indeed, in the class of equilibria found in section 3.3 both over- and under-consumption are present in different regions of the state space.

The following section will show how over- and under-consumption are induced dynamically, how they are related to the altruism parameters (α, α'), and the value-functions' cross derivatives $(v_{k'}, v'_k)$.

2.1 Savings incentives: The Euler equation

Our analysis provides new insights on savings incentives in a fully-dynamic setting when altruism is imperfect. In order to see this, we ask how his consumption and transfer decisions affect her consumption-savings behavior.

For the relevant trade-offs consider her Euler equation:[16]

$$\frac{d}{dt}\left[u_c(c_t)\right] = \underbrace{(\rho - r)u_c(c)}_{\text{standard = efficient}} + \underbrace{\left[v_{k'} - \alpha u_c(c')\right]c'_k}_{\text{altruistic-strategic distortion}} + \underbrace{\left[v_{k'} - u_c(c)\right]g'_k}_{\text{transfer-induced incentives}} \quad . \tag{15}$$

Here, c'_k denotes the partial derivative of his consumption policy with respect to her assets and g'_k is the partial derivative of his transfer function (assuming it is a rate, i.e. of flow- and not of mass-type) with respect to her assets.

We gain intuition for the three terms on the right-hand side of (15) – the terms which determine marginal-utility growth – by considering a hypothetical deviation from an equilibrium path in an example framed as a discrete-time counterpart to our setting. There are three periods. At the beginning of each period, agents make all decisions simultaneously. Figure 3 shows the paths for the assets (the solid lines) that result from the players' equilibrium strategies. Consider the following deviation from her equilibrium strategy: She saves one unit more in period one and reverts back to the equilibrium level of assets in period three (the dotted line in the left panel of figure 3). If her equilibrium behavior is optimal, then such a deviation is not

[16]In order to obtain her Euler equation we take the derivative of her HJB (12) with respect to k and use the fact that $\frac{d}{dt}u_c(c_t) = \frac{d}{dt}v_k = \dot{k}v_{kk} + \dot{k}'v_{kk'}$ by the FOC (13).

	period 1	period 2	period 3
standard	$-u'(c_1)$	$+\ \beta R u'(c_2)$	
altruistic-strategic		$\alpha\beta u'(c_2)\frac{\partial c_2'}{\partial k_2}$	$-\beta^2 R v_{k'}(k_3, k_3')\frac{\partial c_2'}{\partial k_2}$
transfer-induced		$\beta u'(c_2)\frac{\partial g_2'}{\partial k_2}$	$-\beta^2 R v_{k'}(k_3, k_3')\frac{\partial g_2'}{\partial k_2}$

Table 1: Marginal costs and benefits in discrete-time Euler equation

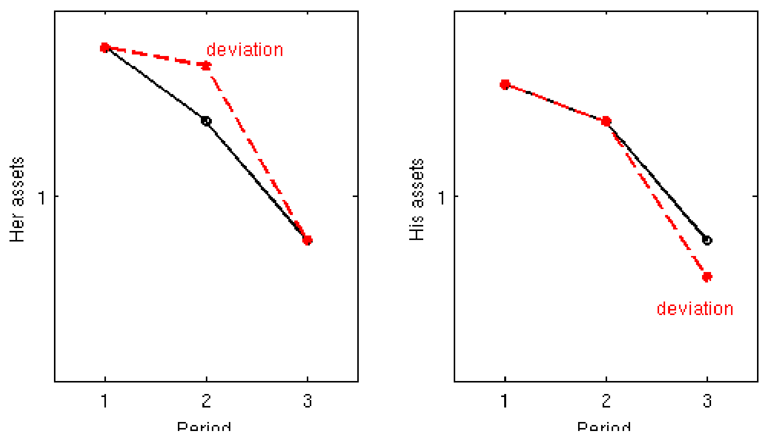

Figure 3: Discrete-time intuition for savings incentives

profitable and its marginal effect on her criterion is zero. Table 1 accounts for all the effects this deviation has on her utility.

First, the two standard terms account for the usual consumption-savings trade-off: Saving one unit more in period one costs marginal utility $u'(c_1)$, but yields $\beta R u'(c_2)$ in period two (β is the discount factor and R the gross interest rate). In continuous time, the two terms correspond to $(\rho - r)u_c(c)$, which coincides with the growth rate of marginal utility that the planner would choose, see equation (10). Thus the additional two terms on the right-hand side of (15) are distortions that make the agent stray from efficient behavior.

Second, the altruistic-strategic distortion is comprised of the terms in the second row in table 1; they stem from his consumption response.

In period two, his consumption will react to her higher asset level, which is captured by $\partial c_2'/\partial k_2$. Suppose for now that $\partial c_2'/\partial k_2 > 0$, which seems intuitive: If she has higher assets, he can count on receiving larger transfers from her in the future and/or is less likely to have to give transfers to her, so he can consume more today.

Because his consumption increases, she realizes the gain $\alpha\beta u'(c_2)\partial c_2'/\partial k_2$ in period two; in continuous time this shows up as an immediate effect $-\alpha u_c(c')c_k'$. This constitutes an additional incentive to save, and, therefore, enters with the same sign as the interest rate R (or r) does in the standard term.

However, his increased consumption comes at a cost: In period two he saves less and enters period three with fewer assets (the dotted line in the right panel of figure 3). Hence, in period three the equilibrium path is left and the economy goes to an equilibrium where he has $R\partial c_2'/\partial k_2$ less assets. Since we can expect $v_{k'}$ to be positive, the term $-\beta^2 R v_{k'}(k_3, k_3)\partial c_2'/\partial k_2$ is negative and acts as a disincentive to save. In continuous time, the corresponding term $v_{k'}c_k'$ enters the Euler equation with the same sign as ρ, and so discourages savings.

Which of the two terms forming part of the altruistic-strategic distortion dominates is directly related to his under- or over-consumption: The bracket $[v_{k'} - \alpha u_c(c')]$ in (15) is negative whenever he consumes less than she desires, i.e. if and only if $c' < \tilde{c}'^*$; it is positive otherwise. As long as $c_k' > 0$ and he is over-consuming, the altruistic-strategic consideration acts as a disincentive to save for her. She responds with front-loading consumption to his over-consumption. If he is under-consuming, the opposite is true: She saves more since this might induce him to consume more.

The third row in table 1 shows the transfer-induced incentives. These two terms only come into play in regions where he gives transfers, i.e. $g' > 0$. Let us suppose that he conditions transfers on her savings behavior by rewarding thrift, i.e. setting $\partial g'/\partial k > 0$.

Since he rewards her for saving more by increasing transfers, she reaps a benefit $\beta u'(c_2)\partial g_2'/\partial k_2$ in period two (in continuous time: $u_c(c)g_k'$). But in order to revert back to the equilibrium level k_3 she immediately consumes all gains from additional transfers. This acts as an incentive to save; it enters with the same sign as R in the standard term. However, there is again a negative effect on his assets in period three $-\beta^2 R v_{k'}(k_3, k_3)\partial g_2'/\partial k_2$ (in continuous time: $v_{k'}g_k'$), which acts as a disincentive for her to save.

To see which of the two terms dominates, note that the bracket $[v_{k'} - u_c(c)]$ is equal to her transfer motive μ, see her FOC (13). Since we assumed that he is giving transfers to her, we expect that she strictly prefers not to give transfers to him, i.e. $\mu < 0$. This means that for the transfer regime under consideration $\mu g_k'$ is negative and thus his transfer schedule acts as an incentive to save for her, as was to be expected.

We now briefly consider the Euler equation (15) for the special cases of selfishness and perfect altruism. This will highlight how distortions disappear when altruism is very weak or very strong and why we expect distortions to be stronger in the intermediate range of values for α and α'.

Under selfishness ($\alpha = \alpha' = 0$) we have that $c'_k = g'_k = 0$; after all, he has no reason to condition his behavior on her assets. Then there are neither altruistic-strategic distortions nor transfer-induced incentives. We are left with the Euler equation from the one-agent world, which says that marginal utility grows at the rate $\rho - r$ on the optimal savings path, which is efficient.

Under perfect altruism ($\alpha = \alpha' = 1$), in equilibrium we will have $v_{k'} = v_k = u_c(c)$: She values an additional unit of assets in his pocket ($v_{k'}$) the same as she values it in her own pocket (v_k), which is equivalent to her transfer motive being zero. So the brackets $[v_{k'} - \alpha u_c(c')]$ and $[v_{k'} - u_c(c)]$ vanish, and distortions are zero because players are in full agreement.

From these two extreme cases we conjecture that distortions will be strongest for intermediate values of the altruism parameters. For values of α and α' close to zero, the behavioral responses c'_k and g'_k should go to zero (this is what the selfish case suggests). When α and α' approach one, instead, the brackets in (15) should go to zero since agents almost agree about allocations, as they would in the case of perfect altruism.

We conclude this section on a technical note. Generally, imperfect altruism and no-commitment forces us to find a solution to a system of PDEs instead of ODEs (ordinary differential equations). When altruism is perfect, absent, or commitment is present, ODEs suffice. Thus, PDEs are a hallmark of subgame perfection (for a short discussion see section A.2 in the appendix).

3 Equilibrium

This section presents our main results. In order to characterize equilibria, it is useful to begin with the special cases of selfishness and perfect altruism as is done in subsection 3.1. When altruism is imperfect, it can be shown that no equilibria exist which are made up of only one smooth region. In order to make progress, subsection 3.2 studies the different candidate regions and how they are connected. Subsections 3.3 and 3.4 then construct and rule out equilibria that result from patching together these regions.

What makes our approach feasible is the fact that the model environment is homogenous, i.e. both players have homothetic utility and income is proportional to assets. While there are currently two state variables, k and k', homogeneity reduces the dimensionality of the state space to one. As a result, the economy can take at most two directions from each point in the state space. This facilitates studying best responding in mass-transfer-type and non-smooth equilibria. Furthermore, the equations characterizing potential equilibria turn from PDEs into ODEs. The number of boundary conditions for the ODEs will provide crucial information on whether to expect zero, a finite number, or a continuum of equilibria for each given equilibrium type. Such predictions would be a formidable task in a higher-dimensional setting with PDEs.

We define the following mapping from pairs (k, k') to pairs (P, K):

$$P = \frac{k}{k + k'}, \qquad K = k + k'. \tag{16}$$

Thus $P \in [0, 1]$ is the fraction of wealth she owns out of the combined wealth K of both players; the bounds $[0, 1]$ on P are due to the no-borrowing constraints the agents face.[17]

We conjecture that the equilibrium consumption and transfer policies are such that all families, rich and poor, are "proportionally alike": Given the same distribution of assets P, all families will choose the same policies as a percentage of total assets K, i.e. we consider strategies $C(\cdot), C'(\cdot), G(\cdot), G'(\cdot)$ such that

$$\begin{aligned} c(k, k') &= C(P)K \qquad\qquad & g(k, k') &= G(P)K \\ c'(k, k') &= C'(P)K \qquad\qquad & g'(k, k') &= G'(P)K. \end{aligned}$$

Appendix A.6 derives the HJBs and Euler equations in P, which is the only remain-

[17]Note that the mapping is ill-defined for the point $k = k' = 0$. This is unproblematic for practical purposes, since agents will never reach this point in "reasonable" equilibria. Recall that if both agents went broke, then they would obtain a payoff of minus infinity.

ing state variable.

3.1 Benchmark cases

As benchmarks for the case of imperfect altruism, it will prove useful to study two reference models: First, a *self-sufficiency* (SS) model in which the possibility of transfers is ruled out; this is linked to players being selfish. Second, a *wealth-pooling* (WP) model without property rights; this is linked to agents being perfectly altruistic.

First, let us consider the self-sufficient allocations, i.e. the allocations players would choose in our setting if we imposed the additional restrictions $G = 0$ and $G' = 0$ on strategies. Optimal consumption is then characterized by the familiar Euler equation

$$\frac{d}{dt}u_c(c_t) = (\rho - r)u_c(c_t).$$

This implies that the optimal policies in the SS problem are given by

$$C_{SS}(P) = \rho P, \qquad C'_{SS}(P) = \rho(1 - P),$$

which are equivalent to the familiar $c(k, k') = \rho k$ and $c'(k, k') = \rho k'$ in k-k'-space. As shown in the planner's problem (11), these policies induce the efficient allocation if the initial P_0 lies in the range of P_η's spanned by the planner's weight $\eta \in [0, 1]$. The following proposition tells us that the SS policies can only be an equilibrium if both players are selfish; they also constitute the unique equilibrium in this case.[18] The proof is given in appendix A.5.

Proposition 1 (Self-sufficient equilibrium) *Consider the self-sufficiency (SS) strategies $C_{SS} = \rho P$, $C'_{SS} = (1 - P)\rho$ and $G_{SS} = G'_{SS} = 0$.*

1. *The SS strategies can be sustained as an equilibrium only if $\alpha = \alpha' = 0$.*

2. *If $\alpha = \alpha' = 0$, then:*

   (a) The SS strategies constitute the unique equilibrium.

[18] The fact that SS policies cannot be supported as an equilibrium would be obvious if invoking lower-boundedness of C, C' under altruism; note that we do not invoke this assumption here.

(b) *This equilibrium is efficient for any initial conditions* P_0.

Under altruism, the equilibrium breaks down at the point where the altruistic player owns all wealth. He would give a mass transfer in this case in order to avoid zero consumption by her, which leads to utility of minus infinity for him.

Next, we study a *wealth-pooling* (WP) model in which players have no property rights over assets. We remove the restrictions $G \geq 0$ and $G' \geq 0$ from our setting, which amounts to players consuming out of a pooled asset stock $K \geq 0$. We again restrict attention to strategies that are linear in K, so players' strategies are characterized by consumption rates C_{WP} and C'_{WP} out of the common asset stock. In appendix A.7 we show that the equilibrium policies of this game are

$$C_{WP} = \frac{\rho}{1+\alpha}, \qquad C'_{WP} = \frac{\rho}{1+\alpha'}.$$

When players are selfish, i.e. $\alpha = \alpha' = 0$, they consume at the same rate ρ as in the SS case, but now out of the common assets K. This is the tragedy of the commons, which is well-known from the resource-extraction literature (e.g. fish wars). On the other hand, under perfect altruism ($\alpha = \alpha' = 1$) – and *only* in this case – consumption corresponds to the (unique) planner's solution. For $0 < \alpha + \alpha' < 2$, altruism alleviates the tragedy of the commons: Allocations are inefficient because extraction from the common stock is too high with respect to efficiency, but in a less pronounced manner than if altruism is absent. Note also that the more altruistic player will take into account to a larger extent the externalities he causes on the other player and consumes less than the less altruistic player.

We now return to our original setting with property rights to see if the wealth-pooling allocation can be sustained there as an equilibrium. It turns out that this is only the case if both agents are perfectly altruistic:

Proposition 2 (Wealth-pooling equilibrium) *Consider the wealth-pooling (WP) strategies* $C(P) = C_{WP}$ *and* $C'(P) = C'_{WP}$.

1. *The WP strategies can be sustained as an equilibrium only if* $\alpha = \alpha' = 1$.

2. *If* $\alpha = \alpha' = 1$, *then:*

   (a) *The WP strategies* $C_{WP} = C'_{WP} = \rho/2$ *are the only consumption strategies that can be sustained in equilibrium.*

(b) Transfer strategies are indeterminate, but must be such that they make WP consumption feasible: $G'(0) \geq \frac{\rho}{2}$ *and* $G(1) \geq \frac{\rho}{2}$.

(c) This equilibrium induces the unique efficient allocation.

A formal proof for proposition 2 is given in appendix A.5. Since the WP allocation gives the globally preferred and thus efficient allocation to both players under perfect altruism, point 2 of the proposition should come as no surprise. To see why the WP consumption plans cannot be supported under imperfect altruism, consider the situation where an imperfectly-altruistic player is left with the entire wealth, say $\alpha < 1$ and $P = 1$. In this case it turns out that she essentially becomes the "family dictator" and can implement her globally preferred allocation, which implies providing less consumption to him than his WP plan would stipulate. So the equilibrium breaks down at this point.

3.2 Characterization of regions

As alluded to before, there are no equilibria consisting of a single type of region when altruism is imperfect (see appendix A.9 for the formal result). We thus turn to studying equilibria which consist of a small number of regions. Section 3.3 will then show a class of such patched equilibria.

An exhaustive listing of the various types of candidate regions is as follows: No-transfer (NT) regions: No player gives transfers; Flow-transfer (FT) regions: Transfers of the flow type occur; Mass-transfer (MT) regions: A mass transfer is given by one player. Furthermore, there are the following two important special types of regions: Self-sufficient (SS) regions, where policies are equal to the self-sufficient ones (a special kind of NT region); and wealth-pooling (WP) regions, where both players' consumption is locally given by WP consumption (a special kind of FT region or, under special circumstances, a NT region). Within a region, the value functions and policy functions are assumed to be continuously differentiable. On the boundaries, however, policy functions may be discontinuous and value functions may have kinks. Value functions still have to be continuous at the boundaries, as we show in appendix A.4. One of our technical contributions is that we study best-responding on non-smooth boundaries and in mass-transfer regions, see appendix A.6.

The key results are the following (for a formal characterization of the regions we refer the reader to appendix A.8):

1. NT regions are characterized by the altruistic-strategic distortions in the Euler equations discussed in section 2.1. NT regions are left in finite time (almost always). If the economy stays in a NT region forever, policies must be of the SS type.

2. FT regions are characterized by the transfer-induced incentives discussed in section 2.1. FT regions are left in finite time (almost always). If the economy stays in a FT region forever, policies must be of WP type (unless one player is broke).

3. MT regions are always left immediately.

It turns out that the results on "transitoriness" paired with the sharp characterizations on SS and WP regions provide us with the boundary conditions that we need to obtain equilibria.

We now turn to two results which rely on local arguments at the important point where one player is broke.

The first result arises when a NT region borders the point where one of the players is broke and transfers flow only there, i.e. $g' > 0$ when $P = 0$ and $g = g' = 0$ on some interval $(0, P_1)$. We denote the limiting policies by $C_{lim} = \lim_{P \to 0} C(P)$ and do the same for the law of motion $\dot{P}_{lim}$. The following theorem is an analogous characterization of the Samaritan's dilemma in the infinite-horizon setting to the Samaritan's Dilemma known from two-period models. Furthermore, it provides new insights.

Theorem 1 (Party Theorem) *If $\alpha' > 0$ and assumptions 1 and 2 are satisfied, then any equilibrium where a NT-region borders $P = 0$ has the following properties:*

1. $\dot{P}_{lim} < 0$ *and* $\dot{P}_0 = 0$*: Her being broke is an absorbing state.*

2. $C(0) = \alpha' C'_{WP} = \frac{\alpha' \rho}{1+\alpha'}$*: When she is broke, his preferred allocation is played.*

3. $C_{lim} \equiv \exp\left(\frac{1-\alpha\alpha'}{1+\alpha'}\right) C(0) > C(0)$*:* (Party) *On reaching $P = 0$, her consumption path has a downward jump unless $\alpha = \alpha' = 1$.*

4. $V_P'(0) > 0$*: He strictly prefers her to be broke to her returning to be unconstrained.*

A proof is given in appendix A.5, for the formal technical assumption see A.3.

A striking feature, which at first glance seems at odds with optimizing behavior, is that the recipient's consumption path exhibits a discontinuity. The future recipient of transfers over-consumes relative to the efficient level, a phenomenon analogous to the Samaritan's dilemma[19] in a two-period model.

The intuition for why a jump in consumption is indeed optimal can already be obtained in the absence of altruism, as the following simple example demonstrates: Consider a consumer with wealth $k_0 > 0$ and no flow income. A government provides a means-tested benefit in the form of a payment g' handed out only if $k_t = 0$. For simplicity assume that $\rho = r$, which implies that the optimal consumption path has to be constant while assets are positive: $c_t = \bar{c}$ for some constant $\bar{c}$. The agent will be able to consume $\bar{c}$ over an interval $t \in [0, T(\bar{c})]$, where $T(\bar{c}) \in (0, \infty]$ is the insolvency time implied by the consumption plan.

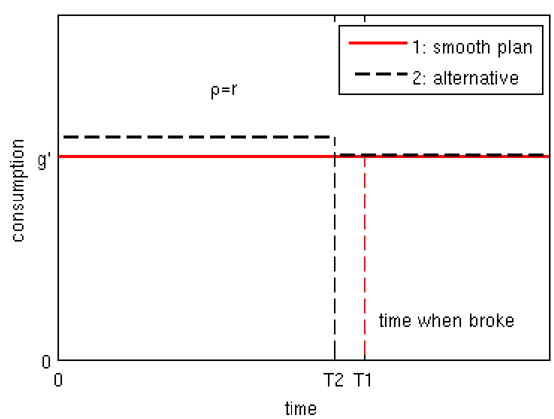

Figure 4: Discontinuity in consumption path with means-tested benefit

Consider the two consumption paths depicted in figure 4. A smooth consumption path implies $\bar{c} = g'$. As is obvious from the figure, any plan with $\bar{c} > g'$ does better than this, so a smooth consumption plan cannot be optimal. In technical terms, the

[19] The Samaritan's dilemma states the following in a two-period model: If an agent receives a transfer from an altruistic donor in period 2, then her consumption is inefficiently high in period 1, see Lindbeck & Weibull (1988).

usual theorems from control theory fail because the law of motion $\dot{k}_t = rk_t - c_t + g' I_{k_t=0}$ is discontinuous at zero. In terms of marginal cost-benefit analysis, there is an additional cost of saving here that is not present in a standard setting: When postponing bankruptcy, one diminishes the net present value of government transfers. For an altruistic recipient, the situation is similar. However, the situation is not quite as pronounced since she takes the effects of her behavior on the donor into account.

Theorem 1 offers the following three new (to the best of our knowledge) insights: First, the inefficiencies in consumption-savings decisions are not limited to the instant before receiving transfers, as is the case in the standard Samaritan's dilemma in a two-period model. The effects propagate further back in time due to the inefficiencies caused by altruistic-strategic distortions in the Euler equation (15). Second, in the time span prior to the actual transfer flow, there are also inefficiencies in the *donor's* consumption behavior, which is in contrast to the donor's Euler equation being the efficient one in the two-period model.[20] Third, we see that the party – i.e. the discontinuity of the recipient's consumption path – constitutes an inefficiency of a higher order than the inefficiencies occurring before. The inefficiencies before are characterized by consumption not growing at the efficient rate, but the path still being continuous.

The second result is that the plausible conjecture of an equilibrium where the donor lifts the recipient out of poverty with a mass transfer and both remain self-sufficient ever after proves wrong. As a matter of fact, there cannot be any equilibrium in which a mass transfer goes to the broke player.

Theorem 2 (The Prodigal-Son Dilemma: No MT when broke) *There cannot be a mass transfer by him to her at $P = 0$ (neither by her to him at $P = 1$) unless $\alpha = \alpha' = 1$.*

For the proof see section A.8.5 in the appendix.

The intuition for the result is as in the prodigal-son parable[21]: He cannot commit to not-provide transfers after having made the initial mass transfer. She would then

[20] The donor's consumption being efficient in the two-period model corresponds to the donor's consumption plan being continuous at $P = 0$ (in contrast to the recipient's).

[21] The prodigal son is one of the most famous parables from the New Testament: A wealthy father has two sons. The younger one asks to be paid out his share of the estate to start a new life in another town. He goes away and squanders all the money. After a spell of living in poverty as a swineherd, he decides to return to his father and to become a servant at his estate. However, his father welcomes the lost son with great festivities, forgives him and re-instates him as an heir equal to the elder brother,

consume the transfer, come back and ask for more. Of course, if he had the ability to commit to not-give transfers, an equilibrium of this type could be supported, as pointed out in section 1.3.

We conclude this section by noting that Bergstrom (1989) has a discussion on the prodigal son. However, the author uses this term interchangeably with the Samaritan's dilemma. Our framework clearly distinguishes between these two forms of behavior: The Samaritan's dilemma refers to the final transfers that flow when the recipient is broke and the recipient's party before this; the Prodigal-Son dilemma refers to the donor's decision of not giving a large transfer in anticipation of this already in the beginning.

3.3 Tragedy-of-the-commons-type equilibrium

We now proceed to describe the only type of equilibrium that we have found in our setting (that is, unless a shock is introduced, see subsection 3.4). The equilibria are non-smooth, the sequence of regions being FT'-WP-FT. When the asset distribution is imbalanced, the poor player receives an increasing transfer schedule that gives her incentives to save herself out of poverty. The economy always winds up in a WP region in which players essentially pool their wealth. For any parameter constellation with two-sided altruism there exists a continuum of such equilibria.

Figure 5 displays one such equilibrium. When the initial asset allocation is tilted in his favor, he provides her with flow transfers (see the blue dotted line in region FT'). In order to provide incentives to her to save herself out of poverty, transfers are increasing in her wealth share. The economy moves to the right, as the solid black line, which depicts $\dot{P}$, suggests. As the solid red line shows, her consumption in FT' is lower than the donor's consumption (the solid blue line). Once the asset distribution is sufficiently balanced, the players essentially pool their assets and play the WP strategies forever. This can be seen in region WP in the middle, where both players consume the same (which is the case since $\alpha = \alpha'$ in this example).

From the right panel in the figure we see that his value function is flat throughout regions FT' and WP, so that he is indifferent between these two regimes. Indeed, as the construction of equilibrium in appendix A.10 shows, she under-consumes in FT'

who had staid hard-working at the estate the entire time, and who is understandably angry about the father's decision.

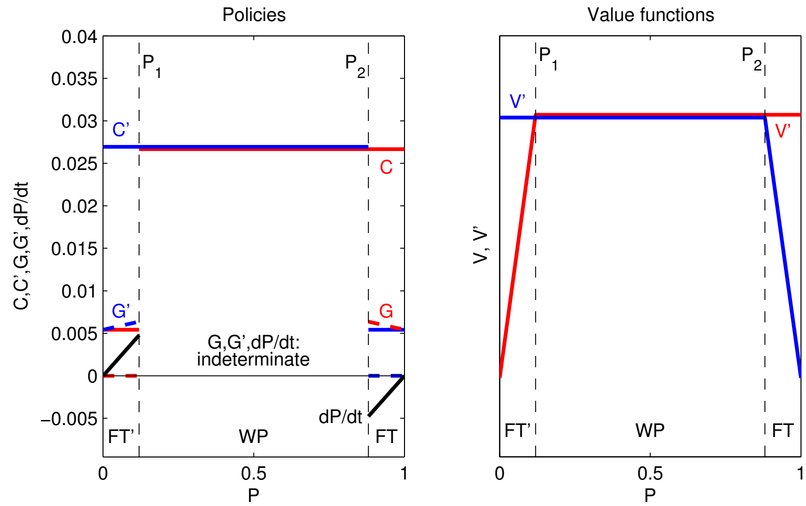

Figure 5: Tragedy-of-the-commons-type equilibrium: $\alpha = \alpha' = 0.5$, $P_1 = P_{max}(0.5, 0.5)$, $P_2 = 1 - P_{max}(0.5, 0.5)$

by *exactly* as much as she over-consumes in WP. So for him, there are no incentives to return to the FT' region once the wealth-pooling regime is reached.

There is a continuum of such equilibria for each pair (α, α') since the boundaries (P_1, P_2) between the regions can be chosen from a range of numbers. This range depends on the levels of altruism, see figure 9 in appendix A.10. There is no boundary condition on policies at the points $P \in \{0, 1\}$ since the economy is moving away from there. This means that no information is fed from the margins into the equilibrium. Due to the lack of such boundary conditions, (P_1, P_2) can be chosen on a continuum, which explains the multiplicity of equilibria. Economically speaking, the entire WP region is the unique steady state of the economy, and there are many transfer-incentive schemes that make players want to reach this region.

A surprising fact of this equilibrium is that it exists even for arbitrarily small values of α and α'. One would conjecture that a very rich player prefers self-sufficiency to the WP allocation. But this is not true any more when the poor agent's wealth becomes very low. Since there is an Inada condition on the other player's consumption in the utility function, players will avoid zero consumption by the other player at any cost and so prefer WP to SS at some point. In this sense, there is a discontinuity of the player's criterion at $\alpha = 0$.

Also, one might wonder why the rich player is not able to implement his preferred allocation at $P = 0$. What would happen if he gave her exactly the amount that he desires her to consume, namely, $\alpha' C'_{WP}$? The answer is that, because she is saving at $P = 0$, he is not controlling her consumption. This is unlike in the case of the party theorem, where she is constrained and the donor has the power to implement his preferred allocation. If he increased G', she would just save the additional amount provided, which would not make him better off. After all, his value function is flat on FT'.

The following theorem formally states this subsection's result; for the construction of this equilibrium and the characterization of the equilibrium policies we refer the reader to appendix A.10.

Theorem 3 (Continuum of tragedy-of-the-commons-type equilibria) *If and only if $\alpha > 0$ and $\alpha' > 0$ there exists a continuum of equilibria of the following type:*

1. *He gives transfers on $[0, P_1)$.*

2. *There is a WP-region $[P_1, P_2]$.*

3. *She gives transfers on $(P_2, 1]$.*

We have $P_1 \in (0, P_{max}(\alpha, \alpha')]$ and $P_2 \in [1 - P_{max}(\alpha', \alpha), 1)$, where $P_{max}(\cdot)$ is given in (33).

This type of equilibrium has, to the best of our knowledge, not been found in the existing literature on altruism. In the WP region, an inefficiency occurs that is akin to the tragedy of the commons, which was discussed in section 3.1. An interesting feature of this equilibrium is that a tragedy of the commons occurs despite property rights being intact. An additional twist is that there is also *under*-consumption by the poor agent in the FT-region, a feature not known from finite-horizon settings.

A problematic point with this equilibrium is that it predicts transfers to flow primarily to recipients who are not liquidity-constrained. But empirical evidence suggests that transfers typically flow to liquidity-constrained individuals.

Furthermore, from a theoretical point of view, this equilibrium is unstable in the following three ways: First, it breaks down if we allow agents to give in-kind transfers (i.e. transfers that cannot be saved but have to be consumed immediately): Returning to the above example, if he had access to this instrument, he would choose to give an in-kind transfer of $\alpha' C'_{WP}$ instead of monetary transfers at $P = 0$. This would lead to his globally-preferred allocation being played, so it would clearly be a profitable deviation. She would always consume the in-kind transfer immediately – it is clearly not optimal to let it rot. Second, value functions must have a kink to support the consumption-switching by the poor player. This kink would disappear when shocks in the form of Brownian motion were introduced into the setting, and the equilibrium would likely disappear. Finally, this equilibrium would break down in a finite-horizon setting: In the final period imperfectly-altruistic agents would not play the WP allocation. Instead, they would withhold transfers since their own consumption provides them with a higher value than obtained from the other's consumption.

In subsection 3.4 we will see that the transfer-when-constrained equilibrium with a shock is not vulnerable to these objections.

3.4 Transfer-when-constrained equilibrium

We now turn to the equilibrium candidate that is empirically most plausible: As mentioned in the introduction, inter-vivos transfers tend to flow from well-off to less-well-off family members, and, in particular, when the recipient family member is liquidity-constrained. In our setting, agents are only constrained at the points $P \in \{0, 1\}$. The party theorem thus suggests an equilibrium in which immiseration of the poorer agent occurs and the richer agent holds back transfers until he owns all assets. Transfers are delayed until the recipient is constrained since then the donor has control over the recipient's consumption behavior and can implement her preferred allocation.

The spirit of this transfer-when-constrained structure is actually very similar to a type of equilibrium that has been studied in a two-period setting: Lindbeck & Weibull (1988), among others, show that for certain initial endowments an equilibrium exists where transfers are only given in the second period. The recipient's savings are inefficiently low in the first period, i.e. there is a Samaritan's Dilemma. In the second period, as in the static altruism setting, transfers only flow in regions where one player is poor relative to the other.[22]

The key message of this section is that a transfer-when-constrained equilibrium

[22] Note that Lindbeck & Weibull (1988) do not consider the possibility of transfers in the first period, which makes their framework more restrictive than ours.

does not exist, at least when restricting attention to the two simplest conceivable cases in a deterministic setting. However, once a shock is introduced into the setting, the equilibrium does exist and is actually the unique equilibrium that is found computationally by backward induction in a finite game.[23]

We will first consider the possibility that there is an equilibrium consisting of a single NT region. As for the special case of a SS equilibrium, we already saw in section 3.1 that it only exists if $\alpha = \alpha' = 0$. With altruism, transfers could still flow at the points $P \in \{0, 1\}$, being absent otherwise. In this case, the party theorem gives us four boundary conditions for consumption policies: (C_{lim}, C'_{lim}) at $P = 0$ and $P = 1$. However, as seen in section A.8.1, there are only two ODEs that have to be fulfilled on $\mathcal{P}_{NT} = (0, 1)$. So we cannot expect a solution to exist generically. Indeed, numerical calculations show that such an equilibrium does not obtain, at least in the case of symmetric altruism.

The economic reason for why this equilibrium does not exist is that NT regimes are always transitory unless they are SS. Therefore, immiseration must occur: One agent will (almost) always end up broke in the end, so that $P \in \{0, 1\}$ constitute the only two stable steady states of the economy. However, it is not clear to which of these the economy should go from any given P_0. This creates a strong tension. One can imagine that each agent wants to be the one who over-consumes and is given transfers in the end. This tension cannot be resolved unless a shock is introduced, a scenario we will study below.

However, it is still conceivable that equilibria exist in which two NT regions enclose a third region in the middle. This middle region can only be of NT, SS or WP type – recall that we are restricting attention to transfer-when-constrained equilibria in this section. We will not study more complicated cases in which NT regions enclose more than one region.

First consider the cases in which the middle region is of SS- or WP-type. These

[23] The curious reader might ask her/himself why the transfer-when-constrained equilibrium does not obtain by backward induction in a finite game. After all, section 2 pointed out that equilibria in stage games are unique. However, this statement was only made for smooth regions. When solving backward in time, it is not obvious how the boundary between two regions evolves. Also, even if equilibria in stage games are unique, they may fail to settle down to a stationary equilibrium. This is indeed what happened in our numerical calculations when solving the (deterministic) model backward in time: Limit cycles occurred in which players carried out "transfer wars" in the region between immiseration and SS.

two cases are special since the value functions are pinned down in these regions. This is intimately related to the fact that they are (or can be, in the case of WP) absorbing. We try to construct the equilibrium as follows: The party theorem tells us the limiting consumption policies of the NT regions. We can then solve the ODEs (28) and (17) for NT regions as explained in section A.8.1 in P and check if one of the following two results occur: The value functions converge at some free boundary to 1) the SS value functions or 2) the WP value functions. There is now one free boundary, but two value-matching conditions. This suggests that such an equilibrium is extremely unlikely to exist. Indeed, we have conducted computations for the special case of symmetric altruism ($\alpha = \alpha'$) on the entire range $\alpha \in (0, 1)$ and found that none of the two convergence results occurs for any α. The intuition for the non-existence results is as in the case of one large NT region: Since there are essentially two steady states bordering each NT region, the economy might converge to either one from each point in NT; it is extremely unlikely that the values of the two possibilities are the same for both players, which leads to conflicts that cannot be resolved in a deterministic setting.

Next, consider the case of a potential NT-NT-NT equilibrium with two free boundaries. Since value functions are not pinned down in NT regions, it is more likely to find an equilibrium with this structure – we will now see why. To simplify matters, we will look for a symmetric equilibrium with $\alpha = \alpha'$ and two boundaries $(P_1, 1 - P_1)$ with $P_1 \in (0, \frac{1}{2})$ as a free parameter.

Our equilibrium-construction strategy is as follows: Given the limiting consumption policies at $P = 0$ from the party theorem, we can solve the ODEs for consumption on NT up to the boundary P_1. We then infer consumption policies on the other side of P_1. Section 2 of our supplemental paper derives the implications of value-matching on boundaries between two adjacent NT-regions and shows how to find the limiting consumption policies on one side of P_1 when knowing the limiting consumption policies on the other side. Using again the ODEs for NT regions, we proceed to solve for consumption policies up to the point $P = \frac{1}{2}$. If his and her consumption policy are equal at this point, we have found an equilibrium. If not, we vary the free boundary P_1 until consumption policies are equal at $P = \frac{1}{2}$.[24] This amounts

[24]Note that $C(\frac{1}{2}) = C'(\frac{1}{2})$ implies $V_P(\frac{1}{2}) = -V'_P(\frac{1}{2})$, and since then $\dot{P} = 0$ at $P = \frac{1}{2}$ it also implies $V(\frac{1}{2}) = V'(\frac{1}{2})$.

to the problem being exactly identified: We are varying one free parameter P_1 to meet one criterion at $P = \frac{1}{2}$.

Indeed, using this procedure we can find symmetric value functions and consumption policies for each $\alpha \in (0,1)$ that satisfy the HJBs and Euler equations inside the regions and that are consistent with value-matching at P_1, see figure 6 for an example.[25] The figure shows the consumption rates (C, C') in the upper-left and

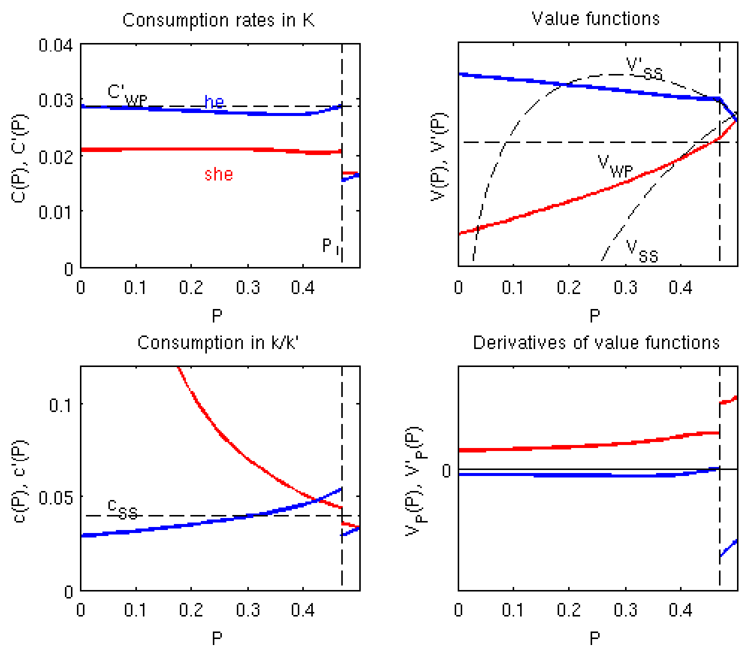

Figure 6: Symmetric equilibrium candidate: Transfers only in bankruptcy ($\alpha = \alpha' = 0.4, \rho = 0.04$).

the value functions (V, V') in the upper-right panel. The vertical dashed line represents the boundary P_1. We see that value-matching is fulfilled at P_1. In the lower left panel, we have plotted players' consumption rates (c, c') out of their own assets (k, k'). As mentioned before, $\dot{P} > 0$ if and only if $c' > c$ in NT regions. So the economy steers towards P_1 locally from both sides, so P_1 becomes an additional steady state. Again, we have the problem that there are "too many steady states" to which the economy can go. However, this time the technical reason for ruling out the equilibrium is different from the cases before: Corollary 1 in the supplemental material

[25]We found that for values of α above 0.4, the transfer motive becomes positive just left of P_1, which is a second reason why this equilibrium does not exist for high values of α.

says that there cannot be mutually best-responding policies at P_1 if it is an attracting boundary.

The technical reason for why there cannot be steady-state supporting policies on the boundary is the following: If players used the right-hand-side strategies at P_1, then the economy would move to the left of P_1. But on the left side the marginal value of assets is different from the right side, as is apparent from the kinks in the value functions. Therefore, the right-hand-side policies cannot be optimal. The same argument applies to the left-hand side policies. Finally, it is also possible to rule out any other policy combinations at P_1 as equilibria.[26]

The economic intuition for why the equilibrium breaks down at P_1 is the following: She (the player with the locally convex value function) wants to steer the economy away from P_1 once the boundary is left. She does not bear all the downside consequences of consuming a lot: If she is profligate and becomes poor, he will provide for her giving transfers in the end. If she is frugal at P_1, however, she will not receive transfers and thus reap all the benefits from savings herself. For him, the situation is exactly the opposite: He is locally risk-averse (V' is locally concave) and tries to contain the economy at the boundary. If she consumes a lot, he does not want to be the nice guy who is frugal, watches her party and then pays transfers in the end, so he prefers to also be profligate. This steers the economy back to P_1. If she is frugal, he also has incentives to be frugal since he will never have to give transfers.

If players had lotteries (or risky assets) at their disposition, then the locally risk-loving agent would make use of these at P_1.[27] This cannot be "counter-acted" in any way by the risk-averse agent, unlike in the deterministic case, and enable the risk-loving agent to steer the economy away from P_1. We will now see how the introduction of a shock into the setting does indeed resolve the tensions: Chance decides to which side the economy moves at the critical point.

Section 4 of our supplemental material extends the current model and adds a shock: Agents face an idiosyncratic shock to assets, which can be either interpreted as idiosyncratic savings risk or as shocks to expenditures (such as costs of house

[26] We can also show that there is no equilibrium on the boundary if we allow for mixed strategies, details are available from the authors upon request.

[27] Laitner (1988) follows this route: He introduces a full set of lotteries into an altruistic OLG setting in which generations overlap for one period in order to remove non-concavities in the value functions.

repair, medical treatment etc.). We solve for the equilibrium by iterating value functions backward in time.[28] Figure 7 shows the resulting equilibrium.

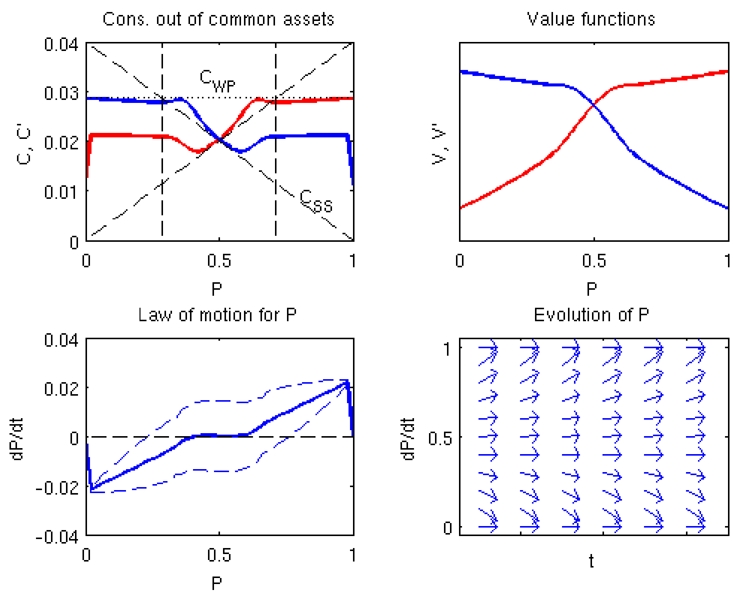

Figure 7: transfer-when-constrained equilibrium with shock ($\alpha = \alpha' = 0.4$, $\rho = 0.04$)

The upper-left panel shows players' consumption C and C'. Around the middle of the state space they are very similar to SS consumption (the diagonal dashed lines). When the asset distribution is imbalanced, the donor's consumption is close to WP consumption (the horizontal dashed lines). The vertical dashed lines indicate the value of P where transfers would start to flow in a static altruism model. The discontinuity of the poor player's consumption at the constraint highlights the third point of the party theorem.

As for the value functions in the upper-right panel, we see that risk-lovingness and risk-aversion follow the pattern already pointed out for the NT-NT-NT structure. The lower two panels show the dynamics of the state variable P. In the lower left panel, the drift of P is represented as a solid line, and 1-standard-deviation bands as

[28]The transfer-when-constrained equilibrium obtains as the unique limit. We have checked many different assumptions for the final period, and the equilibrium is stable with respect to this. Also, the shock has to be of a certain size. If this is not the case, the transfer motive turns positive at the critical point. See our follow-up paper for more on this.

dashed lines. In the neighborhood of $P = 0.5$, we see that the economy is basically stationary. However, the only absorbing states are $P \in \{0, 1\}$, which are the only points in the state space where the shocks do not influence the asset distribution. Once one player is broke, no shock can bring him away from there. The lower-right panel shows the expected evolution of P for various starting values P_0. We can see that when the initial asset distribution is imbalanced, immiseration of the poorer player is likely to occur.

The strength of this equilibrium, in addition to being empirically plausible and unique, is that it is stable with respect to the objections mentioned before in section 3.3: It can be maintained under a finite horizon, survives the introduction of a shock and the introduction of in-kind transfers.

Technically, the problem of over-identification that we faced in the deterministic setting with a single NT region disappears for the following reason: We are still left with four boundary conditions for consumption at $P \in \{0, 1\}$, but the ODEs for consumption on the NT-region are now of second instead of first order because we introduced Brownian motion (the shock). This makes the system exactly identified, so a unique equilibrium is the likely outcome. Economically speaking, the economy can now end up in either of the two steady states from (almost) any starting point, so information from both sides enters the allocation for any $P \in (0, 1)$. As described above, randomness resolves the directional conflicts that arose before.

4 Conclusions

We have studied a parsimonious dynamic model of voluntary transfers with two-sided altruism. In the deterministic setting, the only class of equilibria that we find are tragedy-of-the-commons-type equilibria. Once a shock is added to the setting, we find a transfer-when-constrained equilibrium. This equilibrium is uniquely obtained by backward induction as the unique limit of a finite game, has other desirable stability properties and is empirically plausible. For these reasons, we argue that further research should focus on this type of equilibrium.

During the development of our research agenda we have found that the tensions that we identify in our simple setting also surface in more complex settings, e.g. when introducing idiosyncratic labor-earnings risk.[29] The reason is the following: For rich-enough families, the present value of labor earnings becomes insignificant relative to capital income. For these super-rich families, players' optimization problems look very similar to the ones in our model.

In a follow-up paper, we include idiosyncratic income risk into the present setting and focus on the transfer-when-constrained equilibrium. We argue that the framework provided by our research agenda provides a building block for heterogeneous-agents models with altruistic agents. Barczyk (2011) has already successfully used this building block in a quantitative study of Ricardian equivalence in an OLG setting with imperfectly-altruistic agents.

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[29] This is at least the case as long as labor earnings are bounded and the return to savings is deterministic.

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A Appendix (for online publication only)

A.1 His HJBs, Euler equations etc.

For the convenience of the reader, in this section we state his HJBs, Euler equations etc. The equations are mirror-symmetric copies of the respective equations for her, which are given in the main text.

His HJB in k-k'-space, analogous to her (6), is:

$$\rho v' = \max_{c',g'} \left\{ u(c') + \alpha' u(c) + (rk' - c' - g' + g)v'_k + (rk - c - g + g')v'_{k'} \right\}.$$

The FOC is $u_c(c') = v'_{k'}$ and the Euler equation is, analogous to (13),

$$\frac{d}{dt}\big[u_c(c_t)\big] = (\rho - r)u_c(c') + \big[v'_k - \alpha' u_c(c)\big]c_{k'} + \big[v'_k - u_c(c')\big]g_{k'}.$$

In P-K-space, his HJB, the analogon to (25)), is

$$\begin{aligned}\rho V' =& \alpha' \ln C - C\frac{1+\alpha'}{\rho} - C(1-P)V'_P - GV_P + \\ &+ \max_{C'\geq 0}\left\{ \ln C' - C'\frac{1+\alpha'}{\rho} + C'PV'_P \right\} + \max_{G'\geq 0}\{G'V'_P\},\end{aligned}$$

the FOC is

$$\frac{1}{C'} = \frac{1+\alpha'}{\rho} - PV'_P$$

and the Euler equation, the analogon to (28), is

$$\frac{d}{dt}V_P'(T) = \big[PC' - (1-P)C - G + G'\big]V_{PP}' = \\ = \big[\rho - C - C'\big]V_P' + \left[\frac{1}{C'} - \frac{\alpha'}{C} + V_P'\right]C_P + G_P V_P'. \tag{17}$$

The ODE for his transfers in an FT region, the analogon to (29), is

$$G_P' = \frac{\alpha'}{1+\alpha'}\rho - C. \tag{18}$$

The value function in a WP region for him (see section A.8.4) is

$$\rho V'^{WP} = \ln C_{WP}' + \alpha' \ln C_{WP} - (C_{WP}' + C_{WP})\frac{1+\alpha}{\rho}.$$

A.2 Technical note on subgame-perfection

It is a hallmark of subgame perfection that the partial derivatives c_k',$c_{k'}$,g_k', $g_{k'}$ are present in the Euler equations. These derivatives tell us about the other player's "threats" in case one deviated from the equilibrium policy. These threats have to be credible in the sense of subgame perfection, i.e. agents' policies must be mutual best responses on these neighboring paths as well. But this implies that both agents HJBs (and thus Euler equations) have to hold in an entire neighborhood of the path under consideration. Indeed, our equilibrium concept requires that both agents' HJBs be fulfilled for *every* point in the state space, so we have to find a solution for the system of PDEs given by his and her HJB on the entire k-k'-plane. This is related to the fact that the usual classical calculus-of-variations arguments do not apply: We cannot construct a deviation from the optimal path that reverts to the optimal path, as becomes clear from figure 3.

Only in the special cases of $\alpha = \alpha' = 0$ and $\alpha = \alpha' = 1$ do the partial derivatives c_k' etc. disappear. Then, we can solve an ODE (and not a PDE) for consumption along the equilibrium path in the spirit of Pontryagin's maximum principle. In this case, we do not have to take into account information from neighboring equilibrium paths.

A.3 Formal statement of our technical assumptions

The following are our technical assumptions:

Assumption 1 (Consumption lower-bounded under altruism) *If $\alpha > 0$, then $C'(P) > \epsilon$ for all $P \in [0,1]$ for some $\epsilon > 0$. If $\alpha' > 0$, then $C(P) > \epsilon$ for all $P \in [0,1]$ for some $\epsilon > 0$.*

We introduce this assumption since it is very hard to rule out equilibria where one player's consumption is zero for some $P \in \mathcal{P}$. For example, if we have $C(0) = 0$, then it is a best response for him to set $G(0) = 0$ – giving her transfers would not help, since she would consume nothing anyway and both players would obtain utility of minus infinity. For her, since he is not giving any transfer, the deviation $C(0) > 0$ does not pay since she would still be left with zero consumption, so this pair of policies is consistent with equilibrium. However, this is clearly not in the spirit of the altruism framework because both players have strong incentives to avoid such situations. We thus exclude this case from our analysis.

Note that we do *not* restrict her consumption to be lower-bounded if he is not altruistic towards her. To see why, consider the case $\alpha = \alpha' = 0$: Here, self-sufficiency with her consuming $C(P) = \rho P$ is clearly an equilibrium where $C(P) \to 0$, which is reasonable since he has no incentives to help her out when $\alpha' = 0$.

The second, purely technical, assumption is:

Assumption 2 (Limit-consumption exists) *For each region $\mathcal{P}_i$, the limits of consumption on the boundaries of the region exist: $C(P_{i-1})_{lim} \equiv \lim_{P \to P_{i-1}^+} C(P)$ and analogously for $C'(P_{i-1})_{lim}$, $C(P_i)_{lim}$ and $C(P_i)_{lim}$.*

A.4 Properties of the value function

We will now prove some global properties of the value functions. The following results will later enable us to restrict the set of candidates for equilibrium.

First, and most importantly, observe that V must be weakly increasing in P. If the transfer motive was positive (i.e. $V_P < 0$) on some interval (P_0, P_1), then she could improve her value by setting a mass transfer to reach P_0 for all $P \in (P_0, P_1)$. Since a mass transfer immediately brings the economy to P_0, she obtains the value $V(P_0)$ instantaneously. Therefore $V(P) \geq V(P_0)$, which is a contradiction to what we assumed before. Formally we have:

Proposition 3 (V weakly increasing) *$V(P)$ is weakly increasing in P, and $V'(P)$ is weakly decreasing in P.*

We now make two weak technical assumptions to prove the next results: C must be lower-bounded if he is altruistic towards her, and the limits of policy functions must exist on the border of regions. See appendix A.3 for a formal statement of these assumptions. We can now show the following (for the proof, see appendix A.5):

Proposition 4 (Value functions continuous) *Both players' value functions are continuous at the boundaries between regions and thus continuous throughout.*

The intuition for the proof is the following: Suppose her value function had an upward jump at a boundary. Then she could realize large immediate gains steering the economy from the left of the kink to the right of it; note that this is always possible by being more frugal or cutting transfers. But then, the value function on the left side of the boundary cannot be discretely lower than on the right side, a contradiction.

If both players are altruistic, we can also show that the value functions are bounded (see Lemma 1 in the appendix).

Finally, we will demonstrate that the two essential parameters in our setting are α and α'). We can easily compute the equilibria for arbitrary values of (ρ, r) once we know them for one particular pair. First, note that r does not appear in the HJB (25), so the same V must solve the HJB for any value of r. As for ρ, it is not hard to show that best responses must be linear in this parameter.

Proposition 5 (Equilibrium independent of r and linear in ρ) *Let $(\tilde{\rho}, \tilde{r}) \neq (1,1)$. Then $\{C_1(\cdot)$, $C_1'(\cdot)$, $G_1(\cdot)$, $G_1'(\cdot)\}$ are supported as equilibrium strategies $(\rho, r) = (1,1)$ if and only if $\{\tilde{\rho}C_1(\cdot), \tilde{\rho}C_1'(\cdot), \tilde{\rho}G_1(\cdot), \tilde{\rho}G_1'(\cdot)\}$ are supported as equilibrium strategies for $(\rho, r) = (\tilde{\rho}, \tilde{r})$.*

For a formal proof of this statement see appendix A.5. The economic intuition for this result is the following: The interest rate r does not matter since the income and substitution effect cancel out exactly under log-utility, so savings choices are not affected by the interest rate. As for ρ, consider the following argument: Suppose we have fixed one unit of time to be a year in the model and have found an equilibrium. When changing the (nominal) time unit to one month, then we should divide the discount rate by 12 to maintain the agents' preferences the same. When also dividing all consumption (and transfer) rates by 12, we obtain the exact same allocation as before, which must of course also be an equilibrium – but now with different numbers for the discount rate and equilibrium policies.

A.5 Further results and proofs

A.5.1 Value functions bounded

Lemma 1 *Let consumption functions satisfy assumption 1. Then, if $\alpha > 0$ and $\alpha' > 0$, then there exist numbers $\underline{M} > -\infty$ and $\bar{M} < \infty$ such that $\underline{M} < V(P) < \bar{M}$ and $\underline{M} < V'(P) < \bar{M}$.*

Proof: Clearly, her value function is upper-bounded by the value of the Pareto problem where the planner puts full weight on her. To find a lower bound, notice that her flow-utility is lower bounded when we invoke assumption 1. So clearly also the value function must be lower-bounded. ■

A.5.2 Proof for proposition 4 (Value functions continuous):

Proof: Suppose that the value function is discontinuous at P_i for her. Since his policies C' and G' converge to finite positive numbers at P_i from both sides by assumptions 1 and 2, she can always choose her policy C such that $\dot{P} < 0$ (or $\dot{P} > 0$) in a neighborhood around P_i by the law of motion for P in (22)[30]. Now, if $V(P_i^+) > V(P_i^-)$ (these two denoting the right- and left-side limits at P_i), then the inequality

$$V(P_i - \epsilon) \geq [\ln(C) + \alpha \ln(C'(P_i^-)]\Delta t + e^{-\rho \Delta t} V(P_i) + o(\Delta t)$$

can be violated for some small $\epsilon > 0$ and some C. Δt is the amount of time it takes to reach P_i under the given policies, which vanishes as $\epsilon \to 0$. In other words, she could obtain a higher value than $V(P_i - \epsilon)$ by steering the economy to P_i. The case $V(P_i^+) < V(P_i^-)$ is entirely analogous. ■

A.5.3 Proof for proposition 5: Equilibrium independent of r and linear in ρ

Proof: Fix some arbitrary $\tilde{\rho} > 0$. We now write down the HJB for $\rho = \tilde{\rho}$ and for $\rho = 1$:

$$\rho V^{\tilde{\rho}} = \max_{C,G} \Big\{ \ln C + \alpha \ln C' - (C + C')\frac{1+\alpha}{\rho} + \big[(1+P)C' - PC + G' - G\big] V_P^{\tilde{\rho}} \Big\}, \tag{19}$$

$$V^1 = \max_{C,G} \Big\{ \ln C + \alpha \ln C' - (C + C')(1+\alpha) + \big[(1+P)C' - PC + G' - G\big] V_P^1 \Big\}, \tag{20}$$

where we use the super-index $\tilde{\rho}$ that this is the value function related to $\tilde{\rho}$. Since r does not show up in the value functions, we see immediately that it is irrelevant.

We now guess that the value function for arbitrary ρ is related to the value function for $\rho = 1$ as follows:[31]

$$\rho V^{\rho} = (1+\alpha) \ln \rho + V^1$$

[30] In the special case where $P_i = 0$ this must also be true since $G'(0) > 0$ by our assumption that $C(0) > 0$. Then $\dot{P} > 0$ may be achieved by setting $C < G'(0)$. A similar argument applies if $P_i = 1$.

[31] We can arrive at this guess in the same way as we obtained the guess in section A.6, when we found the form of the value function in K

Under this guess, the optimal policies for $\rho = \tilde{\rho}$ and $\rho = 1$ are related as claimed in the proposition, i.e. $C^{\rho} = \rho C^1$ etc.: Use the FOC and use the fact that $\rho V_P^{\rho} = V_P^1$. Indeed, when we use these policies on the right-hand side of (19) and the guess for the form of V^{ρ} on the left-hand side of (19), we see that (19) simplifies to (20): The two are equivalent. So $\{V^{\tilde{\rho}}, C^{\tilde{\rho}}, \dots\}$ are an equilibrium if and only if $\{V^1, C^1, \dots\}$ are (by our definition of equilibrium), which concludes the proof. ■

A.5.4 Proof for proposition 1 (SS equilibrium)

Proof: Consider first the case where at least one player is altruistic, say $\alpha > 0$. Then, she would obtain a value of minus infinity when he is broke since $C'_{SS} = 0$. So she should respond with a mass transfer at this point, which shows that self-sufficiency is not an equilibrium when $\alpha + \alpha' > 0$.

Consider now the case where $\alpha = \alpha' = 0$: Given that the other player never gives transfers, the best response is obviously to respond with zero transfers and follow the consumption rule of an SS saver. Thus the SS policies constitute an equilibrium.

Finally, we have to establish that the SS-policies are the unique equilibrium when $\alpha = \alpha' = 0$. Note that the SS policies are feasible for any initial conditions, so the value function for each player in equilibrium is lower-bounded by the SS value function. But since any SS allocation is Pareto-efficient when $\alpha = \alpha' = 0$ (see the discussion in 1.3), there does not exist any allocation that is feasible and gives at least one of the players a higher value, which concludes the proof.■

First, it should be pointed out that this proof also applies to non-Markovian strategies, so there are no tit-for-tat strategies either that could implement non-SS equilibria under selfishness. Second, it is interesting to note that the proof for uniqueness of the SS equilibrium does not go through in a stochastic setting, when potential gains from mutual insurance arise: Then there are possible Pareto improvements from risk sharing which can raise value functions above the SS levels.

A.5.5 Proof for proposition 2 (WP equilibrium only under perfect altruism)

Proof: We first show point 1 of the proposition.

Consider first the case where $\alpha = \alpha' = 1$: For any pair of transfer strategies that make the wealth-pooling (consumption) allocation feasible, both agents are clearly best-responding since they obtain their globally preferred (feasible) allocation. So the wealth-pooling strategies can be sustained as an equilibrium in this case.

Second, consider the case where $\alpha + \alpha' < 2$ and the situation where the more-altruistic agent is bankrupt: $k = 0$. Then he should set transfers to $\alpha' C'_{WP} K$, which lowers her consumption to $\frac{\alpha'}{1+\alpha'} K < C_{WP} = \frac{1}{1+\alpha} K$ (the inequality holds since $\frac{\alpha'}{1+\alpha'} < \frac{1}{2} \leq \frac{1}{1+\alpha}$). This implements his globally preferred allocation, which dominates the wealth-pooling outcome. So we have found a profitable deviation and wealth-pooling cannot be sustained in equilibrium.

We now proceed to show point 2 of the proposition. Since the criteria by which players rank allocations (given in (3) and (4)) coincide in the case of perfect altruism, it follows that $V(P) = V'(P)$ for all P. But since $V(\cdot)$ is weakly increasing in P and $V'(P)$ is weakly decreasing in P, it follows that $V(P)$ and $V'(P)$ must be constant in P. Thus $C(P) = C'(P) = \rho/2$ by the FOC (27). Transfers have to be such that these consumption policies are feasible also at $P = 0$ and $P = 1$; if this was not the case, the efficient allocation could not be reached at these points, which would make the value functions drop below the efficient level – a contradiction to the value function being constant. Transfers are indeterminate on $P \in (0, 1)$ since agents are indifferent on how wealth is distributed. ■

It is possible to show that under perfect altruism the wealth-pooling equilibrium is indeed the unique equilibrium even when attention is not restricted to$K$-linear strategies; the proof is more involved and carries no additional economic intuition, please contact the authors for details.

A.5.6 Proof for Theorem 1 (Party Theorem)

Proof: It is first convenient to rule out that $\dot{P}_0 > 0$.[32] Note that $\dot{P}_0 > 0$ clearly implies $G'(0) > 0$ and also $V'_P(0) = 0$ by his FOC for transfers. Thus, his consumption is continuous and given by $C'(0) = C'_{lim} = C'_{WP}$. Continuity of the value function (see proposition 4) tells us that $V_{lim} \equiv \lim_{P \to 0} V(P) = V(0)$, which implies by the HJB a value-matching condition, which becomes $V_P(0)\dot{P}_0 = V_P(0)\dot{P}_{lim}$ (both consumption rates drop out since they are the same on both sides: The recipient must choose $C(0) = C_{lim}$ if $\dot{P}_0 > 0$, since the relevant derivative in the Hamiltonian is the same at both $P = 0$ and slightly to the right of it). Since $V_P > 0$, this implies $\dot{P}_0 = \dot{P}_{lim} < 0$, a contradiction (If it was the case that $V_P = 0$, she would choose $C(0) = C_{WP}$, to which he would clearly respond with setting $G'(0) = \alpha' C_{WP}$, thus making it impossible for her to save. If $\alpha = \alpha' = 1$, this argument

[32] Although $\dot{P}_{lim} < 0$, this remains a possibility under our recursive equilibrium definition. The ODE for the path of P_t would not be solvable in this case, but our equilibrium definition does not require this; one could interpret the resulting equilibrium path as zig-zagging to and away from zero very quickly.

thus not apply, but then we have a unique WP equilibrium which is also covered by our proposition).

So we can proceed under the assumption that $\dot{P}_0 = 0$, which establishes point 1 of the proposition. We first note that his consumption must be continuous at zero and given by $C'_{lim} = C'(0) = C'_{WP}$ by his FOC (note that it is impossible that V'_P tends to infinity – this would mean that his consumption goes to zero, which is clearly not optimal). Now, define the function $H_{WP}(C) = \ln C - C\frac{1+\alpha}{\rho}$ and re-write her value-matching condition at $P = 0$ as

$$H_{WP}(C_0^*) = H_{WP}(C_{lim}) + \dot{P}_{lim} V_P(0).$$

If $\alpha + \alpha' < 2$, then we know that that $V_P(0) > 0$ and $\dot{P}_{lim} < 0$. Then the fact that $H_{WP}(\cdot)$ is a strictly increasing function on $[0, C_{WP}]$, this implies that $C_0^* < C_{lim}$ if $\alpha + \alpha' < 2$, which is what we term a *party* before bankruptcy.

Now define $J_{WP}(C) = \alpha' \ln C - C\frac{1+\alpha'}{\rho}$ and re-write his value-matching condition as

$$J_{WP}(C_0^*) = J_{WP}(C_{lim}) + \dot{P}_{lim} V'_P(0).$$

Since $\dot{P}_{lim} < 0$ and $V'_P(0) \leq 0$, it is clear that $\dot{P}_{lim} V'_P(0) \geq 0$ and thus $J_{WP}(C_0^*) \geq J_{WP}(C_{lim})$. Observe that $J_{WP}(\cdot)$ is strictly increasing on $(0, \alpha' C'_{WP}]$ and strictly decreasing on $[\alpha' C'_{WP}, \infty)$. By the ordering $C_0^* < C_{lim}$ from before, it must therefore be that we are to the right of $\alpha' C'_{WP}$ and that we have $C_{lim} > \alpha' C'_{WP}$. In intuitive terms, she is over-consuming, he dislikes the party and prefers the bankruptcy consumption rates to it – if this was not the case, his value-matching cannot be fulfilled. Again, we note that the reasoning always goes through as long as $\alpha + \alpha < 2$; for perfect altruism value-matching is fulfilled by the WP policies anyway.

We now will see that her optimal policy must be $C(0) = C_{lim}$[33]. The relevant terms in C in her Hamiltonian at $P = 0$ are $\ln C^* + (G'(0) - C^*)V_P(0)$, which are clearly optimized by setting $C(0) = C_{lim}$. Now, since $C_{lim} > \alpha' C'_{WP}$, he can set $G'(0) = \alpha' C'_{WP}$ to obtain his globally preferred allocation. He needs to be altruistic ($\alpha' > 0$) for this, if not the transfer is not given (and also the above reasoning fails since $V_{SS}(0) = -\infty$). Thus, $C_0^* = \min\{C(0), G'(0)\} = \alpha' C'_{WP}$, which is point 2 of the proposition.

Now, using again her value matching and the FOC for C_{lim} in (27), can find the closed-form expression for $C_{lim}(0)$ given in point 3 of the proposition. Finally, using his value-matching condition again, we see that $V'_P(0) < 0$ since $G_{WP}(C_0^*) > G_{WP}(C_{lim}(0))$, which

[33]Recall that the *policy* $C(0)$ may be different from the *realized* consumption rate $C^*(0) = \min\{C(0), G'(0)\}$

proves point 4. ■

A.5.7 Proof for Theorem 2 (the Prodigal-Son Dilemma)

Proof: By way of contradiction, suppose that he gave a mass transfer in the region $[0, P_1)$. We will first show that her optimal policy at the kink is $C(P_1) = C_{WP}$. The terms in the relevant Hamiltonian that contain C are

$$\ln C - \frac{1+\alpha}{\rho}C - (1-P)V_P C,$$

which is decreasing in V_P. Since $V_P^+(P_1) \geq V_P^-(P_1) = 0$, she obviously maximizes the Hamiltonian by going to the left and choosing $C(P_1) = C_{WP}$ (note that going left can always be achieved by choosing a large-enough G). So being profligate and counting on more transfers is always optimal.

Second, note that her "threat consumption" inside $[0, P_1)$ must be the same as on the boundary: $C(P) = C_{WP}$ for all $P \in [0, P_1)$. This is the case because our definition of best responding in MT regions in (26) says that the relevant marginal value of savings is to be taken at the point one is "shot" to.

We will now check a potential deviation by him at $P = 0$: If he sets a flow transfer $G'(0) = \frac{\alpha'}{1+\alpha'}$ when she is broke, this leads to his globally preferred allocation with value $V'_{\eta=0}$. The equilibrium strategy, which is to carry out the transfer, must at least yield this value: $V'(P_1) \geq V'_{\eta=0}$. But we also have $V'(P_1) \leq V'_{\eta=0}$, since $V'_{\eta=0}$ is the highest-possible value attainable by any allocation. So it must be that $V'(P_1) = V'_{\eta=0}$.

But note that $V'_{\eta=0}$ can only be attained if his preferred consumption rates $C = \frac{\alpha'\rho}{1+\alpha'}$ and $C' = C'_{WP}$ are played forever, since his preferred allocation is unique. But this is at odds with her playing C_{WP} at P_1, since $C_{WP} = \frac{\rho}{1+\alpha} \geq \frac{\rho}{2} \geq \frac{\rho\alpha'}{1+\alpha'}$ with one of the inequalities being strict since $\alpha + \alpha' < 2$. Formally, we have $V'_{\eta=1} > V'(P_1) = V'_{WP}$, where the second equality follows from his HJB. This is a contradiction. ■

However, note that a MT region is still conceivable if it does not include the points 0 and 1: For example, we could connect two WP regions by a MT region.

A.6 Exploiting homogeneity

P and K are as defined in (16). The laws of motion for P and K as a function of policies are

$$\dot{K} = \left[r - C - C'\right] K \tag{21}$$
$$\dot{P} = -(1-P)C + PC' + \left[G' - G\right]. \tag{22}$$

Her HJB in (P, K)-space is given by

$$\rho\tilde{V} = \max_{C\geq 0, G\geq 0} \Big\{ \ln(CK) + \alpha \ln(C'K) + (r - C - C')K\tilde{V}_K + \\ + \left[PC' - C(1-P) + G' - G\right]\tilde{V}_P \Big\}. \tag{23}$$

It is now natural to conjecture that the value function $\tilde{V}$ is additively separable in P and K and logarithmic in K. Indeed, we show in section 1 of our supplemental material that K-linear strategies imply it must be of the form

$$\tilde{V}(P, K) = \frac{1+\alpha}{\rho}\left[\frac{r}{\rho} + \ln(K)\right] + V(P). \tag{24}$$

The terms in K are known and represent the *value of common assets* K to her. $V(\cdot)$ remains to be determined; it represents how she feels about the distribution of assets between him and her.

Using (24) in her HJB (23) now enables us to drop terms in K and write

$$\rho V = \alpha \ln C' - C'\frac{1+\alpha}{\rho} + PC'V_P + G'V_P + \\ + \max_{C\geq 0}\left\{\ln C - C\frac{1+\alpha}{\rho} - C(1-P)V_P\right\} + \max_{G\geq 0}\{-GV_P\}. \tag{25}$$

This HJB is an ODE in P, which is an important simplification with regard to (k, k') state variables, in which the HJB (12) was a PDE.

It is now also less cumbersome to state what best-responding means at boundaries between regions since there are only two possible directions to take in the state space. Formally, our strategy space consists of piecewise C_1 functions, where the function values at the discontinuity points may differ from both the left- and right-hand-side limit.

To be specific, we need to introduce some more notation: The state space $\mathcal{P} = [0, 1]$ is divided into finitely many *regions* (intervals) $\mathcal{P}_i = (P_{i-1}, P_i)$, $i = 1, \ldots, n$, with *boundaries* $0 = P_0 < P_1 < \cdots < P_{n-1} < P_n = 1$. Inside each region, the value functions $V(\cdot), V'(\cdot)$

and policy functions $C(\cdot)$, $C'(\cdot)$, $G(\cdot)$ and $G'(\cdot)$ are continuously differentiable. On the boundaries, value functions may have kinks and policy functions may be discontinuous. We also allow that policies exactly on the kink differ from their right- and left-hand side limit, i.e. it is possible that $\lim_{P \nearrow P_i} C(P) < C(P_i) < \lim_{P \searrow P_i} C(P)$, for example.[34] We will classify and characterize the different kinds of regions in section A.8.

We now specify exactly what we mean by best-responding on a boundary P_i. Let us denote flow utility by $U(C, C') = \ln C + \alpha \ln C' - (C + C')\frac{1+\alpha}{\rho}$. The law of motion is given by the function $f(P; C, G; C', G') = PC' - (1-P)C + G' - G$. Denote the left and right derivatives of V at P_i as V_P^+ and V_P^-, and fix his (candidate) equilibrium strategy $\sigma'(P_i) = (C'(P_i), G'(P_i))$ on the boundary P_i. In order to evaluate deviations by her, we need to consider the entire set of feasible tuples (C, G). Define the Hamiltonian as a function of her actions, fixing his strategy:

$$H(C, G; P_i, V_P^+, V_P^-, \sigma') = U(C, C') + \begin{cases} f(P_i; C, G; \sigma'(P_i))V_P^+ & \text{if } f(\cdot) \geq 0 \\ f(P_i; C, G; \sigma'(P_i))V_P^- & \text{if } f(\cdot) < 0 \end{cases}$$

This Hamiltonian gives us her payoff if we fix his and her actions over a short amount of time in the spirit of the Δt-problem in (5). The Hamiltonian uses the left derivative of the value function to evaluate marginal effects of policies if the respective policies steer the economy to the left, and it uses the right derivative otherwise. As mentioned before, this approach has been shown to have good stability properties in the optimal-control literature.

The second important case where we have to be specific about what we mean by the Hamiltonian is the case where she gives a mass transfer δG that catapults the state from P_t to $\bar{P}$ (so $G = \bar{P} - P_t$). The Δt-problem (5) tells us that we have to evaluate both the consumption and the transfer policy at the current state P_t. So the evolution of the state is given by

$$P_{t+\Delta t} = \underbrace{P_t + G}_{=\bar{P}} + \big[P_t C'(P_t) - (1 - P_t)C\big]\Delta t,$$

and flow utility is $U(C, C'(P_t))$. When taking first-order approximations of the value func-

[34]In this sense, our setting allows for a larger set of strategies than is usually considered in the differential-games literature. Usually, strategies are restricted to be such that the law of motion is Lipschitz-continuous in order to ensure existence and uniqueness of the ODE for the state. Note that our equilibrium concept, which is based on best-responding in the sense of (26), makes no reference to the path of P and the ODE for it. It can happen that the economy tends toward P_i from both sides and that the policies at P_i imply that it moves to, say, the right side at the boundary. We interpret this seemingly incoherent behavior as “jittering” around the boundary; we give examples where this concept gives us satisfactory equilibria in a note, which is available on request from the authors.

tion at $t + \Delta t$ to evaluate the marginal effects of consumption on assets, we obviously have to take these at the *new* state $\bar{P}$ (recall the lollipop example from section 1.2):

$$V_{t+\Delta t} = V(\bar{P}) + \left[P_t C'(P_t) - (1-P_t)C\right]\Delta t V_P(\bar{P}) + o(\Delta t),$$

where again $V_P(\bar{P})$ has to be taken in the appropriate direction if the right- and left-derivative of V at $\bar{P}$ do not coincide.

To conclude this discussion, the HJB that we will work with to define best-responding is then

$$\rho V(P) = \max_{C,G} H(P; C, G; C', G'; V_P^+, V_P^-), \tag{26}$$

which coincides with (12) whenever V is differentiable at P. On a boundary P_i, the directional derivatives become important. At points where a mass transfer is given, we have to evaluate the directional derivatives at the point the economy is "shot" to, as explained above. Note that in the special case of kinks in the value function, it is not necessarily true anymore that her response is independent of his actions (as is true in smooth regions); also, there might be multiple "local" equilibria at these boundaries.

When she is unconstrained (i.e. $P > 0$) and V is smooth, her FOC for consumption is given by:

$$\frac{1}{C} = \frac{1+\alpha}{\rho} + (1-P)V_P, \quad \text{for} \quad P \in (0,1]. \tag{27}$$

This equation says that she sets the marginal utility of consumption equal to the marginal value of savings. The marginal value of savings can be decomposed into two components: First, $(1+\alpha)/\rho = \tilde{V}_K/K$ measures the (proportional) *marginal value of common assets*, i.e. the value obtained if K is increased by 1% while leaving the distribution unchanged. Second, $-V_P = (v_{k'} - v_k)K = \mu K$ measures the (proportional) *transfer motive*: $-V_P$ is the value to her when 1% of total assets K are transfered to him from her while holding total assets K unchanged.

Analogously to the k-k'-space, (25) tells us that no transfers will flow whenever $V_P > 0$ and any transfer rate is consistent with optimality when $V_P = 0$. The case $V_P < 0$ – a strictly positive transfer motive – will be ruled out in equilibrium, as will be shown in the following subsection.

Finally, by taking derivatives of her HJB (25) in smooth regions we obtain her Euler equation in P:[35]

$$\frac{d}{dt}V_P(t) = \left[PC' - (1-P)C - G + G'\right]V_{PP} =$$
$$= \left[\rho - C - C'\right]V_P + \left[\frac{1}{C} - \frac{\alpha}{C'} - V_P\right]C'_P - G'_P V_P \tag{28}$$

A.7 Solving the wealth-pooling (WP) model

Consider the WP game laid out in section 3.1. Players' strategies are characterized by consumption rates C_{WP} and C'_{WP} out of the common asset stock. It can actually be shown that such linear strategies constitute the unique equilibrium of this modified game, details are available from the authors upon request.

The law of motion for K is then

$$\dot{K}_t = rK_t - C_{WP}K_t - C'_{WP}K_t.$$

Let $V^{WP}(K)$ be her value function. Her HJB then reads:

$$\rho V^{WP} = \alpha \ln(C'_{WP}K) + (r - C'_{WP})V_K^{WP} + \max_{C \geq 0}\left\{\ln(CK) - CKV_K^{WP}\right\}.$$

The first-order condition (FOC) for C is $C_{WP}^{-1} = V_K^{WP}K$, which gives us V_K^{WP} as a function of C_{WP}. Take the derivative of the HJB with respect to K (which yields the Euler equation) and then use the FOC and symmetry between the players to establish that

A.8 Characterization of regions

We classify regions by the transfer decision; the following is an exhaustive listing of region types:

- No-transfer (NT) region: $G(P) = G'(P) = 0$ for all P in region $\mathcal{P}_i$.
- Flow-transfer (FT) region: $G(P) > 0$ for all $P \in \mathcal{P}_i$ (or equivalently $G'(P) > 0$ for him).
- Mass-transfer (MT) region: $G(P) = (P_i - P)\delta$ for all $P \in \mathcal{P}_i$ (or equivalently $G(P) = (P - P_{i-1})\delta$ for him).

[35]The term $G_P(P)V_P(P)$ vanishes in regions where $G = 0$ since $G_P = 0$ and in regions where $G > 0$ since $V_P = 0$; a similar argument shows that the term $G(P)V_{PP}(P)$ vanishes in both regions. Of course, the same arguments apply for the respective terms in the HJB for $V'(\cdot)$.

Furthermore, there are the following two important special types of regions: *Self-sufficient* (SS) regions, where policies are *locally* equal to the self-sufficient ones (a special kind of NT region); and *wealth-pooling* (WP) regions, where both players' consumption is locally given by WP consumption (a special kind of FT region or, under special conditions, a NT region). These regions may still be part of a patched equilibrium, even though – as we have seen in section 3.1 – they cannot be supported globally as an equilibrium. There is also the possibility that a NT region occurs at the margin of the state space and transfers flow only at the point where the recipient is broke. This case is treated by the party theorem (Theorem 1).

A.8.1 No-transfer (NT) regions

We first characterize a typical NT region $\mathcal{P}_{NT}$, working with the agents' Euler equations (28) and (17), in which all terms in G and G' are set to zero. This yields a system of two non-linear ODEs of first order for the consumption policies $C(\cdot)$ and $C'(\cdot)$ on $\mathcal{P}_{NT}$.

In order to learn more about the properties of such regions, it will be useful to study steady states of the economy inside a NT region. To do this, we first express the law of motion for P in terms of (c, c'), i.e. consumption policies out of agents own assets (k,k'):

$$\dot{P} = P(1 - P)(c' - c).$$

This equation says that the asset distribution evolves in her favor if and only if he consumes at a higher rate out of his own assets than she does. At a steady state P^*, we must have $c' = c = c^*$. In section 3 of our supplemental material, we show that if $c^* \neq \rho$ (i.e. the agents' policies are not SS, which is a special case) we have

$$c_P(P^*) < 0, \qquad c'_P(P^*) > 0.$$

This implies that the dynamics in the NT region must be as depicted in figure 8: At P^*, the two consumption functions c and c' intersect and the economy is in a steady state. To the right of P^*, he consumes more out of k' than she consumes out of k, so that the economy moves to the right. To the left of P^*, the opposite is true. So P^* is an unstable steady state. Furthermore, there cannot be a second steady state in $\mathcal{P}_{NT}$: If c and c' intersected again, by continuity of the consumption functions they would have to do so in a way violating the steady-state dynamics derived above. This implies that the economy moves out of an NT region from all but at most one point.[36]

[36]Technically, we need here that $c - c'$ is bound away from zero on the closure of $\mathcal{P}_{NT}$, which is fulfilled if we consider the continuation of c and c' on this closure. Also, we have to exclude

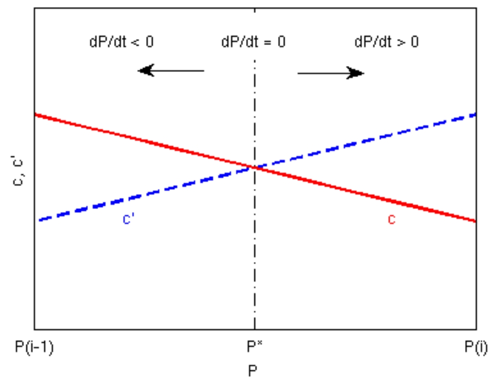


Figure 8: Dynamics in the NT region

Proposition 6 (NT-regime transitory) *Consider a no-transfer region that is not SS, in the sense that* $\{P \in \mathcal{P}_{NT} : c(P) = c'(P) = \rho\} = \emptyset$*. Then, for all but at most one point* $P^* \in \mathcal{P}_{NT}$*, the following holds:* $P_0 \in \mathcal{P}_{NT} \setminus P^*$ *implies* $P_t \notin \mathcal{P}_{NT}$ *for some* $t < \infty$*.*

This also means that if there is a smooth equilibrium that consists entirely of NT, i.e. $\mathcal{P}_{NT} = (0, 1)$, then one agent must own all assets in the economy at some point (for all but at most one initial condition P_0). We will see whether such an equilibrium can be supported in section 3.4.

We now turn our attention to the case where an NT region borders the state where she is broke: $\mathcal{P}_{NT} = (0, P_1)$. Because we have assumed that her consumption is bounded away from zero if $\alpha' > 0$ (see assumption 1), it must be that he gives transfers when she is broke: $G(0) > 0$. Since in this case her income tends to zero, the economy must also move towards zero, i.e. $\dot{P}_{lim} \equiv \lim_{P \to 0} \dot{P} < 0$; thus, her being broke must be an absorbing state. However, there are further implications.

A.8.2 Self-sufficient (SS) regions

A self-sufficient (SS) region $\mathcal{P}_{SS}$ is one in which policies are equal to the SS policies C_{SS} and C'_{SS} defined in 3.1. This type of region is special because it is the only NT region that can be absorbing: The law of motion is $\dot{P} = 0$ for all $P \in \mathcal{P}_{SS}$, as we can see from (22).

Since all points are steady states, we can derive the value functions on $\mathcal{P}_{SS}$. The HJBs

the case that consumption rates approach the SS consumption rates at the boundaries of the region, which is ensured by the reasoning in section 3.1 of our supplemental material. Studying this case is mathematically more involved since we have to deal with a singularity in Euler equations.

are

$$\rho V^{(SS)} = (1+\alpha)(\ln\rho - 1) + \ln P + \alpha \ln(1-P),$$
$$\rho V'^{(SS)} = (1+\alpha')(\ln\rho - 1) + \ln(1-P) + \alpha' \ln P.$$

Taking the derivative in P gives us

$$\rho V_P^{(SS)} = \frac{1}{P} - \frac{\alpha}{1+P}, \qquad \rho V_P'^{(SS)} = \frac{-1}{1-P} + \frac{\alpha'}{P}.$$

In this case, the Euler equations and the FOCs contain the same information. Furthermore, $G = G' = 0$ implies the inequalities $V_P^{(SS)} \geq 0$ and $V_P'^{(SS)} \leq 0$, from which we obtain the restriction

$$P \in \left[\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}\right] \qquad \text{for all } P \in \mathcal{P}_{SS}.$$

Thus, any SS region has to be contained in this interval. The intuition for this result is very simple: If one player becomes too poor, the marginal utility of helping the other out becomes higher than the marginal utility of her own consumption. Observe that the interval corresponds to the region in the static altruism model with log-utility in which transfers are zero. Finally, it shrinks to zero as altruism increases, and extends over the entire state space as the parameter values approach the selfish ones.

The results are summarized in the following proposition:

Proposition 7 (SS region absorbing; bounds) *For any SS region $\mathcal{P}_{SS}$, we have $\mathcal{P}_{SS} \subseteq [\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}]$, which is the region in which agents stay SS in the static altruism model with log-utility. Also, $P_t = P_0$ whenever $P_0 \in \mathcal{P}_{SS}$.*

Given our results on NT and SS regions, we can make an interesting observation regarding patching together possible equilibria: If agents do not give transfers to each other from some initial point P_0 on, then the path P_t must end up in an SS region, since NT regions are transitory (that is, unless they started exactly from an unstable steady state inside an NT region).

A.8.3 Flow-transfer (FT) regions

In the second class of regions, $\mathcal{P}_{FT}$, transfers are given in the form of a flow. As we will see now, in such regions players' consumption rates must be constant, transfers are governed by a simple ODE, and FT regions must be transitory.

Consider an open interval $\mathcal{P}_{FT}$ on which she gives transfers, that is, $G > 0$ for all $P \in \mathcal{P}_{FT}$; he might or might not give (flow) transfers. Obviously, $G > 0$ implies $V_P = 0$. Her

FOC (27) then says that her consumption throughout $\mathcal{P}_{FT}$ has to be $C = C_{WP}$. Furthermore, her HJB (25) implies that *his* consumption C' must also be constant on $\mathcal{P}_{FT}$. Otherwise, if C' would vary, so would the function $\alpha \ln C' - C'[(1+\alpha)/\rho - PV_P]$, violating her HJB (note that all other terms of her HJB are constant in P).[37]

Now, his Euler equation (17) says that

$$V'_P(\rho - C_{WP} - C' + G_P) = 0.$$

Observe that G_P enters his Euler equation with the same sign as does ρ (impatience). Thus, setting transfers *increasing* in P, creates a *disincentive* to save. On the flip side, setting transfers *decreasing* in P, i.e. *increasing* in his wealth share, provides *incentives* to save. As intuition suggests, rewarding thrift induces savings.

His Euler equation leads to the following two relevant cases:

1. $V'_P = 0$: Then, $C = C' = C_{WP}$ on P and transfers are indeterminate – we might indeed have $G' > 0$.

2. $V'_P < 0$: His transfers are zero throughout $\mathcal{P}_{FT}$.

In the first case $\mathcal{P}_{FT}$ would be a WP region, a region that we will study separately below in A.8.4.

From now on, we will refer to the second case as FT-regions, that is, a region where only one player gives transfers. His Euler equation then says

$$G_P = C_{WP} + C' - \rho = C' - \underbrace{\frac{\alpha}{1+\alpha}\rho}_{=\alpha C_{WP}}. \tag{29}$$

Think of it in the following way: It tells her the G_P that is needed to induce some consumption rate C' for him. The lower the recipient's consumption rate the donor wants to induce, the lower G_P has to be, i.e. the more she must make transfers increasing in *his* wealth share to provide strong-enough incentives to save.

Also, note that the last expression in (29) involves αC_{WP}, which is the consumption rate that she would choose for him, if she could do so. If she wants him to consume this desired amount, she must make transfers invariant in his wealth. If she makes them increasing in his assets (i.e. $G_P < 0$), then he will consume below this desired rate. This under-consumption will actually be part of the class of equilibria we find in section 3.3.

[37]We would arrive at the same conclusion by studying her Euler equation.

Furthermore, his Euler equation (29) is a very simple ODE for $G(\cdot)$ – it tells us that the slope of $G(\cdot)$ is constant on $\mathcal{P}_{FT}$. Since C and C' are constant on $\mathcal{P}_{FT}$, this is the only ODE we have to solve when provided with boundary values $G(P_i)$ and $C'(P_i)$ to determine policies throughout $\mathcal{P}_{FT}$

Finally, we state the following result which tells us that FT regions are always transitory (except for points with measure zero on the state space), which seems intuitive:

Proposition 8 (FT-regime transitory unless WP) *Consider a flow-transfer region $\mathcal{P}_{FT}$ that is not of wealth-pooling type, i.e. $C = C_{WP}$ and $C' \neq C'_{WP}$. Then, for all but at most one point $P^* \in \mathcal{P}_{FT}$ the following holds: $P_0 \in \mathcal{P}_{FT} \setminus P^*$ implies $P_t \notin \mathcal{P}_{FT}$ for some $t < \infty$.*

Proof: Using the results from above for the law of motion $\dot{P}$ on $\mathcal{P}_{FT}$, we obtain that the law of motion is a linear function of P on $\mathcal{P}_{FT}$:

$$\frac{d}{dP}\dot{P} = 2(C_{WP} + C') - \rho = const.$$

Note that if $\dot{P} < 0$ or $\dot{P} > 0$ throughout $\mathcal{P}_{FT}$, the region is obviously left. Also, if $\frac{d}{dP}\dot{P} > 0$ and the economy has a steady state for some P^* in the region, the dynamics are unstable and the region is left for all $P \neq P^*$. So we only have to rule out the case $\frac{d}{dP}\dot{P} \leq 0$.

If this was the case, the dynamics would be stable and we would stay inside the region starting from any initial state. Since both consumption rates are constant inside the region, consumption rates would be the same constants for all time. But this implies that both agents obtain the same continuation value starting from any $P_0 \in \mathcal{P}_{FT}$, meaning that their value functions are constant on $\mathcal{P}_{FT}$. But this would mean that $\mathcal{P}_{FT}$ is of WP-type and the recipient should play WP-consumption too, a contradiction to our assumptions. ■

A.8.4 Wealth-pooling (WP) regions

As mentioned above, we now treat the case of WP regions separately.

In a WP region, $\mathcal{P}_{WP}$, consumption policies are given by $C = C_{WP}$ and $C' = C'_{WP}$. Both value function must be flat, i.e. $V_P = V'_P = 0$ throughout $\mathcal{P}_{WP}$. Transfers are indeterminate. As was the case for SS regions, we can actually back out the value functions on $\mathcal{P}_{WP}$. Her HJB (25) tells us that this value function must be the same as in the WP model we studied in section 3.1:

$$\rho V^{(WP)} = \ln C_{WP} + \alpha \ln C'_{WP} - (C_{WP} + C'_{WP})\frac{1+\alpha}{\rho} = const,$$

We now establish that there can be at most one WP region. Suppose that there were two (or more) such regions. Observe that players value functions must be equal to the constants $V^{(WP)}$ and $V'^{(WP)}$ in both (or all) WP-regions. Since the value functions are globally monotonic by proposition 3, it must be that the value functions are also equal to the wealth-pooling level *between* both (all) WP-regions for both players. Then, we also have $V_P = V'_P = 0$ and thus $C = C_{WP}$ as well as $C' = C'_{WP}$ in between, so we can pool both (all) WP-regions into one large WP-region.[38]

Finally, since transfers are indeterminate, the dynamics $\dot{P}$ are not restricted. We conclude:

Proposition 9 (At most one WP-region: Absorbing or transitory) *There can be at most one wealth-pooling region $\mathcal{P}_{WP}$ in equilibrium. It can be absorbing or transitory.*

A.8.5 Mass-transfer regions (MT)

The last type of region that remains to be characterized is the mass-transfer (MT) type: $\mathcal{P}_{MT} = [P_{i-1}, P_i]$. It will again become apparent that allowing for mass transfers enables us to make interesting formal statements using relatively simple arguments. Our concept of best-responding over short horizons will allow us to pin down consumption policies ("threats") inside $\mathcal{P}_{MT}$ that are credible, in the spirit of subgame perfection.

Suppose she gives the mass transfer: $G(P) = (P_i - P)\delta$ for all $P \in \mathcal{P}_{MT}$. Since $\mathcal{P}_{MT}$ is left instantaneously, we have $V(P) = V(P_i)$ and $V'(P) = V'(P_i)$ for all $P \in \mathcal{P}_{MT}$; this implies $V_P = V'_P = 0$ throughout $\mathcal{P}_{MT}$. However, one may argue, that because no time is spent in $\mathcal{P}_{MT}$, the consumption policies in this region should not matter since they never enter the agents' criterion. But this kind of reasoning opens the door to the following unreasonable "suicide threats": He could threaten to eat nothing in $\mathcal{P}_{MT}$,[39] which would force her to give a mass transfer if the economy started anywhere at $P_0 \in \mathcal{P}_{MT}$. After all, if she did not give the transfer, she would obtain utility of minus infinity. But his threat is clearly not credible, i.e. it is not in the spirit of subgame perfection.

We will now see how our equilibrium concept rules out this kind of blackmailing. Equation (26) tells us that policies have to be reasonable in the following sense: If a small amount of time is spent in $\mathcal{P}_{MT}$ before the recipient is "shot" out of the region, then the recipient's

[38] There might also be a mass-transfer (MT) region between two WP regions, but this would induce the same consumption behavior over time within the large region and thus not constitute an essential difference to a pure WP region. So we would still call the large region WP in our language.

[39] Technically, we have ruled this out with assumption 1, so the reader may replace the word "nothing" by ϵ: The only important point is that ϵ must be small enough so that her utility is pressed below the levels she would obtain in the event of giving in to his threat.

consumption policy should still be optimal. In the previous example this means that he would obtain flow utility of minus infinity over Δt when carrying out the suicide threat, which he would clearly avoid.[40] In order to determine the optimal consumption plan inside $\mathcal{P}_{MT}$, players have to take into account the marginal value of assets at the point they are shot to, i.e. P_i, as explained in the lollipop example in section 1.2.

A.9 No smooth equilibria

We first show that there cannot be any equilibrium consisting of a single FT or MT region unless $\alpha = \alpha' = 1$. To see why suppose that she is the donor in such an equilibrium. Clearly, she would set $C = C_{WP}$ throughout. But he would not tolerate this when he has the power to do so. By an argument analogous to the one we saw in the prodigal-son dilemma (theorem 2), he would set the transfer lower than her WP consumption at $P = 0$; or formally: $C^*(0) = G(0) = \frac{\alpha'\rho}{1+\alpha'} \leq \frac{\rho}{2} \leq \frac{\rho}{1+\alpha} = C_{WP}$, where one of the inequalities is strict if $\alpha + \alpha' < 2$.

We are left with the possibility that there is an equilibrium consisting of a single NT region, but this case has been ruled out in section 3.4.

We have now ruled out smooth equilibria that consist of *one* type of region. However, there might still be patched equilibria that are smooth on the boundaries in the sense that policies are continuously differentiable. Note that this cannot happen between FT and NT regions since transfer functions are linear and positive on FT but zero on NT. It could however be that NT regions smoothly turn into WP regions (recall that transfers are indeterminate in those). Since there can only be one WP region (proposition 9), it must be that this WP region is then enclosed by two NT-regions which extend to the boundaries of the state space. But now again the problem of 4 boundary conditions for 2 ODEs arises, and we cannot expect to find equilibria generically. As before, we did not find such an equilibrium in numerical calculations.

To sum up, while we can be fairly sure that there exist no smooth equilibria, we cannot provide a formal proof for this statement since there might exist non-generic exceptions.

[40]Note that the equilibrium candidate of *both* players eating nothing is ruled out by our assumption 1 that consumption policies are bound away from zero.

A.10 Construction of the tragedy-of-the-commons-type equilibrium

Here we construct the FT'-WP-FT equilibrium formally. There are three regions: $\mathcal{P}_{FT'} = [0, P_1)$ (he gives flow transfers), $\mathcal{P}_{WP} = [P_1, P_2]$ (wealth-pooling) and $\mathcal{P}_{FT} = (P_2, 1]$ (she gives flow transfers). By the properties of the respective regions (see sections A.8.3 and A.8.4), we have $C = C_{WP}$ on $\mathcal{P}_{WP} \cup \mathcal{P}_{FT}$ and $C' = C'_{WP}$ on $\mathcal{P}_{FT'} \cup \mathcal{P}_{WP}$. It remains to pin down transfers in all regions and recipients' consumption in the FT regions.

First, we will determine her consumption $C_{FT'}$ on $\mathcal{P}_{FT'}$. Define $J(C) = \alpha' \ln C - C\frac{1+\alpha'}{\rho}$. Value-matching at P_1 (i.e. continuity of his value function, see lemma 4) and the HJBs imply that C_{FT} must solve

$$J(C_{FT}) = \alpha' \ln C_{FT} - C_{FT}\frac{1+\alpha'}{\rho} = \alpha' \ln C_{WP} - C_{WP}\frac{1+\alpha'}{\rho} = J(C_{WP}) \qquad (30)$$

One solution to this equation is obviously $C_{FT} = C_{WP}$. However, this would mean that FT is also a WP-region, which is inconsistent with our construction. If $\alpha' > 0$, then there is also a second solution to (30). Since $J(\cdot)$ is then concave, uniquely maximized at $\alpha' C_{WP}$ and $\lim_{C\to 0} J(C) = -\infty$, there must exist exactly one further solution $C_{FT} \in (0, \alpha' C_{WP})$, where she under-consumes in his eyes.[41] This solution is linear in ρ, as is easily verified: Let $C_{FT}(1, \alpha, \alpha')$ denote the lower solution to (30) for $\rho = 1$ and given (α, α'). Then $C_{FT}(\rho, \alpha, \alpha') = \rho C_{FT}(1, \alpha, \alpha')$ solves (30) for any $\rho > 0$. For him, we can obtain a solution $C'_{FT}(1, \alpha, \alpha')$ on $\mathcal{P}_{FT}$ in the same fashion if $\alpha > 0$.

To pin down transfers on $\mathcal{P}_{FT'}$, we now use her value-matching condition at P_1. Define $L(C) = \ln C - \frac{1+\alpha}{\rho} C$ and notice that the terms in C' cancel out since C' is continuous at P_1 to find

$$L(C_{FT}) + \left[P_1 C'_{WP} - (1 - P_1) C_{FT} + G'_1\right] V_P^{(FT')} = L(C_{WP})$$

where we define $G'_1 = \lim_{P \to P_1} G'(P)$ and where $V_P^{(FT')}$ denotes the slope of her value function on $\mathcal{P}_{FT'}$. We can now solve for G'_1:

$$G'_1 = \frac{L(C_{WP} - L(C_{FT})}{V_P^{FT'}} + C_{FT} - P_1(C_{WP} + C_{FT}).$$

Notice that G'_1 can always be made positive when letting $P_1 \to 0$: The first term on the right-hand side is positive since L is maximized at C_{WP} (and finite since $V_P^{FT'} > 0$, i.e. FT' is not WP), the second term C_{FT} is also positive and the last term can be made arbitrarily

[41] This second solution coincides with the first one in the case $\alpha = \alpha' = 1$: Then, $C_{FT'} = C_{WP}$.

small when P_1 becomes small.

We can now use her FOC for consumption (27) to eliminate the derivatives V_P and solve for transfers at the boundary P_1 as a function of C_{WP} and C_{FT} only:

$$G_1' = (1-P_1)\rho \underbrace{\frac{(1+\alpha)C_{FT}(1,\alpha,\alpha') - \ln C_{FT}(1,\alpha,\alpha') - 1 - \ln(1+\alpha)}{\frac{1}{C_{FT}(1,\alpha,\alpha')} - (1+\alpha)}}_{\equiv K(\alpha,\alpha')} + \tag{31}$$

$$+ C_{FT} - P_1(C_{WP}' + C_{FT}). \tag{32}$$

Transfers on the remainder of $\mathcal{P}_{FT'}$ can then be backed out from the ODE for transfers in FT-regions (29); note that they always must be increasing in P since the recipient's consumption is lower than the level desired by the donor.

It remains to check that the transfers we found are such that the recipient's consumption plan is feasible: We need $G'(0) \geq C_{FT'}$. First, note that for the limiting case when the boundary approaches zero ($P_1 = 0$), we always have $G_1' > C_{FT}$, as we can see from equation (30) and using the fact that $K(\alpha, \alpha') > 0$, as argued before. By continuity, there must thus always be some small-enough $P_1 > 0$ such that $G'(0) \geq C_{FT}$. Since the right-hand side of (31) is linearly decreasing in P_1, we can back out the maximally-possible P_1 that can sustain a FT-WP equilibrium from the equation

$$G'(0) = G_1'(P_{max}) - G_P' P_{max} = C_{FT}.$$

Using (31) and (18), we find that this value is independent of ρ and given by

$$P_{max}(\rho, \alpha, \alpha') = \frac{K(\alpha, \alpha')}{K(\alpha, \alpha') + 1}, \tag{33}$$

where $K(\alpha, \alpha') > 0$ is defined in (31). Figure 9 shows P_{max} as a function of (α, α'). We see that the largest range of equilibria can be supported when he is very altruistic and she is selfish. As is to be expected, the range of equilibria that can be supported becomes extremely small when the donor's (his) altruism approaches zero.

Finally, observe that best responding at the kink P_1 is unproblematic: We set both consumption policies at their WP-levels and his transfer such that the economy is steered into the WP-region. Since his value function is flat in both directions, $C'(P_1) = C_{WP}'$ and any $G'(P_1) \geq 0$ are clearly optimal. For her, $C(P_1) = C_{WP}$ is optimal since it is the global maximum of the Hamiltonian for *any* V_P, as is easily verified. Transfers inside $\mathcal{P}_{WP}$ are indeterminate. This establishes theorem 3.

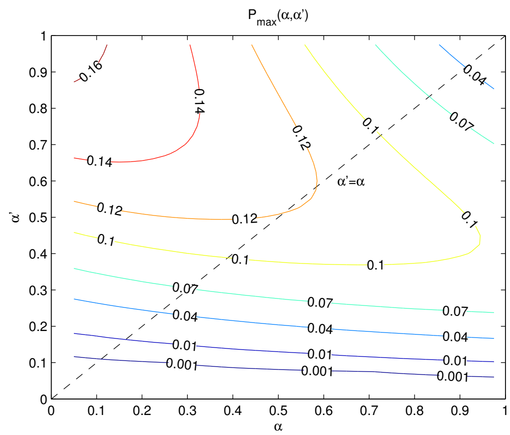


Figure 9: Maximal boundary in FT-WP-equilibrium

A noteworthy feature of this equilibrium is the following: Since transfers G' are linearly increasing in P on FT, we see from the law of motion for P in (24) that $\dot{P}$ linearly increases in P. This means that the economy is moving out of FT at increasing speed as she becomes richer. If $P_1 < P_{max}$, then the equilibrium is such that FT is left in finite time for any starting value P_0, even when $P_0 = 0$. For $P_1 = P_{max}$, however, the initial time spent in FT increases without bound as $P_0 \to 0$ and is indeed infinite when $P_0 = 0$: Then $\dot{P} = 0$ and the economy is stuck at $P = 0$ forever.

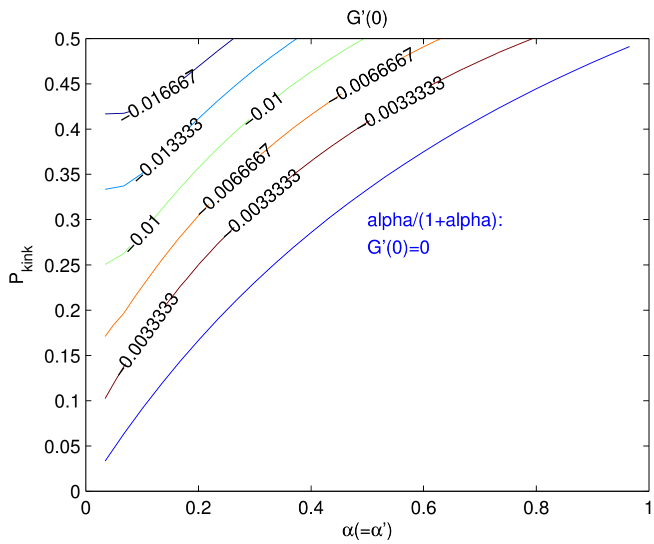
A.11 Transfer-to-SS structure

Lemma 2 *Suppose that $\alpha' > 0$. If there is an SS region $\mathcal{P}_{SS} = (\frac{\alpha'}{1+\alpha'}, P_2)$, then there cannot exist a flow-transfer region $\mathcal{P}_{FT'} = [0, \frac{\alpha'}{1+\alpha'})$ in equilibrium. Analogously, supposing that $\alpha > 0$, if there is an SS region $(P_{N-2}, \frac{1}{1+\alpha})$, then there cannot exist a flow-transfer region $\mathcal{P}_{FT} = (\frac{1}{1+\alpha}, 1]$.*

Proof: His value-matching condition at $\frac{\alpha'}{1+\alpha'}$ implies that her consumption in the FT-region is $C_{FT} = \frac{\alpha'\rho}{1+\alpha'}$. Then, her value-matching condition implies that $G'(\frac{\alpha'}{1+\alpha'}) = 0$. The ODE for transfers in a FT-region implies that $G'_P = 0$ throughout $\mathcal{P}_{FT'}$, which in turn implies $G'(0) = 0$. But this makes her consumption zero at $P = 0$, which means he is clearly not best-responding. ■

We cannot show formally that $G'(0) \leq 0$ when we set the lower boundary of $\mathcal{P}_{SS}$ higher than $\frac{\alpha'}{1+\alpha'}$. Numerical exercises show that indeed we always have $G'(0) \leq 0$ on the entire square $(\alpha, \alpha') \in (0,1)^2$.[42] Figure 10 shows the implied $G'(0)$ for the symmetric case $\alpha = \alpha' \in (0,1)$ for all possible boundaries $P_{kink} \in [\frac{\alpha}{1+\alpha}, \frac{1}{2}]$.

Figure 10: Implied $G'(0)$ with FT-SS structure



[42]Details are available from the authors upon request.